Multivariate

Discrete

Let, $x_1, x_2, \ldots x_n$ be random sample from n observations & $P(x_1), P(x_2) \ldots P(x_n)$ be the prob. associated with these random variables. Then, PMF is defined as,

$$\sum_{i=1}^{n} P(X=x_i) = 1 \quad ; \quad P(X=x_i) \geq 0$$

Continuous

Let $X : (b, a)$, $b < a$ is a random variable contains infinite values s.t.

$$f_X(x) = \lim_{h \to 0} \frac{(x-h < X < x+h)}{h}$$

$$\Rightarrow \int_a^b f_X(x) = 1$$

$\underline{X}$ - A r. v. which contain a number of random variable.
= vector X.

For Vector $\underline{X}$ - $\underline{x} : (b, a)$, $b < a$

Let, $X_1 = \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}_{p\times 1}$ $X_2 = \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}_{p\times 1}$ $\ldots$ $X_n = \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}_{p\times 1}$

$$f_{\underline{X}}(\underline{x}) = \lim_{h \to 0} \frac{P(x_1 - h < X_1 < x_1 + h,\ x_2 - h < X_2 < x_2 + h, \ldots + x_p - h < X_p < x_p + h)}{h}$$

Discrete

$\underline{X} : \underset{X_1}{(\ldots)} \underset{X_2}{(\ldots)} \ldots \underset{X_p}{(\ldots)}$

$$F_X(x) = \sum_{i < x} P(X = i)$$

$$F_{\underline{X}}(\underline{x}) = \sum_{i_1 < x_1} \sum_{i_2 < x_3} \cdots \sum_{i_p < x_p} P(X_1 = i_1, X_2 = i_2, \ldots X_p = i_p)$$

Continuous

$\underline{X}_{p\times 1} \in (\quad)_{p\times 1}$

$$F_X(x) = \int P(X \leq x)$$

$$F_{\underline{X}}(\underline{x}) = \iint \cdots \int$$

Univariate to multivariate -

$$\frac{d}{dx} F_X(x)$$

$$\frac{\partial}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(x)$$

Marginal -

$$f_{x_i}(x_i) = \int_{x_1} \cdots \int_{x_p} f_{\underline{X}}(\underline{x})\, dx_1, \cdots dx_p$$

except x_i

Marginal

$$f_{x_i, x_j}(x_i, x_j) = \sum_{x_1} \cdots \sum_{x_p} f_{\underline{X}}(\underline{x_i})$$

except x_i

Conditional

$$p(x_1, x_2 \ldots x_p \mid x_i, x_j) = \frac{\text{Joint}}{\text{marginal}}$$

$$\phi_x(t) = E(e^{itx})$$

$$= \sum_x e^{itx} p(X = x_i)$$

$$\phi_x(t) = E(e^{itx})$$

$$= \int_R e^{itx} f_x(x)\, dx$$

$$\phi_{\underline{X}}(\underline{t}) = E(e^{i\underline{t}'\underline{x}})$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p1} & \sigma_{2p} & \cdots & \sigma_p^2 \end{bmatrix}$$

→ Variance is positive definite

→ Variance is symmetric $(A = A^T)$

$$\sigma_{12} = \sigma_{21}$$

$$\sigma_1^2 = \sum_{i=1}^{n} (x_{i1} - \cancel{\mu_1})^2$$

$$\sigma_{12} = \cancel{\sum_{i \neq j} \sum (x_{i1} - \mu_1)(x_{j2} - \mu_2)} = \frac{1}{n} \sum_{i=1}^{n} (x_{i1} - \mu_1)(x_{i2} - \mu_2)$$

→ Expectation is used in R.V. but summation not.

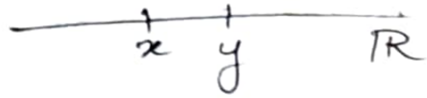
$$\text{cov}(x,y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

Distribution Function –

$$F_X(x) = P(X \le x) \quad \forall\, x \in \mathbb{R}$$

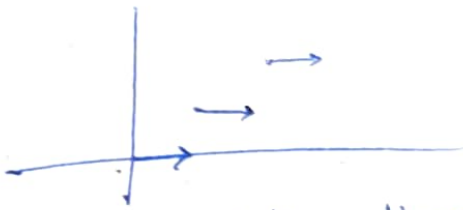
1. – Monotonically non decreasing function

$$F_X(x) < F_X(y) \quad \forall\, x \le y$$



2. $$F_X(-\infty) = \lim_{x \to -\infty} F_X(x) = 0$$

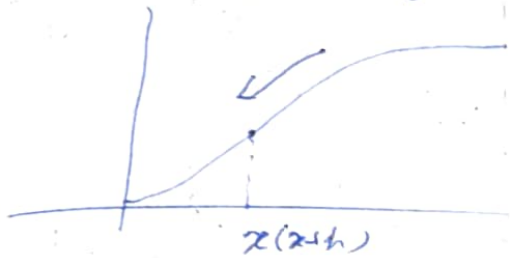
$$F_X(\infty) = \lim_{x \to \infty} F_X(x) = 1$$



