Microeconomics III - Problem Set 3

Tragedy of the commons and mixed strategy nash equilibria

1) Show that for each of the following two games, the only Nash equilibrium is in pure strategies. Describe the intuition for this result. What do these two games have in common?

	L	R
U	5, 5	1, 6
D	6, 1	2, 2

ne = {D, R}

	L	C	R
U	1, 0	1, 2	0, 1
D	0, 3	0, 1	2, 0

ne = {U, C}

We need to argue why there is a unique Nash equilibria in pure strategies in the reported games above. The easiest way to do this is to use proposition A from appendix 1.1.C in 'A primer in game theory'

Theorem 1 (IESDS and unique Nash Equilibrium).
In the n-player normal form game $G = \{S_1, S_2, ..., S_n; u_1, u_2, ..., u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*, s_2^*, ..., s_n^*)$, then these strategies are the unique Nash equilibria of the game.

This is relevant, because this states that we can just solve the games above using IESDS and if we find a unique outcome then we know that this is the only nash equilibrium in both pure and mixed strategies.

Game 1:

	L	R
U	5, 5	1, 6
D	6, 1	2, 2

we see that the red numbers is strictly largest for strategy D and that the blue number is strictly largest for D, therefor we have that for player 1 strategy D strictly dominated strategy U.

	L	R
D	6, 1	2, 2

now it is clear that player 2 will recieve the highest payoff from playing R and therefor, she will never play L.

	R
D	2, 2

so in this game we have a unique solution from IESDS, which therefor is a unique nash equilibria, and we don't have any nash equilibria in mixed strategies

$$\text{NE} = \{D, R\}$$

Game 2:

	L	C	R
U	1, 0	1, 2	0, 1
D	0, 3	0, 1	2, 0

Player 2 will get a strictly higher payoff from playing C compared to playing R, no matter what strategy player 1 chooses to play. Therefor player 2, will never play strategy R and we can remove it from the game

	L	C
U	1, 0	1, 2
D	0, 3	0, 1

now player 1 will get a strictly higher payoff from playing U compared to playing D, and therefor he will never play D. Removing strategy D from the game

	L	C
U	1, 0	1, 2

player 2 will get the highest payoff from playing C and therefor we remove L

	C
U	1, 2

in this game we end up with a unique solution to IESDS, and therefor we know that this is a unique nash equilibra, and there is no nash equilibria in mixed strategies.

$$\text{NE} = \{U, C\}$$

What does these two games have in common?
What the two games have in common, is that they both have a unique solution to IESDS, which we therefor can use to rule out the possibility of mixed strategy nash equilibria.

2) Consider the following two-player game:

	a	b	b
A	2, 2	0, 0	−1, 2
B	0, 0	0, 0	0, 0
C	2, −1	0, 0	1, 1

Solve for all pure strategy Nash equilibria. Which equilibrium do you find most reasonable?

We start by noticing that there are no strictly dominating strategies, so we cannot use IESDS to find Nash equilibria.

Instead lets plot in for the best responses

- If player 1 plays A, then player 2's best response is to play either a or c.
- If player 1 plays B, then player 2's best response is to play either a, b or c.
- If player 1 plays C then player 2's best response is to play c.
- If player 2 plays a, then player 1's best reponse is to play either A or C.
- If player 2 plays b, then player 1's best response is to play either a, b or c.
- If player 2 plays c, then player 1's best response is to play c.

marking these best responses in a bimatrix, with an underline

$$
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & 2,2 & 0,0 & -1,2 \\ \hline
B & 0,0 & 0,0 & 0,0 \\ \hline
C & 2,-1 & 0,0 & 1,1 \\ \hline
\end{array}
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & 2,\underline{2} & 0,0 & -1,\underline{2} \\ \hline
B & 0,0 & 0,0 & 0,0 \\ \hline
C & 2,-1 & 0,0 & 1,1 \\ \hline
\end{array}
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & 2,\underline{2} & 0,0 & -1,\underline{2} \\ \hline
B & 0,\underline{0} & 0,\underline{0} & 0,\underline{0} \\ \hline
C & 2,-1 & 0,0 & 1,1 \\ \hline
\end{array}
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & 2,\underline{2} & 0,0 & -1,\underline{2} \\ \hline
B & 0,\underline{0} & 0,\underline{0} & 0,\underline{0} \\ \hline
C & 2,-1 & 0,0 & 1,\underline{1} \\ \hline
\end{array}
$$

$$
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & \underline{2},\underline{2} & 0,0 & -1,\underline{2} \\ \hline
B & 0,\underline{0} & 0,\underline{0} & 0,\underline{0} \\ \hline
C & \underline{2},-1 & 0,0 & 1,\underline{1} \\ \hline
\end{array}
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & \underline{2},\underline{2} & \underline{0},0 & -1,\underline{2} \\ \hline
B & 0,\underline{0} & \underline{0},\underline{0} & 0,\underline{0} \\ \hline
C & \underline{2},-1 & \underline{0},0 & 1,\underline{1} \\ \hline
\end{array}
\to
\begin{array}{c|c|c|c|}
 & a & b & c \\ \hline
A & \underline{2},\underline{2} & \underline{0},0 & -1,\underline{2} \\ \hline
B & 0,\underline{0} & \underline{0},\underline{0} & 0,\underline{0} \\ \hline
C & \underline{2},-1 & \underline{0},0 & 1,\underline{1} \\ \hline
\end{array}
$$

which clearly illustrates that, we have 3 nash equilibria in pure strategies in the game.

$$\text{PSNE} = \{(A, a), (B, b), (C, c)\}$$

I find the equilibria (A, a), the most likely, because in this equilibrium both players' gets their highest possible payoff.
For the equilibria (B, b) both players are certain of their payoff no matter what the other player does, because they can only get 0.
The equilibrium (C, c), is the least risky equilibrium, because of the other player chooses to deviate, then the player will receive a higher payoff.

3) We have seen in the lectures that IESDS never eliminates a Nash Equilibrium. However, we saw in Problem Set 2 that this is not true if we do iterated elimination of weakly dominated strategies (IEWDS.)

Go through the proof in the slides from lecture 2 and identify the step that is no longer true if we replace IESDS by IEWDS. That is, explain why the proof is no longer true when we replace 'strict domination' by 'weak domination'.

Lets assume that (s_1^*, s_2^*) is a nash equilibrium.

Lets carry out IEWDS. s_1^* is the first Nash equilibria, which is the first strategy we would like to eliminate in the n rounds of elimination.

If we want to eliminate s_1^* then there must be a strategy $s_1' \neq s_1^*$ that strictly dominates s_1^* i.e.

$$\forall s_2 \in S_2^n : u_1(s_1^*, s_2) \leq u_1(s_1', s_2) \tag{1}$$

and at least one of these inequalities holds strictly.

S_2^n is the set of player 2's strategies, that has not been eliminated in round $1, 2, 3, ..., n-$.

We still have not eliminated s_2^* and therefor we still have $s_2^* \in S_2^n$, such that inequality (1) also implies

$$u_1\left(s_1^{*0}, s_2^*\right) \leq u_1\left(s_1', s_2^*\right) \tag{2}$$

but because we know $\left(s_1^*, s_2^*\right)$ is a nash equilibria, then by definition we have

$$\forall s_1 \in S_1 : u_1\left(s_1^*, s_2^*\right) \geq u_1\left(s_1', s_2^*\right) \tag{3}$$

now the two inequalities (2) and (3) will hold if

$$u_1\left(s_1^*, s_2^*\right) = u_1\left(s_1', s_2^*\right) \tag{4}$$

and there is no contradiction.

s_1^* can be a NE and can be eliminated by IEWDS, if there exist another strategy s_1', that gives player 1 the same payoff when player 2 plays s_2^* (and strictly higher payoff for at least on other strategy of player 2, for example $s_2'' \in S_2^n : u_1\left(s_1^*, s_2^*\right) < u_1\left(s_1, s_2''\right)$).

1 Qustion 4)

Note: I (Anders) have written a separate solution to this sub-question.

