Micro B: Problem Set 7.b

Repeated Games*

Anders Munk-Nielsen

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Question 1. Recall the static pure Bertrand duopoly model (where products are perfect substitutes): the firms name prices simultaneously; demand is

$$q_i = \begin{cases} a - p_i & \text{if } p_i < p_j \\ \frac{a-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

Marginal costs are $c < a$.
Consider the infinitely repeated game based on this stage game.

(a) Find the static Nash equilibrium of the game. (You did this in a previous problem set but the practice is good...)

(b) Solve the optimal cartel price (hint: solve the monopolist's problem).

(c) Show that a grim trigger strategy can sustain the monopoly price level as a subgame perfect Nash equilibrium if $\delta \geq \frac{1}{2}$. (Hint: The grim trigger plays the desired collusive price in initial periods and all periods except if a deviation has ever occurred; then it plays the static Nash equilibrium forever.)

Question 2. The next exercises use the following game G:

	L	M	R
L	$10, 10$	$3, 15$	$0, 7$
M	$15, 3$	$7, 7$	$-4, 5$
R	$7, 0$	$5, 4$	$-15, -15$

G

Suppose that the players play the infinitely repeated game $G(\infty)$ and that they would like to support as a SPNE the 'collusive' outcome in which (L, L) is played every round.

(a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

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Question 3 (Optimal Punishment)**.** We continue analyzing $G(\infty)$. Consider the strategy profile (OP, OP), where OP stands for optimal punishment. The dynamic strategy OP is fully specified by the following rules:

- **Initial play:** Play (L, L) in round 1.
- **Subsequent play:**
  - **Cooperation phase:** If (L, L) was played in the previous round, play (L, L).
  - If player i deviated, switch to **punishment phase**.
  - **Punishment phase:** Define the strategies for the punishing (P) player and the deviating (D) player:

$$Q^P = \{\text{play } R, \text{ then } M \text{ in all subsequent subgames}\}$$
$$Q^D = \{\text{play } M, \text{ then } L \text{ in all subsequent subgames}\}$$

    If player 1 deviated from L:

    (i) Play switches to $\left(Q^D, Q^P\right)$
    (ii) If player 1 deviates again, play $\left(Q^D, Q^P\right)$ again
    (iii) If player 2 deviates, switch to punishing 2.

    If player 2 deviated from L:

    (i) Play switches to $\left(Q^P, Q^D\right)$
    (ii) If player 2 deviates again, begin $\left(Q^P, Q^D\right)$ again
    (iii) If player 1 deviates, switch to punishing 2

Consider the punishment path $\left(Q^D, Q^P\right)$, which (OP, OP) says players should start after any round where player 1 deviates. Notice that this punishment becomes more lenient over time. That is, if player 1 accepts his punishment (and plays according to Q^D), then he earns -4 in the first round, but then 3 in all subsequent rounds for all eternity. (Any further deviation simply incurs the opponent to play the harsh R again.)

(a) Discuss how this increased leniency over time may give player 1 an incentive to accept his punishment (and actually play according to Q^D, rather than deviate again).

(b) How does your answer relate to the following quote from Wikipedia?

> *The "carrot and stick" approach is an idiom that refers to a policy of offering a combination of rewards and punishment to induce behavior. It is named in reference to a cart driver dangling a carrot in front of a mule and holding a stick behind it. The mule would move towards the carrot because it wants the reward of food, while also moving away from the stick behind it, since it does not want the punishment of pain, thus drawing the cart.*

Question 4 (Proving that OP is SPNE)**.** Prove that (OP, OP) is a subgame perfect Nash equilibrium when $\delta \geq \frac{4}{7}$. To check this, you need to verify that there are no profitable deviations in all possible histories:

(a) It is not profitable to deviate from the cooperation phase.

(b) It is not profitable to deviate from the first period of the punishment phase,

(c) It is not profitable to deviate from the second period of the punishment phase.

Hint: You must check for both the deviant and the offended player for both (a) and (b). Suppose 1 deviated originally: then the first period is supposed to be (M, R) (yielding payoffs (-4,5)), followed by (L, M) forever (yielding (3,15)). Note that when a player is considering to deviate, she assumes that the opponent will keep true to the strategy.
To solve the problem, it will prove useful to compute the expected discounted continuation value from being in the punishment phase forever to both players: V^P, V^D.