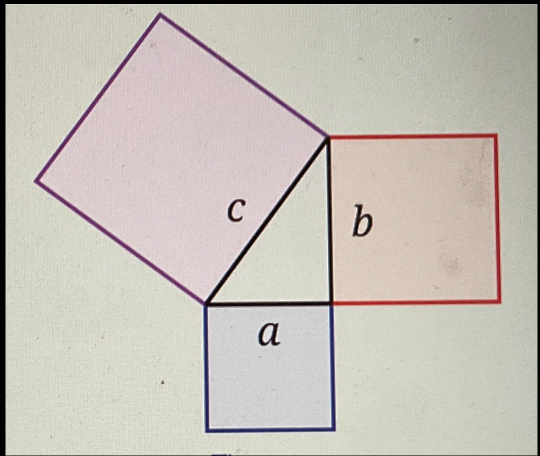
Half Spaces and Distances

[Linear Algebra]

- Trigonometry Basics
- Recap and motivation
- Distance from a hyperplane
- Half space

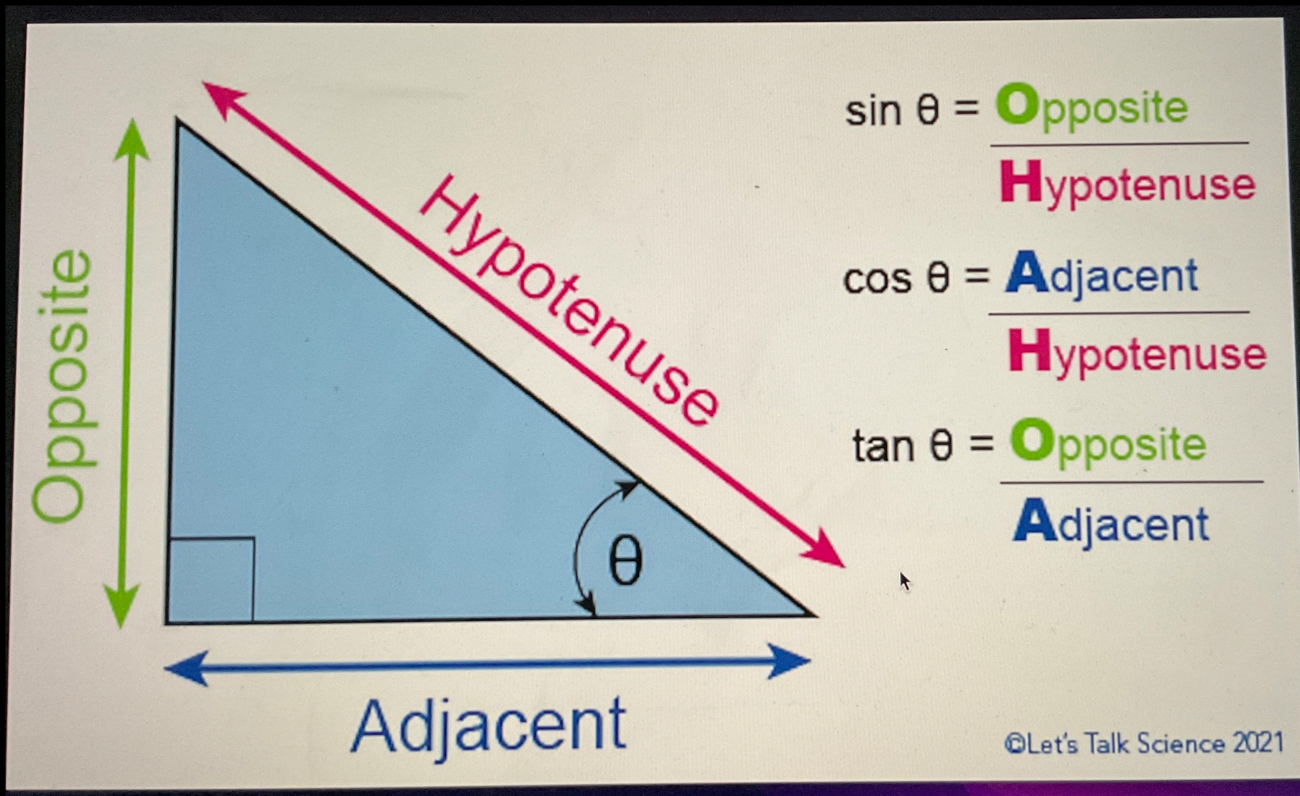
Trignometry Basics



Pythagorean Theorum

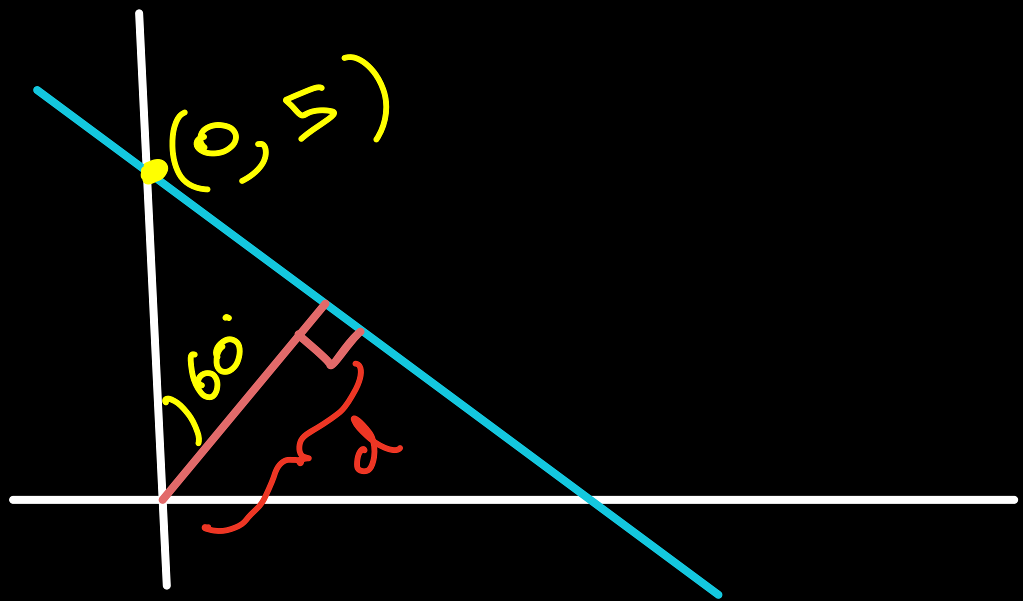
$$c^2 = a^2 + b^2$$

