Problem Solving – Probability

Recap: Sample space outcomes {1, 2, 3, 4, 5, 6}

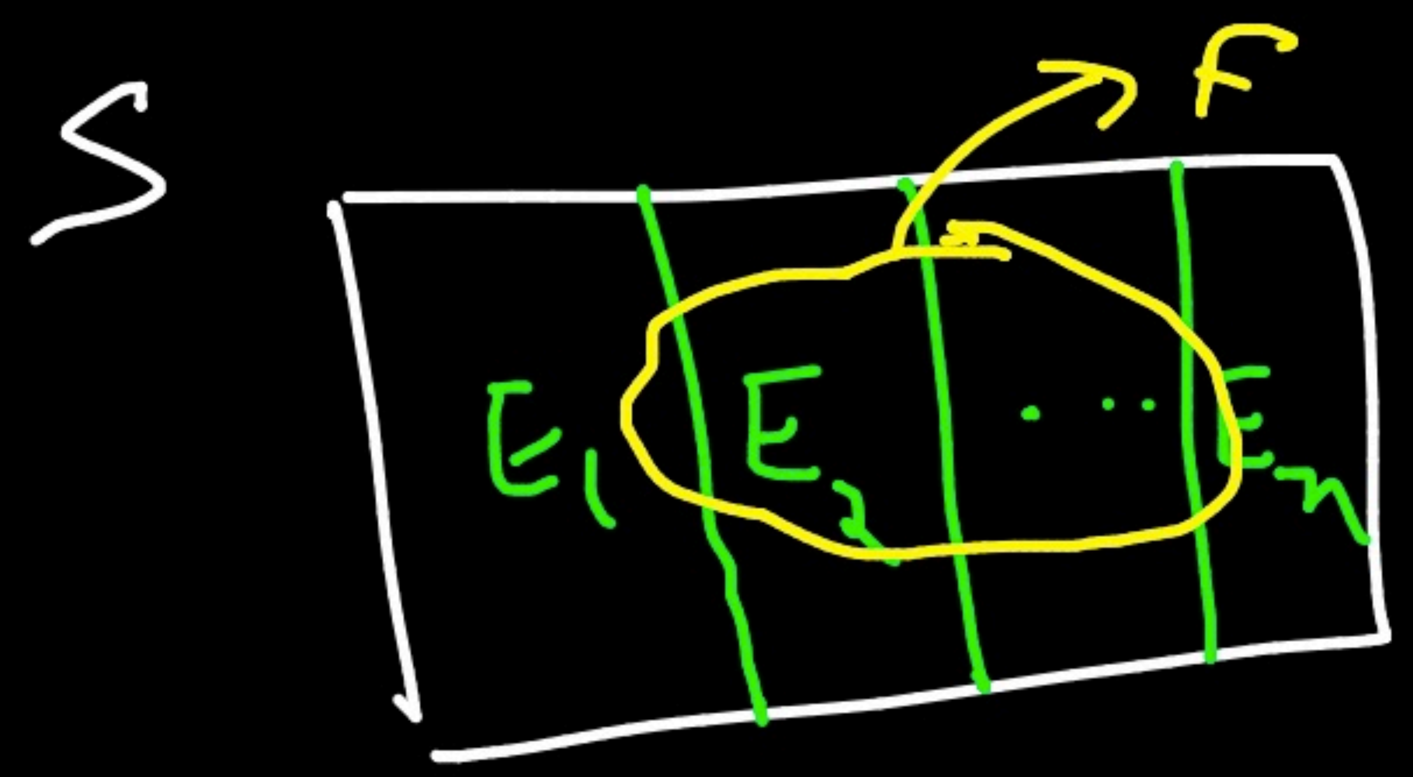
Events

Complement Intersection, union # Mutually exclusive exhaustive

Conditional prob

Independence

Baye's theorem



there are n event

$E_i \cap E_j = \{\}$ → all i, j $i \neq j$

"mutually exclusive"

$E_1 \cup E_2 \cdots \cup E_n = S$ "exhaustive"