4) Consider price competition between two firms when some consumers are informed about prices and others are not. Firms have zero marginal cost and they set price simultaneously; for the sake of this example, assume each price can only take one of the following values: 80, 54, 38. The market consists of two consumers. The uninformed consumer will visit a firm at random (probabilities 1/2,1/2) and buy from it, regardless of the price. The informed consumer will visit the firm with the lowest price and buy from it. If both firms set the same price, assume that the informed consumer picks a firm at random (probabilities 1/2,1/2).

a) Argue that this game can be represented by the following bimatrix

	$p_2 = 80$	$p_2 = 54$	$p_2 = 38$
$p_1 = 80$	80, 80	40, 81	40, 57
$p_1 = 54$	81, 40	54, 54	27, 57
$p_1 = 38$	57, 40	57, 27	38, 38

Let state the normal form representation of the game

Players	Two players firm 1 (Player 1) and firm 2 (Player 2)
Strategy Sets	The strategy sets are to choose between one of three prices $p_i = \{38, 54, 80\}$
Payoff's	The two players payoffs can be expressed in a bracket

$$u_i(s_i, s_j) = \begin{cases} p_i + \frac{1}{2}p_i & \text{if } p_i < p_j \\ \frac{1}{2}p_i + \frac{1}{2}p_i & \text{if } p_i = p_j \\ 0 + \frac{1}{2}p_i & \text{if } p_i > p_j \end{cases}$$

instead of stating this like this, we could rewrite the normal form representation as a Bi-matrix

	$p_2 = 80$	$p_2 = 54$	$p_2 = 38$
$p_1 = 80$	$\frac{1}{2}\cdot 80 + \frac{1}{2}\cdot 80, \frac{1}{2}\cdot 80 + \frac{1}{2}\cdot 80$	$0 + \frac{1}{2}\cdot 80, 54 + \frac{1}{2}\cdot 54$	$0 + \frac{1}{2}\cdot 80, 38 + \frac{1}{2}\cdot 38$
$p_1 = 54$	$54 + \frac{1}{2}\cdot 54, 0 + \frac{1}{2}\cdot 80$	$\frac{1}{2}\cdot 54 + \frac{1}{2}\cdot 54, \frac{1}{2}\cdot 54 + \frac{1}{2}\cdot 54$	$0 + \frac{1}{2}\cdot 54, 38 + \frac{1}{2}\cdot 38$
$p_1 = 38$	$38 + \frac{1}{2}\cdot 38, 0 + \frac{1}{2}\cdot 80$	$38 + \frac{1}{2}\cdot 38, 0 + \frac{1}{2}\cdot 54$	$\frac{1}{2}\cdot 54 + \frac{1}{2}\cdot 54, \frac{1}{2}\cdot 54 + \frac{1}{2}\cdot 54$

doing the algebra we get the Bi-matrix we are asked to find

	$p_2 = 80$	$p_2 = 54$	$p_2 = 38$
$p_1 = 80$	$80, 80$	$40, 81$	$40, 57$
$p_1 = 54$	$81, 40$	$54, 54$	$27, 57$
$p_1 = 38$	$57, 40$	$57, 27$	$38, 38$

b) Show that there is no Nash equilibria in pure strategies

Lets plug in for the best responses

- If player 1 plays $p_1 = 80$ then player 2's optimal choice is to play $p_2 = 54$
- If player 1 plays $p_1 = 54$ then player 2's optimal choice is to play $p_2 = 38$
- If player 1 plays $p_1 = 38$ then player 2's optimal choice is to play $p_2 = 80$
- If player 2 plays $p_2 = 80$ then player 1's optimal choice is to play $p_1 = 54$
- If player 2 plays $p_2 = 54$ then player 1's optimal choice is to play $p_1 = 38$
- If player 2 plays $p_2 = 38$ then player 1's optimal choice is to play $p_1 = 80$

and marking the best responses with an underline

	$p_2 = 80$	$p_2 = 54$	$p_2 = 38$
$p_1 = 80$	$80, 80$	$40, \underline{81}$	$\underline{40}, 57$
$p_1 = 54$	$\underline{81}, 40$	$54, 54$	$27, \underline{57}$
$p_1 = 38$	$57, \underline{40}$	$\underline{57}, 27$	$38, 38$

it is clear that there are no Nash equilibria in pure strategies

c) Confirm that the following strategy profile is a Nash equilibrium: each firm plays price 80 with probability 0.232, price 54 with probability 0.361, and price 38 with probability 0.407.

We need to show that the game has a Nash equilibria in mixed strategies, where firms play $p_1 = 80$ with probability $r_1 = 0,232$ and $p_2 = 54$ with probability $r_2 = 0,361$, that is we need to show that a specific strategy profile is a Nash equilibria in mixed strategies. We do this by using proposition 116.2 from

Proposition 1 (Characterization of mixed strategy Nash equilibria in finite games).
A mixed strategy profile α^* *in a static game with* νNM *preferences in which each player has a finitely number of actions is a mixed strategy Nash equilibrium if an only if, for each player*

- *The expected payoff given* α^*_{-i} *to every action to which* α^*_i *assigns a positive probability is the same*

- *The expected payoff given* α^*_{-i} *to every action to which* α^*_i *assigns zero probability is at most the expected payoff to any other action to which* α^*_i *assigns positive probability.*

Each players expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability.

This means that we need to check if the expected payoff for the two players for the 3 strategies gives the same expected payoff.

Player 1

$$\mathbb{E}[p_1 = 80, (0.232, 0.361)] = 0.232 \cdot 80 + 0.361 \cdot 40 + (1 - 0.232 - 0.361) \cdot 40 = 18.56 + 14.44 + 16.28 \approx 49.3$$
$$\mathbb{E}[p_1 = 54, (0.232, 0.361)] = 0.232 \cdot 81 + 0.361 \cdot 54 + (1 - 0.232 - 0.361) \cdot 57 = 18.79 + 19.49 + 10.99 \approx 49.3$$
$$\mathbb{E}[p_1 = 38, (0.232, 0.361)] = 0.232 \cdot 57 + 0.361 \cdot 57 + (1 - 0.232 - 0.361) \cdot 38 = 13.22 + 20.58 + 15.47 \approx 49.3$$

and because of symmetry between the two firms, this also holds for player 2, and we have a mixed strategy nash equilibria

$$\text{MSNE} = \left\{ (r_1^*, r_2^*, q_1^*, q_2^*) = (0.232, 0.361, 0.232, 0.361) \right\}$$

Lets now also show how we can find the probabilities our self, by assigning a probability to each action in the game

		q_1	q_2	$1 - q_1 - q_2$
		$p_2 = 80$	$p_2 = 54$	$p_2 = 38$
r_1	$p_1 = 80$	80, 80	40, 81	40, 57
r_2	$p_1 = 54$	81, 40	54, 54	27, 57
$1 - r_1 - r_2$	$p_1 = 38$	57, 40	57, 27	38, 38

the two players are symmetric, so if we solve for when one of them is indifferent then the same holds for the other player.

Player 1's expected payoffs are

$$\mathbb{E}[p_1 = 80] = 80 \cdot q_1 + 40 \cdot q_2 + 40 \cdot (1 - q_1 - q_2) = 80q_1 + 40q_2 + 40 - 40q_1 - 40q_2 = 40 + 40q_1$$
$$\mathbb{E}[p_1 = 54] = 81 \cdot q_1 + 54 \cdot q_2 + 27 \cdot (1 - q_1 - q_2) = 81q_1 + 54q_2 + 27 - 27q_1 - 27q_2 = 27 + 54q_1 + 27q_2$$
$$\mathbb{E}[p_1 = 38] = 57 \cdot q_1 + 57 \cdot q_2 + 38 \cdot (1 - q_1 - q_2) = 57q_1 + 38q_2 + 38 - 38q_1 - 38q_2 = 38 + 19q_1 + 19q_2$$

Now we need to figure out for what values of

$$\mathbb{E}[p_1 = 80] = \mathbb{E}[p_1 = 54] \Longrightarrow 40 + 40q_1 = 27 + 54q_1 + 27q_2 \Leftrightarrow 14q_1 + 27q_2 = 13$$

$$\mathbb{E}[p_1 = 80] = \mathbb{E}[p_1 = 38] \Longrightarrow 40 + 40q_1 = 38 + 19q_1 + 19q_2 \Leftrightarrow -21q_1 + 19q_2 = 2$$

$$\mathbb{E}[p_1 = 54] = \mathbb{E}[p_1 = 38] \Longrightarrow 27 + 54q_1 + 27q_2 = 38 + 19q_1 + 19q_2 \Leftrightarrow 35q_1 + 8q_2 = -11$$

Now this is 3 equations with 2 unknowns, which we solve by solving to equations with two unknowns and trying to see if the solution holds for the last equation as well.

