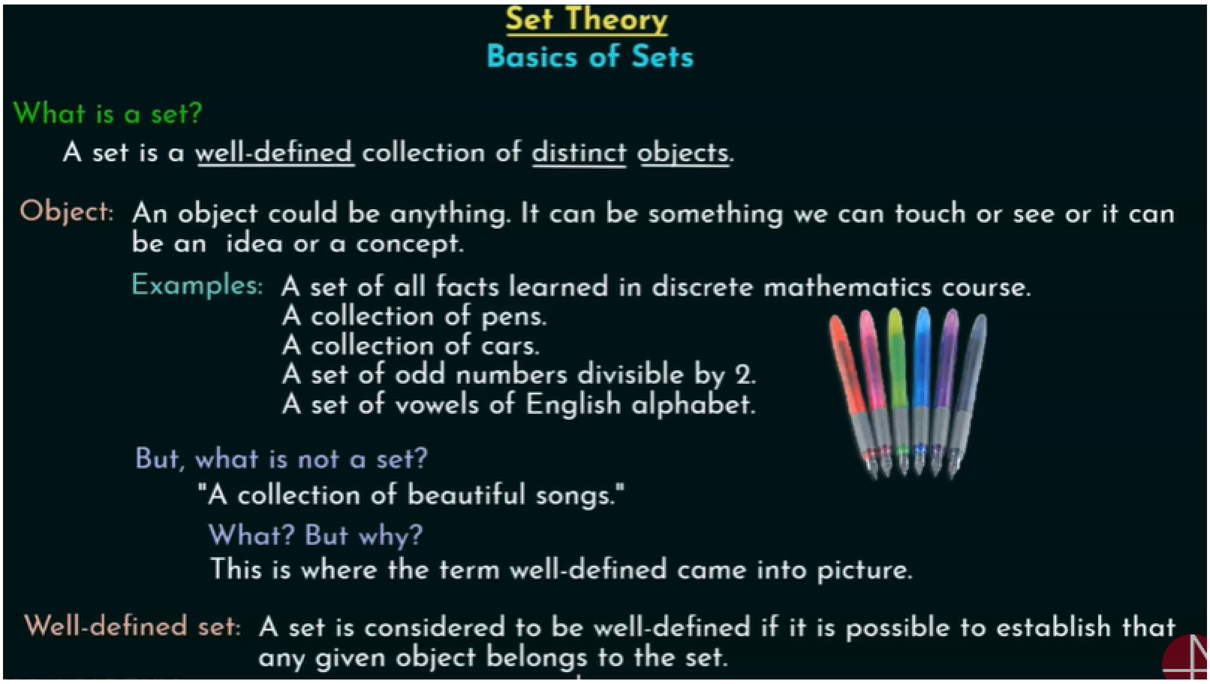
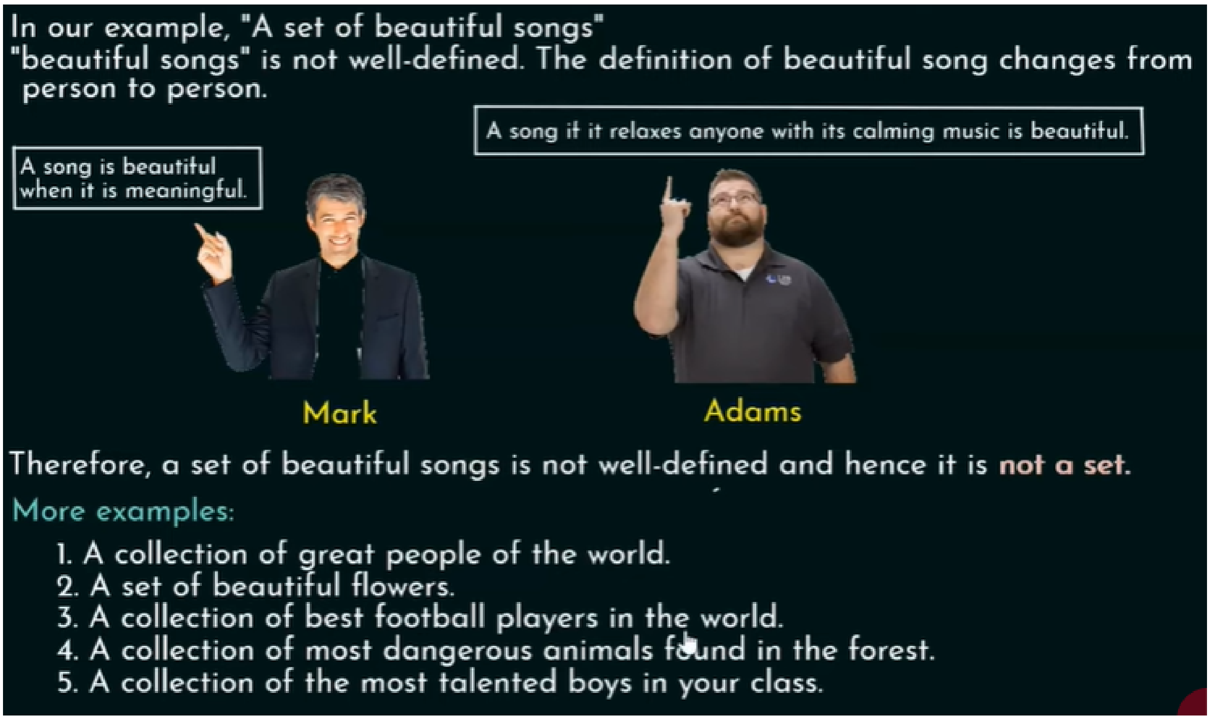
Discrete Mathematics - Set





Distinct objects:

We can have a set with duplicate objects.
But a set with duplicate objects is similar to a set with distinct objects.

For example: $A = \{1, 2, 2, 3, 3, 3\}$
$B = \{1, 2, 3\}$ $\qquad A = B$

Eventually, we end up with a set without duplicate elements.
This is the reason why the definition

"A set is a well-defined collection of distinct objects" holds true.

Set Membership

Any object belonging to a set is called a <u>member</u> or an <u>element</u> of that set.

We will represent sets by uppercase letters and lowercase letters will be used to represent the elements of the set.

If a is an element of set A then

$a \in A$ or a is in A

"belongs to" symbol.

If there exist an element b that does not belong to set A, then we express this fact by

$b \notin A$ or b is not in A

"does not belong to" symbol

Set Representation

Three ways to represent a set:

1. List representation.
2. Predicate representation.
3. Missing element representation.

1. List representation:

Let us suppose we have a set A with elements 1, 2, 3, a and b.
Generally, a set is represented by listing all the elements of it. Here, set A is represented by

$$A = \{1, 2, 3, a, b\}$$

Here elements are simply listed
within the pair of brackets ({})

2. Predicate representation:

In this representation, a set is defined by a predicate. This representation is more convenient then list representation.

For example: $B = \{x \mid x \text{ is an odd positive integer}\}$

Let us suppose that $P(x)$ denotes "x is an odd positive integer" then

$$B = \{x \mid P(x)\}$$

If we want to tell that some element b belongs to a set B then for this $P(b)$ has to be true.

For example: $1 \in B$ because 1 is an odd positive integer.
but $2 \notin B$ because 2 is not an odd positive integer.

The sets which are usually specified by listing elements can also be specified by predicates.

For Example: $A = \{1, 2, 3, a, b\}$ is equivalent to
$\{x \mid (x = 1) \vee (x = 2) \vee (x = 3) \vee (x = a) \vee (x = b)\}$

3. Missing element representation

Sometimes it is convenient to represent sets by missing element representation.

Example: $B = \{x \mid x \text{ is an odd positive integer}\}$
$B = \{1, 3, 5, ...\}$

Other examples: $C = \{2, 4, 6, 8, ...\}$
$D = \{1, 3, 5, ... , 17\}$

<u>Inclusion and Equality</u>

<u>Inclusion</u>:

Let A and B are two sets. If every element of A is an element of B, then A is called a <u>subset</u> of B or A is said to be <u>included</u> in B.

$A \subseteq B$ (A is a subset of B)
or
$B \supseteq A$ (B is a superset of A)

Example: $A = \{1, 2, 3\}$ $\quad$ $A \subseteq B$ but $B \supseteq A$
$B = \{1, 2, 3, 4, 5\}$ $\quad$ (Note: $B \not\subseteq A$)

Note: $A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ is true.

Why?

Example: $A = \{1, 2\}$ $\quad$ Consider all elements of A
$B = \{1, 3, 5\}$ $\quad$ $1 \in A$ and $1 \in B$
$2 \in A$ but $2 \notin B$

$\therefore x \in A \rightarrow x \in B$ is not true for all elements of A.

Important properties of set inclusion:

1. Reflexivity: $A \subseteq A$

   Example: $A = \{1, 2, 3\}$
   It is true that A is itself the subset of A.

2. Transitivity: $(A \subseteq B) \wedge (B \subseteq C) \rightarrow (A \subseteq C)$

   Example: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 5\}$, and $C = \{1, 2, 3, 5, 7\}$
   it is clear that $A \subseteq B$ and $B \subseteq C$
   Also, it is clear that $A \subseteq C$

$\therefore$ set inclusion is both reflexive and transitive.

Equality:

Two sets A and B are said be equal if $A \subseteq B$ and $B \subseteq A$.

$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A) \quad \text{OR} \quad A = B \Leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$$

Example:
1. $A = \{1, 2, 4\}$, $B = \{1, 2, 2, 4\}$
   $A = B$
2. $A = \{\{1, 2\}, 3\}$, $B = \{1, 2, 3\}$
   $A \neq B$ because $\{1, 2\} \in A$ and $\notin B$

Important properties:
1. Reflexive: $A = A$
2. Symmetric: $A = B \rightarrow B = A$ if $A = B$ is true then $B = A$ is also true.
3. Transitive: $(A = B) \wedge (B = C) \rightarrow (A = C)$

$\therefore$ Equality of sets is reflexive, symmetric and transitive.

Proper Subset:

A set A is said to be a proper subset of B if $A \subseteq B$ and $A \neq B$.
It is represented by $A \subset B$.

$$A \subset B \Leftrightarrow (A \subseteq B \wedge A \neq B)$$

For example: $A = \{1, 2, 4\}$
$B = \{1, 2, 4, 5\}$
then $A \subset B$

Important Properties:

Transitivity: $(A \subset B) \wedge (B \subset C) \Rightarrow (A \subset C)$

Note that proper subset is not reflexive.

Inclusion Vs Membership

Lets try to understand the difference between inclusion and membership with the help of an example.

Example: Let $A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

Which of the following is true?

$1 \in B$? true.
$1 \in A$? false.

Let us assume that $\{1, 2\}$ is represented by the name "Set A_1"

Try to understand this analogy.
We have a box named "Set A." After opening the box, we can see 4 different objects. One is a box named "Set A_1" and the rest are the elements 3, 4 and 5.

So, opening the box is associated with knowing the members of the set.

Here, we have opened the box named "Set A". After opening the box, we can only see the box "Set A_1" and not its elements. Hence, we don't know what is there inside set A_1

So, is it true that $1 \in A$?
No. The elements of Set A are Set A_1, 3, 4, 5.
But $1 \in A_1$

More questions:

$A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

>> $\{1, 2\} \in A$?

True. $\{1, 2\}$ is a set within set A. Therefore, $\{1, 2\} \in A$.

>> $\{3, 4\} \subseteq A$?

True. Whenever it is required to answer if a particular set is a subset of a different set, see the elements of the given set and compare it with the elements of the other set.

Here, the given set is $\{3, 4\}$.
Ask this: $3 \in A$? Yes
$4 \in A$? Yes
$\therefore \{3, 4\} \subseteq A$

>> $\{1, 6\} \subseteq B$?

False. Ask yourself: $1 \in B$? Yes.
$6 \in B$? No.
$\therefore \{1, 6\} \not\subseteq B$

>> $1 \subseteq B$? $A = \{\{1, 2\}, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$

False. 1 is not a set itself.

>> $1 \in B$?

True. 1 is the element in B

$\therefore 1 \in B$

>> $\{1, 2\} \subseteq A$?

False. Ask yourself: $1 \in A$? No.

(Note: 1 belongs to set $\{1, 2\}$ contained within set A. It does not belong to A)

$2 \in A$? No.

$\therefore \{1, 2\} \not\subset A$

>> $\{\{1, 2\}\} \subseteq A$?

True. Ask yourself: $\{1, 2\} \in A$? Yes.

$\therefore \{\{1, 2\}\}$ is the subset of A.

>> $\{\{1, 2\}, 3, 4, 5\} \subseteq A$?

True. In fact, the given set is equal to A.

Universal Set, Null Set, and Singleton Set

Universal Set:

A universal set is a set which includes every set under consideration.
A universal set is represented by E.
For any predicate, P(x)

$$E = \{x \mid P(x) \lor \neg P(x)\}$$

The universal set is same as universe of discourse.

Null Set:

A set which does not contain any element is called a null set or empty set.
It is denoted by ϕ or {}.

$$\phi = \{x \mid P(x) \land \neg P(x)\}$$

For example: A set of all positive integers which are both even and odd.

Singleton Set:

A singleton set is a set with exactly one element.

For example: $A = \{2\}$
$B = \{\phi\}$

please note that $\{\phi\} \neq \phi$

$\{\phi\}$ consists of one element which is the null set itself while there is no element inside ϕ.

Null Set
(Solved Problem)

Determine whether the following statements are true or false.

a) $\phi \in \{\phi\}$

True. ϕ is an empty set and is also a member of set $\{\phi\}$.

b) $\phi \in \{\phi, \{\phi\}\}$

True.

c) $\{\phi\} \in \{\phi\}$

False. $\{\phi\}$ is not a member of the $\{\phi\}$ because there is only one element in the set $\{\phi\}$ which is ϕ not $\{\phi\}$.

d) $\{\phi\} \in \{\{\phi\}\}$

True. $\{\phi\}$ is the member of the set $\{\{\phi\}\}$.

e) $\{\phi\} \subset \{\phi, \{\phi\}\}$

True. Ask yourself: $\phi \in \{\phi, \{\phi\}\}$? Yes. ϕ is an element of the set $\{\phi, \{\phi\}\}$

f) $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$

True. Ask yourself: $\{\phi\} \in \{\phi, \{\phi\}\}$? Yes. $\{\phi\}$ is an element of the set $\{\phi, \{\phi\}\}$

g) $\{\{\phi\}\} \subset \{\{\phi\}, \{\phi\}\}$

False. The above statement is equivalent to $\{\{\phi\}\} \subset \{\{\phi\}\}$.

Important Theorem on Non-Empty Set

Theorem: Every non-empty set A is guaranteed to have at least two subsets, the empty set and the set A itself i.e. $\phi \subseteq A$ and $A \subseteq A$.

Proof: (i) $\phi \subseteq A$

If ϕ is a subset of A, then from the definition of subset:
$\forall x(x \in \phi \rightarrow x \in A)$

$x \in \phi$ is always false because no element can be a member of an empty set. Therefore, the implication $x \in \phi \rightarrow x \in A$ will always be true.

