Microeconomics B Problem Set 5

Subgame Perfect Nash Equilibria

1) Consider the dynamic game shown in extensive form below. Solve it by backwards induction.

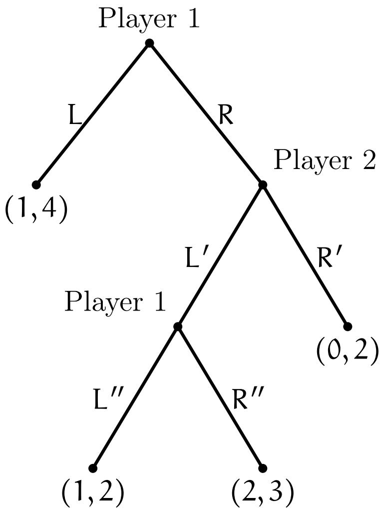
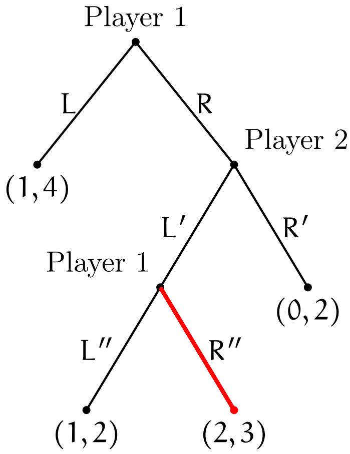


Figure 1: Two-player game with two moves

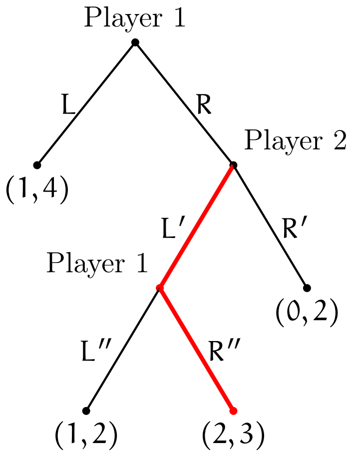
Solution

When solving for the backwards induction outcome we need to solve the last decision node for an arbitrary level of earlier outcomes i.e. in the game above, we start by solving player 1's choice between L'' and R''.

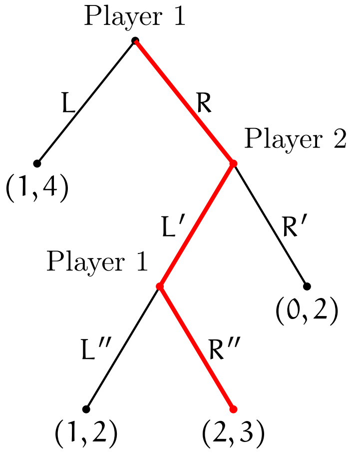
In the 3rd decision node player 1 chooses between L'' for a payoff of 1 and R'' for a payoff of 2, and it is therefor optimal to choose R''



now lets solve for the second decision node, given the choice of the 3rd decision node. Player 2 can choose between a payoff of 3 and a payoff of 2, and therefor optimal chooses L'



now in the first decision node player 1 chooses between a payoff of 1 and 2, and therefor chooses R



therefor the backwards induction outcome of this game is

$$\text{Backwards Induction Outcome: } R, L', R''$$

2) Consider the game in figure 2

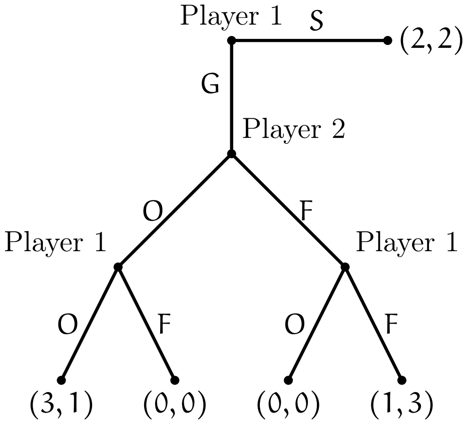


Figure 2: Extended Battle of the Sexes Game

a) Write up the strategy sets for the players

Lets start by defining a strategy in a dynamic game

Definition 1 (A Strategy in Dynamic Games).
Let $\mathcal{H}_i$ denote the collection of player i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subset \mathcal{A}$ the set of actions possible at information set H. A strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

That is when we write up a strategy for each of the two players, we need to specify the action for the player for each of his decision nodes.

A strategy for player 1 will therefor specify 3 actions.

The strategy sets are

$$S_1 = \left\{ (G,O,O), (G,O,F), (G,F,O), (G,F,F), (S,O,O), (S,O,F), (S,F,O), (S,F,F) \right\}$$
$$S_2 = \left\{ O, F \right\}$$

b) Write up the normal form

The normal form of a game is defined as

Definition 2 (The Normal Form Representation).
The normal form representation of an n-player game specifies the players' strategy spaces $S_1, S_2, ..., S_n$ and their payoff functions $u_1, u_2, ..., u_n$. We denote this game by $G = \{S_1, S_2, ..., S_n; u_1, u_2, ..., u_n\}$.

and therefor specifies the players, their strategies and the payoffs. We can illustrate the normal form game in a bi-matrix

		Player 2	
		O	F
Player 2	GOO	3,1	0,0
	GOF	3,1	1,3
	GFO	0,0	0,0
	GFF	0,0	1,3
	SOO	2,2	2,2
	SOF	2,2	2,2
	SFO	2,2	2,2
	SFF	2,2	2,2

c) Find the Nash Equilibria

We can find the nash equilibria but plugging in for the best responses

- If player 1 plays GOO then player 2's optimal choice is O
- If player 1 plays GOF then player 2's optimal choice is F
- If player 1 plays GFO then player 2's optimal choice is either O or F
- If player 1 plays GFF then player 2's optimal choice is F
- If player 1 plays SOO then player 2's optimal choice is either O or F
- If player 1 plays SOF then player 2's optimal choice is either O or F
- If player 1 plays SFO then player 2's optimal choice is either O or F

