Microeconomics B - Problem Set 4

Dynamic Games of Perfect Information

1) Find all equilibria (pure and mixed) in the following games, first analytically and then through plotting the best-response functions

a)

	L	R
T	3, 3	0, 0
B	0, 0	4, 4

We start by checking if there is any strictly dominating strategies, because if there is then we don't have a mixed strategy equilibrium with the dominated strategy.

We see that player 1 and player 2 don't have any dominating strategies.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays T then players 2's best option is to play L
- If player 1 plays B then players 2's best option is to play R
- If player 2 plays L then players 1's best option is to play L
- If player 2 plays R then players 1's best option is to play B

$$\begin{array}{c|c|c|} & L & R \\ \hline T & 3,3 & 0,0 \\ \hline B & 0,0 & 4,4 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & 3,\underline{3} & 0,0 \\ \hline B & 0,0 & 4,4 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & 3,\underline{3} & 0,0 \\ \hline B & 0,0 & 4,\underline{4} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & \underline{3},\underline{3} & 0,0 \\ \hline B & 0,0 & 4,\underline{4} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & L & R \\ \hline T & \underline{3},\underline{3} & 0,0 \\ \hline B & 0,0 & \underline{4},\underline{4} \\ \hline \end{array}$$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0,0), (1,1)\}$ which corresponds to (B, R) and (T, L) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

$$\begin{array}{cc|c|c|} & & q_1 & 1-q_1 \\ & & L & R \\ \hline p_1 & T & 3,3 & 0,0 \\ \hline 1-p_1 & B & 0,0 & 4,4 \\ \hline \end{array}$$

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers T $(p = 1)$	**Player 1 indifferent** $(p \in [0,1])$	**Player 1 prefers** B $(p = 0)$
$$\begin{gathered} E[T] > E[B] \\ q \cdot 3 + (1-q) \cdot 0 > q \cdot 0 + (1-q) \cdot 4 \\ 3q > 4 - 4q \\ 7q > 4 \\ 1 > q > \frac{4}{7} \end{gathered}$$	$$\begin{gathered} E[T] = E[B] \\ q \cdot 3 + (1-q) \cdot 0 = q \cdot 0 + (1-q) \cdot 4 \\ 3q = 4 - 4q \\ 7q = 4 \\ q = \frac{4}{7} \end{gathered}$$	$$\begin{gathered} E[T] < E[B] \\ q \cdot 3 + (1-q) \cdot 0 < q \cdot 0 + (1-q) \cdot 4 \\ 3q < 4 - 4q \\ 7q < 4 \\ 0 < q < \frac{4}{7} \end{gathered}$$

and lets look at when player 1 is indifferent, and when he prefers L and R

Player 2 prefers L $(q = 1)$	**Player 2 indifferent** $(q \in [0,1])$	**Player 2 prefers** R $(q = 0)$
$$\begin{gathered} E[L] > E[R] \\ p \cdot 3 + (1-p) \cdot 0 > p \cdot 0 + (1-p) \cdot 4 \\ 3p > 4 - 4p \\ 7p > 4 \\ 1 > p > \frac{4}{7} \end{gathered}$$	$$\begin{gathered} E[L] = E[R] \\ p \cdot 3 + (1-p) \cdot 0 = p \cdot 0 + (1-p) \cdot 4 \\ 3p = 4 - 4p \\ 7p = 4 \\ p = \frac{4}{7} \end{gathered}$$	$$\begin{gathered} E[L] < E[R] \\ p \cdot 3 + (1-p) \cdot 0 < p \cdot 0 + (1-p) \cdot 4 \\ 3p < 4 - 4p \\ 7p < 4 \\ 0 < p < \frac{4}{7} \end{gathered}$$

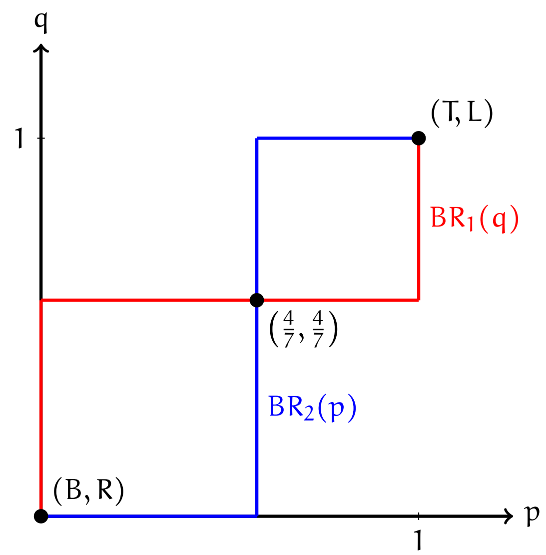
such that the two players best response functions are

$$BR_1(q) = \begin{cases} p = 1 & \text{when } \frac{4}{7} < q < 1 \\ p \in [0,1] & \text{when } q = \frac{4}{7} \\ p = 0 & \text{when } 0 < q < \frac{4}{7} \end{cases} \qquad BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{4}{7} < p < 1 \\ q \in [0,1] & \text{when } p = \frac{4}{7} \\ q = 0 & \text{when } 0 < p < \frac{4}{7} \end{cases}$$

therefor we have mixed and pure strategy nash equilibria in

$$(MS)NE = \left\{ (p^*, q^*) = (0,0), \left(\frac{4}{7}, \frac{4}{7}\right), (1,1) \right\}$$

we can illustrate this in a figure



and we will have nash equilibria in the points where these to best response functions intersect. If this is with not fractions, then that is mixed strategy nash equilibria.

b)

	L	R
T	1, 1	0, 0
B	1, 0	2, 1

We start by checking if there is any strictly dominating strategies, because if there is then we don't have a mixed strategy equilibrium with the dominated strategy.

