Hotelling's - T^2.

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If x is univariate normal with mean μ & s.d. σ, then $U = \frac{\sqrt{n}(x-\mu)}{\sigma} \sim N(0,1)$ &

$V = \frac{1}{\sigma^2} \sum_i (x_i - \bar{x})^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$, where s^2 is the sample variance from a sample of size n. If U & V are independently distributed, then Student's - t is defined as.

$$t = \frac{U}{\sqrt{V/(n-1)}} = \frac{\sqrt{n}(\bar{x}-\mu)/\sigma}{\sqrt{(n-1)s^2/(n-1)\sigma^2}} = \frac{\sqrt{n}(\bar{x}-\mu)}{s} \sim t_{n-1}$$

The multivariate analogue of Student's-t is Hotelling's T^2.

If x_α $(\alpha = 1, 2, \ldots, n)$ is an independent sample of size n from $N_p(\underline{\mu}, \Sigma)$, and if $\bar{x}$ is the sample mean vector, S is the matrix of variance-covariance, then the Hotelling's-T^2 is defined by the relation

$$T^2 = n(\bar{x} - \underline{\mu})' S^{-1} (\bar{x} - \underline{\mu}).$$

Distribution of Hotelling's-T^2-

$T^2/(n-1)$ is the ratio of a non-central χ^2_p to an ind. $\chi^2_{(n-p)}$, i.e.

$$\frac{T^2}{n-1} \times \frac{n-p}{p} = \frac{\chi^2_p(\lambda^2)/p}{\chi^2_{n-p}/(n-p)} \sim F_{p,n-p}(\lambda^2)$$

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Properties of Hotelling's T^2 -

① T^2 statistic as a function of likelihood ratio criterion -

Let $\underline{x}_\alpha$ $(\alpha = 1, 2, \dots, n > p)$ be a random sample from $N_p(\underline{\mu}, \Sigma)$. The likelihood fun. is,

$$L(\underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp\left[-\frac{1}{2} \sum_\alpha (\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1} (\underline{x}_\alpha - \underline{\mu})\right]$$

and the likelihood ratio criterion,

$$\lambda = \frac{\max L_\omega(\underline{\mu}, \Sigma)}{\max L_\Omega(\underline{\mu}, \Sigma)} = \frac{\max L_0}{\max L}$$

In the parameter space Ω, the maximum of L occurs when the parameters $\underline{\mu}$ & Σ are ~~associated~~ estimated by their mles. i.e. $\hat{\underline{\mu}} = \overline{x}$ & $\hat{\Sigma} = A/n$. In the space ω, we have $\underline{\mu} = \underline{\mu}_0$ & $\hat{\Sigma} = \frac{1}{n} \sum_\alpha (\underline{x}_\alpha - \underline{\mu}_0)(\underline{x}_\alpha - \underline{\mu}_0)'$, therefore

$$\max L_\Omega = \frac{1}{(2\pi)^{np/2} \left|\frac{A}{n}\right|^{n/2}} \exp\left[-\frac{1}{2} \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \overline{\underline{x}})' \left(\frac{A}{n}\right)^{-1} (\underline{x}_\alpha - \overline{\underline{x}})\right]$$

$$= \frac{n^{np/2}}{(2\pi)^{np/2} |A|^{n/2}} \exp\left[-\frac{1}{2} n \cdot p\right]$$

Similarly.

$$\max L_\omega = \frac{1}{(2\pi)^{np/2} \left|\frac{1}{n} \sum_\alpha (\underline{x}_\alpha - \underline{\mu}_0)(\underline{x}_\alpha - \underline{\mu}_0)'\right|^{n/2}} \times$$

$$\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{n}(\underline{x}_\alpha-\underline{\mu}_0)'\left(\frac{\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)'}{n}\right)^{-1}(\underline{x}_\alpha-\underline{\mu}_0)\right]$$

$$= \frac{1}{(2\pi)^{np/2}\left|\frac{1}{n}\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)'\right|^{n/2}}\exp\left[-\frac{1}{2}n\,tr\left[\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)'\right]^{-1}\left[\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)'\right]\right]$$

