Microeconomics B Problem Set 9

Bayesian Nash equilibrium

1) Review the intuition from the 'Doctor' example in lecture 7, and then use Bayes' rule to solve the following problem:

> *"A cab was involved in a hit and run accident at night. 85% of the cabs in the city are Green and* 15% *are Blue. A witness later recalls that the cab was Blue, and we know that this witness' memory is reliable* 80% *of the time. Given the statement from the witness, calculate the probability that the cab involved in the accident was actually Blue."*

Lets define bayes rule

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Lets define the events of the case

A: The cab is Green

B: The cab is Blue

and lets write up the probabilities

Probability that a cab was blue	$P(B) = 0.15$
Probability that a cab was green	$P(A) = 0.85$
Probability that a witness remember color correctly	$P(\text{obs } A \mid A) = 0.80$
Probability that a witness remember color incorrectly	$P(\text{obs } B \mid A) = 0.20$

Now we can use Bayes' rule to find the probability that a cab from the accident is blue, as the witness testifies

$$\begin{aligned} P(B \mid \text{obs } B) &= \frac{P(\text{obs } B \mid B) \cdot P(B)}{P(\text{obs } B)} \\ &= \frac{P(\text{obs } B \mid B) \cdot P(B)}{P(\text{obs } B \mid B) \cdot P(B) + P(\text{obs } B \mid G) \cdot P(G)} \\ &= \frac{0.8 \cdot 0.15}{0.8 \cdot 0.15 + 0.2 \cdot 0.85} = 0.414 \approx 41\% \end{aligned}$$

2) Consider the following static game, where a is a real number:

	L	R
U	$2, 1$	$0, a$
D	$0, 1$	$1, a$

a) Suppose that $a = 2$. Does any player have a dominant strategy? What about when $a = -2$?

Lets look at the two cases seperately

Case 1: $a = 2$

	L	R
U	$2, 1$	$0, 2$
D	$0, 1$	$1, 2$

with $a = 2$, player 2 has a strictly dominating strategy in R.
Player 1 does not have a strictly dominating strategy.

Case 1: $a = -2$

	L	R
U	$2, 1$	$0, -2$
D	$0, 1$	$1, -2$

with $a = -2$, player 2 has a strictly dominating strategy in L.
Player 1 does not have a strictly dominating strategy.

b) **Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.**

The definition of a static Bayesian game is

Definition 1 (The Normal Form Representation of an n-player Bayesian Game).
The normal form representation of an n-player static Bayesian game specifies the players' action spaces $A_1, A_2, ..., A_n$, their type spaces $T_1, T_2, ..., T_n$, their beliefs $p_1, p_2, ..., p_n$, and their payoff functions $u_1, u_2, ..., u_n$. Player i's type t_i is privately known by player i, determines player i's payoff function $u_i(a_1, a_2, ..., a_n)$, and is a member of the set of possible types T_i. Player i's beliefs $p_i(t_{-i} \mid t_i)$ describes i's uncertainty about the $n-1$ other players possible types t_{-i} given i's own type t_i. We denote this game by $G = \{A_1, A_2, ..., A_n; T_1, T_2, ..., T_n; p_1, p_2, ..., p_n; u_1, u_2, ..., u_n\}$.

The normal form representation of an n-player Bayesian game specifies ...

Lets setup the game as a normal form Bayesian game

Players Player 1 and player 2

Type spaces $T_1 : \{t\}$
$T_2 : \{t_{a=2}, t_{a=-2}\}$

Action Sets $A_1 : \{U, D\}$
$A_2 : \{L, R\}$

Strategy Sets $S_1 : \{U, D\}$
$S_2 : \{LL, LR, RL, RR\}$

Beliefs
$p_1(t_{a=2} \mid t) = \frac{1}{2}$
$p_1(t_{a=-2} \mid t) = \frac{1}{2}$
$p_2(t_{a=2} \mid t_{a=2}) = 1$
$p_2(t_{a=-2} \mid t_{a=-2}) = 1$
$p_2(t_{a=2} \mid t_{a=-2}) = 0$
$p_2(t_{a=-2} \mid t_{a=2}) = 0$

