# ISyE 6761 — Fall 2012

## Homework #2 Solutions

1. The joint p.m.f. of $X$ and $Y$ is

| $f(x,y)$ | $x=1$ | $x=2$ | $x=3$ |
|---|---|---|---|
| $y=1$ | 1/9 | 1/3 | 1/9 |
| $y=2$ | 1/9 | 0 | 1/18 |
| $y=3$ | 0 | 1/6 | 1/9 |

   (a) Find $\mathsf{E}[X|Y=y]$ for $y=1,2,3$.

   (b) Find $\mathsf{E}[\mathsf{E}[X|Y]]$.

   (c) Are $X$ and $Y$ independent?

**Solution:** (a) By definition of conditional expectation,

$$\mathsf{E}[X|Y=y] \;=\; \sum_x x\mathsf{Pr}(X=x|Y=y) \;=\; \frac{\sum_x xf(x,y)}{f_Y(y)}.$$

Now,

$$f_Y(1) \;=\; \sum_x f(x,1) \;=\; 5/9.$$

Similarly, $f_Y(2)=1/6$ and $f_Y(3)=5/18$.

Plug these results into the first equation to get $\mathsf{E}[X|Y=1]=2$, $\mathsf{E}[X|Y=2]=5/3$, and $\mathsf{E}[X|Y=3]=12/5$. $\diamondsuit$

(b) $\mathsf{E}[\mathsf{E}[X|Y]] = \sum_y \mathsf{E}[X|y]f_Y(y) = 37/18$. $\diamondsuit$

Check: Note that

$$f_X(1) \;=\; \sum_y f(1,y) \;=\; 2/9.$$

Similarly, $f_X(2)=1/2$ and $f_X(3)=5/18$. Thus, $\mathsf{E}[\mathsf{E}[X|Y]] = \mathsf{E}[X] = \sum_x xf_X(x) = 37/18$, which matches the answer above! $\diamondsuit$

(c) Adding in the p.m.f.'s, the original joint table becomes

| $f(x,y)$ | $x=1$ | $x=2$ | $x=3$ | $f_Y(y)$ |
|---|---|---|---|---|
| $y=1$ | $1/9$ | $1/3$ | $1/9$ | $5/9$ |
| $y=2$ | $1/9$ | $0$ | $1/18$ | $1/6$ |
| $y=3$ | $0$ | $1/6$ | $1/9$ | $5/18$ |
| $f_X(x)$ | $2/9$ | $1/2$ | $5/18$ | $1$ |

Notice, e.g., that $f(1,3) = 0 \neq f_X(1)f_Y(3)$. Thus, $X$ and $Y$ are not independent. $\diamondsuit$

2. Show in the discrete case that if $X$ and $Y$ are independent, then $\mathsf{E}[X|Y=y] = \mathsf{E}[X]$ for all $y$.

   **Solution:** Since $X$ and $Y$ are independent, we have

$$\mathsf{E}[X|Y=y] \;=\; \sum_x x\mathsf{Pr}(X=x|Y=y) \;=\; \sum_x x f_X(x) \;=\; \mathsf{E}[X]. \quad \diamondsuit$$

3. Suppose the joint p.d.f. of $X$ and $Y$ is

$$f(x,y) \;=\; \frac{e^{-x/y}e^{-y}}{y}, \quad \text{for } x \text{ and } y > 0.$$

   Find $\mathsf{E}[X|Y=y]$.

