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$X \rightarrow$ random variable $\qquad E(X) = \mu$

$\underset{p\times 1}{\underline{X}}$ $\qquad E(\underline{X}) = \underline{\mu}_{p\times 1}$

Distribution function

$$F_X(x) = P(X \le x).$$

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, X_2 \le x_2, \dots, X_p \le x_p).$$

for continuous,

$$F_{\underline{X}}(\underline{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_p} f_{\underline{X}}(\underline{x})\, dx_1\, dx_2 \dots dx_p.$$

for discrete

$$F_{\underline{X}}(\underline{x}) = \sum_{i_1=-\infty}^{x_1} \dots \sum_{i_p=-\infty}^{x_p} P(X_1 = i_1, \dots, X_p = i_p)$$

Joint pdf $\quad \dfrac{\partial^p F_{\underline{X}}(\underline{x})}{\partial x_1 \partial x_2 \dots \partial x_p}.$

for discrete

$$P(X = a) = F(a) - F(a^-).$$

$x_i \in \underline{X}_{p\times 1}$

Marginal of $X_i \rightarrow$

$$F_{X_i}(x_i) = \int_{-\infty}^{x_i} \dots \int_{-\infty}^{x_p} F_{\underline{X}}(\underline{x}) \prod_{\substack{j=1 \\ \text{except } x_i}}^{p} dx_j$$

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Conditional of X_i = $\frac{\text{Joint pdf}}{\text{Marg.}}$

$$\underline{X}_{p\times 1} = \begin{bmatrix} X^{(1)}_{q\times 1} \\ X^{(2)}_{(p-q)\times 1} \end{bmatrix}$$

$$E(\underline{X}) = \underline{\mu} = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$\Sigma_{11} \rightarrow$ var. cov. Matrix of q components considered in $X^{(1)}$.

$$\Sigma_{11} = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(1)} - \underline{\mu}^{(1)})'.$$

$$\Sigma_{12} = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})$$

$$\Sigma_{22} = E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(X^{(2)} - \mu^{(2)})'.$$

If $\Sigma_{12} = 0$, two variable are ~~donot~~ necessarily be independent.

The two variables will be independent only iff

$$f_{\underline{X}^{(1)}\underline{X}^{(2)}}(\underline{x}^{(1)}, \underline{x}^{(2)}) = f_1(\underline{x}^{(1)})\, f_2(x^{(2)})$$

Univariate Normal Distribution.

p.d.f. $\rightarrow$ $$f_X(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}.$$

; $x, \mu \in \mathbb{R}$
$\sigma^2 > 0.$

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

; $x_1, x_2, \mu_1, \mu_2 \in \mathbb{R}$
$\sigma_1, \sigma_2 \geq 0$
$-1 < \rho < +1$

$$\frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu})\right\}.$$

$$\underset{2\times 1}{\underline{X}} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\underset{2\times 1}{\underline{\mu}} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\left\{ \sigma_{12} = \rho\sigma_1\sigma_2 \quad \therefore \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} \right.$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12}' & \sigma_2^2 \end{bmatrix}$$

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$$\Sigma^{-1} = \frac{\operatorname{adj}\Sigma}{|\Sigma|}$$

$$|\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2).$$

$$\Sigma^{-1} = \frac{1}{(1-\rho^2)}\begin{bmatrix} 1/\sigma_1^2 & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{bmatrix}$$

p-variate $X_{p\times 1}$

$$f_{X_1,\dots X_p}(x_1,\dots x_p) = K\exp\left\{-\frac{1}{2}(\underline{x}-\underline{b})' A(\underline{x}-\underline{b})\right\} \quad ; \underline{x}, \underline{\mu} \in \mathbb{R}^p. \quad \text{—} (1)$$

A is a (+) ve definite matrix.

By the same analogy, we may define the density function of a multivariate normal random vector in the form (eq - (1)) where K is greater than zero a normalising constant to be determine such that the integral over entire p-dimensional ~~eucl~~ euclidean space of $X_1, X_2 \dots X_p$ is unity.

We also assume that matrix A is positive definite and symmetric.

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$$f_{\underline{X}}(\underline{x}) = K \exp\left\{-\frac{1}{2}(\underline{x}-\underline{b})' A (\underline{x}-\underline{b})\right\}$$

If A is some positive definite matrix, then there always exists a matrix C such that : $C'AC = I$ — (1).

$$f_{\underline{X}}(\underline{x}) > 0 \quad \Leftarrow (\underline{x}-\underline{b})' A (\underline{x}-\underline{b}) > 0$$

Non singular Transformation

$$\underline{X} - \underline{b} = C\underline{Y} \qquad , |C| \neq 0.$$

Jacobian $\rightarrow |J| = |C| =$ abs value of $|C|$.

Transforming in y using Jacobian,

$$g_{\underline{Y}}(\underline{y}) = K \exp\left\{-\frac{1}{2}\underline{Y}'C'AC\underline{Y}\right\}|C|.$$

using (1), we get.

$$g_{\underline{Y}}(\underline{y}) = K \exp\left\{-\frac{1}{2}\underline{Y}'\underline{Y}\right\}|C|.$$

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{p\text{-fold}} g_{\underline{y}}(\underline{y})\,d\underline{y} = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{p\text{-fold}} K \exp\left\{-\frac{1}{2}\underline{Y}'\underline{Y}\right\} = 1.$$

$$\underset{1\times p}{Y'}\ \underset{p\times 1}{Y} = [y_1, y_2 \ldots y_p]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = y_1^2 + y_2^2 + y_3^2 + \ldots + y_p^2 = \sum_{i=1}^{p} y_i^2$$

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$$\Rightarrow \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{p\text{-fold}} |c|\, K\, e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}\, dy_1 \ldots dy_p = 1.$$

$$\Rightarrow K|c| \int \cdots \int \prod_{i=1}^{p} e^{-\frac{1}{2}y_i^2}\, dy_i = 1.$$

$$\Rightarrow K|c| \prod_{i=1}^{p} \int e^{-\frac{1}{2}y_i^2}\, dy_i = 1.$$

$$\Rightarrow K|c|(2\pi)^{p/2} = 1$$

$$\Rightarrow K = \frac{1}{(2\pi)^{p/2}|c|}.$$

$$\Rightarrow \boxed{K = \frac{1}{|c|(2\pi)^{p/2}}} \quad \text{--- (ii)}$$

putting (ii) in p.d.f.

$$g_{\underline{Y}}(\underline{y}) = \frac{1}{|c|(2\pi)^{p/2}} |c|\, e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}$$

$$= \frac{1}{(2\pi)^{p/2}}\, e^{-\frac{1}{2}\sum_{i=1}^{p} y_i^2}.$$

$$g_{\underline{Y}}(\underline{y}) = \frac{1}{(2\pi)^{p/2}}\, e^{-\frac{1}{2}\sum y_i^2}$$

$$= \frac{1}{(2\pi)^{p/2}} \prod_{i=1}^{p} e^{-\frac{1}{2}y_i^2}$$

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$$= \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y_i^2}$$ (In standard form)

$$= \prod_{i=1}^{p} f_{Y_i}(y_i).$$

$$g_{\underline{Y}}(\underline{y}) = \prod_{i=1}^{p} f_i(y_i) = f_1(y_1) \cdots f_p(y_p).$$

The joint density is written in the form of product of marginal.
y_i's are independent.

$$Y_i \sim N(0,1), \quad ; \; i=1,2 \ldots p.$$

$$E[\underline{Y}] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{p \times 1}$$

$$\Sigma_Y = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{p \times p}$$

$$E[\underline{Y}] = \underline{0}$$

$$E[C^{-1}(\underline{X} - \underline{b})] = 0$$

$$C^{-1}[E(\underline{X}) - \underline{b}] = 0$$

$$\boxed{E(\underline{X}) = \underline{b}} = \underline{\mu}_x$$

$$\Sigma_y = E[\underline{Y} - \underline{0}][\underline{Y} - \underline{0}]' = I$$

$$= E[\underline{Y}\underline{Y}']$$

$$= E[C^{-1}(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})' C^{-1'}]$$

$$= C^{-1} E[(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'] C^{-1'}$$

$$\Sigma_y = C^{-1} \Sigma C^{-1'} \quad \text{—— (3)}$$

$$C' A C = I$$

$$C'^{-1} C' A C C^{-1} = C'^{-1} I C^{-1}$$

$$A = (C C')^{-1} \quad \text{—— (4)}$$

From equation (3), $\underline{C^{-1} \Sigma C^{-1'}} = I$

$$C C^{-1} \Sigma C^{-1'} C' = C I C'$$

$$\Sigma = C C' \quad \text{—— (5)}$$

From (4) & (5), $\boxed{A = \Sigma^{-1}}$

when x wa

$$|C' A C| = |I|$$

$$|C'|\,|A|\,|C| = 1$$

$$|A| = \frac{1}{|C'|\,|C|} = \frac{1}{|C|^2}$$

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$$|A|^{1/2} = \frac{1}{|C|}$$

$$|\Sigma^{-1}|^{1/2} = \frac{1}{|C|} \Rightarrow |C| = |\Sigma|^{1/2}$$

$$K = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}}$$

A random vector $X_{p\times 1} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$ taking values $x_1, x_2 \ldots x_p$. $\to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$. in E^p.

euclidian space is said to have a normal distribution if its p.d.f can be written as

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu}) \right\}$$

where $\underline{X}$, $\mu \in \mathbb{R}^p$ and Σ is a symmetric positive definite matrix of order p.

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Theorem

If the variance covariance matrix of p-variate normal random vector $X_{p\times 1}$, is a diagonal matrix then the components of X are independently normally distributed random variables.

$$\longrightarrow \quad f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})\right\} \quad \text{--- (1)}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_p^2 \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/\sigma_p^2 \end{bmatrix}$$

$$|\Sigma| = \prod_{i=1}^{p} \sigma_i^2$$

kernal part $\rightarrow$

$$(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu}) =$$

$$= \begin{bmatrix} (x_1-\mu_1) & (x_2-\mu_2) & \cdots & (x_p-\mu_p) \end{bmatrix} \begin{bmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1/\sigma_p^2 \end{bmatrix}$$

$$\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_p-\mu_p \end{bmatrix}$$

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$$= \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 + \cdots + \left(\frac{x_p-\mu_p}{\sigma_p}\right)^2$$

$$= \sum_{i=1}^{p} \left(\frac{x_i-\mu_i}{\sigma_i}\right)^2$$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^{p} \sigma_i} \exp\left\{ \frac{-1}{2} \sum_{i=1}^{p} \left(\frac{x_i-\mu_i}{\sigma_i}\right)^2 \right\}.$$

$$= \prod_{i=1}^{p} \frac{1}{(2\pi)^{1/2}\sigma_i} \exp\left\{ \frac{-1}{2} \left(\frac{x_i-\mu_i}{\sigma_i}\right)^2 \right\}.$$

$$= \prod_{i=1}^{p} f_{X_i}(x_i).$$

$$f_{\underline{X}}(\underline{x}) = f_{X_1}(x_1) \cdots f_{X_p}(x_p).$$

$\Rightarrow$ $\therefore$ $X_1, X_2, \ldots X_p$ are independently normally distributed random variable.

Theorem

If X is a p-variate normal random vector then $\underline{Y} = CX$ (a non-singular transformation) is distributed as normal $(N_p(C\mu, C\Sigma C'))$.