$$= \frac{1}{(2\pi)^{np/2}\left|\frac{1}{n}\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)'\right|^{n/2}}\exp\left[-\frac{1}{2}np\right] \quad \text{--- (i)}$$

$$\sum_\alpha(\underline{x}_\alpha-\underline{\mu}_0)(\underline{x}_\alpha-\underline{\mu}_0)' = \sum_\alpha(\underline{x}_\alpha-\bar{\underline{x}}+\bar{\underline{x}}-\underline{\mu}_0)(\underline{x}_\alpha-\bar{\underline{x}}+\bar{\underline{x}}-\underline{\mu}_0)'$$

$$= \sum_\alpha(\underline{x}_\alpha-\bar{\underline{x}})(\underline{x}_\alpha-\bar{\underline{x}})' + n(\bar{\underline{x}}-\underline{\mu}_0)(\bar{\underline{x}}-\underline{\mu}_0)'$$

$$= A + n(\bar{\underline{x}}-\underline{\mu}_0)(\bar{\underline{x}}-\underline{\mu}_0)'$$

$\Rightarrow$ By eq. (i).

$$\max L_\omega = \frac{n^{np/2}}{(2\pi)^{np/2}\left|A+n(\bar{\underline{x}}-\underline{\mu}_0)(\bar{\underline{x}}-\underline{\mu}_0)'\right|^{n/2}}\exp\left[-\frac{1}{2}np\right]$$

Thus the likelihood criterion is.

$$\lambda = \frac{|A|^{n/2}}{|A+n(\bar{\underline{x}}-\underline{\mu}_0)(\bar{\underline{x}}-\underline{\mu}_0)'|^{n/2}}$$

$$\text{or } \lambda^{2/n} = \frac{|A|}{\begin{vmatrix} 1 & -\sqrt{n}(\bar{\underline{x}}-\underline{\mu}_0)' \\ \sqrt{n}(\bar{\underline{x}}-\underline{\mu}_0) & A\end{vmatrix}} = \frac{|A|}{|A|\,|1+n(\bar{\underline{x}}-\underline{\mu}_0)'A^{-1}(\bar{\underline{x}}-\underline{\mu}_0)|}$$

since $|\Sigma| = |\Sigma_{22}|\,|\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|$

$$\Rightarrow \lambda^{2/n} = \frac{1}{1+n(\bar{\underline{x}}-\underline{\mu}_0)'A^{-1}(\bar{\underline{x}}-\underline{\mu}_0)} = \frac{1}{1+\frac{n}{n-1}(\bar{\underline{x}}-\underline{\mu}_0)'S^{-1}(\bar{\underline{x}}-\underline{\mu}_0)}$$

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$$\Rightarrow \lambda^{2/n} = \frac{1}{1+\frac{T^2}{n-1}}$$

where, $T^2 = n(\bar{\underline{x}} - \underline{\mu}_0)' S^{-1} (\bar{\underline{x}} - \underline{\mu}_0)$

The L.R.T. is defined by the C.R. $\lambda \le \lambda_0$ where λ_0 is chosen s.t. $P(\lambda \le \lambda_0 | H_0) = \alpha$.

Thus $\lambda^{2/n} \le \lambda_0^{2/n}$

or $\dfrac{1}{1+T^2/(n-1)} \le \lambda_0^{2/n}$

or $1+\dfrac{T^2}{n-1} \ge \lambda_0^{-2/n}$

or $T^2 \ge (n-1)(\lambda_0^{-2/n} - 1) = T_0^2$ (say)

Therefore $P(T^2 \ge T_0^2 | H_0) = \alpha$.

② <u>Invariance property</u>:

Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then $T_x^2 = n(\bar{\underline{x}} - \underline{\mu}_{0x})' S_x^{-1} (\bar{\underline{x}} - \underline{\mu}_{0x})$

where $S_x = \dfrac{1}{n-1} \sum_\alpha (\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})'$

$= \dfrac{1}{n-1} (\underline{x}_1 \underline{x}_1' + \cdots + \underline{x}_n \underline{x}_n' - n \bar{\underline{x}} \bar{\underline{x}}')$

Make a non-singular transformation.

