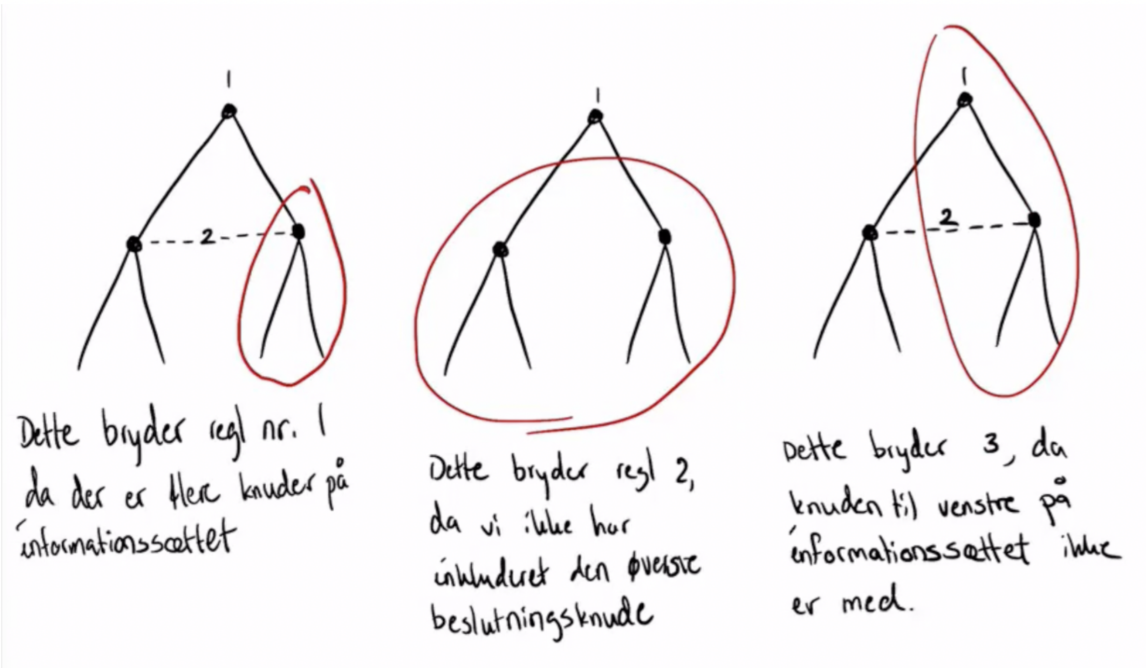
Microeconomics B Problem Set 7

Repeated Games

1) Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information must satisfy the three properties

1. It begins at a decision node n that is a singleton information set.
2. It includes all the following decision and terminal nodes following n in the game
3. It does not cut any information sets.



2) Let G be the following game

	C	D
A	$27, -3$	$-0, 0$
B	$6, 6$	$-2, 7$

Consider the repeated game $G(T)$, where G is repeated T times and the outcomes of each round are observed by both players before the next round.

a) If $T = 2$, is there a Subgame Perfect Nash Equilibria such that (B, C) is played during the first round?

Lets start by defining a finite repeated game

Definition 1 (Finite Repeated Game).
Given a stage game G*, let* $\mathsf{G}(\mathsf{T})$ *denote the finitely repeated game in which* G *is played* T *times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for* $\mathsf{G}(\mathsf{T})$ *are simply the sum of the payoffs from the* T *stage games.*

and we have a theorem if a stage game have a unique nash equilibria

Theorem 1 (Unique SPNE in finite repeated game).
Assuming the stage game G *has a unique Nash equilibria, then for any finite* T*, the finite repeated game* $\mathsf{G}(\mathsf{T})$ *has a unique subgame-perfect nash equilibria where the Nash equilibria of* G *is played in every stage.*

Lets look for Nash equilibria in the stage game, and lets use IESDS to see if we can find a unique Nash equilibria.

In the stage game, player 1 can get strictly higher payoff's by playing A compared to B, so we can eliminate B from the game

	C	D
A	27, −3	−0, 0

now it is optimal for player 2, to play D so we can eliminate C

	D
A	−0, 0

such that the outcome to survive IESDS is (A,D), and the unique Nash equilibrium is

$$\mathsf{NE} = (\mathsf{A},\mathsf{D})$$

this also means that the finite repeated game has a unique subgame-perfect nash equilibria where the nash equilibria is played in every round.