Lets use the two top equations, and lets solve this using matrix (See for yourself that the remaining 2 remaining 2 equations with 2 unknowns problems will result in the same solution as this one.

$$\begin{bmatrix} 14 & 27 \\ -21 & 19 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 27 \\ -21 & 19 \end{bmatrix}^{-1} \begin{bmatrix} 14 & 27 \\ -21 & 19 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 14 & 27 \\ -21 & 19 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{14 \cdot 19 - 27 \cdot -21} \begin{bmatrix} 19 & -27 \\ -21 & 14 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{833} \begin{bmatrix} 19 & -27 \\ 21 & 14 \end{bmatrix} \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{19}{833} & -\frac{27}{833} \\ \frac{21}{833} & \frac{14}{833} \end{bmatrix} \begin{bmatrix} 13 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{19}{833} \cdot 13 - \frac{27}{833} \cdot 2 \\ \frac{21}{833} \cdot 13 + \frac{14}{833} \cdot 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{193}{833} \\ \frac{301}{833} \end{bmatrix}$$

such that q_1 and q_2 will be

$$q_1 = \frac{193}{833}$$
$$q_2 = \frac{301}{833}$$

Lets check if these probabilites makes the expected values equal

$$\mathbb{E}[p_1 = 80] = 80 \cdot \frac{193}{833} + 40 \cdot \frac{301}{833} + 40 \cdot \left(1 - \frac{193}{833} - \frac{301}{833}\right) = \frac{41040}{833} \approx 49.3$$
$$\mathbb{E}[p_1 = 54] = 81 \cdot \frac{193}{833} + 54 \cdot \frac{301}{833} + 27 \cdot \left(1 - \frac{193}{833} - \frac{301}{833}\right) = \frac{41040}{833} \approx 49.3$$
$$\mathbb{E}[p_1 = 38] = 57 \cdot \frac{193}{833} + 57 \cdot \frac{301}{833} + 38 \cdot \left(1 - \frac{193}{833} - \frac{301}{833}\right) = \frac{41040}{833} \approx 49.3$$

and they are the same, such that these probabilities are a nash equilibria.

d) Why do you think the equilibrium is so different from the standard Bertrand pricing game (i.e. where competition drives price down to marginal cost)?

The main difference from a standard Bertrand model is that we have introduced uninformed consumers who is insensitive to prices. This decreases the effect from price competition

- More uninformed customers will lead to higher prices
- More competing firms will make it less likely for the uninformed consumer to randomly choose the high priced firms goods which will lead to lower prices.

5) Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican (V) and Denmark (D) are preparing an attack on Luxembourg's nuclear

research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_V s_D$$

where $s_i \in [0,1]$, **is the share of its military capacity that country** i ($i \in \{V, D\}$) **uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country** i

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost

a) **Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium of this game.**

We need to find the nash equilibria, to do this we need to maximize the

$$\begin{aligned}\max_{s_i} U_i\left(s_i, s_j\right) &= p(s_i, s_j) - c_i(s_i) \\ &= s_i + s_j - s_i s_j - s_i^2\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial U_i\left(s_i, s_j\right)}{\partial s_i} &= 1 - s_j - 2s_i = 0 \\ \Leftrightarrow 2s_i &= 1 - s_j \\ \Leftrightarrow s_i &= \frac{1 - s_j}{2} \equiv R_i\left(s_j\right)\end{aligned}$$

equation (2) is country i's best response function to s_j.

We know that the two countries are symmetric, such that the best response for country j is symmetrically identical to equation (2)

$$R_j\left(s_i\right) \equiv s_j = \frac{1 - s_i}{2}$$

to find the nash equilibrium of this game, we need to solve the first order condition for s_i, taking s_j as the best response

$$\begin{aligned}s_i = \frac{1 - R_j\left(s_i\right)}{2} &= \frac{1 - \frac{1-s_i}{2}}{2} \\ \Leftrightarrow 2s_i &= 1 - \frac{1 - s_i}{2} \\ \Leftrightarrow 4s_i &= 1 - (1 - s_i) \\ \Leftrightarrow 3s_i &= 1 \\ \Leftrightarrow s_i &= \frac{1}{3}\end{aligned}$$

This could also be solved using that the two countries are identical and therefor in a nash equilibrium will choose the same s_i. To do this we set $s_i = s_j$ in the best response for country i

$$s_i = \frac{1 - \overbrace{s_i}^{s_j}}{2} \Leftrightarrow 2s_i = 1 - s_i$$
$$\Leftrightarrow 3s_i = 1$$
$$\Leftrightarrow s_i = \frac{1}{3}$$

b) Find the social optimum under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

Lets find the social planners optimal s under the assumption that the two countries have the same s.

The social planner maximizes the utility as a whole

$$\max_{\bar{s}} U_s(\bar{s}) = \bar{s} + \bar{s} - \bar{s} \cdot \bar{s} - \bar{s}^2$$
$$= 2\bar{s} - 2\bar{s}^2$$

taking the first order condition

$$\frac{\partial U_s(\bar{s})}{\partial \bar{s}} = 2 - 4\bar{s} = 0$$
$$\Leftrightarrow 4\bar{s} = 2$$
$$\Leftrightarrow \bar{s} = \frac{1}{2}$$

it is clear that the share s is larger for the social planner than in the cournot competition. The share is higher in the social optimum because in the duopoly there is an incentive to free ride because positive cost if you participate in the attack i.e. because the best response functions are downwards sloping.

6) There are three identical firms in an industry. Their production quantities are denoted q_1, q_2 and q_3. The inverse demand function is

$$p = a - Q$$

where $(a = 1)$ and

$$Q = q_1 + q_2 + q_3$$

the marginal production cost is zero $(c = 0)$

a) Compute the quantities in the Cournot equilibrium, i.e., the Nash Equilibrium of the game where the firms simultaneously choose quantities.

Lets solve this game with an arbitrary a and$c$, and then afterwards insert the specific values.

The three firms is identical so we know that in optimum they will produce the same amount, and the best response function for firm 1 is the same as for firm 2 and 3.

The profit maximization problem for firm 1

$$\begin{aligned}\max_{q_1} \pi_1 &= p(Q)\cdot q_1 - c\cdot q_1 \\ &= (a-(q_1+q_2+q_3))\cdot q_1 - c\cdot q_1\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} = -q_1 + a - (q_1+q_2+q_3) - c = 0 \\ \Leftrightarrow 2q_1 = a-(q_2+q_3)-c \\ \Leftrightarrow q_1 = \frac{a-(q_2+q_3)-c}{2}\end{aligned}$$

now lets use the fact that the three firms are identical and set $q_1 = q_2 = q_3$ and isolate for q_1

$$\begin{aligned}q_1 = \frac{a-(q_1+q_1)-c}{2} &\Leftrightarrow 2q_1 = a-(q_1+q_1)-c \\ &\Leftrightarrow 4q_1 = a-c \\ &\Leftrightarrow q_1^* = \frac{a-c}{4} = q_2^* = q_3^*\end{aligned}$$

b) What is the price in the Cournot-equilibrium?