$\underline{Y} = C\underline{X} \Rightarrow \underline{y}_\alpha = C\underline{x}_\alpha$

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$$S_y = \frac{1}{n-1} \sum_{\alpha} (\underline{y}_\alpha - \bar{\underline{y}})(\underline{y}_\alpha - \bar{\underline{y}})'$$

$$= \frac{1}{n-1} (\underline{y}_1 \underline{y}_1' + \cdots + \underline{y}_n \underline{y}_n' - n\bar{\underline{y}}\bar{\underline{y}}')$$

$$= \frac{1}{n-1} (C\underline{x}_1 \underline{x}_1' C' + \cdots + C\underline{x}_n \underline{x}_n' C' - nC\bar{\underline{x}}\bar{\underline{x}}'C')$$

$$= C\left[\frac{1}{n-1} (\underline{x}_1 \underline{x}_1' + \cdots + \underline{x}_n \underline{x}_n' - n\bar{\underline{x}}\bar{\underline{x}}')\right] C'$$

$$= C S_x C'$$

By def^n :-

$$T_y^2 = n(\bar{\underline{y}} - \underline{\mu}_{0y})' S_y^{-1} (\bar{\underline{y}} - \underline{\mu}_{0y})$$

$$= n(C\bar{\underline{x}} - C\underline{\mu}_{0x})'(C S_x C')^{-1}(C\bar{\underline{x}} - C\underline{\mu}_{0x})$$

$$= n(\bar{\underline{x}} - \underline{\mu}_{0x})' C' C'^{-1} S_x^{-1} C^{-1} C (\bar{\underline{x}} - \underline{\mu}_{0x})$$

$$= n(\bar{\underline{x}} - \underline{\mu}_{0x})' S_x^{-1} (\bar{\underline{x}} - \underline{\mu}_{0x})$$

$$= T_x^2.$$

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Uses of T^2 - statistic

1.) One sample problem -

Let $x_1, x_2, \ldots, x_n$ be a r.s. from $N_p(\underline{\mu}, \Sigma)$; Σ unknown. Suppose we are required to test

$$H_0 : \underline{\mu} = \underline{\mu_0}$$

Let $\underline{Y} = \sqrt{n}\,(\underline{\bar{x}} - \underline{\mu_0})$, then $E(\underline{Y}) = \underline{0}$ under H_0 and $\Sigma_{\underline{Y}} = \Sigma$. Thus $\underline{Y} \sim N_p(\underline{0}, \Sigma)$ and

$$(n-1)S = \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\bar{x}})(\underline{x}_\alpha - \underline{\bar{x}})' = A = \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}_\alpha', \text{ with}$$

$\underline{Z}_\alpha \sim N_p(\underline{0}, \Sigma)$

Therefore, by definition.

$$T^2 = \underline{Y}' S^{-1} \underline{Y} \quad \text{and} \quad \frac{T^2}{n-1} \cdot \frac{n-p}{p} \sim F_{p, n-p}$$

Therefore the null hyp. is rejected if

$$T^2 \geq T_0^2, \text{ where } T_0^2 = \frac{(n-1)p}{n-p} F_{p, n-p}(\alpha).$$

2.) Two sample problem -

Let $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$ be a r.s. from $N_p(\underline{\mu}^{(i)}, \Sigma)$; $i = 1, 2$ & Σ equal but unknown. The hyp. of interest is $H_0 : \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$.

Let $\underline{\bar{x}}^{(i)} = \frac{1}{n_i} \sum_{\alpha=1}^{n_i} x_\alpha^{(i)}$ be the sample mean vector

and $\underline{\bar{x}}^{(i)} \sim N_p\left(\underline{\mu}^{(i)}, \frac{\Sigma}{n_i}\right)$, then

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$\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)} \sim N_p\left(\underline{0}, \left(\frac{1}{n_1}+\frac{1}{n_2}\right)\Sigma\right)$, under H_0.