$P(A|B) = P(A \cap B)/P(B)$

$$P[F] = P[F \cap E_1] + P[F \cap E_2] + \dots + P[F \cap E_n]$$

$$= P(F|E_1)\,P[E_1] + \cdot \quad + P[F|E_n]\,P[E_n]$$

Family has 2 children. Atleast one is a girl. What is the probability that both are girls? $\longrightarrow$ only $\underline{\underline{28\%}}$ got this right

$$S = \{bb,\ \underline{bg,\ g,b,\ \underline{\underline{g,g}}}\}$$

$\longrightarrow$ numerator

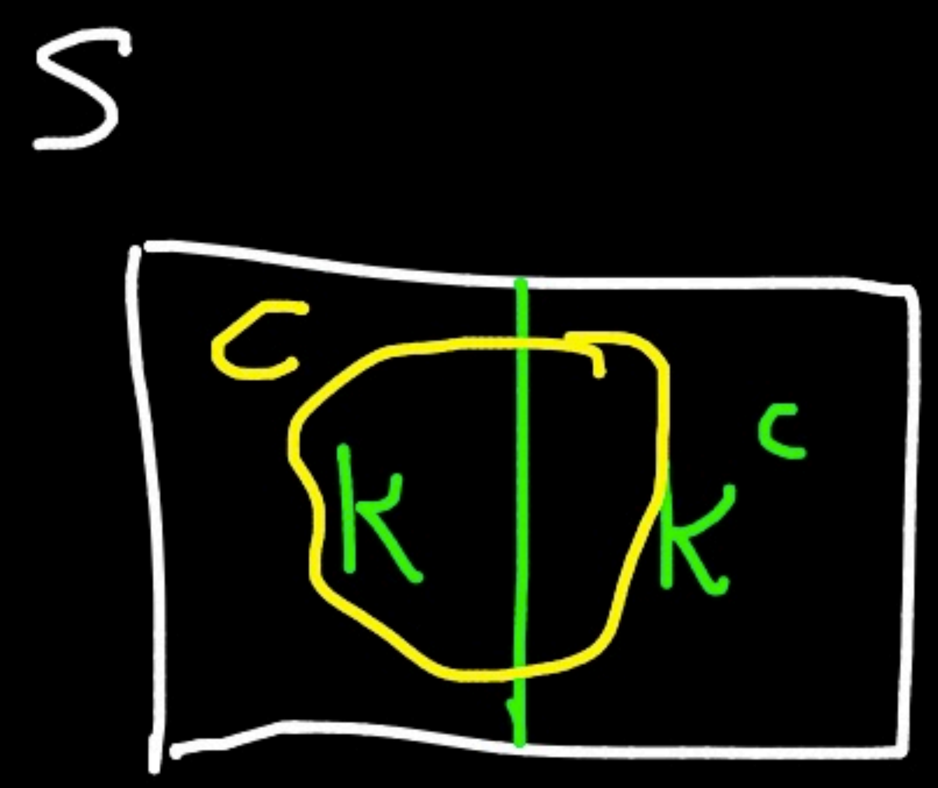
$$P[\underbrace{\{gg\}}_{A} \mid \underbrace{\{bg, gb, gg\}}_{B}] = \frac{P[\{gg\}]}{P[\{bg, gb, gg\}]} = \frac{1/4}{3/4} = \frac{1}{3}$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

4 options

$K \rightarrow$ Knows $\rightarrow 0.8$ $K^c \rightarrow$ guess $\rightarrow 0.2$ $\boxed{25\%}$

$C \rightarrow$ correct $C^c \rightarrow$ wrong



$$P[K|C] = \frac{P[K \cap C]}{P[C]} = \frac{\left(\overbrace{P[C|K]}^{\rightarrow 1}\right)\left(\overbrace{P[K]}^{\rightarrow 0.8}\right)}{P[C]}$$

