Discrete Mathematics

What is Discrete Mathematics?

>> Discrete Mathematics is the study of discrete objects.

Discrete means: "distinct or not connected".

>> It is not a branch of Mathematics. It is rather a description of set of branches that have one common property - that they are "discrete" and not "continuous".

What is Logic?

>> Logic is the science of reasoning.
It helps us to understand and reason about different mathematical statements.

>> With rules of logic, we would be able to think about mathematical statements and finally we would be able to prove or disprove those mathematical statements precisely.

Purpose of logic is to construct valid arguments (or proofs).
Once we prove a mathematical statement is TRUE then we call it a Theorem.
and this is the basis of whole mathematics.

Propositional Logic:

What is proposition?

Proposition is a <u>declarative sentence</u> (a sentence that is declaring a fact or stating an argument) which can be either TRUE or FALSE but cannot be both.

For example:
(1) Delhi is the capital of India.
(2) Water Froze this morning.
(3) 1 + 1 = 2.

Sentences which are not propositions:
(1) What time is it?
(2) x + 1 = 2 (can be both TRUE or FALSE)
(3) Send us your resume before 11 PM.
(4) I request you to please allow me a day off.
(5) Fetch my umbrella!

2. Compound Proposition : Two or more simple proposition when combined by various connectivities into a single composite sentence is called compound proposition.

Example :

1. The earth is round and revolves around the sun.
2. A triangle is equilateral iff its three sides are equal.

Compound proposition: A compound proposition is formed by combining two or more simple propositions (called **components**) using the logical operators (**connectives**).

Propositional logic, Propositional Variables and Compound Propositions

Propositional Logic

area of logic that studies ways of joining and/or modifying propositions to form more complicated propositions and it also studies the logical relationships and properties derived from these combined/altered propositions.

What does this definition really mean?

Statement 1 - "Adam is good in playing football"

Statement 2 - "Adam is good in playing football and this time he is representing his college at National level."

"Adam is good in playing football and this time he is representing his college at National level."

proposition 1

joining two propositions with logical connective

proposition 2

Statement 3 - "I enjoy watching Television."

Statement 4 - "It is not the case that I enjoy watching Television."

Modifying the statement using negation

Fact: propositional logic is sometimes called as "sentential logic" or "statement logic"

Why do we need compound propositions?

because most of the mathematical statements are constructed by combining one or more than one propositions. As simple as that!

Propositional Variables:

if p = Adam is good in playing football

q = this time he is representing his college at National level.

$p \wedge q \leftarrow$ p and q

Definition: Variables that are used to represent propositions are called Propositional Variables.

Negation, Conjunction and Disjunction

Negation:

Let p be a proposition. $\neg p$ is called negation of p which simply states that

"It is not the case that p"

if p is true then $\neg p$ is false. if p is false then $\neg p$ is true.

Example: let p be a proposition - "Adam and Eve lived together for many years."

then $\neg p$ will be:

"It is not the case that Adam and Eve lived together for many years."

OR

"Adam and Eve haven't lived together for many years."

p	¬p
T	F
F	T

Conjunction:

Let p and q be two propositions. Conjunction of p and q is denoted by $p \wedge q$

When both p and q are true then only the compound proposition $p \wedge q$ is true.

Example: "12 is $\overset{T}{\text{divisible}}$ by 3 and 3 is a $\overset{T}{\text{prime}}$ number" = T

Important Note: Sometimes we use 'but' instead of 'and'.

Example: "12 is divisible by 3 but 3 is a prime number"

↓

and

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

Let p and q be propositions. Disjunction of p and q is denoted by $p \vee q$
When both p and q are false then only the compound proposition $p \vee q$ is false.

Example: "16 - 4 = 10 or 4 is an even number"

F T = T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operator - Exclusive OR

Consider the following statement:

"In order to get a job in this multinational company, experience with C++ or Java is mandatory."

p, q

Inclusive OR (Disjunction).

Experience with C++ ✓

" " Java ✓

both ✓

p or q or both

Consider the following statement:

"When you buy a car from XYZ company, you get \$2500 cashback or accessories worth \$2500."

p, q

Exclusive OR. (XOR)

cashback ✓

Accessories ✓

both ✗

p or q but not both.

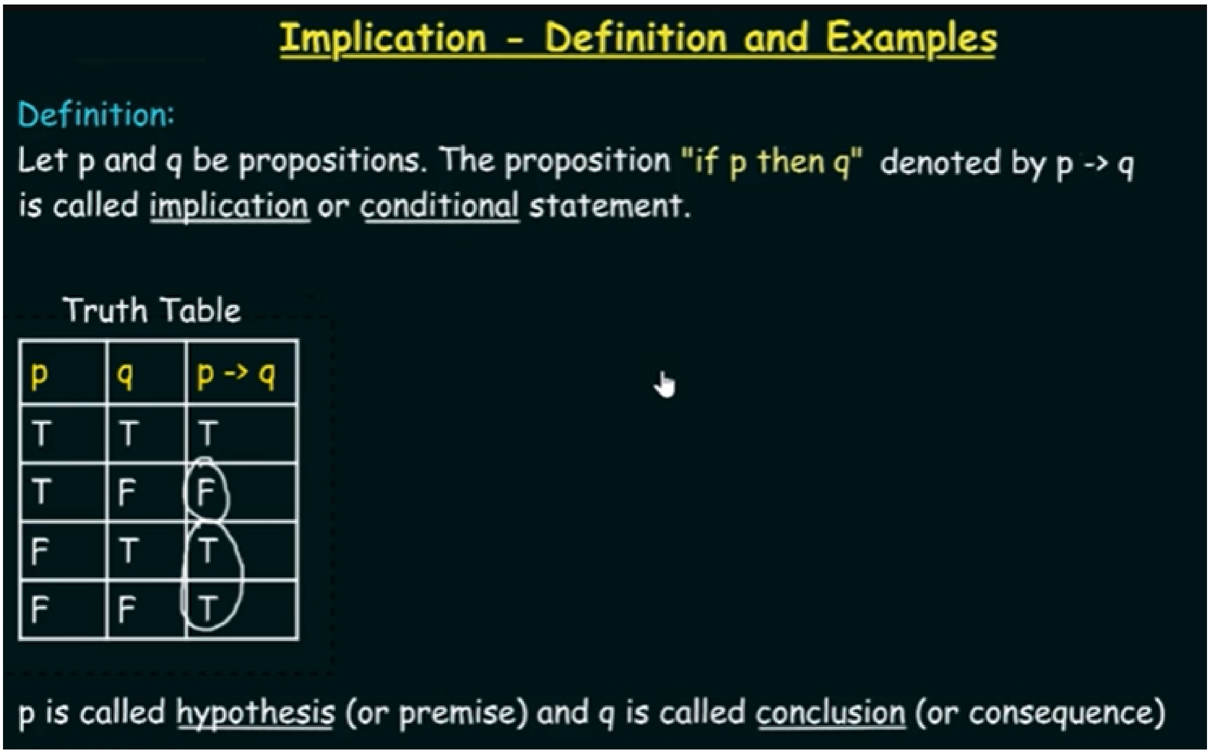
Definition: Let p and q be two propositions. The exclusive OR of p and q (denoted by $p \oplus q$) is a proposition that simply means exactly one of p and q will be true but both cannot be true.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Inclusive disjunction: The connective "or" is used in the inclusive sense "and/or" to mean at least one, maybe both. Such a disjunction is an inclusive disjunction.

Exclusive disjunction: The connective "or" is used in the exclusive sense to mean at least one, but not both. Such a disjunction is an exclusive disjunction.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F



Example 1:

"If you try hard for your exam, then you will succeed".

p = you tried hard for your exam.
q = you succeed

Case 1: "You tried hard for your exam" and "you succeed"

p = True   q = True

Compound proposition p -> q is True

Case 2: "You tried hard for your exam" but "you failed"

p = True   q = False

Compound proposition p -> q is False.

Case 3: "You haven't tried hard for your exam" and "You succeeded"

p = False   q = True

Compound proposition p -> q is True. Why?
because you can make the compound proposition false only when you satisfy the first condition itself i.e. p. If that itself not satisfied then we cannot make compound proposition False. Not False means True.

