Linear Algebra - 1

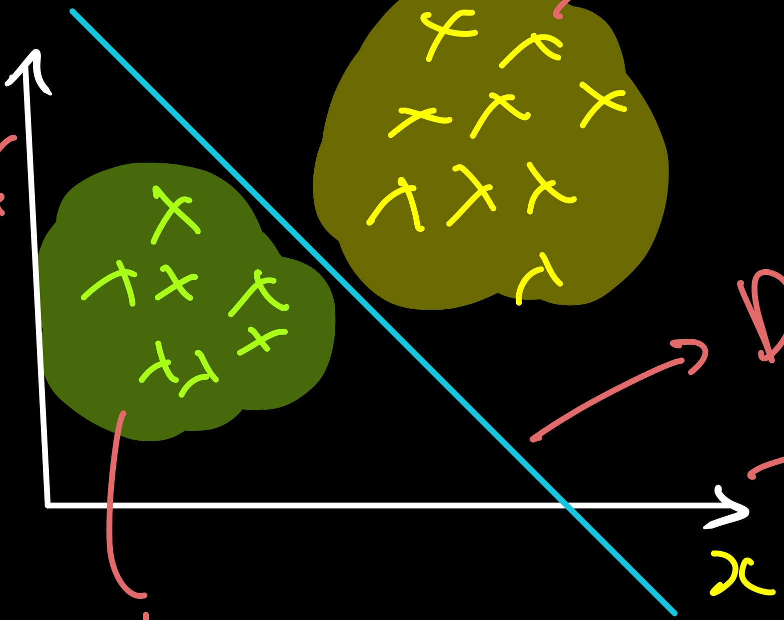
[Math for ML]

- → Notations
- → Distances
- → Angles

Recap

y

feature 2



x

Class-1

$y = mx + c$

Decision boundary

feature-1

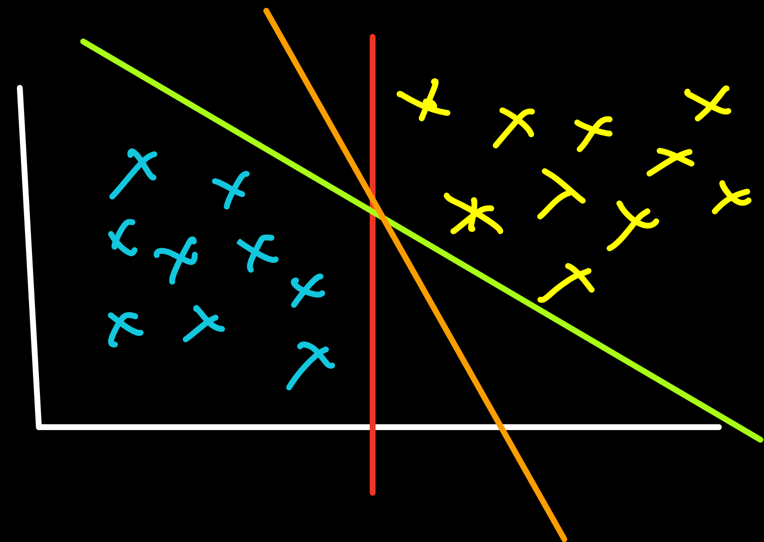
class 2

→ Classification

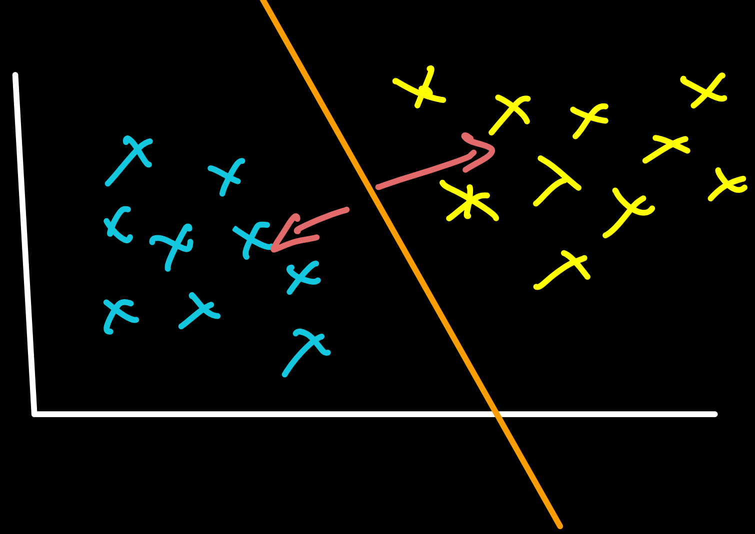
Q: Which line is the best decision boundary?

a) green   b) orange

c) red



↳ Ans **Orange**



→ Equal distances from both classes → safest

→ SSE, MSE, etc later.

Q: What next mathematical tool do we need?