$\Rightarrow \sqrt{\frac{n_1 n_2}{n_1+n_2}}\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right) \sim N_p(\underline{0}, \Sigma)$.

Let $\underline{y} = \sqrt{\frac{n_1 n_2}{n_1+n_2}}\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right)$, then.

$E(\underline{y}) = \underline{0}$, under H_0. & $\Sigma_y = \frac{n_1 n_2}{n_1+n_2} \times E\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right)\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right)' = \Sigma$

$\Rightarrow \underline{y} \sim N_p(\underline{0}, \Sigma)$

Let $S^{(i)} = \frac{1}{n_i - 1}\sum_{\alpha=1}^{n_i}\left(\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)}\right)\left(\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)}\right)'$

and.

$$S = \frac{1}{n_1+n_2-2}\sum_{i=1}^{2}\sum_{\alpha=1}^{n_i}\left(\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)}\right)\left(\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)}\right)'$$

$$= \frac{(n_1-1)S^{(1)} + (n_2-1)S^{(2)}}{n_1+n_2-2}$$

$$\Rightarrow (n_1+n_2-2)S = (n_1-1)S^{(1)} + (n_2-1)S^{(2)}$$

$$= A^{(1)} + A^{(2)}$$

$$= \sum_{\alpha=1}^{n_1+n_2-2} Z_\alpha Z_\alpha' \; ; \; \underline{Z}_\alpha \sim N_p(\underline{0}, \Sigma)$$

Hence (n_1+n_2-2) is distributed as $\sum_{\alpha=1}^{n_1+n_2-2} \underline{Z}_\alpha \underline{Z}_\alpha'$

by defn.

$$T^2 = \underline{y}' S^{-1} \underline{y} = \frac{n_1 n_2}{n_1+n_2}\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right)' S^{-1}\left(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}\right)$$

and $\frac{T^2}{n_1+n_2-2} \cdot \frac{n_1+n_2-2-(p-1)}{p} \sim F_{p,\, n_1+n_2-p-1}$

H_0 is rejected if $T^2 > T_0^2$ where $T_0^2 = \frac{(n_1+n_2-2)p}{n_1+n_2-p-1} F_{p,\, n_1+n_2-p-1}(\alpha)$

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Mahalanobis's Δ^2 distribution-

The Mahalanobis distance is a measure of the distance between a point P and a distribution D. It is a multivariate generalization of the idea of measuring how many standard deviations away P is from the mean of D. The distance is zero for P at the mean of D.

Let $D \in \mathbb{R}^N$ with mean $\underline{\mu} = (\mu_1, \dots \mu_N)'$ and var-cov. matrix S, the Mahalanobis' distance of a point $\underline{x}$ from D is

$$d_M(\underline{x}, D) = \sqrt{(\underline{x}-\underline{\mu})' S^{-1} (\underline{x}-\underline{\mu})}$$

A similar quantity denoted by Δ^2 was proposed by Mahalanobis as a measure of distance between two populations, $N_p(\underline{\mu}^{(1)}, \Sigma)$ & $N_p(\underline{\mu}^{(2)}, \Sigma)$.

$$\Delta^2 = (\underline{\mu}^{(1)} - \underline{\mu}^{(2)})' \Sigma^{-1} (\underline{\mu}^{(1)} - \underline{\mu}^{(2)})$$

If the parameters are replaced by their unbiased estimates, we write the same as

$$D^2 = (\underline{x}^{(1)} - \underline{x}^{(2)}) S^{-1} (\underline{x}^{(1)} - \underline{x}^{(2)})$$

where $S = \dfrac{(n_1 - 1) S^{(1)} + (n_2 - 1) S^{(2)}}{n_1 + n_2 - 2}$

$$= \frac{1}{n_1 + n_2 - 2} \sum_{i=1}^{2} \sum_{\alpha=1}^{n_i} (\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)}) (\underline{x}_\alpha^{(i)} - \bar{\underline{x}}^{(i)})'$$

where $\bar{\underline{x}}^{(i)} = \dfrac{1}{n_i} \sum_{\alpha=1}^{n_i} \underline{x}_\alpha^{(i)}$; $i = 1, 2$.