Payoffs We can illustrate the payoffs in two payoff matrices

Type $t_{a=2}$

	L	R
U	$2, 1$	$0, 2$
D	$0, 1$	$1, 2$

Type $t_{a=-2}$

	L	R
U	$2, 1$	$0, -2$
D	$0, 1$	$1, -2$

notice that these matrixes are payoff matrixes and not the normal form representation of the game. The normal form representation, specifies all of the above.

c) Find the Bayes-Nash equilibrium of the game described in b).

Lets definde a Bayesian Nash equilibria

Definition 2 (A Pure Strategy Bayesian Nash Equilibria).
In the static Bayesian game $G = \{A_1, A_2, ..., A_n; T_1, T_2, ..., T_n; p_1, p_2, ..., p_n; u_1, u_2, ..., u_n\}$, *the strategies* $s^* = \left(s_1^*, s_2^*, ..., s_n^*\right)$ *are a pure strategy Bayesian Nash equilibrium if for each player* i *and for each of* i*'s types* t_i *in* T_i, $s_i^*(t_i)$ *solves*

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(i_1), s_2^*(t_2), ..., s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..., s_n^*(t_n); t\right) p_i(t_{-i} \mid t_i)$$

Thts is, no player wants to change his or her strategy, even if the change involves only one action by one type.

We can setup a bi-matrix specifying the expected payoff,

	LL	LR	RL	RR
U				
D				

to calculate the payoff's in the different scenarios, lets start by giving a letter to each payoff's to know where each payoffs comes from

Type $t_{a=2}$

	L	R
U	$2^a, 1^b$	$0^c, 2^d$
D	$0^i, 1^j$	$1^k, 2^l$

Type $t_{a=-2}$

	L	R
U	$2^e, 1^f$	$0^g, -2^h$
D	$0^m, 1^n$	$1^o, -2^p$

calculating the expected payoffs Lets calculate the expected payoffs for player 1 playing T

$$u_1(U, LL) = p_1(t_{a=2} \mid t) \cdot 2^a + (1 - p_1(t_{a=2} \mid t)) \cdot 2^e = \frac{1}{2} \cdot 2^a + \frac{1}{2} \cdot 2^e = 1 + 1 = 2$$
$$u_2(U, LL) = p_1(t_{a=2} \mid t) \cdot 1^b + (1 - p_1(t_{a=2} \mid t)) \cdot 1^f = \frac{1}{2} \cdot 1^b + \frac{1}{2} \cdot 1^f = \frac{1}{2} + \frac{1}{2} = 1$$

$$u_1(U, LR) = p_1(t_{a=2} \mid t) \cdot 2^a + (1 - p_1(t_{a=2} \mid t)) \cdot 0^g = \frac{1}{2} \cdot 2^a + \frac{1}{2} \cdot 0^g = 1 + 0 = 1$$
$$u_2(U, LR) = p_1(t_{a=2} \mid t) \cdot 1^b + (1 - p_1(t_{a=2} \mid t)) \cdot \left(-2^h\right) = \frac{1}{2} \cdot 1^b - \frac{1}{2} \cdot 2^h = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$u_1(U, RL) = p_1(t_{a=2} \mid t) \cdot 0^c + (1 - p_1(t_{a=2} \mid t)) \cdot 2^e = \frac{1}{2} \cdot 0^c + \frac{1}{2} \cdot 2^e = 0 + 1 = 1$$
$$u_2(U, RL) = p_1(t_{a=2} \mid t) \cdot 2^d + (1 - p_1(t_{a=2} \mid t)) \cdot 1^f = \frac{1}{2} \cdot 2^d + \frac{1}{2} \cdot 1^f = 1 + \frac{1}{2} = \frac{3}{2}$$

