Micro B: Problem Set 9

Bayesian Nash Equilibrium*

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Question 1 (Bayes' Rule). Use Bayes' rule to solve the following problem:

> *"A cab was involved in a hit and run accident at night. 85% of the cabs in the city are Green and* 15% *are Blue. A witness later recalls that the cab was Blue, and we know that this witness' memory is reliable* 80% *of the time."*

Calculate the probability that the cab involved in the accident was actually Blue.

Question 2 (Two-player Bayesian Game). Consider the following static game, where a is a real number:

	L	R
U	$2, 1$	$0, a$
D	$0, 1$	$1, a$

(a) Suppose that $a = 2$. Does any player have a dominant strategy? What about when $a = -2$?

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(c) Find the Bayes-Nash equilibrium of the game described in b).

Question 3 (Cournot with Imperfect Information about Demand). Consider the following Cournot games:

- Players: two firms, $N = \{1, 2\}$,
- Actions: quantities $q_1, q_2 \in [0; \infty) = S_1 = S_2$,
- Payoffs: common marginal cost $c > 0$ and market demand

$$P(Q; a) = a - Q, \quad a > 0, Q \equiv q_1 + q_2,$$

  yielding the *ex post* profit function

$$\pi_i(q_1, q_2; a) = [P(q_1 + q_2; a) - c]q_i, \quad i = 1, 2.$$

- Types: Firms are uncertain about *market demand*, parameterized by a. For simplicity, assume that a can take two values, a_L and a_H, $a_L < a_H$, which occurs with probabilities $\Pr(a = a_H) = \theta$, and $\Pr(a = a_L) = 1 - \theta$.

*For prior contributions, thanks to Jeppe Dinsen.

  - Firm 1 observes a before choosing its price,
  - Firm 2 does not and must choose a price only knowing the distribution of a.

(a) Solve for the Bayesian Nash equilibrium.

Question 4 (Python)**.** Solve question 3 using Python. To make the game numerically workable, assume that

- $a \in A \equiv \{0.1, 0.5, 1.0, 2.0\}$ with equal probabilities. So $\Pr(a) = \frac{1}{4}$ for all $a \in A$.
- $c = 0.1$
- Firm action spaces are bounded to the unit interval: $q_1, q_2 \in [0; 1] = S_1 = S_2$.

Solve the following problems:

(a) Code up the *ex post* profit functions for both players (i.e. conditional on a's realization). *Hint: This is already done for you in* `uncertain_demand.ipynb`

(b) Code up the *interrim* expected profit functions.

- Firm 1: This takes three scalar inputs, q_1 and q_2 and a and returns a scalar: $\pi_1(q_1, q_2; a)$.
- Firm 2: This takes the two firms' strategies as inputs: the scalar q_2 for firm 2 and the vector $q_1 = \Big(q_1(0.1), q_1(0.5), q_1(1.0), q_1(2.0)\Big)$ for firm 1. It returns the expected profit (a scalar):

$$\mathbb{E}_a[\pi_2(q_1, q_2; a)] = \sum_{a \in A} \Pr(a)\pi_2(q_1(a), q_2; a).$$

(c) Code up the best response functions for both players:

- Firm 1: Takes two scalars, q_1 and a: $q_1(a) = BR_1(q_2, a)$.
- Firm 2: Takes the $|A|$-vector, $(q_1(a))_{a \in A}$ and A: $q_2 = BR_2\Big(\big(q_1(a)\big)_{a \in A}\Big)$.

(d) Find the pure strategy Bayesian Nash equilibrium as the pair of quantities that solve

$$\begin{aligned} q_1^*(a) &= BR_1(q_2^*, a), \quad \forall a \in A \\ q_2^* &= BR_2(q_1^*). \end{aligned}$$

In other words,

$$q^* = BR(q^*),$$

where BR is the "stacked" best response functions operating on $q \in \mathbb{R}^5$.

Hint: You can either use the Iterated Best Response algorithm or `fsolve` *on the mapping* $q \mapsto BR(q) - q$.

Question 5 (Nature chooses the game)**.** Consider the following static Bayesian game:

(i) Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.

(ii) Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.

(iii) The players must choose actions simultaneously.

Find all the pure-strategy Bayesian Nash equilibria of this game.

Game 1 (probability $\frac{1}{2}$)

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Game 2 (probability $\frac{1}{2}$)

	L	R
T	$0,0$	$0,0$
B	$0,0$	$2,2$

Question 6 (Python)**.** Solve question 5 using Python.
Hint: Use the Python approach with "wide matrices". You may also want to consider the first problem from the 2021 exam.

If time permits

Question 7 (Repeated Game)**.** Consider the following game G:

	X	Y	Z
A	$6,6$	$0,8$	$0,0$
B	$7,1$	$2,2$	$5,1$
C	$0,0$	$1,6$	$4,5$

Suppose that G is repeated infinitely many times, so that we have $G(\infty,\delta)$. Define trigger strategies such that the outcome of all stages is (A,X). Find the smallest value of δ such that these strategies constitute a SPNE.

Question 8. Consider the public goods game from lecture 7. Suppose now instead that there is two-sided incomplete information. In particular, the cost of writing the reference is uniformly distributed between 0 and 2:

$$c_i \sim u(0,2) \text{ for } i = 1,2$$

In this setting, we can show that the players optimally follow a 'cutoff' strategy. Thus, the equilibrium strategies take the form

$$s_1^*(c_1) = \begin{cases} \text{Write} & \text{if } c_1 \leq c_1^* \\ \text{Don't} & \text{if } c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \text{Write} & \text{if } c_2 \leq c_2^* \\ \text{Don't} & \text{if } c_2 > c_2^* \end{cases}$$

(a) Let $z_{-i}^* = \Pr\left(s_{-i}^* = \text{Write}\right)$, i.e. the probability that the other players plays Write in equilibrium. Argue that

$$1 - c_i^* = z_{-i}^*$$

Hint: Calculate i's expected payoff from writing the reference and from not writing the reference, conditional on z_{-i}^*

(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\Pr(x < a) = \frac{a}{2}$. Use this to find z_i^* .
Hint: Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

(c) Use the result from the previous question together with (1) to find (c_1^*, c_2^*).

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?