Case 4: "You haven't tried hard for your exam" and "You failed"

p = False   q = False

Compound proposition p -> q is True. (Same reason as above)

Example 2:

"If you have connection with seniors, then you will get promoted."

T	T	= T
T	F	= F
F	T or F	= T

Example 3:

"If you get 100% marks on the final exam, then you will be awarded a Trophy."

T	F	= F
F		= T

Remember this golden point: If p is false then it doesn't matter what will be the truth value of q, p->q is always TRUE.

Implication - Representations

Different ways to represent conditional statements:

"if p then q"
"p implies q"
"q when p"
"q whenever p"
"q follows from p"

"p only if q"
"q is necessary for p"
"p is sufficient for q"
"q unless ¬p"

Very Important

"p only if q"

How "if p then q" and "p only if q" can be same?

Example: "I will stay at home only if I'm sick."

Let p = "I will stay at home" and let q = "I'm sick"

Above statement is of the form p only if q

According to the above statement, becoming sick is the necessary condition that will make you stay at home.

This means "if you're not sick then, you cannot stay at home at any cost."

In order to falsify the above statement, q must be FALSE and p must be TRUE i.e. you are not sick and you still stay at home.

Proof idea: truth value of p and q must be same in order to falsify the statement.

if p then q: "If I'll stay at home then I'm sick"

The only way to falsify the above statement is by making p TRUE and q FALSE. Therefore, p only if q is equivalent to if p then q

HENCE PROVED

Implication, Converse, Contrapositive and Inverse

Implication or conditional statement: $p \to q$

Converse - $q \to p$

Contrapositive - $\neg q \to \neg p$

Inverse - $\neg p \to \neg q$

Example: "If it rains today then, I will stay at home."

Converse - If I will stay at home then it rains today.

Contrapositive - If I will not stay at home, then it does not rain today.

Inverse - If it does not rain today then, I will not stay at home.

FACTS:

>> Implication and contrapositive both are equivalent.

>> converse and inverse both are equivalent.

>> Neither converse nor inverse is equivalent to Implication.

TRUTH TABLE

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg q \to \neg p$	$\neg p \to \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

Biconditional Operator

Definition: let p and q be two propositions. The biconditional statement of the form $p \leftrightarrow q$ is the proposition "p if and only if q".
$p \leftrightarrow q$ is true whenever the truth values of p and q are same.

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

How "p if and only if q" make sense?

"p if and only if q" composed of two statements -
"p if q" and "p only if q".

"p only if q" = if p then q and "p if q" = if q then p

$$(p \to q) \wedge (q \to p) \equiv p \leftrightarrow q$$

Representations:

1. p is necessary and sufficient for q and vice versa"
2. if p then q, and conversely
3. p iff q

Example: let p be a proposition "You get promoted" and let q be a proposition "You have connections"

then $p \leftrightarrow q$ is the statement: "You get promoted if and only if you have connections."

Precedence of Logical Operators

Precedence of operators helps us to decide which operator will get evaluated first in a complicated looking compound proposition.

For example: $(p \to (q \wedge (\neg p)))$

p = true   q = false

$p \to q \wedge \neg p$

$T \quad \underbrace{F \wedge F}_{F}$

$= F$

Operators	Names	Precedence
¬	Negation	1
^	Conjunction	2
v	Disjunction	3
→	Implication	4
↔	Biconditional	5

Problem: Construct the truth table for the compound proposition below:

$p \to \underbrace{(\neg q \wedge \neg p)}_{r} \leftrightarrow \neg q$

s

p	q	¬p	¬q	r	p → r	s ↔ ¬q
T	T	F	F	F	F	T
T	F	F	T	F	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Translating English Sentences into Logical Expressions

Reasons:

1. Removes ambiguity
2. Easy manipulation
3. Able to solve puzzles

Example: "You are not allowed to watch adult movies if your age is less than 18 years or you have no age proof."

Step 1: Find the connectives which are connecting the two propositions together.

Step 2: Rename the propositions

Let q = "You are allowed to watch adult movies"
r = "Your age is less than 18 years"
s = "You have age proof"

$(r \vee \neg s) \rightarrow \neg q$

Problem: Are these system specifications consistent?

"The system is in multiuser state if and only if it is operating normally."
"If the system is operating normally, then the kernel is functioning."
"The kernel is not functioning or the system is in interrupt mode."
"If the system is not in multiuser state, then it is in interrupt mode."
"The system is not in interrupt mode."

Solution: Let p = "The system is in multiuser state."
q = "The system is operating normally."
r = "The kernel is functioning."
s = "The system is in interrupt mode."

1. $p \leftrightarrow q$
2. $q \rightarrow r$
3. $\neg r \vee s$
4. $\neg p \rightarrow s$
5. $\neg s$

Consistent means assigning truth values to propositional variables in such a way that finally we would be able to make all specifications "true".

1. $\overset{T}{p} \leftrightarrow \overset{F}{q} = F$
2. $\overset{F}{q} \rightarrow \overset{F}{r}$ $\quad q = F$
3. $\overset{T}{\neg r} \vee \overset{F}{s}$ $\quad r = F$
4. $\overset{F}{\neg p} \rightarrow \overset{F}{s}$ $\quad p = T$
5. $\neg s$ $\quad s = F$

NOT CONSISTENT

Logical Puzzle #1

Problem: In an island there are two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. Determine, if possible, what A and B are if they address you in the ways described.

(a) A says "B is a knight" and B says "The two of us are opposite types".

Solution: Let p = A is knight and q = B is knight. $\neg p$ = A is Knave $\quad \neg q$ = B is Knave.

① A is Knight. $p = T \quad q = T$ ✗

$(p \wedge \neg q) \vee (\neg p \wedge q)$

$(T \wedge F) \vee (F \wedge T)$

$F \vee F = F$

② B is Knight. $q = T$ ✗

$\underbrace{(p \wedge \neg q)}_{F} \vee (\neg p \wedge q)$

$p = F$

A is Knave

B is Knave

③ A is Knave. $p = F$

B is Knave. $q = F$

$(p \wedge \neg q) \vee (\neg p \wedge q)$

$(F \wedge T) \vee (T \wedge F)$

$F \vee F = F$

④ B is Knave $\quad q = F$ ✓

$(p \wedge \neg q) \vee \underbrace{(\neg p \wedge q)}_{F}$

$p = F$

A is Knave

A is Knave and B is Knave

Tautology, Contradiction, Contingency and Satisfiability

Tautology: Compound proposition which is always TRUE

Example:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Contradiction: Compound proposition which is always FALSE.

Example:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Contingency: Compound proposition which is sometimes TRUE and sometimes FALSE.

Example:

p	q	p^q
T	T	T
T	F	F
F	T	F
F	F	F

Satisfiability: A compound proposition is satisfiable if there is at least one TRUE result in its truth table.

Unsatisfiability: not even a single TRUE result in its truth table.

Valid: A compound proposition is valid when it is a tautology.

Invalid: A compound proposition is invalid when it is either a contradiction or contingency.

Important take aways:

>> Tautology is always satisfiable but satisfiable is not always tautology.

>> Invalid not only mean a compound proposition is always FALSE. If a compound proposition is sometimes TRUE and sometimes FALSE, then also it is said to be invalid.

Summary:

Tautology	Contradiction	Contingency
always TRUE	always FALSE	Sometimes TRUE or FALSE
Satisfiable	Unsatisfiable	Satisfiable
Valid	Invalid	Invalid

Logical Equivalences

Definition: The compound propositions p and q are said to be logically equivalent if $p \leftrightarrow q$ is a Tautology. Logical Equivalence is denoted by $\equiv$ or $\Leftrightarrow$

Example: $p \wedge T \equiv p$

p	T	p ^ T
T	T	T
F	T	F

Most common and famous Logical Equivalences:

1. Identity Laws: (a) $p \wedge T \equiv p$ (b) $p \vee F \equiv p$
2. Domination Laws: (a) $p \vee T \equiv T$ (b) $p \wedge F \equiv F$
3. Idempotent Laws: (a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$
4. Double Negation Law: $\neg(\neg p) \equiv p$
5. Commutative Laws: (a) $p \vee q \equiv q \vee p$ (b) $p \wedge q \equiv q \wedge p$
6. Associative Laws: (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7. Distributive Laws: (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8. De Morgan's Laws: (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption Laws: (a) $p \vee (p \wedge q) \equiv p$ (b) $p \wedge (p \vee q) \equiv p$
10. Negation Laws: (a) $p \vee \neg p \equiv T$ (b) $p \wedge \neg p \equiv F$

NORMAL FORM

Let $A(p_1, p_2, p_3, \ldots p_n)$ be a statement formula then the construction of truth table may not be practical always.