- If player 1 plays SFF then player 2's optimal choice is either O or F
- If player 2 plays O then player 1's optimal choice is either GOO or GOF
- If player 2 plays F then player 1's optimal choice is either SOO, SOF, SFO or SFF

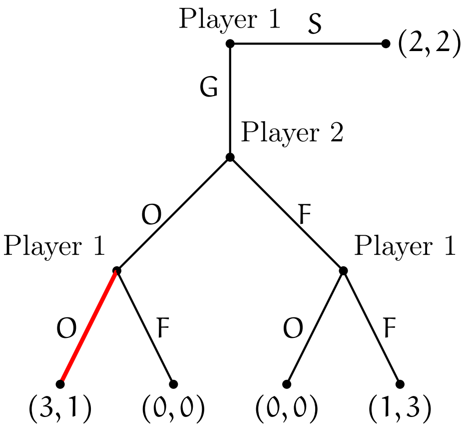
Player 2

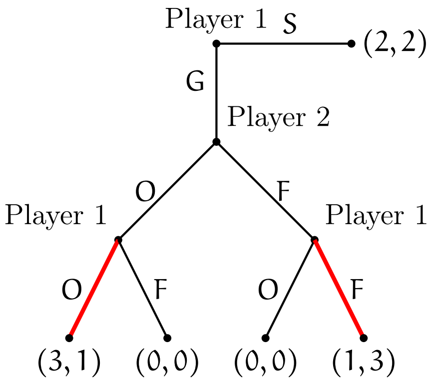
		O	F
Player 2	GOO	$\underline{3},\underline{1}$	$0,0$
	GOF	$\underline{3},1$	$1,\underline{3}$
	GFO	$0,\underline{0}$	$0,\underline{0}$
	GFF	$0,0$	$1,\underline{3}$
	SOO	$2,\underline{2}$	$\underline{2},\underline{2}$
	SOF	$2,\underline{2}$	$\underline{2},\underline{2}$
	SFO	$2,\underline{2}$	$\underline{2},\underline{2}$
	SFF	$2,\underline{2}$	$\underline{2},\underline{2}$

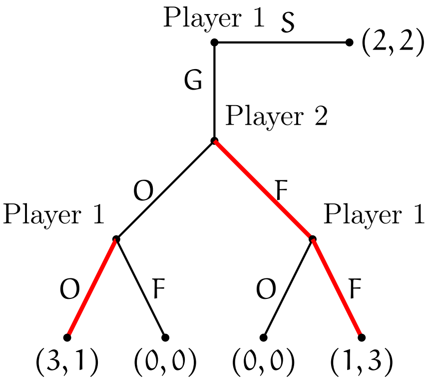
so we have 5 nash equilibria in pure strategies

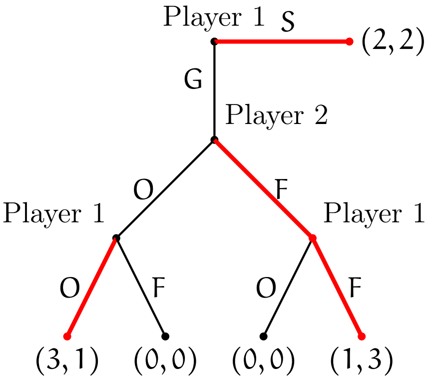
$$\text{PSNE} = \{ (\text{GOO}, \text{O}), (\text{SOO}, \text{F}), (\text{SOF}, \text{F}), (\text{SFO}, \text{F}), (\text{SFF}, \text{F}) \}$$

d) Find the backwards induction outcome









Consider the Stackelberg game we saw in the lecture. We solved for the backwards induction outcome. But if we were to look for Nash Equilibria, there are many. In fact there is an infinite amount. But they rely on 'empty threats'. To see why, consider the following equilibrium. The follower says to the leader: I want you to produce $\hat{q}_1$ (where $\hat{q}_1 < a$) and then I will produce $\hat{q}_2 = BR_2(\hat{q}_1)$. If you produce $q_1 \neq \hat{q}_1$ then I will set $q_2 = a - q_1$ such that you make zero profit.

a) **Write up the equilibria described above formally, using for the strategies the notation of the 'Simple Dynamic Game'.**

Lets start by showing the results also shown in the lecture. Consider an economy with two firms that both choose a non negative quantity to produce. The market in the economy is specified with an inverse demand function $P(Q) = a - Q$, where $Q = q_1 + q_2$ is the aggregate demand and q_1, q_2 is the quantity produced by firm 1 and firm 2 respectively. Assuming both firms have constant marginal cost of c and zero fixed costs. We assume the timing of the game as follows

(1) Firm 1 chooses the quantity $q_1 \geq 0$ to produce

(2) Firm 2 observes q_1 and then chooses their quantity to produce $q_2 \geq 0$

(3) The payoffs to firm i is their profit functions

that is we have assumed firm 1 to be called the stackelberg leader and firm 2 to be the stackelberg follower.

<u>Backwards Induction Outcome:</u>
Firm 1 moves as the leader and must choose the quantity to produce given their best guess on firm 2's quantity i.e. using firm 2's best response. Therefor we need firm 2's best response function.

Firm 2's best response function is found by maximizing profit for an arbitrary level of q_1

$$\begin{aligned}\max_{q_2} \pi_2(q_1, q_2) &= P(Q) \cdot q_2 - c \cdot q_2 \\ &= (a - (q_1 + q_2)) \cdot q_2 - c \cdot q_2 \qquad (1)\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi_2}{\partial q_2} &= -q_2 + (a - (q_1 + q_2)) - c = 0 \\ &\Leftrightarrow -2q_2 + a - q_1 - c = 0 \\ &\Leftrightarrow 2q_2 = a - q_1 - c \\ &\Leftrightarrow q_2 = \frac{a - q_1 - c}{2} \equiv R_2(q_1) \qquad (2)\end{aligned}$$

which is the best response for firm 2.