$$u_1(U, RR) = p_1(t_{a=2} \mid t) \cdot 0^c + (1 - p_1(t_{a=2} \mid t)) \cdot 0^g = \frac{1}{2} \cdot 0^c + \frac{1}{2} \cdot 0^g = 0 + 0 = 0$$
$$u_2(U, RR) = p_1(t_{a=2} \mid t) \cdot 2^d + (1 - p_1(t_{a=2} \mid t)) \cdot \left(-2^h\right) = \frac{1}{2} \cdot 2^d - \frac{1}{2} \cdot 2^h = 1 - 1 = 0$$

and the expected payoff for player 1 playing B

$$u_1(B, LL) = p_1(t_{a=2} \mid t) \cdot 0^i + (1 - p_1(t_{a=2} \mid t)) \cdot 0^m = \frac{1}{2} \cdot 0^i + \frac{1}{2} \cdot 0^m = 0 + 0 = 0$$
$$u_2(B, LL) = p_1(t_{a=2} \mid t) \cdot 1^j + (1 - p_1(t_{a=2} \mid t)) \cdot 1^n = \frac{1}{2} \cdot 1^j + \frac{1}{2} \cdot 1^n = \frac{1}{2} + \frac{1}{2} = 1$$

$$u_1(B, LR) = p_1(t_{a=2} \mid t) \cdot 0^i + (1 - p_1(t_{a=2} \mid t)) \cdot 1^o = \frac{1}{2} \cdot 0^i + \frac{1}{2} \cdot 1^o = 0 + \frac{1}{2} = \frac{1}{2}$$
$$u_2(B, LR) = p_1(t_{a=2} \mid t) \cdot 1^j + (1 - p_1(t_{a=2} \mid t)) \cdot (-2^p) = \frac{1}{2} \cdot 1^j - \frac{1}{2} \cdot 2^p = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$u_1(B, RL) = p_1(t_{a=2} \mid t) \cdot 1^k + (1 - p_1(t_{a=2} \mid t)) \cdot 0^m = \frac{1}{2} \cdot 1^k + \frac{1}{2} \cdot 0^m = \frac{1}{2} + 0 = \frac{1}{2}$$
$$u_2(B, RL) = p_1(t_{a=2} \mid t) \cdot 2^l + (1 - p_1(t_{a=2} \mid t)) \cdot 1^n = \frac{1}{2} \cdot 2^l + \frac{1}{2} \cdot 1^n = 1 + \frac{1}{2} = \frac{3}{2}$$

$$u_1(B, LR) = p_1(t_{a=2} \mid t) \cdot 1^k + (1 - p_1(t_{a=2} \mid t)) \cdot 1^o = \frac{1}{2} \cdot 1^k + \frac{1}{2} \cdot 1^o = \frac{1}{2} + \frac{1}{2} = 1$$
$$u_2(B, LR) = p_1(t_{a=2} \mid t) \cdot 2^l + (1 - p_1(t_{a=2} \mid t)) \cdot (-2^p) = \frac{1}{2} \cdot 2^l - \frac{1}{2} \cdot 2^p = 1 - 1 = 0$$

such that the expected payoff matrix will be

	LL	LR	RL	RR
U	$2, 1$	$1, -\frac{1}{2}$	$1, \frac{3}{2}$	$0, 0$
D	$0, 1$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{3}{2}$	$1, 0$

to find nash equilibria, we plug in for the best responses.