We see that player 1 and player 2 don't have any dominating strategies.

Lets solve for the pure strategy Nash equilibria, by plugging in for the best responses in the bimatrix

- If player 1 plays T then players 2's best option is to play L
- If player 1 plays B then players 2's best option is to play R
- If player 2 plays L then players 1's best option is to play either T or B
- If player 2 plays R then players 1's best option is to play B

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,1 & 0,0 \\ \hline
B & 1,0 & 2,1 \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,\underline{1} & 0,0 \\ \hline
B & 1,0 & 2,1 \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & 1,\underline{1} & 0,0 \\ \hline
B & 1,0 & 2,\underline{1} \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & \underline{1},\underline{1} & 0,0 \\ \hline
B & \underline{1},0 & 2,\underline{1} \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & L & R \\ \hline
T & \underline{1},\underline{1} & 0,0 \\ \hline
B & \underline{1},0 & \underline{2},\underline{1} \\ \hline
\end{array}$$

We see that there are two nash equilibria in pure strategies, so whenever we look for mixed strategies, we should also find the equilbiria where $\{(p^*, q^*) = (0,0), (1,1)\}$ which corresponds to (B, R) and (T, L) respectively.

Now lets take a look at the mixed strategy nash equilibria. We assign a probability for each player and for each action

$$\begin{array}{cc|c|c|}
 & & q_1 & 1-q \\
 & & L & R \\ \hline
p_1 & T & 1,1 & 0,0 \\ \hline
1-p & B & 1,0 & 2,1 \\ \hline
\end{array}$$

Then we need to look at for what probabilities player 1 and player 2 is indifferent between playing both actions, and when they prefer one of the two actions.

Player 1 prefers T $(p = 1)$

$$E[T] > E[B]$$
$$q \cdot 1 + (1-q) \cdot 0 > q \cdot 1 + (1-q) \cdot 2$$
$$q > q + 2 - 2q$$
$$2q > 2$$
$$q > 1$$

Not possible $q \in [0, 1]$

Player 1 indifferent $(p \in [0, 1])$

$$E[T] = E[B]$$
$$q \cdot 1 + (1-q) \cdot 0 = q \cdot 1 + (1-q) \cdot 2$$
$$q = q + 2 - 2q$$
$$2q = 2$$
$$q = 1$$

Player 1 prefers B $(p = 0)$

$$E[T] < E[B]$$
$$q \cdot 1 + (1-q) \cdot 0 < q \cdot 1 + (1-q) \cdot 2$$
$$q < q + 2 - 2q$$
$$2q < 2$$
$$0 < q < 1$$

and lets look at when player 1 is indifferent, and when he prefers L and R

Player 2 prefers L $(q = 1)$	**Player 2 indifferent** $(q \in [0,1])$	**Player 2 prefers R** $(q = 0)$
$$\begin{gathered} E[L] > E[R] \\ p \cdot 1 + (1-p) \cdot 0 > p \cdot 0 + (1-p) \cdot 1 \\ p > 1-p \\ 2p > 1 \\ 1 > p > \frac{1}{2} \end{gathered}$$	$$\begin{gathered} E[L] = E[R] \\ p \cdot 1 + (1-p) \cdot 0 = p \cdot 0 + (1-p) \cdot 1 \\ p = 1-p \\ 2p = 1 \\ p = \frac{1}{2} \end{gathered}$$	$$\begin{gathered} E[L] < E[R] \\ p \cdot 1 + (1-p) \cdot 0 < p \cdot 0 + (1-p) \cdot 1 \\ p < 1-p \\ 2p < 1 \\ 0 < p < \frac{1}{2} \end{gathered}$$

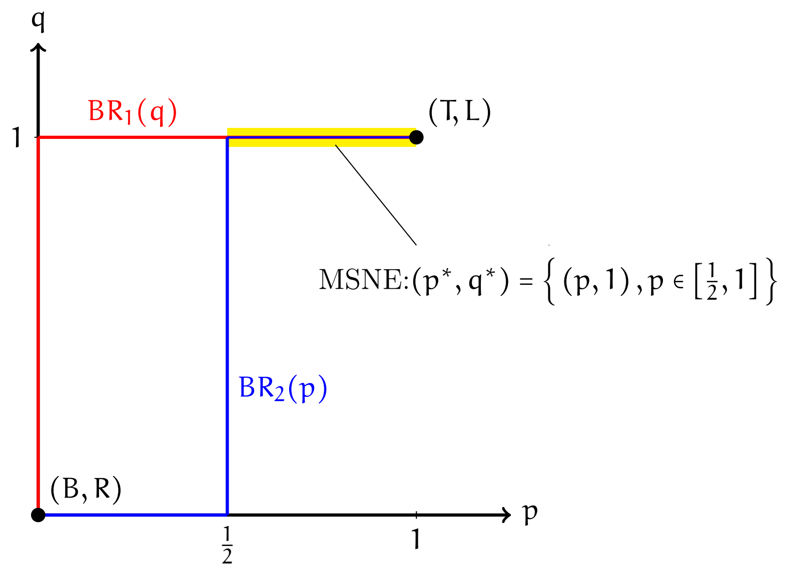
such that the two players best response functions are

$$BR_1(q) = \begin{cases} p \in [0,1] & \text{when } q = 1 \\ p = 0 & \text{when } 0 < q < 1 \end{cases} \qquad BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{1}{2} < p < 1 \\ q \in [0,1] & \text{when } p = \frac{1}{2} \\ q = 0 & \text{when } 0 < p < \frac{1}{2} \end{cases}$$

therefor we have mixed and pure strategy nash equilibria in

$$(MS)NE = \left\{ (p^*, q^*) = (0,0), \left\{ (p,1), p \in \left[\frac{1}{2}, 1\right] \right\} \right\}$$

we can illustrate this in a figure



and we will have nash equilibria in the points where these to best response functions intersect. If this is with not fractions, then that is mixed strategy nash equilibria.

