Discrete Mathematics (CSA103)

Ganga Ram

DST-Centre for Interdisciplinary Mathematical Sciences

Institute of Science, Banaras Hindu University

gangaiitr11@gmail.com; gangacims@bhu.ac.in

Mathematical Logic: Propositional and Predicate Logic, Propositional Equivalences, Normal Forms, Predicates and Quantifiers, Nested Quantifiers, Rules of Inference.

Sets and Relations: Set Operations, Representation and Properties of Relations, Equivalence Relations, Partially Ordering. Counting,

Mathematical Induction and Discrete Probability: Basics of Counting, Pigeonhole Principle, Permutations and Combinations, Inclusion- Exclusion Principle, Mathematical Induction, Probability, Bayes Theorem.

Group Theory: Groups, Subgroups, Semi Groups, Product and Quotients of Algebraic Structures, Isomorphism, Homomorphism, Automorphism, Rings, Integral Domains, Fields, Applications of Group Theory.

Graph Theory: Simple Graph, Multigraph, Weighted Graph, Paths and Circuits, Shortest Paths in Weighted Graphs, Eulerian Paths and Circuits, Hamiltonian Paths and Circuits, Planner graph, Graph Coloring, Bipartite Graphs, Trees and Rooted Trees, Prefix Codes, Tree Traversals, Spanning Trees and Cut-Sets.

Boolean Algebra: Boolean Functions and its Representation, Simplifications of Boolean Functions.

Optimization: Linear Programming - Mathematical Model, Graphical Solution, Simplex and Dual Simplex Method, Sensitive Analysis; Integer Programming, Transportation and Assignment Models,

PERT-CPM: Diagram Representation, Critical Path Calculations, Resource Levelling, Cost Consideration in Project Scheduling.

Books Recommended:

- J.P. Trembley and R.P.Manohar, Discrete Mathematical Structures with Applications to Computer Science, McGraw Hill.
- Dornhoff and Hohn, Applied Modern Algebra, McMillan.
- N. Deo, Graph Theory with Applications to Engineering and Computer Science, PHI.
- C.L. Liu, Elements of Discrete Mathematics, McGraw-Hill.
- Kenneth H Rosen, Discrete Mathematics, Tata McGraw Hill.
- K.L.P. Mishra, N. Chandrasekaran, Theory of Computer Science: Automata, Languages and Computation, PHI.

Logic: Logic is a field of study that deals with the method of reasoning. Logic provides rules by which we can determine whether a given argument or reasoning is valid (correct) or not.

Proposition (or a Statement): A declarative sentence that is either true or false, but not both, is a proposition.
Consider, for example, the following sentences:

- Ice floats in water.
- China is in Europe.
- $2 + 2 = 4$
- Where are you going?
- Do your homework.
- If $1 = 2$, then roses are red.

Propositional/Predicate calculus: The area of logic that deals with proposition/predicates is called propositional/predicate calculus.

Truth Value: The truthfulness or falsity of a proposition is called its truth value, denoted by T(true) and F(false), respectively.

Compound proposition: A compound proposition is formed by combining two or more simple propositions (called **components**) using the logical operators (**connectives**).

Logical operators (connectives)

Negation: The negation of a proposition p is “It is not the case that p,” denoted by $\neg p$.

p	¬ p
T	F
F	T

Conjunction: The conjunction of two arbitrary propositions p and q, denoted by $p \wedge q$, is the proposition p and q.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction: The disjunction of two arbitrary propositions p and q, denoted by $p \vee q$, is the proposition p or q.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Types of disjunction

- Inclusive disjunction
- Exclusive disjunction

Inclusive disjunction: The connective "or" is used in the inclusive sense "and/or" to mean at least one, maybe both. Such a disjunction is an inclusive disjunction.

Exclusive disjunction: The connective "or" is used in the exclusive sense to mean at least one, but not both. Such a disjunction is an exclusive disjunction.

p	q	p $\oplus$ q
T	T	F
T	F	T
F	T	T
F	F	F

Implication: The combined form: "If p, then q" is an implication, denoted by $p \to q$. It is also called a **conditional statement.** The component p is the **hypothesis (or premise)** of the implication and q the **conclusion**.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Implications occur in a variety of ways. The following are some commonly known occurrences:

- If p, then q.
- If p, q.
- p implies q.
- p only if q.
- q if p.
- p is sufficient for q.
- q is necessary for p.

Converse, Inverse, and Contrapositive: From an implication we can form three new implications–its converse, inverse, and contrapositive- as defined below.