$$\therefore \quad c = \sqrt{a^2 + b^2}$$



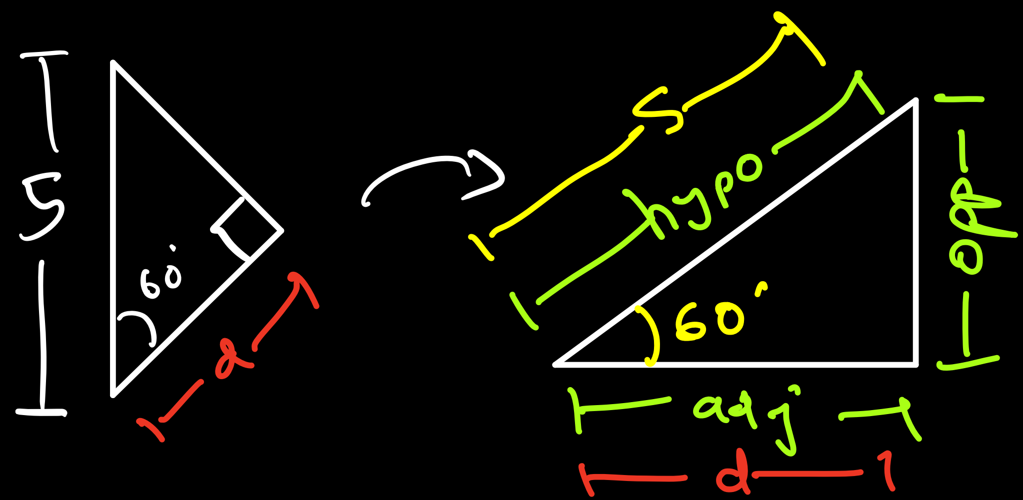
$$\therefore \; A = H \cdot \cos\theta$$

Quiz: Find length 'd'



θ	0° (or) 0	30° (or) $\frac{\pi}{6}$	45° (or) $\frac{\pi}{4}$	60° (or) $\frac{\pi}{3}$	90° (or) $\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Not Defined