   **Solution:** First of all,

$$f_Y(y) \;=\; \int_{-\infty}^{\infty} f(x,y)\,dx \;=\; \frac{e^{-y}}{y}\int_0^{\infty} e^{-x/y}\,dx \;=\; e^{-y}.$$

   Now,

$$\begin{aligned}
\mathsf{E}[X|Y=y] &= \int_{\Re} x f(x|y)\,dx \\
&= \frac{1}{f_Y(y)}\int_{-\infty}^{\infty} x f(x,y)\,dx \\
&= e^y \int_0^{\infty} \frac{x e^{-x/y}e^{-y}}{y}\,dx \\
&= \int_0^{\infty} \frac{x e^{-x/y}}{y}\,dx \\
&= y,
\end{aligned}$$

   since the integral is the expected value of an Exp$(1/y)$ p.d.f. $\diamondsuit$

4. Suppose $X \sim \text{Exp}(\lambda)$. Find $\mathsf{E}[X|X > x]$ for $x > 0$.

   **Solution:** I'll use $y$ as a dummy variable. The conditional p.d.f. of $X$ given that $X > x$ is

$$\begin{aligned} f_{X|X>x}(y) &= \frac{d}{dy}\mathsf{Pr}(X \le y|X > x) \quad \text{for } y \ge x \\ &= \frac{d}{dy}\frac{\mathsf{Pr}(x < X \le y)}{\mathsf{Pr}(X > x)} \\ &= \frac{\frac{d}{dy}(F(y) - F(x))}{\mathsf{Pr}(X > x)} \\ &= \frac{f(y)}{\mathsf{Pr}(X > x)} \\ &= \frac{\lambda e^{-\lambda y}}{e^{-\lambda x}}. \end{aligned}$$

   This implies that

$$\mathsf{E}[X|X > x] = \int_x^\infty y f_{X|X>x}(y)\,dy = \int_x^\infty y\lambda e^{\lambda x}e^{-\lambda y}\,dy = x + \frac{1}{\lambda}. \quad \diamondsuit$$

   (Could you have gotten the same thing via inspection?)

5. Suppose $X \sim \text{Unif}(0, 1)$. Find $\mathsf{E}[X|X < x]$ for $0 < x < 1$.

   **Solution:** Again, I'll use $y$ as a dummy variable. Similar to the above work, we get

$$f_{X|X<x}(y) = \frac{f(y)}{\mathsf{Pr}(X < x)} = \frac{1}{x}, \quad \text{for } 0 < y < x.$$

   This implies that

$$\mathsf{E}[X|X < x] = \int_0^x y f_{X|X<x}(y)\,dy = \int_0^x \frac{y}{x}\,dy = x/2. \quad \diamondsuit$$

   (Could you have gotten the same thing via inspection?)

6. Suppose the joint p.d.f. of $X$ and $Y$ is

$$f(x, y) = \frac{e^{-y}}{y}, \quad \text{for } 0 < x < y < \infty.$$

Find $\mathsf{E}[X^2|Y=y]$.

**Solution:** First of all,

$$f_Y(y) \;=\; \int_{-\infty}^{\infty} f(x,y)\,dx \;=\; \int_0^y \frac{e^{-y}}{y}\,dx \;=\; e^{-y}.$$

Then

$$f(x|y) \;=\; \frac{f(x,y)}{f_Y(y)} \;=\; \frac{1}{y}, \quad \text{for } 0<x<y<\infty.$$

Thus,

$$\mathsf{E}[X^2|Y=y] \;=\; \int_{-\infty}^{\infty} x^2 f(x|y)\,dx \;=\; \int_0^y \frac{x^2}{y}\,dx \;=\; \frac{y^2}{3}. \quad \diamondsuit$$

7. Suppose $Y \sim \text{Gamma}(\alpha,\lambda)$. Also suppose that the conditional distribution of $X$ given that $Y=y$ is $\text{Pois}(y)$. Find the conditional distribution of $Y$ given that $X=x$.

   **Solution:** We have that the required conditional p.d.f. is

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)} \\ &= \frac{1}{f_X(x)}\cdot\frac{e^{-y}y^x}{x!}\cdot Ce^{-\lambda y}y^{\alpha-1} \\ &= Ke^{-y}y^x e^{-\lambda y}y^{\alpha-1} \\ &= Ke^{-(1+\lambda)y}y^{\alpha+x-1}, \end{aligned}$$

   where $C$ and $K$ don't depend on $y$. This is simply the $\text{Gamma}(\alpha+x, 1+\lambda)$ p.d.f. $\diamondsuit$

8. A miner is trapped in a mine with three doors. The first door leads to a tunnel that takes him to safety after two hours. The second leads back to the mine after three hours. The third leads back to the mine after five hours.