We can find the equilibrium price by inserting the cournot quantities into the inverse demand function

$$\begin{aligned}p(q_1^*, q_2^*, q_3^*) &= a - q_1^* - q_2^* - q_3^* \\ &= a - \frac{a-c}{4} - \frac{a-c}{4} - \frac{a-c}{4} \\ &= \frac{4a}{4} - \frac{a-c}{4} - \frac{a-c}{4} - \frac{a-c}{4} \\ &= \frac{4a-a+c-a+c-a+c}{4} \\ &= \frac{a+3c}{4}\end{aligned}$$

for the next question lets also find the profit to each of the three firms in the cournot equilibrium

$$\begin{aligned}\pi_1 &= p(q_1^*, q_2^*, q_3^*)\cdot q_1^* - c\cdot q_1^* \\ &= \frac{a+3c}{4}\cdot\frac{a-c}{4} - c\cdot\frac{a-c}{4} \\ &= \frac{(a+3c)(a-c)}{16} - \frac{ac-c^2}{4} \\ &= \frac{a^2-ac+3ac-3c^2}{16} - \frac{ac-c^2}{4} \\ &= \frac{a^2+2ac-3c^2}{16} - \frac{ac-c^2}{4} \\ &= \frac{a^2+2ac-3c^2}{16} - \frac{4ac-4c^2}{16} \\ &= \frac{a^2+2ac-3c^2-4ac+4c^2}{16} \\ &= \frac{a^2-2ac+c^2}{16}\end{aligned}$$

c) Show that if two of the three firms merge (transforming the industry into a duopoly), the profits of the merging firms decrease. Explain.

The market has now transformed into a duopoly, and the two firms is now identical. We solve firm 1's profit maximization

$$\begin{aligned}\max_{q_1} \pi_1 &= p(Q)\cdot q_1 - c\cdot q_1\\ &= (a-(q_1+q_2))\cdot q_1 - c\cdot q_1\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} = -q_1 + a - (q_1+q_2) - c = 0&\\ \Leftrightarrow 2q_1 &= a-(q_2)-c\\ \Leftrightarrow q_1 &= \frac{a-(q_2)-c}{2}\end{aligned}$$

now lets use the fact that the three firms are identical and set $q_1 = q_2$ and isolate for q_1

$$\begin{aligned}q_1 = \frac{a-(q_1)-c}{2} &\Leftrightarrow 2q_1 = a-(q_1)-c\\ &\Leftrightarrow 3q_1 = a-c\\ &\Leftrightarrow q_1^* = \frac{a-c}{3} = q_2^*\end{aligned}$$

and calculating the duopoly price

$$\begin{aligned}p(q_1^*, q_2^*) &= a - q_1^* - q_2^*\\ &= a - \frac{a-c}{4} - \frac{a-c}{4}\\ &= \frac{3a}{3} - \frac{a-c}{3} - \frac{a-c}{3}\\ &= \frac{3a-a+c-a+c}{3}\\ &= \frac{a+2c}{3}\end{aligned}$$

such that the profits becomes

$$\begin{aligned}\pi_1 &= p(q_1^*, q_2^*)\cdot q_1^* - c\cdot q_1^*\\ &= \frac{a+2c}{3}\cdot\frac{a-c}{3} - c\cdot\frac{a-c}{3}\\ &= \frac{(a+2c)(a-c)}{9} - \frac{c(a-c)}{3}\\ &= \frac{a^2-ac+2ac-2c^2}{9} - \frac{ac-c^2}{3}\\ &= \frac{a^2+ac-2c^2}{9} - \frac{ac-c^2}{3}\\ &= \frac{a^2+ac-2c^2}{9} - \frac{3ac-3c^2}{9}\\ &= \frac{a^2+ac-2c^2-3ac+3c^2}{9}\\ &= \frac{a^2-2ac+c^2}{9}\end{aligned}$$

Lets compare the combined profit for two firms in the 3 firm cournot problem with 1 firm duopoly firm case

	3 Firms		Duopoly
q_i^*	$2 \cdot \frac{a-c}{4}$	$>$	$\frac{a-c}{3}$
p^*	$\frac{a+3c}{4}$	$<$	$\frac{a+2c}{3}$
π^*	$2 \cdot \frac{a^2-2ac+c^2}{16}$	$>$	$\frac{a^2-2ac+c^2}{9}$

which clearly illustrates that the combined profit of 2 firms in the case with 3 firms is higher than the profit of one firm in the duopoly.

d) What happens if all three firms merge?

The market has now transformed to a monopoly marked. Solving the single firms profit maximization problem

$$\begin{aligned}\max_q \pi &= p(Q) \cdot q - c \cdot q \\ &= (a-q) \cdot q - c \cdot q\end{aligned}$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi}{\partial q} = -q + (a-q) - c = 0 \\ \Leftrightarrow 2q = a - c \\ \Leftrightarrow q^* = \frac{a-c}{2}\end{aligned}$$

finding the monopololy price

$$p(q^*) = a - q^* = a - \frac{a-c}{2} = \frac{2a}{2} - \frac{a-c}{2} = \frac{2a-a+c}{2} = \frac{a+c}{2}$$

such that the monopoly profit is

$$\begin{aligned}\pi_M &= p(q^*) \cdot q^* - c \cdot q^* \\ &= \frac{a+c}{2} \cdot \frac{a-c}{2} - c\frac{a-c}{2} \\ &= \frac{a^2 - ac + ac - c^2}{4} - \frac{ac - c^2}{2} \\ &= \frac{a^2 - ac + ac - c^2}{4} - \frac{2ac - 2c^2}{4} \\ &= \frac{a^2 - c^2 - 2ac + 2c^2}{2} \\ &= \frac{a^2 - 2ac + c^2}{4}\end{aligned}$$

Lets set the information from all the different cases up in a table

	Quantity	Price	Profit
1 Firm	$\frac{a-c}{4}$	$\frac{a+c}{2}$	$\frac{a^2-2ac+c^2}{4}$
2 Firms	$\frac{a-c}{3}$	$\frac{a+2c}{3}$	$\frac{a^2-2ac+c^2}{9}$
3 Firms	$\frac{a-c}{2}$	$\frac{a+3c}{4}$	$\frac{a^2-2ac+c^2}{16}$

it is clear that as the number of firms increase, the produced quantity decrease and the profit fir the individual firm decrease that is we move towards perfect competition.

7) Plot the mixed best reponses of each player (in a (p, q)-diagram - see the textbook) and find all Nash equilibria (pure and mixed) in the games below

a)

	L	R
T	0, 0	0, 0
D	0, 0	1, 1

We start by checking if there is any strictly dominating strategies, because if there is then we don't have a mixed strategy equilibrium with the dominated strategy.

We see that player 1 and player 2 don't have any dominating strategies.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays T then players 2's best option is to play either L or R
- If player 1 plays D then players 2's best option is to play R
- If player 2 plays L then players 1's best option is to play either T or D
- If player 2 plays R then players 1's best option is to play D

$$\begin{array}{c|c|c|} & L & R \\ \hline T & 0,0 & 0,0 \\ \hline D & 0,0 & 1,1 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & 0,\underline{0} & 0,\underline{0} \\ \hline D & 0,0 & 1,1 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & 0,\underline{0} & 0,\underline{0} \\ \hline D & 0,0 & 1,\underline{1} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & \underline{0},\underline{0} & 0,\underline{0} \\ \hline D & \underline{0},0 & 1,\underline{1} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & \underline{0},\underline{0} & 0,\underline{0} \\ \hline D & \underline{0},0 & \underline{1},\underline{1} \\ \hline \end{array}$$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0, 0), (1, 1)\}$ which corresponds to (D, R) and (T, L) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

		q_1	$1 - q_1$
		L	R
p_1	T	0, 0	0, 0
$1 - p_1$	D	0, 0	1, 1

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers T $(p = 1)$	**Player 1 indifferent** $(p \in [0,1])$	**Player 1 prefers** B $(p = 0)$
$E[T] > E[D]$ $q \cdot 0 + (1-q) \cdot 0 > q \cdot 0 + (1-q) \cdot 1$ $0 > 1 - q$ $q > 1$ Not possible $q \in [0,1]$	$E[T] = E[D]$ $q \cdot 0 + (1-q) \cdot 0 = q \cdot 0 + (1-q) \cdot 1$ $0 = 1 - q$ $q = 1$	$E[T] < E[D]$ $q \cdot 0 + (1-q) \cdot 0 < q \cdot 0 + (1-q) \cdot 1$ $0 < 1 - q$ $q < 1$