$\boxed{\text{Hence, } \phi \subseteq A.}$

(ii) $A \subseteq A$

If A is a subset of A, then from the definition of subset:
$\forall x(x \in A \rightarrow x \in A)$

No matter what x we choose, if $x \in A$ is true, $x \in A$ must be true. Therefore, the implication $x \in A \rightarrow x \in A$ will always be true for every element x.

$\boxed{\text{Hence, } A \subseteq A}$

Power Set

For a particular set A, a collection of all subsets of set A is called the power set of set A. Usually, a power set of set A is denoted by P(A) or 2^A.

$$P(A) = \{x \mid x \subseteq A\}$$

Examples: $P(\phi) = \{\phi\}$
$A = \{a\}$, $P(A) = \{\phi, \{a\}\}$
$B = \{a, b\}$, $P(B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$
$C = \{a, b, c\}$, $P(C) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

This is the reason why power set of A is sometimes represented by 2^A.
In set A, we have only 1 element.
Therefore, P(A) has $2^1 = 2$ elements.

In set B, we have 2 elements.
Therefore, P(B) has $2^2 = 4$ elements.

Therefore, if we have n elements in set A, then P(A) must have 2^n elements.

Problem 1: Give the power sets of the following
a) $\{a, \{b\}\}$
b) $\{1, \phi\}$
c) $\{X, Y, Z\}$

Solution: a) Let $A = \{a, \{b\}\}$
$P(A) = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$
b) Let $B = \{1, \phi\}$
$P(B) = \{\phi, \{1\}, \{\phi\}, \{1, \phi\}\}$
c) Let $C = \{X, Y, Z\}$
$P(C) = \{\phi, \{X\}, \{Y\}, \{Z\}, \{X, Y\}, \{Y, Z\}, \{X, Z\}, \{X, Y, Z\}\}$

Problem 2: Find the power set of each of these sets, where a and b are distinct elements.
a) $\{a\}$ b) $\{a, b\}$ c) $\{\phi, \{\phi\}\}$

Solution: a) Let $A = \{a\}$, $P(A) = \{\phi, \{a\}\}$
b) Let $B = \{a, b\}$, $P(B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$
c) Let $C = \{\phi, \{\phi\}\}$, $P(C) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Problem 3: How many elements does each of these sets have where a and b are distinc
elements?
a) $P(\{a, b, \{a, b\}\})$
b) $P(\{\phi, a, \{a\}, \{\{a\}\}\})$
c) $P(P(\phi))$

Solution: a) The set $\{a, b, \{a, b\}\}$ has 3 elements.
$P(\{a, b, \{a, b\}\})$ contains all possible subsets of $\{a, b, \{a, b\}\}$.
Therefore, $P(\{a, b, \{a, b\}\})$ has $2^3 = 8$ elements.

b) Let $A = \{\phi, a, \{a\}, \{\{a\}\}\}$ has 4 elements.
Therefore, P(A) has $2^4 = 16$ elements.

c) The set ϕ has no elements.
$P(\phi)$ has 1 element which is set ϕ.
Therefore, $P(P(\phi))$ must have $2^1 = 2$ elements.

$P(P(\phi)) = P(\{\phi\}) = \{\phi, \{\phi\}\}$

Problem: For a set A, the power set of A is denoted by 2^A. If A = {5, {6}, {7}}, which of the following options are TRUE?

I. $\phi \in 2^A$
II. $\phi \subseteq 2^A$
III. $\{5, \{6\}\} \in 2^A$
IV. $\{5, \{6\}\} \subseteq 2^A$

(A) I and III only.
(B) II and III only.
(C) I, II and III only.
(D) I, II and IV only.

[GATE 2015]

Solution: Power set of A is denoted by 2^A. A = {5, {6}, {7}}

$$P(A) = 2^A = \{\phi, \{5\}, \{\{6\}\}, \{\{7\}\}, \{5, \{6\}\}, \{\{6\}, \{7\}\}, \{5, \{7\}\}, \{5, \{6\}, \{7\}\}\}$$

I. $\phi \in 2^A$
Yes. ϕ is the member of set 2^A.

II. $\phi \subseteq 2^A$
True. $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$.

III. $\{5, \{6\}\} \in 2^A$
True. $\{5, \{6\}\}$ is the member of set 2^A.

IV. $\{5, \{6\}\} \subseteq 2^A$
False. $5 \notin 2^A$
$\{6\} \notin 2^A$

$\therefore$ option (C) is the correct option.

Cardinality of Sets

Cardinality of a set means size of a set.
Cardinality of set A is denoted by |A|.

Example: A set of all odd positive integers less than 10.

A = {1, 3, 5, 7, 9} $\boxed{|A| = 5}$

Example 2: Cardinality of null set is 0. $|\Phi| = 0$

[GATE Problem] [GATE 2015]

The cardinality of the power set of {0, 1, 2, ..., 10} is?

Solution: Let say A = {0, 1, 2, ..., 10} Total 11 elements in set A

Therefore, $|P(A)| = 2^{11} = 2048$.

Cartesian Product

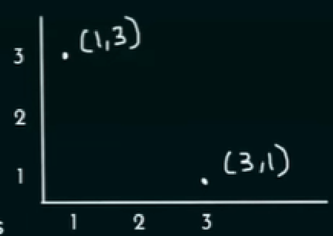
Sometimes, we want a collection of ordered pairs.
Let's say we are interested in finding all possible combinations of X and Y coordinates where X and Y values should not exceed 3.

{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Here, (a, b) for any a and b is called an ordered pair.

Note: (a, b) and (b, a) are not equal.
Therefore, in an ordered pair, order does matter.

Example:



Definition of Cartesian Product:
Let A and B are two sets. The cartesian product of A and B, denoted by AXB, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A X B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example 1: What is the cartesian product of A = {1, 2} and B = {a, b, c}?

Solution: $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

Note 1: $AXB \neq BXA$

Example: A = {1, 2} and B = {3, 4}

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

Note 2: For any two non-empty sets A and B, if there are n elements in A and m elements in B, then the number of ordered pairs in AXB will be nm.

Note 3: $\phi \times A = A \times \phi = \phi$

From the definition,

$\phi \times A = \{(a, b) \mid a \in \phi \text{ and } b \in A\} = \phi$

There is no such pair (a, b) where $a \in \phi$ and $b \in A$.

Similarly, $A \times \phi = \phi$

Therefore, $\phi \times A = A \times \phi = \phi$

Question 3: Show by means of an example that $AXBXC \neq (AXB)XC$

Solution: Let A = {1, 2}, B = {3, 4}, C = {5, 6}

$AXBXC = \{(a, b, c) \mid a \in A \wedge b \in B \wedge c \in C\}$

$AXB = \{(a, b) \mid a \in A \wedge b \in B\}$

$(AXB)XC = \{(x, c) \mid x \in AXB \wedge c \in C\}$

$(AXB)XC = \{((a, b), c) \mid (a \in A \wedge b \in B) \wedge c \in C\}$

$AXBXC = \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\}$

$(AXB)XC = \{((1, 3), 5), ((1, 3), 6), ((1, 4), 5), ((1, 4), 6), ((2, 3), 5), ((2, 3), 6), ((2, 4), 5), ((2, 4), 6)\}$

Set Operations

(Intersection and Union with Venn Diagrams)

Definition of Intersection:

The intersection of any two sets A and B, denoted by $A \cap B$, is the set consisting of all the elements which belong to both A and B.

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Representing Intersection of Two Sets Pictorially using Venn Diagrams

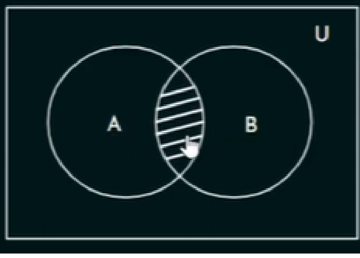
A Venn diagram is a diagram used to illustrate the logical relationship between two or more sets by using overlapping circles or other shapes.

Basic Overview:

A set is usually represented by a circle and the elements of the set lies within the circle.

A universal set or universe of discourse is represented by a rectangle. Every element under consideration lies within the rectangle.

Pictorial Representation of Intersection of sets A and B



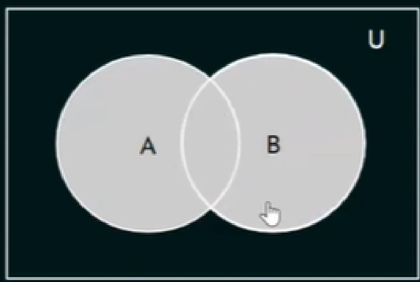
Shaded region is the common area between A and B i.e. $A \cap B$.

Definition of Union

For any two sets A and B, the union of A and B, denoted by $A \cup B$, is the set of all elements which are members of the set A or set B or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Venn Diagram of Union of Sets



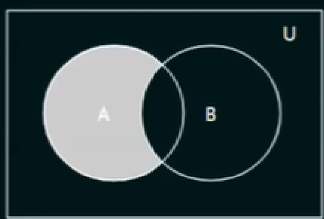
Shaded region is representing union of A and B.

Set Difference and Complement

For any two sets A and B, the difference of A and B, denoted by A - B, is the set containing those elements that are in set A but not in set B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Venn Diagram Representation of A - B



Shaded region represents A but not B

The difference of A and B is also called the complement of B with respect to A.

Example 1: Let say, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

$A - B = \{5\}$ (A but not B)

Example 2: Let say A represents a set of all computer science students at a University and B represents a set of all students who are studying discrete mathematics.

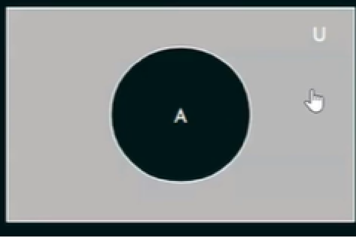
then, A - B will represent a set of all computer science students who are not studying discrete mathematics.

Complement of a Set

Let U be the Universal set.
The complement of set A, denoted by A', is the complement of A with respect to U.
In other words, complement of set A is U - A.

$$A' = \{x \mid x \notin A\}$$

Venn Diagram for the Complement of Set A



Shaded region represents A'

Set Identities

Let's consider some basic set identities.
Let say A, B and C are three sets and these three sets are subsets of the universal set U.

Set Identities:

1. Identity Laws:
   a) $A \cup \phi = A$
   b) $A \cap U = A$

2. Domination Laws:
   a) $A \cup U = U$
   b) $A \cap \phi = \phi$

3. Idempotent Laws:
   a) $A \cup A = A$
   b) $A \cap A = A$

4. Complementation Laws:
   $(A')' = A$

5. Commutative Laws:
   a) $A \cup B = B \cup A$
   b) $A \cap B = B \cap A$

6. Associative Laws:
   a) $A \cup (B \cup C) = (A \cup B) \cup C$
   b) $A \cap (B \cap C) = (A \cap B) \cap C$

7. Distributive Laws:
   a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

8. De Morgan's Laws:
   a) $(A \cup B)' = A' \cap B'$
   b) $(A \cap B)' = A' \cup B'$

9. Absorption Laws:
   a) $A \cup (A \cap B) = A$
   b) $A \cap (A \cup B) = A$

10. Complement Laws:

a) $A \cup A' = U$

b) $A \cap A' = \phi$

Problem: Show that if A and B are sets, then

a) $A - B = A \cap B'$ b) $(A \cap B) \cup (A \cap B') = A$

Solution:

a) $LHS = A - B$
$= \{x \mid x \in A \wedge x \notin B\}$
$= \{x \mid x \in A \wedge \neg(x \in B)\}$
$= \{x \mid x \in A \wedge x \in B'\}$
$= \{x \mid x \in (A \cap B')\}$
$= A \cap B' = RHS$

b) $LHS = (A \cap B) \cup (A \cap B')$
$= \{x \mid x \in (A \cap B) \vee x \in (A \cap B')\}$
$= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in B')\}$
From the distributive law of logical equivalence
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

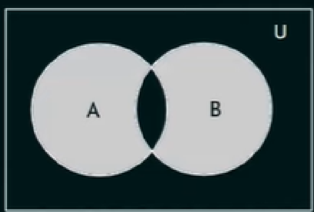
$= \{x \mid x \in A \wedge (x \in B \vee x \in B')\}$
$= \{x \mid x \in A \wedge T\}$
$= \{x \mid x \in A\} = A = RHS$

Symmetric Difference

Symmetric difference of sets A and B, denoted by $A \oplus B$, is the set of all elements which are in either set A or set B, but not in both A and B.

$$A \oplus B = \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$$

Venn diagram representation of $A \oplus B$



Note that:
1. $A \oplus B = (A \cup B) - (A \cap B)$
2. $A \oplus B = (A - B) \cup (B - A)$

Proof: 1. $A \oplus B = (A \cup B) - (A \cap B)$
$LHS = A \oplus B$
$= \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$
$= \{x \mid (x \in (A \cup B)) \wedge \neg(x \in A \wedge x \in B)\}$
$= \{x \mid (x \in (A \cup B)) \wedge \neg(x \in (A \cap B))\}$
$= \{x \mid (x \in ((A \cup B) - (A \cap B))\}$
$= (A \cup B) - (A \cap B) = RHS$

Problem: Show that if A is a subset of Universal set U, then

a) $A \oplus A = \phi$ b) $A \oplus \phi = A$ c) $A \oplus U = A'$ d) $A \oplus A' = U$

Solution:

a) $A \oplus A = \phi$
$LHS = A \oplus A$
$= \{x \mid (x \in A \vee x \in A) \wedge \neg(x \in A \wedge x \in A)\}$
$= \{x \mid x \in A \wedge \neg(x \in A)\}$
$= \{x \mid x \in A \wedge x \notin A\} = \phi = RHS$

b) $A \oplus \phi = A$
$LHS = A \oplus \phi$
$= \{x \mid (x \in A \vee x \in \phi) \wedge \neg(x \in A \wedge x \in \phi)\}$
$= \{x \mid (x \in A \vee F) \wedge \neg(x \in A \wedge F)\}$
$= \{x \mid x \in A \wedge \neg(F)\}$
$= \{x \mid x \in A \wedge T\}$
$= \{x \mid x \in A\} = A = RHS$

Introduction to Relations

Definition: Let A and B be two sets. A binary relation R from A to B is a subset of AXB.

OR

$$R \subseteq A \times B$$

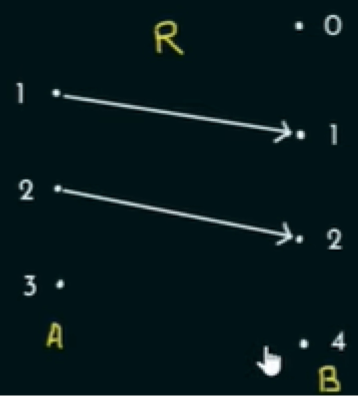
Recall: $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
Usually we use the notation aRb to denote $(a, b) \in R$.

$a \not R b$ is used to denote $(a, b) \notin R$

Example: Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 4\}$
$A \times B = \{(1, 0), (1, 1), (1, 2), (1, 4), (2, 0), (2, 1), (2, 2), (2, 4), (3, 0), (3, 1), (3, 2), (3, 4)\}$

Let say R is the relation where $(a, b) \in R$ if and only if $a = b$ then
$R = \{(1, 1), (2, 2)\}$ and $R \subseteq A \times B$

Graphical representation of ordered pairs.