So, we consider alternate procedure known as reduction to normal form.

(i) Disconjunction Normal Form : A statement form which consist of disjunction between conjunction is called DNF.

Example : (1) $(p \wedge q) \vee r$

(2) $(p \wedge \sim q) \vee (\sim p \wedge r) \vee (r \wedge \sim q)$

Example : Obtain the DNF of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$

Solution : We know that

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

So, $(\sim p \vee q) \wedge (\sim p \wedge q)$

Apply distributive law

$(\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$

$\Rightarrow (\sim p \wedge q) \vee (q \wedge \sim p)$

(ii) Conjunction Normal Form :

A statement form which consists of conjunction between disjunction is called CNF.

Example : (i) $p \wedge q$ (ii) $(\sim p \vee q) \wedge (\sim p \vee r)$

Example : Obtain CNF of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Solution : $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Using distributive law

$$(p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$$

$$[(p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$$

$$\Rightarrow (p \vee q) \wedge (p \vee r) \wedge [(q \vee \sim p) \wedge q \wedge (q \vee r)]$$

1 Obtain DNF of $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r))$

Solution :

$$\Leftrightarrow p \vee (\sim p \rightarrow (q \vee (\sim q \vee \sim r)))$$

$$\Leftrightarrow p \vee (p \vee (q \vee (\sim q \vee \sim r)))$$

$$\Leftrightarrow p \vee (p \vee (q \vee \sim q \vee q \vee \sim r)))$$

$$\Leftrightarrow p \vee (p \vee (q \vee \sim r))$$

$$\Leftrightarrow p \vee (p \vee q \vee p \vee \sim r)$$

$$\Leftrightarrow p \vee (p \vee q \vee \sim r)$$

$$\Leftrightarrow p \vee p \vee p \vee q \vee p \vee \sim r$$

$$\Leftrightarrow p \vee q \vee \sim r$$

2

Obtain CNF of $(p \to q) \wedge (q \vee (p \wedge r))$ and determine whether or not it is tautology.

Solution : We know that $p \to q \Leftrightarrow \sim p \vee q$

$\Rightarrow (\sim p \vee q) \wedge (q \vee (p \wedge r))$

$\Rightarrow (\sim p \vee q) \wedge (q \vee p) \wedge (q \vee r)$

Truth table

p	q	r	~p	~p∨q	q∨p	q∨r	(~p∨q)∧(q∨p)∧(q∨r)
T	T	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	F	T	T	F
F	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	F
F	F	F	T	T	F	F	F

which is not tautology.

3

Find ~~DNF~~ CNF of $p \wedge (p \to q)$

Solution : We know that $p \to q \Leftrightarrow \sim p \vee q$

$\Rightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \vee \sim p) \vee (p \wedge q)$

$\Rightarrow (p \wedge q)$

4

Obtain CNF of $(\sim p \to r) \wedge (q \leftrightarrow p)$

Solution : $(p \vee r) \wedge (q \to p) \wedge (p \to q)$

$\Leftrightarrow (p \vee r) \wedge (\sim q \vee p) \wedge (\sim p \vee q)$

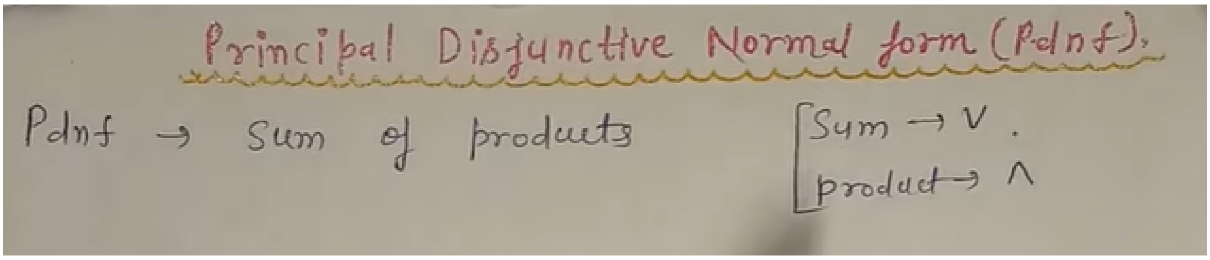
Consider following normal forms

- **Principal Disjunctive Normal Form (Sum-of-product canonical form):** For a given formula, an equivalent formula consisting of disjunctions of **minterms** only.

  $p \vee \neg q \equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

- **Principal Conjunctive Normal Form (Product-of -sums canonical form):** An equivalent formula consisting of conjunctions of **maxterms** only.

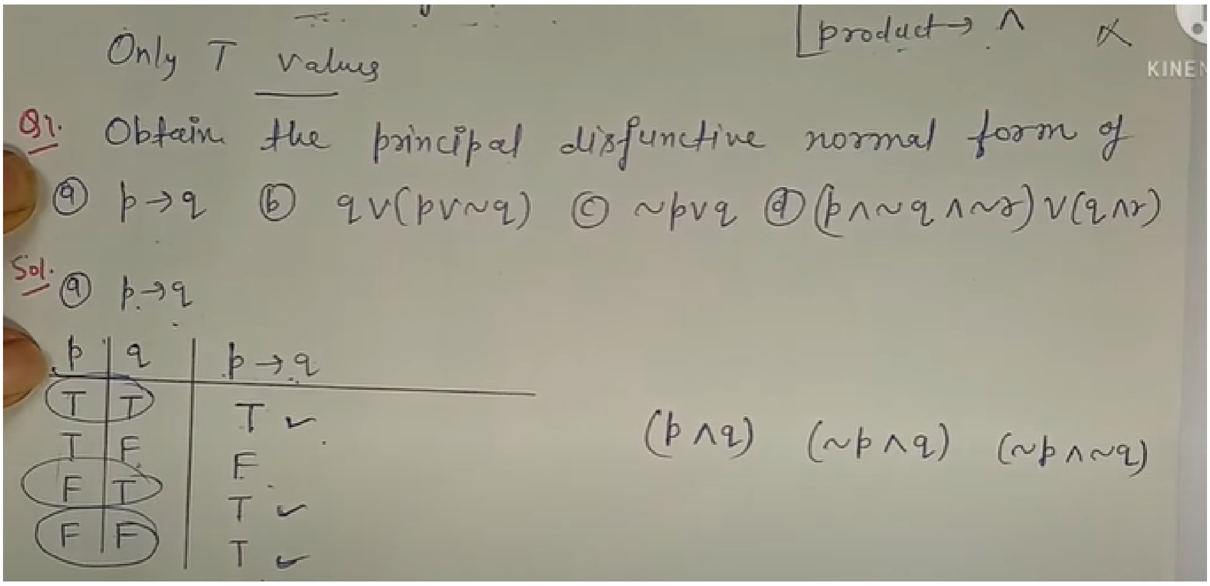
  $(p \leftrightarrow q) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$



Principal Disjunctive Normal form (Pdnf).