- If player 1 plays U, then player 2's best response is to play RL
- If player 1 plays D, then player 2's best response is to play RL
- If player 2 plays LL, then player 1's best response is to play U
- If player 2 plays LR, then player 1's best response is to play U
- If player 2 plays RL, then player 1's best response is to play U
- If player 2 plays RR, then player 1's best response is to play D

	LL	LR	RL	RR
U	$\underline{2}, 1$	$\underline{1}, -\frac{1}{2}$	$\underline{1}, \underline{\frac{3}{2}}$	$0, 0$
D	$0, 1$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \underline{\frac{3}{2}}$	$\underline{1}, 0$

such that we have a one BNE in pure strategies

$$\text{BNE} = \{(\text{U}, \text{RL})\}$$

3)

- Spillere: To virksomheder, $N = \{1, 2\}$,

- Handlinger: mængder $q_1, q_2 \in [0; \infty) = S_1 = S_2$,

- Nytter: ens marginalomkostning $c > 0$ og efterspørgsel:

$$P(Q; a) = a - Q, \quad a > 0, Q \equiv q_1 + q_2,$$

hvilket giver ex post profit funktion:

$$\pi_i (q_1, q_2; a) = [P(q_1 + q_2; a) - c] q_i, \quad i = 1, 2.$$

- Types: Virksomhederne er usikre på efterspørgslen, parametriseret ved a. Vi antager at a kan tage to værdier, a_L and $a_H, a_L < a_H$, Hvilket sker med sandsynlighed $\mathbb{P}(a = a_H) = \theta$, and $\mathbb{P}(a = a_L) = 1 - \theta$. - Virksomhed 1 observerer a før den vælger sin mængde, - Virksomhed 2 observerer ikke a før den vælger sin mængde.

Da firma 2 kun har en type, der er en strategi blot givet ved ét træk. For firma 1, der har to typer, skal man specificerer en handling betinget på hvilken type firma 2 er. Det vil sige $q_1(a_H)$ og $q_1(a_L)$. Vi starter med at se på den optimale strategi for firma 2 .

$$q_1^*(a_H) = \arg\max_{q_2} (a_H - q_1 - q_2 - c) q_1$$

Vi bruger første ordensbetingelsen:

$$(a_H - q_1 - q_2 - c) - q_1 = 0$$

Hvilket giver:

$$2q_1 = a_H - q_2 - c$$
$$q_1^*(a_H) = \frac{a_H - q_2 - c}{2}$$

Ligeledes får vi:

$$q_1^*(a_L) = \frac{a_L - q_2 - c}{2}$$

Vi ser nu på firma 2's optimale strategi. Denne er givet ved:

$$q_2^* = \arg\max_{q_2} \theta (a - q_2 - q_1^*(a_H)) q_2 + (1 - \theta)(a - q_2 - q_1^*(a_L)) q_2$$

Vi benytter så første ordensbetingelsen:

$$\frac{\partial}{\partial q_2}\left[\theta\left(a - q_2 - q_1^*\left(a_H\right)\right) q_2 + (1-\theta)\left(a - q_2 - q_1^*\left(a_L\right)\right) q_2\right] = 0$$

Dette giver:

$$q_2^* = \theta\left(\frac{a - q_2^*\left(a_H\right) - c}{2}\right) + (1-\theta)\left(\frac{a - q_2^*\left(c_L\right) - c}{2}\right) = \frac{a - \left[\theta q_1^*\left(a_H\right) + (1-\theta) q_1^*\left(a_L\right)\right] - c}{2}$$

Vi løser så for denne værdi:

$$\begin{aligned} 2q_2 &= a - \theta\frac{a_H - q_2 - c}{2} - (1-\theta)\frac{a_L - q_2 - c}{2} - c \\ 4q_2 &= 2a - \theta\left(a_H - q_2 - c\right) - (1-\theta)\left(a_L - q_2 - c\right) - 2c \\ 3q_2 &= 2a + c - 2c - \left[\theta a_H + (1-\theta) a_L\right] \\ 3q_2 &= 2a - c - \left[\theta a_H + (1-\theta) a_L\right] \end{aligned}$$

Altså er den optimale mængde for spiller to givet ved:

$$q_2^* = \frac{2a - c - \left[\theta a_H + (1-\theta) a_L\right]}{3}$$

Bemærk at vi for $a_H = a_L = a$ får den sædvanlige løsning. Vi kan så finde de to øvrige løsninger ved at indsætter overstående udtryk i spiller 1's best response.