Arrows are used to represent ordered pairs of relation R.

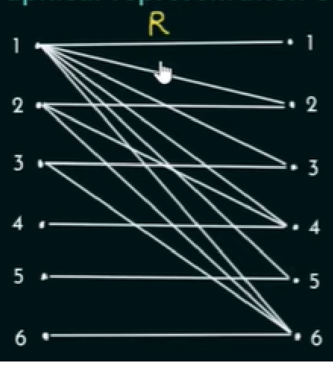
Relation from a set to itself

A relation on a set A is a relation from A to A.

Example: Let $R = \{(a, b) \mid a \text{ divides } b\}$ $\quad A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

Graphical representation of relation R



Number of relations on a set with n elements

A relation on a set A is a subset of AXA. Set A contains n elements and AXA contains n^2 elements

We are interested in listing down all the subsets of AXA means we are interested in finding the power set of AXA.
We know that P(A) is the power set of set A and let say A has n elements then P(A) must have 2^n elements.
Therefore, P(AXA) must have 2^{n^2} elements.

Hence, number of relations on a set A with n elements = $|P(AXA)| = 2^{n^2}$

Types of Relations (Part 1)

1. Reflexive Relation:

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.
In other words, $\forall a((a, a) \in R)$.

Example: Let A = {1, 2, 3, 4}
$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (4, 4)\}$
Relation R_1 is reflexive because it contains all ordered pairs of the form (a, a) for every element $a \in A$ i.e., R_1 has (1, 1), (2, 2), (3, 3), (4, 4)

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 4)\}$
Relation R_2 is not reflexive because the ordered pair (3, 3) is not in R_2.

2. Irreflexive Relation:

A relation R on a set A is called irreflexive if $\forall a \in A, (a, a) \notin R$.

Example: A = {1, 2, 3, 4}
$R_3 = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$ is not irreflexive because (3, 3) and (4, 4) is there in R_3.

$R_4 = \{(1, 2), (2, 1)\}$ is irreflexive because $\forall a \in A, (a, a) \notin R_4$

3. Symmetric Relation:

A relation R on a set A is called symmetric if $(b, a) \in R$ holds when $(a, b) \in R$ for all $a, b \in A$
In other words, relation R on a set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$

Example: Relation $R_5 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is symmetric because for every $(a, b) \in R_5$ $(b, a) \in R_5$
like (1, 2) (2, 1) is in R_5.
There is no need to check for (1, 1), (2, 2).

Relation $R_6 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ is not symmetric because for (1, 2) there is no (2, 1) in R_6. Same is true for (1, 3) and (1, 4).

4. Antisymmetric Relation:

A relation R on a set A is called antisymmetric if $\forall a \forall b((a, b) \in R \wedge (b, a) \in R \rightarrow (a = b))$
Whenever we have (a, b) in R, we will never have (b, a) in R until or unless (a = b)

Example: Relation $R_7 = \{(1, 1), (2, 1)\}$ on set A is antisymmetric because (2, 1) is in R_7 but (1, 2) is not in R_7.

5. Transitive Relation:

A relation R on a set A is called transitive if $\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$

Example: $A = \{1, 2, 3, 4\}$

$R_8 = \{(2, 1), (3, 1), (3, 2), (4, 4)\}$ is transitive because (3, 2), (2, 1), and (3, 1) are there in R_8.

$R_9 = \{(2, 1), (1, 3)\}$ is not transitive as (2, 1) and (1, 3) are there in R_9 but there is no (2, 3) in relation R_9.

6. Asymmetric Relation:

A relation R on a set A is called asymmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$

Example: $A = \{1, 2, 3, 4\}$

$R_{10} = \{(1, 1), (1, 2), (1, 3)\}$ is not an asymmetric relation because of (1, 1).

$R_{11} = \{(1, 2), (1, 3), (2, 3)\}$ is an asymmetric relation.

Summary

Relation	Property
1. Reflexive	$\forall a((a, a) \in R)$
2. Irreflexive	$\forall a((a, a) \notin R)$
3. Symmetric	$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$
4. Antisymmetric	$\forall a \forall b(((a, b) \in R \land (b, a) \in R) \rightarrow (a = b))$
5. Asymmetric	$\forall a \forall b((a, b) \in R \rightarrow (b, a) \notin R)$
6. Transitive	$\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$

Types of Relations (Solved Problem)

Problem: Determine whether relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if
a) $x + y = 0$ b) $x - y$ is a rational number c) $xy = 0$
d) $x = 1$ or $y = 1$

Solution: a) $R = \{(x, y) \mid x + y = 0\}$

1. Reflexive: $\forall a \in \mathbb{R} ((a, a) \in R)$
   Not reflexive. Only true for (0, 0).
2. Symmetric:
   $\forall a \forall b \in \mathbb{R} ((a, b) \in R \rightarrow (b, a) \in R)$
   If $a+b = 0$ then $b+a = 0$.
   Therefore, R is symmetric.
3. Antisymmetric:
   $\forall a \forall b \in \mathbb{R} (((a, b) \in R \land (b, a) \in R) \rightarrow (a=b))$
   Not antisymmetric.
   $1+(-1)=0$ and $-1+1=0$ but $1 \neq -1$
4. Transitive:
   $\forall a \forall b \forall c(((a, b) \in R \land (b, c) \in R) \rightarrow (a, c) \in R)$
   Not Transitive. $1+(-1)=0$ and $(-1)+1 = 0$ but $1+1 \neq 0$

b) $R = \{(x, y) \mid x-y$ is a rational no.$\}$

1. Reflexive: $a - a = 0$ is a rational number.
2. Symmetric:
   if $a-b$ is a rational number then $b-a = -(a-b)$ is also a rational number.
3. Antisymmetric: Not antisymmetric
   $3 - 2$ and $2 - 3$ are both rational numbers but $3 \neq 2$.
4. Transitive:
   If $a - b$ is a rational number and $b - c$ is also a rational number, then $a - c = (a-b)+(b-c)$ is also a rational number.

c) $R = \{(x, y) \mid xy = 0\}$

c) $R = \{(x, y) \mid xy = 0\}$

1. Reflexive: Not reflexive
   $a.a = a^2 = 0$ when $a = 0$.
2. Symmetric:
   $ab = 0$ then $ba = 0$
3. Antisymmetric:
   Not antisymmetric.
   $1 \times 0 = 0$ and $0 \times 1 = 0$ but $1 \neq 0$
4. Transitive: Not transitive
   $-1 \times 0 = 0$ and $0 \times 2 = 0$ but
   $-1 \times 2 \neq 0$.

d) $R = \{(x, y) \mid x = 1 \text{ or } y = 1\}$

1. Reflexive: Not reflexive
   $(0, 0)$ is not in R.
2. Symmetric:
   if $(a, b) \in R$ then $a = 1$ or $b = 1$.
   This means $(b, a) \in R$.
3. Antisymmetric:
   Not antisymmetric.
   if $a = 1$ and $b = 0$ then $(a, b) \in R$ and
   $(b, a) \in R$ but $a \neq b$.

Types of Relations (GATE Problems)

Problem 1: Amongst the properties {reflexivity, symmetry, antisymmetry, transitivity}, the relation $R = \{(x, y) \in N^2 \mid x \neq y\}$ satisfies?

[GATE 1994]

Solution:

1. Reflexivity: $\forall a((a, a) \in R)$
   $a \neq a$ (Not possible).
   Therefore, R is not reflexive.

2. Symmetry: $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$
   $a \neq b$ then $b \neq a$
   Therefore, R is symmetric.

3. Antisymmetry: $\forall a \forall b(((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$
   $(a \neq b \wedge b \neq a) \rightarrow (a = b)$ is false.
   Therefore, R is not antisymmetric.

4. Transitivity: $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$
   $(a \neq b \wedge b \neq c) \rightarrow (a \neq c)$
   Need not be true.
   Let $a = 1$, $b = 2$, and $c = 1$
   $1 \neq 2$ and $2 \neq 1$ but $1 = 1$.
   Therefore, R is not transitive.

Problem 2: The binary relation $S = \varnothing$(empty set) on a set $A = \{1, 2, 3\}$ is
(A) Neither reflexive nor symmetric.
(B) Symmetric and reflexive.
(C) Transitive and reflexive.
(D) Transitive and symmetric.

[GATE 2002]

Solution: $S = \varnothing$

(i) Reflexivity: $\forall a \in A((a, a) \in S)$
S is an empty set and there is not even a single pair (a, a) inside S.
Therefore, S is not reflexive.

(ii) Symmetry: $\forall a \forall b \in A((a, b) \in S \rightarrow (b, a) \in S)$
$S = \varnothing$ is symmetric.
because $(a, b) \in S$ is false. Hence, implication is always true.
Therefore, empty set is symmetric.

(iii) Transitivity: $\forall a \forall b \forall c \in A(((a, b) \in S \wedge (b, c) \in S) \rightarrow (a, c) \in S)$
$S = \varnothing$ is transitive.
because $(a, b) \in S \wedge (b, c) \in S$ is false.
Hence, implication is always true.

Therefore, option (D) is the correct option.

Types of Relations (GATE Problems)

Problem 1: Consider the binary relation $R = \{(x, y), (x, z), (z, x), (z, y)\}$ on the set $\{x, y, z\}$. Which one of the following is TRUE?

(A) R is symmetric but NOT antisymmetric.
(B) R is NOT symmetric but antisymmetric.
(C) R is both symmetric and antisymmetric.
(D) R is neither symmetric nor antisymmetric.

[GATE 2009]

Solution: $R = \{(x, y), (x, z), (z, x), (z, y)\}$

(i) Symmetric: $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R\}$
$(x, y) \in R$ but $(y, x) \notin R$
Therefore, R is not symmetric.

(ii) Antisymmetric: $\forall a \forall b((a, b) \in R \wedge (b, a) \in R \rightarrow (a=b)\}$
$(x, z) \in R \wedge (z, x) \in R$ but $x \neq z$
Therefore, R is not antisymmetric.

Operations on Relations

Let us consider two relations R_1 and R_2 from set A to B.
These relations can be combined in a way two sets can be combined. i.e., $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

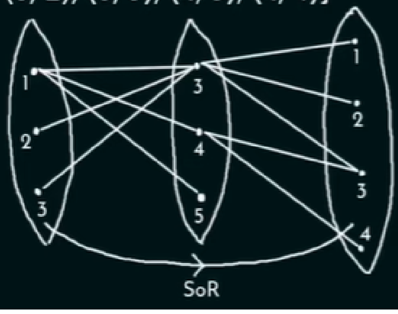
$R_1 = \{(1, 3), (1, 4), (3, 3)\}$
$R_2 = \{(2, 3), (2, 4), (3, 3), (2, 5)\}$
$R_1 \cup R_2 = \{(1, 3), (1, 4), (2, 3), (2, 4), (2, 5), (3, 3)\}$
$R_1 \cap R_2 = \{(3, 3)\}$
$R_1 - R_2 = \{(1, 3), (1, 4)\}$
$R_2 - R_1 = \{(2, 3), (2, 4), (2, 5)\}$
$R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2) = \{(1, 3), (1, 4), (2, 3), (2, 4), (2, 5)\}$

Composition of Relations

Let A, B and C be three sets. Suppose, R is a relation from A to B, and S is a relation from B to C. The composite of R and S, denoted by SoR, is a binary relation from A to C consisting of ordered pairs (a, c) where $a \in A$ and $c \in C$. Also, $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Example: What is the composite of the relations R and S, where R is the relation from set A = {1, 2, 3} to set B = {3, 4, 5} with R = {(1, 3), (1, 4), (1, 5), (2, 3), (3, 3)} and S is the relation from set B = {3, 4, 5} to set C = {1, 2, 3, 4} with S = {(3, 1), (3, 2), (3, 3), (4, 3), (4, 4)}

Solution:



SoR = {(1, 1), (1, 2), (1, 3), (1,4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Note: If R and S are relations from A to B and C to D respectively, then SoR is not defined unless B = C.

Composition of Relation with itself

Problem: Let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2) and (5, 4). Find

a) R^2 b) R^3 c) R^4

Solution: R = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), (5, 4)}

R = {(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), (5, 4)}

a) R^2 is the composite of R with itself and it consists of ordered pairs (a, c) such that $(a, b) \in R$ and $(b, c) \in R$.

R^2 = RoR

RoR = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 4), (2, 5), (2, 2), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}

b) $R^3 = R^2 oR$

= {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}

c) $R^4 = R^3 oR$

R^3 already contains all possible ordered pairs in AxA, R^4 will also contain all possible ordered pairs.

Therefore, $R^4 = R^3$

Representation of Relations

Let say we have two sets A and B.

$A = \{1, 2, 3\}$
$B = \{0, 1, 2, 4\}$

There are multiple ways to represent a relation.

Let say we want to represent a relation R which consists of all ordered pairs (x, y) where $x \in A$ and $y \in B$ and $x \leq y$.

1. Listing Method: $R = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 4)\}$
2. Set Builder Method:

$R_{A \text{ to } B} = \{(x, y) \mid x \leq y\}$ OR $R = \{(x, y) \mid x \in A \wedge y \in B \wedge x \leq y\}$

3. Statement Representation:

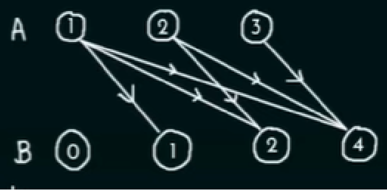
$\forall x \in A \ \forall y \in B, xRy \text{ iff } x \leq y$

4. Matrix Representation:

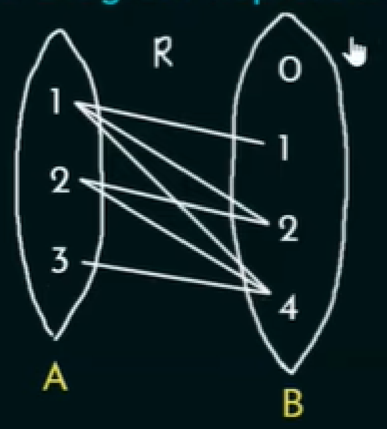
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{m \times n} \end{array}$$

$|A| = m$
$|B| = n$

5 Graph Representation:
Directed Graph



6. Arrow Diagram Representation:



Closure of Relations (Part 1)

Reflexive Closure:

Let say we have a binary relation R.
$R = \{(1, 1), (2, 2), (2, 3)\}$ defined on a set $A = \{1, 2, 3\}$.

The relation R is not reflexive.
Smallest reflexive relation that contains R must include the ordered pair (3, 3).

$R_{New} = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$

Definition: Reflexive closure of a binary relation R on a set A is the smallest reflexive relation of the set A that contains R.
Reflexive closure of R is usually denoted by R_r^+.

$R_r^+ = R \cup \{(a, a) \mid a \in A\}$

Problem: Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the reflexive closure of R.

Solution: $R = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$
$A = \{0, 1, 2, 3\}$

Reflexive closure of R

$R_r^+ = R \cup \{(a, a) \mid a \in A\}$

$R_r^+ = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (0, 0), (3, 3)\}$

This relation
1. contains R.
2. is reflexive.
3. is minimal.

