Problem 1 Find the number of ways to place 3 rooks on a 5×5 chess board so that no two of them attack each other.
**In chess, a rook attacks all the pieces in its row and in its column.

Solution 1 Note that for the rooks not to attack each other they must use 3 different rows and 3 different columns. There are $\binom{5}{3}^2$ ways to choose them. Then we have to assign to each row its corresponding column where the rooks are going to be placed.
There are $3!$ ways to do this. Thus there are $\binom{5}{3}^2 . 3! = 600$ ways to place the rooks

Problem 2 A number of persons seat at a round table. It is known that there are 7 women who have a woman to their right and 12 women who have a man to their right. We know that 3 out of each 4 men have a woman to their right. How many people are seated at the table?

Solution 2 Note that there are 19 women. For each woman that has a woman to its right there is a woman that has a woman to its left. Thus there are 12 women that have a man to their left. Hence, there are 12 men that have a woman to their right.
Thus there are 16 men, which gives us a total of 35 persons at the table.

Problem 3 A spider has 8 feet, 8 different shoes and 8 different socks. Find the number of ways in which the spider can put on the 8 socks and the 8 shoes (considering the order in which it puts them on). The only rule is that to put a shoe on the spider must already have a sock on that foot.

Solution 3 First order the shoes in one row and the socks in another. There are $(8!)^2$ ways to do this. To decide the order in which the spider is going to put on the shoes and socks let us write in a list the numbers from 1 to 8 twice in some order. The spider is going to read this list and, according to the number it reads, put a sock or a shoe on that foot (depending on whether it already has a sock on it or not). To write these number note that there are $(16)!$ to order 16 numbers, but we are counting $2!$ times each list because the numbers 1 are indistinguishable, $2!$ times because the numbers 2 are indistinguishable, etc. Thus the number of ways to put the shoes and socks is $\frac{16!(8!)^2}{2^8}$.

Problem 4 A square board with side-length of 8 cm is divided into 64 squares with side-length of 1 cm each. Each square can be painted black or white. Find the total number of ways to colour the board so that every square with side-length of 2 cm formed with 4 small squares with a common vertex has two black squares and two white squares.

Solution 4 Paint the first column in any way. There are 2^8 possibilities. Note that if there are two consecutive squares of the same colour, in the next column they must have the colours swapped. Having these two squares with the colours swapped, the whole next column must have opposite colours to the first column. We can go on this way and the whole board colouring is fixed. If there were no two consecutive squares of the same colour in the first column then it had to be painted alternating colours.
There are only 2 ways to do this. If this happens, then the next column must also be alternating colours, and so on. Thus we only have to choose with which colour each column starts. In the first case there were $2^8 - 2$ colorings, while in the second case there were 2^8.
Thus there's a total of $2^8 + 2^8 - 2 = 2(2^8 - 1)$ possibilities.

Problem 5 Let $A_1, A_2, \ldots, A_{2n}$ be pair-wise different subsets of $\{1, 2, \ldots, n\}$. Determine the maximum value of $\sum_{i=1}^{n} \frac{|A_i \cap A_{i+1}|}{||A_i|.|A_{i+1}||}$

Solution 5 We show that for all $i \neq j$, $\frac{|A_i \cap A_j|}{|A_i|.|A_j|} \leq \frac{1}{2}$.

If A_i and A_j do not intersect, that number is 0. Suppose without loss of generality that $|A_i| \leq |A_j|$.

If they do intersect, note that $|A_j| \geq 2$. Also $|A_i \cap A_j| \leq |A_i|$. Thus $\frac{|A_i \cap A_j|}{|A_i|.|A_j|} \leq \frac{1}{2}$.

This means that the sum we want is at most n. This value is achieved by letting

$A_1 = \{1\}$,

$A_2 = \{1, 2\}$,

$A_3 = \{2\}$,

$A_4 = \{2, 3\}, \ldots, A_{2n-1} = \{n\}, A_{2n} = \{n, 1\}$.

PHP

Problem 6 The sides and diagonals of a regular octagon are coloured black or red. Show that there are at least 7 monochromatic triangles with vertices in the vertices of the octagon.

Solution 6 We will show that there are at least 8 monochromatic triangles. Each vertex is part of 7 lines. The largest possible number of pairs of lines of different colours that use this vertex is $4 \cdot 3 = 12$.

Hence, the number of pairs of lines of different colour that share a vertex is at most $8 \cdot 12$.

Each triangle that is not monochromatic uses exactly two of these pairs, so there are at most $8 \cdot 6 = 48$ non-monochromatic triangles.

Thus the number of monochromatic triangles is at least $\binom{8}{3} - 48 = 8$.

Problem 7 Show that if an infinite number of points in the plane are joined with blue or green segments, there is always an infinite number of those points such that all the segments joining them are of only one colour.

Solution 7 Let v_0 be any point. Since an infinite number of segments come from v_0, by the infinite pigeonhole principle there is an infinite number of green segments or an infinite number of blue segments from v_0.