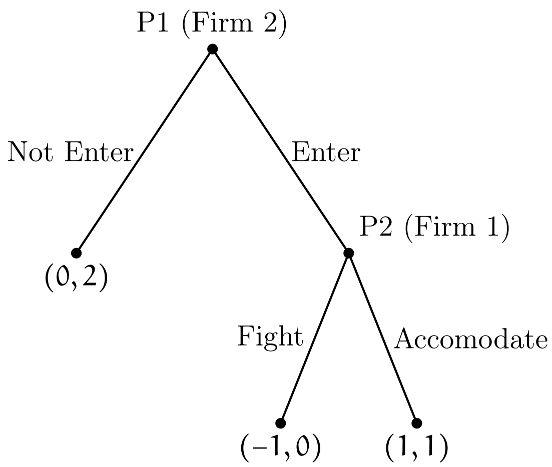
2) [Entry deterrence] Consider the following dynamic game: firm 1 owns a shop in town A. Firm 2 decides whether to enter the market in town A. If firm 2 enters, firm 1 chooses whether to fight or accommodate the entrant. If firm 2 does not enter, firm 1 receives a profit of 2 and firm 2 gets 0. If firm 2 enters and firm 1 accommodates, they share the market and each of them receives a profit of 1. If firm 2 enters and firm 1 decides to fight, firm 2 suffers a loss of 1 (so that the payoff is -1), but fighting is costly for firm 1, lowering its payoff to 0.

a) **Draw the game tree**

Lets state the dynamics of the game

1. Firm 2 decides to Enter a market or $\mathsf{Not\ Enter}$ a market, if firm 2 does not enter the game ends.
2. Firm 1 observes Firm 2's choice, and chooses between $\mathsf{Accomodating}$ or $\mathsf{Fighting}$ the entrants.
3. Payoffs are recieved

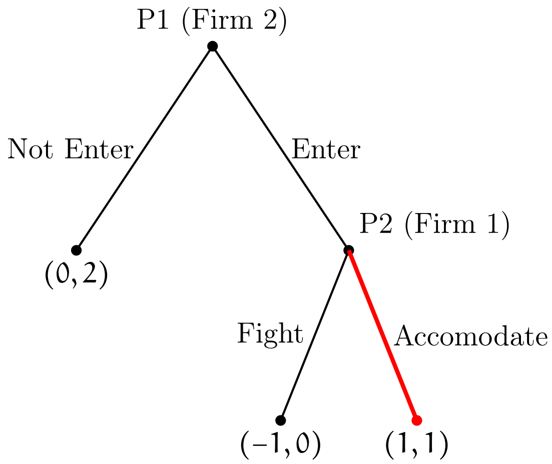
We are asked to illustrate this game in a game tree



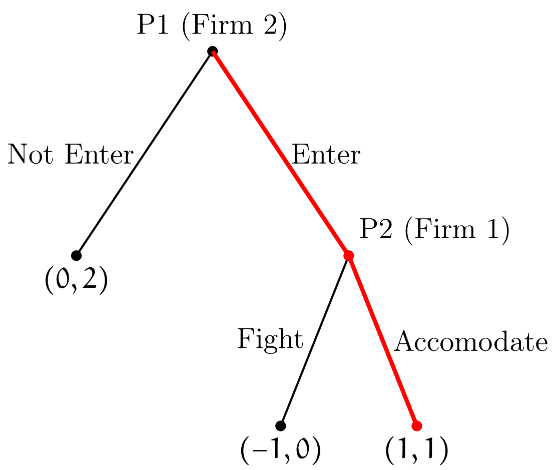
b) Solve the game by backwards induction

When solving a game by backwards induction we need to solve the game in its last stage first taken the action earlier on in the game as given.

Solving firm 1's problem. Firm 1 i.e. player 2 chooses between fighting or accomodating the entrants of the second firm. Player 2 gets the highest payoff of 1 by accomodating so he chooses to accomodate



Now we need to solve player 1's decision between entering and not entering, knowing that if he enters then player 2 will choose to accomodate. Therefor player 1's choise is between a payoff of 1 by entering and a payoff of 0 by not enteringer. It is therefor optimal for player 1 to enter the market



such that the backwards iduc

3) Consider the following Generalized Battle of the Sexes Game, with $N > 1$:

	C1	C2
C1	N, 1	0, 0
C2	0, 0	1, N

a) How can you interpret the parameter N?

An additional utility factor for your own most preferred outcome, if you are together with the other person.

Higher N gives the two players a higher conflict of interest.

b) Solve for the mixed strategy Nash equilibrium. When N becomes very large, what happens to the probability of successful coordination?