(B,C) is not this nash equilibria, and therefor it is not possible to have a SPNE where (B,C) is played in the first round when $\mathsf{T} = 2$.

b) What if $\mathsf{T} = 42$

The theorem holds for any finite number of T, and therefor also for $\mathsf{T} = 42$. There can therefor not be a SPNE where (B,C) is played in the first round.

3) Consider the two times repeated game where the stage game is

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

a) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (B,Y)**. Make sure to write down the full equilibrium.**

We start by finding Nash equilibria in pure strategies in the stage game G, by plugging in for the best responses

- · If player 1 plays A, then player 2's best response is to play Y
- · If player 1 plays B, then player 2's best response is to play Y
- · If player 1 plays C, then player 2's best response is to play Z
- · If player 2 plays X, then player 1's best response is to play B
- · If player 2 plays Y, then player 1's best response is to play B
- · If player 2 plays Z, then player 1's best response is to play C

	X	Y	Z
A	6, 6	0, 8	0, 0
B	7, 1	2, 2	1, 1
C	0, 0	1, 1	4, 5

Such that we have two Nash equilibria in pure strategies in the stage game

$$\text{PSNE} = \{(B, Y), (C, Z)\}$$

we have two Nash equilibria in the stage game of a finite repeated game, and therefor we are able to use one of the two Nash equilibria as a credible threat.

Lets assume that the Nash equilibria (B, Y) is played in the first stage such that the outcome of the two stages is

	X	Y	Z
A	$2+6, 2+6$	$2+0, 2+8$	$2+0, 2+0$
B	$2+7, 2+1$	$2+2, 2+2,$	$2+1, 2+1$
C	$2+0, 2+0$	$2+1, 2+1$	$2+4, 2+5$

$\longrightarrow$

	X	Y	Z
A	8, 8	2, 10	2, 2
B	9, 3	4, 4	3, 3
C	2, 2	3, 3	6, 7

Lets find Nash equilibria in this stage of the game

- · If player 1 plays A, then player 2's best response is to play Y
- · If player 1 plays B, then player 2's best response is to play Y
- · If player 1 plays C, then player 2's best response is to play Z
- · If player 2 plays X, then player 1's best response is to play B
- · If player 2 plays Y, then player 1's best response is to play B
- · If player 2 plays Z, then player 1's best response is to play C

	X	Y	Z
A	8, 8	2, 10	2, 2
B	9, 3	4, 4	3, 3
C	2, 2	3, 3	6, 7

b) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (C, Z). Make sure to write down the full equilibrium.

Lets assume that the nash equilibria (C, Z) is played in the first round, and then calculate the payoffs for the two stages

	X	Y	Z
A	$4+6, 5+6$	$4+0, 5+8$	$4+0, 5+0$
B	$4+7, 5+1$	$4+2, 5+2,$	$4+1, 5+1$
C	$4+0, 5+0$	$4+1, 5+1$	$4+4, 5+5$

$\longrightarrow$

	X	Y	Z
A	10, 11	4, 13	4, 5
B	11, 6	6, 7	5, 6
C	4, 5	4, 6	8, 10

Lets find Nash equilibria in this stage of the game

· If player 1 plays A, then player 2's best response is to play Y
· If player 1 plays B, then player 2's best response is to play Y
· If player 1 plays C, then player 2's best response is to play Z
· If player 2 plays X, then player 1's best response is to play B
· If player 2 plays Y, then player 1's best response is to play B
· If player 2 plays Z, then player 1's best response is to play C

	X	Y	Z
A	10, 11	4, 13	4, 5
B	11, 6	6, 7	5, 6
C	4, 5	4, 6	8, 10

c) Can you find a subgame perfect Nash equilibrium such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

4) Consider the situation of two flatmates. They both prefer having a clean kitchen, but cleaning is a tedious task, so that it is individually rational not to clean regardless of what the other does. This results in the following game G:

	Clean	Do not clean
Clean	4, 4	0, 6
Do not clean	5, 0	1, 1

Now consider the situation where the two flatmates have to decide every day whether to clean or not, i.e. consider the infinitely repeated game $G(\infty, \delta)$.

a) Define trigger strategies such that the outcome of all stages will be $(Clean, Clean)$.

In a infinite repeated game a trigger strategy can constitutde an SPNE even if the stage game only has one Nash equilibria, because the stage game Nash equilibrium can be used as a credible punishment.

We find the Nash equilibrium in pure strategies in the stage game G.