3. It is right continuity.

$$\lim_{h \to 0^+} F_X(x+h) = F_X(x)$$

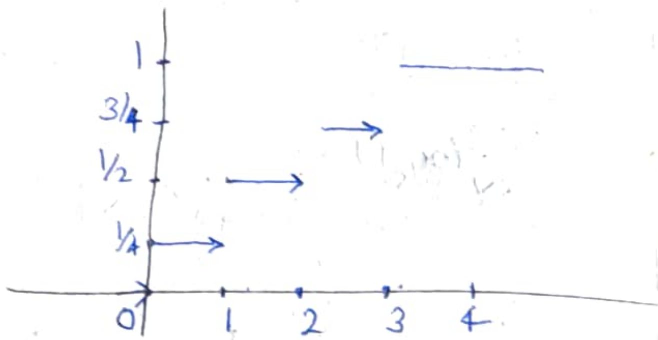
right continuous in case of discrt



$$P(a < x \le b) = F(b) - F(a)$$
$$= P(X \le b) - P(X \le a)$$

E.g. –

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 1/2, & 1 \le x < 2 \\ 3/4, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$



Note

In case of Bivariate – Right continuity

$$\lim_{h \to 0^+} F_{XY}(x+h, y) = \lim_{h \to 0^+} F_{XY}(x, y+h)$$

$$N(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{— Univariate Normal Distn}$$

$$x, \mu \in \mathbb{R}, \ \sigma^2 > 0$$

Bivariate Normal Distribution-

$$f_{XY}(x,y) = \frac{1}{(2\pi)^{2/2}\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right]$$

$$, x, y, \mu_1, \mu_2 \in \mathbb{R}, \ \sigma_1^2\sigma_2^2 > 0 \quad \text{—— (1)}$$
$$-1 < \rho < 1$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$
$$= \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1 \end{bmatrix}$$

from (1)

$$f_{XY}(x,y) = \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}\begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}'_{1\times 2} \Sigma^{-1}_{2\times 2} \begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}_{2\times 1}\right]$$

$$= \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

$$= K \exp\left[-\frac{1}{2}(\underline{x}-\underline{b})'A(\underline{x}-\underline{b})\right]$$

where K is positive const. & A is positive definite & symmetric. since $\alpha' A \alpha \geq 0 \Rightarrow (x-b)'A(x-b) \geq 0$

$f(x) \geq 0$, $f(x)$ is bounded above.

$$CX \to Y$$

C is non singular if $|C| \neq 0$

Since A is positive definite there exists a non singular matric C s.t. $C'AC = I$.

Let us make a non singular transformation

$$\underline{X} - \underline{b} = C\underline{Y}$$

Since it is a linear transformation, so introducing

Jacobian $|J| = \frac{\partial \underline{X}}{\partial \underline{Y}} = ||C|| = |C|$, where $|C|$ denote the absolute value of C.

$$g_Y(\underline{y}) = K \exp\left[-\frac{1}{2} (C\underline{Y})' A (C\underline{Y})\right] |C|$$

$$= K \exp\left[-\frac{1}{2} (\underline{Y}'C'AC\underline{Y})\right] |C|$$

$$= K \exp\left[-\frac{1}{2} (\underline{Y}'I\underline{Y})\right] |C|$$

$$= K \exp\left[-\frac{1}{2} \underline{Y}'_{1\times 1} Y_{1\times 1}\right] |C|$$

$$= K \exp\left[-\frac{1}{2} (y_1^2 + y_2^2 + \cdots + y_p^2)\right] |C|$$

Since $g_{\underline{Y}}(\underline{y})$ is the pdf, we must

$$\int\int \cdots \int g_{\underline{Y}}(y)\, dy_1\, dy_2 \cdots dy_p = 1$$

$$\Rightarrow K|C| \int\int \cdots \int e^{-\frac{1}{2}(y_1^2 + y_2^2 + \cdots y_p^2)}\, dy_1 \cdots dy_p = 1$$

$$\Rightarrow K|C| \prod_{i=1}^{p} \int e^{-\frac{y_i^2}{2}}\, dy_i = 1$$

$$\Rightarrow K|C| \prod_{i=1}^{p} (2\pi)^{1/2} = 1$$

$$\Rightarrow K = \frac{1}{|C|(2\pi)^{p/2}}$$

$\left\{ \because N = \frac{1}{(2\pi)^{1/2}\sigma^2} \exp\left(\cancel{\quad} \frac{y_i^2}{2}\right) = 1 \right.$

$$f_{\underline{X}}(\underline{x}) = K \exp\left[-\frac{1}{2}(\underline{x}-\underline{b})' A (\underline{x}-\underline{b})\right]$$

$$g_{\underline{Y}}(\underline{y}) = K \exp\left[-\frac{1}{2}\, Y'_{1\times 1} Y_{1\times 1}\right] |C|$$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{1}{2}(y_1^2 + y_2^2 + \cdots y_p^2)\right)$$

$$= \prod_{i=1}^{p} \frac{1}{(2\pi)^{1/2}} e^{-y_i^2/2}$$

$$g_{\underline{Y}}(\underline{y}) = g(y_1) \cdot g(y_2) \cdots g(y_p)$$

$$Y \sim N_p(\underline{0}, I)$$

$$E(\underline{Y}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{p\times p}$$

$$E(\underline{Y}) = 0$$

$$E(\underline{X} - \underline{b}) = 0$$

$$E(\underline{X}) = \underline{b} = \mu_X$$

$$\Sigma_{\underline{Y}} = E(\underline{Y} - E(\underline{Y}))(\underline{Y} - E(\underline{Y}))'$$

$$= E(\underline{Y}\underline{Y}')$$

$$= E\left[C^{-1}(\underline{X}-b)(\underline{X}-\underline{b})'(C^{-1})'\right]$$