→ Computing distances from a line/hyperplane

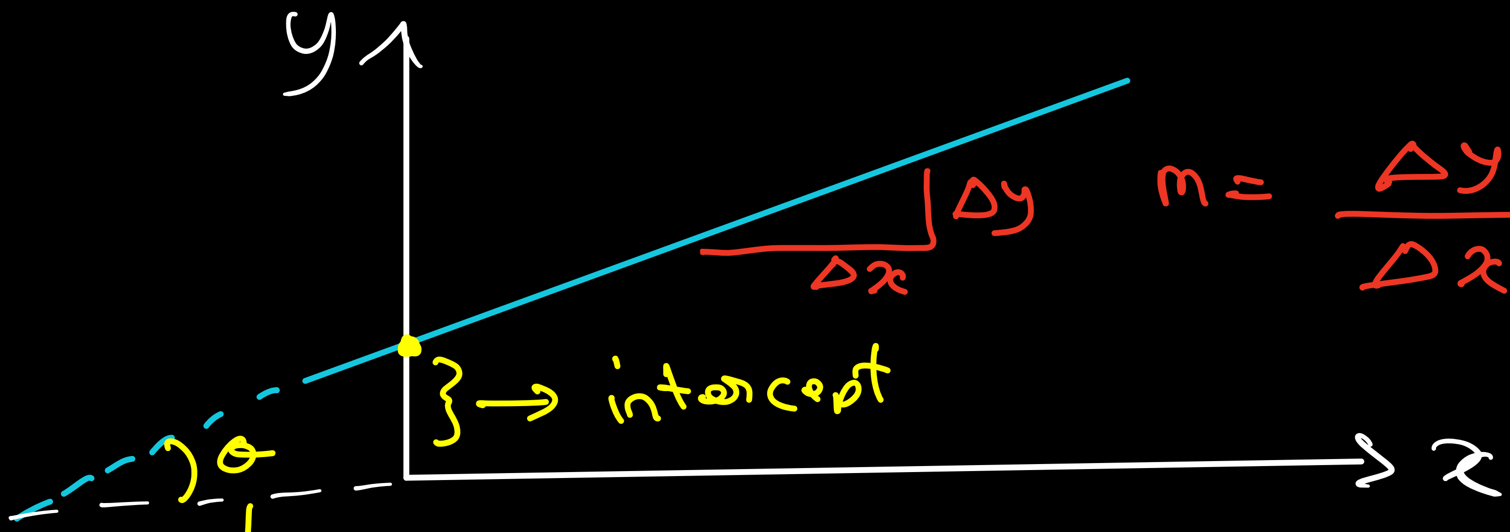
→ To calc distance we need to learn a lot of things

Equation of a Line

$$y = mx + c$$

$m \downarrow$ slope

$c \downarrow$ intercept



$$m = \frac{\Delta y}{\Delta x}$$

$\theta = \tan^{-1}(m)$; $m = \tan\theta$

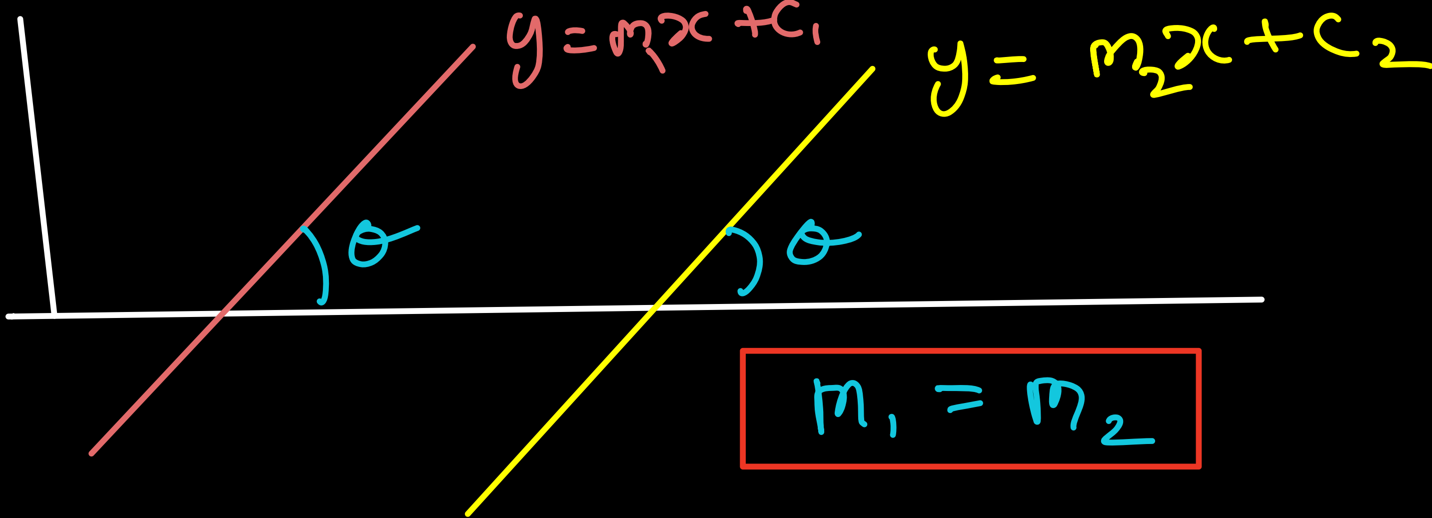
Quiz: Find the value of m if the

$y = mx + 3$ is parallel to $0.5y - x = 0$.