Suppose there is an infinite number of blue segments and let A_0 be the set of the other endpoints of these segments. If in A_0 there is a point with an infinite number of blue segments to points of A_0, call it v_1 and define A_1 as the set of all other endpoints of such blue segments. If we can iterate this process indefinitely, then we get a sequence of points $v_0, v_1, v_2, \ldots$.. and they are all joined by blue segments.

If we cannot construct some v_{k+1}, that means that in A_k all vertices are joined with blue segments only to a finite number of points in A_{k+1}.

Let u_0 be vertex of A_k and repeat the process with the green segments (constructing the sequences u_r and V_r). If again there is a u_{r+1} that we cannot construct, then there is a V_r such that all the points in V_r are connected with green segments only to a finite number of points in V_r. Since V_r is a subset of A_k, the same happens for blue segments.

Thus V_r should be finite, but it is infinite by construction. Thus we can always find at least one of these two sequences

Problem 8 In the congress, three disjoint committees of 100 congressmen each are formed. Every pair of congressmen may know each other or not. Show that there are two congressmen from different committees such that in the third committee there are 17 congressmen that know both of them or there are 17 congressmen that know neither of them.

Solution 8 Consider the triples *(a, b, c)* such that *a*, *b*, *c* are in different committees and either both *a* and *c* know *b* or neither knows *b*. We consider the triples *(a, b, c)* and *(c, b, a)* as identical.
Note that if we pick one congressman from each committee, they form at least one of these triples. Thus the number of triples is at least 100^3. Each triple has its "central" person from some committee, so there are at least $\frac{100^3}{3}$ of these triples with the central person from the same committee.
Each of these triples uses one pair of persons, one of each of the other two committees, of which there are 100^2 possible choices. Thus there are at least $\frac{100}{3}$ of these triples that use the same pair of these two committees. One triple can be one where the central person knows the other two or one where he does not know any of the other two.
Thus we have at least $\frac{100}{6}$ triples of the same type and with the same "exterior" pair.
Since $\frac{100}{6} > 16$, this means there are at least 17 such pairs. The common exterior pair consists of the two congressmen we want, and the 17 triples correspond to the persons that know both of them or neither of them.

Problem 9 Let *G* be the set of points *(x, y)* in the plane such that *x* and *y* are integers in the range $1 \leq x, y \leq 2011$. A subset *S* of *G* is said to be *parallelogram-free* if there is no proper parallelogram with all its vertices in *S*.
Determine the largest possible size of a parallelogram-free subset of *G*.
Note: A proper parallelogram is one whose vertices do not all lie on the same line.

Solution 9 We will show that the largest size is 4021. Consider the subset S_0 of all points *(x, y)* such that at least one of *x* and *y* is equal to 2011. For any 4 points of S_0 either 3 of them lie on the same line or they define a quadrilateral with two orthogonal opposite sides. In either case they do not form a proper parallelogram.
Thus S_0 has 4021 points and is parallelogram-free.
Suppose that *S* is a subset of at least 4022 points. We show that *S* contains a parallelogram with two sides parallel to the *x*-axis. For this, given two points of *S* on the same row of *G*, consider their distance. If there are two pairs on different rows with the same distance, they form a proper parallelogram.
Assign to each row the different distances you can find among its points. Note that if there are *m* points on the same row, they define at least $m - 1$ different distances.
Let $m_1, m_2, \ldots, m_{2011}$ the number of points on each row.
The number of assigned distances is at least
$\sum_{i=1}^{2011}(m_i - 1) = \sum_{i=1}^{201} m_i - 2011 \geq 2011.$
However, there are only 2010 possible distances to assign. By the pigeonhole principle, there is at least one distance that was assigned twice, as we wanted to prove

Problem 10 Let *n* be a positive integer. A board of size $N = n^2 + 1$ is divided into unit squares with *N* rows and *N* columns. The N^2 squares are coloured with one of *N* colours in such a way that each colour was used *N* times.

Show that, regardless of the colouring, there is a row or a column with at least *n* + 1 different colours.

Solution 10 Let a_i and b_i be the number of rows with color $\boldsymbol{i}$ and the number of columns with color $\boldsymbol{i}$, respectively. Note that there are at most $\boldsymbol{a_i b_i}$ squares of color $\boldsymbol{i}$. Then we have that $\boldsymbol{N \leq a_i b_i \leq \left(\frac{a_i+b_i}{2}\right)^2}$. But, $\boldsymbol{N > n^2}$, we have that $\boldsymbol{a_i + b_i > 2n}$.

Summing over $\boldsymbol{i}$, we obtain $\boldsymbol{\sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i > 2nN}$.

Thus, by the pigeonhole principle, either $\sum_{i=1}^{N} a_i > nN$ *or* $\sum_{i=1}^{N} b_i > nN$.

Suppose without loss of generality that the first case holds. Note that $\sum_{i=1}^{N} a_i$ is the number of pair *(C, i)* where *C* is a column that has a square of color $\boldsymbol{i}$. There are $\boldsymbol{N}$ possible columns and more than $\boldsymbol{nN}$ pairs. Thus there must be a column in more than $\boldsymbol{n}$ pairs, and that is what we wanted.