$$I = C^{-1}\Sigma_X (C^{-1})'$$

$$C I C' = C C^{-1} \Sigma_X C^{-1'} C'$$

$$\boxed{CC' = \Sigma_X}$$

$$c'AC = I$$

$$\Rightarrow (C^{-1})'C'ACC^{-1} = (C^{-1})'IC^{-1}$$

$$\Rightarrow A = (C^{-1})'C^{-1}$$

$$\Rightarrow A = (CC')^{-1}$$

$$\Rightarrow A = \Sigma_X^{-1}$$

$$ACC' = \Sigma_X \Sigma_X^{-1}$$

$$\boxed{\cancel{ACC' = I}}$$

$$\cancel{C'AC = I} \quad c'AC = I$$

$$\Rightarrow |c'AC| = |I|$$

$$\Rightarrow \cancel{|c'||AC|}$$

$$\Rightarrow |c'||AC| = 1$$

$$\Rightarrow |c'||c||A| = 1$$

$$\Rightarrow |c|^2 = \frac{1}{|A|}$$

$$\Rightarrow |A| = \frac{1}{|c|^2}$$

$$\boxed{|c| = |A|^{-1/2}}$$

$$\Rightarrow |c| = |\Sigma_X^{-1}|^{-1/2}$$

$$\boxed{|c| = |\Sigma_X|^{1/2}}$$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma_X|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu_x})'\Sigma_x^{-1}(\underline{x}-\mu_x)\right] \quad (1)$$

A random vector $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}_{p\times 1}$ taking values $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1}$

in ~~eclidian~~ euclidean space of dimension p is said to have a p variate normal distribution if its pdf can be written as eqn (1) where $\underline{\mu}_X = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}_{p\times 1} \in R^p$ &

Σ_X is a positive definite symmetric of order p.

Properties of Multivariate Normal Distⁿ -

Theorem 1 - If the variance - covariance matrix of a p ~~variate~~ normal random vector $\underline{x}_{p\times 1}$ is a diagonal matrix then the components of x are independently normally distributed random variables.

$$\Sigma_X = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_p^2 \end{bmatrix}$$

Imp

Theorem 2 - If $\underline{X} \sim N_p(\underline{\mu}_X, \Sigma)$ then $\underline{Y} = C\underline{X}$ (C being non singular) is distributed as $N_p(C\underline{\mu}_X, C\Sigma C')$

Proof -

$$\underline{Y} = C\underline{X}$$

$$\underline{X} = C^{-1}\underline{Y}$$

$$|J| = |C^{-1}| = \left|\frac{\partial Y}{\partial \underline{X}}\right|$$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma_X|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x} - \underline{\mu}_X)' \Sigma_X^{-1}(\underline{x} - \underline{\mu}_x)\right] \quad \text{—(1)}$$

$$g_{\underline{y}}(\underline{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma_X|^{1/2}} \exp\left[-\frac{1}{2}(C^{-1}\underline{Y} - \mu_X)' \Sigma_X^{-1}(C^{-1}\underline{Y} - \underline{\mu}_x)\right]|C^{-1}|$$

$$= K \exp\left[-\frac{1}{2}(\underline{Y} - C\underline{\mu}_X)' C^{-1'} \Sigma_X^{-1} C^{-1}(\underline{Y} - C\underline{\mu}_X)\right]|C^{-1}|$$

$$= K \exp\left[-\frac{1}{2}(\underline{Y} - C\mu_X)'(C\Sigma_X C')^{-1}(\underline{Y} - C\mu_X)\right]|C^{-1}|$$

$$= \frac{|\Sigma_X|^{1/2}}{(2\pi)^{p/2}|\Sigma_X|^{1/2}} \times \frac{1}{|C\Sigma_X C'|^{1/2}} \exp\left[-\frac{1}{2}(\underline{Y} - C\underline{\mu}_X)'(C\Sigma_X C')^{-1}(\underline{Y} - C\underline{\mu}_X)\right]$$

$C\Sigma C'$

$$= \frac{1}{(2\pi)^{p/2}|C\Sigma_X C'|^{1/2}} \exp\left[-\frac{1}{2}(\underline{Y} - C\underline{\mu}_X)'(C\Sigma_X C')^{-1}(\underline{Y} - C\underline{\mu}_X)\right] \quad \text{—(2)}$$

$$|C^{-1}| = \frac{1}{|C|} = \frac{1}{|C|^{1/2}|C'|^{1/2}}$$

$$= \frac{|\Sigma_x|^{1/2}}{|C|^{1/2}|\Sigma_x|^{1/2}|C'|^{1/2}}$$

$$= \frac{|\Sigma_x|^{1/2}}{|C\Sigma_x C'|^{1/2}}$$

comparing eqn ② with multivariate Normal distribution we see that mean $\mu_x = C\mu_{x'}$ & variance $\Sigma = C\Sigma C'$

<u>Theorem-3</u>:- Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, a necessary & sufficient condition that a subset $\underline{X}^{(1)}$ component of $\underline{X}$ be independent of the subset $\underline{X}^{(2)}$ consisting of remaining component of $\underline{X}$ is that the co-variance between each component of $\underline{X}^{(1)}$ with a component of $\underline{X}^{(2)}$ is zero.