$\rightarrow$ Non-singular transformation $\rightarrow$ $\underline{Y} = C\underline{X}$.

$$\frac{dx}{dy} = \frac{1}{C} \Rightarrow |J| = |C^{-1}|.$$

Jacobian

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$$g_{\underline{y}}(\underline{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}(C^{-1}\underline{y} - \underline{\mu})' \Sigma^{-1}(C^{-1}\underline{y} - \underline{\mu})\right\} |C^{-1}|. \quad — (2)$$

$$Q = (C^{-1}\underline{y} - C^{-1}C\underline{\mu})'\Sigma^{-1}(C^{-1}\underline{y} - C^{-1}C\underline{\mu}).$$

$$= [C^{-1}(\underline{y} - C\underline{\mu})]'\Sigma^{-1}[C^{-1}(\underline{y} - C\underline{\mu})].$$

$$= (\underline{y} - C\underline{\mu})' C'^{-1}\Sigma^{-1}C^{-1}(\underline{y} - C\underline{\mu})$$

$$= (\underline{y} - C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y} - C\underline{\mu}) \quad — (3)$$

$$|C^{-1}| = \frac{1}{|C|} = \sqrt{\frac{1}{|C|^2}} = \sqrt{\frac{1}{|C|\,|C'|}} = \sqrt{\frac{|\Sigma|}{|C||\Sigma||C'|}}$$

$$= \frac{|\Sigma|^{1/2}}{|C\Sigma C'|^{1/2}} \quad — (4)$$

Substituting (3) & (4) in (2),

$$g_{\underline{y}}(\underline{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}(\underline{y} - C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y} - C\underline{\mu})\right\} \cdot \frac{|\Sigma|^{1/2}}{|C\Sigma C'|^{1/2}}$$

$$= \frac{1}{(2\pi)^{p/2}(C\Sigma C')^{1/2}} \exp\left\{\frac{-1}{2}(\underline{y} - C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y} - C\underline{\mu})\right\}.$$

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$$\Rightarrow \underline{y}_{p\times 1} \sim N_p(C\mu, C\Sigma C')$$

Theorem

Let $X_{p\times 1}$ be a normal random vector, then a necessary and sufficient condition that a subset $\underline{X}^{(1)}$ of the components of $\underline{X}$ be independent of the subset $\underline{X}^{(2)}$ of the remaining components of $\underline{X}$ is that the covariance between each component of $\underline{X}^{(1)}$ with a component of $\underline{X}^{(2)}$ is zero.

$$\rightarrow \quad \underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} \quad \begin{matrix} X^{(1)}_{q\times 1} \\ \\ \underline{X}^{(2)}_{(p-q)\times 1} \end{matrix}$$

$$\underline{X}^{(1)} = \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} \qquad \underline{\mu}^{(1)} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_q \end{bmatrix}$$

$$\underline{X}^{(2)} = \begin{bmatrix} X_{q+1} \\ \vdots \\ X_p \end{bmatrix} . \qquad \underline{\mu}^{(2)} = \begin{bmatrix} \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix}$$

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Σ_{11} = variance covariance matrix of $X^{(1)}$
$= E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(1)} - \underline{\mu}^{(1)})'$

Σ_{22} = variance covariance matrix of $X^{(2)}$.
$= E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'$

Σ_{12} = variance covariance matrix of $X^{(1)}$ & $X^{(2)}$
$= E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(X^{(2)} - \underline{\mu}^{(2)})'$.

$\Sigma_{21} = \Sigma_{12}'$

→ Necessary part

Let $\underline{X}^{(1)}$ & $\underline{X}^{(2)}$ be independent.

By definition, $\Sigma_{12} = 0$,

$$E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})' = \underset{\sim}{0}$$

This means all the components are uncorrelated and this implies that all the covariances are zero.

→ Sufficient part

Let all the covariances are zero.

$\Rightarrow \Sigma_{12} = \underset{\sim}{0}$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22} \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{11}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22}^{-1} \end{bmatrix}$$

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$$|\Sigma| = \begin{vmatrix} \Sigma_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22} \end{vmatrix} = \begin{vmatrix} \Sigma_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & I \end{vmatrix} \begin{vmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22} \end{vmatrix}$$

$$|\Sigma| = |\Sigma_{11}|\,|\Sigma_{22}|$$

$$f_{\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)}}(x^{(1)}, x^{(2)}) = \frac{1}{(2\pi)^{q/2}(2\pi)^{(p-q)/2}\,|\Sigma_{11}|^{1/2}\,|\Sigma_{22}|^{1/2}}$$

$$\exp\left\{\frac{-1}{2}\left[(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})'(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})'\right] \begin{bmatrix} \Sigma_{11}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{bmatrix}\right.$$

$$= \frac{1}{(2\pi)^{q/2}(2\pi)^{(p-q)/2}\,|\Sigma_{11}|^{1/2}\,|\Sigma_{22}|^{1/2}} \exp\left\{\frac{-1}{2}\left[(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})'\,\Sigma_{11}^{-1}\right.\right.$$

$$\left.(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})'\,\Sigma_{22}^{-1}\right] \begin{bmatrix} \underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)} \end{bmatrix}$$

$$= \frac{1}{(2\pi)^{q/2}(2\pi)^{(p-q)/2}\,|\Sigma_{11}|^{1/2}\,|\Sigma_{22}|^{1/2}} \exp\left\{\frac{-1}{2}\left[(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})'\,\Sigma_{11}^{-1}\right.\right.$$

$$\left.\left.(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}) + (\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})'\,\Sigma_{22}^{-1}(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})\right]\right\}.$$

$$= \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11}|^{1/2}} \exp\left[\frac{-1}{2}(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})'\,\Sigma_{11}^{-1}(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)})\right].$$

$$\times \frac{1}{(2\pi)^{(p-q)/2}\,|\Sigma_{22}|^{1/2}} \exp\left[\frac{-1}{2}(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})'\,\Sigma_{22}^{-1}(\underset{\sim}{x}^{(2)} - \underset{\sim}{\mu}^{(2)})\right]$$

$$= f_{\underset{\sim}{X}^{(1)}}(x^{(1)}) \times f_{\underset{\sim}{X}^{(2)}}(x^{(2)})$$

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Theorem

\# If $\underset{p\times 1}{X}$ is a $p\times 1$ normal random vector with $p\times 1$ mean vector μ and var. cov matrix Σ, Then the marginal distribution of any set of components of X is multivariate normal with mean and covariances obtained by taking corresponding components of μ and Σ respectively.

$$\underline{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} \qquad \underline{X}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} x_{q+1} \\ \vdots \\ x_p \end{bmatrix}$$

$$\underline{\mu} = \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix} ; \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)}, \quad \underline{Y}^{(2)} = \underline{X}^{(2)}$$

where B is chosen such that $\underline{Y}^{(1)}$ and $Y^{(2)}$ are independent.

Therefore,

$$E[\underline{Y}^{(1)}] = \mu^{(1)} + B\mu^{(2)}.$$

$$\text{and } E[Y^{(2)}] = \underline{\mu}^{(2)}.$$

$$\Rightarrow E[\underline{Y}^{(1)} - E(\underline{Y}^{(1)})][Y^{(2)} - E(\underline{Y}^{(2)})]' = 0$$

$$E[\underline{X}^{(1)} + B\underline{X}^{(2)} - \underline{\mu}^{(1)} - B\underline{\mu}^{(2)}][X^{(2)} - \underline{\mu}^{(2)}]' = 0$$

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$$E\left[\left(\underline{X}^{(1)} - \underline{\mu}^{(1)}\right) + B\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)\right]\left[\underline{X}^{(2)} - \underline{\mu}^{(2)}\right]' = 0.$$

$$E\left[\left(\underline{X}^{(1)} - \underline{\mu}^{(1)}\right)\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' + BE\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)\left(\underline{X}^{(2)} - \underline{\mu}^{(2)}\right)' = 0$$

$$\Sigma_{12} + B\Sigma_{22} = 0 \Rightarrow \boxed{B = -\Sigma_{12}\Sigma_{22}^{-1}}.$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}^{(2)}$$

$$\underline{Y}^{(2)} = \underline{X}^{(2)}$$

$$\begin{bmatrix} \underline{Y}^{(1)} \\ \underline{Y}^{(2)} \end{bmatrix} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix}$$

$$\underline{Y} = C\underline{X}$$

$$E(\underline{Y}) = CE(\underline{X})$$

$$E(\underline{Y}) = C\underline{\mu} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\underline{\mu}_y = \begin{bmatrix} \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$\cancel{\Sigma_y =}$

variance - covariance matrix of Y :-

$$D(Y) = C^2 D(X)$$

$$= C\Sigma C'$$

$$= \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ \left(-\Sigma_{12}\Sigma_{22}^{-1}\right)' & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -\left(\Sigma_{12}\Sigma_{22}^{-1}\right)' & I \end{bmatrix}$$

$$\cancel{D(Y)} = \left[\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + (\Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22})(-\Sigma_{12}\Sigma_{22}^{-1}) \right.$$

$$= -\Sigma_{12}\Sigma_{12}\Sigma_{22}^{-1} + \Sigma_{12}\Sigma_{12}\Sigma_{22}^{-1}$$

$$= 0$$

$\cancel{D(Y) =}$

$$\{ \Sigma_{12}' = \Sigma_{21} \}$$

$$(-\Sigma_{12}\Sigma_{22}^{-1})' = -\Sigma_{22}^{-1'}\Sigma_{21}$$

$\therefore$

$$D(Y) = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\begin{bmatrix} \underline{I} & \underline{0} \\ -\Sigma_{22}^{-1}\Sigma_{21} & \underline{I} \end{bmatrix}$$

$$D(Y) = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

$$Y \sim N_p(\underline{\mu}_y, \Sigma_y).$$

As $Y^{(1)}$ and $Y^{(2)}$ are independent

$$g_Y(y) = g_1(y^{(1)}) \times g_2(y^{(2)}).$$

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$$\Sigma_{\underline{Y}}^{-1} = \begin{bmatrix} \Sigma_{11.2}^{-1} & \underline{0} \\ \underline{0} & \Sigma_{22}^{-1} \end{bmatrix}.$$

$$\underline{Y} \sim N_p(\underline{\mu}_{Y_p}, \Sigma_{\underline{Y}})$$

$$|\Sigma_{\underline{Y}}| = |\Sigma_{11.2} \; \Sigma_{22}|$$

$$= |\Sigma_{11.2}| \, |\Sigma_{22}|.$$

$$g_{\underline{Y}}(\underline{y}) = g_1(\underline{y}^{(1)}) \times g_2(\underline{y}^{(2)}).$$

$$= \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{1/2}} \exp\left\{ \frac{-1}{2} (\underline{y}^{(1)} - \underline{\mu}^{*})' \right.$$

$$\left. \Sigma_{11.2}^{-1} (\underline{y}^{(2)} - \underline{\mu}^{(2)}) \right\} \times \frac{1}{(2\pi)^{(p-q)/2} |\Sigma_{22}|^{1/2}}$$

$$\cancel{\exp\left\{ \frac{-1}{2} (\underline{y}^{(2)} - \underline{\mu}^{(*)}) \right\}}$$

$$\exp\left\{ \frac{-1}{2} (y^{(2)} - \mu^{(2)})' \Sigma_{22}^{-1} (y^{(2)} - \mu^{(2)}) \right\}.$$

$$f_{X^{(2)}}(x^{(2)}) = \int \cdots \int g_X(y) \, dy^{(1)} \sim N_{p-q}(\mu^{(2)}, \Sigma_{22})$$

So, Normal vector of component of X is also a multivariate normal vector.

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Conditional distribution of $X^{(1)} / X^{(2)}$

$$P(\underline{X}^{(1)} / \underline{X}^{(2)} = \underline{x}^{(2)}) = \frac{P(\underline{X}^{(1)}, \underline{X}^{(2)})}{P(\underline{X}^{(2)})} = \frac{f_{\underline{X}^{(1)}, \underline{X}^{(2)}}(\underline{x}^{(1)}, \underline{x}^{(2)})}{f_{\underline{X}^{(2)}}(\underline{x}^{(2)})}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} \underline{X}^{(2)}$$

$$\underline{Y}^{(2)} = \underline{X}^{(2)}.$$

$$g_{\underline{Y}}(\underline{y}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11 \cdot 2}|^{1/2}} \exp\left\{ \frac{-1}{2} (\underline{y}^{(1)} - \underline{\mu}^{*})' \Sigma_{11 \cdot 2}^{-1} (\underline{y}^{(1)} - \underline{\mu}^{*}) \right\} \times f_{\underline{X}^{(2)}}(\underline{x}^{(2)})$$

$$f_{\underline{X}^{(1)}, \underline{X}^{(2)}}(\underline{x}^{(1)}, \underline{x}^{(2)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11 \cdot 2}|^{1/2}}$$

$$\exp\left\{ \frac{-1}{2} (\underline{x}^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} \underline{x}^{(2)} - \underline{\mu}^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} \underline{\mu}^{(2)})' \Sigma_{12}^{-1} \right.$$

$$\left. (\underline{x}^{(1)} - \Sigma_{12} \Sigma_{12}^{-1} \underline{x}^{(2)} - \underline{\mu}^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} \underline{\mu}^{(2)}) \right\} \times f_{\underline{X}^{(2)}}(\underline{x}^{(2)})$$

$$= \frac{1}{(2\pi)^{q/2} |\Sigma_{11 \cdot 2}|^{1/2}} \exp\left\{ \frac{-1}{2} (\underline{x}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12} \Sigma_{12}^{-1} (\underline{x}^{(2)} - \underline{\mu}^{(2)}))' \right.$$

$$\left. \Sigma_{11 \cdot 2}^{-1} (\underline{x}^{(1)} - \underline{\mu}^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}^{(2)} - \underline{\mu}^{(2)})) \right\} \times f_{\underline{X}^{(2)}}(\underline{x}^{(2)})$$

$$N_q(\underline{\mu}^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}^{(2)} - \underline{\mu}^{(2)}), \Sigma_{11 \cdot 2})$$

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Theorem :-

If $X \sim N_p(\mu, \Sigma)$ and $\underline{Y} = D\underline{X}$ where D is $q \times p$ matrix such that $q < p$. the rank of D is q then $\underline{Y} \sim N_q(D\underline{\mu}, D\Sigma D')$

→ Given ~~D~~ Matrix $D_{q \times p}$; $r(q)$.

$$\underline{Y} = D\underline{X}$$

$$\Rightarrow E(\underline{Y}) = E(D\underline{X})$$
$$= D\,E(\underline{X})$$
$$= D\mu$$

$$\Sigma_{\underline{Y}} = E(\underline{Y} - E(\underline{Y}))(\underline{Y} - E(\underline{Y}))'$$
$$= E(D\underline{X} - D\underline{\mu})(D\underline{X} - D\underline{\mu})'$$

$$\Sigma_{\underline{Y}} = E[D(\underline{X}-\underline{\mu})][D(\underline{X}-\underline{\mu})]'$$

$$= D\,E[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})']D'$$

$$= D\Sigma D'$$

Since rank of $D = q$, This means q rows are independent here.
we know that a set of q independent ~~rows~~ vectors can be extended to form a basis of the p-dimensional vector space by adding to ~~it~~ it $(p-q)$ vectors.

