GEOMETRY

1. Let $\Delta \mathbf{ABC}$ be an equilateral triangle with side length 1 and let τ the circle tangent to $\mathbf{AB}$ and $\mathbf{AC}$ at $\mathbf{B}$ and $\mathbf{C}$, respectively. Let $\mathbf{P}$ be on side $\mathbf{AB}$ and $\mathbf{Q}$ be on side $\mathbf{AC}$ so that $\mathbf{PQ} \parallel \mathbf{BC}$, and the circle through $\mathbf{A}, \mathbf{P}$, and $\mathbf{Q}$ is tangent to τ. If the area of $\Delta \mathbf{APQ}$ can be written in the form $\frac{\sqrt{a}}{b}$ for positive integers $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ is not divisible by the square of any prime, find $\mathbf{a}+\mathbf{b}$.

19

2. Let $\mathbf{ABCD}$ be a square with side length $\mathbf{8}$. Let $\mathbf{M}$ be the midpoint of $\mathbf{BC}$ and let $\boldsymbol{\omega}$ be the circle passing through $\mathbf{M}, \mathbf{A}$, and $\mathbf{D}$. Let $\mathbf{O}$ be the center of $\boldsymbol{\omega}$, $\mathbf{X}$ be the intersection point (besides $\mathbf{A}$) of $\boldsymbol{\omega}$ with $\mathbf{AB}$, and $\mathbf{Y}$ be the intersection point of $\mathbf{OX}$ and $\mathbf{AM}$. If the length of $\mathbf{OY}$ can be written in simplest form as $\frac{m}{n}$, compute $\mathbf{m}+\mathbf{n}$.

36

3. Let $\Delta \mathbf{ABC}$ be a triangle with integer side lengths such that $\mathbf{BC} = \mathbf{2016}$. Let $\mathbf{G}$ be the centroid of $\Delta \mathbf{ABC}$ and I be the in-center of $\Delta \mathbf{ABC}$. If the area of $\Delta \mathbf{BGC}$ equals the area of $\Delta \mathbf{BIC}$, find the largest possible length of AB.

3023

4. Let $\mathbf{D}, \mathbf{E}$, and F respectively be the feet of the altitudes from $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ of acute triangle $\Delta \mathbf{ABC}$ such that $\mathbf{AF} = \mathbf{28}$, $\mathbf{FB} = \mathbf{35}$ and $\mathbf{BD} = \mathbf{45}$. Let $\mathbf{P}$ be the point on segment $\mathbf{BE}$ such that $\mathbf{AP} = \mathbf{42}$. Find the length of $\mathbf{CP}$.

14

5. Triangle $\Delta \mathbf{ABC}$ is isosceles and $\angle \mathbf{ACB} = \mathbf{90}^{\circ}$. The point $\mathbf{D}$ is on the line $\mathbf{AC}$ beyond $\mathbf{C}$, and the point $\mathbf{E}$ is on the line $\mathbf{CB}$ beyond $\mathbf{B}$. Show that $|\mathbf{CD}| = |\mathbf{CE}|$ if line $\mathbf{BD}$ is perpendicular to line $\mathbf{AE}$.

6. In acute triangle $\mathbf{ABC}$ let $\mathbf{A_1}, \mathbf{B_1}$ and $\mathbf{C_1}$ be the midpoints of sides $\mathbf{BC}, \mathbf{CA}$ and $\mathbf{AB}$, respectively. The radius of its circumscribed circle, with centre $\mathbf{O}$, is $\mathbf{1}$.
Prove that $\frac{1}{|0A_1|} + \frac{1}{|0B_1|} + \frac{1}{|0C_1|} \geq \mathbf{6}$.

7. Prove that any triangle can be decomposed into $\mathbf{n}$ isosceles triangles for every positive integer $\mathbf{n} \geq \mathbf{4}$.

8. Let $\mathbf{ABC}$ be a triangle with $|\mathbf{AC}| < |\mathbf{BC}|$ and denote its circum-circle by $\boldsymbol{\kappa}$. Let $\mathbf{E}$ be the midpoint of the arc $\mathbf{AB}$ that contains the point $\mathbf{C}$ and let $\mathbf{D}$ be a point on the segment $\mathbf{BC}$, such that $|\boldsymbol{BD}| = |\boldsymbol{AC}|$. The line $\mathbf{DE}$ meets the circle $\boldsymbol{\kappa}$ again in $\mathbf{F}$. Prove that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{F}$ are the vertices of an isosceles trapezoid.