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_q \\ x_{q+1} \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} X^{(1)}_{q\times 1} \\ X^{(2)}_{(p-q)\times 1} \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} \begin{matrix} \mu^{(1)} \\ \\ \mu^{(2)} \end{matrix}$$

$$\& \ \Sigma = \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \end{array}$$
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1q} & \sigma_{1,q+1} & \cdots & \sigma_{1p} \\ \vdots & & & \vdots & & & \vdots \\ \sigma_{q,1} & \cdots & & \sigma_{qq} & \sigma_{q,q+1} & \cdots & \sigma_{qp} \\ \sigma_{q+1} & \cdots & & \sigma_{q+1,q} & \sigma_{q+1,q+1} & \cdots & \sigma_{q+1,p} \\ \vdots & & & & & & \vdots \\ \sigma_{p} & \cdots & & \sigma_{p,q} & \sigma_{p,q+1} & & \sigma_{pp} \end{bmatrix}$$
$$\begin{array}{cc} \Sigma_{21} & \Sigma_{22} \end{array}$$

$$\Sigma_{11} = E(X^{(1)} - \mu^{(1)})(X^{(1)} - \mu^{(1)})'$$

$$\Sigma_{22} = E(X^{(2)} - \mu^{(2)})(X^{(2)} - \mu^{(2)})'$$

$$\Sigma_{12} = E(X^{(1)} - \mu^{(1)})(X^{(2)} - \mu^{(2)})'$$

$$\Sigma = \begin{bmatrix} \Sigma_{11\, q\times q} & \Sigma_{12\, q,(p-q)} \\ \Sigma_{21\,(p-q)\times q} & \Sigma_{22\,(p-q)\times(p-q)} \end{bmatrix}_{p}$$

Property - If $\underline{X} \sim N_p(\mu, \Sigma)$ the marginal distribution of any set of component $\underline{X}$ is multivariate normal with means & co-variances obtained by taking corresponding components of μ & Σ respectively.

Proof -

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ \vdots \\ X_p \end{bmatrix} \begin{matrix} \underline{X}^{(1)} \\ \\ \underline{X}^{(2)} \end{matrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Let us make a non singular linear transformation to subvectors $\underline{Y}^{(1)} = \underline{X}^{(1)} + B\underline{X}^{(2)}$

$$\underline{Y}^{(2)} = \underline{X}^{(2)}$$

where B is choosen such that $\underline{Y}^{(1)}$ & $\underline{Y}^{(2)}$ are independent.

$$\therefore \operatorname{cov}(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = 0$$

$$E[(\underline{Y}^{(1)} - E(\underline{Y}^{(1)}))(\underline{Y}^{(2)} - E(\underline{Y}^{(2)}))'] = 0$$

$$\Rightarrow E(\underline{X}^{(1)} + B\underline{X}^{(2)} - \underline{\mu}^{(1)} - B\underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})' = 0$$

$$\Rightarrow E(\underline{X}^{(1)} - \underline{\mu}^{(1)} + B(\underline{X}^{(2)} - \underline{\mu}^{(2)}))(\underline{X}^{(2)} - \underline{\mu}_2^{(2)})' = 0$$

$$\Rightarrow E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}_2^{(2)})' + B\,E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(\underline{X}^{(2)} - \underline{\mu}_2^{(2)})' = 0$$

$$\Rightarrow \Sigma_{12} + B\Sigma_{22} = 0 \Rightarrow \boxed{B = -\Sigma_{12}\Sigma_{22}^{-1}}$$

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}^{(2)}$$

$$\underline{Y}^{(2)} = \underline{X}^{(2)}$$

$$\begin{bmatrix} \underline{Y}^{(1)} \\ \underline{Y}^{(2)} \end{bmatrix} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \underset{\sim}{0} & I \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix}$$

$$\underline{Y} = \underset{\substack{\uparrow \\ \text{non singular}}}{C}\,\underline{X}$$

From the theorem 2

$$\underline{Y} \sim N_p(C\underline{\mu}, C\Sigma C')$$

$$E(\underline{Y}) = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \underset{\sim}{0} & I \end{bmatrix} \begin{bmatrix} \underline{\mu}^{(1)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)} \\ \underline{\mu}^{(2)} \end{bmatrix}$$

$$\Sigma_Y = C\Sigma C'$$

$$= \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \underset{\sim}{0} & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & \underset{\sim}{0} \\ (-\Sigma_{12}\Sigma_{22}^{-1})' & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & \underset{\sim}{0} \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \underset{\sim}{0} \\ \Sigma_{21} - \Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11\cdot2} & \underset{\sim}{0} \\ \underset{\sim}{0} & \Sigma_{22} \end{bmatrix} \qquad \det|\Sigma| = \begin{vmatrix} \Sigma_{11\cdot2} & 0 \\ 0 & \Sigma_{22} \end{vmatrix} = |\Sigma_{11\cdot2}|\,|\Sigma_{22}|$$

$$g_{\underline{Y}}(\underline{y}) = \frac{\exp\left[-\frac{1}{2}(\underline{Y}-\underline{\mu}_Y)' \Sigma_Y^{-1} (\underline{Y}-\underline{\mu}_Y)\right]}{(2\pi)^{q/2}(2\pi)^{p-q/2}|\Sigma_{11\cdot2}|^{1/2}|\Sigma_{22}|}$$

since, Y_1 & Y_2 are independent.

$$g_{\underline{Y}}(\underline{y}) = f_{y^{(1)}}(y^{(1)}) \times f_{y^{(2)}}(y^{(2)})$$

since $\underline{Y}^{(2)} = \underline{X}^{(2)} \sim N_{p-q}(\underline{\mu}^{(2)}, \Sigma_{22})$

$$= \exp\left[\frac{-1}{2}\begin{bmatrix} Y^{(1)} - \mu_{Y^{(1)}} \\ Y^{(2)} - \mu_Y^{(2)} \end{bmatrix}'_{1\times2}\begin{bmatrix} \Sigma_{11\cdot2}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix}_{2\times2}\begin{bmatrix} \underline{Y}^{(1)} - \mu_Y^{(1)} \\ \underline{Y}^{(2)} - \mu_Y^{(2)} \end{bmatrix}_{2\times1}\right]$$

$$= \exp\left[\frac{-1}{2}\left[(\underline{Y}^{(1)}-\underline{\mu}_Y^{(1)})'\Sigma_{11\cdot2}^{-1} \quad (\underline{Y}^{(2)}-\underline{\mu}_Y^{(2)})'\Sigma_{22}^{-1}\right]_{1\times2}\begin{bmatrix} \underline{Y}^{(1)} - \mu_Y^{(1)} \\ \underline{Y}^{(2)} - \mu_Y^{(2)} \end{bmatrix}_{2\times1}\right]$$

$$= \exp\left[-\frac{1}{2}\left[(\underline{Y}^{(1)}-\underline{\mu}_Y^{(1)})'\Sigma_{11\cdot2}^{-1}(\underline{Y}^{(1)}-\mu_Y^{(1)}) + (\underline{Y}^{(2)}-\mu_Y^{(2)})'\Sigma_{22}^{-1}(\underline{Y}^{(2)}-\mu_Y^{(2)})\right]\right]$$