· If player 1 plays C, then player 2's best response is to play N
· If player 1 plays N, then player 2's best response is to play N
· If player 2 plays C, then player 1's best response is to play N
· If player 2 plays N, then player 1's best response is to play N

	C	N
C	4, 4	0, $\underline{6}$
N	$\underline{5}$, 0	$\underline{1}$, $\underline{1}$

and we have a unique[1] Nash equilibrium in the stage game at $NE: \{(N, N)\}$.

Lets now define a trigger strategy, such that the outcome of all rounds will be (C, C).

Trigger Strategy:

Play (C, C) in round 1

If the result of the earlier rounds was (C, C) then play (C, C) if not play (N, N)

b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

[1] We could solve the game using IESDS, and we find a unique solution. In the full game, player 1's strategy N, strictly dominates C. Then player 2's optimal strategy will be N, and we end up in the unique solution

We need to define the discounted payoff from staying with the trigger strategy, and the discounted payoff from deviating. Then we need to find a discount rate δ, such that the payoff from the trigger strategy is weakly larger than the discounted payoff from deviating.

A player will follow the trigger strategy, if the sum of the discounted payoff/utility of playing the trigger strategy is higher than the sum of the discounted payoff.

It would be enough to just check for player 2, because the payoff from (C, C) is the same, and player 2 has the highest incentive to deviate. I will for illustrative purposes check for both players.

Player 1's discounted payoff from playing the trigger strategy

$$u_{1,TS} = 4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots = \sum_{t=0}^{\infty} 4\delta^t = 4\sum_{t=0}^{\infty} \delta^t = \frac{4}{1-\delta}$$

Player 1's discounted payoff from deviating from the the trigger strategy

$$u_{1,D} = 5 + 1\delta + 1\delta^2 + 1\delta^3 + \ldots = \sum_{t=1}^{\infty} 5 + 1 \cdot \delta^t = 5 + \sum_{t=1}^{\infty} \delta^t = 5 + \sum_{t=0}^{\infty} \delta^t\delta = 5 + \frac{\delta}{1-\delta}$$

player 1 will follow the trigger strategy when

$$\begin{aligned} u_{1,TS} \geq u_{1,D} \Rightarrow \frac{4}{1-\delta} &\geq 5 + \frac{\delta}{1-\delta} \\ \Leftrightarrow 4 &\geq 5 - 5\delta + \delta \\ \Leftrightarrow 4\delta &\geq 1 \\ \Leftrightarrow \delta &\geq \frac{1}{4} \end{aligned}$$

whenever $\delta \geq \frac{1}{4}$ player 1 will follow the trigger strategy.

Lets move on to player 2. Player 2 will follow the trigger strategy, whenever the discounted payoff from following the trigger strategy is larger than the discounted payoff from deviating.

Player 2's discounted payoff from playing the trigger strategy

$$u_{2,TS} = 4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots = \sum_{t=0}^{\infty} 4\delta^t = 4\sum_{t=0}^{\infty} \delta^t = \frac{4}{1-\delta}$$

Player 2's discounted payoff from deviating from the the trigger strategy

$$u_{2,D} = 6 + 1\delta + 1\delta^2 + 1\delta^3 + \ldots = \sum_{t=1}^{\infty} 6 + 1 \cdot \delta^t = 6 + \sum_{t=1}^{\infty} \delta^t = 6 + \sum_{t=0}^{\infty} \delta^t\delta = 5 + \frac{\delta}{1-\delta}$$

player 2 will follow the trigger strategy when

$$\begin{aligned} u_{2,TS} \geq u_{2,D} \Rightarrow \frac{4}{1-\delta} &\geq 6 + \frac{\delta}{1-\delta} \\ \Leftrightarrow 4 &\geq 6 - 6\delta + \delta \\ \Leftrightarrow 5\delta &\geq 2 \\ \Leftrightarrow \delta &\geq \frac{2}{5} \end{aligned}$$

whenever $\delta \geq \frac{2}{5}$ player 2 will follow the trigger strategy.

Lets compare the two, to see which of the two values is the binding one for the trigger strategy

$$\frac{2}{5} > \frac{1}{4} \Leftrightarrow 8 > 5 \tag{True}$$

such that we know that for a discount factor $\delta \geq \frac{2}{5}$, we will have that both players follows the trigger strategy, and the trigger strategy will constitute a SPNE.