$$P[C] = P[C|K]\,P[K] + \underbrace{P[C|K^c]}\,P[K^c]$$

$$(1)(0.8) + \left(\frac{1}{4}\right)(0.2)$$

$P[C|K^c] \rightarrow$ How likely are you to get it right if you guess

$P(E) = 0.6$

$P[E|F] = \dfrac{P[E \cap F]}{P[F]}$

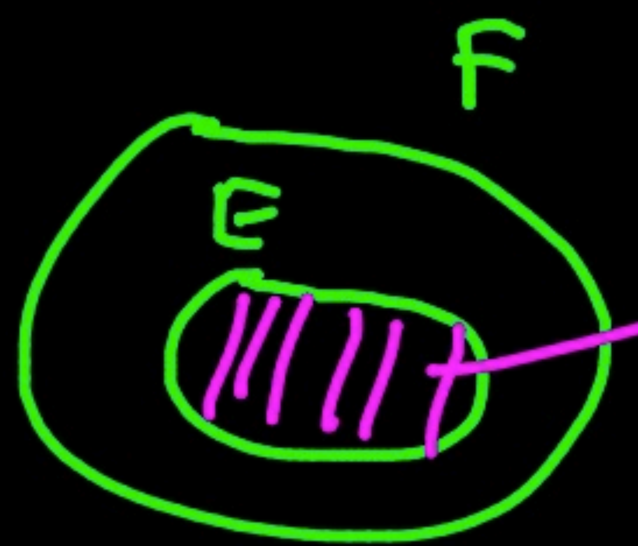
$S = \{1,2,3,4,5,6\}$

1) $E \cap F = \{\ \} \rightarrow P[E \cap F] = 0 \xRightarrow{\text{"implies"}} P[E|F] = 0$

$E = \{1,2\}$

$F = \{3,4\}$

2) $E \subseteq F \rightarrow P[E|F] = \dfrac{P[E \cap F]}{P[F]} = \dfrac{(P[E])}{\underbrace{P[F]}_{\leq 1}} \geq P[E]$



$E \cap F = E$

$P(E) = 1/6$

$E = \{1\}$

$F = \{1,2\}$

$P[E|F] = \dfrac{1/6}{2/6} = \dfrac{1}{2}$

3) $F \subseteq E \rightarrow P[E|F] = \dfrac{P[E \cap F]}{P[F]} = \dfrac{P[F]}{P[F]} = 1$

$E \cap F = F$

$E = \{1,2\}$

$F = \{1\}$

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$F \to$ fair $\qquad F^c \to$ 2-headed

$H \to$ Heads $\qquad T \to$ tails

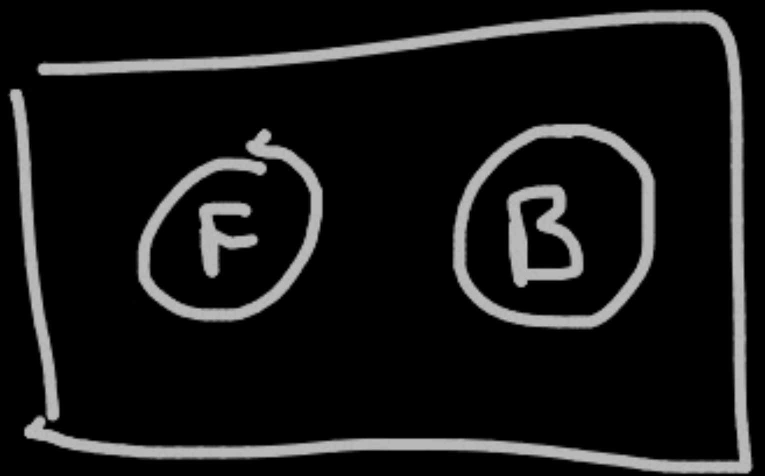
$$P[F \mid H] = \frac{P[H \mid F]\,P[F]}{P[H \mid F]\,P[F] + P[H \mid F^c]\,P[F^c]} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{1}{3}$$

5 8.1.