Closure of Relations (Part 2)

Symmetric closure: Symmetric closure of a binary relation R on a set A is the smallest symmetric relation on a set A that contains R.

$$R_s^+ = R \cup \{(b, a) \mid (a, b) \in R\}$$

Problem: Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the symmetric closure of R.

Solution: $R = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$
$A = \{0, 1, 2, 3\}$

$R_s^+ = R \cup \{(b, a) \mid (a, b) \in R\}$
$R_s^+ = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (1, 0), (2, 1), (0, 2), (0, 3)\}$
$R_s^+ = \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0)\}$

Transitive closure: Transitive closure of a binary relation R on a set A is the smallest transitive relation on a set A that contains R.

$$R_t^+ = R \cup \{(a, c) \mid (a, b) \in R \wedge (b, c) \in R\}$$

Problem: Let R be the relation on a set {1, 2, 3} containing the ordered pairs (1, 1), (2, 3), and (3, 1). Find the transitive closure of R.

Solution: $R = \{(1, 1), (2, 3), (3, 1)\}$
$A = \{1, 2, 3\}$

$R_t^+ = R \cup \{(a, c) \mid (a, b) \in R \wedge (b, c) \in R\}$
$R_t^+ = \{(1, 1), (2, 3), (3, 1), (2, 1)\}$

Closure of Relations (Solved Problems)

Problem 1: Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

Solution: $R = \{(a, b) \mid a \neq b\}$
Relation R is defined on set of integers.

$R_r^+ = R \cup \{(a, a) \mid a \in A\}$ [A is the set of all integers]
$= \{(a, b) \mid a \neq b\} \cup \{(a, b) \mid a = b\}$
This means all pairs of integers must be included in the reflexive closure of R.
$= \{(a, b) \mid a, b \in Z\}$
$= ZXZ$

Therefore, $R_r^+ = ZXZ$

Problem 2: Let R be the relation $\{(a, b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

Solution: $R = \{(a, b) \mid a \text{ divides } b\}$

A = set of integers = Z

$$
\begin{aligned}
R_S^{+} &= R \cup \{(b, a) \mid (a, b) \in R\} \\
&= \{(a, b) \mid a \text{ divides } b\} \cup \{(b, a) \mid a \text{ divides } b\} \\
&= \{(a, b) \mid a \text{ divides } b\} \cup \{(a, b) \mid b \text{ divides } a\} \\
&= \{(a, b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\}
\end{aligned}
$$

[Warshall's Algorithm (Finding the Transitive Closure)]

NOT in SIR PDF

Warshall's Algorithm

(Finding the Transitive Closure)

Finding the transitive closure using Warshall's Algorithm

Sometimes it is difficult to find all the ordered pairs in transitive closure of a relation. Warshall's Algorithm is considered an efficient method in finding the transitive closure of a relation.

Example: By using Warshall's algorithm, find the transitive closure of the relation $R = \{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$ on set $A = \{1, 2, 3, 4\}$.

Solution: First, we will represent the relation R in matrix form.

$$
R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}
$$

Understand that there are 4 elements in set A. Therefore, 4 steps are required in order to find the transitive closure of relation R [According to Warshall's Algorithm].

>> In Step 1, we will consider 1st column and 1st row of the above matrix i.e., C_1 and R_1.

Write all positions where 1 is present in column 1.

$C_1 = \{2, 3, 4\}$

Also, write all position where 1 is present in row 1

$R_1 = \varnothing$

$$
R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}
$$

Now, take the cross product of C_1 and R_1.

$C_1 X R_1 = \varnothing$.

Therefore, no new additions.

>> In Step 2, we will consider 2nd column and 2nd row of the above matrix.

C_2	R_2
$\varnothing$	$\{1, 3\}$

$C_2 X R_2 = \varnothing$

Therefore, no new additions.

>> In Step 3, we will consider 3rd column and 3rd row.

C_3	R_3
$\{2, 4\}$	$\{1, 4\}$

$C_3 X R_3 = \{(2, 1), (2, 4), (4, 1), (4, 4)\}$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}_{4\times 4}$$

New Matrix

>> In Step 4, we will consider 4th column and 4th row of the above matrix.

C_4	R_4
{2, 3, 4}	{1, 3, 4}

$C_4 X R_4 = \{(2, 1), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}_{4\times 4}$$

New Matrix

R_t^+

$R_t^+ = \{(2, 1), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$

R_t^+ is the transitive closure of relation R.

Closure of Relations (Solved Problems)
(Set 2)

Problem 1: Find the smallest relation containing the relation {(1, 2), (1, 4), (3, 3), (4, 1)} defined on a set A = {1, 2, 3, 4} that is reflexive and transitive.

Solution: Reflexive closure of R

$R_r^+ = R \cup \{(a, a) \mid a \in A\}$

$R_r^+ = \{(1, 2), (1, 4), (3, 3), (4, 1)\} \cup \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$= \{(1, 1), (1, 2), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)\}$

Transitive closure of R

R = {(1, 2), (1, 4), (3, 3), (4, 1)}

$$R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

Step 1:

C_1	R_1
{4}	{2, 4}

$C_1 X R_1 = \{(4, 2), (4, 4)\}$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{array}$$

Step 2: C_2 R_2

$\{1, 4\}$ $\varnothing$

$C_2 \times R_2 = \varnothing$

Step 3: C_3 R_3

$\{3\}$ $\{3\}$

$C_3 \times R_3 = \{(3, 3)\}$

Step 4: C_4 R_4

$\{1, 4\}$ $\{1, 2, 4\}$

$C_4 \times R_4 = \{(1, 1), (1, 2), (1, 4), (4, 1), (4, 2), (4, 4)\}$

$$R_t^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$R_t^+ = \{(1, 1), (1, 2), (1, 4), (3, 3), (4, 1), (4, 2), (4, 4)\}$

$R_r^+ = \{(1, 1), (1, 2), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)\}$

$R_r^+ \cup R_t^+ = \{(1, 1), (1, 2), (1, 4), (2, 2), (3, 3), (4, 1), (4, 2), (4, 4)\}$

Problem 2: Find the transitive closure of relation $R = \{(a, c), (b, d), (c, a), (d, b), (e, d)\}$ on set $A = \{a, b, c, d, e\}$

Solution: Let say $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 5$.

$A = \{1, 2, 3, 4, 5\}$

$R = \{(1, 3), (2, 4), (3, 1), (4, 2), (5, 4)\}$

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 1: C_1 R_1

$\{3\}$ $\{3\}$

$C_1 \times R_1 = \{(3, 3)\}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2: C_2 R_2

$\{4\}$ $\{4\}$

$C_2 \times R_2 = \{(4, 4)\}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 3: C_3 R_3

$\{1, 3\}$ $\{1, 3\}$

$C_3 \times R_3 = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 4: C_4 R_4

$\{2, 4, 5\}$ $\{2, 4\}$

$C_4 \times R_4 = \{(2, 2), (2, 4), (4, 2), (4, 4), (5, 2), (5, 4)\}$

$$R_t^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 5: C_5 R_5

$\varnothing$ $\{2, 4\}$

$C_5 \times R_5 = \varnothing$

$R_t^+ = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4), (5, 2), (5, 4)\}$

$R_t^+ = \{(a, a), (a, c), (b, b), (b, d), (c, a), (c, c), (d, b), (d, d), (e, b), (e, d)\}$

Equivalence Relation

Definition: A relation R on a set A is an equivalence relation iff R is reflexive, symmetric, and transitive.

Example: $A = \{0, 1, 2, 3\}$

$R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Is R_1 an equivalence relation?

Yes.

$R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

Is R_2 an equivalence relation?

Is R_2 reflexive?

No. Because (1, 1) is not a member of R_2.

$R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

Is R_3 an equivalence relation? Ask yourself: Is R_3 reflexive? Yes

Is R_3 symmetric? Yes

Is R_3 transitive? Yes

$R_4 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Is R_4 an equivalence relation? Ask yourself: Is R_4 reflexive? Yes

Is R_4 symmetric? No.

Therefore, R_4 is not an equivalence relation.

$R_5 = \varnothing$ Is R_5 an equivalence relation?

Ask yourself: Is R_5 reflexive? No

Is R_5 symmetric?

Therefore, R_5 is not an equivalence relation.

$R_6 = A \times A$ Is R_6 an equivalence relation?

Ask yourself: Is R_6 reflexive? Yes

Is R_6 symmetric? Yes

Is R_6 transitive? Yes

Therefore, R_6 is an equivalence relation.

Equivalence Relation (Solved Problems)

Problem 1: Let us assume that R is a relation on the set of integers defined by aRb if and only if a - b is an integer. Prove that R is an equivalence relation.

Solution: $A = \{..., -2, -1, 0, 1, 2, ...\}$ (A set of all integers)
R is defined on set A
aRb iff a - b is an integer.

(i) Reflexivity: $\forall a \in A \ (a, a) \in R$
$(a, a) \in R$ means a - a is an integer = 0 is an integer.
Therefore, R is reflexive.

(ii) Symmetry: $\forall a,b \in A \ [(a, b) \in R \rightarrow (b, a) \in R]$
$(a, b) \in R$ means a - b is an integer.
$(b, a) \in R$ means b - a is an integer.
We know that a - b is an integer.
$b - a = -(a - b)$ is also an integer.
Therefore, R is symmetric

(iii) Transitivity: $\forall a,b,c \in A \ [((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R]$
$(a, b) \in R$ means a - b is an integer.
$(b, c) \in R$ means b - c is an integer.
$(a, c) \in R$ means a - c is an integer.

$(a - b) + (b - c)$ is also an integer [Addition of two integers gives an integer]
$= a - c$ is an integer.
Therefore, R is transitive.

Therefore, R is an equivalence relation.

Problem 2: Suppose A is a finite set with n elements. The number of elements in the largest equivalence relation of A is

(A) n ~~(B)~~ n^2 (C) 1 (D) n + 1 [GATE 1998]

Solution: Largest equivalence relation of set A is AXA.
If A has n elements then AXA will have n^2 elements.
Therefore, option (B) is the correct option.

Equivalence Relations (GATE Problems)

Problem 1: Let R_1 and R_2 be two equivalence relations on a set.
Consider the following assertions:-
(i) $R_1 \cup R_2$ is an equivalence relation.
(ii) $R_1 \cap R_2$ is an equivalence relation.

Which of the following is correct?
(A) Both assertions are true.
(B) Assertion (i) is true but assertion (ii) is not true.
(C) Assertion (ii) is true but assertion (i) is not true.
(D) Neither (i) nor (ii) is true. [GATE 1998]

Solution: If R_1 and R_2 are equivalence relations on a particular set (Let say A) then

(i) $R_1 \cup R_2$ is not an equivalence relation.
(ii) $R_1 \cap R_2$ is an equivalence relation.

Proof: (ii) $R_1 \cap R_2$ is an equivalence relation. R_1 and R_2 are equivalence relations defined on set A. This means R_1 and R_2 are reflexive, symmetric and transitive.

We have to prove the following:

(a) $R_1 \cap R_2$ is reflexive.

(b) $R_1 \cap R_2$ is symmetric.

(c) $R_1 \cap R_2$ is transitive.

(a) $R_1 \cap R_2$ is reflexive.

We know that R_1 and R_2 are equivalence relations and thus they are reflexive.

Reflexivity: $\forall x \in A\ (x, x) \in R$

$\forall x \in A\ ((x, x) \in R_1 \wedge (x, x) \in R_2) \Rightarrow \forall x \in A\ (x, x) \in R_1 \cap R_2$

Therefore, $R_1 \cap R_2$ is reflexive.

(b) $R_1 \cap R_2$ is symmetric.

R_1 and R_2 are symmetric.

Symmetry: $\forall x, y \in A\ [(x, y) \in R \rightarrow (y, x) \in R]$

Let say $(x, y) \in R_1 \cap R_2$

this means $(x, y) \in R_1$ and $(x, y) \in R_2$ and we know that both R_1 and R_2 are symmetric. Therefore, $(y, x) \in R_1$ and $(y, x) \in R_2 \Rightarrow (y, x) \in R_1 \cap R_2$

Therefore, $(x, y) \in R_1 \cap R_2 \Rightarrow (y, x) \in R_1 \cap R_2$

Therefore, $R_1 \cap R_2$ is symmetric.

(c) $R_1 \cap R_2$ is transitive.

Transitivity: $\forall x, y, z \in A\ [((x, y) \in R \wedge (y, z) \in R) \rightarrow (x, z) \in R]$

Lets consider

$(x, y) \in R_1 \cap R_2 \wedge (y, z) \in R_1 \cap R_2$

Since, R_1 is transitive

$(x, y) \in R_1 \wedge (y, z) \in R_1 \Rightarrow (x, z) \in R_1$

and since R_2 is transitive.

$(x, y) \in R_2 \wedge (y, z) \in R_2 \Rightarrow (x, z) \in R_2$

It is clear that $(x, z) \in R_1 \wedge (x, z) \in R_2$

Therefore, $(x, z) \in R_1 \cap R_2$

$(x, y) \in R_1 \cap R_2 \wedge (y, z) \in R_1 \cap R_2 \Rightarrow (x, z) \in R_1 \cap R_2$

$R_1 \cap R_2$ is transitive.

Hence, $R_1 \cap R_2$ is an equivalence relation.

Problem 2: Let S be a set of n elements. The number of ordered pairs in the largest and the smallest equivalence relations on S are

(A) n and n ✓(B) n^2 and n (C) n^2 and 0 (D) n and 1 [GATE 2007]

Solution: An equivalence relation must be reflexive, symmetric, and transitive.
A reflexive relation is also symmetric and transitive relation.
Every equivalence relation must at least include all pairs of the form (a, a) where a belongs to some set.

R is defined on a set of n elements. Let say R is defined on set $S = \{1, 2, 3, ..., n\}$ then R must include all pairs (1, 1), (2, 2), (3, 3), ..., (n, n).

Therefore, the smallest equivalence relation is a reflexive relation.
Hence, the size of the smallest equivalence relation is n.

The largest equivalence relation includes every possible pair of AXA.

Therefore, the largest equivalence relation must include n^2 elements.

Equivalence Relation (GATE Problem)

Problem: Consider the following relations:
R_1 (a, b) iff (a+b) is even over the set of integers.
R_2 (a, b) iff (a+b) is odd over the set of integers.
R_3 (a, b) iff $a.b > 0$ over the set of non-zero rational numbers.
R_4 (a, b) iff $|a-b| \le 2$ is over the set of natural numbers.

Which of the following statements is correct?
(A) R_1 and R_2 are equivalence relations, R_3 and R_4 are not.
(B) R_1 and R_3 are equivalence relations, R_2 and R_4 are not.
(C) R_1 and R_4 are equivalence relations, R_2 and R_3 are not.
(D) R_1, R_2, R_3, and R_4 all are equivalence relations. [GATE 2001]

Solution: (i) $R_1 = \{(a, b) \mid (a+b) \text{ is even}\}$

R_1 is an equivalence relation.