5) Consider again the game from exercise 5

	C	NC
C	4, 4	0, 6
NC	5, 0	1, 1

a) Define a tit-for-tat strategy such that the outcome of all stages will be $(Clean, Clean)$.

b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean".

c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

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6) *(Tit-for-tat strategies):* Consider again the game from exercise 5

	C	NC
C	4,4	0,6
NC	5,0	1,1

a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).

b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''.

c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.

d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

Again first we find the NE in the stage game

	C	NC
C	4,4	0,6
NC	5,0	1,1

$NE = \{(NC, NC)\}$

a)

Tit-For-Tat

In round 1 play (C,C)

In subsequent rounds play what your opponent played in last round

b)

Vi ser kun på spiller 2, da spiller 2's gevinst ved at afvige er størst

	C	NC
C	4,4	0,6
NC	5,0	1,1

Forventede payoffs (spiller 2, hvor p1 altid følger TFT)

$$U_{2,TFT} = 4 + 4\delta + 4\delta^2 = \sum_{t=0}^{\infty} 4\delta^t = \frac{4}{1-\delta}$$

$$U_{2,NC} = 6 + 1\cdot\delta + 1\cdot\delta^2 = 6 + \sum_{t=1}^{\infty} \delta^t = \dots = 6 + \frac{1}{1-\delta}$$

$$\Rightarrow U_{2,TFT} \geq U_{2,NC} \Rightarrow \delta \geq \frac{2}{5}$$

c

Vi ser kun på P2, da vi ved at δ for P2 vil være højest grundet større incitament til at afvige

	C	NC
C	4, 4	0, 6
NC	5, 0	1, 1

TFT

$$U_{2,TFT} = 4 + 4\delta + 4\delta^2 + \ldots = \sum_{t=0}^{\infty} 4\delta^t = \frac{4}{1-\delta}$$

start NC, NC → Derefter TFT (P1 spiller TFT)

$$U_{2,NC,TFT} = 6 + 0 \cdot \delta + 6 \cdot \delta^2 + 0 \cdot \delta^3 + 6 \cdot \delta^4 + \ldots = \sum_{t=0}^{\infty} 6\delta^{2t}$$

$$= 6 \sum_{t=0}^{\infty} \delta^{2t}$$

$$= \frac{6}{1-\delta^2}$$

Hvornår $U_{2,TFT} \geq U_{2,NC,TFT}$

$$\Rightarrow \frac{4}{1-\delta} \geq \frac{6}{1-\delta^2} \Leftrightarrow \ldots \geq \ldots \Leftrightarrow -2\delta^2 + 3\delta - 1 \geq 0$$

Andengradsfkt

$$\delta = \frac{-b \pm \sqrt{4 \cdot ac}}{2 \cdot a} = \frac{-3 \pm \sqrt{4 \cdot (-2) \cdot (-1)}}{2 \cdot -2}$$

$$= \frac{-3 \pm \sqrt{9-8}}{-4} = \frac{-3 \pm 1}{-4} \geq \begin{cases} 1 \\ 1/2 \end{cases}$$

d

Vi har nu set på TFT mod forskellige værdier. hver gang har det laveste delta været $\delta \geq \frac{2}{5}$

Det er derfor ikke muligt at finde bedre strategier, da alle andre typer af strategier vil være kombinationer af TFT og trigger som ikke vil give mindre δ.

CHECK off EQ PATH

NEJ $\Rightarrow$ Fordi fælles da bruges en NE som trudsel

6) Exercise 2.13 in Gibbons (p. 135). Recall the static Bertrand duopoly model (with homogeneous products) from Problem 1.7: the firms name prices simultaneously; demand for firm i's product is $a - p_i$, if $p_i < p_j$ is 0 if $p_i > p_j$, and is $(a - p_i)/2$ if $p_i = p_j$; marginal costs are $c < a$. Consider the infinitely repeated game based on this stage game. Show that the firms can use trigger strategies (that switch forever to the stage-game Nash equilibrium after any deviation) to sustain the monopoly price level in a subgame-perfect Nash equilibrium if and only if $\delta \geq 1/2$.

We are looking at a Bertrand model, with homogeneous (Identical) goods. Two indentical firms, decides prices simultaneously, based on the demand function

$$D_i(p_i) = \begin{cases} a - p_i & \text{if} \quad p_i < p_j \\ \frac{1}{2}(a - p_i) & \text{if} \quad p_i = p_j \\ 0 & \text{if} \quad p_i > p_j \end{cases}$$

with a marginal cost $c < a$.

We repeat the game an infinite amount of times $G(\infty, \delta)$.