a) 5 b) 10 c) 2.5 d) $\sqrt{5}$

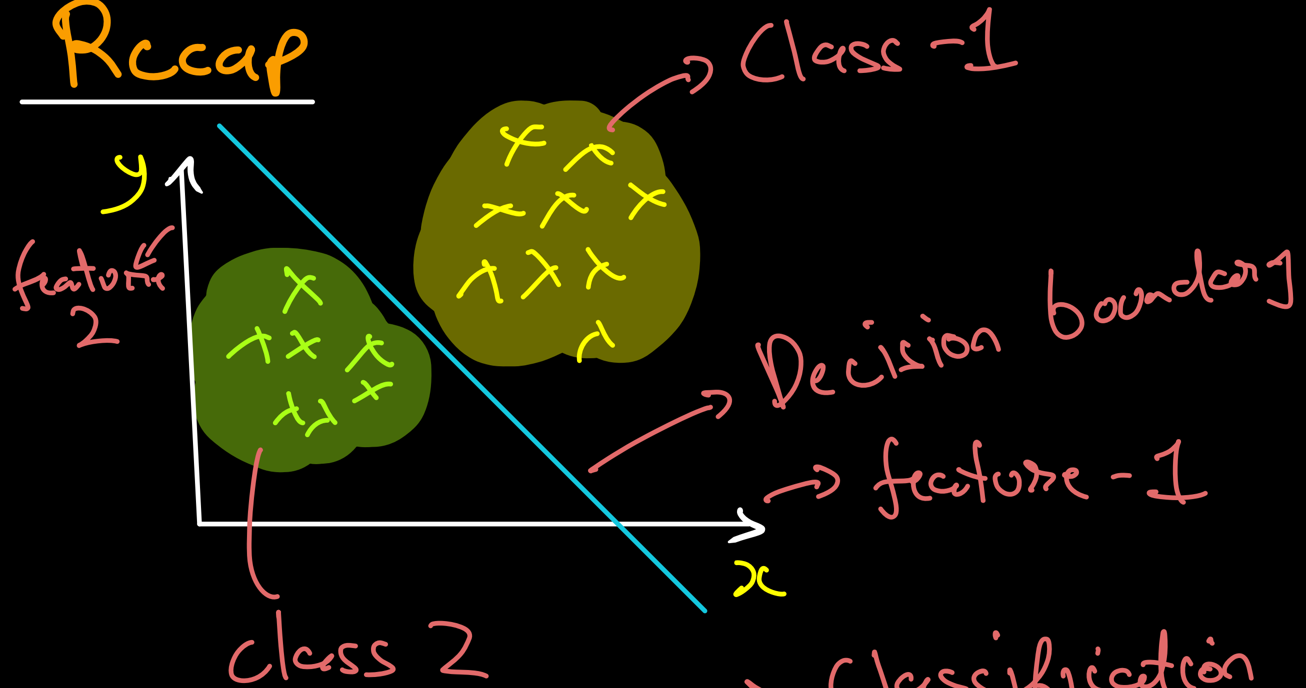


$\rightarrow$ adj = hyp $(\cos\theta)$

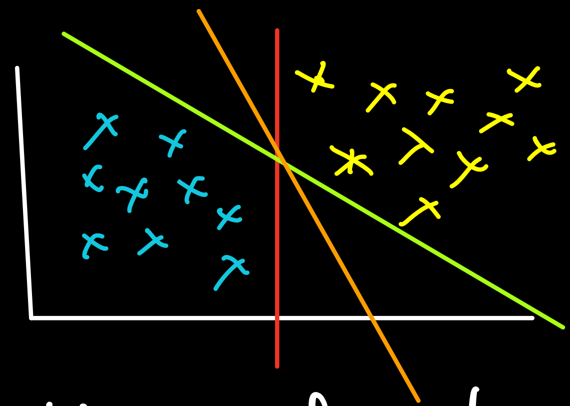
$= 5 \cdot \cos(60^\circ)$

$= 5 \cdot \frac{1}{2} = \boxed{2.5}$

Recap



→ Classification

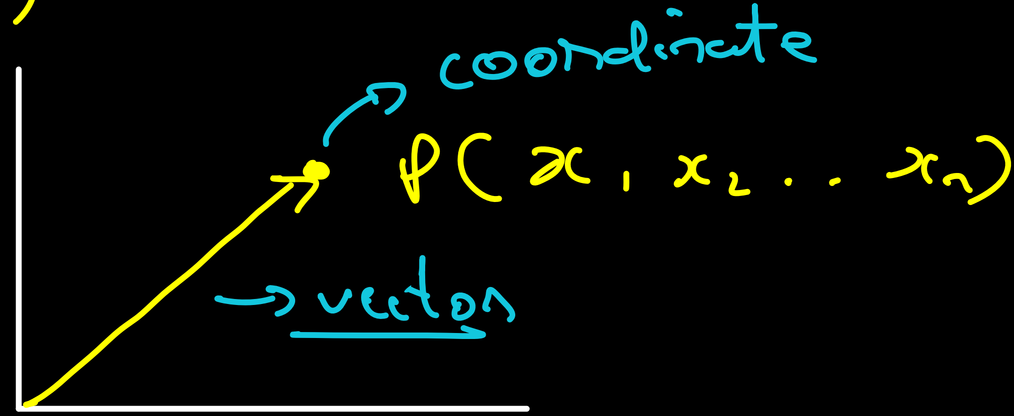


We need to calc distances to decide best line

- $y = mx + c$
- $ax + by + c = 0$
- $w_1 x_1 + w_2 x_2 + w_3 x_3 \dots w_n x_n + w_0 = 0$
- $\bar{w}^T \bar{x} + w_0 = 0$