Lets start by looking at Nash equilibria in pure strategies, by plugging in for the best responses

- If player 1 plays C1 then player 2's optimal choice is to play C1
- If player 1 plays C2 then player 2's optimal choice is to play C2
- If player 2 plays C1 then player 1's optimal choice is to play C1
- If player 2 plays C2 then player 1's optimal choice is to play C2

	C1	C2
C1	$\underline{N}, \underline{1}$	0, 0
C2	0, 0	$\underline{1}, \underline{N}$

Lets find Nash equilibria in mixed strategies by assigning a probability to each players actions

		q	$1-q$
		C1	C2
p	C1	N, 1	0, 0
$1-p$	C2	0, 0	1, N

player 1 is indifferent between the two strategies and prefers

Player 1 prefers C1 ($p = 1$)

$$
\begin{aligned}
E[C1] &> E[C2] \\
q \cdot N + (1-q)\cdot 0 &> q \cdot 0 + (1-q)\cdot 1 \\
Nq &> 1-q \\
(1+N)\,q &> 1 \\
1 > q &> \frac{1}{1+N}
\end{aligned}
$$

Player 1 indifferent ($p \in [0,1]$)

$$
\begin{aligned}
E[C1] &= E[C2] \\
q \cdot N + (1-q)\cdot 0 &= q \cdot 0 + (1-q)\cdot 1 \\
Nq &= 1-q \\
(1+N)\,q &= 1 \\
q &= \frac{1}{1+N}
\end{aligned}
$$

Player 1 prefers C2 ($p = 0$)

$$
\begin{aligned}
E[C1] &< E[C2] \\
q \cdot N + (1-q)\cdot 0 &< q \cdot 0 + (1-q)\cdot 1 \\
Nq &< 1-q \\
(1+N)\,q &< 1 \\
0 < q &< \frac{1}{1+N}
\end{aligned}
$$

player 2 is indifferent between the two strategies and prefers

Player 1 prefers C1 ($q = 1$)

$$
\begin{aligned}
E[C1] &> E[C2] \\
p \cdot 1 + (1-p)\cdot 0 &> p \cdot 0 + (1-p)\cdot N \\
p &> N - Np \\
(1+N)\,p &> N \\
1 > p &> \frac{N}{1+N}
\end{aligned}
$$

Player 1 indifferent ($q \in [0,1]$)

$$
\begin{aligned}
E[C1] &= E[C2] \\
p \cdot 1 + (1-p)\cdot 0 &= p \cdot 0 + (1-p)\cdot N \\
p &= N - Np \\
(1+N)\,p &= N \\
p &= \frac{N}{1+N}
\end{aligned}
$$

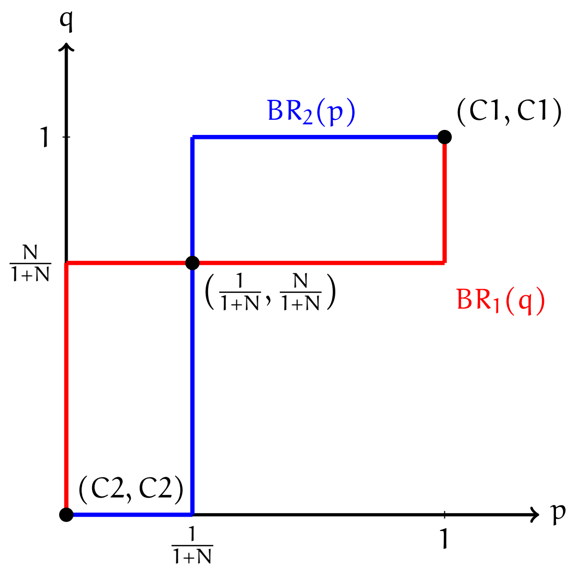
Player 1 prefers C2 ($q = 0$)

$$
\begin{aligned}
E[C1] &< E[C2] \\
p \cdot 1 + (1-p)\cdot 0 &< p \cdot 0 + (1-p)\cdot N \\
p &< N - Np \\
(1+N)\,p &< N \\
0 < p &< \frac{N}{1+N}
\end{aligned}
$$

we can now state the best response functions for the two players

$$
BR_1(q) = \begin{cases} p = 1 & \text{when } \frac{1}{1+N} < q < 1 \\ p \in [0,1] & \text{when } q = \frac{1}{1+N} \\ p = 0 & \text{when } 0 < q < \frac{1}{1+N} \end{cases}
\qquad
BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{N}{1+N} < p < 1 \\ q \in [0,1] & \text{when } p = \frac{N}{1+N} \\ q = 0 & \text{when } 0 < p < \frac{N}{1+N} \end{cases}
$$

illustrating this in a figure



the mixed strategy nash equilibria are

$$\text{MSNE} = \left\{(p^*, q^*) = (0,0), \left(\frac{1}{1+N}, \frac{N}{1+N}\right), (1,1)\right\}$$

What happens as $N \to \infty$:

Lets take the limit to the probabilities giving a mixed strategy Nash equilibria

$$\lim_{N\to\infty}(p^*) = \lim_{N\to\infty}\left(\frac{1}{1+N}\right) = \frac{1}{\infty} \overbrace{=}^{\text{L'Höpital}} \lim_{N\to\infty}\left(\frac{0}{1}\right) = 0$$
$$\lim_{N\to\infty}(q^*) = \lim_{N\to\infty}\left(\frac{N}{1+N}\right) = \frac{\infty}{\infty} \overbrace{=}^{\text{L'Höpital}} \lim_{N\to\infty}\left(\frac{1}{1}\right) = 1$$

such that player 1 will often play $C1$ and player 2 will often play $C2$.

This is because whenever $N \to \infty$, it becomes harder and harder to coordinate, because the payoffs become to large.