$$P[F \mid HH] = \frac{\underline{P[HH \mid F]}\,P[F]}{P[HH \mid F]\,P[F] + P[HH \mid F^c]\,P[F^c]} = \frac{\underline{\frac{1}{4}}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}} = \frac{1}{5}$$

$$P[F \mid HHT] = 1$$

Box

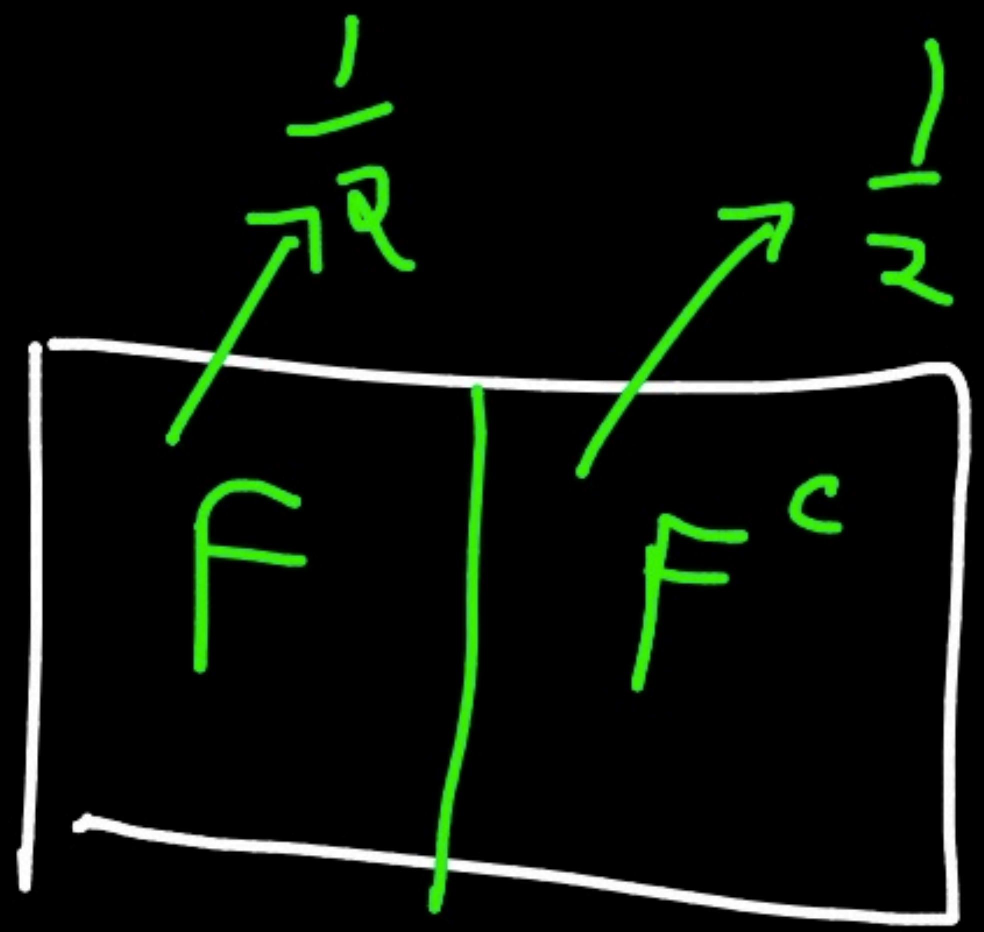


F : Fair coin

F^c) B : Biased (2-headed)