- The converse of the implication $p \to q$ is $q \to p$ (switch the premise and the conclusion in $p \to q$).
- The inverse of $p \to q$ is $\neg p \to \neg q$ (negate the premise and the conclusion).
- The contrapositive of $p \to q$ is $\neg q \to \neg p$ (negate the premise and the conclusion, and then switch them).

Biconditional Statement: Two propositions p and q can be combined using the connective "if and only if." The resulting proposition, "p if and only if q," is the conjunction of two implications:

- p only if q,
- p if q,
- that is, $p \to q$ and $q \to p$.

It is called a biconditional statement, symbolized by $p \leftrightarrow q$.

p	q	p $\leftrightarrow$ q
T	T	T
T	F	F
F	T	F
F	F	T

Order of Precedence: To evaluate complex logical expressions, you must know the order of precedence of the logical operators. The order of precedence from the highest to the lowest is:

$$(1)\ \neg \quad (2)\ \wedge \quad (3)\ \vee \quad (4)\ \rightarrow \quad (5)\ \leftrightarrow$$

Note: that parenthesized subexpressions are always evaluated first; if two operators have equal precedence, the corresponding expression is evaluated from left to right.

For example: The expression $(p \to q) \wedge \neg q \to \neg p$ is evaluated as

$$[(p \to q) \wedge (\neg q)] \to (\neg p),$$

And $p \to q \leftrightarrow \neg q \to \neg p$ is evaluated as

$$(p \to q) \leftrightarrow [(\neg q) \to (\neg p)].$$

Tautology: If the compound statement is always true, regardless of the truth values of its components. Such a compound proposition is a tautology.
For example, $p \vee \neg p$

Contradiction: A compound statement that is always false is a contradiction.
For example, $p \wedge \neg p$

Contingency: A compound proposition that is neither a tautology nor a contradiction is a contingency.
For example, $p \wedge p$

Logically equivalences Propositions: Two compound propositions p and q, although they may look different, can have identical truth values for all possible pairs of truth values of their components. Such statements are logically equivalent, symbolized by $p \equiv q$ otherwise, we write $p \not\equiv q$.

Example: $p \to q \equiv \neg p \vee q$.

Laws of Logic: Let p, q, and r be any three propositions. Let t denote a tautology and f a contradiction. Then:

- Idempotent laws: $p \wedge p \equiv p$
- Identity laws $p \wedge t \equiv p$
- Inverse laws $p \wedge (\neg p) \equiv f$
- Domination laws $p \wedge f \equiv f$
- Commutative laws $p \wedge q \equiv q \equiv p$
- Double negation $\neg(\neg p) \equiv p$
- Associative laws $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
- Distributive laws $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- De Morgan's laws $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Implication conversion law $p \to q \equiv \neg p \vee q$
- Contrapositive law $p \to q \equiv \neg q \to \neg p$
- Reductio ad absurdum law $p \to q \equiv (p \wedge \neg q) \to f$

Assignment: Write laws of logic for disjunction operator $\vee$.

Normal Form: A well formed formula (wff) of propositional logic, called propositional normal form is a string consisting of propositional, connectives and parentheses used in the proper manner.

- **Disjunctive Normal Form:** A formula which is equivalent to a given formula and consists of a sum of elementary products.

  $(p \to q) \wedge \neg q \equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q)$

- **Conjunctive Normal Form:** "Product of elementary sums."

  $(p \to q) \wedge \neg q \equiv \neg(p \vee q) \wedge \neg q$

Consider following normal forms

- **Principal Disjunctive Normal Form (Sum-of-product canonical form):** For a given formula, an equivalent formula consisting of disjunctions of **minterms** only.

  $p \vee \neg q \equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

- **Principal Conjunctive Normal Form (Product-of -sums canonical form):** An equivalent formula consisting of conjunctions of **maxterms** only.

  $(p \leftrightarrow q) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Quantifiers: Proposition contains a word indicating quantity such as all, every, none, some, and one. Such words, called quantifiers.

There are two different quantifiers.

- The first is "all", the universal quantifier, denoted by $\forall$. It may read $\forall$ as for all, for each, or for every.
- The second quantifier is some, the existential quantifier, denoted by $\exists$. It may read $\exists$, for some, there exists a, or for at least one.
- Note that the word some means at least one.

Example: Let x be any apple. Then the sentence All apples are green can be written as For every x, x is green. Using the universal quantifier $\forall$, this sentence can be represented symbolically as $(\forall x)\ (x\ \ is\ \ green)$ or $(\forall x)P(x)$ where $P(x)$: x is green.

Predicate: A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

Example: $(\forall x)P(x)$ where $P(x)$: x is green. Here $P(x)$, called a predicate, states the property the object x has.

