Differential Calculus

[optimisation]

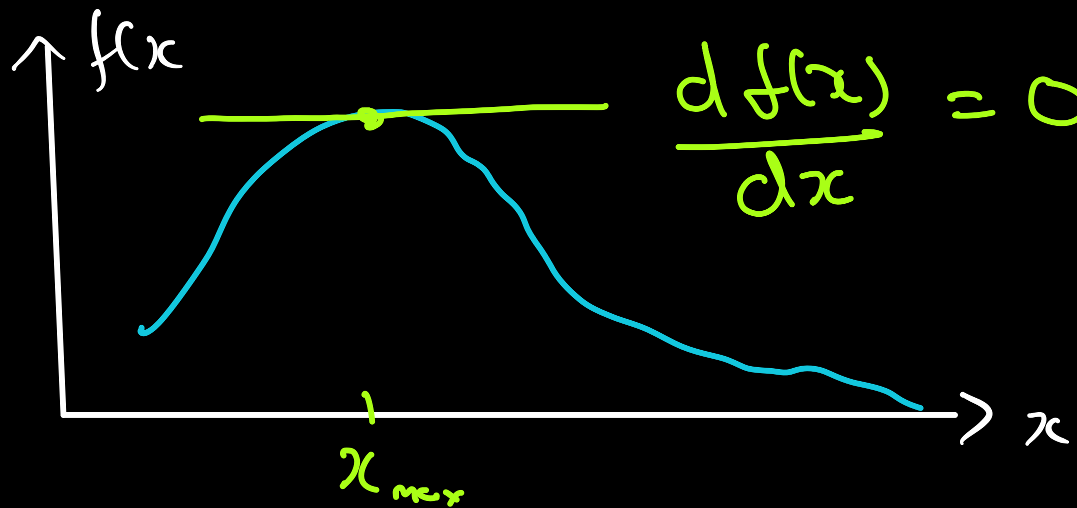
→ In depth discussion on differential calculus

Recap

→ Optimisation problem :

$$\max_{\vec{w},\, w_0} \sum_{i=1}^{n} y_i \cdot \frac{w^T x_i + w_0}{\|w\|}$$

→ We need some calculus :

