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CS6363.004 Design and Analysis of Computer Algorithms.

HOMEWORK-1

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1) Let $a_1 = 2$, $a_2 = 9$ and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$
Show that $a_n \leq 3^n$ for all positive integers.

Sol: Given that

$$a_1 = 2$$
$$a_2 = 9$$
$$a_n = 2a_{n-1} + 3a_{n-2} \text{ for } n \geq 3.$$

To Prove: $a_n \leq 3^n$ for all $n > 0$.

Let us prove this by "Proof by Induction"

Basis step:

When $n=1$; 3^n becomes 3^1

For $a_1 = 2$ (given)

$$a_1 = 2 < 3^1$$

$\therefore$ It is true for $n=1$

Hence, it is true for base step.

Inductive Hypothesis:

Assume that

$$a_n \leq 3^n \text{ for all } n \leq k$$

i.e for $n = 1, 2, \dots k$; $a_k \leq 3^k$.

Now we need to show that for all n, the hypothesis is true.

Let us check for ~~k=2~~ $n=2$;

$$a_2 = 9 \text{ (given)}$$

$$3^n = 3^2 = 9$$

$$\therefore \; a_2 = 9 \leq 3^2 \text{ (Hence true for } n=2)$$

Let us check for $k+1$.

$$a_{k+1} = 2a_k + 3a_{k-1} \text{ [From given]}$$

$$\leq 2 \cdot 3^k + 3 \cdot 3^{k-1} \text{ [Replacing } a_k \text{ \& } a_{k-1} \text{ from the hypothesis]}$$

$$\leq 2 \cdot 3^k + 3^k.$$

$$\leq 3^k(2+1)$$

$$\leq 3^k \cdot 3$$

$$\leq 3^{k+1}.$$

$$\therefore \; a_{k+1} \leq 3^{k+1} \text{ [Here we proved that hypothesis is true for } k+1]$$

Similarly, if we check for n

$$a_n = 2a_{n-1} + 3a_{n-2}$$

$$\leq 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} \text{ [From Hypothesis]}$$

$$\leq 2 \cdot 3^{n-1} + 3^{n-1}$$

$$\leq 3^{n-1}[2+1]$$

$$\leq 3^{n-1} \cdot 3$$

$$\leq 3^n.$$

$$\therefore \; a_n \leq 3^n \text{ [Hence proved for all } n]$$

Therefore, it is proved that

$$a_n \leq 3^n \;\; \forall \; n > 0.$$

(2) 3-1.8 We can extend our notation to the case of two parameters n and m that can go to infinity independently at different rates. For a given function $g(n,m)$, we denote by $O(g(n,m))$ the set of functions

$$O(g(n,m)) = \{f(n,m) : \text{there exist positive constants } c, n_0 \text{ and } m_0 \text{ such that } 0 \le f(n,m) \le cg(n,m) \text{ for all } n \geq n_0 \text{ or } m \geq m_0\}$$

Give corresponding definitions for $\Omega(g(n,m))$ and $\Theta(g(n,m))$

Sol: We can denote $\Omega(g(n,m))$ the set of functions

$$\Omega(g(n,m)) = \{f(n,m) : \text{there exist positive constants } c, n_0 \,\&\, m_0 \; [c>0, n_0>0, m_0>0] \text{ such that } \boxed{0 \le c \cdot g(n,m) \le f(n,m)} \text{ for all } n \geq n_0 \text{ or } m \geq m_0\}$$

We can denote $\Theta(g(n,m))$ the set of functions

$$\Theta(g(n,m)) = \{f(n,m) : \text{there exist positive constants } c_1, c_2, n_0, m_0 \; [c_1>0, c_2>0, n_0>0, m_0>0] \text{ such that } \boxed{0 \le c_1 \cdot g(n,m) \le f(n,m) \le c_2 \cdot g(n,m)} \text{ for all } n \geq n_0 \text{ or } m \geq m_0\}$$

③ 4.3-3 We saw that the solution of $T(n) = 2T(\lfloor n/2 \rfloor) + n$ is $O(n \lg n)$. Show that the solution of this recurrence is also $\Omega(n \lg n)$. Conclude that the solution is $\Theta(n \lg n)$.

Sol: Given that,

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

To Prove:

Solution of the recurrence is $\Omega(n \lg n)$.

This implies, we need to show that there exists some +ve c, n_0 such that $T(n) \geq c \cdot n \lg n$ for all $n \geq n_0$.