Conditional Distribution of Any sets of components

Transformation.

$$\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}X^{(2)}$$

$$Y^{(2)} = X^{(2)}$$

$$\underline{Y} = C\underline{X} \Rightarrow \underline{X} = C^{-1}\underline{Y}$$

$$|J| = \left|\frac{d\underline{X}}{d\underline{Y}}\right| = |C^{-1}|$$

$$= \begin{vmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ \underset{\sim}{0} & I \end{vmatrix}$$

$$= 1$$

Rewitting the joint density of $\underline{X}^{(1)}$ & $\underline{X}^{(2)}$ by substituting $\underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}^{(2)}$ for $Y^{(1)}$ & $\underline{X}^{(2)}$ for $\underline{Y}^{(2)}$

& multiplying by Jacobian of the transformation. which is one, we get:

$$f_{\underline{X}^{(1)},\underline{X}^{(2)}}(\underline{x}^{(1)}, \underline{x}^{(2)}) = \frac{1}{(2\pi)^{q/2}|\Sigma_{11\cdot 2}|^{1/2}} \exp\left[-\frac{1}{2}\left\{\underline{x}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}^{(2)} - \underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)}\right)'\Sigma_{11\cdot 2}^{-1}(\underline{x}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{x}^{(2)} - \underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)})\right]$$

$$\times \frac{\exp\left[-\frac{1}{2}(\underline{x}^{(2)} - \underline{\mu}^{(2)})'\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)})\right]}{2\pi^{(p-q)/2}|\Sigma_{22}|^{1/2}} \times 1$$

$$= \frac{\exp\left[-\frac{1}{2}\left\{(\underline{x}^{(1)} - (\underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)})))'\Sigma_{11\cdot 2}^{-1}(\underline{x}^{(1)} - (\underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)})))\right\}\right]}{(2\pi)^{q/2}|\Sigma_{11\cdot 2}|^{1/2}} \times f_{\underline{X}^{(2)}}(\underline{x}^{(2)})$$

$$h(x^{(1)} | x^{(2)} = x^{(2)}) = \frac{f(\underline{x}^{(1)}, \underline{x}^{(2)})}{f(\underline{x}^{(2)})}$$

$$= \frac{\exp\left[-\frac{1}{2}\left\{(\underline{x}^{(1)} - (\underline{\mu}_1^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)})))'\Sigma_{11\cdot 2}^{-1}(\underline{x}^{(1)} - \underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)}))\right\}\right]}{(2\pi)^{q/2}|\Sigma_{11\cdot 2}|^{1/2}}$$

$$\therefore \underline{X}^{(1)} | \underline{X}^{(2)} = \underline{x}^{(2)} \sim N_q(\underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)}), \Sigma_{11\cdot 2})$$

1 → If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ & $\underline{Y} = D\underline{X}$ where D is a ~~p×q~~ q×p matrix ($q<p$) of rank q then $\underline{Y} \sim N_q(D\underline{\mu}, D\Sigma D')$.

⇒ Transformation $\underline{Y} = D\underline{X}$

$$E(\underline{Y}) = D\,E(\underline{X})$$
$$= D\underline{\mu}$$

$$\Sigma_{\underline{Y}} = E\left((\underline{Y} - E(\underline{Y}))(\underline{Y} - E(\underline{Y}))'\right)$$
$$= E\left((D\underline{X} - D\underline{\mu})(D\underline{X} - D\underline{\mu})'\right)$$
$$= D\,E(\underline{X} - \underline{\mu})(\underline{X} - \underline{\mu})'D'$$
$$= D\Sigma D'$$

Since rank of D is q this means q rows of D are independent. We know that a set of q independent vectors can be extended to form a basis of p-dimensional vector space by adding to it $(p-q)$ vectors.

Let us define a new vector $C = \begin{bmatrix} D_{q\times p} \\ E_{(p-q)\times p} \end{bmatrix}_{p\times p}$

Now, C is non singular.
Let us make the transformation,

$$\underline{Z} = C\underline{X}$$

which implies, ∴ $\underline{Z} \sim N_p(C\underline{\mu}, C\Sigma C')$

$$\underline{Z} = \begin{bmatrix} D \\ E \end{bmatrix}' \underline{X} = \begin{bmatrix} D\underline{X} \\ E\underline{X} \end{bmatrix}$$

But $D\underline{X}$ being the partition vector of $\underline{X}$ & ~~also~~ as ~~being~~ the marginal q variate normal distribution

$\therefore \underline{Y} \sim N_q(D\underline{\mu}, D\Sigma D')$

Corolary - This property tells us that if $X \sim N_q(\mu, \Sigma)$ then every linear transformation of the component of X has a univariate normal distribution.