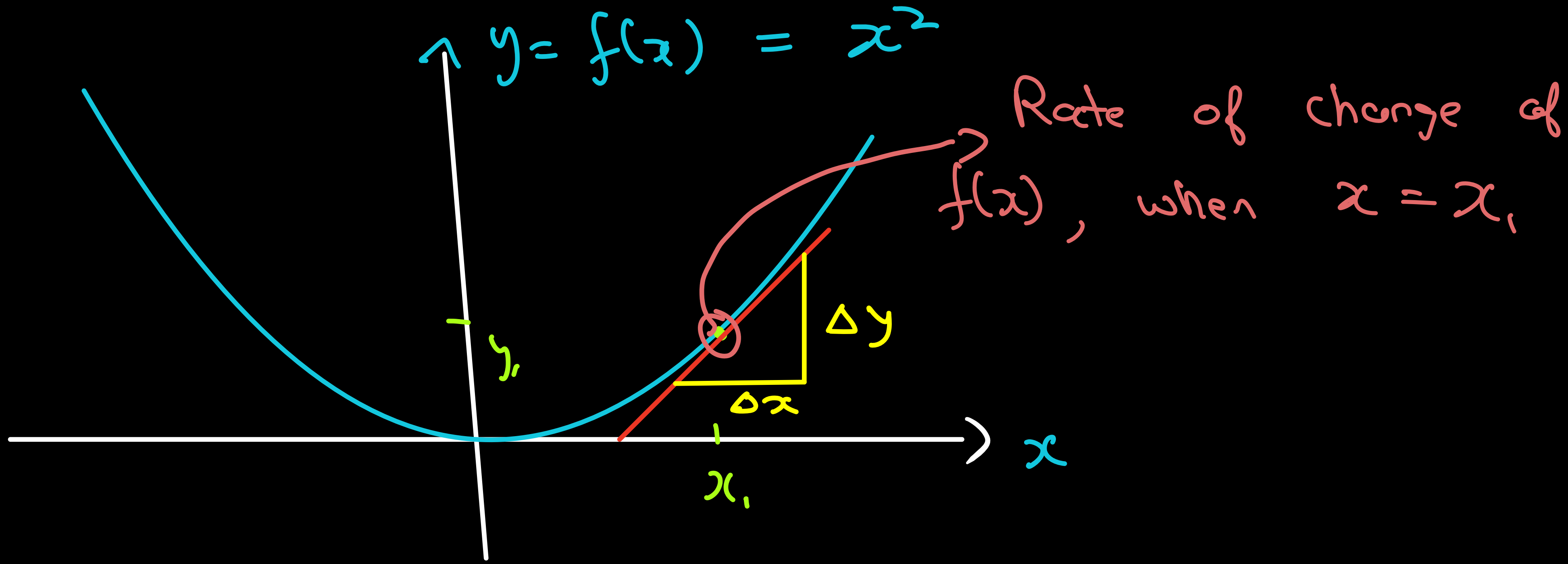


$$\frac{d f(x)}{dx} = 0$$

$\therefore f'(x_{max}) = 0$ | Solve for x_{max}

Differentiation

↳ Rate of change on any function w.r.t one or more variable



→ The angle of this red line will change w.r.t x

$\therefore \dfrac{d f(x)}{dx}$ is a function of $x \rightarrow f'(x)$

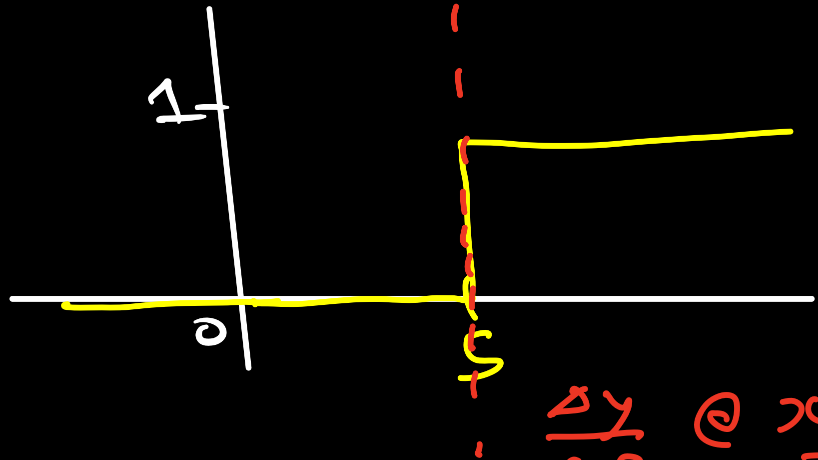
Differentiable functions

Let's start with a non-continuous function

$x'(4) = ? \rightarrow 0$

$x'(7) = ? \rightarrow 0$

$x'(5) = ?$ slope of line is ∞



$\frac{\Delta y}{\Delta x}$ @ $x = 5$

$$\frac{f(5.00000\ldots1) - f(4.99999\ldots9)}{0.0000\ldots1}$$

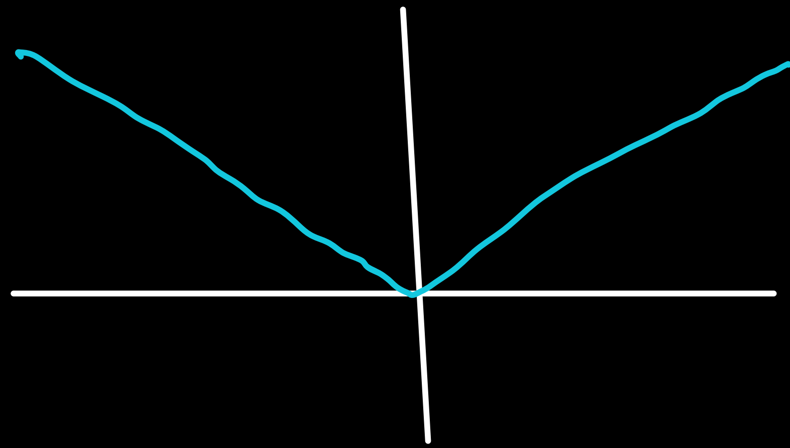
$$= \frac{1 - 0}{0.00000\ldots1} \sim \infty$$

The funcⁿ suddenly changes

Differentiable functions are continuous

Eg 2

$f(x) = |x|$



Q: Is this continuous?