9. The circles $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ of different radii meet at $\mathbf{A}_1$ and $\mathbf{A}_2$. Let $\boldsymbol{t}$ be the common tangent of the two circles, such that the distance from $\boldsymbol{t}$ to $\mathbf{A}_1$ is shorter than the distance from $\boldsymbol{t}$ to $\mathbf{A}_2$. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be the points in which $\boldsymbol{t}$ touches $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, respectively.
Let $\boldsymbol{\kappa}_3$ and $\boldsymbol{\kappa}_4$ be the circles with radii $|\mathbf{A}_1\mathbf{B}_1|$ and $|\mathbf{A}_1\mathbf{B}_2|$ and the centre $\mathbf{A}_1$. The circles $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_3$ meet again at $\mathbf{C}_1$, while the circles $\boldsymbol{\kappa}_2$ and $\boldsymbol{\kappa}_4$ meet again at $\mathbf{C}_2$. Denote the intersection of the lines $\mathbf{B}_1\mathbf{C}_1$ and $\mathbf{B}_2\mathbf{C}_2$ by $\mathbf{D}$ and let $\mathbf{E}$ be the intersection of $\mathbf{B}_1\mathbf{C}_1$ and $\boldsymbol{\kappa}_4$ which lies on the same side of the line $\mathbf{B}_2\mathbf{C}_2$ as $\mathbf{C}_1$. Show that $\mathbf{A}_1\mathbf{D}$ is perpendicular to $\mathbf{EC}_2$.

10. Let $\mathbf{ABCD}$ be a trapezoid with $\mathbf{AB}$ parallel to $\mathbf{CD}$ and $|\mathbf{AB}| > |\mathbf{CD}|$. Let $\mathbf{E}$ and $\mathbf{F}$ be the points on segments $\mathbf{AB}$ and $\mathbf{CD}$, respectively, such that $\left|\frac{\mathbf{AE}}{\mathbf{EB}}\right| = \left|\frac{\mathbf{DF}}{\mathbf{FC}}\right|$.
Let $\mathbf{K}$ and $\mathbf{L}$ be two points on the segment EF such that
$\angle\mathbf{AKB} = \angle\mathbf{DCB}$ and $\angle\mathbf{CLD} = \angle\mathbf{CBA}$.
Show that $\mathbf{K}, \mathbf{L}, \mathbf{B}$ and $\mathbf{C}$ are con-cyclic.

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NUMBER THEORY

1. Let $x = 0.a_1a_2a_3a_4 \dots\dots$ and $y = 0.b_1b_2b_3b_4 \dots\dots$
be the decimal representations of two positive real numbers.
The equality $b_n = a_{2^n}$ holds for all positive integers n. Given that x is a rational number, show that y is rational, too.

2. We have eight cubes with digits $1, 2, 3, 4, 5, 6, 7, 9$ (each cube has one digit written on one of its faces). In how many ways can we create four two-digit primes from the cubes?
Four possible sets of primes:
$\{41, 67, 23, 59\}, \{41, 67, 29, 53\}, \{61, 47, 23, 59\}, \{61, 47, 29, 53\}$.

3. (a) Find all positive integers n such that both of the numbers $2^n - 1$ and $2^n + 1$ are primes.
(b) Find all primes p such that both of the numbers $4p^2 + 1$ and $6p^2 + 1$ are primes.

4. Find all positive integers n such that $n + 200$ and $n - 269$ are cubes of integers.
$\{1997\}$

5. Find all triples of integers x, y, z satisfying $2^x + 3^y = z^2$.

$(x, y, z) = (0, 1, \pm 2), (3, 0, \pm 3), (4, 2, \pm 5)$.

6. Prove that the number $\underbrace{111\dots111}_{1997}\underbrace{222\dots222}_{1998}5$ is a perfect square.

7. For $a \in \mathbb{R}$, $\{a^{-1}\} = \{a^2\}, 2 \le a^2 \le 3$, $\{.\}$ denote the fractional part, then evaluate $a^{12} - 144a^{-1}$.
$\{233\}$

8. Integers x and y greater than 1 satisfy the relation $2x^2 = 1 + y^{15}$
(a) Prove that x is divisible by five.
(b) Are there such Integers x and y greater than 1 satisfy the relation $2x^2 = 1 + y^{15}$? Could you find all such numbers?
$\{1,1\}$

9. Let a, b and x be positive integers such that $x^2 - bx + a - 1 = 0$. Prove that $a^2 - b^2$ is not a prime number.

10. The tens digit of the $\mathbf{4}$-digit integer $\boldsymbol{n}$ is nonzero. If we take the first $\mathbf{2}$ digits and the last $\mathbf{2}$ digits as two $\mathbf{2}$-digit integers, their product is a divisor of $\boldsymbol{n}$. Determine all $\boldsymbol{n}$ with this property.