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extended

$$D = \begin{array}{c|c} & 1 \cdots p \\ \begin{matrix}1\\ \vdots \\ q\end{matrix} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

$$\underline{C} = \begin{bmatrix} D_{q \times p} \\ E_{(p-q) \times p} \end{bmatrix}, \quad C \text{ is non-singular.}$$

$$\cancel{\underline{Z} = C \underline{X}}$$

Let us make the transformation $\underline{Z} = C\underline{X}$ which implies

$\Rightarrow$ $Z \sim N_p(C\mu, C \Sigma C')$.

$$\underline{Z} = C\underline{X} = \begin{bmatrix} D \\ E \end{bmatrix} \underline{X} = \begin{bmatrix} D\underline{X} \\ E\underline{X} \end{bmatrix}$$

but $D\underline{X}$ being the partition vector of Z as a marginal q variate normal distribution therefore,

$$Y = D\underline{X} \sim N_q(D\underline{\mu}, D \Sigma D').$$

Property

This theorem tells us that if $X \sim N_p(\mu, \Sigma)$ then every linear combination of the components of X has a univariate normal distribution.

$$D_{1 \times p} = (d_1, d_2 \ldots d_p)$$

$$D\underline{X} = (d_1, d_2 \ldots d_p) \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix}$$

$$\phi_X(t) = e^{i\mu t - \frac{1}{2}t^2\sigma^2}$$

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$$= d_1 X_1 + d_2 X_2 + \dots + d_p X_p$$

$$\sim N(\Sigma d_i \mu_i, \Sigma d_i^2 \sigma_i^2)$$

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Characteristic function

$$\phi_X(t) = E(e^{itX})$$

$$\boxed{\phi_{\underline{X}}(\underline{t}) = E(e^{i\underline{t}'\underline{X}})}$$; t is a vector of reals. $i = \sqrt{-1}$.

$$t' = (t_1, t_2, \dots, t_p)$$

$$\phi_{\underline{X}}(\underline{t}) = E(e^{i(t_1 X_1 + t_2 X_2 + \dots + t_p X_p)})$$

Theorem

Let X be a $p \times 1$ normal random vector with mean μ and variance-covariance matrix Σ, vector then the characteristic function of X is

$$\cancel{\phi_X(t) = e^{i\mu t - \frac{1}{2}t^2}}$$

$$\boxed{\phi_{\underline{X}}(\underline{t}) = e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}}$$

where $t' = (t_1, t_2, \dots, t_p)$ which is a vector of reals.

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let us consider the transformation

$\underline{X} - \underline{\mu} = C\underline{Y}$

$C'\Sigma^{-1}C = I$

the joint density of $\underline{X}$ is

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}(\underline{X}-\mu)'\Sigma^{-1}(x-\mu)\right\} \times \frac{|C|}{\cancel{|C|}}$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}(C\underline{Y})'\Sigma^{-1}(C\underline{Y})\right\} \times |C^2|^{1/2}$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}Y'(C'\Sigma^{-1}C)Y\right\} \times |C^2|^{1/2}$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}Y'IY\right\} \times \frac{1}{|C|} \quad |C|$$

$$= \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{\frac{-1}{2}Y'Y\right\} \frac{1}{|C|}$$

since

$\Rightarrow Y$

$|C'C|^{1/2} \times |C'C|^{1/2}$

~~using~~ $\cancel{|C^{-1}| = \frac{1}{|C|} = \sqrt{\frac{1}{\quad}}}$

~~using~~ $\cancel{|C^{-1}| = \frac{|\Sigma|^{1/2}}{|C'\Sigma C'|^{1/2}}}$

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$$\cancel{= \frac{1}{(2\pi)^{k/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} y'y\right\} \times \frac{|C\Sigma C'|^{1/2}}{|\Sigma|^{1/2}}}$$

$$= \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} y'y\right\} \frac{|C'\Sigma^{-1}C|^{1/2}}{|\Sigma^{-1}|^{1/2}}$$

$$= \frac{1}{(2\pi)^{p/2} \cancel{|\Sigma|^{1/2}}} \exp\left\{\frac{1}{2} y'y\right\} \; I \times \cancel{|\Sigma|^{1/2}}$$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left\{\frac{1}{2} y'y\right\}.$$

since $y' = [y_1, y_2, y_3 \ldots y_p]$ $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix}$

$\Rightarrow y'y = \sum y_i^2.$

$$\frac{1}{|C|} = \sqrt{\frac{1}{|C|^2}} = \sqrt{\frac{1}{|C||C'|}}$$

$$= \sqrt{\frac{|\Sigma|}{|C||\Sigma||C'|}}$$

$|C'C|^{1/2}$

$$= \frac{|\Sigma|^{1/2}}{|C\Sigma C'|^{1/2}}$$

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$$g_{\underline{Y}}(\underline{y}) = \prod_{i=1}^{p} \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2} y_i^2\right\} = \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{1}{2} y'y\right\}$$

$$y_i \sim N(0,1) \quad \forall \; i = 1, 2, \ldots p$$

$$\phi_{y_i}(y_i) = e^{it \cdot 0 - \frac{1}{2} t^2 \cdot 1}.$$

$$\phi_{y_i}(y_i) = e^{-\frac{1}{2} t^2}$$

$$g_{\underline{Y}}(\underline{y}) = \cancel{\exp\left\{\frac{-1}{2}\right.} \frac{1}{(2\pi)^{p/2}} \exp\left\{\frac{-1}{2} y'y\right\}$$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left\{-\frac{1}{2} (y_1^2 + \ldots + y_p^2)\right\}.$$

$$\phi_{\underline{Y}}(\underline{t}) = \prod_{i=1}^{p} e^{-\frac{1}{2} t_i^2}$$

$$= e^{-\frac{1}{2} \Sigma t_i^2}$$

$$\boxed{\phi_{\underline{Y}}(\underline{t}) = e^{-\frac{1}{2} \underline{t}' \underline{t}}}$$

$$\phi_{\underline{X}}(\underline{t}) = E\left(e^{i\underline{t}'\underline{X}}\right)$$

$$= E\left(e^{i\underline{t}'(C\underline{Y} + \underline{\mu})}\right)$$

$$= E\left(e^{i\underline{t}'C\underline{Y} + i\underline{t}'\underline{\mu}}\right)$$

$$= E\left(e^{i\underline{t}'\underline{\mu}} \cdot e^{i\underline{t}'C\underline{y}}\right)$$

$E\left(e^{i(\underline{t}'C')'\underline{y}}\right)$

$\uparrow \quad \phi_{\underline{y}}(\underline{t}'C') \; ; \; \underline{u} = \underline{t}'C'$

$= \phi_{\underline{y}}(\underline{u})$

$y'y\}$

$$= e^{i\underline{t}'\underline{\mu}} E\left(e^{i\underline{t}'C\underline{y}}\right) \; ; \; \underline{u}' = (\underline{t}'C)'$$

$$= e^{i\underline{t}'\underline{\mu}} \quad \phi_{\underline{y}}(\underline{u})$$

$$= e^{i\underline{t}'\underline{\mu}} \quad e^{-\frac{1}{2}\underline{u}'\underline{u}}$$

$$= e^{i\underline{t}'\underline{\mu}} \quad e^{-\frac{1}{2}\underline{t}'C(\underline{t}'C)'}$$

$$= e^{i\underline{t}'\underline{\mu}} \quad e^{-\frac{1}{2}\underline{t}'.CC'\underline{t}}$$

$\Big\{$ using directly $C'\Sigma^{-1}C = I$

$$\boxed{\phi_{\underline{x}}(\underline{t}) = e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}}$$

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$X_1, X_2, \ldots X_n \sim N(\mu, \sigma^2)$

characteristics	Individuals $1 \quad 2 \ldots \alpha \ldots n$	Mean
X_1	$x_{11} \quad x_{12} \ldots x_{1\alpha} \ldots x_{1n}$	$\bar{x}_1$
X_2	$x_{21} \quad x_{22} \ldots x_{2\alpha} \ldots x_{2n}$	$\bar{x}_2$
X_3	$x_{31} \quad x_{32} \ldots x_{3\alpha} \ldots x_{3n}$	$\bar{x}_3$
$\vdots$	$\vdots$	$\vdots$
X_p	$x_{p1} \quad x_{p2} \ldots x_{p\alpha} \ldots x_{pn}$	$\bar{x}_p$
Notation $\longrightarrow$ (NO significance)	$\underline{x}_1 \quad \underline{x}_2 \ldots \underline{x}_\alpha \ldots x_n$	$\bar{\underline{x}}$

Sample Mean Vector

$$\bar{\underline{x}} = \frac{1}{n}\sum_{\alpha=1}^{n} \underline{x}_\alpha = \frac{1}{n}\begin{bmatrix} \sum_{\alpha=1}^{n} x_{1\alpha} \\ \sum_{\alpha=1}^{n} x_{2\alpha} \\ \sum_{\alpha=1}^{n} x_{i\alpha} \\ \vdots \\ \sum_{\alpha=1}^{n} x_{p\alpha} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_i \\ \vdots \\ \bar{x}_p \end{bmatrix}$$

~~$\bar{x}_i =$~~

$i = 1, 2 \ldots p$

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Sample V-Covariance Matrix.

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ S_{21} & S_{22} & \cdots & S_{2p} \\ \vdots & \vdots & & \vdots \\ S_{p1} & S_{p2} & \cdots & S_{pp} \end{bmatrix}_{p \times p}$$

$$S_{ij} = \frac{1}{(n-1)} \underbrace{\sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}_{A}$$

$$S = \frac{A}{n-1}$$

$$A = \sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$$

$$A = \sum_{\alpha=1}^{n} x_{i\alpha} x_{i\alpha} - n \bar{x}_i \bar{x}_j$$

Result

(1) $Q = \underline{x}' A \underline{x}$

$$\frac{\partial Q}{\partial \underline{x}} = 2A\underline{x}$$

3) If $Q = (\underline{x} - \underline{b})' A (\underline{x} - \underline{b})$

$= (\underline{b} - \underline{x})' A (\underline{b} - \underline{x})$

A is symmetric

$$\frac{\partial Q}{\partial \underline{x}} = 2A(\underline{x} - \underline{b}), \quad \frac{\partial Q}{\partial \underline{b}} = 2A(\underline{b} - \underline{x})$$

(2) If $B = \underline{y}' A \underline{x} = \underline{x}' A' \underline{y}$

then $\frac{\partial B}{\partial \underline{y}} = A\underline{x}$ and $\frac{\partial B}{\partial \underline{x}} = A'\underline{y}$

Q Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$x = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$

Derive the result using the above matrices.