and lets look at when player 1 is indifferent, and when he prefers L and R

Player 2 prefers L $(q = 1)$	**Player 2 indifferent** $(q \in [0,1])$	**Player 2 prefers** B $(q = 0)$
$E[L] > E[R]$ $p \cdot 0 + (1-p) \cdot 0 > p \cdot 0 + (1-p) \cdot 1$ $0 > 1 - p$ $p > 1$ Not possible $p \in [0,1]$	$E[L] = E[R]$ $p \cdot 0 + (1-p) \cdot 0 = p \cdot 0 + (1-p) \cdot 1$ $0 = 1 - p$ $p = 1$	$E[L] < E[R]$ $p \cdot 0 + (1-p) \cdot 0 < p \cdot 0 + (1-p) \cdot 1$ $0 < 1 - p$ $p < 1$

such that the two players best response functions are

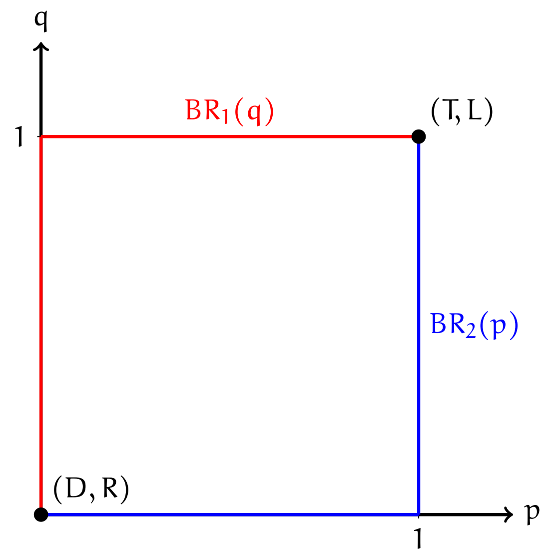
$$BR_1(q) = \begin{cases} p = 1 & \text{when } q > 1 \text{ i.e. Never} \\ p \in [0,1] & \text{when } q = 1 \\ p = 0 & \text{when } 0 < q < 1 \end{cases} \qquad BR_2(p) = \begin{cases} q = 1 & \text{when } p > 1 \text{ i.e. Never} \\ q \in [0,1] & \text{when } p_1 = 1 \\ q = 0 & \text{when } 0 < p < 1 \end{cases}$$

therefor we have mixed and pure strategy nash equilibria in

$$(MS)NE = \{(p^*, q^*) = (0,0), (1,1)\}$$

which really is just the pure strategy nash equilibria, and the game does not have anyu mixed strategy nash equilibria.

we can illustrate this in a figure



and we will have nash equilibria in the points where these to best response functions intersect. If this is with not fractions, then that is mixed strategy nash equilibria.

b)

	L	R
T	1, 3	1, 0
D	1, 1	5, 5

We start by checking if there is any strictly dominating strategies, because if there is then we don't have a mixed strategy equilibrium with the dominated strategy.

We see that player 1 and player 2 don't have any dominating strategies.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays T then players 2's best option is to play either L
- If player 1 plays D then players 2's best option is to play R
- If player 2 plays L then players 1's best option is to play either T or D
- If player 2 plays R then players 1's best option is to play D

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,3 & 1,0 \\ \hline
D & 1,1 & 5,5 \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,\underline{3} & 1,0 \\ \hline
D & 1,1 & 5,5 \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,\underline{3} & 1,0 \\ \hline
D & 1,1 & 5,\underline{5} \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & \underline{1},\underline{3} & 1,0 \\ \hline
D & \underline{1},1 & 5,\underline{5} \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & \underline{1},\underline{3} & 1,0 \\ \hline
D & \underline{1},1 & \underline{5},\underline{5} \\ \hline
\end{array}$$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0, 0), (1, 1)\}$ which corresponds to (D, R) and (T, L) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

$$
\begin{array}{cc|c|c|}
 & & q & 1-q \\
 & & L & R \\
\hline
p & T & 1,3 & 1,0 \\
\hline
1-p & D & 1,1 & 5,5 \\
\hline
\end{array}
$$

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers T $(p = 1)$

$$
\begin{aligned}
E[T] &> E[D] \\
q \cdot 1 + (1-q) \cdot 1 &> q \cdot 1 + (1-q) \cdot 5 \\
q + 1 - q &> 1 + 5 - 5q \\
1 &> 6 - 5q \\
5q &> 5 \\
q &> 1
\end{aligned}
$$

Not possible $q \in [0, 1]$

Player 1 indifferent $(p \in [0, 1])$

$$
\begin{aligned}
E[T] &= E[D] \\
q \cdot 1 + (1-q) \cdot 1 &= q \cdot 1 + (1-q) \cdot 5 \\
q + 1 - q &= 1 + 5 - 5q \\
1 &= 6 - 5q \\
5q &= 5 \\
q &= 1
\end{aligned}
$$

Player 1 prefers B $(p = 0)$

$$
\begin{aligned}
E[T] &< E[D] \\
q \cdot 1 + (1-q) \cdot 1 &< q \cdot 1 + (1-q) \cdot 5 \\
q + 1 - q &< 1 + 5 - 5q \\
1 &< 6 - 5q \\
5q &< 5 \\
0 < q &< 1
\end{aligned}
$$

And lets look at when player 1 is indifferent, and when he prefers L and R

Player 2 prefers L $(q = 1)$

$$
\begin{aligned}
E[L] &> E[R] \\
p \cdot 3 + (1-p) \cdot 1 &> p \cdot 0 + (1-p) \cdot 5 \\
3p + 1 - p &> 5 - 5p \\
2p + 1 &> 5 - 5p \\
7p &> 4 \\
1 > p &> \frac{4}{7}
\end{aligned}
$$

Player 2 indifferent $(q \in [0, 1])$

$$
\begin{aligned}
E[L] &= E[R] \\
p \cdot 3 + (1-p) \cdot 1 &= p \cdot 0 + (1-p) \cdot 5 \\
3p + 1 - p &= 5 - 5p \\
2p + 1 &= 5 - 5p \\
7p &= 4 \\
p &= \frac{4}{7}
\end{aligned}
$$

Player 2 prefers B $(q = 0)$

$$
\begin{aligned}
E[L] &< E[R] \\
p \cdot 3 + (1-p) \cdot 1 &< p \cdot 0 + (1-p) \cdot 5 \\
3p + 1 - p &< 5 - 5p \\
2p + 1 &< 5 - 5p \\
7p &< 4 \\
0 < p &< \frac{4}{7}
\end{aligned}
$$

such that the two players best response functions are

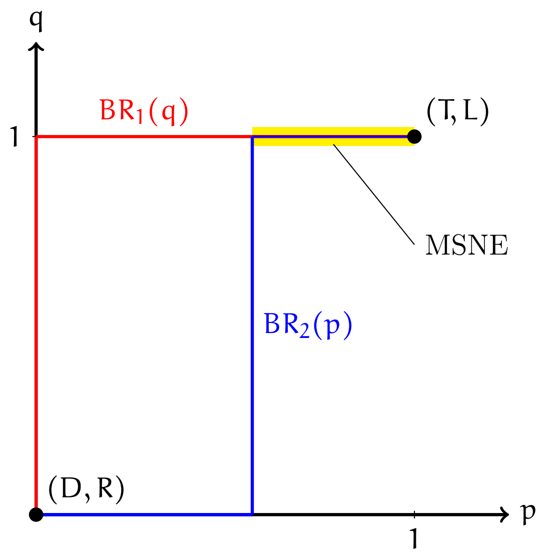
$$
BR_1(q) = \begin{cases} p = 1 & \text{when } q > 1 \text{ i.e. Never} \\ p \in [0, 1] & \text{when } q = 1 \\ p = 0 & \text{when } 0 < q < 1 \end{cases}
\qquad
BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{4}{7} < p < 1 \\ q \in [0, 1] & \text{when } p_1 = \frac{4}{7} \\ q = 0 & \text{when } 0 < p < \frac{4}{7} \end{cases}
$$

therefor we have mixed and pure strategy nash equilibria in

$$
(MS)NE = \left\{ (p^*, q^*) = (0, 0), \left\{ (p, 1) \mid p \in \left[\frac{4}{7}, 1 \right] \right\} \right\}
$$

so this game has an infinite amount of mixed strategy nash equilibria.

we can illustrate this in a figure



c)

	L	R
T	3, 2	1, 2
D	0, 1	1, 2

We start by checking if there is any strictly dominating strategies, because if there is then we don't have a mixed strategy equilibrium with the dominated strategy.