a) 2 ✓ b) 0.5 c) -0.5 d) 1

Parallel lines

↳ slope is same, intercept is different

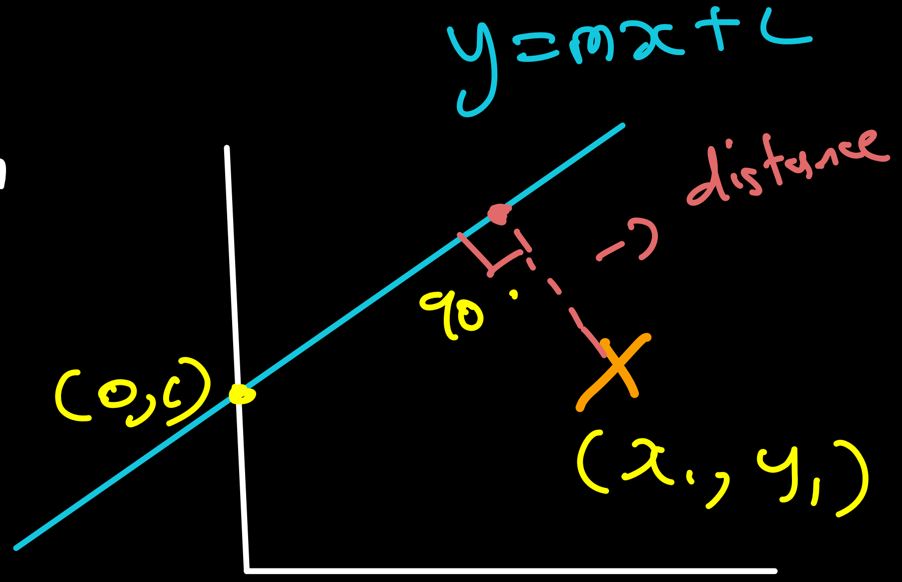


$$\boxed{m_1 = m_2}$$

$$\theta = \tan^{-1}(m)$$

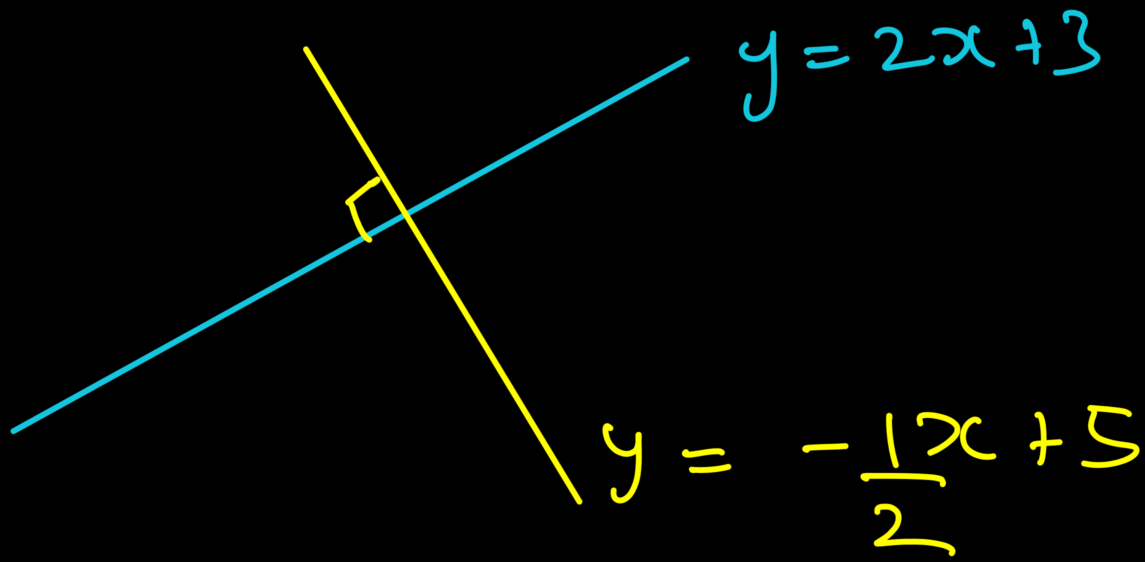
Perpendicular Lines

Distance of a point from a line is the lenght of the perpendicular line segment between line and point.