$\lim_{x \to 0^+} |x| = 0$

$\lim_{x \to 0^-} |x| = 0$ ✓

$$|x| = \begin{cases} x & ; x > 0 \\ -x & ; x < 0 \end{cases}$$

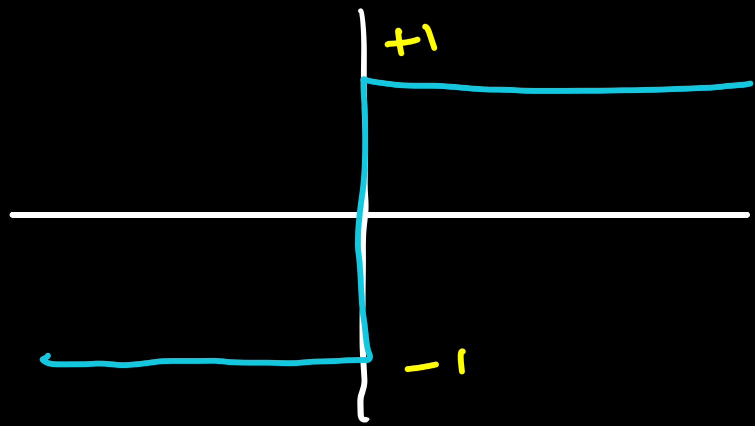
$\frac{d|x|}{dx}$ @ $x = 0$?