We see that player 1 and player 2 don't have any dominating strategies.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays T then players 2's best option is to play either L or R
- If player 1 plays D then players 2's best option is to play R
- If player 2 plays L then players 1's best option is to play T
- If player 2 plays R then players 1's best option is to play either T or D

$$\begin{array}{ccc} & L & R \\ T & 3,2 & 1,2 \\ D & 0,1 & 1,2 \end{array} \rightarrow \begin{array}{ccc} & L & R \\ T & 3,\underline{2} & 1,\underline{2} \\ D & 0,1 & 1,2 \end{array} \rightarrow \begin{array}{ccc} & L & R \\ T & 3,\underline{2} & 1,\underline{2} \\ D & 0,1 & 1,\underline{2} \end{array} \rightarrow \begin{array}{ccc} & L & R \\ T & \underline{3},\underline{2} & 1,\underline{2} \\ D & 0,1 & 1,\underline{2} \end{array} \rightarrow \begin{array}{ccc} & L & R \\ T & \underline{3},\underline{2} & \underline{1},\underline{2} \\ D & 0,1 & \underline{1},\underline{2} \end{array}$$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0,0), (1,0), (1,1)\}$ which corresponds to $(D, R), (T, R)$ and (T, L) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

		q	$1-q$
		L	R
p	T	3, 2	1, 2
$1-p$	D	0, 1	1, 2

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers T ($p = 1$)

$$\begin{aligned} E[T] &> E[D] \\ q \cdot 3 + (1-q) \cdot 1 &> q \cdot 0 + (1-q) \cdot 1 \\ 3q + 1 - q &> 1 - q \\ 2q + 1 &> 1 - q \\ 1 > q &> 0 \end{aligned}$$

Player 1 indifferent ($p \in [0,1]$)

$$\begin{aligned} E[T] &= E[D] \\ q \cdot 3 + (1-q) \cdot 1 &= q \cdot 0 + (1-q) \cdot 1 \\ 3q + 1 - q &= 1 - q \\ 2q + 1 &= 1 - q \\ q &= 0 \end{aligned}$$

Player 1 prefers D ($p = 0$)

$$\begin{aligned} E[T] &< E[D] \\ q \cdot 3 + (1-q) \cdot 1 &< q \cdot 0 + (1-q) \cdot 1 \\ 3q + 1 - q &< 1 - q \\ 2q + 1 &< 1 - q \\ q &< 0 \end{aligned}$$

which is not possible because $q \in [0, 1]$

And lets look at when player 1 is indifferent, and when he prefers L and R

Player 2 prefers L ($q = 1$)

$$\begin{aligned} E[L] &> E[R] \\ p \cdot 2 + (1-p) \cdot 1 &> p \cdot 2 + (1-p) \cdot 2 \\ 2p + 1 - p &> 2p + 2 - 2p \\ p + 1 &> 2 \\ p &> 1 \end{aligned}$$

which is not possible because $p \in [0, 1]$

Player 2 indifferent ($q \in [0,1]$)

$$\begin{aligned} E[L] &= E[R] \\ p \cdot 2 + (1-p) \cdot 1 &= p \cdot 2 + (1-p) \cdot 2 \\ 2p + 1 - p &= 2p + 2 - 2p \\ p + 1 &= 2 \\ p &= 1 \end{aligned}$$

Player 2 prefers R ($q = 0$)

$$\begin{aligned} E[L] &< E[R] \\ p \cdot 2 + (1-p) \cdot 1 &< p \cdot 2 + (1-p) \cdot 2 \\ 2p + 1 - p &< 2p + 2 - 2p \\ p + 1 &< 2 \\ 0 < p &< 1 \end{aligned}$$

such that the two players best response functions are

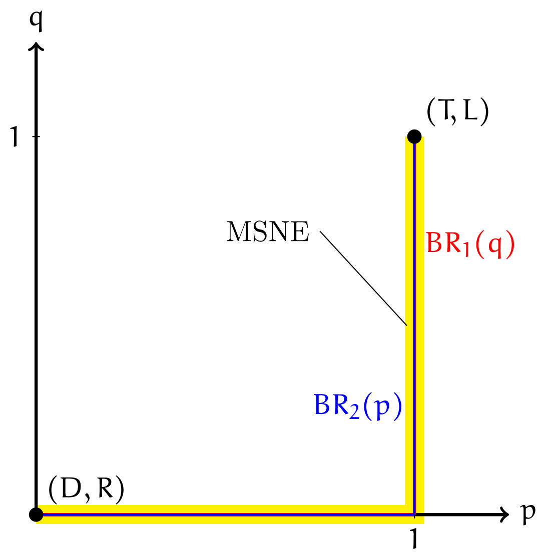
$$BR_1(q) = \begin{cases} p = 1 & \text{when } 0 \leq q \leq 1 \\ p \in [0,1] & \text{when } q = 0 \end{cases} \qquad BR_2(p) = \begin{cases} q \in [0,1] & \text{when } p = 1 \\ q = 0 & \text{when } 0 \leq p \leq 1 \end{cases}$$

therefor we have mixed and pure strategy nash equilibria in

$$(MS)NE = \left\{ (p^*, q^*) = \left\{ (p, 0), p \in [0,1] \right\} \bigcup \left\{ (1, q), q \in [0,1] \right\} \right\}$$

so this game has an infinite amount of mixed strategy nash equilibria.

we can illustrate this in a figure



d)

	t_1	t_2
s_1	2, 1	3, 0
s_2	1, 2	4, 3
s_3	0, 1	0, 3

We start by checking if there is any strictly dominating strategies, and we see that strategy s_1 strictly dominates strategy s_3, therefor player 1 will never play strategy s_3 with any probabilities, and it is not part of a nash equilibria in both pure or mixed strategies.

Removing strategy s_3 from the game

	t_1	t_2
s_1	2, 1	3, 0
s_2	1, 2	4, 3

Now there is no more strictly dominated strategies, and we cannot reduce the game more.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays s_1 then players 2's best option is to play either t_1 or R
- If player 1 plays s_2 then players 2's best option is to play t_2
- If player 2 plays t_1 then players 1's best option is to play s_1
- If player 2 plays t_2 then players 1's best option is to play either t_2 or D

	t_1	t_2
s_1	2, 1	3, 0
s_2	1, 2	4, 3

→

	t_1	t_2
s_1	$2, \underline{1}$	3, 0
s_2	1, 2	4, 3

→

	t_1	t_2
s_1	$2, \underline{1}$	3, 0
s_2	1, 2	$4, \underline{3}$

→

	t_1	t_2
s_1	$\underline{2}, \underline{1}$	3, 0
s_2	1, 2	$4, \underline{3}$

→

	t_1	t_2
s_1	$\underline{2}, \underline{1}$	3, 0
s_2	1, 2	$\underline{4}, \underline{3}$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0,0), (1,1)\}$ which corresponds to (s_3, t_2) and (s_1, t_1) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

		q	$1-q$
		t_1	t_2
p	s_1	$2,1$	$3,0$
$1-p$	s_2	$1,2$	$4,3$

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers s_1 $(p=1)$

$$\begin{gathered} E[s_1] > E[s_2] \\ q\cdot 2 + (1-q)\cdot 3 > q\cdot 1 + (1-q)\cdot 4 \\ 2q + 3 - 3q > q + 4 - 4q \\ 3 - q > 4 - 3q \\ 2q > 1 \\ 1 > q > \frac{1}{2} \end{gathered}$$

Player 1 indifferent $(p \in [0,1])$

$$\begin{gathered} E[s_1] = E[s_2] \\ q\cdot 2 + (1-q)\cdot 3 = q\cdot 1 + (1-q)\cdot 4 \\ 2q + 3 - 3q = q + 4 - 4q \\ 3 - q = 4 - 3q \\ 2q = 1 \\ q = \frac{1}{2} \end{gathered}$$