$\boldsymbol{n} \in \{\mathbf{1352}, \mathbf{1734}\}$

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COMBINATORICS

1. A nine-member committee was formed to select a chief of the HBCSE. There are three candidates for the chief. Each member of the committee orders the candidates and gives 3 points to the first one, 2 points to the second one and 1 point to the last one. After summing the points of the candidates it turned out that no two candidates have the same number of points, hence the order of the candidates is clear. Someone noticed that if every member of the committee selected only one candidate, the resulting order of candidates would be reversed. How many points did the candidates get?

$\{54\}$

2. The inhabitants of a certain island speak a language in which every word can be written with the following letters: a, b, c, d, e, f, g. A word is said to ***produce*** another one if the second word can be formed from the first one applying any of the following rules as many times as needed:
(i) Replace a letter by two letters according to one of the substitutions:
$a \to bc$, $b \to cd$, $c \to de$, $d \to ef$, $e \to fg$, $f \to ga$, $g \to ab$.
(ii) If only one letter is between two letters that are the same, these two letters can be eliminated. Example: $dfd \to f$
As another example, cafed produces $bfed$, since $cafed \to cbcfed \to bfed$.
Prove that every word on this island produces any other word.

3. Mr. X's math class needs to choose a committee consisting of two girls and two boys. If the committee can be chosen in 3630 ways, how many students are there in Mr. X's math class?

{23}

4. We have 15 cards numbered $1, 2, \ldots, 15$. How many ways are there to choose some (at least one) cards so that all numbers on these cards are greater than or equal to the number of cards chosen?

Hints: the answer is $f(15) = F_{17} - 1$. By straightforward computations, we find $F_{17} = 1597$ so $f(15) = 1596$.

5. In how many ways can $\mathbf{100}$ be written as a sum of non-negative powers of $\mathbf{3}$?

402

6. A set of cards with positive integers on them is given, and the sum of these integers is $\mathbf{2017}$. For any integer $\boldsymbol{k} = \mathbf{1, 2, \ldots, 2016}$, there is only one way to choose some of these cards so that the sum of the numbers on them is $\boldsymbol{k}$. How many such sets of cards are there?

Hints: There are thus four suitable sets of cards:

- $\mathbf{2017}$ cards, all of which are $\mathbf{1's}$;
- $\mathbf{1013}$ cards which are $\mathbf{1's}$ and a $\mathbf{1004}$;
- $\mathbf{511}$ cards which are $\mathbf{1's}$, a $\mathbf{502}$, and a $\mathbf{1004}$;
- $\mathbf{260}$ cards which are $\mathbf{1's}$, a $\mathbf{251}$, a $\mathbf{502}$, and a $\mathbf{1004}$.

8

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ALGEBRA

1. Find all polynomials $P(x)$ satisfying $(x-16)P(2x) = 16(x-1)P(x)$.

$\{P(x) = a(x-16)(x-8)(x-4)(x-2)\}$

2. Let $f(x) = x^3 + px^2 + qx + r$ and $g(x) = x^3 + qx^2 + rx + p$, where p, q, r are integers with $r \neq 0$. Suppose that the following conditions hold:
(a) $f(1) = 0$;
(b) the roots of $g(x)$ are squares of the roots of $f(x)$.
Find the possible values of $p^n + q^n + r^n$, where n be any integer.

$\{-1, 3\}$

3. Show that there exists a polynomial $P(x)$ with integer coefficients such that the equation $P(x) = 0$ has no integer solutions but for each positive integer n there is an $x \in \mathbb{Z}$ such that $n \mid P(x)$.

4. If x, y and z are whole numbers and $xyz + xy + 2yz + xz + x + 2y + 2z = 28$ find $x + y + z$.

$\{8, 10, 14, 28\}$

5. Find x such that $4x^2 - 40[x] + 51 = 0$.

6. How many pairs of integers (a, b) satisfy $a^2b^2 = 4a^5 + b^3$?

$(a, b) \in \{(125, 3125), (27, 486), (54, 972), (32, 512)\}$

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