4) Penalty kicks - No solution

5) North-Atlantic, 1943. An allied convoy, counting 100 ships, is heading east and it can choose between a northern route where icebergs are known to be numerous or a more southern route. The northern route is dangerous - because of the icebergs - and it is estimated that 6 ships will get lost due to icebergs. Below the surface, the wolf-pack lures. If the u-boats catch the convoy on the southern route, it is a field day, and 40 ships from the convoy are estimated to get lost. If the u-boats catch the convoy on the northern route, they do not have as much time hunting down the convoy - due to petrol shortages - and they are only expected to be able to sink 20 ships from the convoy. The wolf-pack does not have time to check both locations, north and south. Each headquarter (allied or nazi) has to decide whether to go north or south. Unfortunately, there is no radar etc, so one cannot observe the move of the enemy before taking a decision. Each headquarter has a simple payoff function. For the allied headquarter it equals the number of ships making it across the Atlantic. For the nazi headquarter payoff equals the number of ships lost by the allies.

a) Write down this strategic situation in a bi-matrix

We need to state this game in a normal form

Players	Two players Allies (Player 1) and Nazis (Player 2)
Strategy Sets	The strategy Sets are $S_i = \{\text{"North"}, \text{"South"}\}$ for $i = N, A$.
Payoff's	The players' payoffs, in all outcomes in all the different scenarios

$$u_A\left(s_A, s_N\right) = \begin{cases} 100 - 20 - 6 = 74 & \text{if } s_A = s_N = \text{"North"} \\ 100 - 40 = 60 & \text{if } s_A = s_N = \text{"South"} \\ 100 - 6 = 94 & \text{if } s_A = \text{"North"} \neq s_N = \text{"South"} \\ 100 & \text{if } s_A = \text{"South"} \neq s_N = \text{"North"} \end{cases}$$

$$u_N\left(s_A, s_N\right) = \begin{cases} 20 + 6 = 26 & \text{if } s_A = s_N = \text{"North"} \\ 40 & \text{if } s_A = s_N = \text{"South"} \\ 6 & \text{if } s_A = \text{"North"} \neq s_N = \text{"South"} \\ 0 & \text{if } s_A = \text{"South"} \neq s_N = \text{"North"} \end{cases}$$

lets state this in a bimatrix

		Nazis	
		North	South
Allies	North	$74, 26$	$94, 6$
	South	$100, 0$	$60, 40$

b) Find the Nash Equilibrium (equilibria?)

We start by noticing that there are no strictly dominating strategies, and we cannot eliminate any strategies from the game.

Let plug in for the best responses to find the Nash equilibria in pure strategies

- If player 1 (Allies) plays "North", then player 2 (Nazis) optimal choice is to play "North"
- If player 1 (Allies) plays "South", then player 2 (Nazis) optimal choice is to play "South"
- If player 2 (Nazis) plays "North", then player 1 (Allies) optimal choice is to play "North"
- If player 2 (Nazis) plays "South", then player 1 (Allies) optimal choice is to play "South"

illustrating these best responses in the bi-matrix

$$\begin{array}{c|c|c|} & N & S \\ \hline N & 74,26 & 94,6 \\ \hline S & 100,0 & 60,40 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & N & S \\ \hline N & 74,\underline{26} & 94,6 \\ \hline S & 100,0 & 60,40 \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & N & S \\ \hline N & 74,\underline{26} & 94,6 \\ \hline S & 100,0 & 60,\underline{40} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & N & S \\ \hline N & 74,\underline{26} & 94,6 \\ \hline S & \underline{100},0 & 60,\underline{40} \\ \hline \end{array} \rightarrow \begin{array}{c|c|c|} & N & S \\ \hline N & 74,\underline{26} & \underline{94},6 \\ \hline S & \underline{100},0 & 60,\underline{40} \\ \hline \end{array}$$

it is clear that there are no Nash equilibria in pure strategies.

Lets assign a probability to each action

		q	$1-q$
		"North"	"South"
p	"North"	$74, 26$	$94, 6$
$1-p$	"South"	$100, 0$	$60, 40$

Lets look, at when player 1 is indifferent

Player 1 prefers North ($p = 1$)

$$\begin{aligned} E[N] &> E[S] \\ q \cdot 74 + (1-q) \cdot 94 &> q \cdot 100 + (1-q) \cdot 60 \\ 74q + 94 - 94q &> 100q + 60 - 60q \\ -20q + 94 &> 40q + 60 \\ 34 &> 60q \\ 17 &> 30q \\ \frac{17}{30} > q &> 0 \end{aligned}$$

	$\begin{aligned} E[N] &= E[S] \\ q\cdot 74 + (1-q)\cdot 94 &= q\cdot 100 + (1-q)\cdot 60 \\ 74q + 94 - 94q &= 100q + 60 - 60q \\ -20q + 94 &= 40q + 60 \\ 34 &= 60q \\ 17 &= 30q \\ \frac{17}{30} &= q \end{aligned}$	$\begin{aligned} E[N] &< E[S] \\ q\cdot 74 + (1-q)\cdot 94 &< q\cdot 100 + (1-q)\cdot 60 \\ 74q + 94 - 94q &< 100q + 60 - 60q \\ -20q + 94 &< 40q + 60 \\ 34 &< 60q \\ 17 &< 30q \\ \frac{17}{30} &< q < 1 \end{aligned}$
Player 1 indifferent $(p \in [0,1])$	**Player 1 prefers** South $(p = 0)$	