Player 1 prefers s_2 $(p=0)$

$$\begin{gathered} E[s_1] < E[s_2] \\ q\cdot 2 + (1-q)\cdot 3 < q\cdot 1 + (1-q)\cdot 4 \\ 2q + 3 - 3q < q + 4 - 4q \\ 3 - q < 4 - 3q \\ 2q < 1 \\ 0 < q < \frac{1}{2} \end{gathered}$$

And lets look at when player 1 is indifferent, and when he prefers t_1 and t_2

Player 2 prefers t_1 $(q=1)$

$$\begin{gathered} E[t_1] > E[t_2] \\ p\cdot 1 + (1-p)\cdot 2 > p\cdot 0 + (1-p)\cdot 3 \\ p + 2 - 2p > 3 - 3p \\ 2 - p > 3 - 3p \\ 2p > 1 \\ 1 > p > \frac{1}{2} \end{gathered}$$

which is not possible because $p \in [0,1]$

Player 2 indifferent $(q \in [0,1])$

$$\begin{gathered} E[t_1] = E[t_2] \\ p\cdot 1 + (1-p)\cdot 2 = p\cdot 0 + (1-p)\cdot 3 \\ p + 2 - 2p = 3 - 3p \\ 2 - p = 3 - 3p \\ 2p = 1 \\ p = \frac{1}{2} \end{gathered}$$

Player 2 prefers t_2 $(q=0)$

$$\begin{gathered} E[t_1] < E[t_2] \\ p\cdot 1 + (1-p)\cdot 2 < p\cdot 0 + (1-p)\cdot 3 \\ p + 2 - 2p < 3 - 3p \\ 2 - p < 3 - 3p \\ 2p < 1 \\ 0 < p < \frac{1}{2} \end{gathered}$$

such that the two players best response functions are

$$BR_1(q) = \begin{cases} p = 1 & \text{when } \frac{1}{2} < q < 1 \\ p \in [0,1] & \text{when } q = \frac{1}{2} \\ p = 0 & \text{when } 0 < q < \frac{1}{2} \end{cases} \qquad BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{1}{2} < p < 1 \\ q \in [0,1] & \text{when } p = \frac{1}{2} \\ q = 0 & \text{when } 0 < p < \frac{1}{2} \end{cases}$$

therefor we have mixed and pure strategy nash equilibria in

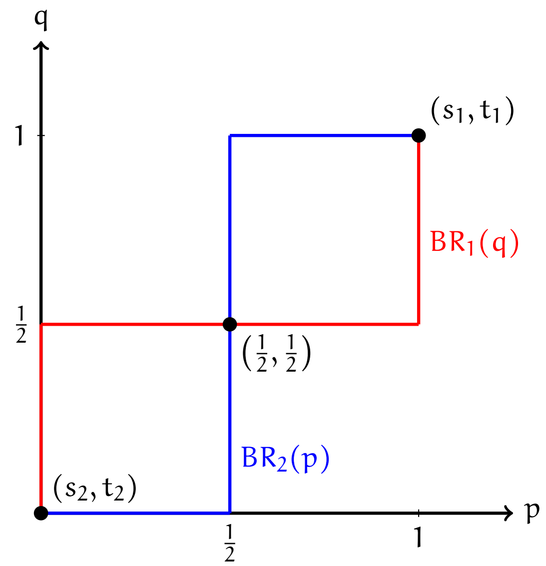
$$(MS)NE = \left\{ (p^*, q^*) = (0,0), \left(\frac{1}{2}, \frac{1}{2}\right), (1,1) \right\}$$

and the probability 0, to s_3, such that the NE for the full game is

$$(MS)NE = \left\{ (p_1^*, p_2^*, q^*) = (0,1,0), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), (1,0,1) \right\}$$

so this game has an infinite amount of mixed strategy nash equilibria.

we can illustrate this in a figure



8) Find all (pure and mixed) Nash equilibria in the following game

	L	R	C
T	4, 1	2, 3	0, 4
D	2, 3	1, 2	5, 0

We start by noticing that there are no strictly dominating strategies for both players, and we cannot remove any strategies.

Let plug in for the best responses and find the pure strategy nash equilibria

- If player 1 plays T, then player 2's best respoonse is to play C
- If player 1 plays D, then player 2's best response is to play L
- If player 2 plays L, then player 1's best response is to play T
- If player 2 plays C, then player 1's best response is to play T

- If player 2 plays R, then player 1's best response is to play B

	L	C	R
T	4, 1	2, 3	0, 4
D	2, 3	1, 2	5, 0

→

	L	C	R
T	4, 1	2, 3	$0, \underline{4}$
D	2, 3	1, 2	5, 0

→

	L	C	R
T	4, 1	2, 3	$0, \underline{4}$
D	$2, \underline{3}$	1, 2	5, 0

→

	L	C	R
T	$\underline{4}, 1$	2, 3	$0, \underline{4}$
D	$2, \underline{3}$	1, 2	5, 0

→

	L	C	R
T	$\underline{4}, 1$	$\underline{2}, 3$	$0, \underline{4}$
D	$2, \underline{3}$	1, 2	5, 0

→

	L	C	R
T	$\underline{4}, 1$	$\underline{2}, 3$	$0, \underline{4}$
D	$2, \underline{3}$	1, 2	$\underline{5}, 0$

we see that there are no nash equilibria in pure strategies.

The game is a static game of a finite number of players, and therefor we know that there is a nash equilibria either in pure strategies or in mixed strategies, and because there was no nash equilibria in pure strategies, we know that there are at least one nash equilibria in mixed strategies.

Lets assign a probability to each action

		q_1	q_2	$1-q_1-q_2$
		L	C	R
p	T	4, 1	2, 3	0, 4
$1-p$	D	2, 3	1, 2	5, 0

Lets calculate the two players expected returns

$$\mathbb{E}[T] = 4 \cdot q_1 + 2 \cdot q_2 + 0 \cdot (1 - q_1 - q_2) = 4q_1 + 2q_2$$
$$\mathbb{E}[D] = 2 \cdot q_1 + 1 \cdot q_2 + 5 \cdot (1 - q_1 - q_2) = 2q_1 + q_2 + 5 - 5q_1 - 5q_2 = 5 - 3q_1 - 4q_2$$

$$\mathbb{E}[L] = 1 \cdot p + 3 \cdot (1 - p) = p + 3 - 3p = 3 - 2p$$
$$\mathbb{E}[C] = 3 \cdot p + 2 \cdot (1 - p) = 3p + 2 - 2p = p + 2$$
$$\mathbb{E}[R] = 4 \cdot p + 0 \cdot (1 - p) = 4p$$

Now we need to figure out when the two players are indifferent.

Lets look, at when player 1 is indifferent

Player 1 prefers T $(p = 1)$

$$\mathbb{E}[T] > \mathbb{E}[D]$$
$$4q_1 + 2q_2 > 5 - 3q_1 - 4q_2$$
$$7q_1 + 6q_2 > 5$$
$$q_1 + \frac{6}{7}q_2 > \frac{5}{7}$$

Player 1 indifferent $(p \in [0, 1])$

$$\mathbb{E}[T] = \mathbb{E}[D]$$
$$4q_1 + 2q_2 = 5 - 3q_1 - 4q_2$$
$$7q_1 + 6q_2 = 5$$
$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

Player 1 prefers D $(p = 0)$

$$\mathbb{E}[T] < \mathbb{E}[D]$$
$$4q_1 + 2q_2 < 5 - 3q_1 - 4q_2$$
$$7q_1 + 6q_2 < 5$$
$$q_1 + \frac{6}{7}q_2 < \frac{5}{7}$$

lets write up the best response for player 1

$$BR_1(q_1,q_2) = \begin{cases} p = 1 & \text{when } q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ p \in [0,1] & \text{when } q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ p = 0 & \text{when } q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

now lets take a look at player 2, and calculate when player 2 is indifferent.

