Micro B: Problem Set 5

Dynamic Games of Perfect Information*

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Question 1 (Dynamic game with imperfect information)**.** Consider the dynamic game shown in extensive form below. Solve it by backwards induction.

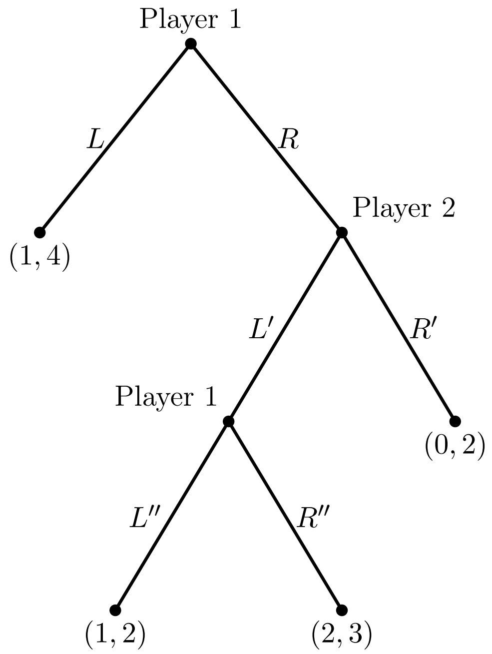


Figure 1: Two-player game with two moves

Question 2 (Extended "battle of the sexes")**.** Consider the game in figure 2. Answer each of the following questions:

(a) Write up the strategy sets for the players

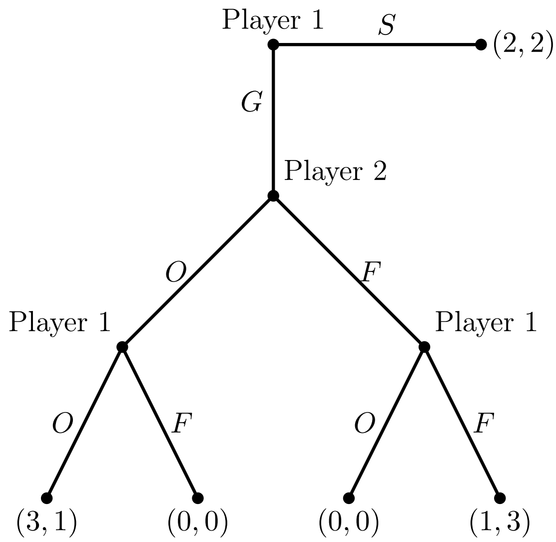
(b) Write up the normal form (in a table)

(c) Find the Nash Equilibria

(d) Find the backwards induction outcome

*For prior contributions, thanks to Jeppe Dinsen.

Figure 2: Extended Battle of the Sexes Game



Question 3 (Stackelberg: NE vs. SPNE)**.** Consider the Stackelberg game we saw in the lecture. We solved for the backwards induction outcome. But if we were to look for Nash Equilibria, there are many. In fact there is an infinite amount. But they rely on 'empty threats'. To see why, consider the following equilibrium. The follower (player 2) says to the leader: I want you to produce $\hat{q}_1$ (where $\hat{q}_1 < a$) and then I will produce $\hat{q}_2 = BR_2(\hat{q}_1)$. If you deviate from this and produce some $q_1 \neq \hat{q}_1$ then I will set $q_2 = a - q_1$ such that you make zero profit.

a) Write up the equilibria described above formally, using for the strategies the notation of the 'Simple Dynamic Game'.

b) Explain why this kind of equilibrium does not survive backwards induction unless $\hat{q}_1 = q_1^*$, where q_1^* is the Stackelberg outcome we derived in the lecture.

Question 4 (Cournot, Stackelberg, and the social optimum)**.** Two students, $N = \{1, 2\}$, are working together on the next assignment. Student i exerts an effort $y_i \geq 0$. The resulting quality of the assignment is

$$q(y_1, y_2) = y_1 y_2$$

Exerting effort is costly, but the costs differ, since one student likes game theory more than the other. More precisely, the cost functions are

$$C_1(y_1) = \frac{1}{3}(y_1)^3$$
$$C_2(y_2) = (y_2)^2$$

The payoff for student i, U_i, is equal to the quality of the assignment less his cost of effort.

$$U_1(y_1, y_2) = q(y_1, y_2) - C_1(y_1) \qquad U_2(y_1, y_2) = q(y_1, y_2) - C_2(y_1)$$

(a) Consider the game where both of them choose their effort levels simultaneously and independently. Derive the best response functions. Find the (pure strategy) Nash equilibrium $\left(y_1^{NE}, y_2^{NE}\right)$ with $y_1^{NE}, y_2^{NE} > 0$.