↳ = dot product $\langle w, x \rangle = \vec{w} \cdot \vec{x} = \sum w_i x_i$

- vector $\rightarrow$ $[x_1, x_2, \ldots x_n]$



- Distance formula:

$$Dist(\bar{a}, \bar{b}) =$$

$$= \sqrt{\sum (x_{ai} - x_{bi})^2}$$

$$= \sqrt{(x_{1a} - x_{1b})^2 + (x_{2a} - x_{2b})^2 \ldots}$$

Distance from origin

$= \text{norm} \rightarrow \|x\|$

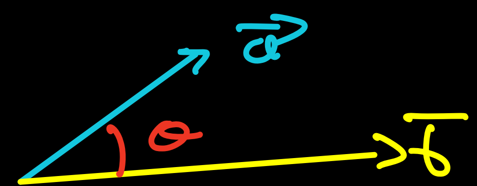
$= \text{magnitude}$

$= \text{modulus or mod}$

$$= \sqrt{\sum x_i^2} = \sqrt{x_1^2 + x_2^2 \ldots}$$

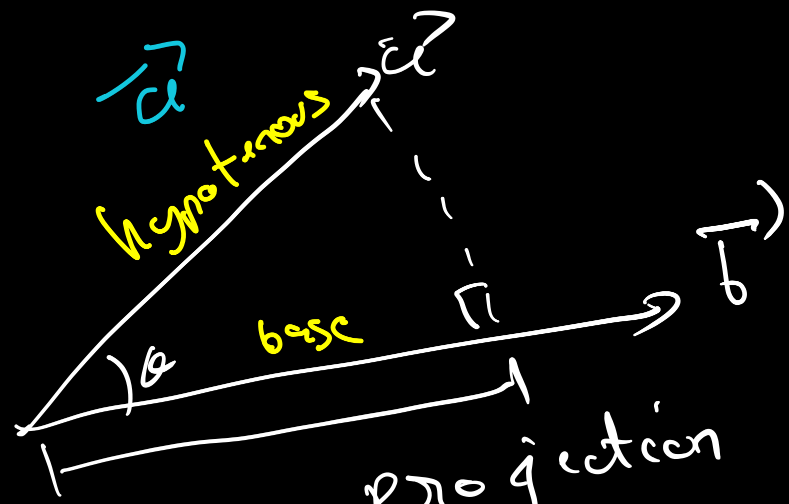
$$= \sqrt{\bar{x} \cdot \bar{x}} = \sqrt{x^T x}$$

Angle b/w vectors



$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|a\| \cdot \|b\|}$$

Proof for angle [Optional]



projection of $\vec{a}$ on $\vec{b}$ $\rightarrow \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$

$$\cos\theta = \frac{\text{base}}{\text{hypotenous}}$$

$$= \frac{\vec{a}\cdot\vec{b}}{\underbrace{\|\vec{b}\|}_{\text{base}} \cdot \underbrace{\|\vec{a}\|}_{\text{hypotenous}}}$$