$\rightarrow$ $x' = \begin{bmatrix} x_{11} & x_{21} \end{bmatrix}$

$$x'A = \begin{bmatrix} x_{11} & x_{21} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}a_{11} + x_{21}a_{21} & x_{11}a_{12} + x_{21}a_{22} \end{bmatrix}$$

$$Q = x'Ax = \begin{bmatrix} a_{11}x_{11} + a_{21}x_{21} & x_{11}a_{12} + a_{22}x_{21} \end{bmatrix}_{1\times 2} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}_{2\times 1}$$

$$Q = x'Ax = \begin{bmatrix} (a_{11}x_{11}^2 + a_{21}x_{21}x_{11}) + (a_{12}x_{11}x_{21} + a_{22}x_{21}^2) \end{bmatrix}_{1\times 1}$$

$$Q = x'Ax = \begin{bmatrix} a_{11}x_{11}^2 + a_{21}x_{11}x_{21} + a_{12}x_{11}x_{21} + a_{22}x_{21}^2 \end{bmatrix}_{1\times 1}$$

$\cancel{\frac{\partial(x'Ax)}{\partial x}} =$ Take x_{11} as x_1 and x_{21} as x_2

$$\frac{\partial(x'Ax)}{\partial x_1} = \frac{\partial}{\partial x_1}\left[a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 \right]$$

$$= \left[2a_{11}x_1 + a_{21}x_2 + a_{12}x_2 \right]$$

$$\frac{\partial(x'Ax)}{\partial x_1} = 2a_{11}x_1 + a_{21}x_2 + a_{12}x_2$$

$$= 2a_{11}x_1 + 2a_{21}x_2$$

$$\frac{\partial(x'Ax)}{\partial x_2} = 2a_{21}x_1 + a_{12}x_1 + 2a_{22}x_2$$

$$= 2a_{22}x_2 + 2a_{12}x_1.$$

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \end{bmatrix}_{2\times 1}$$

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} 2a_{11}x_1 + 2a_{21}x_2 \\ 2a_{22}x_2 + 2a_{12}x_1 \end{bmatrix}_{2\times 1}$$

$$\text{R.H.S.} = 2Ax = 2\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

$$\cancel{Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}\begin{bmatrix} x_{11} & x_{21} \end{bmatrix}}$$

$$\cancel{Ax = \begin{bmatrix} a_{11}x_{11} \end{bmatrix}}$$

$$2Ax = 2\begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} \\ a_{21}x_{11} + a_{22}x_{21} \end{bmatrix} = \begin{bmatrix} 2a_{11}x_{11} + 2a_{12}x_{21} \\ 2a_{21}x_{11} + 2a_{22}x_{21} \end{bmatrix}$$

Taking i.e. $x_{11} \rightarrow x_1$ and $x_{12} \rightarrow x_2$

$$2Ax = 2\begin{bmatrix} 2a_{11}x_1 + 2a_{12}x_2 \\ 2a_{21}x_1 + 2a_{22}x_2 \end{bmatrix}$$

Result ① proved

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* A sub-matrix of A is a rectangular array obtained from A by deleting rows and columns.

* A minor is the determinant of square sub-matrix of A. $|A| = \sum_{i=1}^{p} a_{ij} A_{ij} = \sum_{j=1}^{p} a_{kj} A_{kj}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where a_{ij} is the $(-1)^{i+j}$ times minor of a_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Minor of } a_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

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MLE of μ

~~x_1~~ (random sample)

let $x_1, x_2 \dots x_\alpha \dots x_n \sim N_p(\underline{\mu}, \Sigma)$
$(n > p)$

$$L = \prod_{\alpha=1}^{n} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1} (\underline{x}_\alpha - \underline{\mu}) \right\}.$$

$$= \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp\left\{ -\frac{1}{2} \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1} (\underline{x}_\alpha - \underline{\mu}) \right\}$$

$$\log L = -\frac{np}{2} \log 2\pi - \frac{n}{2} \log |\Sigma| - \frac{1}{2} \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1} (\underline{x}_\alpha - \underline{\mu})$$

$$\frac{\partial \log L}{\partial \underline{\mu}} = 0 - 0 - \frac{1}{2} \sum_{\alpha=1}^{n} 2 \Sigma^{-1} (\underline{\mu} - \underline{x}_\alpha) = \underline{0}$$

$$\Rightarrow \sum_{\alpha=1}^{n} (\underline{\mu} - \underline{x}_\alpha) = \underline{0}$$

$$\Rightarrow \hat{\underline{\mu}} = \frac{1}{n} \sum_{\alpha=1}^{n} \underline{x}_\alpha.$$

$$\Rightarrow \boxed{\hat{\underline{\mu}} = \bar{\underline{x}}}$$

Quiz Questions

1 Find the mean vector μ and variance-covariance matrix Σ of the following density function.

(i) $$f(x,y) = \frac{1}{2\pi} \exp\left[\frac{-1}{2}\left\{ (x-1)^2 + (y-2)^2 \right\}\right] \quad \text{— (I)}$$

Sol^n:- Bivariate normal density function is :-

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)} \exp\left[\frac{-1}{2\sqrt{1-\rho^2}}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right] \quad \text{— (II)}$$

Comparing (I) & (II), we get

$\sigma_1 = 1 \quad \sigma_2 = 1 \quad \rho = 0 \quad \mu_1 = 1 \quad \mu_2 = 2$

The mean vector is $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The variance covariance matrix Σ is

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) $f(x,y) = \dfrac{1}{2.4\pi} \exp\left[\dfrac{-1}{0.72}\left\{\dfrac{x^2}{4} - \dfrac{1.6xy}{2} + y^2\right\}\right]$

Solⁿ :- $\cancel{f(x,y)} = \dfrac{1}{\cancel{2\pi(4)}\ 2.4\pi} \exp\left[\dfrac{-1}{2}\left\{\dfrac{x^2}{4\times 0.36} - \dfrac{1.6xy}{2\times 0.36} + \dfrac{y^2}{0.36}\right\}\right]$

$\Rightarrow \cancel{f(x,y) = \dfrac{1}{2\pi(4)} \exp\left[\dfrac{-1}{2}\left[x^2 - 0.\right.\right.}$

$$f(x,y) = \frac{1}{2.4\pi} \exp\left[\frac{-1}{0.72}\left\{\left(\frac{x-0}{2}\right)^2 + \left(\frac{y-0}{1}\right)^2 - \left(2\times 0.8\left(\frac{x-0}{2}\right)\left(\frac{y-0}{1}\right)\right)\right\}\right] \quad \text{— (iii)}$$

Comparing the terms in (ii) & (iii), we get

$2\pi\sigma_1\sigma_2\sqrt{1-\rho^2} = 2.4\pi$

$\sigma_1 = 2 \qquad \sigma_2 = 1 \qquad \rho = 0.8.$

$\mu_1 = 0, \mu_2 = 0.$

$2(1-\rho^2) = 0.72$

Hence, The mean vector $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The var. covariance matrix $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 4 & 0.16 \\ 0.16 & 1 \end{bmatrix}$$

Maximum Likelihood Estimate of variance Covariance Matrix (Σ) :-

~~let $\Sigma^{-1} = \sigma$~~

$$L = \prod_{\alpha=1}^{n} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{ \frac{-1}{2} (\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1} (\underline{x}_\alpha - \underline{\mu}) \right\}$$

$$\Sigma = (\sigma_{ij})_{p \times p}$$

$$\Sigma_{ij} = (-1)^{i+j} \text{ times minor } \Sigma$$

$$\Sigma^{-1} = (\sigma^{ij}) = \frac{\Sigma_{ij}}{|\Sigma|}$$

Correspondingly,

$$\frac{\Sigma^{\sigma ij}}{|\Sigma^{-1}|} = \Sigma = \sigma_{ij}$$

$$|\Sigma^{-1}| = \sigma^{i1} \Sigma^{i1} + \sigma^{i2} \Sigma^{i2} + \dots + \sigma^{ip} \Sigma^{ip} \quad \forall\, i$$

eg

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} A_{11} + a_{12} A_{12} + \dots$$

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$$\log L = \frac{-np}{2}\log 2\pi + \frac{n}{2}\log|\Sigma^{-1}| - \frac{1}{2}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu})' \Sigma^{-1}(\underline{x}_\alpha - \underline{\mu})$$

$$\log L = \frac{-np}{2}\log 2\pi + \frac{n}{2}\log\left(\sigma^{i1}\Sigma^{i1} + \dots + \sigma^{ij}\Sigma^{ij} + \dots + \sigma^{ip}\Sigma^{ip}\right)$$

$$-\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{p}\sigma^{ij}(x_{i\alpha}-\mu_i)(x_{j\alpha}-\mu_j).$$

$$\left\{\text{since } \underline{x}' A \underline{x} = \sum_{i,j} a_{ij} x_i x_j\right\}$$

differentiating w.r.to σ^{ij} and equating to zero, we get:

$$\frac{\partial \log L}{\partial \sigma^{ij}} = 0 \Rightarrow \frac{n}{2}\frac{\Sigma^{ij}}{|\Sigma^{-1}|} - \frac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha}-\mu_i)(x_{j\alpha}-\mu_j) = 0$$

because $\frac{\partial}{\partial u}\log f(u) = \frac{f'(u)}{f(u)}$

$$\frac{n}{\cancel{2}}\frac{\Sigma^{ij}}{|\Sigma^{-1}|} = \frac{1}{\cancel{2}}\sum_{\alpha=1}^{n}(x_{i\alpha}-\mu_i)(x_{j\alpha}-\mu_j).$$

$$n\,\sigma_{ij} = \sum_{\alpha=1}^{n}(x_{i\alpha}-\mu_i)(x_{j\alpha}-\mu_j)$$

$$\hat{\sigma}_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha}-\mu_i)(x_{j\alpha}-\mu_j)$$

Hence, $\boxed{\hat{\Sigma} = \frac{A}{n}}$

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Theorem

Given $X_1, X_2, X_3, \ldots, X_n$ be an independent random sample of size n where $n > p$ from $N_p(\mu, \Sigma)$. Then $\bar{x} = \frac{1}{n} \sum_{\alpha} x_\alpha$.

and $A = \sum_{\alpha} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$.

Thus, $\bar{x}$ and A are independently distributed. and $\bar{x} \sim N_p\left(\mu, \frac{\Sigma}{n}\right)$.

Proof: →

Since $\bar{X}$ is a linear combination of a multivariate normal random vector it is also a multivariate normal vector.

Now, the mean vector of $\bar{x}$,

we know that $\bar{X} = \frac{1}{n} \sum_{\alpha=1}^{n} x_\alpha$.

$$E(\bar{X}) = \frac{1}{n} \sum_{\alpha=1}^{n} E(x_\alpha) = \frac{1}{n} \times n\mu$$

$$\boxed{E(\bar{X}) = \mu.}$$

Now, var-covariance Matrix,

$$\Sigma_{\bar{x}} = E\left[(\bar{x} - \mu)(\bar{x} - \mu)'\right]$$

$$= E\left[\left\{\frac{1}{n}(x_1 + x_2 + \ldots + x_n) - \mu\right\} \left\{\frac{1}{n}(x_1 + x_2 + \ldots + x_n) - \mu\right\}'\right]$$

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$$= \frac{1}{n^2} E\left[\left(\sum_{\alpha=1}^{n} x_\alpha - n\mu\right)\left(\sum_{\alpha=1}^{n} x_\alpha - n\mu\right)'\right]$$

$$= \frac{1}{n^2} E\left[\left\{(\underline{x}_1 - \underline{\mu}) + (\underline{x}_2 - \underline{\mu}) + \dots + (\underline{x}_n - \underline{\mu})\right\}\left\{(\underline{x}_1 - \underline{\mu}) + (\underline{x}_2 - \underline{\mu}) + \dots + (\underline{x}_n - \underline{\mu})\right\}'\right]$$

$$= \frac{1}{n^2} E\left[(\underline{x}_1 - \underline{\mu})(\underline{x}_1 - \underline{\mu})' + (\underline{x}_2 - \underline{\mu})(\underline{x}_2 - \underline{\mu})' + \dots + \text{Cross product term}\right]$$

vanishes because x_i's are iid.

$$= \frac{1}{n^2}\left[\Sigma + \Sigma + \dots + \Sigma + 0\right] = \frac{n\Sigma}{n^2} = \frac{\Sigma}{n}.$$

$$\boxed{\Sigma_{\underline{x}} = \frac{\Sigma}{n}}$$

Theorem

$\frac{A}{n-1}$ is an unbiased estimate of Σ_x.