   (a) Assuming that he's equally likely at any time to choose any one of the three doors, what's the expected length of time $X$ until he reaches safety?

(b) Let $N$ denote the total number of doors (attempts) before he reaches safety. Let $T_i$ be the travel time corresponding to the $i$th choice $i \geq 1$. Give an identity that relates $X$ to $N$ and the $T_i$'s.

(c) What is $\mathsf{E}[N]$?

(d) What is $\mathsf{E}[T_N]$?

(e) What is $\mathsf{E}[\sum_{i=1}^{N} T_i | N = n]$?

(f) Using the above, what is $\mathsf{E}[X]$?

**Solution:** (a) Let $Y$ denote the door he initially chooses. Then

$$\mathsf{E}[X] \;=\; \mathsf{E}[\mathsf{E}[X|Y]] \;=\; \sum_{y=1}^{3} \mathsf{E}[X|Y=y]\mathsf{Pr}(Y=y) \;=\; \frac{1}{3}\sum_{y=1}^{3} \mathsf{E}[X|Y=y].$$

Meanwhile,

$$\mathsf{E}[X|Y=1] \;=\; 2, \quad \mathsf{E}[X|Y=2] \;=\; 3+\mathsf{E}[X], \quad \text{and} \quad \mathsf{E}[X|Y=2] \;=\; 5+\mathsf{E}[X].$$

Plugging into the above, we have

$$\mathsf{E}[X] \;=\; \frac{10+2\mathsf{E}[X]}{3},$$

leading to $\mathsf{E}[X] = 10$. $\diamondsuit$

(b) $X = \sum_{i=1}^{N} T_i$. $\diamondsuit$

(c) $N \sim \text{Geom}(1/3)$. Thus, $\mathsf{E}[N] = 3$. $\diamondsuit$

(d) $T_N$ is the travel time corresponding to the final choice, i.e., the choice leading to freedom. Therefore, $T_N = 2$, and so $\mathsf{E}[T_N] = 2$. $\diamondsuit$

(e) Since $N = n$, this means that the travel times $T_1, T_2, \ldots, T_{n-1}$ are each equally likely to be 3 or 5, while $T_n = 2$. Therefore,

$$\begin{aligned}
\mathsf{E}\Big[\sum_{i=1}^{N} T_i | N = n\Big] &= \mathsf{E}\Big[\sum_{i=1}^{n-1} T_i | N = n\Big] + \mathsf{E}[T_n | N = n] \\
&= (n-1)\frac{(3+5)}{2} + 2 \\
&= 4n - 2. \quad \diamondsuit
\end{aligned}$$

(f) By (b), (c), and (e), we have

$$\mathsf{E}[X] \;=\; \mathsf{E}\Big[\mathsf{E}\Big(\sum_{i=1}^{N} T_i|N\Big)\Big] \;=\; \mathsf{E}[4N-2] \;=\; 4\mathsf{E}[N]-2 \;=\; 10. \quad \diamondsuit$$

9. A coin having probability $p$ of heads is flipped until the two most-recent flips are heads, Let $N$ be the number of flips. Find $\mathsf{E}[N]$.