Firm 1 moved as the leader before firm 2, so it maximizes its profit with the assumption that firm 2 produces giving their best response function

$$\begin{aligned}\max_{q_1} \pi_1(q_1, R_2(q_1)) &= P(q_1, R_2(q_1)) \cdot q_2 - c \cdot q_2 \\ &= (a - (q_1 + R_2(q_1))) \cdot q_1 - c \cdot q_1 \\ &= \left(a - \left(q_1 + \frac{a - q_1 - c}{2}\right)\right) \cdot q_1 - c \cdot q_1 \qquad (3)\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= -q_1 + \frac{1}{2}q_1 + \left(a - \left(q_1 + \frac{a - q_1 - c}{2}\right)\right) - c = 0 \\ &\Leftrightarrow -\frac{1}{2}q_1 + a - q_1 - \frac{a - q_1 - c}{2} - c = 0 \\ &\Leftrightarrow -q_1 + 2a - 2q_1 - a + q_1 + c - 2c = 0 \\ &\Leftrightarrow 2q_1 = a - c \\ &\Leftrightarrow q_1^* = \frac{a - c}{2} \qquad (4)\end{aligned}$$

which is the quantity produced by the stackelberg leader.

Firm 2 that is the stackelerberg follower observes q_1^* and chooses its quantity.

$$\begin{aligned}q_2^* &= R_2(q_1^*) \\ &= \frac{a - q_1^* - c}{2} \\ &= \frac{a - \frac{a-c}{2} - c}{2} \\ &= \frac{\frac{2a}{2} - \frac{a-c}{2} - \frac{2c}{2}}{2} \\ &= \frac{\frac{a-c}{2}}{2} \\ &= \frac{a - c}{4} \qquad (5)\end{aligned}$$

which is the quantity the stackelberg follower optimally chooses to produce.

a) Write up the equilibria described above formally, using for the strategies the notation of the 'Simple Dynamic Game'.

Player 1's strategy is to choose an action

Player 2's strategy, is a contigency plan for all the possible actions of player 1.

Therefor the proposed equilibrium is

$$q_1^* = \hat{q}_1, \text{ where } q_1 < a$$

$$q_2^* = \begin{cases} \hat{q}_2 \equiv R_2(\hat{q}_1) & \text{if } q_1 = \hat{q}_1 \\ a - q_1 & \text{if } q_1 \neq \hat{q}_1 \end{cases}$$

b) Explain why this kind of equilibrium does not survive backwards induction unless $\hat{q}_1 = q_1^*$, where q_1^* is the Stackelberg outcome we derived in the lecture.

All other equilibriums than $q_1 = q_2^{\text{Stackelberg}}$, are equilibrium's are based on non-credible threats.

Lets assume that player 1 wants to deviate to $q_1 \neq q_2$ then player 2's best strategy is that player 2 plays $q_2^* = a - q_1$, which results in rice of $p = 0$ and zero profit.

What if player 2 instead plays $R_2(\hat{q}_1)$ and obtain a profit higher than zero, that is $q_2^* = a - q_1$ when $q_1 \neq \hat{q}_2$ is a non-credible threat.

4) Two students are working together on the next assignment. Student i, $i = 1,2$, exerts an effort $y_i \geq 0$. The resulting quality of the assignment is

$$q(y_1, y_2) = y_1 y_2$$

Exerting effort is costly, but the costs differ, since one student likes game theory more than the other. More precisely, the cost functions are

$$C_1(y_1) = \frac{1}{3}(y_1)^3$$

$$C_2(y_2) = (y_2)^2$$

The payoff for student i, U_i, is equal to the quality of the assignment less his cost of effort.

$$U_i(y_i, y_j) = q(y_i, y_j) - C_i(y_i)$$

for $i, j = 1, 2$ and $i \neq j$

a) Consider the game where both of them choose their effort levels simultaneously and independently. Derive the best response functions. Find the (pure strategy) Nash equilibrium (y_1^{NE}, y_2^{NE}) with $y_1^{NE}, y_2^{NE} > 0$.

This is a similar to a standard Cournot game, where the two players simultaneously chooses their effort.

The two players are not identical, so we need to find each players best response function, and maximize each players utility given the other players best response.

Player 1 maximizes his payoff

$$\begin{aligned}\max_{y_1} U_1(y_1, y_2) &= q(y_1, y_2) - C_1(y_1) \\ &= y_1 y_2 - \frac{1}{3}(y_1)^3\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial U_1}{\partial y_1} &= y_2 - y_1^2 = 0 \\ \Leftrightarrow y_1^2 &= y_2 \\ \Leftrightarrow y_1 &= (y_2)^{\frac{1}{2}} \equiv R_1(y_2)\end{aligned}$$

now lets solve for player 2's best response function

$$\begin{aligned}\max_{y_2} U_2(y_1, y_2) &= q(y_1, y_2) - C_2(y_2) \\ &= y_1 y_2 - (y_2)^2\end{aligned}$$

lets take the first order condition

$$\begin{aligned}\frac{\partial U_2}{\partial y_2} &= y_1 - 2y_2 = 0 \\ \Leftrightarrow 2y_2 &= y_1 \\ \Leftrightarrow y_1 &= \frac{1}{2}y_1 \equiv R_2(y_1)\end{aligned}$$

lets find player 1's optimal level of effort, by inserting player 2's best response function in player 1's best response function

$$\begin{aligned}y_1 = R_1(R_2(y_1)) = (R_2(y_1))^{\frac{1}{2}} &= \left(\frac{1}{2}y_1\right)^{\frac{1}{2}} \\ \Leftrightarrow (y_1)^2 &= \frac{1}{2}y_1 \\ \Leftrightarrow y_1^* &= \frac{1}{2}\end{aligned}$$

so it is optimal for player 1 to use $1/2$ hours of effort.

Inserting this in player 2's best response

$$\begin{aligned}y_2^* &= R_2(y_1^*) \\ &= \frac{1}{2}y_1^* \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4}\end{aligned}$$

such that player 2's optimal level of hours is $1/4$.