Pdnf → Sum of products

Sum → $\vee$.
product → $\wedge$



product → $\wedge$

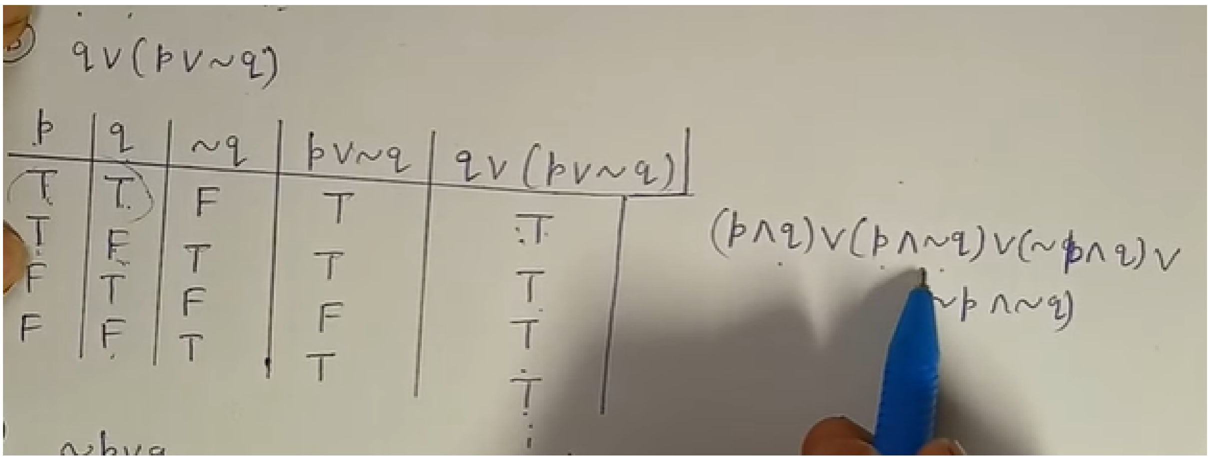
Only T values

Q1. Obtain the principal disjunctive normal form of
(a) $p \to q$ (b) $q \vee (p \vee \sim q)$ (c) $\sim p \vee q$ (d) $(p \wedge \sim q \wedge \sim r) \vee (q \wedge r)$

Sol. (a) $p \to q$

p	q	$p \to q$
T	T	T ✓
T	F	F
F	T	T ✓
F	F	T ✓

$(p \wedge q)$ $(\sim p \wedge q)$ $(\sim p \wedge \sim q)$



(b) $q \vee (p \vee \sim q)$

p	q	$\sim q$	$p \vee \sim q$	$q \vee (p \vee \sim q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee$
$(\sim p \wedge \sim q)$

$\sim p \vee q$

PDNF & PCNF Without Using Truth Table

$p, q \rightarrow$ propositional variables

* **Minterms:** $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$
* **Maxterms:** $p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$
* **PDNF:** Sum of minterms
* **PCNF:** Product of maxterms

DNF → Sum of elementary product

CNF → Product of elementary sum

ethod:

Obtain DNF / CNF

Introduce missing factor

Method:

1. Obtain DNF / CNF ≡ $\rightarrow$, $\leftrightarrow$, $\sim(p \wedge q)$
2. Introduce missing factor
3. Avoid duplications.

Avoi...

- Obtain the principal disjunctive normal form of

(a) $p \vee \sim q$ (b) $p \rightarrow q$ (c) $q \vee (p \vee \sim q)$

(d) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$x - 2 + 2$ $\quad$ $x + 2 - 2$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(a) $p \vee \sim q \equiv [p \wedge (q \vee \sim q)] \vee [\sim q \wedge (p \vee \sim p)]$

$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$

(b) $p \rightarrow q \equiv \sim p \vee q$

$\equiv [\sim p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)]$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)$

$\equiv (p \wedge q \wedge \ldots) \wedge (q \vee \sim q)$

2. Obtain the conjunctive normal form of

(a) $p \wedge \sim q$ (b) $p \leftrightarrow q$ (c) $p \wedge q$ (d) $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

Sol. (a) $p \wedge \sim q \equiv [p \vee (q \wedge \sim q)] \wedge [\sim q \vee (p \wedge \sim p)]$

$\equiv [(p \vee q) \wedge (p \vee \sim q)] \wedge (\sim q \vee p) \wedge (\sim q \vee \sim p)$

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim q \vee \sim p)$

(b) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Logical Equivalences involving conditional statements:

1. $p \rightarrow q \equiv \neg p \vee q$ (Imp)

$T \rightarrow F = F$ $\quad$ $F \vee F = F$

2. $p \rightarrow q \equiv \neg q \rightarrow \neg p$

3. $p \vee q \equiv \neg p \rightarrow q$

$\neg(\neg p) \vee q = p \vee q$

4. $p \wedge q \equiv \neg(q \rightarrow \neg p)$

$\neg(\neg q \vee \neg p)$

$= q \wedge p$

$= p \wedge q$

5. $\neg(p \rightarrow q) \equiv p \wedge \neg q$

6. $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(\neg p \vee q) \wedge (\neg p \vee r)$

$= \neg p \vee \underbrace{(q \wedge r)}_{s}$

$= \neg p \vee s = p \rightarrow s$

$= p \rightarrow (q \wedge r)$

7. $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

8. $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

9. $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences involving biconditionals:

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$= (p \to q) \wedge (q \to p)$

$= (\neg q \to \neg p) \wedge (\neg p \to \neg q)$

$= (\underbrace{\neg p}_{s} \to \underbrace{\neg q}_{r}) \wedge (\neg q \to \neg p)$

$= (s \to r) \wedge (r \to s) = s \leftrightarrow r$

$= \neg p \leftrightarrow \neg q$

3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$= (p \to q) \wedge (q \to p)$

$= (\neg p \vee q) \wedge (\neg q \vee p)$ $\quad (a+b)(c+d) = ac + ad + bc + bd$

$= (\neg p \wedge \neg q) \vee \underbrace{(\neg p \wedge p)}_{F} \vee \underbrace{(q \wedge \neg q)}_{F} \vee (q \wedge p)$

$= (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p) = (\neg p \wedge \neg q) \vee \underbrace{(q \wedge p)}_{p \wedge q}$

4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Propositional Logic (Solved Problem #2)

Problem: P and Q are two propositions. Which of the following Logical expressions are equivalent?

1. $P \vee \sim Q$ 2. $\sim(\sim P \wedge Q)$ 3. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$
4. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

(a) Only 1 and 2 ✓(b) Only 1, 2 and 3 (c) Only 1, 2 and 4
(d) All of these [GATE 2008: 2 Marks]

Solution: $\sim(\sim P \wedge Q) = P \vee \sim Q$.

3. $\wedge(\cdot)$, $\vee(+)$, $\sim(')$

$P \cdot Q + P \cdot Q' + P' \cdot Q'$

$= P(Q + Q') + P'Q'$

$= P(T) + P'Q'$

$= P + P'Q' = (P + P')(P + Q')$

$= (T)(P + Q')$

$= P + Q' = P \vee \sim Q$

4. $P \cdot Q + P \cdot Q' + P' \cdot Q$

$= P(Q + Q') + P'Q$

$= P + P'Q$

$= (P + P')(P + Q)$

$= (T)(P + Q)$

$= P + Q = P \vee Q$

Propositional Logic (Solved Problem #3)

Problem: Which one of the following boolean expression is NOT a tautology?

x (a) ((a -> b) ^ (b -> c)) -> (a -> c)

✓ (b) (a ↔ c) -> (~b -> (a ^ c))

(c) (a ^ b ^ c) -> (c v a)

(d) a -> (b -> a)

[GATE 2014 (set-2): 2 Marks]

Solution:

(a) $((a \to b) \wedge (b \to c)) \to (a \to c)$

$T \to F = F$, F, F; T, F

$F \to F = T$

(b) $(a \leftrightarrow c) \to (\sim b \to (a \wedge c))$

F, F $= T$; T; $\underbrace{F \quad F}_{F}$

$T \to F = F$

Propositional Logic (Solved Problem #4)

Problem: The following propositional statement is

(P -> (Q v R)) -> ((P ^ Q) -> R)

✓ (a) satisfiable but not valid

x (b) valid

(c) a contradiction

(d) None of the above

[GATE 2004: 2 Marks]

Solution:

$(P \to (Q \vee R)) \to ((P \wedge Q) \to R)$

T, $\underbrace{T \quad F}_{T}$; T, T, F

T ... F $= F$

$(P \to (Q \vee R)) \to ((P \wedge Q) \to R)$

$\underbrace{F \quad T \quad T}_{T}$; $\underbrace{\underbrace{F \quad T}_{F} \quad T}_{T}$ $= T$

NESO ACADEMY

Propositional Logic (Solved Problem #5)

Problem: Let P, Q and R be three atomic propositional assertions. Let X denote $(P \vee Q) \rightarrow R$ and Y denote $(P \rightarrow R) \vee (Q \rightarrow R)$. Which one of the following is a tautology?