**Solution:**

$$\begin{aligned}
\mathsf{E}[N] &= \mathsf{E}[N|H]p + \mathsf{E}[N|T]q \\
&= \mathsf{E}[N|HH]p^2 + \mathsf{E}[N|HT]pq + \mathsf{E}[N|T]q \\
&= 2p^2 + (\mathsf{E}[N]+2)pq + (\mathsf{E}[N]+1)q
\end{aligned}$$

where $q = 1-p$. After a little algebra, we get

$$\mathsf{E}[N] \;=\; \frac{2p^2+2pq+q}{1-pq-q} \;=\; \frac{2-q}{p^2}. \quad \diamondsuit$$

10. You have two opponents with whom you alternate play. Whenever you play $A$, you win with probability $p_A$; whenever you play $B$, you win with probability $p_B$, where $p_B > p_A$. If your objective is to minimize the expected number of games you need to play to win two in a row, should you start with $A$ or $B$?

**Solution:** Let $N_A$ [$N_B$] denote the number of games needed if you start with $A$ [$B$]. We'll condition on the outcome of the first game, where $w$ denotes a win and $\ell$ denotes a loss.

$$\mathsf{E}[N_A] \;=\; \mathsf{E}[N_A|w]p_A + \mathsf{E}[N_A|\ell](1-p_A).$$

We can also condition on the outcome of the next game,

$$\begin{aligned}
\mathsf{E}[N_A|w] &= \mathsf{E}[N_A|ww]p_B + \mathsf{E}[N_A|w\ell](1-p_B) \\
&= 2p_B + (2+\mathsf{E}[N_A])(1-p_B) \\
&= 2 + (1-p_B)\mathsf{E}[N_A].
\end{aligned}$$

Furthermore,

$$\mathsf{E}[N_A|\ell] \;=\; 1 + \mathsf{E}[N_B].$$

Plugging these results into the first equation gives

$$\mathsf{E}[N_A] \;=\; \Big(2 + (1-p_B)\mathsf{E}[N_A]\Big)p_A + (1+\mathsf{E}[N_B])(1-p_A).$$

By symmetry,

$$\mathsf{E}[N_B] \;=\; \Big(2 + (1-p_A)\mathsf{E}[N_B]\Big)p_B + (1+\mathsf{E}[N_A])(1-p_B).$$

Subtract, and plow thru some algebra. Since $p_B > p_A$, you'll eventually see that $\mathsf{E}[N_A] < \mathsf{E}[N_B]$ (so playing $A$ first is better). $\diamondsuit$

11. A manuscript is sent to a typing firm that has typists $A$, $B$, and $C$. If it's typed by $A$, then the number of errors made is Pois(2.6). If $B$, then the number is Pois(3). If $C$, then the number is Pois(3.4). Let $X$ denote the number of errors in the typed manuscript, and assume that each typist is equally likely to do the work. What are $\mathsf{E}[X]$ and $\mathsf{Var}(X)$?

**Solution:**

(a) $\mathsf{E}[X] = (2.6+3+3.4)/3 = 3$. $\diamondsuit$

(b) Before we begin, recall that if $Y \sim \text{Pois}(\lambda)$, then $\mathsf{E}[Y^2] = \lambda + \lambda^2$. This implies that

$$\mathsf{E}[X^2] \;=\; [(2.6+2.6^2)+(3+3^2)+(3.4+3.4^2)]/3 \;=\; 12.1067,$$

and so $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 3.11$. $\diamondsuit$

12. Suppose $U \sim \text{Unif}(0,1)$. Suppose that $n$ trials are to be performed and that, conditional on $U = u$, these trials will be independent with a common success probability $u$. Compute the mean and variance of the number of successes that occur in these trials.

**Solution:** Suppose $X$ is the number of successes. Then $(X|U=u) \sim \text{Bin}(n,u)$. So

$$\mathsf{E}[X] \;=\; \mathsf{E}[\mathsf{E}(X|U)] \;=\; \mathsf{E}[nU] \;=\; n/2.$$