$$Y = d_1 X_1 + d_2 X_2 + \cdots + d_p X_p$$

$$\underline{Y} = [d_1 \ d_2 \ \cdots \ d_p]_{1\times p} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}_{p\times 1}$$

$$\cancel{D\underline{\mu} = [d_1 \ d_2 \ \cdots \ d_p] \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \sim N(\Sigma d_i \mu_i, \Sigma d_i^2 \sigma_i^2)}$$

$$\underline{Y} \sim N_p\left(\sum_{j=1}^{p} d_j \mu_j, \left(\sum_{j=1}^{p} d_j^2\right)\Sigma\right)$$

Characteristic Function -

Let $\underline{X} \sim N_p(\mu, \Sigma)$ then the characteristic function of $\underline{X}$ given by

$$\phi_{\underline{X}}(\underline{t}) = E(e^{i\underline{t}'\underline{x}})$$
$$= e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

$$\left\{ \begin{array}{l} \phi_X(t) = E(e^{itx}) \\ = e^{i\mu t - \frac{1}{2}\sigma^2 t^2} \end{array} \right.$$

where $\underline{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_p \end{bmatrix}$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left[\frac{-1}{2}(x-\mu)'\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

Let us make a non-singular transformation

$$\underline{x} - \underline{\mu} = C\underline{Y}$$

s.t. $C'\Sigma^{-1}C = I \Rightarrow \Sigma^{-1} = CC' \Rightarrow |C| = |\Sigma|^{1/2}$

& $|J| = \left|\frac{d\underline{X}}{d\underline{Y}}\right| = |C|$

$$g_{\underline{Y}}(\underline{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\underline{y}'C'\Sigma^{-1}C\underline{y})\right]|C|$$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left[\frac{-1}{2}(\underline{y}'I\underline{y})\right]$$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left[\frac{-1}{2}(y_1^2 + y_2^2 + \cdots + y_p^2)\right]$$

$$= \prod_{i=1}^{p} \frac{\exp\left[\frac{-1}{2}\sum_{i=1}^{p} y_i^2\right]}{(2\pi)^{1/2}}$$

$\Rightarrow y_i \sim N(0,1)$ & y_i $(i = 1, 2, \ldots p)$ are independent characteristic function of y_i - as.

$$\phi_{y_i}(t_i) = e^{-\frac{1}{2}t_i^2}$$

$$\left\{ \begin{aligned} \phi_x(t) &= E(e^{itx}) \\ &= e^{\mu t i - \frac{1}{2}\sigma^2 t^2} \end{aligned} \right.$$

$$\phi_{\underline{y}}(\underline{y}) = \phi_{y_1}(y_1) + \cancel{\phi_{y_2}(y_2) + \dots + \phi_{y_p}(y_p)} \quad \mu = 0 \;\&\; \sigma = 1$$

$$= \cancel{\prod_{i=1}^{p} e^{-\frac{1}{2}\mu_1^2}}$$

$$\phi_{\underline{Y}}(\underline{\mu}) = \phi_{Y_1}(\mu_1) + \phi_{Y_2}(\mu_2) + \dots + \phi_{Y_p}(\mu_p)$$

$$= \prod_{i=1}^{p} e^{-\frac{1}{2}\mu_i^2} = e^{-\frac{1}{2}\sum_{i=1}^{p} u_i^2}$$

$$= e^{-\frac{1}{2}\underline{u}'\underline{u}}$$

$$\phi_{\underline{X}}(\underline{t}) = E(e^{i\underline{t}'\underline{x}})$$

$$= E(e^{i\underline{t}'(C\underline{Y}+\underline{\mu})})$$

$$= E(e^{i\underline{t}'C\underline{Y}} \cdot e^{i\underline{t}'\underline{\mu}})$$

$$= e^{i\underline{t}'\underline{\mu}} E(e^{i\underline{V}'\underline{Y}}) \quad ; \quad \underline{V}' = \underline{t}'C$$

$$= e^{i\underline{t}'\underline{\mu}} \cdot e^{-\frac{1}{2}\underline{V}'\underline{V}}$$

$$= e^{i\underline{t}'\underline{u}} \cdot e^{-\frac{1}{2}\underline{t}'CC'\underline{t}}$$

$$= e^{i\underline{t}'u} \cdot e^{-\frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

$$\phi_{\underline{X}}(\underline{t}) = e^{i\underline{t}'u - \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

$\phi(x+y)(t) =$

Multi Random Sampling (Estimation of parameters

1.) Let $X_1, X_2, \ldots X_\alpha \ldots X_n$ be a r.s. of size n from $N_p(\underline{\mu}, \Sigma)$ where $(n>p)$ & $\underline{X}_\alpha$ is a $p\times 1$ ~~r.v.~~ vector where $1 \le \alpha < n$.

sample / Variable / characteristic	1	2	....	α	....	n	mean
X_1	x_{11}	x_{12}	...	$x_{1\alpha}$	...	x_{1n}	$\bar{x}_1$
X_2	x_{21}	x_{22}	--	$x_{2\alpha}$	--	x_{2n}	$\bar{x}_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	
X_p	x_{p1}	x_{p2}	...	$x_{p\alpha}$	--	x_{pn}	$\bar{x}_p$

$$\underline{\bar{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{\alpha=1}^{n} x_{1\alpha} \\ \vdots \\ \frac{1}{n}\sum_{\alpha=1}^{n} x_{p\alpha} \end{bmatrix}$$

$$\underline{\bar{x}} = \frac{1}{n}\sum_{\alpha=1}^{n} \underline{x}_\alpha$$

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & & & \vdots \\ \vdots & & \ddots & \vdots \\ s_{p1} & \cdots & & s_{pp} \end{bmatrix}$$