Proof :- Given that

$$A = \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})'.$$

$$= \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu} + \underline{\mu} - \bar{\underline{x}})(\underline{x}_\alpha - \underline{\mu} + \underline{\mu} - \bar{\underline{x}})'.$$

$$= \sum_{\alpha=1}^{n} \left[(\underline{x}_\alpha - \underline{\mu}) - (\bar{\underline{x}} - \underline{\mu})\right]\left[(\underline{x}_\alpha - \underline{\mu}) - (\bar{\underline{x}} - \underline{\mu})\right]'$$

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$$= \sum_{\alpha=1}^{n} (x_\alpha - \mu)(x_\alpha - \mu)' - \sum_{\alpha=1}^{n} (\bar{x} - \mu)(x_\alpha - \mu)'$$

$$- \sum_{\alpha=1}^{n} (x_\alpha - \mu)(\bar{x} - \mu)' + \sum_{\alpha=1}^{n} (\bar{x} - \mu)(\bar{x} - \mu)'$$

Since $\sum_{\alpha=1}^{n} (x_\alpha - \mu) = n\bar{x} - n\mu$

$$A = \sum_{\alpha=1}^{n} (x_\alpha - \mu)(x_\alpha - \mu)' - n(\bar{x} - \mu)(\bar{x} - \mu)'$$

$$- \cancel{n(\bar{x} - \mu)(\bar{x} - \mu)'} + \cancel{n(\bar{x} - \mu)(\bar{x} - \mu)'}.$$

$$A = \sum_{\alpha=1}^{n} (x_\alpha - \mu)(x_\alpha - \mu)' - n(\bar{x} - \mu)(\bar{x} - \mu)'$$

$$E(A) = \sum_{\alpha=1}^{n} \Sigma - \frac{n\Sigma}{n}.$$

$$E(A) = n\Sigma - \Sigma = (n-1)\Sigma.$$

$$\boxed{E\left(\frac{A}{n-1}\right) = \Sigma}$$

21/05/2022

Theorem

Given a random sample $X_1, X_2, \ldots X_n$ from $N_p(\mu, \Sigma)$. where $\bar{X} = \frac{1}{n}\sum_{\alpha=1}^{n} X_\alpha$ and $A = \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})$.

prove that $\bar{X}$ and A are independently

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distributed.

Proof:- $Y = CX$

$CC' = I$ orthogonal transformation

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & \cdots & c_{(n-1)n} \\ 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \end{bmatrix}$$

$$CC' = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{(n-1)1} & c_{(n-1)2} & \cdots & c_{(n-1)n} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{(n-1)1} & 1/\sqrt{n} \\ c_{12} & c_{22} & \cdots & c_{(n-1)2} & 1/\sqrt{n} \\ \vdots & & & & \\ c_{1n} & c_{2n} & \cdots & c_{(n-1)n} & 1/\sqrt{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$c_{11}c_{11} + c_{12}c_{12} + \cdots + c_{1n}c_{1n} = 1.$

$$\sum_{k=1}^{n} c_{ik}^2 = 1 \qquad \forall\ i = 1, 2, \ldots n-1$$

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$$\frac{1}{\sqrt{n}}(c_{11}+c_{12}+\dots+c_{1n})=0 \Rightarrow \sum_{k=1}^{n} c_{ik}=0$$

$$i=1,2\dots(n-1) \quad \text{---} ①$$

$$c_{11}c_{21}+c_{12}c_{22}+\dots+c_{1n}c_{2n}=0.$$

$$c_{11}c_{31}+c_{12}c_{32}+\dots+c_{1n}c_{3n}=0.$$

$$\vdots$$

$$c_{11}.c_{(n-1)1}+c_{12}\,c_{(n-2)1}+\dots+c_{1n}\,c_{(n-1)n}=0$$

$$\rightarrow \sum_{k=1}^{n} c_{ik}c_{jk} = \begin{cases} 0 & \forall\ i\neq j=1,2,\dots n-1 \\ 1 & \forall\ i=j=1,2,\dots n-1 \end{cases} \quad \text{---} ②$$

$$\underline{Y}=C\underline{X} \Rightarrow \begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n}\\ c_{21} & c_{22} & \dots & c_{2n}\\ \vdots & \vdots & \dots & \vdots\\ c_{(n-1)1} & c_{(n-1)2} & & c_{(n-1)n}\\ 1/\sqrt{n} & 1/\sqrt{n} & & 1/\sqrt{n}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

$$y_i = c_{i1}\underline{x}_1 + c_{i2}\underline{x}_2+\dots+c_{in}\underline{x}_n$$

$$E(y_i) = E\left(\sum_{k=1}^{n} c_{ik}\underline{x}_k\right) = \mu\sum_{k=1}^{n} c_{ik}=0 \quad \forall i=1,2..(n-1)$$

$$y_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\underline{x}_k = \frac{1}{\sqrt{n}}\times n\bar{x} = \sqrt{n}\,\bar{x}.$$

$$\operatorname{cov}(y_i,y_j) = E\left[\left[(y_i-E(y_i))\right]\left[(y_j-E(y_j))'\right]\right]$$

$$= E\left[(c_{i1}\underline{x}_1+c_{i2}\underline{x}_2+\dots+c_{in}\underline{x}_n)(c_{j1}\underline{x}_1+\dots+c_{jn}\underline{x}_n)\right]$$

$$= c_{i1}c_{j1}E(x_1x_1') + c_{i2}c_{j2}E(x_2x_2')+\dots+c_{in}c_{jn}E(x_nx_n')$$

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Since x_i's and x_j's are iid random sample so cross product term vanishes.

so, $cov(y_i, y_j) = (c_{i1}c_{j1} + c_{i2}c_{j2} + \dots + c_{ik}c_{jk})\Sigma$

$$= \begin{cases} 0 & , i \neq j \\ \Sigma & , i = j \end{cases}$$

non diagonal terms of the $\hat{\Sigma}$ matrix are 0 and diagonal terms are Σ.
y_i & y_j are independently distributed (covariance)

Now, $cov(y_i, y_n) = E[(y_i - E(y_i))(y_n - E(y_n))]$

$$= E\left[(c_{i1}x_1 + c_{i2}x_2 + \dots + c_{ik}x_k)\cdot\left(\frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \dots + \frac{x_n}{\sqrt{n}} - \sqrt{n}\mu\right)\right]$$

$= E$

..., y_i and y_j are independent of each other.

$$\Sigma y_i y_i' = \sum_{i=1}^{n} (c_{i1}x_1 + \dots + c_{in}x_n)(c_{i1}x_1 + \dots + c_{in}x_n)$$

$$= \sum_{i=1}^{n} c_{i1}^2 x_1 x_1' + \dots + \sum_{i=1}^{n} c_{in}^2 x_n x_n' + \sum_{i=1}^{n} c_{i1}c_{i2} x_1 x_2'$$

$$\dots + \sum_{i=1}^{n} c_{in}c_{in} x_1 x_n' + \sum_{i=1}^{n} c_{i2}c_{i1} x_2 x_1' + \dots + \sum_{i=1}^{n} c_{in}c_{i(n-1)} x_n x_{n-1}'$$

$$= (x_1 x_1' + \dots + x_n x_n') + 0.$$

$$y_i y_i' = \sum_{i=1}^{n} x_i x_i' \Rightarrow \sum_{i=1}^{n-1} y_i y_i' + y_n y_n' = \sum_{i=1}^{n} x_i x_i'.$$

$$y_i y_i + n\bar{x}.\bar{x}' = \sum_{i=1}^{n} x_i x_i'.$$

$\left\{ \begin{array}{l} Y_i \sim \text{Normal} \\ Y_i Y_i \sim \chi^2 \end{array} \right.$

$$\sum y_i y_i' = \sum_{i=1}^{n} x_i x_i' - n\bar{x}\bar{x}'.$$

$$= A \quad \text{—} \quad (**)$$

By eq^n (*) and (**) $\bar{x}$ and A depend on two mutually exclusive set which are independently distributed. Therefore,

i) $\bar{x}$ and A are independent.

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Test for the μ when variance - Covariance Matrix (Σ) is known. —

Given a random sample from $N_p(\mu, \Sigma)$.

Let $H_0 : \mu = \underline{\mu}_0$
where μ_0 is some specified vector then under H_0, then the test statistic

$$n(\bar{x} - \mu_0)' \Sigma^{-1} (\bar{x} - \mu_0) \sim \chi^2_p .$$

$$\bar{x} \sim N_p\left(\underline{\mu}, \frac{\Sigma}{n}\right) = N_p(\underline{\mu}, \Sigma^*)$$

Let $C' \Sigma^{*-1} C = I$

$\Rightarrow \Sigma^* = CC'$

$\bar{x} - \mu_0 = C.y \Rightarrow y = C^{-1}(\bar{x} - \mu_0)$.

$E(y) = C^{-1} E(\bar{\underline{x}} - \mu_0)$.

$E(y) = C^{-1}(\mu_0 - \mu_0)$ under H_0.
$= 0$.

$$\Sigma_y = E[(y - E(y))(y - E(y))']$$
$$= E[C^{-1}(\bar{\underline{x}} - \underline{\mu}_0)(C^{-1}(\bar{\underline{x}} - \mu_0))']$$
$$= C^{-1} E(\bar{\underline{x}} - \underline{\mu}_0)(\bar{\underline{x}} - \underline{\mu}_0)' C^{-1'}$$
$$= C^{-1} \Sigma^* C^{-1'} = (C' \Sigma^{*-1} C)^{-1} = I$$

'$A \rightarrow$ variance - covariance matrix.
(symmetric)

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Therefore, $\underline{Y} \sim N_p(0, I)$.

$\Rightarrow y_i \sim N(0,1) \quad \forall \; i = 1, 2 \ldots p$.

$n(\bar{\underline{x}} - \mu_0)' \Sigma^{-1} (\bar{\underline{x}} - \underline{\mu}_0)$.

$\Rightarrow n(Cy)' \Sigma^{-1} (Cy)$.

$\Rightarrow (Cy)' n \Sigma^{-1} Cy$.

$\Rightarrow y' C' \Sigma^{*-1} Cy$. $\qquad \Sigma^* = \dfrac{\Sigma}{n} \Rightarrow (\Sigma^*)^{-1} = n\Sigma^{-1}$

$\Rightarrow y'y$.

$\sum_{i=1}^{p} y_i^2 \sim \chi^2_p$.

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Testing of Mean vector $\underline{\mu}$ when variance covariance matrix Σ is known (Two sample problem)

Given $\underline{x}_1^{(1)}, \underline{x}_2^{(1)}, \ldots \underline{x}_\alpha^{(1)} \ldots \underline{x}_{n_1}^{(1)} \sim N_p(\underline{\mu}^{(1)}, \Sigma)$

and $\underline{x}_1^{(2)}, \underline{x}_2^{(2)}, \ldots \underline{x}_\alpha^{(2)} \ldots \underline{x}_{n_2}^{(2)} \sim N_p(\underline{\mu}^{(2)}, \Sigma)$

Let the hypothesis to be tested is;

$$H_0 : \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$$

Then the test statistic is

$$\frac{n_1 n_2}{n_1 + n_2} (\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)})' \Sigma^{-1} (\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}) \sim \chi_p^2$$

$$\bar{\underline{x}}^{(1)} \sim N_p\left(\underline{\mu}^{(1)}, \frac{\Sigma}{n_1}\right).$$

$$\bar{\underline{x}}^{(2)} \sim N_p\left(\underline{\mu}^{(2)}, \frac{\Sigma}{n_2}\right)$$

$$\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)} \sim N_p\left((\underline{\mu}^{(1)} - \underline{\mu}^{(2)}), \underbrace{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma}_{\Sigma^*}\right)$$

Since Σ^* is a (+)ve definite, symmetric matrix, we have a non-singular matrix C such that

$$C' \Sigma^{*-1} C = I$$

$$\Rightarrow \quad \Sigma^* = CC' = \frac{n_1 + n_2}{n_1 n_2} \Sigma$$

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Let us make a non-singular transformation

$$(\bar{\underline{x}}^{(1)} - \bar{\underline{x}}^{(2)}) = C\underline{y} \Rightarrow \underline{y} = C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)})$$

$$E(\underline{y}) = C^{-1}E(\bar{x}^{(1)} - \bar{x}^{(2)})$$
$$= \underline{0} \quad \text{under } H_0.$$

$$\Sigma_{\underline{y}} = E(\underline{y} - E(\underline{y}))(\underline{y} - E(\underline{y}))'$$
$$= E(C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)}))(C^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)}))'$$
$$= C^{-1}E(\bar{x}^{(1)} - \bar{x}^{(2)})(\bar{x}^{(1)} - \bar{x}^{(2)})'C^{-1'}$$

$$\Sigma_{\underline{y}} = C^{-1}\Sigma^{*}C^{-1'}$$
$$= (C'\Sigma^{*-1}C)^{-1}$$
$$= I^{-1} = I$$

$$\underline{y} \sim N_p(\underline{0}, I)$$

$$y_i \sim N(0,1) \quad \forall\, i = 1, 2, \ldots p$$

$$\cancel{n_1 + n_2} \quad \frac{n_1 n_2}{n_1 + n_2}(C\underline{y})'\Sigma^{-1}(C\underline{y})$$

$$= \underline{y}'C'\Sigma^{*-1}C\underline{y}$$

$$= \underline{y}'\underline{y}$$

$$= \sum_{i=1}^{p} y_i^2 \sim \chi^2_p$$

Let $\chi^2_p(\alpha)$ be such that $P(\chi^2_p \geq \chi^2_p(\alpha)) = \alpha$
Therefore, under H_0, we use the critical region as

$$\frac{n_1 n_2}{n_1 + n_2}(\bar{x}^{(1)} - \bar{x}^{(2)})'\Sigma^{-1}(\bar{x}^{(1)} - \bar{x}^{(2)}) \sim \chi^2_p$$

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Wishart Distribution

$$\sum_{i=1}^{n-1} z_i^2 \sim \chi^2_{n-1} \; ; \; z_i \sim N(0,1) \; \forall i.$$

$\cancel{(n-1)\, s^2 \sim}$

$\cancel{s^2}$

If $x_1, x_2 \ldots x_n$ are independent observations from a ~~Nor~~ $N(\mu, \sigma^2)$ distribution then

$$\frac{(n-1)\, s^2}{\sigma^2} \sim \chi^2_{n-1}$$

where $(n-1)\, s^2 = \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})^2$

The Multivariate Analog of $(n-1)s^2$ is matrix A and is called Wishart Matrix.