$$S = \{H, T\}$$

$H \rightarrow \frac{3}{4}$, $T \rightarrow \frac{1}{4}$



$$P[H] = P[H \mid F]\,P[F] + P[H \mid F^c]\,P[F^c]$$

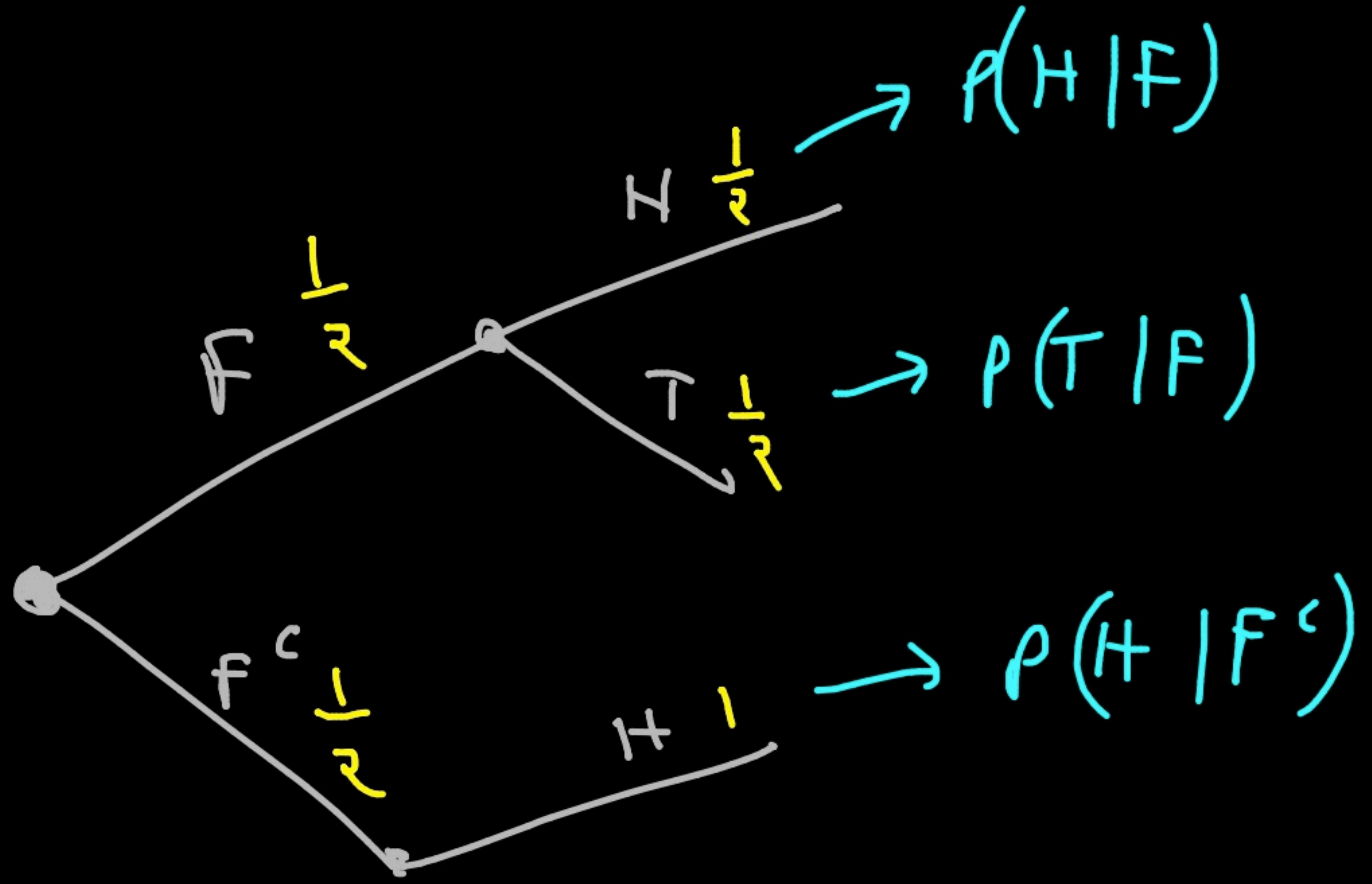
$$= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= \frac{3}{4}$$

$$S = \{H_1, H_2, H_3, T\}$$

set

lost infor


F
1/2
H 1/2
T 1/2
F^c 1/2
H 1
P(H|F)
P(T|F)
P(H|F^c)

Airline

5% of people won't show up. → each person 5% prob of not coming

50 seats. They have sold 52 tickets.