(a) Reflexivity: $a + a = 2a$ is even.
Therefore, R_1 is reflexive.
(b) Symmetry: if a+b is even then b+a is also even.
Therefore, R_1 is symmetric.
(c) Transitivity: if a+b is even and b+c is even then a+c is even.
If (a, b) and (b, c) belongs to R_1 then both of them must be even or odd.
i.e., if a is even then b is even and if b is even then c is even.
If a is odd then b is odd and if b is odd then c is also odd.
Therefore, a+c is even and hence, R_1 is transitive.

(ii) $R_2 = \{(a, b) \mid (a+b) \text{ is odd}\}$

a) Reflexivity: $a+a = 2a$ is not odd.
Therefore, R_2 is not reflexive.
Hence, R_2 is not an equivalence relation.

(iii) $R_3 = \{(a, b) \mid a.b > 0\}$

a) Reflexivity: $a.a > 0$
b) Symmetry: $a.b > 0 \to b.a > 0$
c) Transitivity: If $a.b > 0$ and $b.c > 0$ (Both pairs must be either positive or negative)
Therefore, $a.c > 0$

Hence, R_3 is an equivalence relation.

(iv) $R_4 = \{(a, b) \mid |a-b| \le 2\}$

a) Reflexivity: $|a - a| \le 2$
b) Symmetry: if $|a - b| \le 2$ then $|b - a| \le 2$
c) Transitivity: if $|a - b| \le 2$ and $|b - c| \le 2$ then $|a - c|$ need not be ≤ 2
Example:- $|4 - 2| \le 2$ and $|2 - 1| \le 2$ but $|4 - 1| \not\le 2$

Hence, R_4 is not an equivalence relation.
Option (B) is the correct option.

Equivalence Classes

Equivalence class is the name given to a subset of some equivalence relation R which includes all elements that are equivalent to each other.

Let R be an equivalence relation on a set A. The set of all elements which are related to an element x of set A is called the equivalence class of x.

$$[x] = \{y \mid (x, y) \in R\}$$

Equivalence class of $x \in A$

Example: Let $A = \{1, 2, 3, 4, 5\}$
$R = \{(a, b) \mid a+b \text{ is even}\}$ (Relation R is defined on set A)

First, we have to check whether R is an equivalence relation or not.

(i) Reflexive: $a+a = 2a$
(ii) Symmetric: $a+b$ is even $\to$ $b+a$ is even
(iii) Transitive:
$a+b$ is even and $b+c$ is even $\to$ $a+c$ is even
Both a and b can be either even or odd.
If a is even and b is even
b is even and c is even, then $a+c$ is even.
If a is odd and b is odd
b is odd and c is odd, then $a+c$ is even.

Therefore, R is an equivalence relation.

$[1] = \{1, 3, 5\}$ because 1R1, 1R3, 1R5.
$[2] = \{2, 4\}$
$[3] = \{1, 3, 5\}$
$[4] = \{2, 4\}$
$[5] = \{1, 3, 5\}$

Equivalence class of elements 1, 3, and 5 are same and equivalence class of elements 2 and 4 are same.

$[1] = \{1, 3, 5\}$ because 1R1, 1R3, 1R5.
$[2] = \{2, 4\}$
$[3] = \{1, 3, 5\}$
$[4] = \{2, 4\}$
$[5] = \{1, 3, 5\}$

Equivalence class of elements 1, 3, and 5 are same and equivalence class of elements 2 and 4 are same.

Any element out of 1, 3 and 5 can be chosen as a representative of the equivalence class {1, 3, 5}

Also, any element out of 2 and 4 can be the representative of equivalence class {2, 4}.

Let say 1 is the representative of equivalence class {1, 3, 5} and 2 is the representative of equivalence class {2, 4}.

$[1] = \{1, 3, 5\}$
$[2] = \{2, 4\}$

Equivalence Classes and Partitions

Theorem 1: Let R be an equivalence relation on a set A. Following statements for elements a and b of set A are equivalent.
(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \varnothing$

Conclusion: Theorem 1 shows that the equivalence classes of two elements of set A are either identical or disjoint.

$$[a]_R \cap [b]_R = \varnothing \text{ when } [a]_R \neq [b]_R$$

Also, union of equivalence classes of R is all of A.

$$\bigcup_{a \in R} [a] = A$$

because no matter what, every element exist in its own equivalence class because of reflexivity $[\forall a \in A \ (a, a) \in R]$

Therefore, equivalence classes of set A forms partition of A.

Definition: Partition of set A is a collection of all disjoint non-empty subsets A_i of A where $i \in I$ (I is the index set)

$$A_i \neq \varnothing \quad \forall i \in I$$

$$A_i \cap A_j = \varnothing \text{ when } i \neq j$$

$$\bigcup_{i \in I} A_i = A$$

Example: $A = \{1, 2, 3, 4, 5, 6\}$
$R = \{(a, b) \mid a+b \text{ is even}\}$
R is an equivalence relation
$A_1 = [1] = \{1, 3, 5\}$
$A_2 = [2] = \{2, 4, 6\}$

$A_1 \cap A_2 = \varnothing$
$A_1 \cup A_2 = \{1, 2, 3, 4, 5, 6\} = A$
$A_1 \neq \varnothing$ and $A_2 \neq \varnothing$

Therefore, [1] and [2] forms the partition of A.

Equivalence Classes and Partitions

(Solved Problems)

Problem 1: Which of these collection of subsets are partition of {-3, -2, -1, 0, 1, 2, 3}

a) {-3, -1, 1, 3}, {-2, 0, 2}

b) {-3, -2, -1, 0}, {0, 1, 2, 3}

c) {-3, 3}, {-2, 2}, {-1, 1}, {0}

d) {-3, -2, 2, 3}, {-1, 1}

Solution: a) $S_1 = \{-3, -1, 1, 3\}$ and $S_2 = \{-2, 0, 2\}$

$S_1 \cap S_2 = \varnothing$ and $S_1 \cup S_2 = \{-3, -1, 1, 3, -2, 0, 2\} = S$

Also, $S_1 \neq \varnothing$ and S_2

By the definition of partition, the given collection of subset is a partition.

b) $S_3 = \{-3, -2, -1, 0\}$ and $S_4 = \{0, 1, 2, 3\}$

$S_3 \cap S_4 \neq \varnothing$

Therefore, the given collection of subset is not a partition.

c) $S_5 = \{-3, 3\}$, $S_6 = \{-2, 2\}$, $S_7 = \{-1, 1\}$, and $S_8 = \{0\}$

$S_5 \cap S_6 \cap S_7 \cap S_8 = \varnothing$ and $S_5 \cup S_6 \cup S_7 \cup S_8 = \{-3, 3, -2, 2, -1, 1, 0\} = S$

Also, $S_5 \neq \varnothing$, $S_6 \neq \varnothing$, $S_7 \neq \varnothing$ and $S_8 \neq \varnothing$

By the definition of partition, the given collection of subset is a partition.

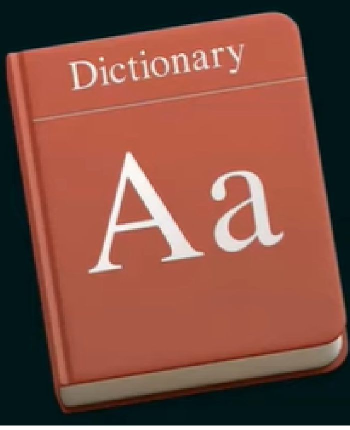
d) $S_9 = \{-3, -2, 2, 3\}$ and $S_{10} = \{-1, 1\}$

$S_9 \cap S_{10} = \varnothing$ and $S_9 \cup S_{10} = \{-3, -2, 2, 3, -1, 1\} \neq S$

Therefore, the given collection of subset is not a partition.

Partial Ordering

Many a time, we use relations to order some or all the elements of a set.

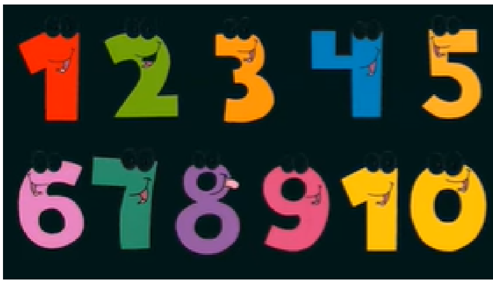


Let say we have a set of all words in the dictionary.

R = {(a, b) | a comes before b}

Partition comes before Relation

(Partition, Relation)



Let say we have a set of all integers.

$R = \{(a, b) \mid a < b\}$

In the previous examples, we have seen how a relation indicates that a certain element precedes the other in the ordering.

These relations are used to order some or all elements of a particular set.

Set of all words in a dictionary

$R_1 = \{(a, b) \mid a \text{ comes before } b\}$

Set of all integers

$R_2 = \{(a, b) \mid a < b\}$

Let say we also add pairs of the form (a, a) in R_1 and R_2

$R_1 = \{(a, b) \mid a \text{ comes before } b \text{ or } a \text{ is equal to } b\}$

$R_2 = \{(a, b) \mid a \leq b\}$

We have obtained relations R_1 and R_2 that are reflexive, antisymmetric and transitive.

Definition of partial ordering

A relation R on set S is called a partial ordering or partial order if it is

1. Reflexive
2. Antisymmetric
3. Transitive

Each element must be related to itself.

No two elements precede each other (Ordering)

First element related to second and second element related to third implies first and third are related.

A set S together with relation R is called a partially ordered set or

POSET

Denoted by (S, R) or (S, $\preceq$)

where $a \preceq b$ means a is related to b

Example 1: Show that the relation $R = \{(a, b) \mid a \subseteq b\}$ defined on the power set of set $S = \{1, 2, 3\}$ is a partial order relation.

Solution: Relation R is said to be a partial ordering iff R is
(i) Reflexive.
(ii) Antisymmetric.
(iii) Transitive.

$S = \{1, 2, 3\}$
$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

$R = \{(a, b) \mid a \subseteq b\}$

(i) Reflexivity: $a \subseteq a$
(ii) Antisymmetry: $a \subseteq b$ and $b \subseteq a$ implies $a = b$.
(ii) Transitivity: $a \subseteq b$ and $b \subseteq c$ implies $a \subseteq c$.

Example 2: Show that the relation $R = \{(a, b) \mid a \text{ divides } b\}$ defined on the set $S = \{1, 2, 3, 4, 6\}$ is a partial order relation.

Solution: Relation R is said to be a partial ordering iff R is
(i) Reflexive.
(ii) Antisymmetric.
(iii) Transitive.

$1 \mid 2 \quad 2 \nmid 1$
$1 \mid 1 \quad 1 \mid 1$

$R = \{(a, b) \mid a \text{ divides } b\}$ $S = \{1, 2, 3, 4, 6\}$

(i) Reflexivity: a divides a
(ii) Antisymmetry: a divides b and b divides a implies $a = b$.
(ii) Transitivity: a divides b and b divides c implies a divides c.

Therefore, R is a partial order relation.

Meaning of partial in partial ordering

The word "partial" in "partial ordering" indicates that not every pair of element in a set is comparable.

Two terms which are important for us to understand:

1) Comparable
2) Incomparable

Difference between comparable and incomparable

Comparable:

The element a and b of poset (S, R) are called comparable if either aRb or bRa.

Incomparable:

The element a and b of poset (S, R) are called incomparable if neither aRb nor bRa.

Remember
We can use $(S, \preceq)$ to represent an arbitrary poset where $a \preceq b$ is used to denote $(a, b) \in R$.

Example: In the poset, $(P(S), \subseteq)$ where $S = \{1, 2, 3\}$

$\{1, 3\} \not\subseteq \{2\}$ or $\{2\} \not\subseteq \{1, 3\}$.
Therefore, $\{1, 3\}$ and $\{2\}$ are incomparable.

On the other hand, $\{1\} \subseteq \{1, 3\}$ are comparable.

Problem 1: Which of the following relations on set {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that relations lack.

a) {(0, 0), (2, 2), (3, 3)}
b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
c) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}

Solution: A = {0, 1, 2, 3}

a) R_1 = {(0, 0), (2, 2), (3, 3)}
Reflexivity: No. R_1 is not reflexive because $(1, 1) \notin R_1$
Antisymmetry: Yes. if $(a, b) \in R_1$ and $(b, a) \in R_1$ then a=b.
Transitivity: Yes.
Therefore, R_1 is not a partial order.

b) R_2 = {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
Reflexivity: Yes. R_2 is reflexive because $\forall a \in A\ ((a, a) \in R_2)$.
Antisymmetry: Yes. if $(a, b) \in R_2$ and $(b, a) \in R_2$ then a=b.
Transitivity: Yes.
Therefore, R_2 is a partial order.

c) R_3 = {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
Reflexivity: Yes. R_3 is reflexive because $\forall a \in A\ ((a, a) \in R_3)$.
Antisymmetry: No. $(0, 1) \in R_3$ and $(1, 0) \in R_3$
Transitivity: No. $(2, 0) \in R_3$ and $(0, 1) \in R_3$ but $(2, 1) \notin R_3$
Therefore, R_3 is not a partial order.

d) R_4 = {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}
Reflexivity: Yes. R_4 is reflexive because $\forall a \in A\ ((a, a) \in R_4)$.
Antisymmetry: Yes.
Transitivity: No. $(1, 2) \in R_4$ and $(2, 0) \in R_4$ but $(1, 0) \notin R_4$
Therefore, R_4 is not a partial order.

Problem 2: Let say S is the set of all people in the world and R is the relation defined on set S such that $(a, b)\in R$, where a and b are people, if a is taller than b. Is (S, R) a poset?

Solution: $R = \{(a, b) \mid a \text{ is taller than } b\}$

(S, R) is a poset?

Reflexivity: $\forall a\in S\ ((a, a)\in R)$.

$(a, a)\in R$ means a is taller than a (Not possible).

Therefore, R is not reflexive.

Antisymmetry: $\forall a,b\in S$ if $(a, b)\in R$ and $(b, a)\in R$ then $a=b$.

i.e., if a is taller than b then b cannot be taller than a.

Therefore, R is antisymmetric.

Transitivity: R is transitive.

Therefore, R is not a partial order because R is not reflexive.

Problem 3: Let say S is the set of all people in the world and R is the relation defined on set S such that $(a, b)\in R$, where a and b are people, if a is not taller than b. Is (S, R) a poset?

Solution: $R = \{(a, b) \mid a \text{ is not taller than } b\}$

(S, R) is a poset?

Reflexivity: $\forall a\in S\ ((a, a)\in R)$.

$(a, a)\in R$ means a is not taller than a (True).

Therefore, R is reflexive.

Antisymmetry: $\forall a,b\in S$ if $(a, b)\in R$ and $(b, a)\in R$ then $a=b$.

i.e., if a is not taller than b and b is not taller than a then it is true that a and b are of same height but it does not necessarily mean that $a=b$.
(a and b are not necessarily the same person)

Therefore, R is not a partial order because R is not antisymmetric.

Problem 1: Which of the following are posets?
a) $(\mathbb{Z}, =)$ b) $(\mathbb{R}, <)$ c) $(\mathbb{Z}, \neq)$ d) $(\mathbb{Z}, \geq)$ e) $(\mathbb{Z}, \nmid)$

Solution: a) $(\mathbb{Z}, =)$

$R = \{(a, b) \mid a=b\}$

(i) Reflexivity: Let say $a \in \mathbb{Z}$, then $a = a$
Therefore, R is reflexive.