Quiz: Does the point $(4, 13)$ belong on the line $3x - y + 7 = 0$

a) Yes b) No

Intersection of 2 lines

At Point of intersection, both lines will have the same coordinates. → Desmos

→ $3x - y + 7 = 0$ → $y = 3x + 7$

→ $2x + 2y = 0$ → $y = -x$

$\therefore 3x + 7 = -x$

$\therefore 4x = -7$

$\therefore x = -7/4 = -1.75$

$y = -x \therefore y = 1.75$

OR

$y = 3x + 7$

$= -3(1.75) + 7$

$= 1.75$

Quiz: Find the point of intersection of

$$x = y, \qquad x + y + 2 = 0$$

a) 1, 1 b) -1, 2 c) 2, 2 d) -1, -1

Geometric meaning of $\vec{w}$

Hyperplane: $\bar{w}^T \bar{x} + w_0 = 0$

$\rightarrow w_1 x_1 + w_2 x_2 + \dots w_n x_n + w_0 = 0$

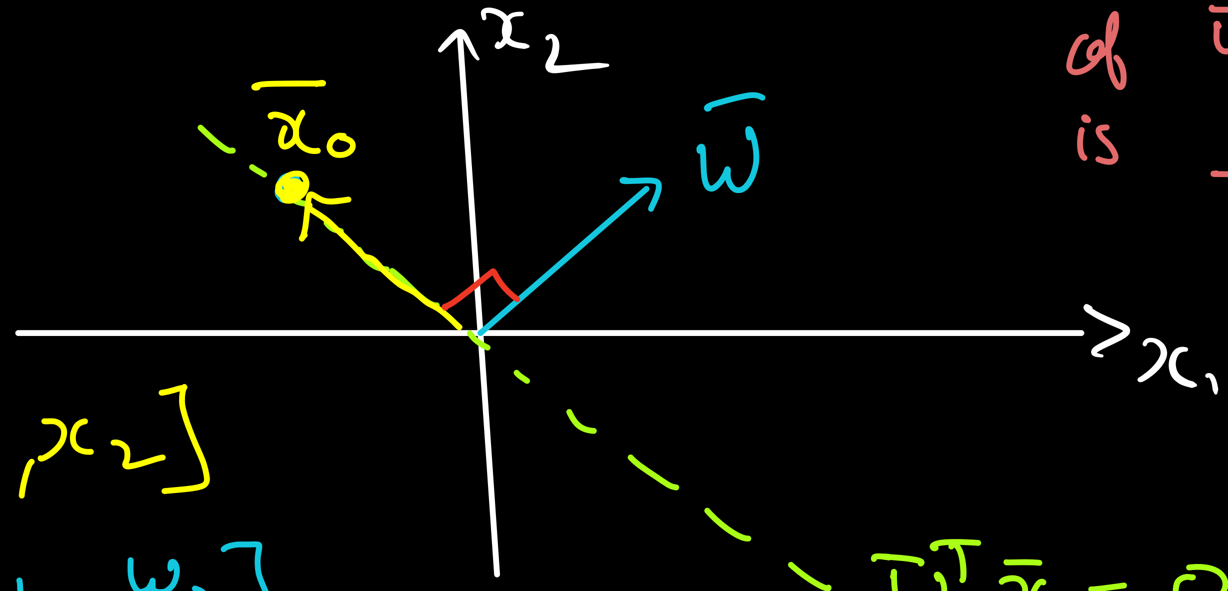
coefficient axes

Effect of $w_0 \rightarrow$ desmos

Changes w_0 affects all intercept $\rightarrow -\dfrac{w_0}{w_i}$

Let's plot the $\vec{w}$ on the same plot as any point x_0 on the line

w_0 moves the line in direction of $\bar{w}$ which is $\perp$ $w^T x + w_0 = 0$



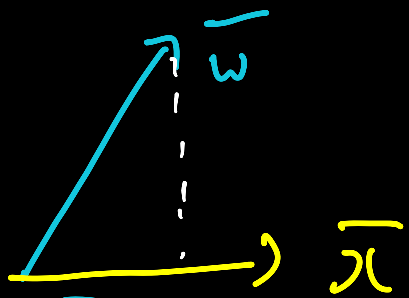
$\bar{x}_0 = [x_1, x_2]$

$\bar{w} = [w_1, w_2]$

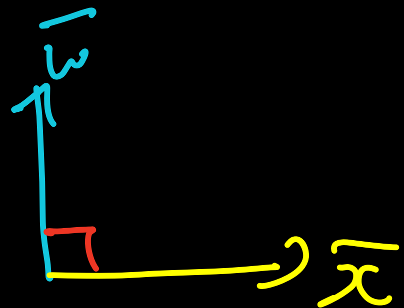
$\underbrace{\bar{w}^T \bar{x}} = 0 \quad (w_0 = 0)$

$\downarrow$

dot product $= 0$



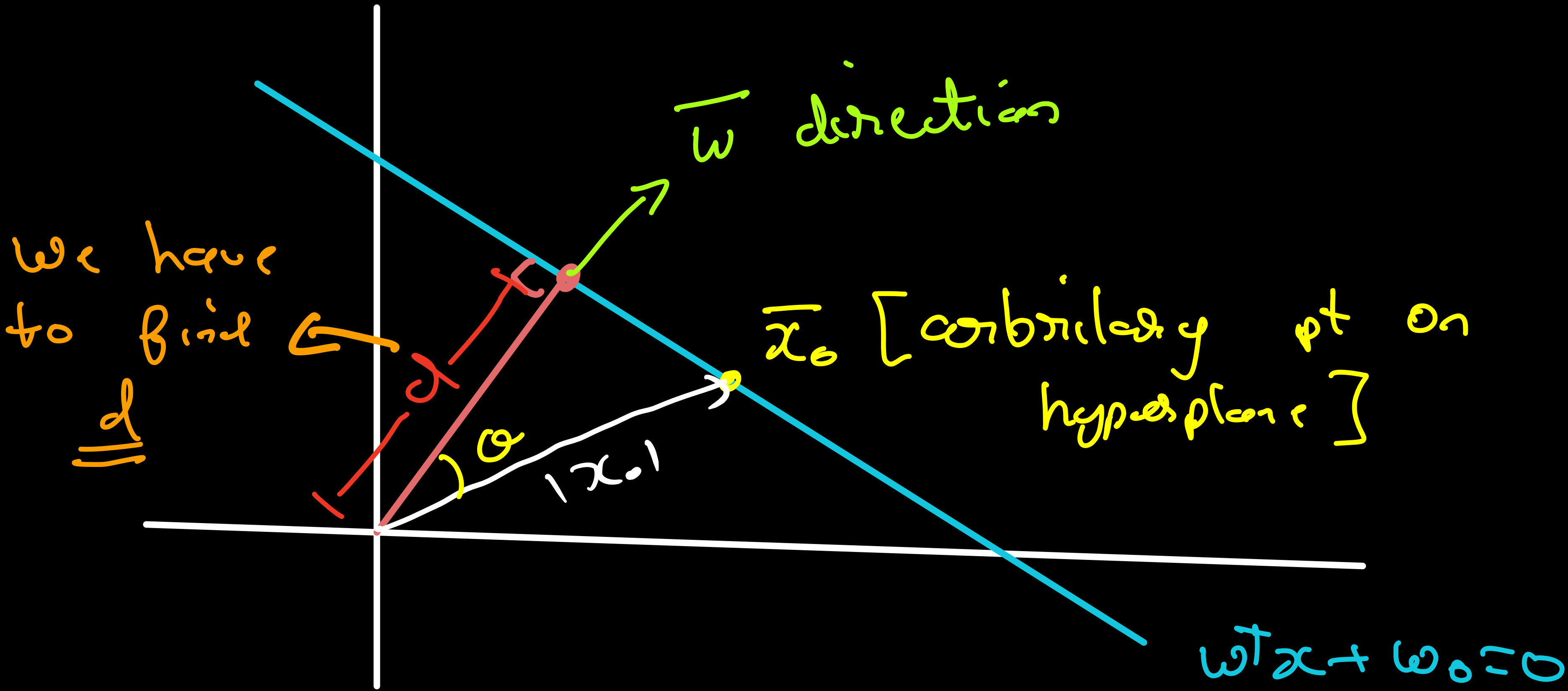
$\frac{\bar{w}\,\bar{x}}{|\bar{x}|} \to 0$