In other words, Wishart matrix is defined as the $p \times p$ symmetric matrix of sum of squares and cross product (of deviations about mean) of sample observations from a p-variate non-singular distribution.

The distribution of A, when the multivariate distribution is assumed Normal, is called Wishart distribution and a generalization of χ^2-distribution in univariate.

Theorem

Let $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_\alpha \sim N_p(\underline{\mu}, I)$ and

let $A = \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})' = \sum_{\alpha=1}^{n-1} \underline{z}_\alpha \underline{z}_\alpha'$,

where $z_\alpha \sim N_p(\underline{0}, I)$ and z_α's are independent. Then the ~~dist~~ density of A is defined as

$$f(A) = \frac{|A|^{(\nu-p-1)/2} \exp\left\{-\frac{1}{2} tr(A)\right\}}{2^{\nu p/2} \prod_{i=1}^{p} \Gamma\left(\frac{\nu-i+1}{2}\right) \pi^{p(p-1)/4}} \; ; \nu = n-1$$

This is the density of Wishart distribution for $\Sigma = I$ and is denoted by $W_p(\nu, I)$

Result :-

$$A = \sum_{\alpha=1} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'.$$

$$= \sum_{\alpha=1}^{n-1} Z_\alpha Z_\alpha' \quad : \cancel{\sim N_p(}\,.$$

where $Z_\alpha \sim N_p(\underline{0}, \Sigma)$.

Then the density of A is :

$$f(A) = \frac{|A|^{(\nu-p-1)/2} \exp\left\{-\frac{1}{2} tr(A)\Sigma^{-1}\right\}}{2^{\nu p/2} \prod_{i=1}^{p} \Gamma\left(\frac{\nu-i+1}{2}\right) \pi^{p(p-1)/4} |\Sigma|^{\nu/2}}$$

$$S = \frac{A}{n-1} = \frac{\sum_{\alpha=1}^{n-1} Z_\alpha Z_\alpha'}{n-1} \quad ; Z_\alpha \sim N_p(\underline{0}, \Sigma)$$

$$= \sum_{\alpha=1}^{n-1} Z_\alpha^* Z_\alpha^{*\prime} \quad ; Z_\alpha^* = \frac{Z_\alpha}{\sqrt{n-1}}$$

and $Z_\alpha^* \sim N_p\left(\underline{0}, \underbrace{\frac{\Sigma}{n-1}}_{\Sigma^*}\right)$

The density of $S = \frac{A}{n-1}$ is :

$$f(S) = \frac{|S|^{(\nu-p-1)/2} \exp\left\{-\frac{1}{2} tr(S)\Sigma^{*-1}\right\}}{2^{\nu p/2} \prod_{i=1}^{p} \Gamma\left(\frac{\nu-i+1}{2}\right) \pi^{p(p-1)/4} |\Sigma^*|^{\nu/2}}$$

$; \nu = n-1$

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Characteristic vector

$$\phi_{x}(t) = E\left(e^{i\underline{t}'\underline{x}}\right)$$

$$\underline{t}'\underline{x} = t_1 x_1 + t_2 x_2 + \dots + t_p x_p$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$$

$$A\Theta = \begin{bmatrix} A_{11}\theta_{11} + A_{12}\theta_{21} & A_{11}\theta_{12} + A_{12}\theta_{22} \\ A_{21}\theta_{11} + A_{22}\theta_{21} & A_{21}\theta_{12} + A_{22}\theta_{22} \end{bmatrix}$$

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$tr\, A\Theta = A_{11}\theta_{11} + A_{12}\theta_{12} + A_{21}\theta_{21} + A_{22}\theta_{22}$

$\underline{t}'\underline{x} = tr\, A\Theta$

$\phi_A(\Theta) = E\left[\exp\{\, i\, tr\, A\Theta\}\right]$

Theorem

If $z_1, z_2, \ldots z_{n-1}$ are independent $\sim N_p(\underline{0}, \Sigma)$.

The characteristic function of matrix A having components A_{ij} i.e., $A = \{A_{ij}\}$. is given by

$$\phi_A(\Theta) = |I - 2i\Theta\Sigma|^{-\nu/2} \quad ; \nu = n-1$$

$$\phi_A(\Theta) = E\left(e^{i\, tr A\Theta}\right)$$

$$= \int_{R_A} e^{i\, tr A\Theta} f(A)\, dA.$$

$$= \int e^{i\, tr A\Theta} \frac{|A|^{(\nu-p-1)/2} e^{-\frac{1}{2} tr A\Sigma^{-1}}\, dA}{2^{\nu p/2} \prod_{i=1}^{p} \Gamma(\nu-i+1)/2\ \pi^{p(p-1)/4}\ |\Sigma|^{\nu/2}}$$

let $k = 1\Big/ 2^{\nu p/2} \prod_{i=1}^{p} \Gamma(\nu-i+1)/2\ \pi^{p(p-1)/4}$

$$\phi_A(\Theta) = \frac{k}{|\Sigma|^{\nu/2}} \int |A|^{(\nu-p-1)/2} \exp\left\{-\frac{1}{2} tr(\Sigma^{-1} - 2i\Theta)A\right\} dA$$

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Now, let $\Sigma^{*^{-1}} = \Sigma^{-1} - 2i\Theta$

$$\phi_A(\Theta) = \frac{k}{|\Sigma|^{\nu/2}} \int |A|^{(\nu-p-1)/2} \exp\left\{\frac{-1}{2} tr\, \Sigma^{*^{-1}} A\right\} dA$$

from density,

$$\Rightarrow \frac{K \int |A|^{(\nu-p-1)/2} \exp\left\{\frac{-1}{2} tr\, \Sigma^{*^{-1}} A\right\} dA}{|\Sigma^*|^{\nu/2}} = 1$$

$$\Rightarrow \int |A|^{(\nu-p-1)/2} \exp\left\{\frac{-1}{2} tr\, \Sigma^{*^{-1}} A\right\} dA = \frac{|\Sigma^*|^{\nu/2}}{K}$$

Now, $\phi_A(\Theta) = \frac{K}{|\Sigma|^{\nu/2}} \int |A|^{(\nu-p-1)/2}$

$$\times \exp\left\{\frac{-1}{2} tr\, \Sigma^{*^{-1}} \cdot A\right\} dA.$$

$$\phi_A(\Theta) = \frac{\cancel{K}}{|\Sigma|^{\nu/2}} \frac{|\Sigma^*|^{\nu/2}}{\cancel{K}} = \frac{|\Sigma^*|^{\nu/2}}{|\Sigma|^{\nu/2}}$$

$$\Sigma^* = \frac{1}{\Sigma^{-1} - 2i\Theta}$$

$$\phi_A(\Theta) = \frac{1}{|\Sigma^{-1} - 2i\Theta|^{\nu/2} |\Sigma|^{\nu/2}} = \frac{1}{|I - 2i\Theta\Sigma|^{\nu/2}}$$

$$\boxed{\phi_A(\Theta) = |I - 2i\Theta\Sigma|^{-\nu/2}}$$

property of wishart distribution

① Suppose A_i ; $(i=1,2)$ are distributed independently according to $W_p(\nu_i, \Sigma)$, respectively, then the distribution of $A_1+A_2 \sim W_p(\nu_1+\nu_2, \Sigma)$.

Proof :→

we know that if A_1 & A_2 are independent,

$$\phi_{A_1+A_2}(\Theta) = \phi_{A_1}(\Theta)\ \phi_{A_2}(\Theta),$$

$$A_1 \sim W_p(\nu_1, \Sigma)$$

$$\phi_{A_1}(\Theta) = |I - 2i\Theta\Sigma|^{-\nu_1/2}$$

$$A_2 \sim W_p(\nu_2, \Sigma)$$

$$\phi_{A_2}(\Theta) = |I - 2i\Theta\Sigma|^{-\nu_2/2}$$

$$\phi_{A_1+A_2}(\Theta) = |I-2i\Theta\Sigma|^{-\nu_1/2}\ |I-2i\Theta\Sigma|^{-\nu_2/2}$$

$$\phi_{A_1+A_2}(\Theta) = |I-2i\Theta\Sigma|^{-(\nu_1+\nu_2)/2}$$

This implies $A_1+A_2 \sim W_p(\nu_1+\nu_2, \Sigma)$

$\left\{ \sum_{i=1}^{n-1} z_i^2 \sim \chi^2_{n-1} \right.$

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Theorem

If $A \sim W_p((n-1), \Sigma)$ then the distribution of $\underline{l}' A \underline{l} \sim (\underline{l}' \Sigma \underline{l}) \chi^2_{n-1}$ where l is $(p \times 1)$ known vector.

Proof :-

$$A = \sum_{\alpha=1}^{n-1} \underline{Z}_\alpha \underline{Z}_\alpha' \quad ; \quad Z_\alpha \sim N_p(\underline{0}, \Sigma)$$

$$\underline{l}' A \underline{l} = \sum_{\alpha=1}^{n-1} \underline{l}' \underline{Z}_\alpha \underline{Z}_\alpha' \underline{l}$$

$$= \sum_{\alpha=1}^{n-1} (\underline{l}' \underline{Z}_\alpha)(\underline{l}' \underline{Z}_\alpha)'$$

$$= \sum_{\alpha=1}^{n} (\underline{l}' Z_\alpha)(\underline{l}' Z_\alpha)'$$

$$= \sum_{\alpha=1}^{n} V_\alpha^2 \sim (\underline{l}' \Sigma \underline{l}) \chi^2_{n-1}$$

$$V_\alpha = \underline{l}' Z_\alpha \sim N(0, \underline{l}' \Sigma \underline{l})$$

Theorem

If $A \sim W_p(n-1, \Sigma)$ and if c is a positive constant then $CA \sim W_p(n-1, C\Sigma)$

Proof :- Given $A \sim W_p(n-1, \Sigma)$

then $\phi_A(\Theta) = |I - 2i\Theta\Sigma|^{-\nu/2}$

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Let's take the transformation $Y = CA$ where C is a +ve constant.

$$\phi_Y(\Theta) = \phi_{CA}(\Theta) = \phi_A(C\Theta)$$

$$= |I - 2iC\Theta\Sigma|^{-\nu/2}$$

$$= |I - 2i\Theta\Sigma^*|^{-\nu/2} \quad \{\Sigma^* = C\Sigma\}$$

$$\Rightarrow \quad Y = CA \sim W_p(\nu, \Sigma^*) = W_p(n-1, C\Sigma)$$

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Theorem

Let A & Σ be partitioned into q & $(p-q)$ rows & columns as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

If A is distributed according to $W_p(\nu, \Sigma)$ then $A_{11} \sim W_q(\nu, \Sigma_{11})$.

Proof:— $A = \sum_{\alpha=1}^{n-1} Z_\alpha Z_\alpha' \quad ; \quad Z_\alpha \sim N_p(0, \Sigma)$

$$Z_\alpha = \begin{array}{c} \begin{bmatrix} Z_\alpha^{(1)} \\ Z_\alpha^{(2)} \end{bmatrix} \end{array} \begin{array}{l} q \\ p-q \end{array}$$

$$Z_\alpha^{(1)} \sim N_q(0, \Sigma_{11})$$

$Z_\alpha^{(1)}$ & $Z_\alpha^{(2)}$ are independent $\because \Sigma_{12} = \Sigma_{21} = 0$

$$A_{11} = \sum_{\alpha=1}^{n-1} Z_\alpha^{(1)} Z_\alpha^{(1)'} \sim W_q(\nu, \Sigma_{11}).$$

Theorem

Let A & Σ be partitioned into q & $(p-q)$ rows & columns as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

If A is distributed according to $W_p(\nu, \Sigma)$, then $\underbrace{A_{11} - A_{12}A_{22}^{-1}A_{21}}_{A_{11\cdot2}}$ is distributed according to $W_q(\nu-(p-q), \Sigma_{11\cdot2})$

Generalized Variance

The Multivariate Analog of the variance σ^2 of a univariate distribution is the covariance matrix Σ and the determinant of this covariance matrix is termed as Generalized variance of the multivariate distribution.