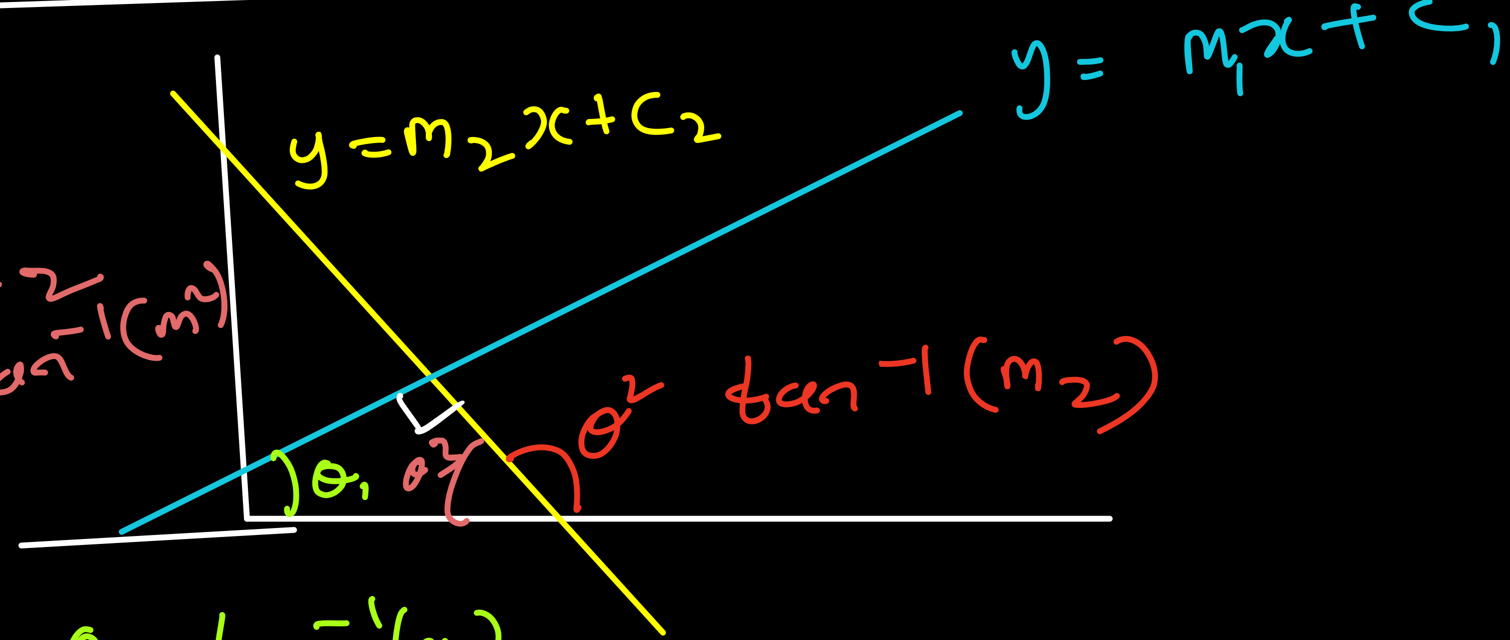
For $\perp$ lines,

$$\boxed{m_1 = -\frac{1}{m_2}}$$



→ desmos

Proof 1 [Extra]



$\theta_3 = 180 - \theta_2$
$= 180 - \tan^{-1}(m_2)$

$\theta = \tan^{-1}(m_1)$

Property: Sum of angles of a triangle is $180°$

$\therefore \theta_1 + \theta_3 + 90 = 180$

$\tan^{-1} + 180 - t$

$\therefore \tan^{-1}\left(\dfrac{m_1 + m_2}{1 + m_1 m_2}\right) = 90$

$\therefore \dfrac{m_1 + m_2}{1 + m_1 m_2} = \tan(90)$

$\therefore \dfrac{m_1 + m_2}{1 + m_1 m_2} = \infty$

$\therefore 1 + m_1 m_2 = 0$

$\therefore \boxed{m_1 m_2 = -1}$
$\therefore \boxed{m_1 = -1/m_2}$

Changing the form:

1. $mx + c = y$

   $mx - 1y + c = 0$

2. $ax + by + c = 0$ ← general form of a line

Both 1 and 2 are important

1. slope $= m$

   intercept $= c$

2. $ax + by + c = 0$

   $by = -ax - c$

$$y = \boxed{\frac{-a}{b}}x \; \boxed{\frac{-c}{b}}$$

↓ slope ↓ intercept

Quiz! What is the slope and intercept of:

$$3x + 2y + 5 = 0$$

a) $-3, 5/2$ b) $3, 5$ c) $-\frac{5}{2}, -\frac{3}{2}$ d) $-\frac{3}{2}, -\frac{5}{2}$

$$ax + by + c = 0$$

$$w_1 x + w_2 y + w_0 = 0$$

Eq^n of a plane

$$\downarrow$$

$$w_1 x + w_2 y + w_3 z + w_0 = 0$$

coefficients / weights

$$\downarrow$$

features

3 $w_1 x_1 + w_2 x_2 + w_3 x_3 \dots + w_n x_n + w_0 = 0$

↳ General Eqn of a hyperplane

This is too long, let's make it shorter.

$$[w_1, w_2, w_3 \ldots w_n] \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + w_0 = 0$$

$1 \times n$ $\quad$ $n \times 1$

where,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}, \quad \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \vec{w}, \quad w^0, \quad 0$$