(ii) Antisymmetry: Let say $a,b \in \mathbb{Z}$.
If $a = b$ and $b = a$ then $a = b$.
Therefore, R is antisymmetric.

(iii) Transitivity: Let say $a,b,c \in \mathbb{Z}$
If $a = b$ and $b = c$ then $a = c$
Therefore, R is transitive.

b) $(\mathbb{R}, <)$

$R = \{(a, b) \mid a<b\}$

(i) Reflexivity: Let say $a \in \mathbb{R}$, then $a < a$ is not True.
Therefore, R is not reflexive.

(ii) Antisymmetry: Let say $a,b \in \mathbb{R}$.
If $a < b$ and $b < a$ then $a = b$ is True
(because False $\Rightarrow$ anything is always True).
Therefore, R is antisymmetric.

(iii) Transitivity: Let say $a,b,c \in \mathbb{R}$.
If $a < b$ and $b < c$ then $a < c$
Homework for you

Hence, $(\mathbb{R}, <)$ is not a poset.

c) $(\mathbb{Z}, \neq)$

$R = \{(a, b) \mid a \neq b\}$

(i) Reflexivity: Let say $a \in \mathbb{Z}$, then $a \neq a$ is False.
Therefore, R is not reflexive.

(ii) Antisymmetry: Let say $a, b \in \mathbb{Z}$.
If $a \neq b$ and $b \neq a$ then $a = b$ is also False.
Therefore, R is not antisymmetric.

(iii) Transitivity: Let say $a, b, c \in \mathbb{Z}$
If $a \neq b$ and $b \neq c$ then $a \neq c$
Not always True.
For example: $1 \neq 2$ and $2 \neq 1$ but $1 = 1$.
Therefore, R is not transitive.

Hence, $(\mathbb{Z}, \neq)$ is not a poset.

d) $(\mathbb{Z}, \geq)$

$R = \{(a, b) \mid a \geq b\}$

(i) Reflexivity: Let say $a \in \mathbb{Z}$, then $a \geq a$ is True.
Therefore, R is reflexive.

(ii) Antisymmetry: Let say $a, b \in \mathbb{Z}$.
If $a \geq b$ and $b \geq a$ then $a = b$ is also True.
Therefore, R is antisymmetric.

(iii) Transitivity: Let say $a, b, c \in \mathbb{Z}$
If $a \geq b$ and $b \geq c$ then $a \geq c$
Always True.
Therefore, R is transitive.

Hence, $(\mathbb{Z}, \geq)$ is a poset.

e) $(\mathbb{Z}, \nmid)$

$R = \{(a, b) \mid a \nmid b\}$

(i) Reflexivity: Let say $a \in \mathbb{Z}$, then $a \nmid a$ is False.

Therefore, R is not reflexive.

(ii) Antisymmetry: Let say $a, b \in \mathbb{Z}$.

If $a \nmid b$ and $b \nmid a$ then $a = b$ is False.

Example: $2 \nmid 5$ and $5 \nmid 2$ but $2 \neq 5$

Therefore, R is not antisymmetric.

(iii) Transitivity: Let say $a, b, c \in \mathbb{Z}$

If $a \nmid b$ and $b \nmid c$ then $a \nmid c$ (Not always true)

Example: $2 \nmid 3$ and $3 \nmid 4$ but $2 \mid 4$

Therefore, R is not transitive.

Hence, $(\mathbb{Z}, \nmid)$ is not a poset.

Problem 2: Determine whether the relations represented by these non-zero matrices are partial orders.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Solution: a) $M_R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

Therefore, R is not a partial order.

(i) Reflexivity: R is reflexive (See the diagonal elements)

(ii) Antisymmetry: R is antisymmetric

if $M_{ij} = 1$ and $M_{ji} = 1$ then $i = j$.

if $M_{ij} = 1$ then $M_{ji} \neq 1$ until $i = j$.

(iii) Transitivity: R is not transitive. $M_{21} = 1$ and $M_{13} = 1$ but $M_{23} = 0$

b) $M_R = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 1 & 0 & 1 \end{array}$

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Therefore, R is a partial order.

(i) Reflexivity: R is reflexive (See the diagonal elements)

(ii) Antisymmetry: R is antisymmetric

if $M_{ij} = 1$ and $M_{ji} = 1$ then $i = j$.

if $M_{ij} = 1$ then $M_{ji} \neq 1$ until $i = j$.

(iii) Transitivity: R is transitive.

c) $$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Therefore, R is not a partial order.

(i) Reflexivity: R is reflexive (See the diagonal elements)

(ii) Antisymmetry: R is antisymmetric

if $M_{ij} = 1$ and $M_{ji} = 1$ then $i = j$.

(iii) Transitivity: R is not transitive.

$(a, c) \in R$ and $(c, d) \in R$ but $(a, d) \notin R$

Problem 3: Find two incomparable elements in these posets

a) $(P(\{0, 1, 2\}), \subseteq)$ b) $(\{1, 2, 4, 6, 8\}, |)$

Solution: a) $(P(\{0, 1, 2\}), \subseteq)$

$P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{1, 2, 3\}\}$

We have to find two incomparable elements in $P(\{0, 1, 2\})$.

Here, two elements a and b are incomparable if neither $a \subseteq b$ nor $b \subseteq a$.

$\{0\} \not\subset \{1\}$ and $\{1\} \not\subset \{0\}$

Therefore, $\{0\}$ and $\{1\}$ are incomparable elements.

b) $(\{1, 2, 4, 6, 8\}, |)$

$4 \nmid 6$ and $6 \nmid 4$

Therefore, 4 and 6 are incomparable elements.

Let say we have a set S = {1, 2, 4, 6, 8} and Relation R is defined on set S.
R = {(a, b) | a|b} or R = {(a, b) | a divides b}

We know that (S, R) is a poset.
(i) Reflexivity: a | a.
(ii) Antisymmetry: a | b and b | a → a = b.
(iii) Transitivity: a | b and b | c → a | c.

Therefore, R is a partial order and (S, R) is a poset.

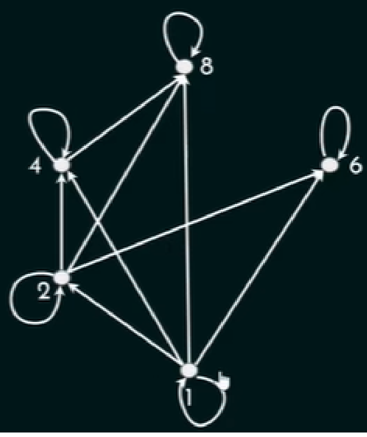
Lets try to represent relation R = {(a, b) | a divides b} using a directed graph.
S = {1, 2, 4, 6, 8}

But why graph?
It is always easy to work with graphs. We can see the benefit of using a graph to represent a partial order in the upcoming lectures.

R = {(1, 1), (2, 2), (4, 4), (6, 6), (8, 8), (1, 2), (1, 4), (1, 6), (1, 8), (2, 4), (2, 6), (2, 8), (4, 8)}

Remove Self Loops

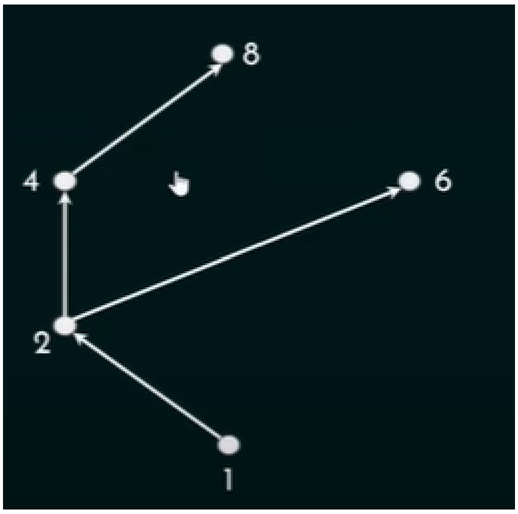
Self loops are quite obvious and they must be present for all vertices because R is reflexive. So we can remove them to simplify the diagram.



Remove Transitive Edges

It is quite obvious that if a | b and b | c then a | c. If 1 | 2 and 2 | 4 then it is obvious that 1 | 4.

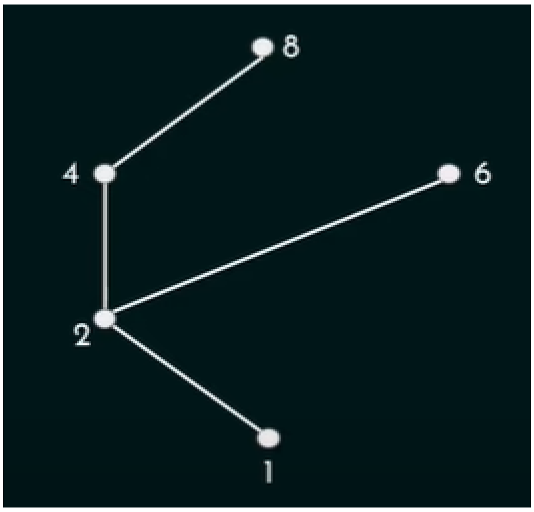
Therefore, there is no need to show the transitive edges in the directed graph to further simplify it.



Remove Arrows

If we assume that all edges are pointing upwards than there is no need to show the directions in the graph.

We can safely remove the arrows in the diagram.



This diagram is called the Hasse Diagram.

This diagram is used to represent partial order relations with sufficient information.

Procedure to draw the Hasse diagram

Step 1: Start with a directed graph.
Step 2: Remove self loops.
Step 3: Remove all transitive edges.

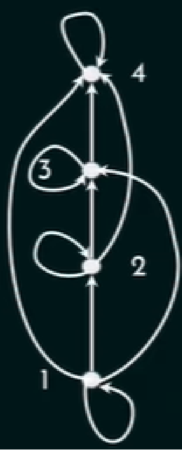
a) Check if (a, b) and (b, c) are in partial ordering.
b) If Yes, then remove the edge between vertices a and c.
c) If (c, d) also belongs to the partial order then remove the edge (a, d) and so on.

Step 4: Arrange each edge so that its initial vertex is below its terminal vertex.
Step 5: Remove all the arrows of the directed graph.

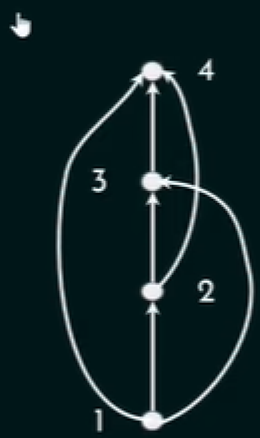
Example: Draw the Hasse diagram representing the partial ordering $R = \{(a, b) \mid a \leq b\}$ on set $S = \{1, 2, 3, 4\}$

Solution: Procedure

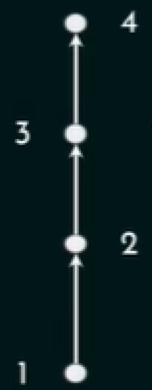
1. Start with a directed graph.



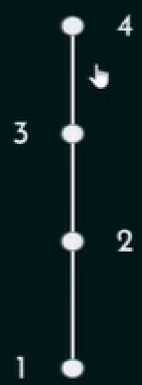
2. Remove self loops.



3. Remove all transitive edges.



3. Remove all arrows.



$R = \{(a, b) \mid a \leq b\}$ is a special type of relation defined on set $S = \{1, 2, 3, 4\}$.

R is a <u>total order</u> or <u>linear order</u> and the poset (S, R) is called a <u>totally ordered set</u> or <u>chain</u>.

R is a total order because in set S, every two elements are comparable.

Problem 1: Draw the Hasse diagram for the "greater than or equal to" relation on set $S = \{0, 1, 2, 3, 4, 5\}$

Solution:

$S = \{0, 1, 2, 3, 4, 5\}$

$R = \{(a, b) \mid a \geq b\}$

R is a partial order and hence (S, R) is a poset.

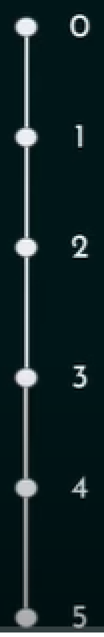
Relation R is $\geq$.

The bottom most vertex in the Hasse diagram must be the largest element in set S.

$S = \{0, 1, 2, 3, 4, 5\}$

$R = \{(a, b) \mid a \geq b\}$

Relation R is a total order.

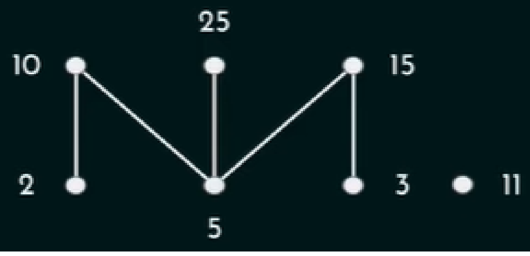


Problem 2: Draw the Hasse diagram for the divisibility on the set $S = \{2, 3, 5, 10, 11, 15, 25\}$

Solution:

$S = \{2, 3, 5, 10, 11, 15, 25\}$

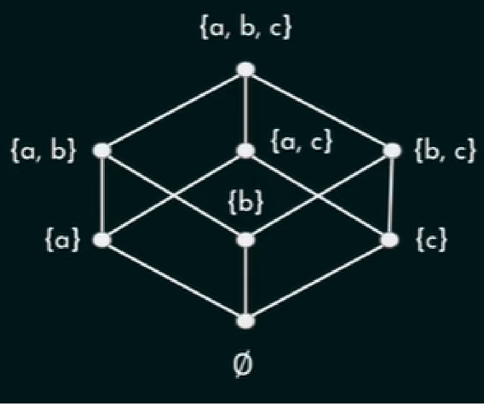
$R = \{(a, b) \mid a \text{ divides } b\}$



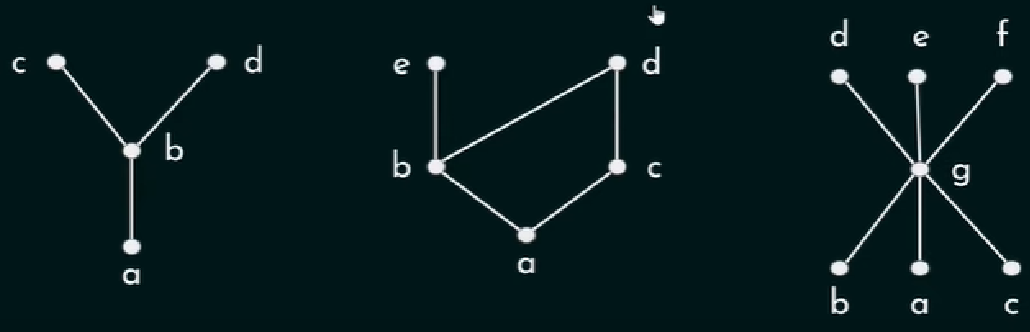
Problem 1: Draw the Hasse diagram for inclusion on the set P(S), where S = {a, b, c}

Solution: S = {a, b, c}

P(S) = {∅, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}}

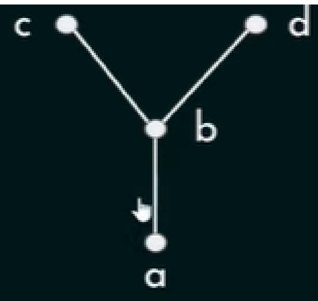


Problem 2: List all the ordered pairs in the partial ordering with the accompanying Hasse diagram.