$$\therefore \frac{d}{dx}(x) = 1 \; ; \; x > 0$$
$$= -1 \; ; \; x < 0$$

The function changes suddenly

→ code

$$f(x) = \begin{cases} 1 \ ; & x > 0 \\ -1 \ ; & x < 0 \end{cases}$$



at $f'(0) = \underbrace{-1 \ , \quad 1}_{??}$

Two possible $\underline{\text{sol}^n}$, → Hence not differentiabl

'0' is just the avg of -1 & 1

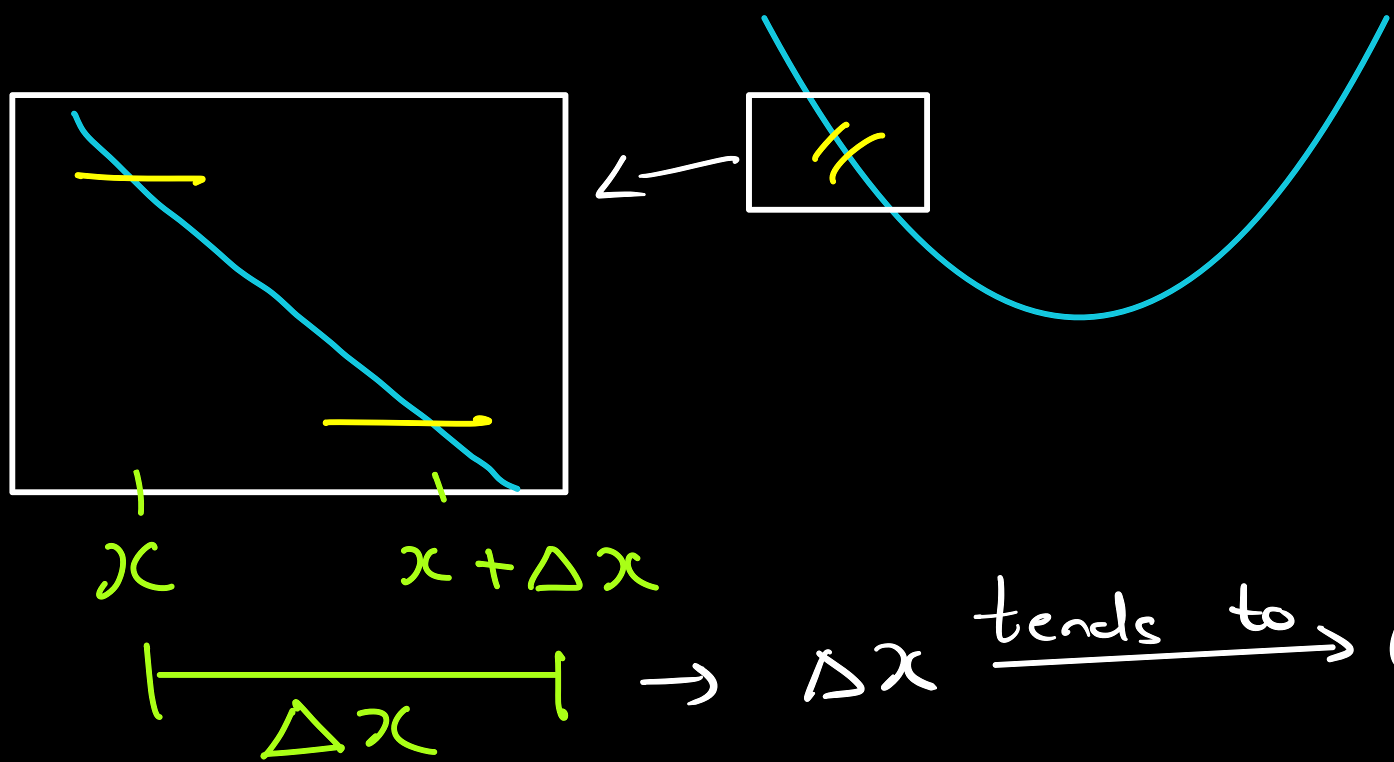
Conditions for differentiability

Function should be smooth

↳ $f(x)$ is continious

↳ $f'(x)$ is continious

Differenciation using First Principals



$\rightarrow \Delta x \xrightarrow{\text{tends to}} 0$

$$\therefore \frac{d\,f(x)}{dx} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\lim\limits_{\Delta x \to 0}$

Put $f(x) = x^2$

$$\therefore \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \frac{\cancel{x^2} + 2\Delta x . x + \Delta x^2 - \cancel{x^2}}{\Delta x}$$

$$\lim_{\Delta x \to 0} : 2x + \Delta x$$

$$= \boxed{2x}$$

$$\therefore \frac{d}{dx} x^2 = 2x$$

This is how we get the power rule:

$$\frac{d}{dx} x^n = n x^{n-1}$$

Similarly other rules have been derived, some however may have used infinite series expansions.

$\frac{d}{dx}c = 0$	Constant Rule
$\frac{d}{dx}x^n = nx^{n-1}$	Power Rule
$\frac{d}{dx}\sin(x) = \cos(x)$	Trigonometric Rules
$\frac{d}{dx}\cos(x) = -\sin(x)$	
$\frac{d}{dx}b^x = b^x \ln(b)$	Exponential Rule
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	Logarithmic Rule

Sum rule	$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Difference rule	$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
Constant multiple rule	$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$
Constant rule	$\frac{d}{dx}k = 0$

Refer to this for assessments

Rules for differenciation

Sum Rule: $\frac{d}{dx}[f(x) + g(x)]$

$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

Eg: $x^2 + 7x + 12$

$= 2x + 7 + 0$

$= \underline{\underline{2x + 7}}$

Product rule: $\frac{d}{dx}[f(x) \cdot g(x)]$

$$= f(x) \cdot \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$$

Eg: $(x+3)(x+4)$

$$= (x+3) \frac{d}{dx}(x+4) + \frac{d}{dx}(x+3) \cdot (x+4)$$

$$= (x+3) \cdot 1 + (x+4) \cdot 1$$

$$= \underline{\underline{2x+7}}$$

Eg: 2 $\quad x\log(x)$

$$= x \cdot \frac{d}{dx}\log(x) + \log(x) \cdot \frac{dx}{dx}$$

$$= x \cdot \frac{1}{x} + \log(x) \cdot 1$$

$$= \boxed{x + \log(x)}$$

Quotient Rule : $\frac{d}{dx} f(x)/g(x)$

↓

Just use product rule $\Rightarrow f(x) \cdot g^{-1}(x)$

$$= \frac{g(x) f'(x) - g'(x) f(x)}{g(x)^2}$$

Eg: $\frac{d}{dx}\left[\frac{\log(x)}{x}\right]$

$$\rightarrow \frac{\log(x)\cdot 1 - \frac{1}{x}\cdot x}{x^2} = \frac{\log(x)-1}{x^2}$$

<u>Chain Rule:</u> $\frac{d}{dx} f(g(x))$

$$= f'(g(x))\cdot g'(x)$$

Eg: e^{5x^2+2}

$\rightarrow g(x) = 5x^2 + 2 = \underline{\underline{y}}$

$$= \frac{d e^y}{dy} \cdot \frac{dy}{dx} = e^y \cdot \frac{dy}{dx}$$

$$= e^{5x^2+2} \cdot \frac{d}{dx}(5x^2+2)$$

$$= \underline{10x \cdot e^{5x^2+2}}$$

Optimisation using calculus

At the point of inflection, the rate of change w.r.t x will become **zero**

infection point: Maxima or Minima

Steps to find optima

→ given : $f(x)$

→ calc : $f'(x)$

→ find the roots of $f'(x)$

∴ solve for x ; $f'(x) = 0$

→ these are all the pt' of inflection or roots.

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→ find $f''(x)$

↳ if $f''(x)$ @ $x_{opt} > 0$ → min

↳ if $f''(x)$ @ $x_{opt} < 0$ → max