Vector, Vector, scalar, scalar

$$\vec{x} \in R^n, \quad \vec{w} \in R^n, \quad w^0 \in R^1$$

R = real number $\rightarrow (-\infty, \infty)$

Hence,

$$\vec{w}^T \vec{x} + w_0 = 0$$

$\leftarrow$ Vector form of hyperplane

Transpose

Note: Conventionally vectors → vertical

For any vector $\vec{V} \to \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{1 \times n}$, $\vec{V}^T = \underset{n \times 1}{[v_1, v_2 \ldots]}$

Recap →

④ $w^T x + w_0 = 0$ ← Vector form

IS EQUAL TO

$\underbrace{w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots w_n x_n}_{\text{Dot product (inner product)}} + w_0 = 0$ ← General form

Dot Product

Dot product of two vectors $\vec{a}$ & $\vec{b}$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots a_n b_n$$

$$\therefore \quad \vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$$

(extra) $\langle \vec{a}, \vec{b} \rangle$

another way to write $\Rightarrow \vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle$

Quiz: Find the dot product of

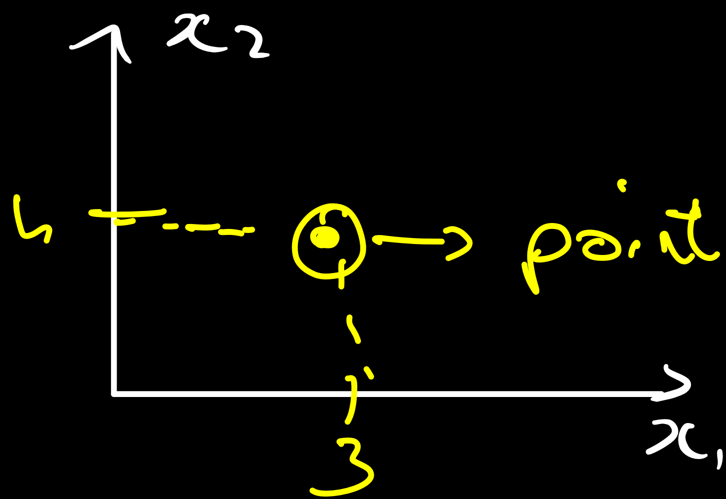
$[1, 2, 3]$ and $[-1, 2, 5]$

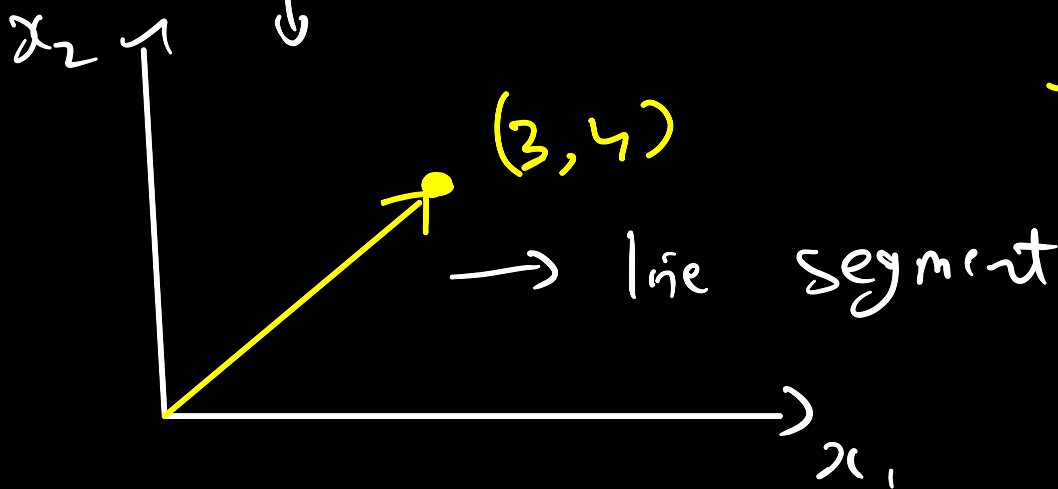
a) 12 b) 0 c) 18 d) 17

Geometric meaning of Vector

$\vec{x} = [x_1, x_2]$

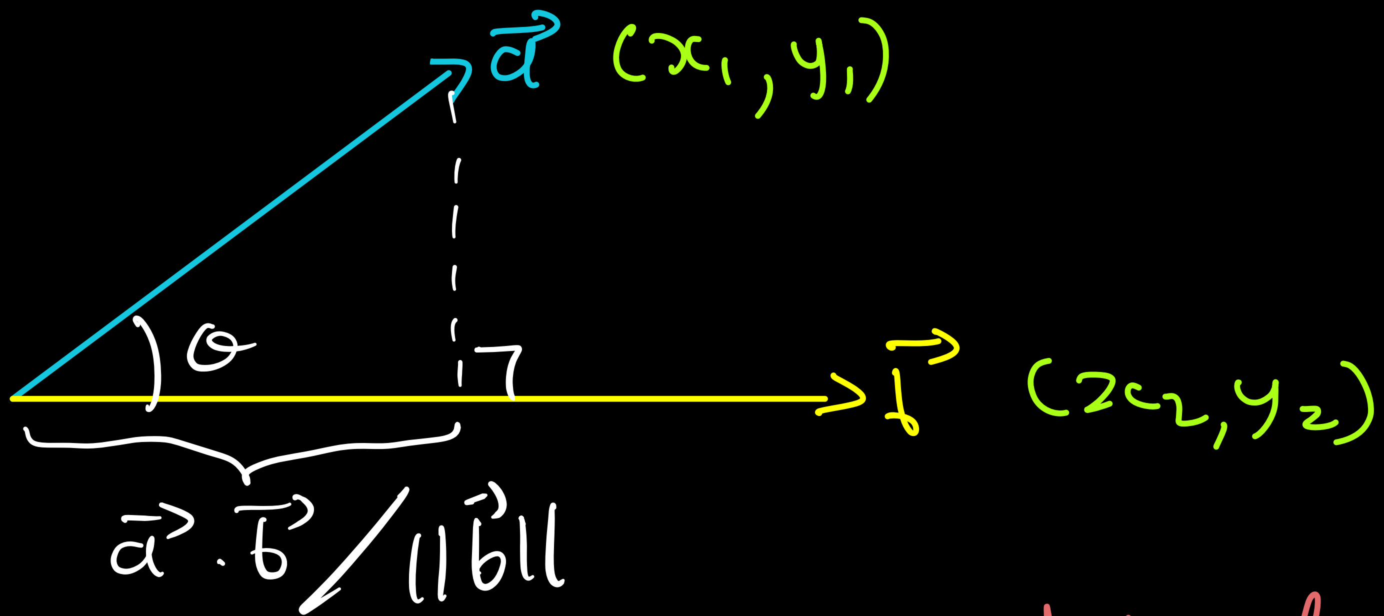
$\vec{x} = [3, 4] \rightarrow$





Vectors are interpreted as coordinates as well as line segment from origin to coordinate

Geometric Meaning of Dot Product



The lenght of the projection of $\vec{a}$ on $\vec{b}$ is called dot product → $\vec{a}\cdot\vec{b} / \|\vec{b}\|$

→ Shadow of $\vec{a}$ on $\vec{b}$

Property: $\boxed{\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}} = \sum_i a_i b_i$

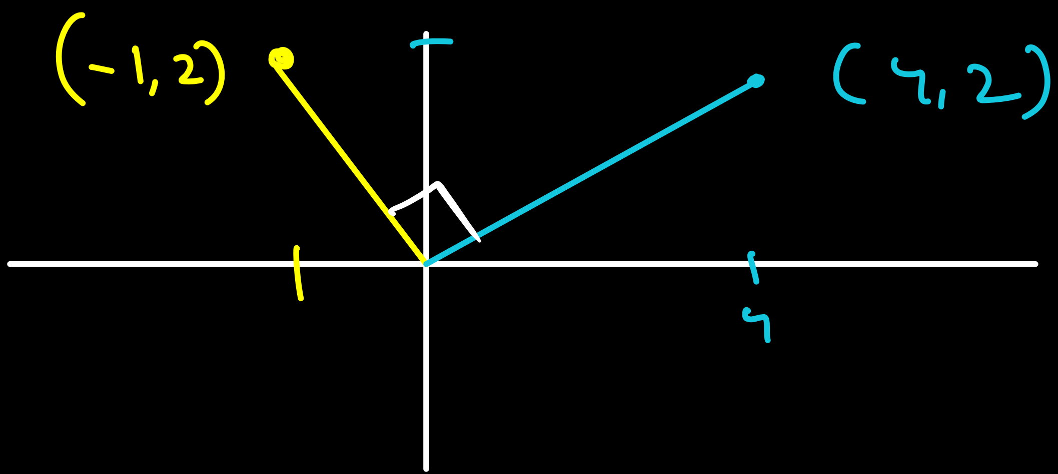
Q: What is the dot product of

$(-1, 2)$ and $(4, 2)$

$$\sum x_i y_i \Rightarrow (-1)(4) + (2)(2)$$

$$= \underline{\underline{0}}$$

when the dot product is zero, the vectors are perpendicular



Property!