(a) $X \equiv Y$ (b) $X \rightarrow Y$ (c) $Y \rightarrow X$ (d) $\sim Y \rightarrow X$

X   ✓   X

[GATE 2005: 2 Marks]

Solution:

$$(P \vee Q) \rightarrow R = \neg(P \vee Q) \vee R$$
$$= (\neg P \wedge \neg Q) \vee R$$
$$= (\neg P \vee R) \wedge (\neg Q \vee R)$$
$$= \underbrace{(P \rightarrow R)}_{A} \wedge \underbrace{(Q \rightarrow R)}_{B} = X$$

$$P \rightarrow Q \equiv \neg P \vee Q$$

$$Y = (P \rightarrow R) \vee (Q \rightarrow R)$$

(b) $X \rightarrow Y$ $A \wedge B \rightarrow A \vee B$

$T \wedge T = T$ $T \vee T = T$

(c) $Y \rightarrow X$ $A \vee B \rightarrow A \wedge B$

$T \vee F = T$ $T \wedge F = F$

$T \rightarrow F = F$

Propositional Logic (Solved Problem #6)

Problem: The binary operation □ is defined as follows:

P	Q	P □ Q
T	T	T
T	F	T
F	T	F
F	F	T

Which one of the following is equivalent to $P \vee Q$?

(a) $\sim Q \,□ \sim P$ (b) $P \,□ \sim Q$

(c) $\sim P \,□\, Q$ (d) $\sim P \,□ \sim Q$

[GATE 2009: 2 Marks]

Solution:

$$P \,□\, Q = PQ + PQ' + P'Q' = P(Q+Q') + P'Q'$$
$$= P \cdot T + P'Q' = P + P'Q'$$
$$= (P+P')(P+Q')$$
$$= P + Q'$$

(a) $Q' \,□\, P' = Q' + (P')' = Q' + P$

(b) $P \,□\, Q' = P + (Q')' = P + Q = P \vee Q$

(c) $P' \,□\, Q = P' + Q'$

(d) $P' \,□\, Q' = P' + (Q')' = P' + Q$

Propositional Logic (Solved Problem #7)

Problem: Consider the following expressions:

~~i.~~ False ~~ii.~~ Q ~~iii.~~ True ~~iv.~~ $P \vee Q$ ~~v.~~ $\neg Q \vee P$

The number of expressions given above that are logically implied by $P \wedge (P \to Q)$ is?

[GATE 2016 (set 2): 1 Mark]

Solution:

$P \Rightarrow Q$ $\quad$ $P \to Q$ is a tautology or valid.

$P \Leftrightarrow Q$ $\quad$ $P \leftrightarrow Q$ " " " "

$$P \wedge (P \to Q) \equiv P(P'+Q) \equiv P.P' + PQ \equiv F + PQ \equiv PQ$$

(i) $PQ \Rightarrow F = F$ (P, Q: T T)

(ii) $PQ \Rightarrow Q$ (P, Q: T T; Q: T); $T \Rightarrow T = T$

(iii) $PQ \Rightarrow T$

(iv) $PQ \Rightarrow P+Q$ (P, Q: T T $= T$; $P+Q$: T + T $= T$); $= T$

(v) $PQ \Rightarrow Q'+P$ (P, Q: T T; $Q'+P$: $T = T$)

Rules of Inference in Propositional Logic
(Basic Terminology and Examples)

Premise: is a proposition on the basis of which we would able to draw a conclusion. You can think of premise as an evidence or assumption.

Therefore, initially we assume something is true and on the basis of that assumption, we draw some conclusion.

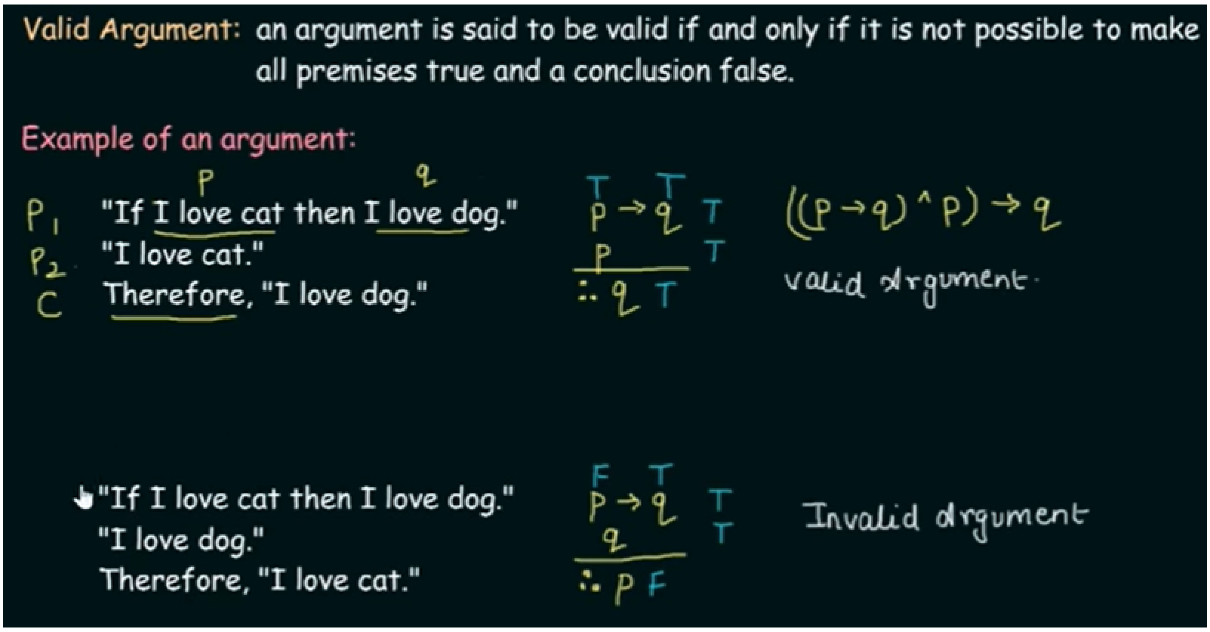
Conclusion: is a proposition that is reached from the given set of premises. You can think of it as the result of the assumptions that we made in an argument.

$\boxed{\text{if premise then conclusion}}$

Argument: Sequence of statements that ends with a conclusion.

OR

it is a set of one or more premises and a conclusion.



Valid and Invalid Arguments: An argument is valid if the conjunction of the hypotheses $H_1, H_2, (1), H_n$, logically implies the conclusion C: that is, the implication

$$H_1 \wedge H_2 \wedge (1), \wedge H_n \to C$$

is a tautology. Otherwise, the argument is invalid, a fallacy.

Thus, an argument is valid if and only if the conclusion is a logical consequence of the hypotheses.

Example: We the following argument.

H_1 : There are more residents in New York City than there are hairs in the head of any resident.
H_2 : No resident is totally bald.

$\therefore$ At least two residents must have the same number of hairs on their heads.