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We can see that for two sample problem

$$T^2 = \frac{n_1 n_2}{n_1+n_2} D^2 = K^2 D^2 \text{ (say)}.$$

Let $\underline{Y} = K(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)})$ then

$$E(\underline{Y}) = K(\underline{\mu}^{(1)} - \underline{\mu}^{(2)}) = \underline{\delta}$$

& $$\Sigma_{\underline{Y}} = K^2 E[(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}) - (\underline{\mu}^{(1)} - \underline{\mu}^{(2)})][(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}) - (\underline{\mu}^{(1)} - \underline{\mu}^{(2)})]$$

$$= K^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma$$

$$= \Sigma$$

Therefore, $\underline{Y} \sim N_p(\underline{\delta}, \Sigma)$

then, $K^2 D^2 = \underline{Y}' S^{-1} \underline{Y}$

Since Σ is p.d. matrix $\exists$ a non-singular matrix C such that

$$C\Sigma C' = I \Rightarrow CC' = \Sigma^{-1}$$

Let

$$\underline{Y}^* = C\underline{Y}, \quad S^* = CSC', \quad \underline{\delta}^* = C\underline{\delta}; \text{ then}$$

$$K^2 D^2 = (C^{-1}\underline{Y}^*)'(C^{-1}S^*C'^{-1})^{-1}(C^{-1}\underline{Y}^*)$$

$$= \underline{Y}^{*\prime} C^{-1\prime} C' S^{*-1} C C^{-1} \underline{Y}^*$$

$$= \underline{Y}^{*\prime} S^{*-1} \underline{Y}^*$$

$$E(\underline{Y}^*) = \underline{\delta}^*$$

$$\Sigma_{\underline{Y}^*} = C\, E(\underline{Y} - E(\underline{Y}))(\underline{Y} - E(\underline{Y}))' C'$$

$$= C\Sigma C'$$

$$= I$$

$$\Rightarrow \underline{Y}^* \sim N_p(\underline{\delta}^*, I) \text{ ∂ } \underline{Y}^{*\prime}\underline{Y}^* \sim \chi_p^2(\underline{\delta}^{*\prime}\underline{\delta}^*)$$

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where $\underline{\delta}^{*'}\underline{\delta}^{*} = \underline{\delta}'C'C\underline{\delta}$
$= \underline{\delta}'\Sigma^{-1}\underline{\delta}$
$= \lambda^2$

Let $(n_1+n_2-2)S = \sum_{\alpha=1}^{n_1+n_2-2} \underline{Z}_\alpha \underline{Z}_\alpha'$, $\underline{Z}_\alpha \sim N_p(\underline{0}, \Sigma)$

$(n_1+n_2-2)S^* = \sum_{\alpha=1}^{n_1+n_2-2} (C\underline{Z}_\alpha)(C\underline{Z}_\alpha)'$, $C\underline{Z}_\alpha \sim N_p(\underline{0}, I)$

$K^2 D^2 = \underline{Y}^* S^{*-1} \underline{Y}^{*'} = (n_1+n_2-2) \dfrac{\chi^2_p(\lambda^2)}{\chi^2_{n_1+n_2-p-1}}$

$\Rightarrow \dfrac{n_1 n_2}{n_1+n_2} \dfrac{(n_1+n_2-p-1)}{(n_1+n_2-2)p} D^2 = F_{p,\, n_1+n_2-p-1}(\lambda^2)$

If $\mu^{(1)} = \mu^{(2)}$, then F is central.

when $T^2 = n(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu)$

$\dfrac{T^2}{n-1} = \dfrac{\chi^2_p(\lambda^2)}{\chi^2_{n-p}}$

when $T^2 = \underline{Y}^* S^{*-1} \underline{Y}^{*'}$

$\dfrac{T^2}{n_1+n_2-2} = \dfrac{\chi^2_p(\lambda^2)}{\chi^2_{n_1+n_2-p-1}}$

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