(a) (b) (c)

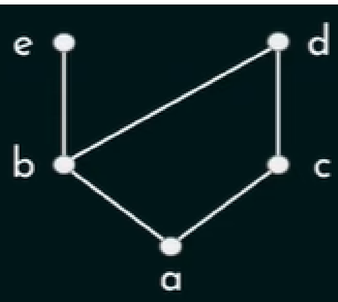
(a)



Because of Reflexivity: (a, a), (b, b), (c, c), (d, d)
Because of Antisymmetry: (a, b), (b, c), (b, d)
Because of Transitivity: (a, c), (a, d)

All ordered pairs: R_1 = {(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, d)}

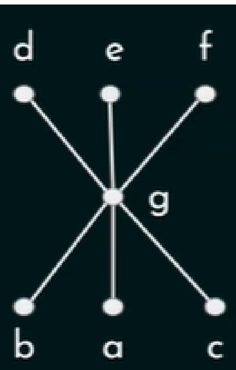
(b)



Because of Reflexivity: (a, a), (b, b), (c, c), (d, d), (e, e)
Because of Antisymmetry: (a, b), (a, c), (b, e), (b, d), (c, d)
Because of Transitivity: (a, e), (a, d)

All ordered pairs: R_2 = {(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, d), (d, d), (e, e)}

(c)



Because of Reflexivity: (a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g)
Because of Antisymmetry: (a, g), (b, g), (c, g), (g, d), (g, e), (g, f)
Because of Transitivity: (a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f)

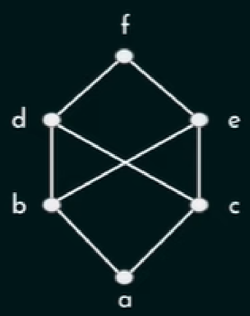
All ordered pairs: R_3 = {(a, a), (a, d), (a, e), (a, f), (a, g), (b, b), (b, d), (b, e), (b, f), (b, g), (c, c), (c, d), (c, e), (c, f), (c, g), (d, d), (e, e), (f, f), (g, d), (g, e), (g, f), (g, g)}

Let say we have a poset (S, R) where S is some arbitrary set and relation R is a partial order relation defined on set S.

Minimal Element:

An element x of a set S is called a minimal element if there is no y∈S such that yRx (or (y, x)∉R) and y≠x.

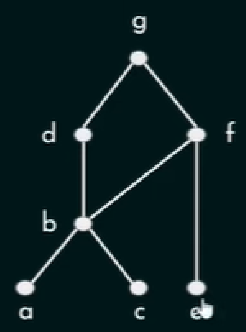
Example 1: Consider the following Hasse Diagram



In the above Hasse diagram, b and c are not minimal because aRb and aRc. Similarly, d and e are not minimal because, bRd and cRd. Also, cRe and bRe. a is the only minimal element because no element is related to a.

Hence, a is the minimal element.

Example 2: Consider the following Hasse Diagram



In the above Hasse diagram, a, c and e are minimal elements because no element is related to a, c, and e.

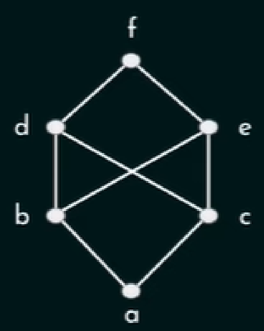
Important Observation: A poset can have more than one minimal element.

Let say we have a poset (S, R) where S is some arbitrary set and relation R is a partial order relation defined on set S.

Maximal Element:

An element x of a set S is called a maximal element if there is no $y \in S$ such that xRy (or $(x, y) \notin R$) and $x \neq y$.

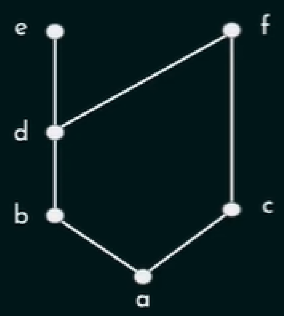
Example 1: Consider the following Hasse Diagram



a is not a maximal element because aRb and aRc.
Similarly, b, c, and e are not maximal because bRd & bRe, cRd & cRe, and dRf & eRf.
But f is not related to any element.

Hence, f is the maximal element.

Example 2: Consider the following Hasse Diagram



In the above Hasse diagram, e and f are maximal elements because e is not related to any element. Similarly, f is also not related to any element.

Important Observation: A poset can have more than one maximal element.

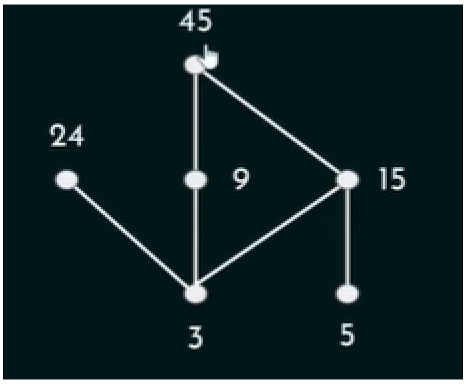
Problem 1: Consider the partial order represented by the following Hasse diagram.

Find the minimal and maximal elements.

Solution:
Maximal elements: l and m.
Minimal elements: a, b and c.

Problem 2: What are the maximal and minimal elements for the poset ({3, 5, 9, 15, 24, 45}, |)

Solution: Let's draw the Hasse diagram for the poset ({3, 5, 9, 15, 24, 45}, |)



Final Result

Maximal Elements: 24 and 45
Minimal Elements: 3 and 5

Let say we have a poset (S, R) where S is some arbitrary set and R is a partial order defined on set S.

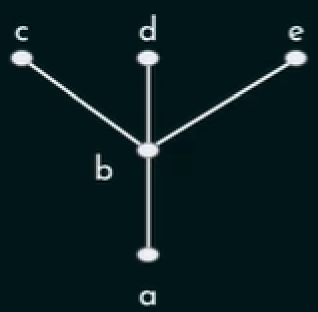
Least element (Minimum element):
An element $x \in S$ is called the least element of S if $\forall y \in S$, xRy.

Note: least element is unique if it exists.

Greatest element (Maximum element):
An element $x \in S$ is called the greatest element of S if $\forall y \in S$, yRx.

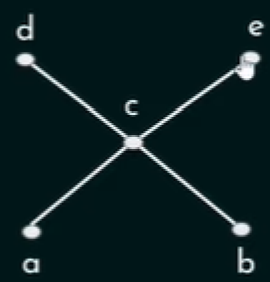
Note: greatest element is unique if it exists.

least element is a.
No greatest element.



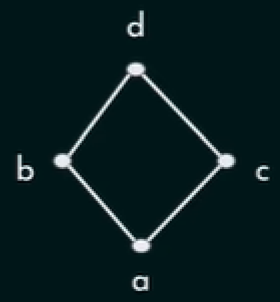
(a)

b) No least element.
No greatest element.



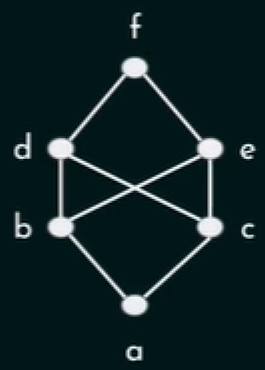
(b)

c) least element is a.
greatest element is d.



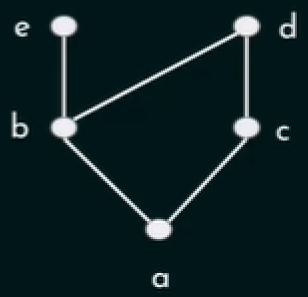
(c)

d) least element is a.
greatest element is f.



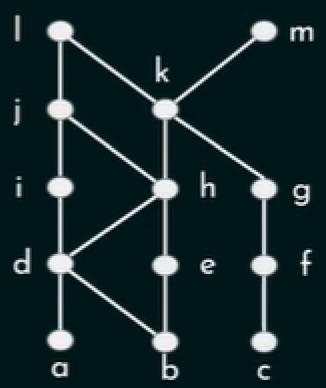
(d)

e) least element is a.
No greatest element



(e)

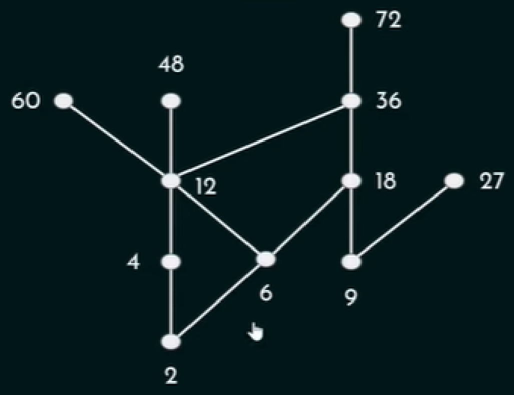
f) No least element.
No greatest element.



(f)

Example 2: Find the greatest element and the least element of poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |}

Solution:



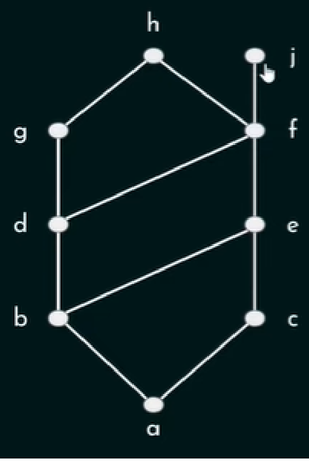
No greatest element.
No least element.

Let say we have a poset (S, R) such that S is an arbitrary set and R is a partial order defined on set S.
Also, let say $T \subset S$.

Lower bound:
An element $x \in S$ is a lower bound of set T if $\forall y \in T\ (x, y) \in R$.

Upper bound:
An element $x \in S$ is an upper bound of set T if $\forall y \in T\ (y, x) \in R$.

Example 1: Find the lower and upper bounds of the subsets {a, b, c}, {j, h}, and {a, c, d, f} in the poset with the Hasse diagram shown below:



Solution:
Lower bound of {a, b, c} is a.
Upper bounds of {a, b, c} are e, f, j and h.

Lower bounds of {j, h} are f, d, e, b, c, a.
Upper bound of {j, h} is Ø.

Lower bound of {a, c, d, f} is a.
Upper bounds of {a, c, d, f} are f, j, h.

Example 2: Consider the following Hasse diagram

Find all the upper bounds of {a, b, c} and all the lower bounds of {f, g, h}.

Solution:

Upper bounds of {a, b, c}

1. k is an upper bound of {a, b, c}
2. l is an upper bound of {a, b, c}
3. m is an upper bound of {a, b, c}

Lower bounds of {f, g, h}

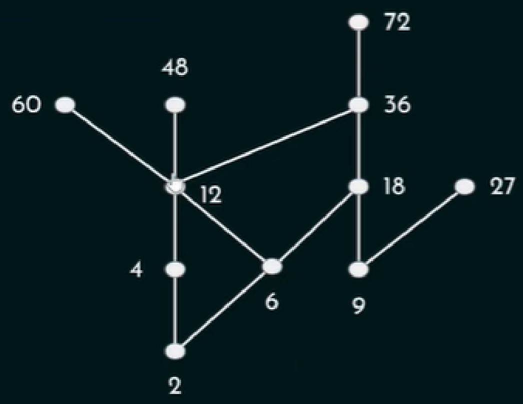
None exists.

Upper bounds of {a, b, c} = k, l, m

Lower bound of {f, g, h} = $\emptyset$

Example 3: Find all upper bounds of {2, 9} and all lower bounds of {60, 72} for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |)

Solution:



Upper bounds of {2, 9}

1. 18 is the upper bound of {2, 9} because 2|18 and 9|18.
2. 36 is the upper bound of {2, 9} because 2|36 and 9|36.
3. 72 is the upper bound of {2, 9} because 2|72 and 9|72.

Lower bounds of {60, 72}

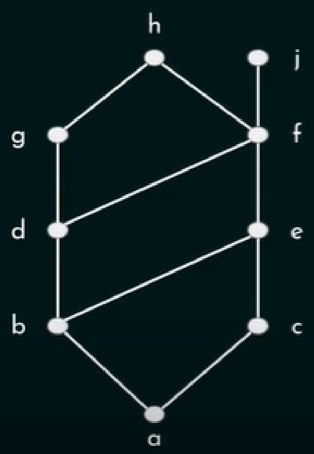
1. 12 is the lower bound of {60, 72} because 12|60 and 12|72.
2. 6 is the lower bound of {60, 72} because 6|60 and 6|72.
3. 4 is the lower bound of {60, 72} because 4|60 and 4|72.
4. 2 is the lower bound of {60, 72} because 2|60 and 2|72.

Let say we have a poset (S, R) such that S is an arbitrary set and R is a partial order defined on set S.
Also, let say $T \subset S$.

Least upper bound (LUB or Supremum or Join or $\vee$):
Let say U is the set of all upper bounds of set T. Then, an element $x \in U$ is called the least upper bound if $\forall y \in U\ (x, y) \in R$.
In other words,
LUB(T) = minimum {UB(T)}

Greatest lower bound (GLB or Infimum or Meet or $\wedge$):
Let say L is the set of all lower bounds of set T. Then, an element $x \in L$ is called the greatest lower bound if $\forall y \in L\ (y, x) \in R$.
In other words,
GLB(T) = maximum {LB(T)}

Example 1: Find the greatest lower bound and the least upper bound of {b, d, g} if they exist in the poset with the Hasse diagram shown below:



Solution: Upper bounds of {b, d, g} are g and h because b is related to g, d is related to g and g is related to g.
Also, b is related to h, d is related to h, and g is related to h.

Out of g and h, minimum element is g. Therefore, g is the least upper bound of {b, d, g}.

Lower bounds of {b, d, g} are a and b.

Out of a and b, greatest element is b. Therefore, b is the greatest lower bound of {b, d, g}.

Example 2: Consider the following Hasse diagram

Find the least upper bound of {a, b, c} and the greatest lower bound of {f, g, h}, if they exist.

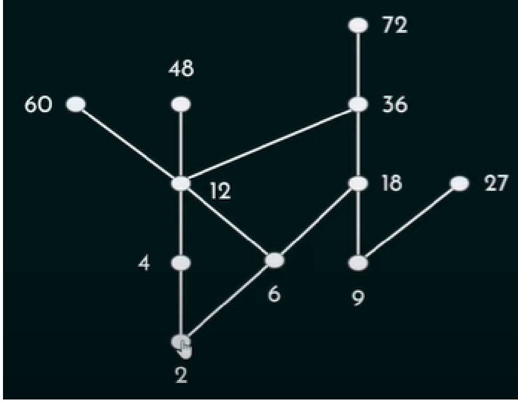
Solution:

Upper bounds of {a, b, c} are k, l, and m.
Least upper bound of {a, b, c} is k.