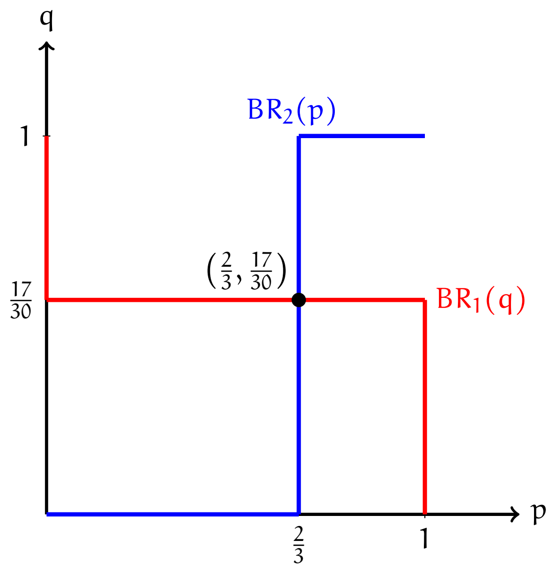
Player 2 is indifferent when

Player 2 prefers North $(q = 1)$	**Player 2 indifferent** $(q \in [0,1])$	**Player 2 prefers** South $(q = 0)$
$\begin{aligned} E[N] &> E[S] \\ p\cdot 26 + (1-p)\cdot 0 &> p\cdot 6 + (1-p)\cdot 40 \\ 26p &> 6p + 40 - 40p \\ 26p &> 40 - 34p \\ 60p &> 40 \\ p &> \frac{40}{60} \\ 1 > p &> \frac{2}{3} \end{aligned}$	$\begin{aligned} E[N] &= E[S] \\ p\cdot 26 + (1-p)\cdot 0 &= p\cdot 6 + (1-p)\cdot 40 \\ 26p &= 6p + 40 - 40p \\ 26p &= 40 - 34p \\ 60p &= 40 \\ p &= \frac{40}{60} \\ p &= \frac{2}{3} \end{aligned}$	$\begin{aligned} E[N] &< E[S] \\ p\cdot 26 + (1-p)\cdot 0 &< p\cdot 6 + (1-p)\cdot 40 \\ 26p &< 6p + 40 - 40p \\ 26p &< 40 - 34p \\ 60p &< 40 \\ p &< \frac{40}{60} \\ 0 < p &< \frac{2}{3} \end{aligned}$

stating the best response functions

$$BR_1(q) = \begin{cases} p = 1 & \text{when } 0 < q < \frac{17}{30} \\ p \in [0,1] & \text{when } q = \frac{17}{30} \\ p = 0 & \text{when } \frac{17}{30} < q < 1 \end{cases} \qquad BR_2(p) = \begin{cases} q = 1 & \text{when } \frac{2}{3} < p < 1 \\ q \in [0,1] & \text{when } p = \frac{2}{3} \\ q = 0 & \text{when } 0 < p < \frac{2}{3} \end{cases}$$

illustrating this in a figure



and the mixed strategy Nash equilibria is

$$\text{MSNE} = \left\{ (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30} \right) \right\}$$

c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

We are asked to find the number of ships that make it across the atlantic, that is we are asked to find the expected payoff for player 1.

We find the expected payoff for player 2 from going north and going south, with the probabilities q, we calculated earlier

$$E[u_A(N)] = 74q^* + 94(1-q^*) = 74 \cdot \frac{17}{30} + 94 \cdot \frac{13}{30} = \frac{1258}{30} + \frac{1222}{30} = \frac{2480}{30} = \frac{248}{3}$$

$$E[u_A(S)] = 100q^* + 60(1-q^*) = 100 \cdot \frac{17}{30} + 60 \cdot \frac{13}{30} = \frac{1700}{30} + \frac{780}{30} = \frac{2480}{30} = \frac{248}{3}$$

the expected number of ships to go over the atlantic is therefor

$$\begin{aligned} \#\,\text{ships} &= p \cdot E[u_A(N)] + (1-p) \cdot E[u_A(S)] \\ &= \frac{2}{3} \cdot \frac{248}{3} + \frac{1}{3} \cdot \frac{248}{3} = \frac{248}{3} \approx 82,66 \end{aligned}$$

6) [Stackelberg] Two neighbors are building a common playground for their children. The time spent on the project by neighbor i is $x_i \geq 0$, $i = 1, 2$. The resulting quality of the playground is

$$q(x_1, x_2) = x_1 + x_2 - x_1 x_2$$

Spending time on the project is costly. More precisely, the cost function of the neighbors are:

$$C_i(x_i) = x_i^2, \quad i = 1, 2.$$

The payoff of neighbor i, U_i is equal to the quality of the playground minus his cost.

a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

This question ask us to solve the model as a cournot case, where the two players move simultaneously.

Lets calculate the two neightbors best responses, this is done by maximizing the profit/payoffs for the two players

$$\begin{aligned} \max_{x_i} \pi_i &= q\left(x_i, x_j\right) - C_i(x_i) \\ &= x_i + x_j - x_1 x_j - x_i^2 \end{aligned}$$

taking the first order condition

$$\begin{aligned} \frac{\partial \pi_i}{\partial x_i} &= 1 - x_j - 2x_i = 0 \\ &\Leftrightarrow 2x_i = 1 - x_j \\ \Leftrightarrow x_i &= \frac{1 - x_j}{2} \equiv R_i\left(x_j\right) \end{aligned} \tag{1}$$

because of symmetry doing the exact same yields the best response function for player j

$$R_j(x_i) = x_j = \frac{1 - x_i}{2} \tag{2}$$

to find the optimal level, we solve for the first order condition given the best response of the other player i.e. insert the best response of j into the best response of i

$$\begin{aligned} x_i &= \frac{1 - R_j(x_i)}{2} \\ \Leftrightarrow x_i &= \frac{1 - \frac{1-x_i}{2}}{2} \\ \Leftrightarrow 2x_i &= 2 - \frac{1 - x_i}{2} \\ \Leftrightarrow 4x_i &= 2 - 1 + x_i \\ \Leftrightarrow 3x_i &= 1 \\ \Leftrightarrow x_i^* &= \frac{1}{3} = x_j^* \end{aligned} \tag{3}$$