Total covariance between two parameters i & j (e.g. if i is height & j is weight)

$$S_{ij} = \frac{1}{n-1}\sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) \quad \forall\ i, j$$

$$S = \frac{1}{n-1}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{p1} & \cdots & & a_{pp} \end{bmatrix}_{p\times p}$$

$$a_{ij} = \sum_{\alpha=1}^{n} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$$

$$S = \frac{1}{n-1} A$$

where A is the matrix of sum of squares & cross product of deviation about the mean.

$$a_{ij} = \sum_{\alpha=1}^{n} x_{i\alpha} x_{j\alpha} - n\bar{x}_i \bar{x}_j \}$$

Note Quardratic eq^n

$$Q = \underline{x}'_{1\times p} A_{p\times p} \underline{x}_{p\times 1}$$

$$\frac{\partial Q}{\partial \underline{x}} = 2A\underline{x}$$

$$Q = (\underline{x}-\underline{b})' A (\underline{x}-\underline{b})$$

$$\frac{\partial Q}{\partial \underline{x}} = 2A(\underline{x}-\underline{b})$$

$$\frac{\partial Q}{\partial \underline{b}} = 2A(\underline{b}-\underline{x})$$

$$\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= \sum_{j=1}^{3} a_{1j} A_{1j}$$

A ~~sub~~ matrix of A is an array obtained from A by redlasing rows & column a minor is the determinant of square sub matrix of A

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pp} \end{bmatrix}$$

$$|A| = \sum_{j=1}^{p} a_{ij} A_{ij} = \sum_{j=1}^{p} a_{jk} A_{jk}$$

where A_{ij} is $(-1)^{i+j}$ minor of a_{ij}

Let a^{ij} be the ij^{th} elements of A^{-1} matrix.

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

$$a^{ij} = \frac{A_{ji}}{|A|}$$

$$a_{ij} = \frac{A^{ji}}{|A^{-1}|}$$

MLE of mean vector -

$$L = \prod_{\alpha=1}^{n} \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}_\alpha - \underline{\mu})'\Sigma^{-1}(\underline{x}_\alpha - \underline{\mu})\right]$$

$$= \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}} \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu})'\Sigma^{-1}(\underline{x}_\alpha - \underline{\mu})\right]$$

$$\log L = -\frac{np}{2}\log 2\pi - \frac{n}{2}\log|\Sigma| - \frac{1}{2}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu})'\Sigma^{-1}(\underline{x}_\alpha - \underline{\mu})$$

$$\frac{\partial \log L}{\partial \underline{\mu}} = -\sum_{\alpha=1}^{n}\frac{2}{2}\Sigma^{-1}(\underline{\mu} - \underline{x}_\alpha) = 0$$

$$\Rightarrow \Sigma^{-1}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu}) = 0$$

$$\Rightarrow \sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu}) = 0$$

$$\Rightarrow \boxed{\underline{\mu} - \frac{1}{n}\sum_{\alpha=1}^{n}x_\alpha = \overline{\underline{x}}}$$

MLE of variance co-variance matrix Σ.

$$\log L = -\frac{np}{2}\log 2\pi + \frac{n}{2}\log|\Sigma^{-1}| - \frac{1}{2}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \underline{\mu})\Sigma^{-1}(\underline{x}_\alpha - \underline{\mu})$$

$$\Sigma^{-1} = \{\sigma^{ij}\}$$

$$|\Sigma^{-1}| = \sigma^{i1}\Sigma^{i1} + \sigma^{i2}\Sigma^{i2} + \cdots + \sigma^{ij}\Sigma^{ij} + \cdots + \sigma^{ip}\Sigma^{ip}$$

$$\log L = -\frac{np}{2}\log 2\pi + \frac{n}{2}\log\left[\sigma^{i1}\Sigma^{i1} + \sigma^{i2}\Sigma^{i2} + \cdots + \sigma^{ij}\Sigma^{ij} + \cdots + \sigma^{ip}\Sigma^{ip}\right]$$
$$-\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{ij=1}^{p}\sigma^{ij}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j)$$

$\left\{ \bar{x}A\underline{x} = \sum_{ij} a_{ij}x_ix_j \right.$

Diff^n w.r.to ijth component

$$\frac{\partial \log L}{\partial \sigma^{ij}} = \frac{n}{2}\frac{\Sigma^{ij}}{\sigma^{i1}\Sigma^{i1} + \sigma^{i2}\Sigma^{i2} + \cdots + \sigma^{ip}\Sigma^{ip}} - \frac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j) = 0$$

$$\Rightarrow \frac{n}{2}\frac{\Sigma^{ij}}{|\Sigma^{-1}|} = \frac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha} - \mu_i)(x_{j\alpha} - \mu_j)$$

$$\left\{ A^{-1} = \frac{\operatorname{adj} A}{|A|} \Rightarrow |\Sigma^{-1}|^{-1} = \frac{\Sigma^{ij}}{|\Sigma^{-1}|} = \sigma_{ji} \right\}$$

$$\Rightarrow \sigma_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \hat{\mu}_i)(x_{j\alpha} - \hat{\mu}_j)$$
$$= \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$$

$$\hat{\Sigma} = \frac{1}{n}\sum_{\alpha=1}^{n}(\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})'$$
$$= \frac{A}{n}$$

Theorem- Given $\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n$ be an independent random samples from $N_p(\underline{\mu}, \Sigma)$ then $\bar{\underline{x}}$ follows $N_p\left(\underline{\mu}, \frac{\Sigma}{n}\right)$