Spørgsmål 6

$$q_i = \begin{cases} a - p_i & \text{if } p_i < p_j \\ \frac{a-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j. \end{cases}$$

a) Se forelæsning 2 slide 32. b) Vi finder monopolist pricen ved:

$$p^M = \arg\max_p p(a-p)$$

Vi får her:

$$\pi(p) = pa - p^2$$

Og første ordensbetingelsen

$$a - 2p = 0$$
$$p^M = \frac{a}{2}$$

c) Vi definerer så trigger strategien ved at starte med $p_i = p^M$ i første runde og derefter:

$$\begin{cases} p^M & \text{hvis } p_{t-1} = (p^M, p^M) \\ c & \text{ellers} \end{cases}$$

Vi udregner så payoff'et ved co-op:

$$\pi^C = \frac{a - \frac{a}{2}}{2} = \frac{a}{4}$$

Der gælder så af payoff'et ved Defection er givet ved:

$$\pi^D = 0$$

[1] Hvis man vælger afvigelsen, hvor man vælger en pris der er marginalt lavere en modstanderen kan man opnå $\hat{\pi} \approx 2\pi^C$ i første runde og vil derefter få en profit på nul. Hvis man ikke vælger at afvige, der er nutidsværdien givet ved $\pi_C + \delta V_C$, hvor $V_C = \frac{\pi^C}{1-\delta}$. Der er så optimalt at co-op, hvis der gælder, at:

$$\pi_C + \delta \frac{\pi^C}{1-\delta} \geq 2\pi_C$$
$$\delta \frac{\pi^C}{1-\delta} \geq \pi_C$$
$$\delta \geq \frac{1}{2}$$

7) The next exercises use the following game G:

	L	M	R
L	$10, 10$	$3, 15$	$0, 7$
M	$15, 3$	$7, 7$	$-4, 5$
R	$7, 0$	$5, 4$	$-15, -15$

Suppose that the players play the infinitely repeated game $G(\infty)$ and that they would like to support as a SPNE the 'collusive' outcome in which (L, L) is played every round.

a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

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8) *(Infinite repeated game):* The next exercises use the following game G:

	L	M	R
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	−4, 5
R	7, 0	5, 4	−15, −15

07.50

Suppose that the players play the infinitely repeated game $G(\infty)$ and that they would like to support as a SPNE the 'collusive' outcome in which (L, L) is played every round.

a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

a)

<u>Trigger</u>

Start med at spille (L,L)
Hvis outcome fra tidligere runde er (L,L) spil da (L,L).
hvis ikke spil så (x_1, x_2)

b)

Vi skal vælge x_1, x_2 til at være en troværdig trudsel, de skal altså være en NE.

	L	M	R
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	−4, 5
R	7, 0	5, 4	−15, −15

pure

En NE = {(M,M)}.

x_1, x_2 skal altså være handlingerne M,M for at være en troværdig trudsel.

c

	L	M	R
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	−4, 5
R	7, 0	5, 4	−15, −15

payoff Trigger

$$U_{i,TS} = 10 + 10\delta + 10\delta^2 + \ldots = \sum_{t=0}^{\infty} 10\delta^t = \frac{10}{1-\delta}$$

payoff Deviate to M

$$U_{i,D} = 15 + 7\delta + 7\delta^2 + 7\delta^3 + \ldots = 15 + \sum_{t=1}^{\infty} 7\delta^t = 15 + \delta\sum_{t=0}^{\infty} 7\delta^t$$

$$= 15 + \frac{7\delta}{1-\delta}$$

$\underbrace{U_{i,TS}(\delta = 4/7)} \geq U_{i,D}(\delta = 4/7)$ Gælder dette?

$$\frac{10}{1-4/7} \geq 15 + \frac{7 \cdot 4/7}{1-4/7}$$

$$\Leftrightarrow \frac{10}{\frac{3}{7}} \geq 15 + \frac{\frac{28}{7}}{\frac{3}{7}}$$

$$\Leftrightarrow \frac{70}{3} \geq 15 + \frac{28}{7} \cdot \frac{7}{3}$$

$$\Leftrightarrow \frac{70}{3} \geq \frac{45}{3} + \frac{28}{3}$$

$$\Leftrightarrow \frac{70}{3} \geq 73/3 \implies \text{Ikke sandt}$$

Det kan betale sig at afvige $\Rightarrow$ EJ SPNE

9) Ingen løsning :(
10) Ingen løsning :(