4) See Python solution

5) Exercise 3.4 in Gibbons (p. 169). Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

a) **Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.**

b) **Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.**

c) **Player 1 chooses either T or B; player 2 simultaneously chooses either L or R.**

d) **Payoffs are given by the game drawn by nature.**

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game 1

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

Game 2

To find the Bayesian Nash equilibriums in this game, we need to setup the expected payoff matrix. In this example player 1 knows, which game is drawn, and therefor needs to specify a strategy for each game, while player 2 does not know which game is played.

Normal-form representation of the Bayesian Game

Players	Player 1 and player 2
Type spaces	$T_1 : \{t_1, t_2\}$ $T_2 : \{t\}$
Action Sets	$A_1 : \{T, B\}$ $A_2 : \{L, R\}$
Strategy Sets	$S_1 : \{TT, TB, BT, BB\}$ $S_2 : \{L, R\}$
Beliefs	$p_1(t_1 \mid t) = 1$ $p_1(t_2 \mid t) = 1$ $p_2(t_1 \mid t) = \frac{1}{2}$ $p_2(t_2 \mid t) = \frac{1}{2}$
Payoffs	We can illustrate the payoffs in two payoff matrices

Type $t = t_1$

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Type $t = t_2$

	L	R
T	$0, 0$	$0, 0$
B	$0, 0$	$2, 2$

notice that these matrixes are payoff matrixes and not the normal form representation of the game. The normal form representation, specifies all of the above.

We setup the expected payoff matrix

	L	R
TT		
TB		
BT		
BB		

to calculate the payoff's in the different scenarios, I will assign a letter to each payoff to know where the payoff comes from

Type $t = t_1$

	L	R
T	$1^a, 1^b$	$0^c, 0^d$
B	$0^i, 0^j$	$0^k, 0^l$

Type $t = t_2$

	L	R
T	$0^e, 0^f$	$0^g, 0^h$
B	$0^m, 0^n$	$2^o, 2^p$

the expected payoff for the players when player 2 plays L

$$u_1(TT, L) = p_1(t_1 \mid t) \cdot 1^a + (1 - p_1(t_1 \mid t)) \cdot 0^e = \frac{1}{2} \cdot 1^a + \frac{1}{2} \cdot 0^e = \frac{1}{2} + 0 = \frac{1}{2}$$
$$u_2(TT, L) = p_1(t_1 \mid t) \cdot 1^b + (1 - p_1(t_1 \mid t)) \cdot 0^f = \frac{1}{2} \cdot 1^b + \frac{1}{2} \cdot 0^f = \frac{1}{2} + 0 = \frac{1}{2}$$

$$u_1(TB, L) = p_1(t_1 \mid t) \cdot 1^a + (1 - p_1(t_1 \mid t)) \cdot 0^e = \frac{1}{2} \cdot 1^a + \frac{1}{2} \cdot 0^m = \frac{1}{2} + 0 = \frac{1}{2}$$
$$u_2(TB, L) = p_1(t_1 \mid t) \cdot 1^b + (1 - p_1(t_1 \mid t)) \cdot 0^n = \frac{1}{2} \cdot 1^b + \frac{1}{2} \cdot 0^f = \frac{1}{2} + 0 = \frac{1}{2}$$

$$u_1(BT, L) = p_1(t_1 \mid t) \cdot 0^i + (1 - p_1(t_1 \mid t)) \cdot 0^e = \frac{1}{2} \cdot 0^i + \frac{1}{2} \cdot 0^e = 0 + 0 = 0$$
$$u_2(BT, L) = p_1(t_1 \mid t) \cdot 0^j + (1 - p_1(t_1 \mid t)) \cdot 0^f = \frac{1}{2} \cdot 0^j + \frac{1}{2} \cdot 0^f = 0 + 0 = 0$$