A → Probability that everyone has a seat available

① → Prob. that all 52 people show up → $(0.95)(0.95)\dots(0.95) = (0.95)^{52}$

② → Prob that 51 people show up → $52\,(0.95)^{51}\,(0.05)$

$$\underbrace{1 - (0.95)^{52} - 52\,(0.95)^{51}\,(0.05)}_{\text{someone won't have a seat}}$$

$\boxed{10.45}$

$A^c \rightarrow \begin{Bmatrix} 51 \\ 52 \end{Bmatrix}$

$H \rightarrow 0.95 \rightarrow$ toss 52 times

$HHH \cdots H \rightarrow (0.95)^{52}$

One tail

$$52 \begin{cases} THHH \cdots H \\ HTHH \cdots H \\ HHTH \cdots H \\ \\ \end{cases} \rightarrow 52\,(0.95)^{51}(0.05)$$

H/T H/T 52 boxes

$\square\ \square\ \square \cdots \square$

2^{52} ways of writing these sequences

$P(H)\,P(H)\,P(T)\,P(H) \cdots P(H)$

$(0.95)(0.95)(0.05)(0.95) \cdots (0.95)$

A and B toss a coin alternatively till one of them gets H. A starts the game. Prob. that A wins

$$S = \{\underbrace{H}_{p},\ TH,\ \underbrace{TTH}_{(1-p)^2 p},\ TTTH,\ \underbrace{TTTTH}_{(1-p)^4 p},\ TTTTTH,\ TTTT\ldots\}$$

$$P[A \text{ wins}] = p + (1-p)^2 p + (1-p)^4 p + (1-p)^6 p$$

$$= p + rp + r^2 p + r^3 p \longrightarrow C$$

$(1-p)^2 = r$

S_1 First serve $\rightarrow$ as fast possible, accuracy is low (miss)