Rules of Inference for Propositional Logic

(Definition and Types of Inference Rules)

Rules of Inference: are the templates for constructing valid arguments

$\downarrow$

$\boxed{\text{deriving conclusions from evidences.}}$

Types of Inference Rules:

1. Modus Ponens:

$$\begin{array}{rl} & \overset{T}{p} \to \overset{T}{q} \quad T \\ & p \quad T \\ \hline \therefore & q \quad T \end{array}$$

OR $[(p \to q) \wedge p] \to q$

2. Modus Tollens:

$$\begin{array}{rl} & \overset{F}{p} \to \overset{F}{q} \quad T \\ & \sim q \quad T \\ \hline \therefore & \sim p \quad T \end{array}$$

OR $[(p \to q) \wedge \sim q] \to \sim p$

3. Hypothetical Syllogism

$$\begin{array}{rl} & \overset{T}{p} \to \overset{T}{q} \quad T \\ T & q \to r \quad T \\ \hline \therefore & \underset{T}{p} \to \underset{T}{r} = T \end{array}$$

OR $[(p \to q) \wedge (q \to r)] \to (p \to r)$

4. Disjunctive Syllogism

$$\begin{array}{rl} F & p \vee \overset{T}{q} \quad T \\ & \sim p \quad T \\ \hline \therefore & q \; T \end{array}$$

OR $[(p \vee q) \wedge \sim p] \to q$

5. Addition

$$\begin{array}{rl} & p \quad T \\ \hline \therefore & \underset{T}{p} \vee q = T \end{array}$$

OR $p \to (p \vee q)$

6. Simplification

$$\begin{array}{rl} & \overset{T}{p} \wedge \overset{T}{q} \quad T \\ \hline \therefore & p \; T \end{array}$$

OR

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & q \; T \end{array}$$

OR $(p \wedge q) \to p$ or $(p \wedge q) \to q$

7. Conjunction

$$\begin{array}{rl} & p \quad T \\ & q \quad T \\ \hline \therefore & \underset{T}{p} \wedge \underset{T}{q} = T \end{array}$$

OR $[(p) \wedge (q)] \to (p \wedge q)$

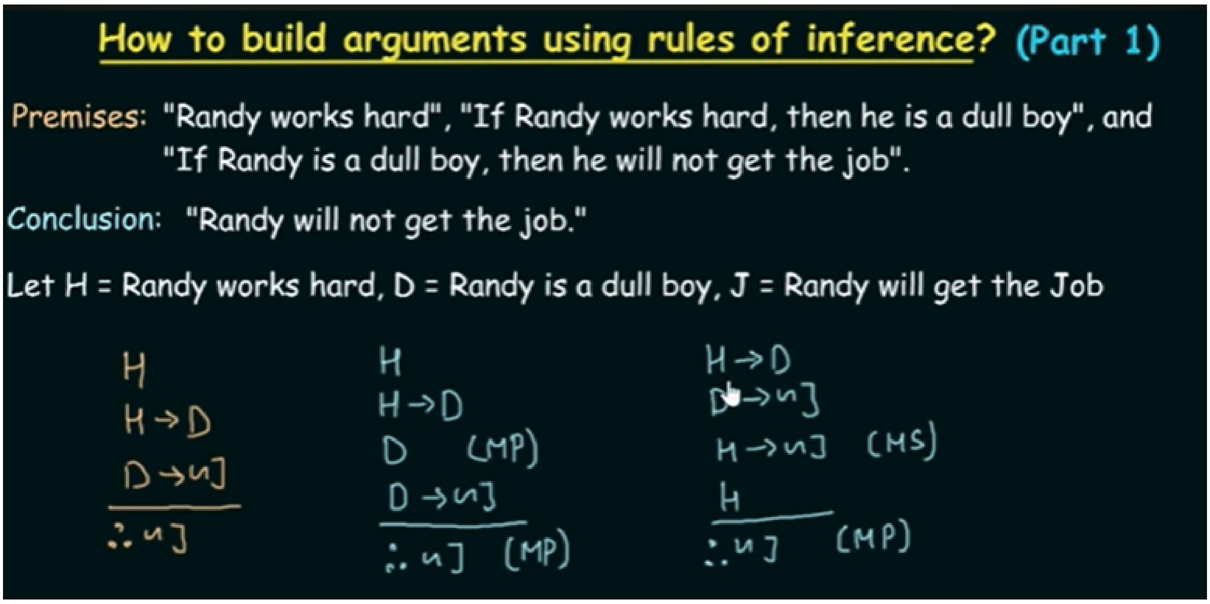
8. Resolution

$$\begin{array}{rl} T/F & p \vee \overset{T}{q} \quad T \\ F/T & \sim p \vee \overset{T}{r} \quad T \\ \hline \therefore & q \vee r = T \end{array}$$

OR $[(p \vee q) \wedge (\sim p \vee r)] \to (q \vee r)$

Inference Rules: If the inference rules given below and/or the laws of logic can be used to reach the given conclusion, then the given argument is valid; otherwise, it is invalid; that is, the argument contains a flaw.

Rule of Inference	**Name**
$[p \wedge (p \to q)] \to q$	Modus ponens
$[\neg q \vee (p \to q)] \to \neg p$	Modus tollens
$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$p \to (p \vee q)$	Addition
$(p \wedge q) \to p$	Simplification
$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution
$[p \wedge (p \to q)] \to q$	Law of detachment
$[(p \to q) \wedge (\neg q)] \to \neg p$	Law of the contrapositive



How to build arguments using rules of inference? (part 2)

Premises: "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on", "If the sailing race is held, then the trophy will be awarded" and "The trophy was not awarded".

Conclusion: "It rained"

Let R = it rains, F = it is foggy, S = the sailing race will be held, D = the lifesaving demonstration will go on and T = trophy will be awarded.

$$(\sim R \vee \sim F) \to S \wedge D$$
$$S \to T$$
$$\sim T$$
$$\therefore R$$

$$S \to T$$
$$\sim T$$
$$\sim S \quad (MT)$$
$$\sim S \vee \sim D = \sim\overbrace{(S \wedge D)}^{q}$$
$$\underbrace{(\sim R \vee \sim F)}_{p} \to \underbrace{S \wedge D}_{q}$$
$$\sim p = \sim(\sim R \vee \sim F) = R \wedge F$$
$$\therefore R$$

Checking the validity of the argument

Problem: Show that the following argument is valid. If today is Tuesday, then I have a test in Mathematics or Economics. If my Economics Professor is sick, then I will not have a test in Economics. Today is Tuesday and my Economics Professor is sick. Therefore I have a test in Mathematics.

Solution: Let T = Today is Tuesday, M = I have a test in Mathematics, E = I have a test in Economics and S = My Economics Professor is sick.

$$\overset{T}{T} \to \overset{F}{M} \vee E^{F} \qquad T \times \quad T \to F = F$$
$$^{T}S \to \sim E^{T} \qquad T \checkmark$$
$$_{T}T \wedge S_{T} \qquad T \checkmark$$
$$\therefore \underset{F}{M} \qquad F \checkmark$$

Valid

Predicate: A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

Example: $(\forall x)P(x)$ where $P(x)$: x is green. Here $P(x)$, called a predicate, states the property the object x has.

The set of all values x can have is called the **universe of discourse** (UD). In the above example, the UD is the set of all apples.

Note that P(x) is not a proposition, but just an expression. However, it can be transformed into a proposition by assigning values to x. The truth value of P(x) is predicated on the values assigned to x from the UD.

Example: Let $Q(x, y)$ denotes the statement "$x = y + 3$"

Note Predicate becomes Proposition if value is assigned to variable | Quantifiers.

Predicate logic (Propositions containing variables)

A logical expression containing some variables that becomes a proposition when we substitute specific values for the variables.

Example- x is less than or equals to y $(x \leq y)$

This is an example of predicate but not proposition

x, y - variable/ arguments $\leq$ - predicate (P)

P(x, y) - It is a propositional function with two variables.

let $x=1, y=2 \Rightarrow P(1,2) = 1 \leq 2$ = True

$P(3,-1) = 3 \leq -1$ = False

Propositional function depends on arguments which reduces to a proposition

Statement		denoted by (let)	
$x < 5$	-	$P(x)$	1 place propositional fn (unary predicate)
$x \le y$	-	$P(x,y)$	2 place " (binary predicate)
$y \le x \le z$	-	$P(x,y,z)$	3 place "
		⋮	
		n-ary or → n place predicate	n place propositional fn

Let $P(x,y)$ denote the statement
$x = y+4$
what are the truth values of the propositions
$P(1,2)$ & $P(4,0)$?

