Now,

$$n(\bar{\underline{x}}-\underline{\mu_0})'\Sigma^{-1}(\bar{\underline{x}}-\underline{\mu_0}) = (\bar{\underline{x}}-\underline{\mu_0})'\Sigma^{*^{-1}}(\bar{\underline{x}}-\underline{\mu_0})$$

$$= (C\underline{y})'\Sigma^{*^{-1}}(C\underline{y})$$

$$= \underline{y}'C'\Sigma^{*^{-1}}C\underline{y}$$

$$= \underline{y}'I\underline{y}.$$

$$= \underline{y}'\underline{y}$$

$$= \sum_{i=1}^{p} y_i^2$$

$$\sim \chi_p^2$$

Let $\chi_p^2(\alpha)$ be the number such that $P[\chi_p^2 \geq \chi_p^2(\alpha)] = \alpha$ then for testing $H_0: \underline{\mu} = \underline{\mu_0}$ we use the critical region

$$n(\bar{\underline{x}}-\underline{\mu_0})'\Sigma^{-1}(\bar{\underline{x}}-\underline{\mu_0}) \geq \chi_p^2(\alpha)$$

Two sample Problem –

Given, $\underline{x}_1^{(1)}, \underline{x}_2^{(1)}, \ldots \underline{x}_\alpha^{(1)} \ldots \underline{x}_{n_1}^{(1)}$ from $N_p(\underline{\mu}^{(1)}, \Sigma)$

& $\underline{x}_1^{(2)}, \underline{x}_2^{(2)}, \ldots \underline{x}_\alpha^{(2)} \ldots \underline{x}_{n_2}^{(2)}$ from $N_p(\underline{\mu}^{(2)}, \Sigma)$

Let the null hypothesis be $H_0: \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$

Then the ~~the~~ test statistics $\frac{n_1 n_2}{n_1+n_2}(\bar{\underline{x}}^{(1)}-\bar{\underline{x}}^{(2)})'\Sigma^{-1}(\bar{\underline{x}}^{(1)}-\bar{\underline{x}}^{(2)}) \sim \chi_p^2$

$$\bar{\underline{x}}^{(1)} \sim N_p\left(\underline{\mu}^{(1)}, \frac{\Sigma}{n_1}\right)$$

$$\bar{\underline{x}}^{(2)} \sim N_p\left(\underline{\mu}^{(2)}, \frac{\Sigma}{n_2}\right)$$

$$\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)} \sim N_p\left(\underline{\mu}^{(1)} - \underline{\mu}^{(2)},\ \Sigma\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\right)$$

$$\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)} \sim N_p\left(0,\ \Sigma\left(\frac{n_1+n_2}{n_1 n_2}\right)\right)$$, under H_0 $\underline{\mu}^{(1)} = \underline{\mu}^{(2)}$

$$\sim N_p(0, \Sigma^*)$$

Let $\left(\frac{n_1+n_2}{n_1 n_2}\right)\Sigma = \Sigma^*$

Since Σ^* is a positive definite symmetric matrix there exists a non singular C s.t. $C' \Sigma^* C = I$

Let the transformation be

$$\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)} = C\underline{y} \Rightarrow \underline{y} = C^{-1}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})$$

$$E(\underline{y}) = C^{-1} E[\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)}]$$

$$= C^{-1} \times 0$$

$$= 0 \quad \text{under } H_0$$

$$\Sigma_{\underline{y}} = \cancel{E(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)} - 0)(x}$$

$$= E(\underline{y} - \underline{0})(\underline{y} - \underline{0})'$$

$$= E\left(C^{-1}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})(C^{-1}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)}))'\right)$$

$$= E\left(C^{-1}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})'(C^{-1})'\right)$$

$$= C^{-1} E(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})(C^{-1})'$$

$$= C^{-1} \Sigma^* (C^{-1})'$$

$$= (C' \Sigma^{*-1} C)^{-1}$$

$$= I$$

$$\underline{y} \sim N_p(\underline{\mu}, \Sigma^*)$$

Now,

$$\frac{n_1 n_2}{n_1+n_2}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})' \Sigma^{-1} (\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})$$

$$= (\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})' \Sigma^{*} (\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})$$

$$= (c\underline{y})' \Sigma^{*} (c\underline{y})$$

$$= \underline{y}' c' \Sigma^{*} c \underline{y}$$

$$= \underline{y}' \underline{y}$$

$$= \sum_{i=1}^{p} y_i^2$$

$$\therefore \frac{n_1 n_2}{n_1+n_2}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})' \Sigma^{-1} (\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)}) \sim \chi_p^2$$