(b) Suppose now that Student 1 chooses his effort first, then sends the assignment on to Student 2. Student 2 observes how much exert Student 1 has exerted, makes his own choice of effort, and then submits. Solve by backwards induction.

(c) Compare the outcomes in (a) and (b) with respect to the payoffs of the students. Which game does each of the two students prefer? Give an intuitive explanation of your answer.

(d) Find the socially optimal levels of effort $\left(y_1^{SO}, y_2^{SO}\right)$, i.e., the levels that maximize the sum of the two students' payoffs. Calculate the payoff that the two students get in the social optimum.

(e) Compare the example to one where the two players are firms investing in a joint project. What is the interpretation of (y_1^{SO}, y_2^{SO}) in that model?

Question 5 (Python)**.** Solve question 4 using Python.
Some hints:

- Write a best response function for each student, `BR1(y2)` and `BR2(y1)`. It takes as input a single float (the price of the opponent), and numerically finds the best response by maximizing the payoff, holding fixed the opponent's action at the level given by the input. You can e.g. use **`minimize`** or **`minimize_scalar`** from **`scipy.optimize`** (giving it the *negative* of the payoff function).
- The Nash equilibrium can be found by using the Iterated Best Response Algorithm (IBR).
- The sequential equilibrium is found by inserting the best response function of one player into the best response function of the other.
- The social optimum is found by maximizing numerically a criterion function which sums the payoffs of the two players.

Question 6 (Dynamic game with perfect information)**.** Consider a game where two evil organizations, rather prosaically named A and B, are battling for world domination. The battle takes the form of a three-stage game. Organization A is on the verge of acquiring a new powerful weapon, the mutated seabass.

- *Stage 1:* Organization A decides whether to acquire the weapon or not. The choice will be observed by organization B afterwards.
- *Stage 2:* Organization B decides whether to attack organization A. If no attack occurs, the game moves to stage 3, otherwise the game ends
- *Stage 3:* Organization A decides whether or not to attack organization B.
- *Payoffs:* If no-one attacks the other, the payoffs to both organizations are 0. If B attacks A, then the payoffs to both organizations are -1. The same if A attacks B, without having acquired the seabass weapon. If, on the other hand, A acquires the weapon, the payoffs from A attacking B are 2 to A and -2 to B.

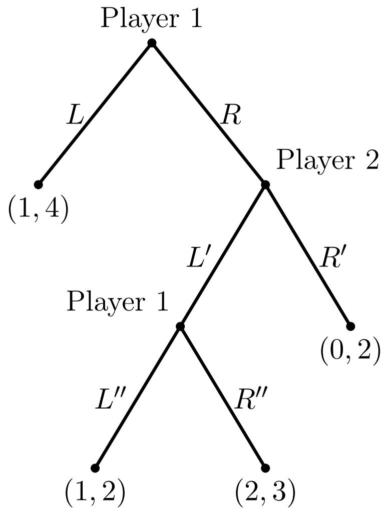
Answer each of the following questions:

(a) Draw the game tree that corresponds to the game. What are the strategies of the players?

(b) What is the backwards induction outcome?

(c) What is the intuition for the outcome? What role do you think it plays that B observes if A acquires the weapon or not?

Question 7 (SPNE)**.** Consider the game in the figure

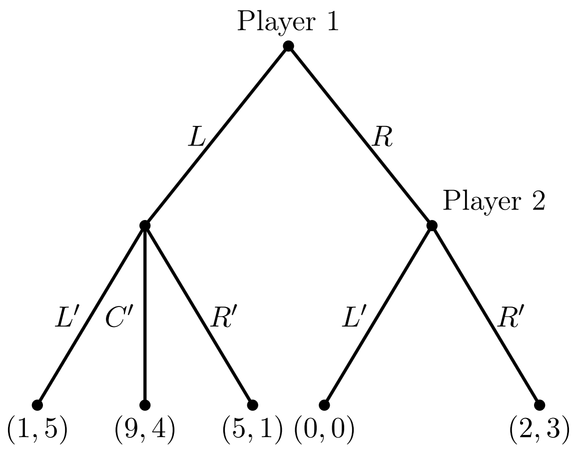


(a) Write down the strategies of the two players. How many proper subgames are there (so not including the entire game itself)?

(b) Write down the Subgame-Perfect Nash Equilibrium (SPNE). Compare the SPNE to the backwards-induction outcome which you found in question 1.

(c) Write down the normal form of the game and find all (pure strategy) Nash equilibria. Compare to the set of SPNE and comment.

Question 8 (SPNE)**.** For the game given in extensive form below, answer the following four questions:



(a) What are the strategy sets of each player?

(b) How many proper subgames are there (i.e. not including the entire game itself)?

(c) Find the backwards induction outcome and write down the SPNE.

(d) Write down this game as a bi-matrix and find all pure strategy Nash equilibria.