Similarly, since $Y \sim \text{Bin}(n,p)$ implies $\mathsf{E}[Y^2] = np(1-p) + n^2p^2$, we have

$$\begin{aligned}
\mathsf{E}[X^2] &= \mathsf{E}[\mathsf{E}(X^2|U)] \\
&= \mathsf{E}[nU(1-U) + n^2U^2] \\
&= \mathsf{E}[nU + (n^2-n)U^2] \\
&= n\mathsf{E}[U] + (n^2-n)\Big(\mathsf{Var}(U) + (\mathsf{E}[U])^2\Big) \\
&= \frac{n}{2} + (n^2-n)\Big(\frac{1}{12} + \frac{1}{4}\Big) \\
&= \frac{n}{6} + \frac{n^2}{3}.
\end{aligned}$$

This implies that $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = \frac{n}{6} + \frac{n^2}{12}$. $\diamondsuit$

13. The number of customers entering a store today is Pois(10). the amount of money spent by a customer is Unif(0, 100). Find the mean and variance of the amount of money that the store takes in on a given day.

**Solution:** Let $N \sim \text{Pois}(10)$ denote the number of customers, and let $X_i \sim \text{Unif}(0,100)$ denote the \$ spent by person $i$. Assume that the $X_i$'s and $N$ are independent.

Then we have

$$\mathsf{E}\Big[\sum_{i=1}^{N} X_i\Big] = \mathsf{E}[N]\mathsf{E}[X_i] = 10 \cdot 50 = 500$$

and

$$\begin{aligned}
\mathsf{Var}\Big(\sum_{i=1}^{N} X_i\Big) &= \mathsf{E}[N]\mathsf{Var}(X_i) + (\mathsf{E}[X_i])^2\mathsf{Var}(N) \\
&= 10(100^2/12) + (50)^2 10 \\
&= 33333. \ \diamondsuit
\end{aligned}$$

14. If $\mathsf{E}[Y|X] = 1$, show that $\mathsf{Var}(XY) \geq \mathsf{Var}(X)$.

**Solution:** First of all, let's write out the following general result. For any reasonable functions $h(X)$ and $g(Y)$, we have

$$\mathsf{E}[h(X)g(Y)] = \mathsf{E}\Big[\mathsf{E}[h(X)g(Y)|X]\Big]$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \mathsf{E}[h(X)g(Y)|X=x] f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} h(x)\mathsf{E}[g(Y)|X=x] f_X(x)\,dx \\
&= \mathsf{E}\Big[h(X)\mathsf{E}[g(Y)|X]\Big].
\end{aligned}
$$

Thus, we have

$$\mathsf{E}[XY] \;=\; \mathsf{E}[X\mathsf{E}[Y|X]] \;=\; \mathsf{E}[X],$$

since, for the current problem, $\mathsf{E}[Y|X] = 1$. Similarly, since $\mathsf{E}[Z^2] \geq (\mathsf{E}[Z])^2$ for any random variable $Z$ (even conditional ones), we have

$$\mathsf{E}[(XY)^2] \;=\; \mathsf{E}[X^2\mathsf{E}[Y^2|X]] \;\geq\; \mathsf{E}[X^2(\mathsf{E}[Y|X])^2] \;=\; \mathsf{E}[X^2].$$

Putting this all together, we get the desired result. $\diamondsuit$

15. $A$ and $B$ play a series of games with $A$ winning each game with probability $p$. The overall winner is the first player to have won two more games than the other.

   (a) Find the probability that $A$ is the overall winner.

   (b) Find the expected number of games played.

   **Solution:** Let $A$ be the event that A is the winner, and let $X$ denote the number of games played. Further, let $Y$ be the number of times A wins in the first two games.