Hence $w \perp x$

Distance of a hyperplane from origin

[Proof → optional]



Given:

$$w^T x + w_0 = 0$$

- Take an arbitrary point on the

H-plane $\underline{\bar{x}_0} \to (x_1, x_2, \ldots x_n)$

- This point satisfies the equa^n

$$\therefore \quad w^T x_0 + w_0 = 0$$

$$\therefore \quad w^T x_0 = -w_0 \quad \text{—} \quad (1)$$

- $d = h \cdot \cos\theta$

$$\therefore \quad d = \underbrace{\|\bar{x}_0\|}_{\text{magnitude / norm of } x_0} \cdot \underbrace{\cos\theta}_{\text{angle between } \vec{x} \text{ and } \vec{w}}$$

$$\therefore \quad d = \cancel{\|\vec{x}_0\|} \cdot \left(\frac{\vec{w}^T \vec{x}_0}{\cancel{\|\vec{x}_0\|} \; \|\vec{w}_0\|} \right)$$

$$\therefore \quad d = \frac{\vec{w}^T \vec{x}_0}{\|w_0\|} \quad - \text{②}$$

Using ①

$$\boxed{d = \frac{-w_0}{\|\vec{w}\|} \sim \frac{|w_0|}{\|\vec{w}\|}}$$

Take abs because distances are +ve

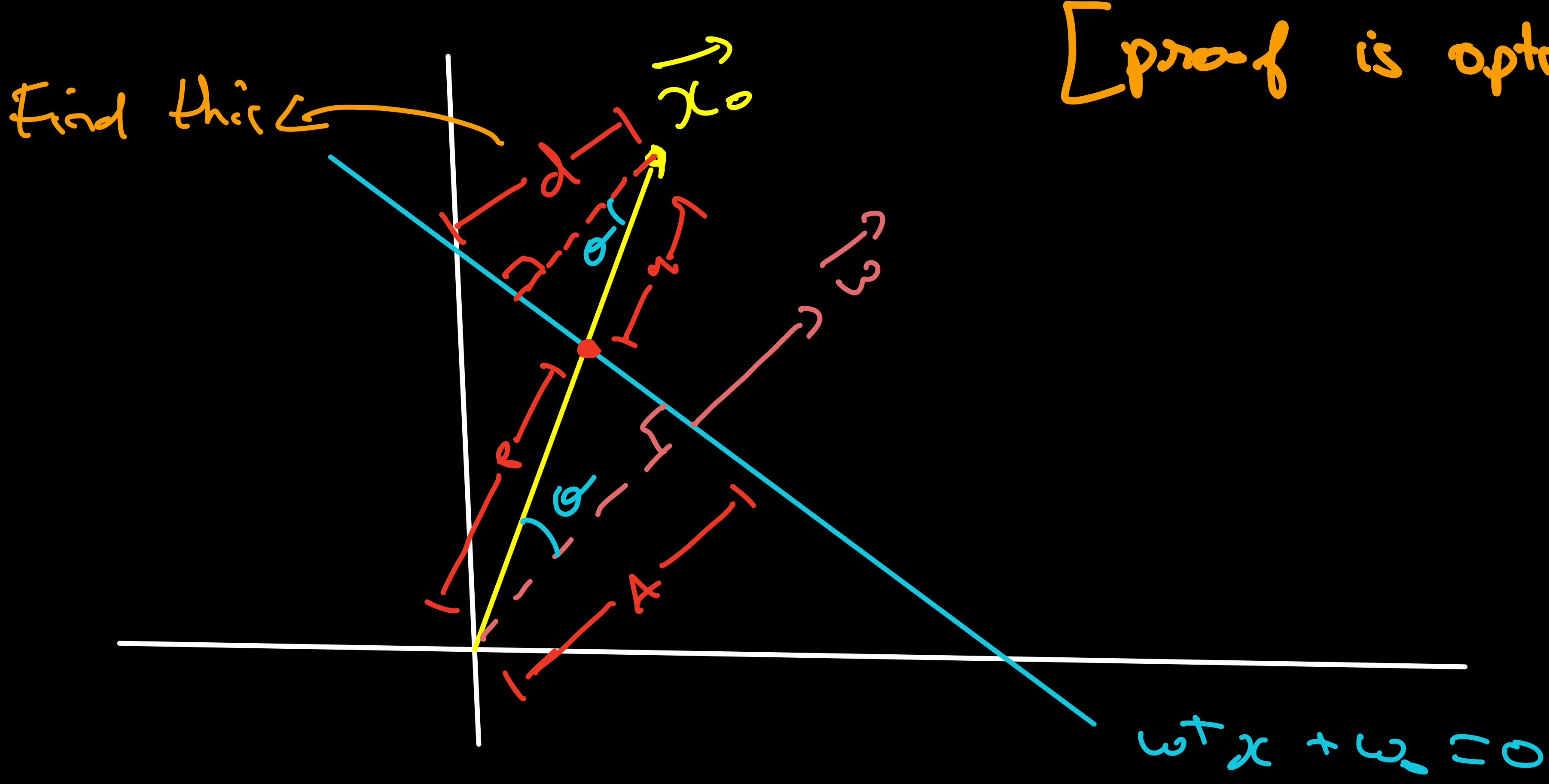
Quiz: Find distance of $3x + 4y + 0z + 5 = 0$ from origin