We can now find the quality of the paper by inserting y_1^*, y_2^*

$$q^*(y_1^*, y_2^*) = y_1^* \cdot y_2^* = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

and the payoff for the two players will be

$$U_1(y_1^*, y_2^*) = q^*(y_1^*, y_2^*) - C_1(y_1^*) = \frac{1}{8} - \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8} - \frac{1}{3} \cdot \frac{1}{8} = \frac{1}{8} - \frac{1}{24} = \frac{3}{24} - \frac{1}{24} = \frac{2}{24} = \frac{1}{12}$$

$$U_2(y_1^*, y_2^*) = q^*(y_1^*, y_2^*) - C_2(y_2^*) = \frac{1}{8} - \left(\frac{1}{4}\right)^2 = \frac{1}{8} - \frac{1}{16} = \frac{2}{16} - \frac{1}{16} = \frac{1}{16}$$

- **b) Suppose now that Student 1 chooses his effort first, then sends the assignment on to Student 2. Student 2 observes how much exert Student 1 has exerted, makes his own choice of effort, and then submits. Solve by backwards induction.**

The game has now become a stackelberg game, where student 1 is the stackelberg leader and student 2 is the stackelberg follower.

Player 1 maximizes his effort level for students 2's best response function

$$\begin{aligned}\max_{y_1} U_1(y_1, R_2(y_1)) &= q(y_1, R_2(y_1)) - C_1(y_1) \\ &= y_1 \cdot R_2(y_1) - \frac{1}{3}(y_1)^3 \\ &= y_1 \cdot \frac{1}{2} y_1 - \frac{1}{3}(y_1)^3 \\ &= \frac{1}{2} y_1^2 - \frac{1}{3} y_1^3\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial U_1(y_1, R_2(y_1))}{\partial y_1} &= y_1 - y^2 = 0 \\ &\Leftrightarrow y_1^2 = y_1 \\ &\Leftrightarrow y_1 = 1\end{aligned}$$

now student 2 observes this level of effort, and solves his maximization problem, that is we can insert for y_1^{**} in the best response function for student 2

$$\begin{aligned}y_2^{**} = R_2(y_1^{**}) &= \frac{1}{2} y_1^{**} \\ &= \frac{1}{2} \cdot (1) \\ &= \frac{1}{2}\end{aligned}$$

such that the quality of the paper becomes

$$q^{**}(y_1^{**}, y_2^{**}) = y_1^{**} \cdot y_2^{**} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

such that the payoffs for the two player are

$$U_1(y_1^{**}, y_2^{**}) = q^{**}(y_1^{**}, y_2^{**}) - C_1(y_1^{**}) = \frac{1}{2} - \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{2} - \frac{1}{3} \cdot 1 = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

$$U_2(y_1^{**}, y_2^{**}) = q^{**}(y_1^{**}, y_2^{**}) - C_2(y_2^{**}) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$

- **c) Compare the outcomes in (a) and (b) with respect to the payoffs of the students. Which game does each of the two students prefer? Give an intuitive explanation of your answer.**

Lets make the comparison by looking at the quality of the paper and the payoffs for the two players in the two cases

	Cournot Game		Stackelberg Game
Quality q	$\frac{1}{8}$	$<$	$\frac{4}{8}$
Player 1 u_1	$\frac{1}{12}$	$<$	$\frac{2}{12}$
Player 2 u_2	$\frac{1}{16}$	$<$	$\frac{4}{16}$

The effort put into the paper by player 1 and player 2 is strategic complements so whenever player 1 uses more time on the assignment, then so does player 2.

Player 1 can in the stackelberg game credible commit to putting in extra time in the assignment, which leads to player 2 putting in extra effort as well, which benefits both players.

d) Find the socially optimal levels of effort $\left(y_1^{SO}, y_2^{SO}\right)$, i.e., the levels that maximize the sum of the two students' payoffs. Calculate the payoff that the two students get in the social optimum.

To find the social optimum, we look at the social planner.

The social planner maximizes the total payoff of the two players

$$\begin{aligned}\max_{y_1,y_2} u\left(y_1,y_2\right) &= u_1\left(y_1,y_2\right)+u_2\left(y_1,y_2\right)\\ &= y_1y_2-\frac{1}{3}y_1^3+y_1y_2-y_2^2\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial u\left(y_1,y_2\right)}{\partial y_1} &= y_2-y_1^2+y_2=0\\ &\Leftrightarrow 2y_1=y_2^2\end{aligned}$$

$$\begin{aligned}\frac{\partial u\left(y_1,y_2\right)}{\partial y_1} &= y_1+y_1-2y_2=0\\ &\Leftrightarrow 2y_1=2y_2\\ &\Leftrightarrow y_1=y_2\end{aligned}$$

Inserting the second first order condition into the first, first order condition to find y_1

$$\begin{aligned}2y_1=y_2^2 \Longrightarrow 2y_1 &= y_1^2\\ \Leftrightarrow y_1 &= y_2=2\end{aligned}$$

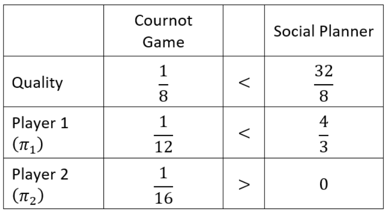
lets find the social optimal quantity

$$q^{SO}\left(y_1^{SO},y_2^{SO}\right)=y_1^{SO}\cdot y_2^{SO}=2\cdot 2=4$$

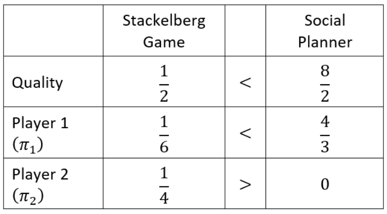
and lets find the payoffs to the two players for the social optimal quantity

$$\begin{aligned}u_1\left(y_1^{SO},y_2^{SO}\right)&=q^{SO}\left(y_1^{SO},y_2^{SO}\right)-c_1\left(y_1^{SO}\right)=4-\frac{1}{3}\cdot(2)^3=4-\frac{8}{3}=\frac{12}{3}-\frac{8}{3}=\frac{4}{3}\\ u_2\left(y_1^{SO},y_2^{SO}\right)&=q^{SO}\left(y_1^{SO},y_2^{SO}\right)-c_2\left(y_2^{SO}\right)=4-(2)^2=4-4=0\end{aligned}$$

lets compare the 3 situations



	Cournot Game		Social Planner
Quality	$\frac{1}{8}$	<	$\frac{32}{8}$
Player 1 (π_1)	$\frac{1}{12}$	<	$\frac{4}{3}$
Player 2 (π_2)	$\frac{1}{16}$	>	0