Proof- We know that any linear combination of the components of a random vector also follows a normal distribution. (Theorem 2)

$$E(\bar{\underline{x}}) = E\left[\frac{1}{n}\sum_{\alpha=1}^{n} x_\alpha\right]$$

$$= \frac{1}{n} E(x_1 + x_2 + \cdots x_n)$$

$$= \frac{1}{n}(\underline{\mu} + \underline{\mu} + \cdots \underline{\mu})$$

$$= \frac{n\mu}{n}$$

$$= \underline{\mu}$$

$$\Sigma_{\bar{\underline{x}}} = E[(\bar{\underline{x}} - E(\bar{\underline{x}}))(\bar{\underline{x}} - E(\bar{\underline{x}}))']$$

$$= E\left[\left(\frac{1}{n}(\underline{x}_1 + \cdots \underline{x}_n) - \underline{\mu}\right)\left(\frac{1}{n}(\underline{x}_1 + \cdots \underline{x}_n) - \underline{\mu}\right)'\right]$$

$$= \frac{1}{n^2} E\left[\left((\underline{x}_1 - \underline{\mu}) + (\underline{x}_2 - \underline{\mu}) + \cdots + (\underline{x}_n - \underline{\mu})\right)\left((\underline{x}_1 - \underline{\mu}) + (\underline{x}_2 - \underline{\mu}) + \cdots (\underline{x}_n - \underline{\mu})\right)'\right]$$

$$= \frac{1}{n^2}\left[E(\underline{x}_1 - \underline{\mu})(\underline{x}_1 - \underline{\mu})' + (\underline{x}_2 - \underline{\mu})(\underline{x}_2 - \underline{\mu})' + \cdots + (\underline{x}_n - \underline{\mu})(\underline{x}_n - \underline{\mu})'\right]$$

$$= \frac{1}{n^2}[\Sigma + \Sigma + \cdots + \Sigma]$$

$$= \frac{n\Sigma}{n^2}$$

$$= \frac{\Sigma}{n}$$

Theorem - $\frac{A}{n-1}$ is an unbiased estimate of Σ

$\Rightarrow$ $A = \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \bar{\underline{x}})(\underline{x}_\alpha - \bar{\underline{x}})'$

$= \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu} + \underline{\mu} - \bar{\underline{x}})(\underline{x}_\alpha - \underline{\mu} + \underline{\mu} - \bar{\underline{x}})'$

$= \sum_{\alpha=1}^{n} \Big[(\underline{x}_\alpha - \underline{\mu})(\underline{x}_\alpha - \underline{\mu})' - (\underline{x}_\alpha - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$
$- (\underline{x}_\alpha - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})' + (\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'\Big]$

$= \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu})(\underline{x}_\alpha - \underline{\mu})' - \sum_{\alpha=1}^{n} (\underline{x}_\alpha - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$

$- (\bar{\underline{x}} - \underline{\mu}) \sum_{\alpha} (\underline{x}_\alpha - \underline{\mu})' + n(\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$

$= n\Sigma + n(\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})' - n(\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$
$- n(\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$

$= n\Sigma - nE(\bar{\underline{x}} - \underline{\mu})(\bar{\underline{x}} - \underline{\mu})'$

$= n\Sigma - \frac{n\Sigma}{n}$

$= (n-1)\Sigma$

Test for $\underline{\mu}$ when Σ is known (one sample prob)

Given a random sample $\underline{x}_1, \underline{x}_2 \ldots \underline{x}_\alpha \ldots \underline{x}_n$ from $N_p(\underline{\mu}, \Sigma)$. Let, $H_0 : \underline{\mu} : \underline{\mu}_0$ where $\underline{\mu}_0$ is specified vector under H_0 the test statistics is $n(\bar{\underline{x}} - \underline{\mu}_0)' \Sigma^{-1} (\bar{\underline{x}} - \underline{\mu}_0)$

$$\left\{ \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim Z \Rightarrow \frac{(\bar{x} - \mu_0)^2}{\frac{\sigma^2}{n}} \sim \chi^2 \Rightarrow n(\bar{x} - \mu_0)' \Sigma^{-1} (\bar{x} - \mu_0) \sim \chi^2_p(\alpha) \right.$$

$\bar{\underline{x}} \sim N_p(\underline{\mu}_0, \frac{\Sigma}{n})$: under H_0

$\therefore \bar{\underline{x}} - \underline{\mu}_0 \sim N_p(\underline{\mu}, \Sigma^*)$ $\quad (\Sigma^* = \frac{\Sigma}{n})$

Since Σ^* is a definite positive symmetric matric, there always exists a non singular C,

s-t $C' \Sigma^{-1} C = I$

Let the transformation

$$\bar{\underline{x}} - \underline{\mu}_0 = C\underline{y} \Rightarrow \underline{y} = C^{-1}(\bar{\underline{x}} - \underline{\mu}_0)$$

$$\begin{aligned}
\therefore E(\underline{y}) &= C^{-1} E(\bar{\underline{x}} - \underline{\mu}_0) \\
&= C^{-1} \times 0 \qquad [\text{ under } H_0 \; E(\bar{\underline{x}}) = \underline{\mu}_0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Sigma_y &= E(\underline{y}\underline{y}') \\
&= E[C^{-1}(\bar{\underline{x}} - \underline{\mu}_0)(\bar{\underline{x}} - \underline{\mu}_0)'(C^{-1})'] \\
&= C^{-1} E(\bar{\underline{x}} - \underline{\mu}_0)(\bar{\underline{x}} - \underline{\mu}_0)'(C^{-1})' \\
&= C^{-1} \Sigma^* (C^{-1})' \\
&= (C' \Sigma^{*-1} C)^{-1} \\
&= I
\end{aligned}$$

$\therefore \underline{y} \sim N_p(\underline{0}, I) \Rightarrow y_i$'s are iid $N(0,1)$