if $\vec{a} \perp \vec{b}$

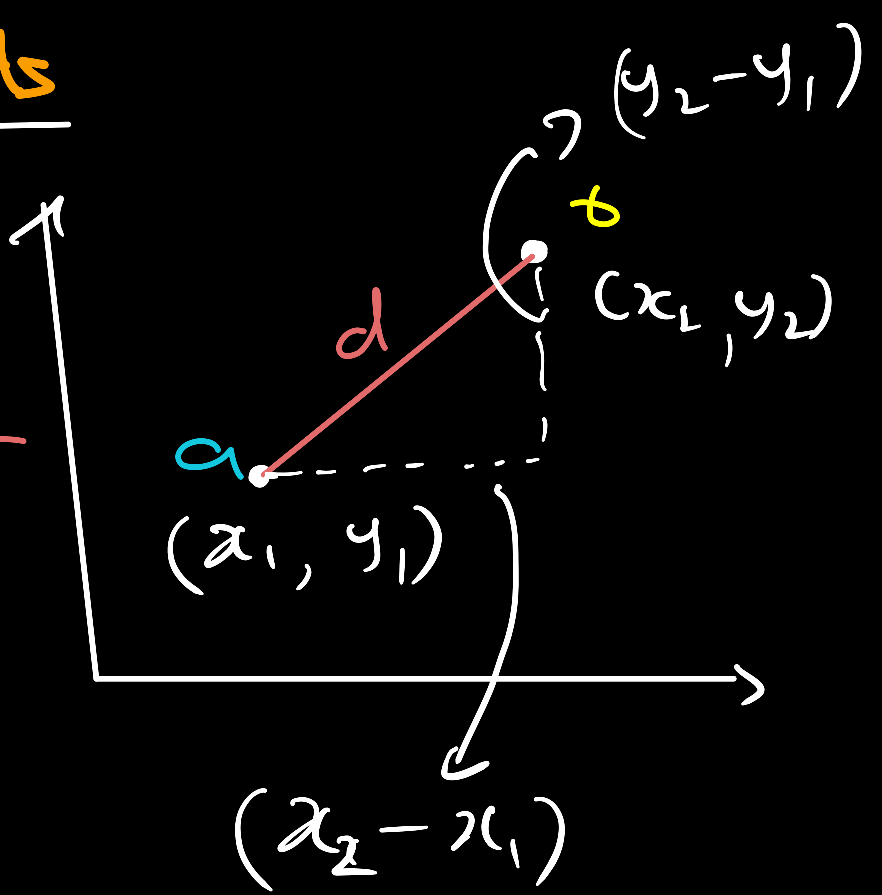
then

$\vec{a} \cdot \vec{b} = \underline{\underline{0}}$

Distance b/w 2 points

Using pythagorous theorem

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



distance of any point from origin

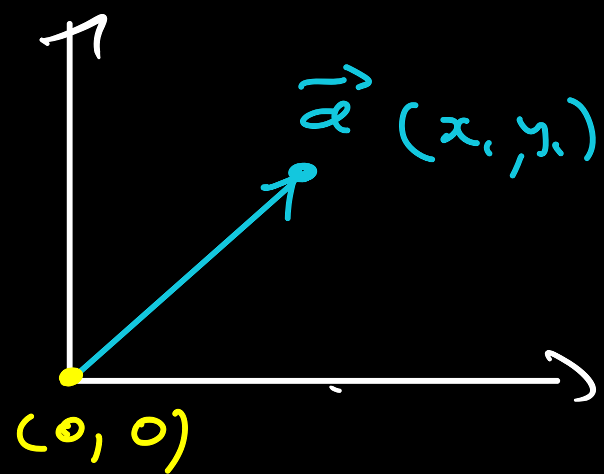
$(x_2, y_2) = \underline{0, 0}$

$$\therefore \quad d = \sqrt{x_1^2 + y_1^2}$$

$$d^2 = [x_1, y_1] \times \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$d^2 = \vec{a} \cdot \vec{a}$$

$$d = \sqrt{\vec{a} \cdot \vec{a}}$$



Sq. root of dot product with itself gives the distance from origin

This is also known as $\boxed{\text{Norm}}$ and $\boxed{\text{magnitude}}$

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \text{length of } \vec{a} \text{ vector}$$

$= \text{distance of } \vec{a}$ coordinate from origin

$= \text{magnitude of } \vec{a}$

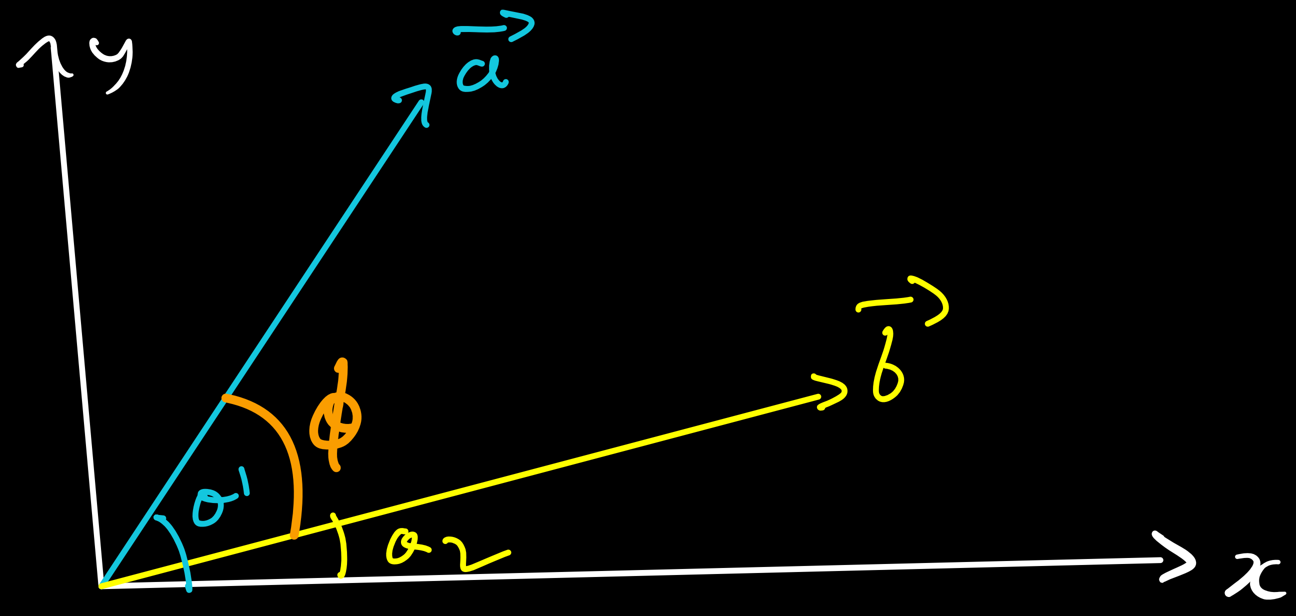
Quiz: Find the distance b/w $(3, 4)$ and $(7, 7)$