	Stackelberg Game		Social Planner
Quality	$\frac{1}{2}$	<	$\frac{8}{2}$
Player 1 (π_1)	$\frac{1}{6}$	<	$\frac{4}{3}$
Player 2 (π_2)	$\frac{1}{4}$	>	0

we have a social optimum where one of the students gets a lower payoff than in the previously seen NEs. This is because the players are asymmetric, and player 2 has a much higher cost of effort than player 1.

5) See Python

6) Consider a game where two evil organizations, rather prosaically named A and B, are battling for world domination. The battle takes the form of a three-stage game. Organization A is on the verge of acquiring a new powerful weapon, the mutated seabass. In stage 1 of the game, they decide whether to acquire the weapon or not. Their choice is observed by organization B. In stage 2, organization B decides whether to attack organization A. If an attack occurs, the game stops. If no attack occurs, it moves to stage 3, where organization A decides whether or not to attack organization B. The payoffs are as follows. If no-one attacks the other, the payoffs to both organizations are 0. If B attacks A, then the payoffs to both organizations are -1. The same if A attacks B, without having acquired the seabass weapon. If, on the other hand, A acquires the weapon, the payoffs from A attacking B are 2 to A and -2 to B.

a) Draw the game tree that corresponds to the game. What are the strategies of the players?

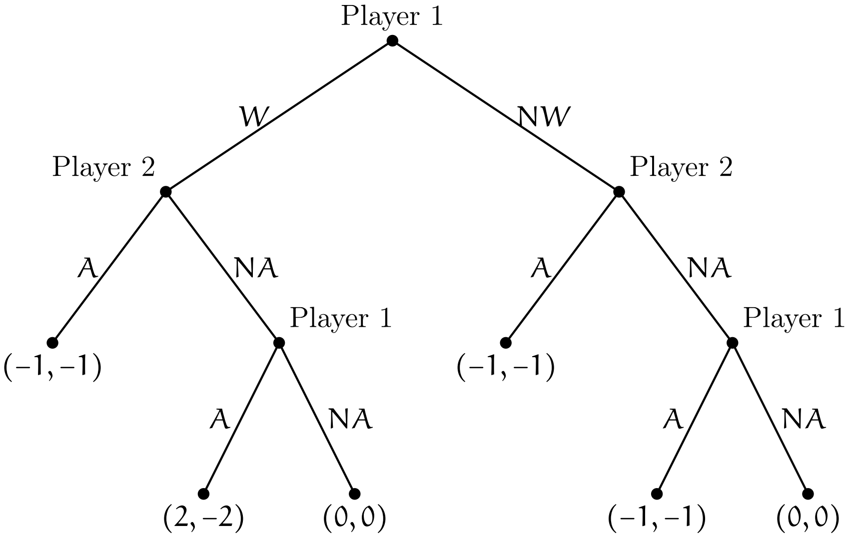
Lets state the timing of the game

(1) In stage 1 organization (A) choose to acquire a weapon or not

(2) Organization (B) observes A's choice, and chooses to attack or not attack

(3) Organization A, oberserves B's choice and chooses to attack back or not

Illustrating this in a game tree



Lets define a strategy in a dynamic game of complete information.

Definition 3 (A Strategy in Dynamic Games).
Let $\mathcal{H}_i$ denote the collection of player i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subset \mathcal{A}$ the set of actions possible at information set H. A strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

that is a strategy is a plan of actions, that specify the players actions at every decision node in the game, i.e. a strategy specifies more than one action.

Player 1 has 3 decision nodes, so a strategy needs to specify an action in each of these decision nodes.

Player 2 has 2 decision nodes so a strategy for player 2 needs to specify an two actions, one for each of the two decision nodes.

The strategies for the two players are

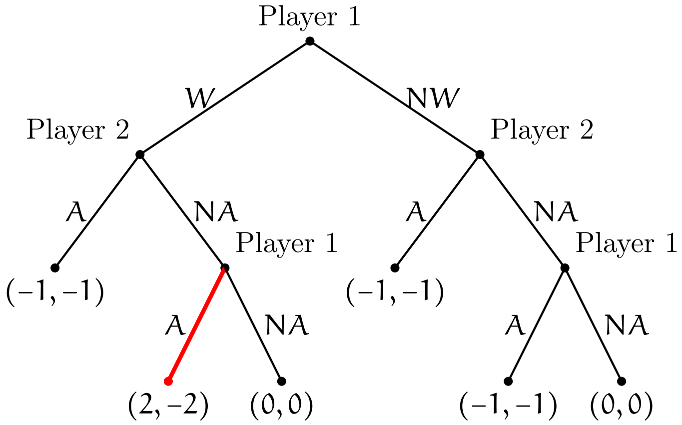
$$S_1 = \{(W, A, A), (W, A, NA), (W, NA, A), (W, NA, NA), (NW, A, A), (NW, A, NA), (NW, NA, A), (NW, NA, NA)\}$$

$$S_2 = \{(A, A), (A, NA), (NA, A), (NA, NA)\}$$

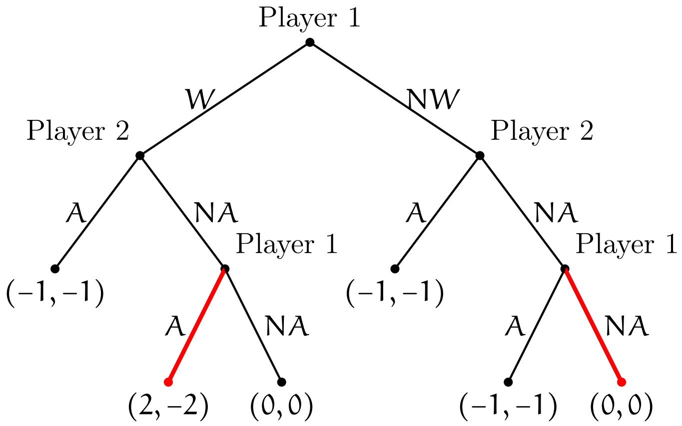
b) What is the backwards induction outcome?