Similarly, the generalized variance of a sample $x_1, x_2, \ldots x_n$ is defined as

$$|S| = \left| \frac{1}{n-1} \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' \right|$$

$$= \frac{|A|}{(n-1)^p} \; ; \text{ where } A = \sum_{\alpha=1}^{n-1} Z_\alpha Z_\alpha'$$

$$\& \; Z_\alpha \sim N_p(\underline{0}, \Sigma)$$

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Distribution of Gen var

Since Σ is a symmetric and positive definite matrix there exist a non-singular Matrix C such that $C\Sigma C' = I$

Make the transformation;

$$Z_\alpha^* = C Z_\alpha \Rightarrow Z_\alpha^* \sim N_p(\underline{0}, I).$$

$$A = \sum_{\alpha=1}^{n-1} Z_\alpha Z_\alpha'.$$

$$CAC' = \sum_{\alpha=1}^{n-1} (CZ_\alpha)(CZ_\alpha)'.$$

$$= \sum_{\alpha=1}^{n-1} Z_\alpha^* Z_\alpha^{*'} = B \sim W_p(\nu, I)$$

$$|CAC'| = |B|$$

$$|C|\,|A|\,|C'| = |B|$$

$$|A|\,|C^2| = |B|$$

$$\Rightarrow |A| = \frac{|B|}{|C^2|}.$$

$$\Rightarrow |A| = |B|.|\Sigma|.$$

But $|C\Sigma C'| = |\overset{I}{\Sigma}|$

$\Rightarrow |C^2| = \dfrac{1}{|\Sigma|}.$

$\Rightarrow \dfrac{1}{|C^2|} = |\Sigma|.$

$$\boxed{|A| = |A_{11.2}|\,|A_{22}|} \Rightarrow |A_{11.2}| = \frac{|A|}{|A_{22}|}$$

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Let $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pp} \end{bmatrix}$.

$$B = B_{11} = \begin{bmatrix} b_{11} & \underline{b}'_{(1)} \\ \underline{b}_{(1)} & B_{22} \end{bmatrix};$$

$$\underline{b}'_{(1)} = [\,b_{12} \;\; b_{13} \;\cdots\; b_{1p}\,]$$

$$B_{ii} = \begin{bmatrix} b_{ii} & \underline{b}'_{(i)} \\ \underline{b}_{(i)} & B_{i+1\,i+1} \end{bmatrix};$$

$$\underline{b}'_{(i)} = [\,b_{i\,i+1} \;\; b_{i\,i+2} \;\cdots\; b_{ip}\,]$$

$$B_{pp} = [\,b_{pp}\,] \qquad B_{11} = B, \quad B_{pp} = b_{pp}$$

$$\{ A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$b_{ii.\,i+1,\ldots,p} = b_{ii} - \underline{b}'_{(i)} B^{-1}_{i+1\,i+1} \underline{b}_{(i)}.$$

$$= \frac{|B_{ii}|}{|B_{i+1\,i+1}|}$$

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$$|B| = \frac{|B_{11}|}{|B_{22}|}\frac{|B_{22}|}{|B_{33}|} \cdots \frac{|B_{p-1\,p-1}|}{|B_{pp}|}|B_{pp}|$$

$$= (b_{11\cdot 2\ldots p})(b_{22\cdot 3\ldots p})\ldots(b_{p-1\,p-1\cdot p})\, b_{pp}.$$

$$\chi^2_{\nu-(p-1)} \quad \chi^2_{\nu-(p-2)} \cdots \chi^2_{\nu-1} \quad \chi^2_{\nu}$$

$\Big\{ b_{11\cdot 2\ldots p} \sim \cancel{\Sigma_{\nu-(p-1)}}\ w_1(\nu-(p-1), \Sigma_{11\cdot 2\ldots p})$

which is same as $\chi^2_{\nu-(p-1)}$

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$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \Rightarrow A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$

$\Rightarrow |A_{11\cdot 2}| = \dfrac{|A|}{|A_{22}|}$

Similarly,

$B_{ii} = \begin{bmatrix} b_{ii} & \underline{b}'_{(i)} \\ \underline{b}_{(i)} & B_{i+1\,i+1} \end{bmatrix} \Rightarrow$

$$\Rightarrow b_{ii\cdot i+1\, i+2\ldots p} = b_{ii} - \underline{b}'_{(i)} B^{-1}_{i+1\,i+1}\underline{b}_{(i)}$$

$$= \frac{|B_{ii}|}{|B_{i+1\,i+1}|}$$

$\underline{x}_1^{(1)}, \underline{x}_2^{(1)}, \ldots \underline{x}_\alpha^{(1)} \ldots \underline{x}_n^{(1)} \sim N_p(\underline{\mu}^{(1)}, \Sigma_1)$

$\underline{x}_\alpha^{(i)} \sim N_p(\underline{\mu}^{(i)}, \Sigma_i)$

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and $n = n_1 + n_2 + \ldots + n_q$ and $A = A_1 + A_2 + \ldots + A_q$

Testing equality of Covariance Matrices

Let $\underline{x}_\alpha^{(i)}$; $\alpha = 1, 2, \ldots, n_i$ ~~and i=1,2..q~~ be observations from $N_p(\underline{\mu}^{(i)}, \Sigma_i)$ where $i = 1, 2 \ldots q$.

and

The hypothesis to be tested is

$$H_0: \Sigma_1 = \Sigma_2 = \ldots = \Sigma_q = \Sigma \text{ (say)}$$

vs $H_A: \Sigma_i \neq \Sigma_j$ for ~~any i, j~~ at least one $i \neq j$.

The likelihood function is defined as

$$\cancel{L = \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i/2}}}$$

$$L = \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i p/2} |\Sigma_i|^{n_i/2}} \exp\left\{-\frac{1}{2} \sum_{\alpha=1}^{n_i} (\underline{x}_\alpha^{(i)} - \overline{\underline{x}}^{(i)})' \Sigma_i^{-1} (\underline{x}_\alpha^{(i)} - \overline{\underline{x}}^{(i)})\right\}$$

and the likelihood criteria is

$$\lambda = \frac{\max\limits_{\text{under } H_0} L}{\max\limits_{\text{under } \Omega} L}$$

V.V.I

$$\underset{1\times p}{X'}\underset{p\times p}{A}\underset{p\times 1}{X} = tr\, A X'X.$$

$$= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots \lambda_p x_p^2$$

where $\lambda_1, \lambda_2, \dots \lambda_p$ are latent roots.

$$\max_{\Omega} L = \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i p/2} \left|\frac{A_i}{n_i}\right|^{n_i/2}} \times \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{n_i} \left(x_\alpha^{(i)} - \bar{x}^{(i)}\right)' \left(\frac{A_i}{n_i}\right)^{-1} \left(x_\alpha^{(i)} - \bar{x}^{(i)}\right)\right]$$

$$= \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i p/2} \left|\frac{A_i}{n_i}\right|^{n_i/2}} \times \exp\left[-\frac{1}{2} tr \left(\frac{A_i}{n_i}\right)^{-1} \sum_{\alpha=1}^{n_i} \left(x_\alpha^{(i)} - x^{(i)}\right)\left(x_\alpha^{(i)} - \bar{x}^{(i)}\right)'\right]$$

$$= \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i p/2} \left|\frac{A_i}{n_i}\right|^{n_i/2}} \times \exp\left[-\frac{1}{2} tr(n_i I)\right].$$

$$= \prod_{i=1}^{q} \frac{1}{(2\pi)^{n_i p/2} \left|\frac{A_i}{n_i}\right|^{n_i/2}} \times \exp\left[-\frac{1}{2} n_i p\right].$$

=

under H_0 $\max_{H_0} L = \prod_{i=1}$

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Some Important Results

$X_1^* = E(X_1 / X_2 = x_2) = a + bx_2$

$\int x_1 f(x_1 / x_2)\, dx_1 = a + bx_2$ — (1)

Multiply by $f(x_2)$ and integrate w.r.t. x_2,

$\int\int x_1\, f(x_1 / x_2)\, f(x_2)\, dx_1\, dx_2$
$= a \int f(x_2)\, dx_2 + b \int x_2\, f(x_2)\, dx_2$

$\Rightarrow \int\int x_1\, f(x_1, x_2)\, dx_1\, dx_2$
$= a + b\mu_2$

$\Rightarrow \int x_1 \left[\int_{x_2} f(x_1, x_2)\, dx_2\right] dx_1 = a + b\mu_2$

$\Rightarrow \int x_1\, f(x_1)\, dx_1 = a + b\mu_2$

Multiplying (1) by $x_2 f(x_2)$ and integrate w.r.to x_2,

$\int\int x_1 x_2\, f(x_1 / x_2)\, f(x_2)\, dx_1\, dx_2$
$= a \int x_2 f(x_2)\, dx_2 + b \int x_2^2 f(x_2)\, dx_2$

$E(X_1 X_2) = a\mu_2 + bE(X_2^2)$

$\{$ since $\mathrm{cov}(X_1, X_2) = E(X_1 X_2) - E(X_1) E(X_2)$
$\Rightarrow E(X_1 X_2) = \mathrm{cov}(X_1 X_2) - E(X_1) E(X_2)$
$\Rightarrow E(X_1 X_2) = \sigma_{12} + \mu_1 \mu_2 \}$

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$$\boxed{\sigma_{12} + \mu_1 \mu_2 = a\mu_2 + b(\sigma_2^2 + \mu_2^2)} \quad \text{--- (3)}$$

$$b = \frac{\sigma_{12}}{\sigma_2^2} = \frac{\rho\sigma_1\sigma_2}{\sigma_2^2} = \frac{\rho\sigma_1}{\sigma_2}$$

by eqn (1). $\boxed{a = \mu_1 - \frac{\rho\sigma_1}{\sigma_2}\mu_2}$

$$X_1^* = \mu_1 - \frac{\rho\sigma_1}{\sigma_2}\mu_2 + \frac{\rho\sigma_1}{\sigma_2}X_2$$

$$\boxed{X_1^* = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(X_2 - \mu_2)}$$

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Multiple Correlation Coefficient

If $X_1^{(1)}$ is the first component of vector X and $X_2^{(2)}$ the vector of remaining $(p-1)$ components.
we express $X^{(1)}$ as a linear combination of vector $X^{(2)}$ defined by the relation

$$X_1^* = \mu_1 + \underline{\beta}'(\underline{X}^{(2)} - \underline{\mu}^{(2)})$$

{we need to estimate $\underline{\beta}'$ such that the difference between X_1 & X_1^* is minimum.}

$$U = E(X_1 - X_1^*)^2$$
$$= E[X_1 - \mu_1 - \underline{\beta}'(\underline{X}^{(2)} - \underline{\mu}^{(2)})]^2$$

$$\frac{\partial U}{\partial \underline{\beta}} = 0.$$

$$\Rightarrow -2E[(\underline{X}^{(2)} - \underline{\mu}^{(2)})(X_1 - \mu_1 - \underbrace{\underline{\beta}'}_{1\times p-1}\underbrace{(X^{(2)} - \underline{\mu}^{(2)})}_{p-1\times 1})] = 0$$

$$\Rightarrow E[(\underline{X}^{(2)} - \underline{\mu}^{(2)})(X_1 - \mu_1)] - E[(\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'\underline{\beta}'] = 0$$

$$\Rightarrow \underline{\sigma}_{12}' - \Sigma_{22}\underline{\beta}' = 0$$

$$\Rightarrow \underline{\sigma}_{12}' = \Sigma_{22}\underline{\beta}'$$

$$\Rightarrow \hat{\underline{\beta}}' = \Sigma_{22}^{-1}\underline{\sigma}_{12}' \quad \Rightarrow \boxed{\hat{\underline{\beta}}' = \sigma_{12}'\Sigma_{22}^{-1}}$$

$$\Rightarrow \hat{X}_1 = X_1^* = \mu_1 + \underline{\sigma}_{12}'\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)})$$