such that the nash equilibria is

$$NE = \left\{(x_1^*, x_2^*) = \left(\frac{1}{3}, \frac{1}{3}\right)\right\}$$

lets calculate the payoff

$$\begin{aligned} \pi_i &= x_i^* + x_j' - x_i^* x_j^* - (x_i)^2 \\ &= \frac{1}{3} + \frac{1}{3} - \frac{1}{3}\frac{1}{3} - \left(\frac{1}{3}\right)^2 \\ &= \frac{2}{3} - \frac{1}{9} - \frac{1}{9} \\ &= \frac{6}{9} - \frac{2}{9} \\ &= \frac{4}{9} \end{aligned} \tag{4}$$

which holds for $i = 1, 2$.

b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

Now the game is solved as a stackelberg game. Lets set player 1 as the stackelberg leader and player 2 as the stackelberg follower.

From above we have the two players best response functions

$$\begin{aligned} R_1(x_2) &= \frac{1 - x_2}{2} \\ R_2(x_1) &= \frac{1 - x_1}{2} \end{aligned}$$

because player 1 is the stackelberg leader, he maximizes his profit given player 2's best response function.

Inserting player 2's best response function in player 1's profit function and maximizing

$$\max_{x_1} \pi_1(x_1, R_2(x_1)) = x_1 + R_2(x_1) - x_1 \cdot R_2(x_1) - x_1^2$$
$$= x_1 + \frac{1-x_1}{2} - x_1 \cdot \frac{1-x_1}{2} - x_1^2$$

taking the first order condition

$$\begin{aligned}\frac{\partial \pi_1}{\partial x_1} = 1\left(\frac{1-x_1}{2}\right) + x_1\left(\frac{-1\cdot 2-(1-x_1)\cdot 0}{2^2}\right) - 1\left(\frac{1-x_1}{2}\right) + x_1\left(\frac{-1\cdot 2-(1-x_1)\cdot 0}{2^2}\right) - 2x_1 &= 0 \\ \Leftrightarrow \frac{1-x_1}{2} + x_1 \cdot \frac{-2}{4} - \frac{1-x_1}{2} + x_1 \cdot \frac{-2}{4} - 2x_1 &= 0 \\ \Leftrightarrow \frac{1}{2} - \frac{1}{2}x_1 - \frac{1}{2}x_1 - \frac{1}{2} + \frac{1}{2}x_1 - \frac{1}{2}x_1 - 2x_1 &= 0 \\ \Leftrightarrow -2x_1 &= 0 \\ \Leftrightarrow x_1 &= 0 \end{aligned} \tag{5}$$

such that the optimal amount for the stackelberg leader is $x_1 = 0$.

The stackelberg follower observes this quantity, and chooses its optimal amount hereefter, i.e. inserting $x_1 = 0$ in the best response

$$R_2(0) = \frac{1-0}{2} = \frac{1}{2} \tag{6}$$

such that the backwards induction outcome is

$$\text{SPNE} = \left\{\left(0, \frac{1}{2}\right)\right\}$$

lets find the payoffs for the two players

$$\pi_1^{SL} = x_1 + x_2 - x_1x_2 - (x_1)^2 = 0 + \frac{1}{2} - 0 \cdot \frac{1}{2} - (0)^2 = \frac{1}{2} \tag{7}$$
$$\pi_1^{SF} = x_1 + x_2 - x_1x_2 - (x_1)^2 = 0 + \frac{1}{2} - 0 \cdot \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \tag{8}$$

c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

Lets compare the two situations

	Cournot		Stackberg
Firm 1	$\frac{4}{9}$	<	$\frac{1}{2}$
Firm 2	$\frac{4}{9}$	>	$\frac{1}{4}$

In the cournot game (question a) then player 2 gets the highest payoff, and in a stackelberg game, then player 1 will get the highest payoff.

In this game the time spent, is strategic substitude, that gives an incentive to free ride.

- Player 1 uses his first mover advantage to obtain the highest payoff, be credible telling he will make a smaller effort.

7) Consider the following 2x2 game where payoffs are monetary

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

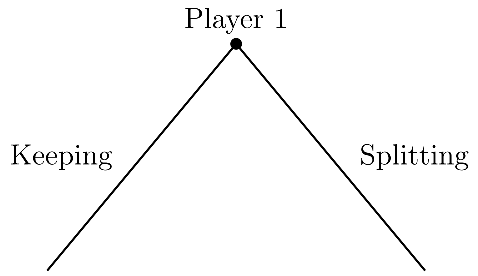
Before this game is played, Player 1 can choose whether, after the game is played, players should keep their own payoffs or split the aggregate payoff evenly between them.

a) Draw the game tree of this two-stage game (assuming that Players 1's choice of whether to split payoffs is revealed to Player 2 before the second stage).

Lets state the stages of the game

(1) Player 1 choose between splitting or keeping payoffs,
(2) Player 2 observes player 1's choice
(3) Player 1 and player 2 simultaneously chooses between their strategies.

