Bishop

(RR)

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Bayesian Modeling of Gaussian Processes

Joint Density func. of $x = \{x_1, x_2, \dots, x_n\}$

$$p(\theta|x) = \frac{p(x|\theta)\,\pi(\theta)}{m(x)}$$

$\pi(\theta)$ → Prior dist. of parameter θ

$m(x)$ ↳ Marginal Likelihood

Ex ⇒

let $X_1, X_2, \dots, X_n \sim$ Exponential dist. with parameter θ
& $\theta \sim G(\alpha, \beta)$
Obtain the Bayes estimate of θ $\quad \theta \in [0, \infty)$

Sol.n ⇒ $\quad f(x|\theta) = \theta e^{-\theta x}$

$$\therefore L(x|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \quad \text{—(1)}$$

$$\pi(\theta) = \frac{\theta^{\alpha-1} \beta^{\alpha} e^{-\theta\beta}}{\Gamma\alpha} \quad \text{—(2)}$$

Now, Posterior -

$$L(x|\theta)\cdot\pi(\theta) = \theta^n e^{-\theta\sum_{i=1}^{n} x_i} \times \frac{\theta^{\alpha-1} \beta^{\alpha} e^{-\theta\beta}}{\Gamma\alpha}$$

$$= \frac{\theta^{n+\alpha-1} \beta^{\alpha} e^{-\theta(\sum_{i=1}^{n} x_i + \beta)}}{\Gamma\alpha}$$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty \theta^{n+\alpha-1} \cdot e^{-\theta(\sum_{i=1}^{n} x_i + \beta)} \, d\theta$$

$\theta(\beta + \sum x_i) = t$

$\theta = \dfrac{t}{(\beta + \sum x_i)}$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty \frac{t^{n+\alpha-1}}{(\beta+\sum x_i)^{n+\alpha}} e^{-t} \, dt$$

$d\theta = \dfrac{dt}{(\beta + \sum x_i)}$

$$= \frac{\beta^{\alpha}}{\Gamma\alpha} \cdot \frac{1}{(\beta+\Sigma x_i)^{n+\alpha}} \int_0^{\infty} t^{n+\alpha-1} e^{-t}\, dt$$

$$= \frac{\beta^{\alpha}\, \Gamma n+\alpha}{\Gamma\alpha\, (\beta+\Sigma x_i)^{n+\alpha}} \quad \text{——— } (3)$$

Since, our objective is to calculate the posterior distribution, we can get it using eq. (1), (2) & (3) -

$$p(\theta|x) = \frac{L(x|\theta)\, \pi(\theta)}{m(x)}$$

$$= \frac{\theta^{n+\alpha-1}\, \dfrac{\cancel{\beta^{\alpha}}\, e^{-\theta(\beta+\Sigma x_i)}}{\cancel{\Gamma\alpha}}}{\dfrac{\cancel{\beta^{\alpha}}\, \Gamma n+\alpha}{\cancel{\Gamma\alpha}\, (\beta+\Sigma x_i)^{n+\alpha}}}$$

$$= \frac{(\beta+\Sigma x_i)^{n+\alpha} \cdot \theta^{n+\alpha-1} \cdot e^{-\theta(\beta+\Sigma x_i)}}{\Gamma n+\alpha}$$

$$p(\theta|x) = \frac{(\beta+\Sigma x_i)^{n+\alpha}}{\Gamma n+\alpha} \cdot \theta^{n+\alpha-1} \cdot e^{-\theta(\beta+\Sigma x_i)}$$

$$\therefore G(n+\alpha,\ \beta+\Sigma x_i)$$

$$E_{(\theta|x)}(\theta) = \int_0^{\infty} \theta\, p(\theta|x)\, d\theta = \frac{n+\alpha}{\beta+\Sigma x_i}$$

$$V_{(\theta|x)}(\theta) = \frac{n+\alpha}{(\beta+\Sigma x_i)^2}$$

$$\boxed{\begin{gathered} \frac{\beta^{\alpha}}{\Gamma\alpha}\, \theta^{\alpha-1} e^{-\theta\beta} \\ \text{OR} \\ \frac{1}{\beta^{\alpha}\Gamma\alpha}\, \theta^{\alpha-1} e^{-\theta\beta} \end{gathered}}$$

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$$\text{Mode} = \frac{n+\alpha-1}{\beta+\Sigma x_i}$$

Q) Calculate Post. Dist. -

i.) $x \sim N(\theta,1)$ & $\theta \sim N(\mu,1)$

ii.) $x \sim N(\theta,\sigma^2)$ & $\theta \sim N(\mu,\tau^2)$, σ^2 is known
$\sigma^2 > 0$, $\tau^2 > 0$

iii.) $x \sim B(n,p)$ & $p \sim \underset{\text{type-1}}{Beta}(\alpha,\beta)$, $0<p<1$