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The correlation coefficient between X_1 and it's best linear estimate in terms of X_2 is called multiple correlation coefficient between X_1 and $X_2, \ldots, X_p$ which is denoted by $\rho_{1(2,\ldots,p)}$,

$$\rho_{1(2,\ldots,p)} = \frac{\text{cov}(X_1, \hat{X}_1)}{\sqrt{V(X_1)\, V(\hat{X}_1)}} \quad ; \; V(X_1) = \sigma_{11}$$

$$V(\hat{X}_1) = E(\hat{X}_1 - E(\hat{X}_1))^2$$
$$= E\left(\mu_1 + \underline{\sigma}_{12}' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)}) - \mu_1\right)^2$$
$$= E\left(\underline{\sigma}_{12}' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})\right)^2$$
$$= E\left(\underline{\sigma}_{12}' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})\right)\left(\underline{\sigma}_{12}' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})\right)'$$
$$= E\left(\underline{\sigma}_{12}' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})' \Sigma_{22}^{-1} \underline{\sigma}_{12}\right)$$
$$= \underline{\sigma}_{12}' \Sigma_{22}^{-1} \Sigma_{22} \Sigma_{22}^{-1} \underline{\sigma}_{12} \qquad \left\{ E\left((\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})'\right) = \Sigma_{22} \right\}$$

$$\Rightarrow \boxed{V(\hat{X}_1) = \underline{\sigma}_{12}' \Sigma_{22}^{-1} \underline{\sigma}_{12}}$$

$$\text{cov}(X_1, \hat{X}_1) = E(X_1 - E(X_1)) \cdot (\hat{X}_1 - E(\hat{X}_1))'$$
$$= E(X_1 - \mu_1)\left(\underline{\sigma}_{12} \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)})\right)'$$
$$= E(X_1 - \mu_1)(\underline{X}^{(2)} - \underline{\mu}^{(2)})' \Sigma_{22}^{-1} \underline{\sigma}_{12}$$

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$$\Sigma = \left[\begin{array}{c|ccc} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \hline \sigma_{21} & & & \\ \vdots & & & \\ \sigma_{p1} & \cdots & & \sigma_{pp} \end{array}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12}' \\ \sigma_{12} & \Sigma_{22} \end{bmatrix}$$

$$Cov(X_1, \hat{X}_1) = \sigma_{12}' \Sigma_{22}^{-1} \sigma_{12}$$

Now, $$\rho_{1(2,\dots p)} = \frac{\sigma_{12}' \Sigma_{22}^{-1} \sigma_{12}}{\sqrt{\sigma_{11} \sigma_{12}' \Sigma_{22}^{-1} \sigma_{12}}}$$

$$= \sqrt{\frac{\sigma_{12}' \Sigma_{22}^{-1} \sigma_{12}}{\sigma_{11}}}$$

$$= \sqrt{\frac{\sigma_{12}' \Sigma_{22}^{-1} \Sigma_{22} \Sigma_{22}^{-1} \sigma_{12}}{\sigma_{11}}}$$

$$\boxed{\rho_{1(2\dots p)} = \sqrt{\frac{\hat{\beta}' \Sigma_{22} \hat{\beta}}{\sigma_{11}}}}$$

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Estimation of Sample Multiple Correlation Coefficient

Let us suppose x_α $(\alpha = 1, 2, \ldots, n)$; $n > p$.

$$\hat{\Sigma} = \frac{A}{n} = \frac{(n-1)S}{n} , \quad A = \sum_\alpha (x_\alpha - \bar{x})(x_\alpha - \bar{x})'.$$

$$\frac{A}{n} = \begin{bmatrix} \frac{a_{11}}{n} & \frac{a_{12}'}{n} \\ \frac{a_{12}}{n} & \frac{A_{22}}{n} \end{bmatrix}$$

$$\hat{\sigma}_{11} = \frac{a_{11}}{n} \qquad \hat{\sigma}_{12}' = \frac{a_{12}'}{n}. \qquad \hat{\Sigma}_{22} = \frac{A_{22}}{n},$$

$$\hat{\sigma}_{12} = \frac{a_{12}}{n},$$

$$R_{1(2,\ldots p)} = \sqrt{\frac{a_{12}' A_{22}^{-1} a_{12}}{a_{11}}}$$

$$1 - R^2_{1(2,\ldots p)} = \frac{a_{11} - a_{12}' A_{22}^{-1} a_{12}}{a_{11}}$$

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Distribution of Sample Multiple Correlation Coefficient in null case ($\rho = 0$).

$$R^2 = \frac{a_{12}' A_{22}^{-1} a_{12}}{a_{11}}$$

$$1 - R^2 = \frac{a_{11} - a_{12}' A_{22}^{-1} a_{12}}{a_{11}}$$

$$\frac{R^2}{1-R^2} = \frac{a_{12}' A_{22}^{-1} a_{12}}{a_{11} - a_{12}' A_{22}^{-1} a_{12}}$$

$$A = \begin{array}{cc} \begin{bmatrix} A_{11} & A_{12}' \\ A_{21} & A_{22} \end{bmatrix} & \begin{array}{l} q \\ p-q \end{array} \end{array}$$

$A \sim W_p(\nu, \Sigma)$

$A_{11} \sim W_q(\nu, \Sigma_{11})$

$A_{11.2} \sim W_q(n-1-(p-q), \Sigma_{11.2})$,

under null case

$a_{11.2} \sim W_1(n-1-(p-1), \sigma_{11})$

$$\frac{a_{11.2}}{\sigma_{11}} \sim \chi^2_{(n-p)}$$

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$\frac{a_{11}}{\sigma_{11}}$ # $\frac{a_{11}}{\sigma_{11}} = \frac{a_{11} - a_{12}' A_{22}^{-1} a_{12}}{\sigma_{11}} + \frac{a_{12}' A_{22}^{-1} a_{12}}{\sigma_{11}}$

$Q = Q_1 + Q_2$

$Q \sim \chi^2_{n-1}$

By Fisher - Cochran's ~~dist~~ Theorem,

$Q_1 \sim \chi^2_{n-p}$

using additive property of χ^2,

$Q_2 \sim \chi^2_{(n-1-(n-p))} \Rightarrow Q_2 \sim \chi^2_{p-1}$

~~Ire the~~ $F = \frac{R^2}{1-R^2} \times \frac{n-p}{p-1} \sim F_{p-1, n-p}$.

Hence, the distribution of F is:

$$df(F) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} F^{\nu_1/2 - 1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left(1 + \frac{\nu_1}{\nu_2} F\right)^{\frac{\nu_1+\nu_2}{2}}} dF.$$

where $\nu_1 = p-1$, $\nu_2 = n-p$, in our case

substituting $F = \frac{R^2}{1-R^2} \cdot \frac{n-p}{p-1}$;

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$$\frac{df}{\phi} = \frac{dR^2(1-R^2) - R^2(\phi-1)\,dR^2}{\phi(1-R^2)^2} \cdot \frac{v_2}{v_1}$$

$$df = \frac{(1-R^2+R^2)\,\phi\, dR^2}{(1-R^2)^2}\,\frac{v_2}{v_1}$$

$$df = \frac{dR^2}{(1-R^2)^2}\,\frac{v_2}{v_1}$$

$$df(R^2) = \frac{\left(\frac{v_1}{v_2}\right)^{v_1/2}\left[\frac{R^2}{1-R^2}\,\frac{v_2}{v_1}\right]^{\frac{v_1}{2}-1}}{B\left(\frac{v_1}{2},\frac{v_2}{2}\right)\left(1+\frac{R^2}{1-R^2}\right)^{\frac{v_1+v_2}{2}}}\,\frac{v_2\,dR^2}{v_1(1-R^2)^2}$$

$$df(R^2) = \frac{\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}-\frac{v_1}{2}+1-1}(R^2)^{\frac{v_1}{2}-1}(1-R^2)^{-\frac{v_1}{2}+1-2+\frac{v_1}{2}+\frac{v_2}{2}}}{B\left(\frac{v_1}{2},\frac{v_2}{2}\right)}$$

$$= \frac{(R^2)^{\frac{v_1}{2}-1}(1-R^2)^{\frac{v_2}{2}-1}\,dR^2}{B\left(\frac{v_1}{2},\frac{v_2}{2}\right)}$$

$$df(R) = \frac{(R)^{v_1-2}(1-R^2)^{\frac{v_2}{2}-1}\,2R\,dR}{B\left(\frac{v_1}{2},\frac{v_2}{2}\right)}$$

$dR^2 = 2R\,dR$.

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$$df(R) = \frac{2R^{\nu_1 - 1} \cdot (1-R^2)^{\frac{\nu_2}{2} - 1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} dR, \quad 0<R<1.$$

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Discriminant Analysis

The problem of Discriminant Analysis deals with assigning an individual to one of several categories on the basis of measurements on a p component vector of variable $\underline{x}$ on that individual.

For example, a patient is to be classified as diabetic or non-diabetic on the basis of certain test such as blood, urine, blood pressure etc, a salesmen is to be classified as successful or unsuccessful based on different psychological test.

procedure of classification into one of the two populations with known probability distributions.

Let R denote the entire p-dimensional space in which the point of observation $\underline{x}$ falls.

~~We~~ We then have to divide R

into two regions R_1 and R_2 so that

→ if $\underline{x}$ falls in R_1, assign the individual to population π_1.

→ if $\underline{x}$ falls in R_2, assign the individual to population π_2.

$$R \quad \boxed{R_1 \mid R_2}$$

$$\underline{x} \qquad R_1 \downarrow \pi_1 \qquad R_2 \downarrow \pi_2$$

obviously with any such procedure an error of misclassification ~~in~~ is inevitable i.e., the rule may assign an individual to π_2 when it really belongs to π_1 and vice versa.

There should be a rule which controls this error of discrimination.

Let $f_1(\underline{x})$ & $f_2(\underline{x})$ be the pdfs of $\underline{x}$ in two populations π_1 & π_2.

Let q_1 = a priori probability that an individual comes from π_1. and

q_2 = a priori probability that an individual comes from π_2.

$P(1/2)$ = Pr(an individual comes from π_2 and is misclassified to π_1)

$$= \int_{R_1} f_2(\underline{x})\, d\underline{x}$$

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Similarly,

$P(2/1)$ = pr (an individual comes from π_1 and is misclassified to π_2)

$$= \int_{R_2} f_1(\underline{x})\, d\underline{x}$$

Now, since the probability of drawing an observation from π_1 is q_1 and from π_2 is q_2, we have probability,

P(drawing an obsⁿ from π_1 and misclassified it as from π_2) =

$= q_1 P(2/1)$.

and similarly,

P(drawing an obsⁿ from π_2 and misclassified it as from π_1) =

$= q_2 P(1/2)$.

Total chances of misclassification say ϕ,

$$\phi = q_1 P(2/1) + q_2 P(1/2)$$

{our objective is to minimize ϕ in any situation}.

$$\Rightarrow \phi = q_1 \int_{R_1} f_2(\underline{x})\, d\underline{x} + q_2 \int_{R_2} f_1(\underline{x})\, d\underline{x}$$

$$\phi = q_1 \int_{R_1} f_2(\underline{x})\, d\underline{x} + q_1 \int_{R_1} f_1(\underline{x})\, d\underline{x}$$

$$- q_1 \int_{R_1} f_1(\underline{x})\, d\underline{x} + q_2 \int_{R_2} f_1(\underline{x})\, d\underline{x}$$

$$\phi = q_1 \int_{R} f_1(\underline{x})\, d\underline{x} + q_2 \int_{R_1} f_1(\underline{x})\, d\underline{x}$$

$$- q_1 \int_{R_1} f_1(\underline{x})\, d\underline{x}$$

$$\Rightarrow \phi = q_1 \int_{R_1} f_1(\underline{x})\, d\underline{x} + \int_{R_1} \underbrace{\left[q_2 f_1(\underline{x}) - q_1 f_1(\underline{x})\right]}\, d\underline{x}$$

(minimum when this becomes negative.)

So, ϕ is minimized when $q_2 f_2(\underline{x}) \le q_1 f_1(\underline{x})$

So, we divide R such that:

$$R_1 = \{ \underline{x} \mid q_2 f_2(\underline{x}) \le q_1 f_1(\underline{x}) \}$$

$$R_1 = \left\{ \underline{x} \mid \frac{f_1(\underline{x})}{f_2(\underline{x})} \ge \frac{q_2}{q_1} \right\}.$$

$$R_2 = \left\{ \underline{x} \mid \frac{f_1(\underline{x})}{f_2(\underline{x})} < \frac{q_2}{q_1} \right\}.$$

Further, if the cost of misclassification is given say, $C(2|1)$ be the cost be misclassification to π_2 when it actually comes from π_1 and $C(1|2)$ be the cost of misclassification to π_1 when it actually comes from π_2.

The total expected cost from misclassification will be

$$\boxed{E(c) = q_1 C(2|1) P(2|1) + q_2 P(1|2) C(1|2)}$$

The classification