Let $\chi_p^2(\alpha)$ be the no. such that $P[\chi_p^2 \geq \chi_p^2(\alpha)] = \alpha$ then for testing $H_0: \mu = \mu_0$ we use the critical region.

$$\frac{n_1 n_2}{n_1+n_2}(\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)})' \Sigma^{-1} (\underline{\bar{x}}^{(1)} - \underline{\bar{x}}^{(2)}) \geq \chi_p^2(\alpha).$$

Q- If $X \sim N_3(0, \Sigma)$ where $\Sigma = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1.0 & 0.8 & -0.4 \\ 2 & 0.8 & 1.0 & -0.56 \\ 3 & -0.4 & -0.56 & 1.0 \end{array}$

i) $X_1 | X_2, X_3$ $\qquad \Sigma = \left[\quad \right]$ $\qquad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \begin{matrix} X^{(1)} \\ X^{(2)} \end{matrix}$

ii) $X_1, X_2 | X_3$ $\qquad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \begin{matrix} X^{(1)} \\ X^{(2)} \end{matrix}$

iii) $X_1 X_3 | X_2$

$X_1^{(1)} = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$

$X_2^{(1)} = [X_2]$

$\Sigma_{11} = \begin{bmatrix} 1.0 & -0.4 \\ -0.4 & 1.0 \end{bmatrix}$

$\underset{(2\times 1)}{\Sigma_{12}} = \begin{bmatrix} 0.8 \\ -0.56 \end{bmatrix}$

$\Sigma_{22} = [1.0]$

i) $\underline{X_1, X_2 | X_3}$ $\qquad X^{(1)} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\qquad X^{(2)} = [X_3]$

$$\Sigma = \left[\begin{array}{cc|c} 1.0 & 0.8 & -0.4 \\ 0.8 & 1.0 & -0.56 \\ \hline -0.4 & -0.56 & 1.0 \end{array}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

The conditional distribution of X_1, X_2 given X_3 is

$N_2(\mu^{(1)*}, \Sigma_{11.2})$

$$\mu^{(1)*} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$$

$$= 0 + \begin{bmatrix} -0.4 \\ -0.56 \end{bmatrix} [1] (X_3 - 0) = \begin{bmatrix} -0.4X_3 \\ -0.56X_3 \end{bmatrix}$$

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix} - \begin{bmatrix} -0.4 \\ -0.56 \end{bmatrix}_{2\times 1} [1]_{1\times 1} \begin{bmatrix} -0.4 & -0.56 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix} - \begin{bmatrix} -0.4 \\ -0.56 \end{bmatrix}_{2\times 1} \begin{bmatrix} -0.4 & -0.56 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix} - \begin{bmatrix} +0.16 & 0.224 \\ +0.224 & 0.3136 \end{bmatrix} = \begin{bmatrix} 0.84 & 0.576 \\ 0.576 & 0.6864 \end{bmatrix}$$

ii) $\underline{X_1, X_3 | X_2}$ $\qquad \underline{x}^{(1)} = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \qquad \underline{x}^{(2)} = \begin{bmatrix} X_2 \end{bmatrix}$

$$\Sigma = \left[\begin{array}{cc|c} 1.0 & -0.4 & 0.8 \\ -0.4 & 1.0 & -0.56 \\ \hline 0.8 & -0.56 & 1.0 \end{array}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

the conditional distribution of X_1, X_3 given X_2 is –

$N_2(\mu^{(1)*}, \Sigma_{11 \cdot 2})$

$$\mu^{(1)*} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}^{(2)} - \underline{\mu}^{(2)})$$

$$= 0 + \begin{bmatrix} 0.8 \\ -0.56 \end{bmatrix} \begin{bmatrix} 1.0 \end{bmatrix} \begin{bmatrix} \underline{x}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8x^{(2)} \\ -0.56x^{(2)} \end{bmatrix}$$

$$\Sigma_{11 \cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \begin{bmatrix} 1.0 & -0.4 \\ -0.4 & 1.0 \end{bmatrix} - \begin{bmatrix} 0.8 \\ -0.56 \end{bmatrix} \begin{bmatrix} 1.0 \end{bmatrix} \begin{bmatrix} 0.8 & -0.56 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & -0.4 \\ -0.4 & 1.0 \end{bmatrix} - \begin{bmatrix} 0.64 & -0.448 \\ -0.448 & 0.3136 \end{bmatrix}$$

$$= \begin{bmatrix} 0.36 & 0.048 \\ 0.048 & 0.6864 \end{bmatrix}$$

Q. Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ and A be a symmetric matrix of order p show that

i) $E[\underset{\sim}{X}\underset{\sim}{X}'] = \Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}'$

ii) $E[\underset{\sim}{X}'A\underset{\sim}{X}] = tr A\Sigma + \underset{\sim}{\mu}'A\underset{\sim}{\mu}$

i) $\Sigma = E[(\underset{\sim}{X}-\underset{\sim}{\mu})(\underset{\sim}{X}-\underset{\sim}{\mu})']$