We solve the game from last decision node

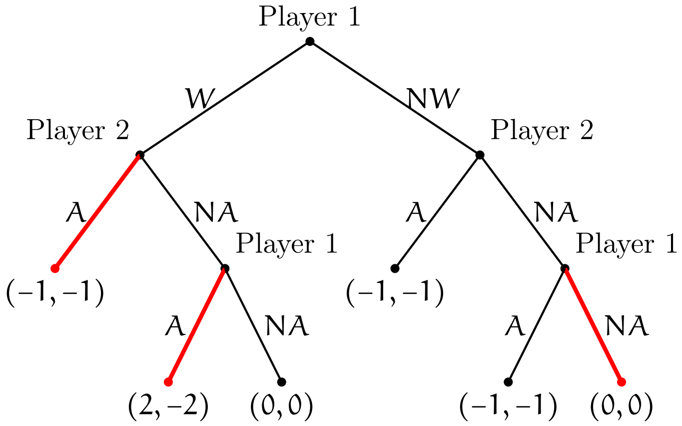
In the bottom left decision node, player 1 chooses between playing A for a payoff of 2 and playing NA for a payoff of 0. Therefor player 1 chooses to play A



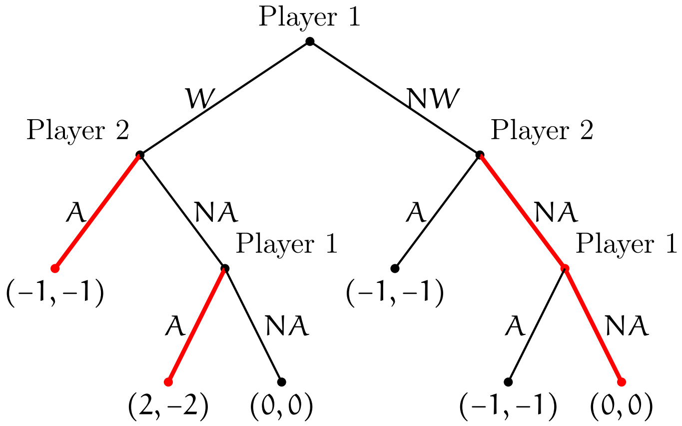
in the bottom right decision node, player 1 chooses between playing A for a payoff of -1 and playing NA for a payoff of 0 and therefor chooses to play NA



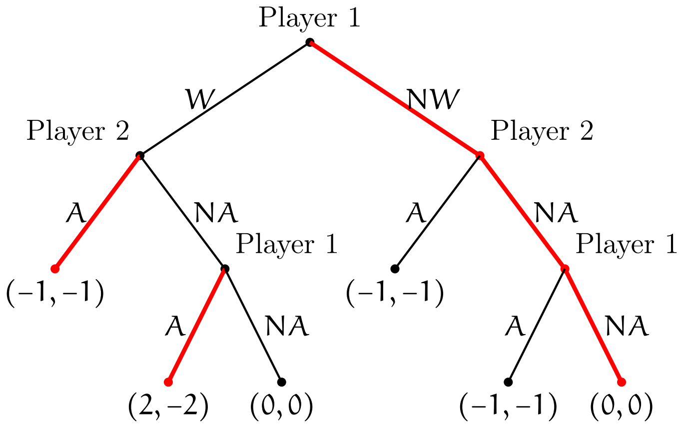
for player 2's left decision node, he chooses between playing A for payoff of -1 and playing NA for a payoff of -2, and therefor chooses to play A



in player 2 right decision node, he chooses between playing A for a payoff of -1 or playing NA for a payoff of 0, and therefor chooses NA



now in player 1's first decision node he can choose between playing W for a payoff of -1 or playing NW for a payoff of 0, and therefor chooses to play NW



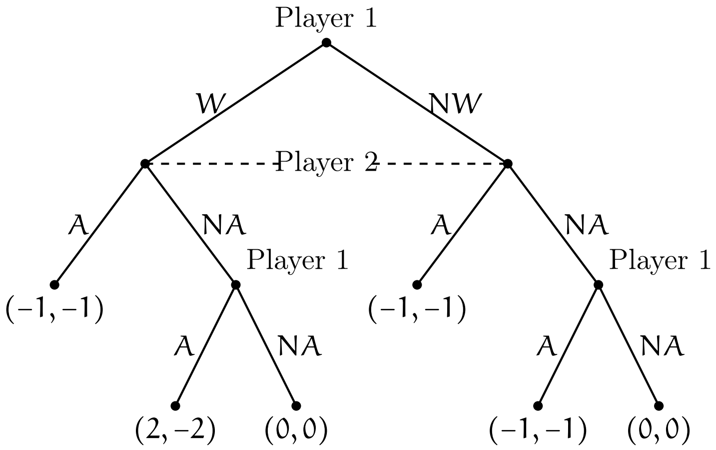
such that the backwards induction outcome is

$$\text{SPNE} = \{(NW, A, NA; A, NA)\}$$

c) What is the intuition for the outcome? What role do you think it plays that B observes if A acquires the weapon or not?

The game shows that a forfeit gives a possibility of a higher payoff in the future .