a) 6 b) 5 c) 25 c) $\sqrt{24}$

Quiz: Find the difference b/w the norm of $(6, 8)$ and magnitude of $(3, 4)$

a) 6 b) 5 c) 25 d) 50

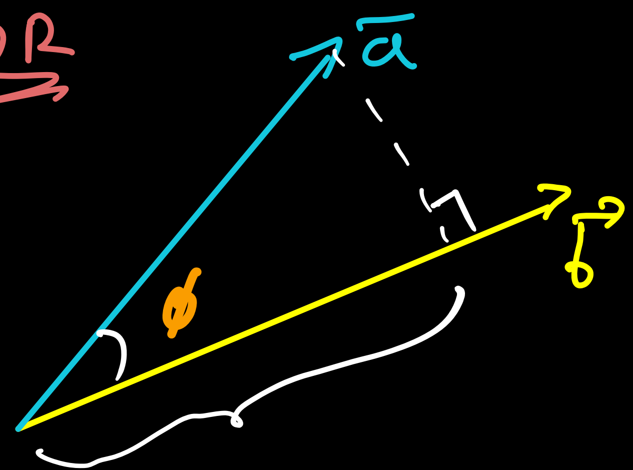
Angle b/w 2 Vectors



$$\phi = \theta_1 - \theta_2$$

You can use slope and find it.

OR

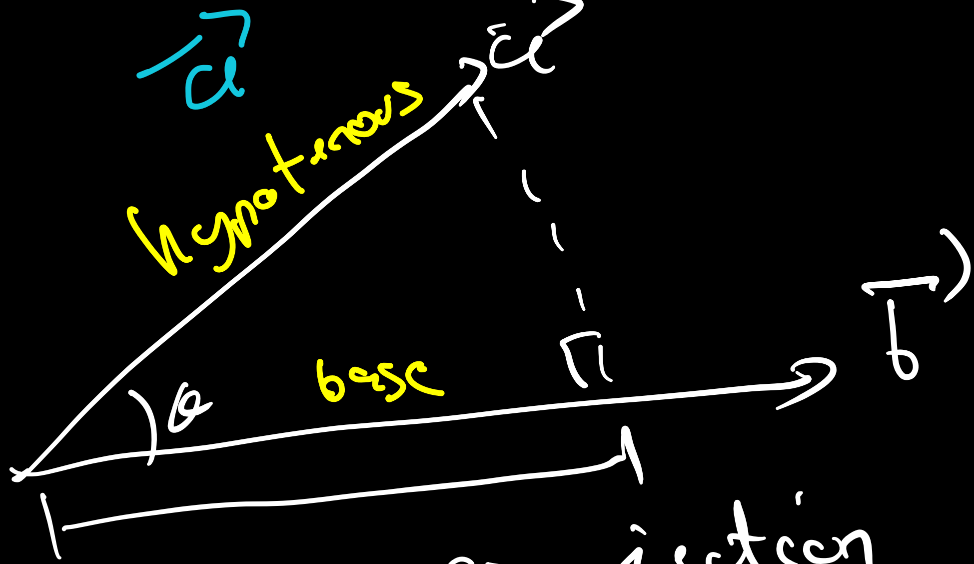


$$\boxed{\cos\phi = \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\| . \|\vec{b}\|}}$$

absolute value

norm

→ Proof:



projection of a on b $\rightarrow \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$

$$\cos\theta = \frac{\text{base}}{\text{hypotenous}}$$

$$= \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{b}\| \cdot \|\vec{a}\|}$$

base

hypotenous

Recap

Line: $y = mx + c$

$ax + by + c = 0$

Hyperplane :

$w_1 x_1 + w_2 x_2 + \dots w_n x_n + w_0 = 0$

$w^T x + w_0 = 0$

Vector

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$, $x^T = [x_1 \; x_2 \dots x_n]$

Dot product

$\vec{x} \cdot \vec{y} = \sum_i x_i y_i$

$$\vec{x} \cdot \vec{x} = x_1^2 + x_2^2 = d^2$$
$$= \text{norm}^2$$
$$= \text{magnitude}^2$$

Distance

(x_1, y_1), (x_2, y_2)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Angle:

$$\cos \phi = \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\| \; \|\vec{b}\|}$$