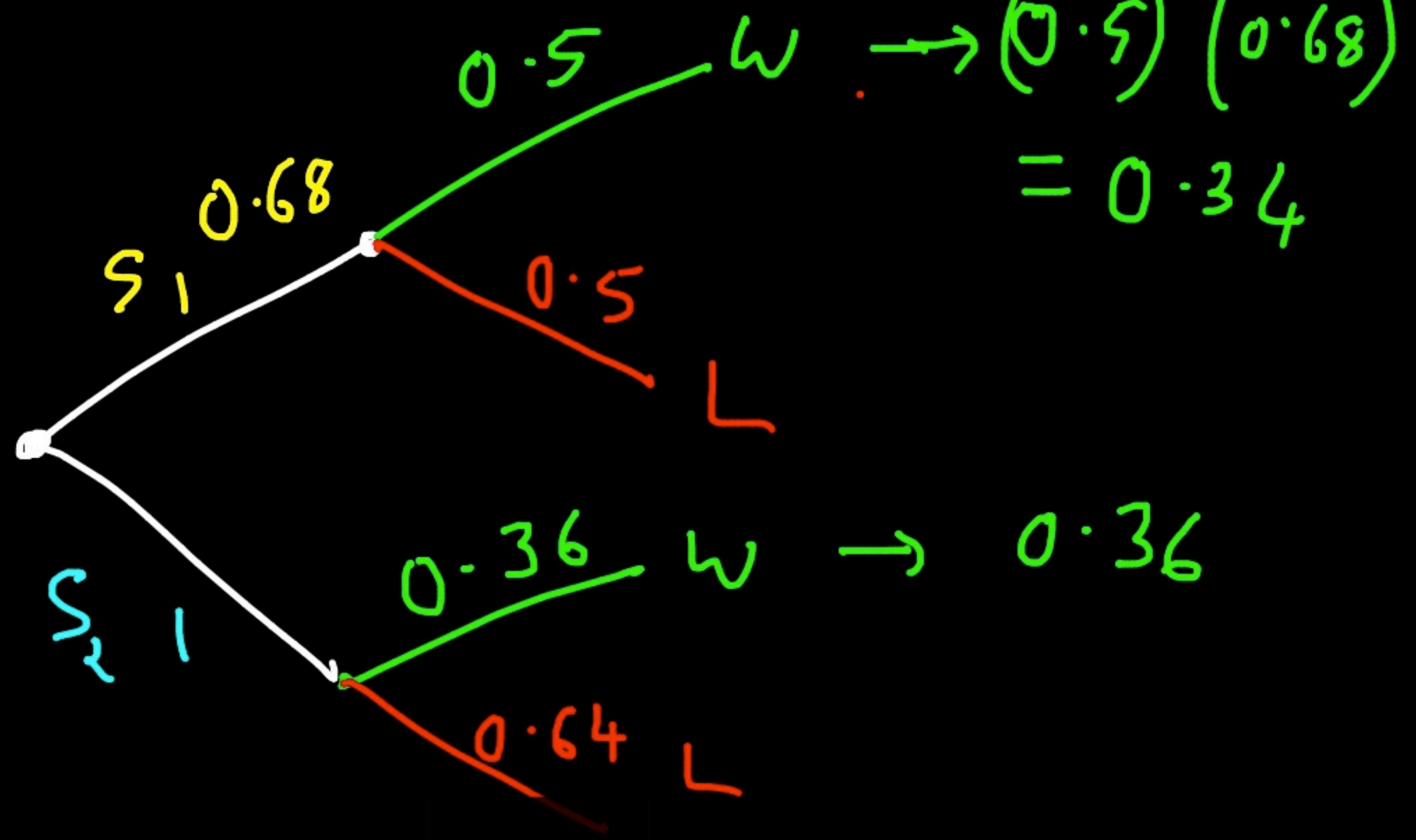
68% inside the box $\rightarrow$ 0.68

S_2 Second serve $\rightarrow$ slow, accuracy high $\rightarrow$ 100%

Win that point

$$P[W \mid S_1] = 0.5 \rightarrow 0.68$$

$$P[W \mid S_2] = 0.36$$



Optional Advanced

Fair Coin : 10 times

Find no. of letter $\leftarrow N$

F_0	F_1	$\dots$	F_{10}

$N=0$ $N=1$ $N=10$

H H T H T T T H T H $\longrightarrow$ one possible seq.

$\times$ $\times$ ... $\times$

0.5 0.5 ... 0.5

X : no. of heads remaining

$$P[X=4] = \sum_{n=0}^{10} \underline{P[X=4 \mid N=n]}\; P[N=n]$$

Given that there are 'n' letters remaining prob of 4 heads

$$= \sum_{n=4}^{10} {}^{n}C_{4} \left(\frac{1}{2}\right)^{n} \; \underline{\underline{{}^{10}C_{n} \left(\frac{1}{2}\right)^{10}}}$$

$$P[N=n] = {}^{10}C_{n} \left(\frac{1}{2}\right)^{10}$$

In 2nd question

$$P[N=n] = \frac{1}{10}$$

5 - 5 in a tiebreaker. $P[\text{Nadal wins a point}] = p$

↳ winning means a difference of 2 (7-5, 8-6, 9-7, 10-8)

A: Nadal winning

$\{wl, ww, lw, ll\}$

w	w	l	l
l	w	w	l

$$P[A] = \underline{P[A \mid wl]}\, P[wl] + P[A \mid ww]\, P[ww] + P[A \mid lw]\, P[lw] + P[A \mid ll]\, P[ll]$$

$$= P[A]\, p(1-p) + 1 \cdot p^2 + P[A]\, p(1-p) + 0$$

$$S = \{\underline{H}, TH, \underline{TTH}, \ldots\} \qquad p + (1-p)^2 p + (\ )^4 p \ldots$$

$$P[A] = P[A|H]\,P[H] + P[A|T]\,P[T]$$

$$= 1 \cdot p + \underbrace{[1 - P(A)]}_{\text{B wins}}(1-p) = p + 1 - p - P(A) + pP(A)$$

$$= \cancel{p} + 1 - \cancel{p} - P(A) + p\,P(A)$$

$$P(A)[1 + 1 - p] = 1$$

$$P(A) = \frac{1}{2-p}$$

$$C = p + \cancel{rp} + \cancel{r^2 p} + \cancel{r^3 p} + \cdots\cdots$$

geometric series

$$Cr = \cancel{rp} + \cancel{r^2 p} + \cancel{r^3 p} + \cdots$$

$$C - Cr = p$$

$$C = \frac{p}{1-r} = \frac{p}{1-(1-p)^2} = \frac{p}{1-\left[1+p^2-2p\right]}$$

$$= \frac{p}{2p-p^2} = \frac{1}{2-p}$$