The set of all values x can have is called the **universe of discourse** (UD). In the above example, the UD is the set of all apples.

Note that P(x) is not a proposition, but just an expression. However, it can be transformed into a proposition by assigning values to x. The truth value of P(x) is predicated on the values assigned to x from the UD.

Example: Let $Q(x, y)$ denotes the statement "$x = y + 3$"

Nested Quantifiers: Two quantifiers are nested if one is within the scope of the others, such as

$$\forall x \forall y P(x, y)$$

$$\forall x \exists y P(x, y)$$

$$\forall x \forall y P(x, y)$$

$$\exists x \exists y P(x, y)$$

Example: Let $Q(x, y)$ denotes the statement $x + y = 0$.

The quantifier $\exists y \forall x Q(x, y)$ denotes "There is a real number y such that for every real x, $Q(x, y)$" is false.

$\forall x \exists y Q(x, y)$

Theory of Inference: Suppose we are given a finite set of propositions (called hypotheses) $H_1, H_2, (1), H_n$, all assumed true. Also assume that from these premises, we can arrive at a conclusion C through reasoning (or argument). Such a discussion can be written in inferential form as follows,

$$\left.\begin{array}{c} H_1 \\ H_2 \\ \vdots \\ H_n \end{array}\right\} \textit{hypotheses}$$

$$\left.\begin{array}{c} \overline{} \\ \therefore C \end{array}\right\} \leftarrow \textit{conclusion}$$

where the symbol $\therefore$ means therefore.

Valid and Invalid Arguments: An argument is valid if the conjunction of the hypotheses $H_1, H_2, (1), H_n$, logically implies the conclusion C: that is, the implication

$$H_1 \wedge H_2 \wedge (1), \wedge H_n \rightarrow C$$

is a tautology. Otherwise, the argument is invalid, a fallacy.

Thus, an argument is valid if and only if the conclusion is a logical consequence of the hypotheses.

Example: We the following argument.

H_1 : There are more residents in New York City than there are hairs in the head of any resident.
H_2 : No resident is totally bald.

$\therefore$ At least two residents must have the same number of hairs on their heads.

Inference Rules: If the inference rules given below and/or the laws of logic can be used to reach the given conclusion, then the given argument is valid; otherwise, it is invalid; that is, the argument contains a flaw.

Rule of Inference	**Name**
$[p \wedge (p \to q)] \to q$	Modus ponens
$[\neg q \vee (p \to q)] \to \neg p$	Modus tollens
$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$p \to (p \vee q)$	Addition
$(p \wedge q) \to p$	Simplification
$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution
$[p \wedge (p \to q)] \to q$	Law of detachment
$[(p \to q) \wedge (\neg q)] \to \neg p$	Law of the contrapositive

A few words of explanation about each rule:

- The conjunction rule says that if both p and q are true, then $p \wedge q$ is true–a fact we already knew.
- According to the simplification rule, if $p \wedge q$ is true, then p is true.
- The addition rule says that if p is true, then $p \vee q$ is true regardless of the truth value of q.
- By the law of detachment, if an implication $p \rightarrow q$ is true and the premise p is true, then you can always conclude that q is also true; in other words, a true premise leads to a true conclusion logically.
- The law of the contrapositive says that if an implication $p \rightarrow q$ is true, but the conclusion q is false, then the premise p must be false.

The other rules can be interpreted similarly.

Example:

If the computer was down Saturday afternoon, then Mary went to a matinee.
Either Mary went to a matinee or took a nap Saturday afternoon.
Mary did not take a nap that afternoon.

$\therefore$ The computer was down Saturday afternoon.

Solution: Let
p : The computer was down Saturday afternoon.
q : Mary went to a matinee Saturday afternoon.
r : Mary took a nap Saturday afternoon.

Then the given argument can be symbolized as follows:

$$\left.\begin{array}{c} H_1 : p \to q \\ H_2 : q \vee r \\ H_3 : \neg r \end{array}\right\} \textit{hypotheses}$$

$$\left.\begin{array}{c} \hline \therefore p \end{array}\right\} \leftarrow \textit{conclusion}$$

Every step in our logical reasoning and the corresponding justification are given below:

1	$\neg r$ is true.	hypothesis H_3
2	$q \vee r$ is true.	hypothesis H_2
3	q is true.	step 1, step 2, and disjunctive syllogism
4	$p \to q$ is true.	hypothesis H_1
5	Then p may be true or false.	step 4, step 5, and definition of implication.

Since our logical conclusion does not agree with the given conclusion, the given argument is invalid.

Questions/Query ?