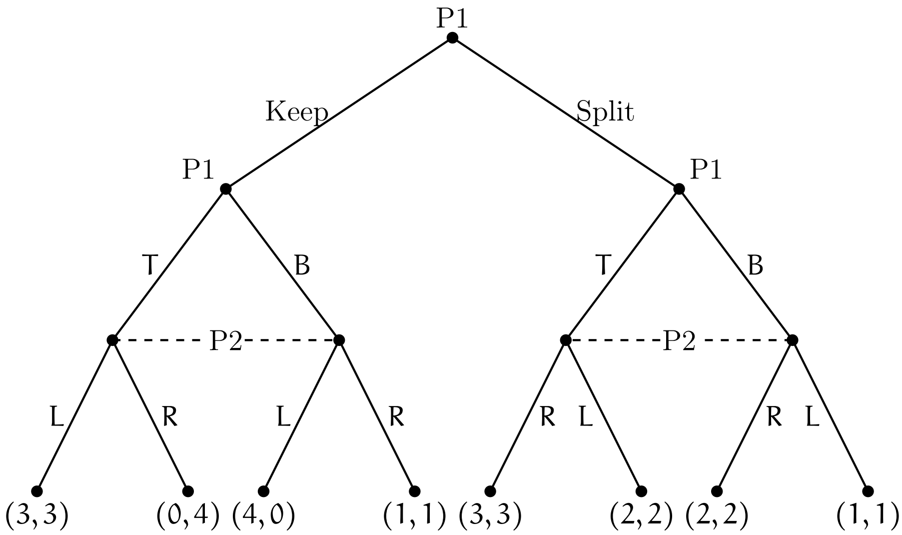
We can illustrate this in game tree in an informal way



	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L	R
T	3, 3	2, 2
B	2, 2	1, 1

but this in an informal way of illustrating the game tree, the formal way to illustrate this is using an information set, which



b) Solve by backwards induction.

We cannot solve the game in all nodes by backwards induction, because in the two bottom parts of the game tree, the two players moves at the same time. Therefor we would need to solve using another concept than backwards induction.

The two bottom parts of the game tree could like illustrated informally be seen as bi-matrices, so lets solve for Nash equilibria in these two bi-matrices

Player 1 play "Keep"

Lets plug in for the best responses in the Keeping bimatrix

- If player 1 plays T, then player 2's optimal choice is to play L.
- If player 1 plays B, then player 2's optimal choice is to play L.
- If player 2 plays L, then player 1's optimal choice is to play T.
- If player 2 plays R, then player 1's optimal choice is to play T.

	L	R
T	$3,3$	$0,\underline{4}$
B	$\underline{4},0$	$\underline{1},\underline{1}$

such that Nash equilibria in "Keep" is

$$\text{PSNE} = (B, R)$$

Player 1 play "Split"

Lets plug in for the best responses in the Keeping bimatrix

- If player 1 plays T, then player 2's optimal choice is to play R.

– If player 1 plays B, then player 2's optimal choice is to play R.

– If player 2 plays L, then player 1's optimal choice is to play B.

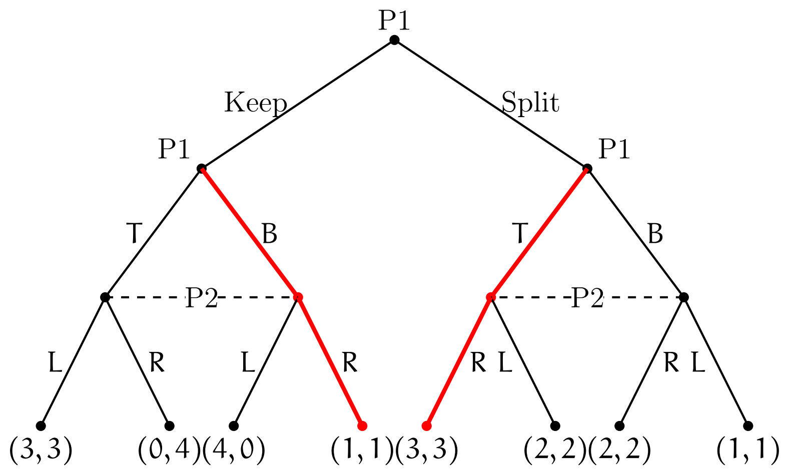
– If player 2 plays R, then player 1's optimal choice is to play B.

	L	R
T	$\underline{3}, \underline{3}$	$\underline{1}, 2$
B	$2, \underline{2}$	$1, 1$

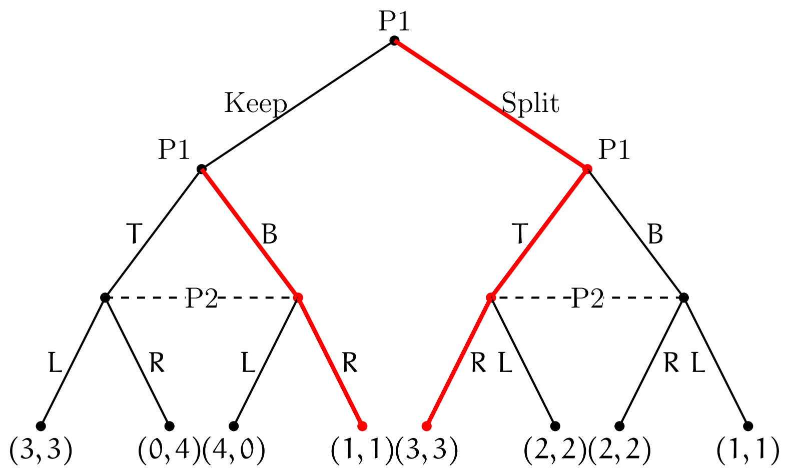
such that Nash equilibria in "Keep" is

$$\text{PSNE} = (T, L)$$

illustrating the two nash equilibria in the game tree



now we can solve for the last decision node from player 1 using backwards induction. It is clear that player 1 gets the highest payoff by playing "split", so he will play "split".



this result is realle a subgame perfect nash equilibria

$$\text{SPNE} = \{(S, B, T), (R, L)\}$$