Player 2 is indifferent between L and C when

$$\mathbb{E}[L] = \mathbb{E}[C] \implies 3 - 2p = 2 + p \Leftrightarrow 1 = 3p \Leftrightarrow p = \frac{1}{3}$$

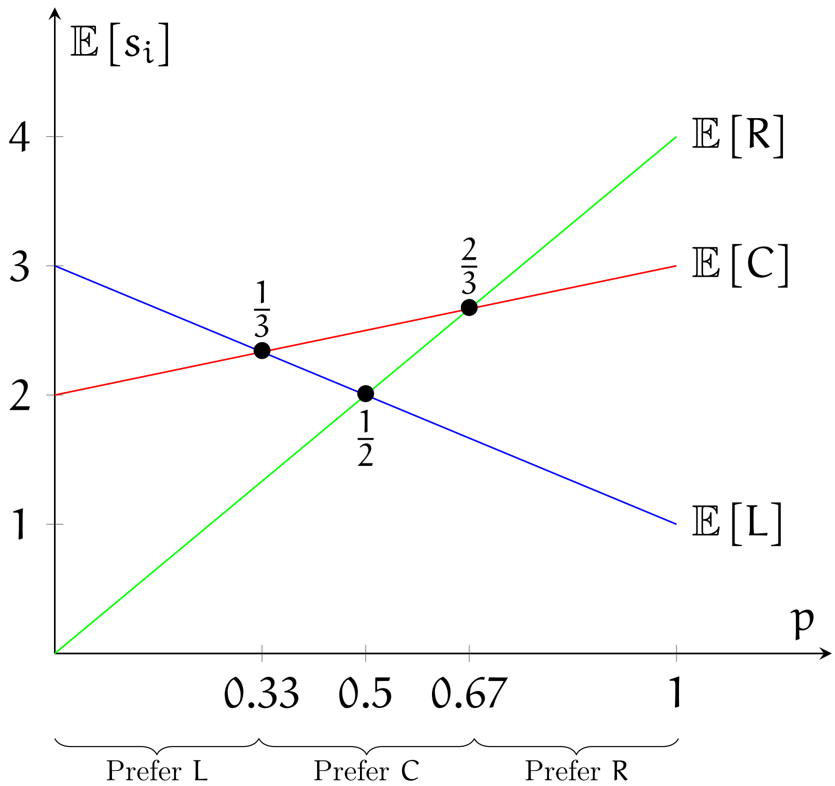
Player 2 is indifferent between L and R when

$$\mathbb{E}[L] = \mathbb{E}[R] \implies 3 - 2p = 4p \Leftrightarrow 3 = 6p \Leftrightarrow p = \frac{1}{2}$$

Player 2 is indifferent between C and R when

$$\mathbb{E}[C] = \mathbb{E}[R] \implies 2 + p = 4p \Leftrightarrow 2 = 3p \Leftrightarrow p = \frac{2}{3}$$

lets illustrate this in a diagram plotting in the expected payoffs, to better understand when a strategy is preffered to another



the figure clearly shows, that the the player will never mix between L and R, because at this point he could get strictly higher payoff by playing C, so we can remove $P = 1/2$ as a possible mixed strategy nash equilibria.

Lets look at the three cases where player 2 is indifferent between two of the three strategies. Writing up the best response for player 2

$$BR_2(p) = \begin{cases} (q_1, q_2) = (0,0) & \text{when } 1 > p > \frac{2}{3} & \text{When player 2 prefers R} \\ (q_1, q_2) = (0,x) : x \in [0,1] & \text{when } p = \frac{2}{3} & \text{When player 2 is indifferent between C \& R} \\ (q_1, q_2) = (0,1) & \text{when } p \in \left(\frac{1}{2}, \frac{2}{3}\right) & \text{When player 2 prefers C} \\ (q_1, q_2) = (x, 1-x) : x \in [0,1] & \text{when } p = \frac{1}{3} & \text{When player 2 is indifferent between C \& L} \\ (q_1, q_2) = (1,0) & \text{when } 0 < p < \frac{1}{3} & \text{When player 2 prefers L} \end{cases}$$

Case $p = \frac{1}{3}$:
When $p = \frac{1}{3}$ then player 2 is indifferent between playing C and L, and therefor the best response is to play $(q_1, q_2) = (x, 1-x)$.

We need to check if these values of q_1 and q_2 fits with the best responses for player 1, where player 1 mixed between strategies (i.e. when $p \in [0,1]$).

$$\begin{aligned} BR_1\left(BR_2\left(p = \frac{1}{3}\right)\right) = x + \frac{6}{7}(1-x) &= \frac{5}{7} \\ \Leftrightarrow x + \frac{6}{7} - \frac{6}{7}x &= \frac{5}{7} \\ \Leftrightarrow \frac{7}{7}x - \frac{6}{7}x &= \frac{5}{7} - \frac{6}{7} \\ \Leftrightarrow \frac{1}{7}x &= -\frac{1}{7} \\ \Leftarrow x &= -1 \end{aligned}$$

but we cannot have a probability which can be outside of $[0,1]$, and therefor there is no Nash equilibria where $p = \frac{1}{3}$.

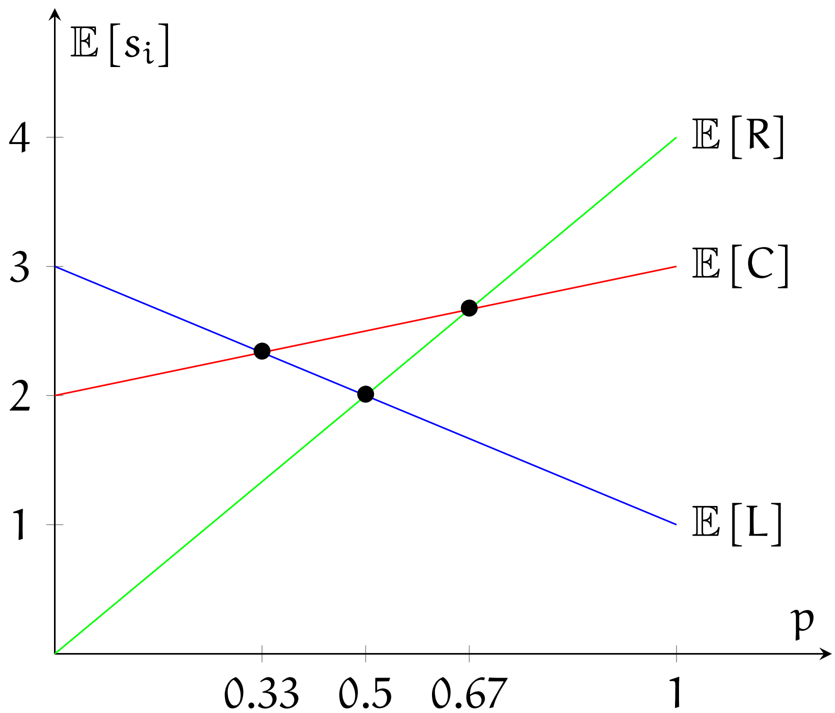
Case $p = \frac{2}{3}$:
When $p = \frac{2}{3}$ then player 2 is indifferent between playing C and R, and therefor the best response is to play $(q_1, q_2) = (0, x)$.

We need to check if these values of q_1 and q_2 fits with the best responses for player 1, where player 1 mixed between strategies (i.e. when $p \in [0,1]$).

$$\begin{aligned} BR_1\left(BR_2\left(p = \frac{2}{3}\right)\right) = 0 + \frac{6}{7}x &= \frac{5}{7} \\ \Leftrightarrow \frac{6}{7}x &= \frac{5}{7} \\ \Leftrightarrow \frac{1}{7}x &= \frac{5}{7 \cdot 6} \\ \Leftrightarrow \frac{1}{7}x &= \frac{5}{42} \\ \Leftrightarrow x &= \frac{35}{42} = \frac{5}{6} \end{aligned}$$

and this is a probability, so we have a mixed strategy nash equilbria in

$$MSEN = \left\{ \left((p, 1-p^*), (q_1^*, q_2^*, 1 - q_1^* - q_2^*) \right) = \left(\left(\frac{2}{3}, \frac{1}{3}\right) \left(0, \frac{5}{6}, \frac{1}{6}\right) \right) \right\}$$


$\mathbb{E}[s_i]$
4
3
2
1
$\mathbb{E}[R]$
$\mathbb{E}[C]$
$\mathbb{E}[L]$
p
0.33
0.5
0.67
1