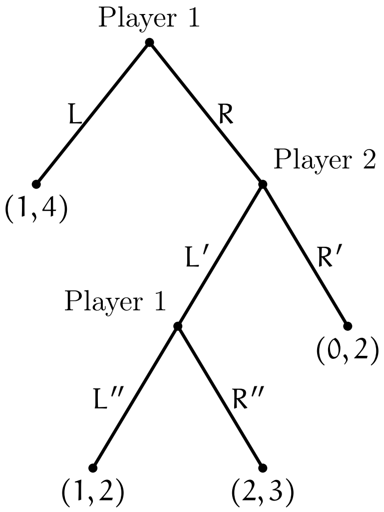
If player 2 was not able to observe the choice made by player 1, then we would need to solve the game a little differently. We can illustrate this case using an information set



we cannot solve for player 1 decision in the first decision node because of the information set, instead we need to write up the game as a bimatrix and solve for nash equilibria.

This will be solved in the next problem set.

7) Consider the game in the figure



a) Write down the strategies of the two players. How many proper subgames are there (so not including the entire game itself)?

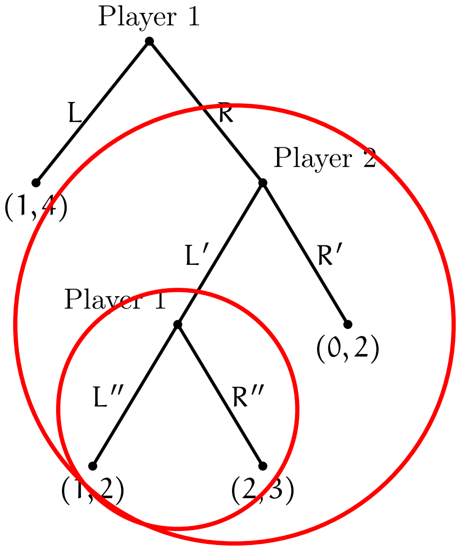
When specifying strategies in dynamic games, we need to specify an action for each decision node. Player 1 has 2 decision nodes each with 2 possible actions, therefor player 1 will have $= 2^2 = 4$ possible strategies.

Player 2 have 1 decision node with 2 possible actions, and therefor has 2 strategies $2^1 = 2$.

The two players strategies are

$$S_1 = \{(L, L''), (L, R''), (R, L''), (R, R'')\}$$
$$S_2 = \{(L'), (R')\}$$

lets mark the proper subgames in the game

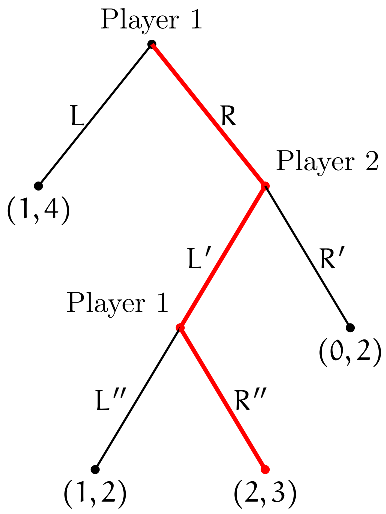


and it is clear that the game has 2 proper subgames.

b) Write down the Subgame-Perfect Nash Equilibrium (SPNE). Compare the SPNE to the backwards-induction outcome which you found in question 1.

To find the subgame perfect Nash equilibria, we need to solve for the Nash equilibria in all subgames of the game. This can be done by using backwards induction

- In stage 3, player 1 chooses to play R'' for a payoff of 2
- In stage 2, player 2 chooses to play L' for a payoff of 3
- In stage 1, player 1 chooses to play R' for a payoff of 2



and the subgame perfect nash equilibria is

$$\mathsf{SPNE} = \left\{\left(\mathsf{R}, \mathsf{R}''; \mathsf{L}'\right)\right\}$$

Compare:
We can use Backwards induction to find SPNE, the same was as if we had found nash equilibria in all the subgames in the game. This is always the case in finite game of complete information.

c) Write down the normal form of the game and find all (pure strategy) Nash equilibria. Compare to the set of SPNE and comment.

Lets state the game in its normal form representation

	L'	R'
LL''	$1, 4$	$1, 4$
LR''	$1, 4$	$1, 4$
RL''	$1, 2$	$0, 2$
RR''	$2, 3$	$0, 2$

lets plug in for the best responses

If player 1 plays LL'', then player 2 is indifferent between playing L' and R'
If player 1 plays LR'', then player 2 is indifferent between playing L' and R'
If player 1 plays RL'', then player 2 is indifferent between playing L' and R'
If player 1 plays RR'', then player 2 prefer to play L'
If player 2 plays L', then player 1 prefers to play RR''
If player 2 plays R', then player 1 is indifferent between playing LL'' and LR''

	L'	R'
LL''	$1, \underline{4}$	$\underline{1}, \underline{4}$
LR''	$1, \underline{4}$	$\underline{1}, \underline{4}$
RL''	$1, \underline{2}$	$0, \underline{2}$
RR''	$\underline{2}, \underline{3}$	$0, 2$

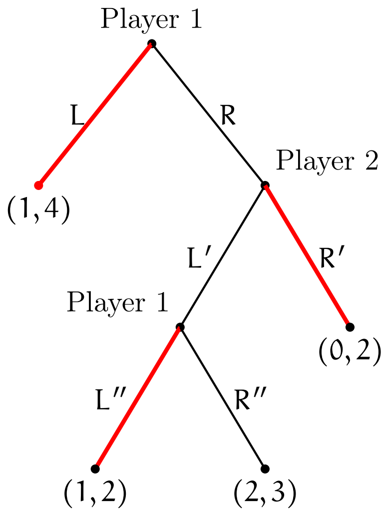
and we have 3 nash equilibria in pure strategies

$$\mathsf{PSNE} = \left\{\left(\mathsf{LL}'', \mathsf{R}'\right), \left(\mathsf{LR}'', \mathsf{R}'\right), \underbrace{\left(\mathsf{RR}'', \mathsf{L}'\right)}_{\text{SPNE}}\right\}$$

We see that there are 3 pure strategy nash equilibria but there is only one subgame perfect nash equilibria, the reason for this is that we have ruled out non-credible threats.