Lower bound of {f, g, h} is Ø.
Greatest lower bound of {f, g, h} is Ø.

Example 3: Find the least upper bound of {2, 9} and the greatest lower bound of {60, 72} for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |)

Solution:



Upper bounds of {2, 9} are 18, 36, 72.
Least upper bound of {2, 9} is 18.

Lower bounds of {60, 72} are 2, 4, 6, 12.
Greatest lower bound of {60, 72} is 12.

Problem 1: Answer the following questions for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆})

a) Find the minimal elements.
b) Find the maximal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of {{2}, {4}}.
f) Find the least upper bound of {{2}, {4}}, if it exists.
g) Find all lower bounds of {{1, 3, 4}, {2, 3, 4}}.
h) Find the greatest lower bound of {{1, 3, 4}, {2, 3, 4}}, if it exists.

The Hasse diagram for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆) is shown below:

a) Minimal elements: {1}, {2}, {4}
Recall: An element x of a set S is called a minimal element if there is no y∈S such that yRx and x≠y.

b) Maximal elements: {1, 2}, {1, 3, 4}, {2, 3, 4}
Recall: An element x of a set S is called a maximal element if there is no y∈S such that xRy and x≠y.

c) Is there a greatest element?
Greatest element does not exist.
Recall: The greatest element exists if there is exactly one maximal element.

d) Is there a least element?
Least element does not exist.
Recall: The least element exists if there is exactly one minimal element.

e) Upper bounds of {{2}, {4}} are {2, 4}, {2, 3, 4}.
Recall: An element x∈S is an upper bound of set T (T ⊂ S) if ∀y∈T (y, x)∈R (where R is a partial order defined on set S)

f) Least upper bound of {{2}, {4}} is {2, 4}.
Recall: least upper bound(T) = minimum(UB(T))

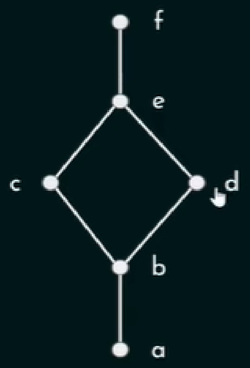
g) Lower bounds of {{1, 3, 4}, {2, 3, 4}} are {4}, {3, 4}.
Recall: An element x∈S is an lower bound of set T (T ⊂ S) if ∀y∈T (x, y)∈R (where R is a partial order defined on set S)

h) Greatest lower bound of {{1, 3, 4}, {2, 3, 4}} is {3, 4}.
Recall: Greatest lower bound(T) = maximum(LB(T))

Consider a poset (S, ≤).

Definition: The poset (S, ≤) is a meet semilattice if ∀x,y∈S, x ∧ y (i,e., GLB(x, y)) must not be empty.

Example 1: Consider the following Hasse diagram.

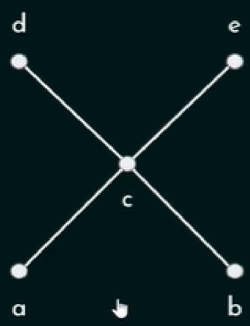


Solution:
In the given Hasse diagram, every pair of element has the greatest lower bound.

For example:
GLB(f, e) = e
GLB(c, d) = b
GLB(e, d) = d
GLB(c, b) = b

Therefore, the given Hasse diagram is a meet semilattice.

Example 2: Consider the following Hasse diagram.



Is the above Hasse diagram a meet semilattice?

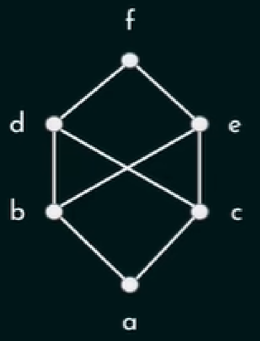
Solution:
Consider the pair (d, e)
LB(d, e) = c, a, b
GLB(d, e) = c

Now,
Consider the pair (a, b)
LB(a, b) = ∅
GLB(a, b) = ∅

There exist a pair in the poset represented by the given Hasse diagram whose greatest lower bound does not exist.

Therefore, the given Hasse diagram is NOT a meet semilattice.

(a)



Consider the pair (b, c)
LB(b, c) = a
GLB(b, c) = a

Consider the pair (d, e)
LB(d, e) = b, c, a
GLB(d, e) = ∅

After tracing the path down from d and e, the first point where they meet are b and c.
There is no single first meeting point.

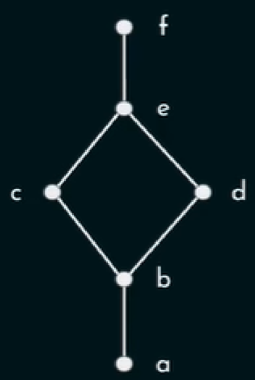
Hence, GLB(d, e) = ∅

Therefore, the given Hasse diagram is NOT a meet semilattice.

Consider a poset (S, ≤).

Definition: The poset (S, ≤) is a join semi lattice if ∀x,y∈S, x ∨ y (i,e., LUB(x, y)) must not be empty.

Example 1: Consider the following Hasse diagram.

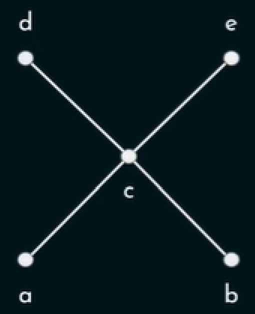


Solution:
In the given Hasse diagram, every pair of element has the least upper bound.

For example:
LUB(f, e) = f
LUB(c, d) = e   [UB(c, d) = e, f]
LUB(e, d) = e
LUB(c, b) = c

Therefore, the given Hasse diagram is a join semilattice.

Example 2: Consider the following Hasse diagram.



Is the above Hasse diagram a join semilattice?

Solution:

Consider the pair (d, e)

UB(d, e) = ∅

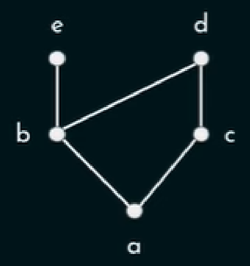
LUB(d, e) = ∅

According to the definition of join semi lattice, The poset (S, ≤) is a join semi lattice if ∀x,y∈S, x ∨ y (i,e., LUB(x, y)) must not be empty.

There exist a pair in the poset represented by the given Hasse diagram whose least upper bound does not exist.

Therefore, the given Hasse diagram is NOT a join semilattice.

(d)



Consider the pair (b, c)

UB(b, c) = d

LUB(b, c) = d

Consider the pair (e, d)

UB(e, d) = ∅

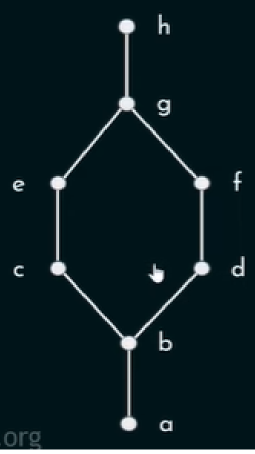
LUB(e, d) = ∅

Therefore, the given Hasse diagram is NOT a join semilattice.

Consider a poset (S, R).

Definition: The poset (S, R) is called a lattice iff it is a meet semilattice and a join semilattice.

Example 1: Is the given Hasse diagram a lattice?



Solution:

We know that a Hasse diagram is called a lattice if it is both meet semilattice and join semilattice.

i.e.,

∀x,y∈S, GLB(x, y) ≠ ∅ and

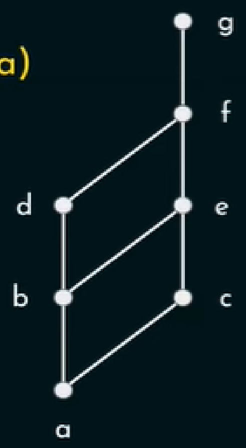
∀x,y∈S, LUB(x, y) ≠ ∅

Consider the incomparable pairs

GLB(f, e) = b	GLB(e, d) = b
LUB(f, e) = g	LUB(e, d) = g
GLB(c, d) = b	GLB(c, f) = b
LUB(c, d) = g	LUB(c, f) = g

The given Hasse diagram is a lattice.

(a)



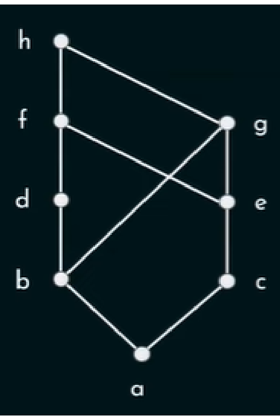
Consider the pair (d, e)
GLB(d, e) = b
LUB(d, e) = f

Consider the pair (b, c)
GLB(b, c) = a
LUB(b, c) = e

For every pair of elements, the greatest lower bound and the least upper bound exists.

Therefore, the given Hasse diagram is a lattice.

(b)



Consider the pair (f, g)
GLB(f, g) = ∅
LUB(f, g) = h

The given Hasse diagram is not a meet semilattice and hence, it is not a lattice.

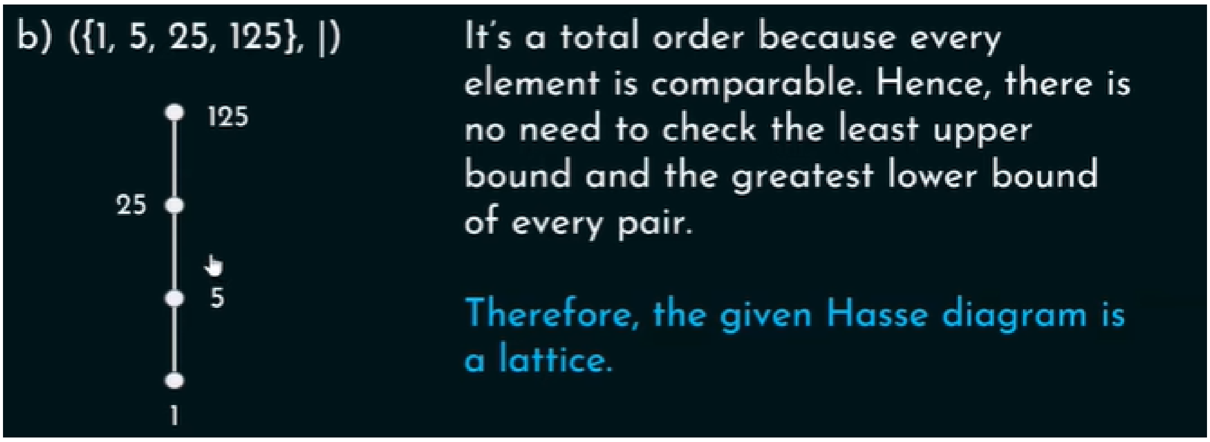
Example 3: Determine whether these posets are lattices.

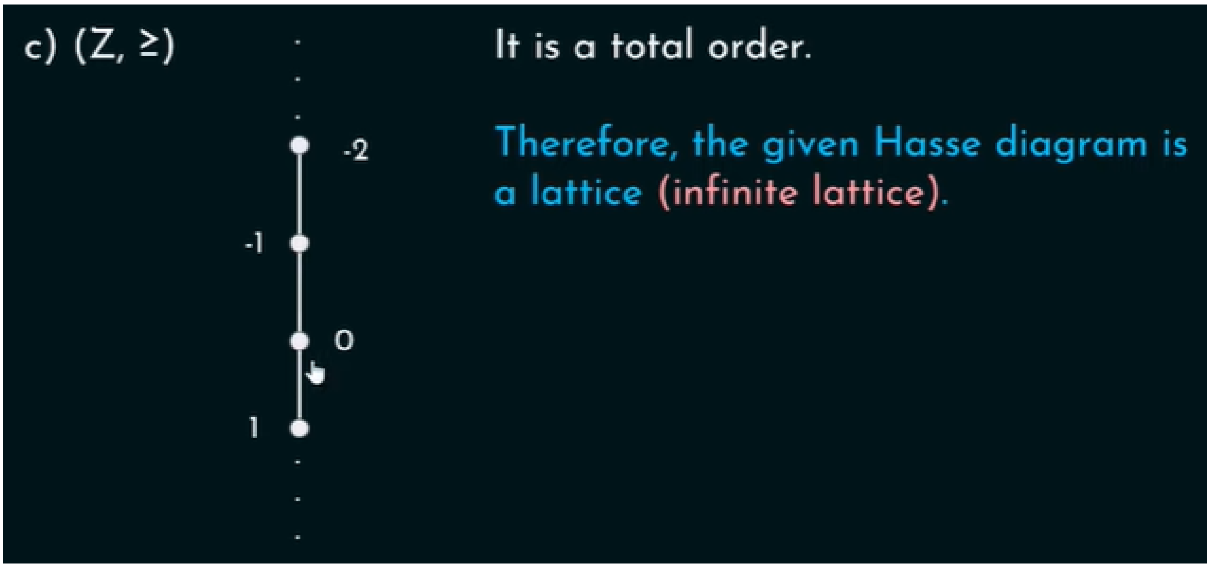
a) ({1, 3, 6, 9, 12}, |)
b) ({1, 5, 25, 125}, |)
c) (Z, ≥)
d) (P(S), ⊇)

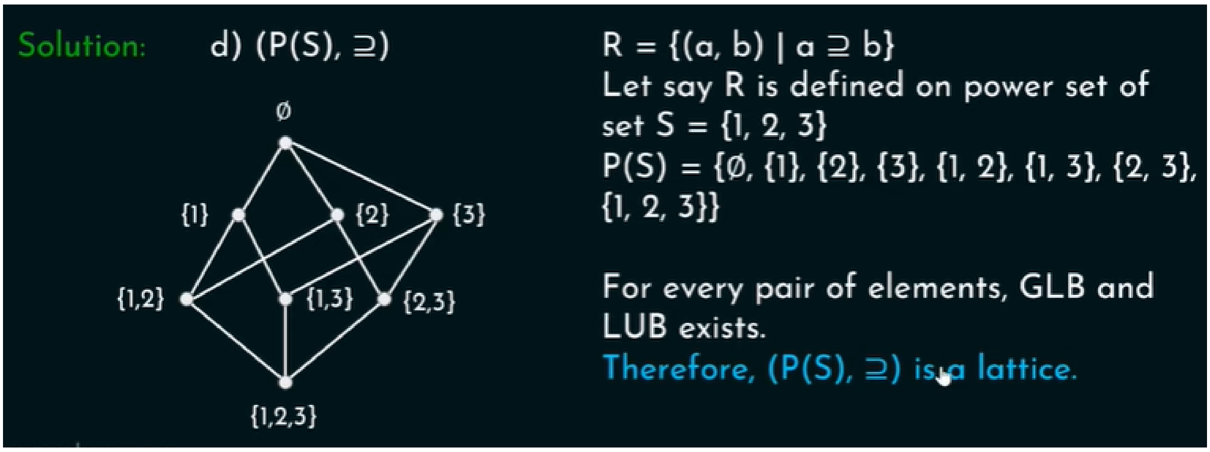
Solution: a) ({1, 3, 6, 9, 12}, |)

Consider the pair (9, 12)
GLB(9, 12) = 3
LUB(9, 12) = ∅

Therefore, the given Hasse diagram is not a lattice.







Complete and Bounded Lattice

Extra laga isliye nhi padha