$= E(\underset{\sim}{X}\underset{\sim}{X}' - \underset{\sim}{\mu}\underset{\sim}{X}' - \underset{\sim}{X}\underset{\sim}{\mu}' + \underset{\sim}{\mu}\underset{\sim}{\mu}')$

$= E[\underset{\sim}{X}\underset{\sim}{X}'] - \underset{\sim}{\mu}E[\underset{\sim}{X}'] - \underset{\sim}{\mu}'E[\underset{\sim}{X}] + \mu\mu'$

$\Rightarrow \Sigma = E[\underset{\sim}{X}\underset{\sim}{X}'] - \underset{\sim}{\mu}\underset{\sim}{\mu}' - \underset{\sim}{\mu}\underset{\sim}{\mu}' + \underset{\sim}{\mu}\underset{\sim}{\mu}'$

$\Rightarrow E[\underset{\sim}{X}\underset{\sim}{X}'] = \Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}'$

ii) $E[\underset{\sim}{X}'A\underset{\sim}{X}] = E[tr(\underset{\sim}{X}'A\underset{\sim}{X})]$

$tr(ABC) = tr(BCA) = tr(CAB) \neq tr(CBA)$

$= E[tr(A\underset{\sim}{X}\underset{\sim}{X}')]$

$= tr A\ E[\underset{\sim}{X}\underset{\sim}{X}']$

$= tr A[\Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}']$

$= tr A.\Sigma + tr A\mu\mu'$

$= tr A\Sigma + tr(\underset{\sim}{\mu}'A\underset{\sim}{\mu})$

$= tr. A\Sigma + \underset{\sim}{\mu}'A\underset{\sim}{\mu}$

Q. Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{0}, I)$ and A & B are real symmetry matrices of order P then

i) $E[\underset{\sim}{X}'A\underset{\sim}{X}] = tr A$

ii) $V[\underset{\sim}{X}'A\underset{\sim}{X}] = 2\, tr A^2$

iii) $Cov(\underset{\sim}{X}'A\underset{\sim}{X}, \underset{\sim}{X}'B\underset{\sim}{X}) = 2\, tr AB$

$\underline{X} \sim N_p(\underline{0}, I)$

If C is a orthogonal matrix then,

$$CC' = I$$

& $CAC' = dig(\lambda_1, \ldots \lambda_p) = \Delta$

$$\Downarrow$$
$$A$$

$\lambda_1, \lambda_2 \ldots \lambda_p$ are eigen value of matrix A.

Let the transformation $\underline{Y} = C\underline{X}$ then,

$$E[\underline{Y}] = C\underline{\mu} = 0$$

$$\Sigma_y = C\Sigma C' = CIC' = CC' = I$$

i) $\underline{X}'A\underline{X} = (C^{-1}\underline{Y})'A\,C^{-1}\underline{Y}$

$$= \underline{Y}'(C^{-1})'A\,C^{-1}\underline{Y}$$

$$= \underline{Y}'(C')'A\,C'\underline{Y}$$

$$= \underline{Y}'(CAC')\underline{Y}$$

$$= \underline{Y}'\Delta\underline{Y}$$

$\left\{ \begin{array}{l} CC' = I \\ C' = C^{-1} \end{array} \right.$

$$E[\underline{X}'A\underline{X}] = E[\underline{Y}'_{1\times p}\Delta_{p\times p}\underline{Y}_{p\times 1}]$$

$$= E[\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_p y_p^2]$$

$$= \lambda_1 E[y_1^2] + \lambda_2 E[y_2^2] + \cdots + \lambda_p E[y_p^2]$$

$$= \lambda_1 + \lambda_2 + \cdots + \lambda_p$$

$$= tr A$$

ii) $V[\underline{X}'A\underline{X}] = V[\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_p y_p^2]$

$$= \lambda_1^2 V[y_1^2] + \lambda_2^2 V[y_2^2] + \cdots + \lambda_p^2 V[y_p^2]$$

$$= 2\lambda_1^2 + 2\lambda_2^2 + \cdots + 2\lambda_p^2$$

$$= 2\,tr A^2$$

$V[y_i^2]$

$\chi_2^2 \sim N(0,1)^2$