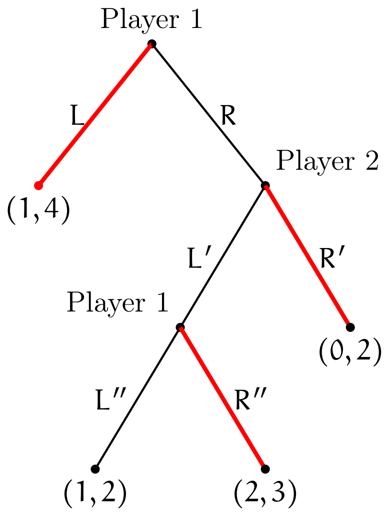
Lets have a look at the two other nash equilbibria in the extensive form,

$\underline{(LL'', R')}$



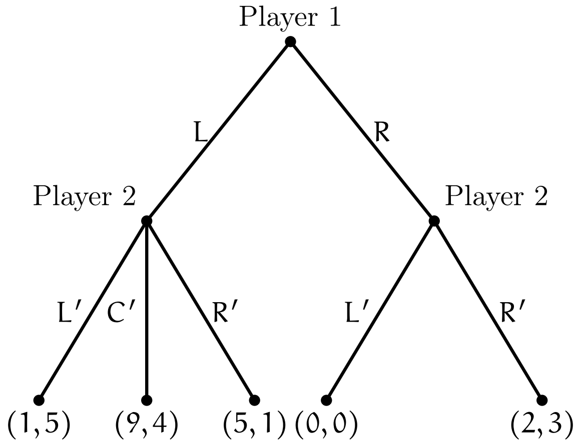
In (LL'', R') : Player 1 acts irrational in his second decision node by getting 1 instead of 2. But he still cannot gain anything from deviating to R''

$\underline{(LR'', R')}$



In (LR'', R') : player 2 choosing R' is a non credible threat, where he forces player 1 to move to L in the first decision node. This makes player 2 better of, but it is non-credible because it would yield player 2 a higher payoff to move to L' and get 3

8) For the game given in extensive form below, answer the following four questions:



a) What are the strategy sets of each player?

When specifying strategies in dynamic games of complete information, we need to specify an action for each decision node.

Player 1 has 1 decision node with two possible actions, and therefor has $2^1 = 2$ possible strategies.

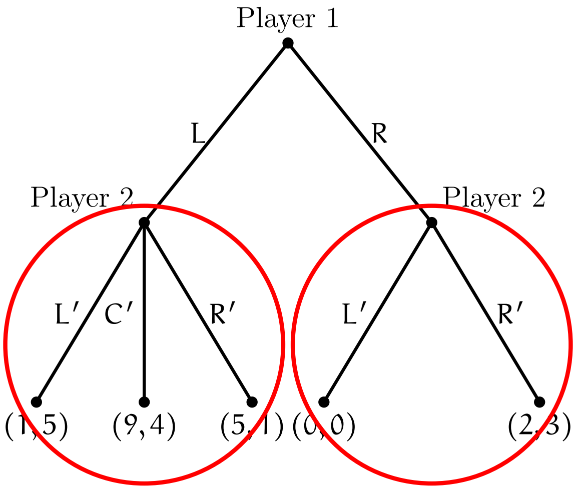
Player 2 has 2 decision nodes, one with 3 actions and one with 2, therefor the number of strategies are $2 \cdot 3 = 6$.

The players strategies are

$$S_1 = \{(L), (R)\}$$
$$S_2 = \{(L'L'), (L'R'), (C'L'), (C'R'), (R'L'), (R'R')\}$$

b) How many proper subgames are there (so not including the entire game itself)?

Lets mark the proper subgames in the game

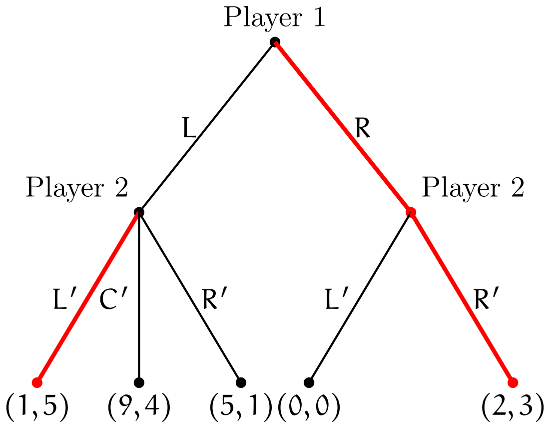


and we see that the game has two proper subgames.

c) Find the backwards induction outcome and write down the SPNE.

In a dynamic game of complete information, we can find the subgame perfect nash equilibria using backwards induction.

- In the bottom left decision node for player 2, it is optimal to play L' for a payoff of 5
- In the bottom right decision node for player 2, it is optimal to play R' for a payoff of 5
- In the first stage of the game, player one chooses to play R for a payoff of 2



such that the subgame perfect nash equilibria is

$$\text{SPNE} = \{(R; L', R')\}$$

d) Write down this game as a bi-matrix and find all pure strategy Nash equilibria.

	L′L′	L′R′	C′L′	C′R′	R′L′	R′R′
L	1,5	1,5	9,4	9,4	5,1	5,1
R	0,0	2,3	0,0	2,3	0,0	2,3

Lets plug in for the best responses

	L′L′	L′R′	C′L′	C′R′	R′L′	R′R′
L	$\boxed{\underline{1},\underline{5}}$	$1,\underline{5}$	$\underline{9},4$	$\underline{9},4$	$\underline{5},1$	$\underline{5},1$
R	$0,0$	$\boxed{\underline{2},\underline{3}}$	$0,0$	$2,\underline{3}$	$0,0$	$2,\underline{3}$

and the nash equilibria in the entire game is

$$\text{PSNE} = \left\{ \left(L, L'L'\right), \left(R, L'R'\right) \right\}$$