Solution

$$P(x,y) \Rightarrow x = y+4$$
$$P(1,2) \Rightarrow 1 = 2+4$$
$$1 = 6 \quad \text{False}$$

$$P(x,y) \Rightarrow x = y+4$$
$$P(4,0) \Rightarrow 4 = 0+4 = \text{True}$$

Quantifiers: Proposition contains a word indicating quantity such as all, every, none, some, and one. Such words, called quantifiers.

There are two different quantifiers.

- The first is "all", the universal quantifier, denoted by ∀. It may read ∀ as for all, for each, or for every.
- The second quantifier is some, the existential quantifier, denoted by ∃. It may read ∃, for some, there exists a, or for at least one.
- Note that the word some means at least one.

Example: Let x be any apple. Then the sentence All apples are green can be written as For every x, x is green. Using the universal quantifier ∀, this sentence can be represented symbolically as
$(\forall x)$ $(x$ is $green)$ or $(\forall x)P(x)$ where $P(x)$: x is green.

Quantifiers

Definition: Quantifiers are words that refer to quantities such as "some" or "all".
It tells for how many elements a given predicate is True.

In **English**, Quantifiers are used to express the quantities without giving an exact number.

Ex: all, some, many, none, few etc.

Sentences like - "Can I have some water?"
"Jack has many friends here."

Types of Quantifiers:

1. Universal Quantifier.
2. Existential Quantifier.

Example: Let $P(x)$ be a statement $x + 1 > x$

$P(1)$: $1 + 1 > 1 = 2 > 1$ (True)
$P(2)$: $2 + 1 > 2 = 3 > 2$ (True)
⋮

for all +ve integers x.

$P(x)$ is true for all +ve integers x. or $\forall x\, P(x)$ Quantifier. $\forall$

for all.

Example 2: Let $Q(x)$ be the statement $x < 2$

$Q(1)$: $1 < 2$ (True)
$Q(2)$: $2 < 2$ (False)
⋮

Q. Is there some value of x for which $Q(x)$ is True?
(If domain is set of +ve integers)

A. Yes. $Q(x)$ is true for $x = 1$.
Therefore, there is an x for which $Q(x)$ is True.

OR

$\exists x\, Q(x)$ There exists some x for which $Q(x)$ is true.

There exists

Universal Quantifier

Definition: The Universal Quantification of $P(x)$ is the statement
"$P(x)$ for all values of x in the domain"

Notation: $\forall x \; P(x)$
→ for all x, $P(x)$
or
for Every x, $P(x)$

$\forall$ is called universal Quantifier

Domain or Domain of Discourse:

A domain specifies the possible values of the variable under consideration.

For example: Let $P(x)$ is the statement "$x+1 > x$" and let us assume that
Domain -> set of all +ve integers.

$P(1) : 1+1 > 1$ True
$P(2) : 2+1 > 2$ True

Note: It is very important to specify the domain of discourse.
without it, universal quantification of a statement is not defined.

From the above example: $\forall x \; P(x)$ is true under the domain of +ve Integers.

Universal Quantifier - Counter Examples

Q. When does $\forall x \; P(x)$ becomes False?

A. $\forall x \; P(x)$ becomes False, when $P(x)$ is not always True
(where x is the domain under consideration)

Example 1: Let $P(x)$ be the statement "$x < 4$". What is the truth value of the quantification $\forall x \; P(x)$ where the domain consists of all +ve integers.

Solution: Finding just one counter example is enough to make $\forall x \; P(x)$ False.

$P(5) : 5 < 4$ False.

As $P(5)$ is False. Therefore, $x = 5$ is one such counterexample for the statement $\forall x \; P(x)$. This shows that $P(x)$ is not True for every x in the domain of +ve integers.

Example 2: Let $Q(x)$ be the statement "$x + 1 > 2x$". If the domain consists of all integers, what is the truth value of $\forall x \; Q(x)$?

Solution: $Q(1) : 1+1 > 2 \times 1 = 2 > 2$ $\forall x \; Q(x)$ is false.

Expressing Quantifications in English

Example: Let $P(x)$ be the statement "x spends more than five hours every weekday in class." where the domain for x consists of all students.
Express $\forall x\, P(x)$ in English.

Solution: $\forall x\, P(x)$:-
for all x $P(x)$
for every x $P(x)$
for each x $P(x)$
for any x $P(x)$
all of x $P(x)$

1. For every x in the domain of all students, x spends more than five hours every weekday in class.
2. Every student spends more than five hours every weekday in class.

Example 2: Let $N(x)$ be the statement "x has visited North Korea" where the domain consists of the students in your school. Express $\forall x\, N(x)$ in English.

Solution:
1. For every x in the domain of the students in your school, x has visited North Korea.
2. All students in your school has visited North Korea.

Existential Quantifier

Definition: The existential Quantification of $P(x)$ is the proposition
"There exists an element x in the domain such that P(x)."

Or

$\exists x\, P(x)$ $\qquad$ $\forall x\, P(x)$

Note:
Specifying the domain is important. Without domain, the statement $\exists x\, P(x)$ has no meaning.

Different ways to say $\exists x\, P(x)$:

"There is an x such that P(x)."
"There is at least one x such that P(x)."
"For some x P(x)."

Example 1: What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution:
$P(x)$: $x^2 > 10$ $\quad$ domain : $\{1, 2, 3, 4\}$ $\quad$ $\forall x\, P(x)$:
$P(1)$: $1^2 > 10$ False $\quad$ $P(4)$: $4^2 > 10$ True $\quad$ $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
$P(2)$: $2^2 > 10$ False $\quad$ $\exists x\, P(x)$ is true.
$P(3)$: $3^2 > 10$ False $\quad$ $\exists x\, P(x)$: $P(1) \vee P(2) \vee P(3) \vee P(4)$

Existential Quantifier - Examples

Example 2: Let P(x) be the statement "$x = x^2$." If the domain consists of the integers. What is the truth value of $\exists x\, P(x)$?

Solution: domain : Integers. $\exists x\, P(x)$? True

$P(x) :\ x = x^2$ $P(1) :\ 1 = 1^2$ True

Example 3: Let Q(x) be the statement "$x + 1 > 2x$." If the domain consists of all integers, What is the truth value of $\exists x\, Q(x)$?

Solution: $Q(x) :\ x+1 > 2x$ domain : Integers.

$Q(-1) :\ -1+1 > -2 \Rightarrow 0 > -2$ $\exists x\, Q(x)$ is true

Example 4: Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x\, (x^3 = -1)$ b) $\exists x\, (x^4 < x^2)$

Solution: a) $\exists x\, (x^3 = -1)$

let $x = -1$ $(-1)^3 = -1$ True

$\exists x\, (x^3 = -1)$ True.

b) $\exists x\, (x^4 < x^2)$

let $x = \frac{1}{2}$ $\left(\frac{1}{2}\right)^4 < \left(\frac{1}{2}\right)^2 \Rightarrow \frac{1}{16} < \frac{1}{4}$ True.

$\exists x\, (x^4 < x^2)$ is true.

It is an operator used to create a proposition from a propositional function P(x), without assigning values to P(x)

Quantifiers

Quantifiers are the symbols which quantify the variables of predicates.

OR Express the extent to which a predicate is true

There are two types of quantifiers -

ii) Universal Quantifier ($\forall$)

$P(x)$: $x < 4$

true → values ~~of x~~ not exceeding 3

$\forall$ ~~range~~ domain / Universe of discourse

Universal quantifier shows a predicate is true for every element under consideration

Universal quantifier states that the statements within its scope are true for every value of the specific variables

- It is denoted by '$\forall$' symbol.

read as for all

$\forall x\, P(x)$ - read as

- for every value of x, $P(x)$ is true
- for every x in domain, $P(x)$
- for all x in domain, $P(x)$

Examples:

If $P(x)$ denotes "$x>0$" & the domain be the integers.

$\forall x P(x)$ is false

$P(x)$: $x>0$ & domain is the positive integers.

$\forall x P(x)$ is true

$P(x)$: x is even & domain - integers.