   (a) Then

   $$\mathsf{Pr}(A) \;=\; \sum_{i=0}^{2} \mathsf{Pr}(A|Y=i)\mathsf{Pr}(Y=i) \;=\; 0 + \mathsf{Pr}(A)2p(1-p) + p^2.$$

   This implies that

   $$\mathsf{Pr}(A) \;=\; \frac{p^2}{1-2p(1-p)}. \quad \diamondsuit$$

   (b) Here we have

   $$
   \begin{aligned}
   \mathsf{E}[X] &= \sum_{i=0}^{2} \mathsf{E}[X|Y=i]\mathsf{Pr}(Y=i) \\
   &= 2(1-p)^2 + (2+\mathsf{E}[X])2p(1-p) + 2p^2 \\
   &= 2 + \mathsf{E}[X]2p(1-p).
   \end{aligned}
   $$

This implies that

$$\mathsf{E}[X] \;=\; \frac{2}{1-2p(1-p)}. \quad \diamondsuit$$

16. Suppose $X$ is Pois($\lambda$). The parameter $\lambda$ is itself Exp(1). Find the p.m.f. of $X$.

**Solution:** $X \sim \text{Pois}(\Lambda)$, where $\Lambda \sim \text{Exp}(1)$. Then by total probability,

$$\mathsf{Pr}(X=n) \;=\; \int_{-\infty}^{\infty} \mathsf{Pr}(X=n|\Lambda=\lambda) f_\Lambda(\lambda)\, d\lambda \;=\; \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!} e^{-\lambda}\, d\lambda \;=\; 1/2^{n+1}$$

after a little calculus. $\diamondsuit$

17. A coin is randomly selected from a group of 10 coins. The $i$th coin has probability $i/10$ of coming up heads. The coin is then flipped until a head appears. Let $N$ be the number of flips necessary. What's the p.m.f. of $N$? Is $N$ geometric? (No!) What would you have to do in order to make $N$ geometric?

**Solution:** By total probability,

$$\begin{aligned} \mathsf{Pr}(N=k) &= \sum_{n=1}^{10} \mathsf{Pr}(N=k|\text{coin } n \text{ selected})\mathsf{Pr}(\text{coin } n \text{ selected}) \\ &= \sum_{n=1}^{10} \left(\frac{10-n}{10}\right)^{k-1} \frac{n}{10}\cdot\frac{1}{10}. \quad \diamondsuit \end{aligned}$$

So $N$ is *not geometric.* $\diamondsuit$

You could make $N$ geometric if the coin were re-selected after each flip. (The probability of success has to be constant from flip to flip.) $\diamondsuit$

18. The number of storms this year is Poisson, but with a parameter that is itself Unif(0,5). Find the probability that there are at least three storms this year.

**Solution:** Suppose $X$ is the number of storms. Then

$$\begin{aligned} P(X\ge 3) &= 1-P(X\le 2) \\ &= 1-\int_0^5 P(X\le 2|\Lambda=x)\frac{1}{5}\,dx \end{aligned}$$

$$
\begin{aligned}
&= 1 - \int_0^5 \left[ e^{-x} + xe^{-x} + \frac{x^2 e^{-x}}{2} \right] \frac{1}{5}\, dx \\
&= \frac{51 e^{-5}}{10} + \frac{2}{5} \doteq 0.434. \quad \diamondsuit
\end{aligned}
$$

19. The Continuous Mapping Theorem (CMT) says that if $Z_n$, $n = 1, 2, \ldots$, and $Z$ are random variables such that $Z_n \xrightarrow{\mathcal{D}} Z$ and $h(\cdot)$ is a continuous function, then $h(Z_n) \xrightarrow{\mathcal{D}} h(Z)$. If the $Z_i$'s are i.i.d. with mean $\mu$ and variance $\sigma^2$, use the CMT and WLLN to show that $S^2 \xrightarrow{\mathcal{P}} \sigma^2$, where $S^2$ is the sample variance of $Z_1, Z_2, \ldots, Z_n$.

20. In addition to the same conditions as in the previous problem, assume that $h(\cdot)$ is twice continuously differentiable at $\mu$. Show that

$$
n\Big( \mathsf{E}[h(\bar{Z}_n)] - h(\mu) \Big) \to \frac{h''(\mu)\sigma^2}{2},
$$

where $\bar{Z}_n$ is the sample mean of the $Z_i$'s.