$$u_1(BB, L) = p_1(t_1 \mid t) \cdot 0^i + (1 - p_1(t_1 \mid t)) \cdot 0^m = \frac{1}{2} \cdot 0^i + \frac{1}{2} \cdot 0^m = 0 + 0 = 0$$
$$u_2(BB, L) = p_1(t_1 \mid t) \cdot 0^j + (1 - p_1(t_1 \mid t)) \cdot 0^n = \frac{1}{2} \cdot 0^j + \frac{1}{2} \cdot 0^n = 0 + 0 = 0$$

and the expected payoff for the players when player 2 plays R

$$u_1(TT, R) = p_1(t_1 \mid t) \cdot 0^c + (1 - p_1(t_1 \mid t)) \cdot 0^g = \frac{1}{2} \cdot 0^c + \frac{1}{2} \cdot 0^g = 0 + 0 = 0$$
$$u_2(TT, R) = p_1(t_1 \mid t) \cdot 0^d + (1 - p_1(t_1 \mid t)) \cdot 0^h = \frac{1}{2} \cdot 0^d + \frac{1}{2} \cdot 0^h = 0 + 0 = 0$$

$$u_1(TB, R) = p_1(t_1 \mid t) \cdot 0^c + (1 - p_1(t_1 \mid t)) \cdot 2^o = \frac{1}{2} \cdot 0^c + \frac{1}{2} \cdot 2^o = 0 + 1 = 1$$
$$u_2(TB, R) = p_1(t_1 \mid t) \cdot 0^d + (1 - p_1(t_1 \mid t)) \cdot 2^p = \frac{1}{2} \cdot 0^d + \frac{1}{2} \cdot w^p = 0 + 1 = 1$$

$$u_1(BT, R) = p_1(t_1 \mid t) \cdot 0^k + (1 - p_1(t_1 \mid t)) \cdot 0^g = \frac{1}{2} \cdot 0^k + \frac{1}{2} \cdot 0^g = 0 + 0 = 0$$
$$u_2(BT, R) = p_1(t_1 \mid t) \cdot 0^l + (1 - p_1(t_1 \mid t)) \cdot 0^h = \frac{1}{2} \cdot 0^l + \frac{1}{2} \cdot 0^h = 0 + 0 = 0$$

$$u_1(BB, R) = p_1(t_1 \mid t) \cdot 0^k + (1 - p_1(t_1 \mid t)) \cdot 2^o = \frac{1}{2} \cdot 0^k + \frac{1}{2} \cdot 2^o = 0 + 1 = 1$$
$$u_2(BB, R) = p_1(t_1 \mid t) \cdot 0^l + (1 - p_1(t_1 \mid t)) \cdot 2^p = \frac{1}{2} \cdot 0^l + \frac{1}{2} \cdot 2^p = 0 + 1 = 1$$

such that the expected payoff matrix will be

	L	R
TT	$\frac{1}{2}, \frac{1}{2}$	$0, 0$
TB	$\frac{1}{2}, \frac{1}{2}$	$1, 1$
BT	$0, 0$	$0, 0$
BB	$0, 0$	$1, 1$

now we can plug in for the best responses

- If player 1 plays TT, then player 2's best response is to play L
- If player 1 plays TB, then player 2's best response is to play R
- If player 1 plays BT, then player 2's best response is to play either L or R
- If player 1 plays BB, then player 2's best response is to play R
- If player 2 plays L, then player 1's best response is to play either TT or TB
- If player 2 plays R, then player 1's best response is to play either TB or BB

	L	R
TT	$\underline{\frac{1}{2}}, \underline{\frac{1}{2}}$	$0, 0$
TB	$\underline{\frac{1}{2}}, \frac{1}{2}$	$\underline{1}, \underline{1}$
BT	$0, \underline{0}$	$0, \underline{0}$
BB	$0, 0$	$\underline{1}, \underline{1}$

such that the game has 3 BNE in pure strategies

$$\mathsf{BNE} = \{(\mathsf{TT}, \mathsf{L}), (\mathsf{TB}, \mathsf{R}), (\mathsf{BB}, \mathsf{R})\}$$

6) See Python