1. (CO 3) Consider the basis $b_1 = [\frac{1}{3}, \frac{3}{2}]^T$ and $b_2 = [\frac{2}{3}, \frac{1}{3}]^T$ of $\mathbf{R}^2$. Suppose a vector has coordinates $[2, 1]^T$ with respect to the basis b_1, b_2, what are its coordinates with respect to the standard basis of $\mathbf{R}^2$? $\boxed{3}$

   Soln: Solve for x, y such that $2b_1 + 1b_2 = xe_1 + ye_2$.

2. (CO 2) Let A be an $n \times n$ matrix. Let $\vec{x}$ and $\vec{y}$ be Eigen vectors of A with Eigen values a and b respectively, $a \neq b$. Show that $\vec{x}$ and $\vec{y}$ are linearly independent. $\boxed{3}$

   Soln: From the definition Eigen vectors, x, y are non-zero vectors. Given $Ax = ax$ and $Ay = by$. Suppose $\alpha x + \beta y = 0$, where α, β are non-zero scalars. then $A(\alpha x + \beta y) = 0$ as well. This gives another equation $a\alpha x + b\beta y =$. Multiplying $\alpha x + \beta y = 0$ with a, we get $a\alpha x + a\beta y = 0$. Subtracting, we get $b\beta y - a\beta y = 0$. That is, $(b - a)0\beta y = 0$. Now, unless $a = b$, we get $y = 0$, a contradiction.

3. (CO 2) Let A be an $n \times n$ **symmetric** matrix. Let $\vec{x}$ and $\vec{y}$ be Eigen vectors of A with Eigen values a and b respectively, $a \neq b$. Show that $\vec{x}$ and $\vec{y}$ are orthogonal (with respect to the standard inner product). $\boxed{3}$

   Soln: By symmetry, the the two inner products (Ax, y) (x, Ay). are equal, where $()$ is the standard inner product. Note that a symmetric matrix has real Eigen vectors and values. Hence, we have:
   $a(x, y) = (ax, y) = (Ax, y) = (x, Ay) = (x, by) = b(x, y)$, or $(a - b)(x, y) = 0$. Since $a \neq b$, $(x, y) = 0$.

4. (CO 2) In R^3, consider the vector $v = [3, 2, 1]^T$. Find a vector u plane $x + y + z = 0$ such that $v - u$ is perpendicular to the plane. $\boxed{3}$

   Soln: This was solved in the class.

5. Let B be an orthogonal real matrix. Show that $det(B) = \pm 1$. $\boxed{3}$

   Soln: By definition $(Bx, By) = (x, y)$ for all vectors x, y. That is, $x^T B^T B y = x^T y$, or $B^T B = I$, where I is the identity matrix. Consequently, $det(B^T B) = det(B^T) det(B) = det(I) = 1$. Noting that $det(B^T) = det(B)$, we get $det(B)^2 = 1$. That is, $det(B) = \pm 1$.

6. (CO 3) Let A be the matrix $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ Suppose we attempt to define an inner product on $\mathbf{R}^2$ by $(\vec{x}, \vec{y}) = \vec{x}^T A \vec{y}$. $\boxed{6}$

   1. Show that the inner product satisfies $(\vec{x}, \vec{x}) > 0$ for every $\vec{x} \neq 0$.
   2. Starting with the standard basis e_1, e_2, use Gram Schmidt process to find an orthonormal basis for $\mathbf{R}^2$ with respect to the inner product.

   Soln:

   1. For the first part, linearity and symmetry follows from the fact that A is a symmetric matrix. To show positivity, $(x, y)A(x, y)^T = [3x + y, x + y][x, y]^T = 3x^2 + 2xy + y^2 = 2x^2 + (x + y)^2 \geq 0$.
   2. Note that $e_1 = [1, 0]^T$ and $e_2 = [0, 1]^T$ are not orthogonal with respect to this inner product as $[1, 0]A[0, 1]^T \neq 0$. We can normalize e_1 and find $b_1 = \frac{e_1}{||e_1||}$, where $||e_1|| = e_1 A e_1^T$. Now, let $\tilde{b}_2 = e_2 - e_2^T A b_1$. Note that $\tilde{b}_2$ will be orthogonal to b_1. Now normalize the length of $\tilde{b}_2$ to get b_2.