$\forall x P(x)$ is false

(2) Existential Quantifier ($\exists$)

Existential Quantifier shows a predicate is true for there is one or more elements under consideration

Existential quantifier states that the statements within its scope are true for some values of the specific variable

It is denoted by '$\exists$' symbol. reads as 'there exists'

$\exists x P(x)$ is read as.

- for some values of x, $P(x)$ is true
- for some x, $P(x)$
- there is an x such that $P(x)$
- for atleast one x, $P(x)$
- for some x in domain, $P(x)$

$P(x)$ is true for some values of x in the universe of discourse

or There exists a value of x in the universe of discourse such that $P(x)$ is true

such that $P(x$

Examples-

$P(x) : x > 0$ & domain - integers

$\exists x P(x)$ is true

$P(x) : x > 0$ domain - positive integers

$\exists x P(x)$ is true

$P(x) : x < 0$ $\exists x P(x)$ is false

Q $P(x) : x^2 < 10$ domain/universe of discourse = $\{0, 1, 2, 3, 4\}$

or positive integers not exceeding 4.

find the truth value of $\forall x P(x)$

$x^2 < 10$

$0^2, 1^2, 2^2, 3^2, 4^2$

$0, 1, 4, 9, 16$

T, T, T, T, F

$\therefore \forall x P(x)$ is false, as proposition should be true for all values.

be true for all

Q find the truth value of $\exists x\, P(x)$

$x^2 < 10$

$0^2 < 10$ - True.

$\exists x\, P(x)$ is true.

Q $P(x)$: $x+3 > x$ domain of discourse - real no.

find truth value of $\forall x\, P(x)$

$\forall x\, P(x)$, $P(x)$ is true.

Nested Quantifiers: Two quantifiers are nested if one is within the scope of the others, such as

$$\forall x \forall y P(x, y)$$

$$\forall x \exists y P(x, y)$$

$$\forall x \forall y P(x, y)$$

$$\exists x \exists y P(x, y)$$

Example: Let $Q(x, y)$ denotes the statement $x + y = 0$.

The quantifier $\exists y \forall x Q(x, y)$ denotes "There is a real number y such that for every real x, $Q(x, y)$" is false.

$\forall x \exists y Q(x, y)$

Introduction to Nested Quantifiers

Defintion: Two quantifiers are said to be nested if one is within the scope of the other.

For example: $\forall x \exists y\ Q(x, y)$

$\exists$ is within the scope of $\forall$

Note: Anything within a scope of the quantifier can be thought of as a propositional function.

$$\forall x \underbrace{\boxed{\exists y\ Q(x, y)}}_{P(x)} \Rightarrow \forall x\ P(x)$$

Different combinations of Nested Quantifiers

Order of quantifiers doesn't matter ← $\forall x \forall y\, Q(x, y)$, $\exists x \exists y\, Q(x, y)$

$\forall x \exists y\, Q(x, y)$, $\exists y \forall x\, Q(x, y)$ → Order of quantifiers does matter

i.e. $\forall x \forall y\, Q(x, y) \equiv \forall y \forall x\, Q(x, y)$
and $\exists x \exists y\, Q(x, y) \equiv \exists y \exists x\, Q(x, y)$

Also, $\forall x \exists y\, Q(x, y) \not\equiv \exists y \forall x\, Q(x, y)$

Nested Quantifiers (Example 1)

Example 1: Let x and y be the real numbers and P(x, y) denotes "x + y = 0."
Find the truth values of

a) $\forall x \forall y\, P(x, y)$
b) $\forall x \exists y\, P(x, y)$
c) $\exists y \forall x\, P(x, y)$
d) $\exists x \exists y\, P(x, y)$

Solution: Domain: All real numbers.

a) $\forall x \forall y\, P(x, y) \equiv \forall x \forall y\, (x + y = 0)$

"For all real numbers x and y, x+y = 0" ← Think of it as "For any combination of x and y, P(x, y)."

Is it True?
No.

Example: $1 + 2 \neq 0$

$\forall y \forall x\, P(x, y) \equiv \forall y \forall x\, (x + y = 0)$

"For all real numbers y and x, x + y = 0" ← Similar to the previous statement

b) $\forall x \exists y\, P(x, y) \equiv \forall x \exists y\, (x + y = 0)$

"For every real number x, there exist a real number y such that x + y = 0"

Is it True?
Yes.

You can choose any real number x; there is always a real number y for that x such that x + y =0.

For example: For

$x = 1$	$x = -1$	$x = \frac{1}{2}$
$y = -1$	$y = 1$	$y = -\frac{1}{2}$

Steps:

1. Consider all different values of x.
2. Find just 1 value of y for each x such that P(x, y) becomes True.

Note: Value of y depends on the value of x.

c) $\exists y \forall x\ P(x, y) \equiv \exists y \forall x\ (x + y = 0)$

"There exist some real number y such that for every real number x, $x + y = 0$"

Is it True?

No.

It is asking us to find a real number y for which P(x, y) becomes true by plugging in every real number x.

What is P(x, y) in our example?

$P(x, y) \Rightarrow x + y = 0$

First, take some real number y

Lets take $y = 1$

$P(x, 1) \Rightarrow x + 1 = 0$

Now, plug in all real numbers x.

$P(\frac{1}{2}, 1)$ is False, $P(1, 1)$ is False and so on.

No matter what y you choose, P(x, y) is always **False** for all real numbers x.

d) $\exists x \exists y\ P(x, y) \equiv \exists x \exists y\ (x + y = 0)$

"There exist some real numbers x and y such that $x + y = 0$"

Obviously. There exist some combination of real numbers x and y exist for which $x + y = 0$.

Take $x = 1$ and $y = -1$ (Many such combinations)

$\boxed{1 + (-1) = 0 \text{ is True.}}$

$\therefore \exists x \exists y\ P(x, y)$ is True.

Also, $\exists x \exists y\ P(x, y) \equiv \exists y \exists x P(x, y)$

Example 2: Let x and y be the real numbers and Q(x, y) denotes "$x.y = 0$."
Find the truth values of the following:

a) $\forall x \forall y\ Q(x, y)$
b) $\forall x \exists y\ Q(x, y)$
c) $\exists y \forall x\ Q(x, y)$
d) $\exists x \exists y\ Q(x, y)$

Solution: a) $\forall x \forall y\ Q(x, y) \equiv \forall x \forall y (x.y = 0)$

"For all real numbers x and y, $x.y = 0$"

Is it True? No.

For example: take $x = 1$ and $y = 2$

$1.2 \neq 0$

$x.y = 0$ is not satisfied for every combination of real numbers x and y.

b) $\forall x \exists y\, Q(x, y) \equiv \forall x \exists y\, (x.y = 0)$

"For every real number x there exist a real number y such that x.y = 0"

Is it True? Yes.

Example:

$x = 1$, $y = 0$	$x = -1$, $y = 0$	$x = \frac{1}{2}$, $y = 0$
$1.0 = 0$	$-1.0 = 0$	$\frac{1}{2}.0 = 0$

No matter what value of x is chosen, one can always take y = 0 to make x.y = 0.

c) $\exists y \forall x\, Q(x, y) \equiv \exists y \forall x\, (x.y = 0)$

"There exist some real number y such that for every real number x, x.y = 0"

Is it True? **Yes.**

I prefer to convert the above sentence in this form:

Find a real number y such that Q(x, y) becomes true by plugging in all real numbers x.

Step 1: Find a real number y.

Let say y = 0.

Step 2: Put y in Q(x, y)

$Q(x, y) = Q(x, 0) \Rightarrow x.0 = 0$

Step 3: Plug in some values of x in Q(x, y) and get the idea.

It isn't easy to plug in all values of x, but one can get the idea that it is not possible to make Q(x, y) False.

In the above example, we can observe that x.0 is always 0 (zero) no matter what value of x could be plugged in.

Therefore, $\exists y \forall x\, Q(x, y)$ is true.

d) $\exists x \exists y\, Q(x, y) \equiv \exists x \exists y\, (x.y = 0)$

"There exist some real numbers x and y such that x.y = 0"

Is it True? Yes.

Take x = 1, y = 0.

$x.y = 0$

$1.0 = 0$

Therefore, $\exists x \exists y\, Q(x, y)$ is true.