a) $\frac{5}{\sqrt{7}}$ b) -1 c) 5 d) 1

Note: Origin is also represented as $\vec{O}$.

Distance of a pt from a h-plane

[proof is optional]



- 'A' = dist from origin

$$A = \frac{-w_0}{\|\vec{w}\|} \quad \text{—} \quad (1)$$

- $'p' = \dfrac{A}{\cos\theta}$

- $'p' + 'q' = \|\vec{x}_0\| - \dfrac{A}{\cos\theta}$

- $d = q \cdot \cos\theta$

$$= \left(\|\vec{x}_0\| - \frac{A}{\cos\theta} \right) \cos\theta$$

$$= \|\vec{x}_0\| \cos\theta - A$$

$$= \cancel{\|\vec{x}_0\|} \, \frac{\vec{w} \cdot \vec{x}_0}{\cancel{\|\vec{x}\|} \, \|\vec{w}\|} - \frac{-w_0}{\|\vec{w}\|}$$

$\Rightarrow |x|$ because $d \to +ve$

$$\therefore \quad \boxed{d = \frac{|w^T x_0 + w_0|}{\|\vec{w}\|}}$$

Very important formula

→ Just put the point in equation of the plane and divide by $\sqrt{\sum coeff^2}$

Q: Find the distance of $x + y + z = 0$ from $(30, 45, 0)$

a) $\frac{75}{\sqrt{3}}$

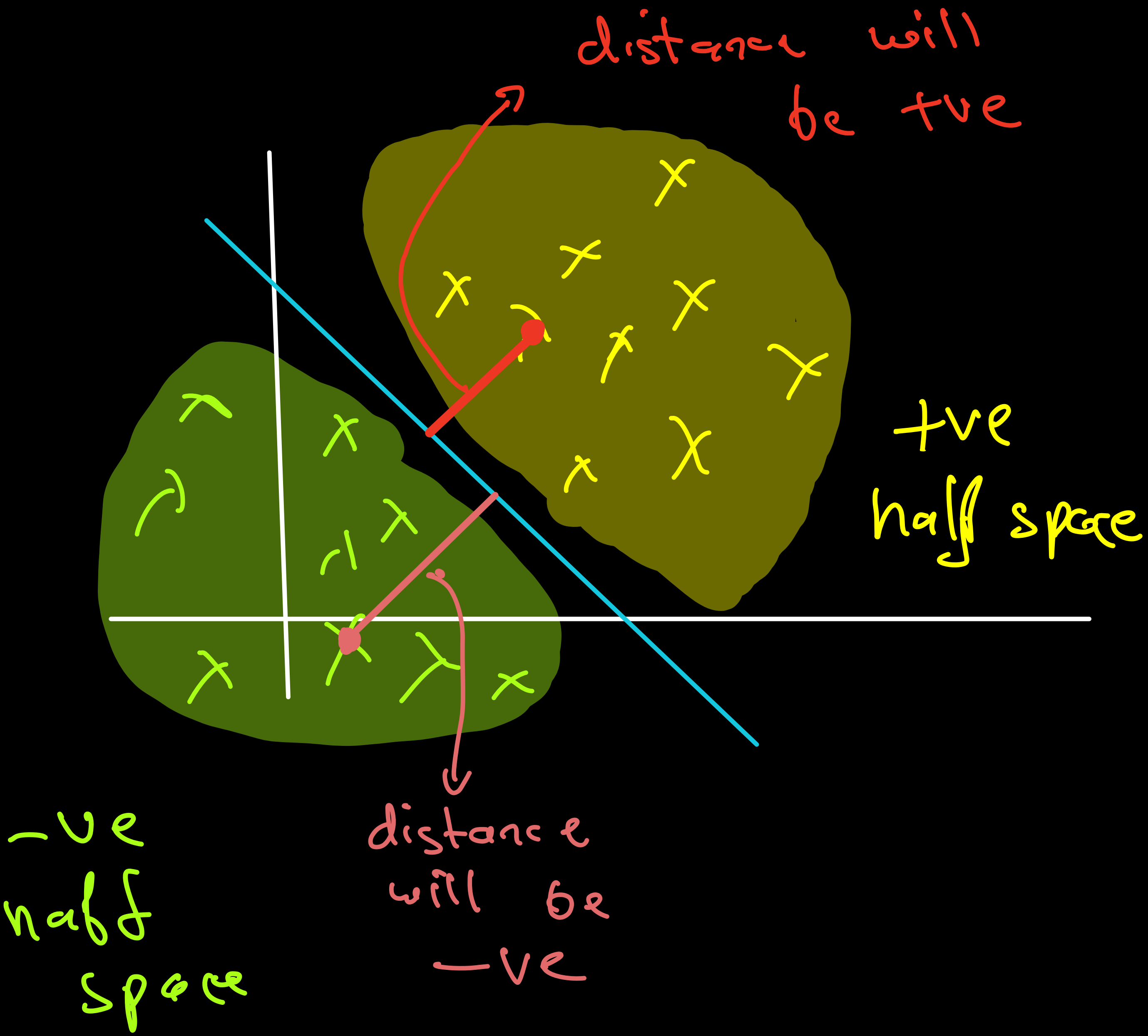
b) $75\sqrt{3}$

c) $\frac{15}{\sqrt{3}}$

d) $15\sqrt{3}$

Half Spaces

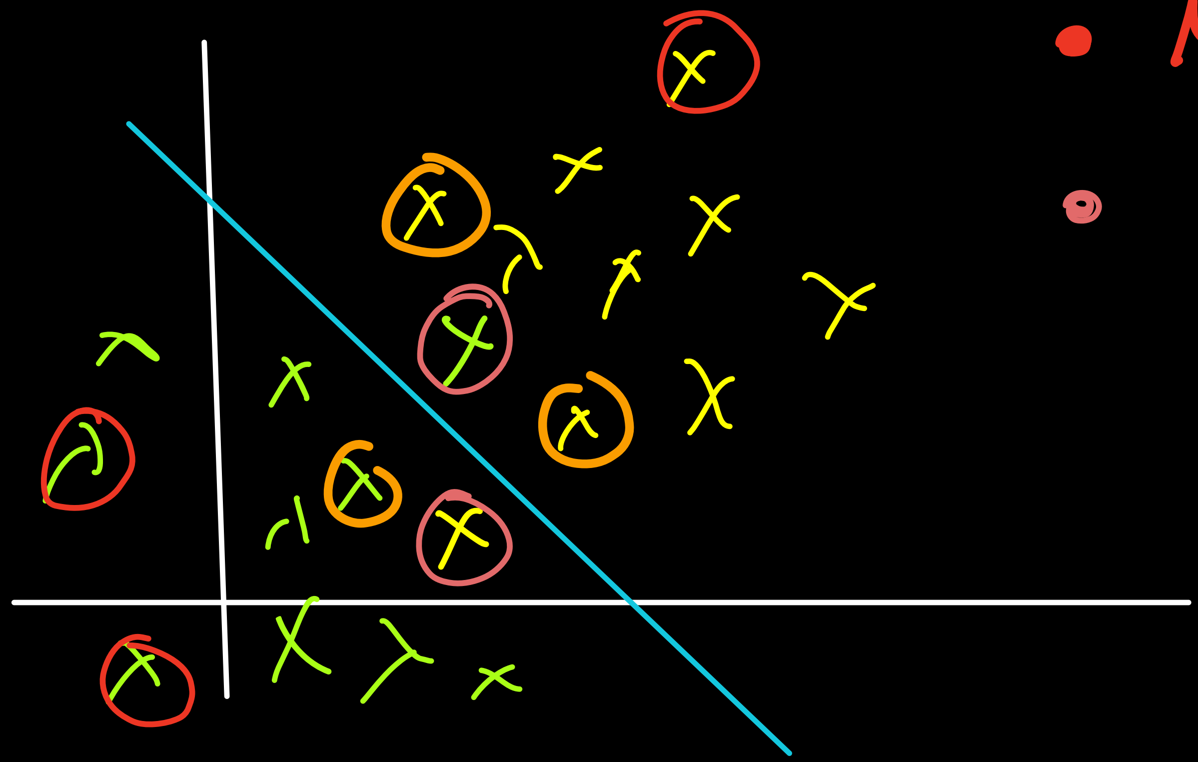
$$d = \frac{w^T x_0 + w}{\|\vec{w}\|} \rightarrow \text{always +ve}$$



Half space = Sign of

$w^T x_0 + w_0$

Confidence



- Less confident
- More confident
- Miss classificn

In the real world, there will be miss-classified data. You will be more confident about points far away from DB