We will try to prove this by "Proof by Induction".

Basis step:

when $n = 1$,

$T(1) = \cancel{1} 1 \lg 1 = 0$

$T(1) = 1 \geq c \cdot 1 \lg 1$ for $c > 0$

This shows that basis is true

Inductive hypothesis:

Let us assume that $T(n) \geq c \cdot n \lg n$ for all $k < n$

i.e $T(k) \geq c \cdot k \lg k$.

Now, we need to show for it is true for n as well

$$T(n) = 2T(\lfloor \tfrac{n}{2} \rfloor) + n$$

$$\geq 2c(\lfloor n/2 \rfloor \lg \lfloor \tfrac{n}{2} \rfloor) + n$$ [From Inductive hypothesis]

$$\geq c(n-1)\lg((n-1)/2) + n$$

$$\geq c(n-1)(\lg(n-1) - \lg 2) + n$$

$$\geq c(n-1)(\lg(n(1-1/n)) - \lg 2) + n$$

$$\geq cn\left[\lg n - 1 + \lg\left(\tfrac{n-1}{n}\right) + \tfrac{1}{c}\right] \underbrace{- c\left(\lg n - 1 + \lg\left(\tfrac{n-1}{n}\right)\right)}_{+ve \text{ for } 0 < c \leq 1}$$

$$\geq cn\left[\lg n - 2 + \tfrac{1}{c} - \underbrace{\lg \tfrac{(n-1)-1}{n}}_{1}\right]$$ [Adding & Subtracting 1]

$$\geq cn\left(\lg n - 3 + \tfrac{1}{c}\right)$$

$$\geq cn\lg n \quad [\text{For } c = 1/3].$$

Hence, we arrived at

$$T(n) \geq cn\lg n$$

$\therefore$ It shows that $T(n) = \Omega(n\lg n)$

Since, $T(n) = O(n\lg n)$ (Given) & $T(n) = \Omega(n\lg n)$

$$T(n) = \Theta(n\lg n).$$

[Because $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ & $f(n) = \Omega(g(n))$]

4) 4.5.1 Use Master theorem to give tight asymptotic bounds for the following recurrences.

a) $T(n) = 2T(n/4) + 1$.

Sol: Master Theorem states that

For $\boxed{T(n) = aT(n/b) + f(n).}$

Case 1:

If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then

$T(n) = \Theta(n^{\log_b a})$

Case 2:

If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3:

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, &

if $af(n/b) \le cf(n)$ for some $c \ge 1$

then $T(n) = \Theta(f(n))$

Here $T(n) = 2T(n/4) + 1$

$a = 2,\ b = 4,\ f(n) = 1.$

$n^{\log_b a} = n^{\log_4 2} = n^{\log_{2^2} 2} = n^{\frac{1}{2}\log_2 2}$

$= n^{1/2} = \sqrt{n}$

$f(n) = 1 = O(n^{1/2 - 1/2})$ for $\epsilon = 1/2$ $[O(n^{\log_b a - \epsilon})]$

$\therefore$ Case 1 of master theorem applies. $f(n) = O(n^{\log_b a - \epsilon})$

Hence, $T(n) = \Theta(n^{\log_4 2}) = \boxed{\Theta(\sqrt{n})}$

b) $T(n) = 2T(n/4) + \sqrt{n}$

Here $a=2$, $b=4$, $f(n) = \sqrt{n}$

$n^{\log_b a} = n^{\log_4 2} = n^{\log_{2^2} 2} = \sqrt{n}$.

Here $f(n) = \sqrt{n} = \Theta(n^{\log_b a}) = \Theta(\sqrt{n})$.

Case 2 of Master Theorem applies.

~~Hence, $T(n) = \Theta(n^{\log_b a}) = \Theta(\sqrt{n})$~~

Hence, $T(n) = \Theta(n^{\log_b a} \lg n) = \boxed{\Theta(\sqrt{n} \lg n)}$

c) $T(n) = 2T(n/4) + n$

Here $a=2$, $b=4$, $f(n) = n$.

$n^{\log_b a} = n^{\log_4 2} = n^{\log_{2^2} 2} = n^{1/2} = \sqrt{n}$

Here $f(n) = n = \Omega(n^{1/2+\epsilon})$ for $\epsilon = 1/2$.

and $af(n/b) = 2f(n/4) = 2 \times n/4 = n/2$

$n/2 \leq c \cdot f(n)$ i.e $n/2 \leq c \cdot n$

for $c = 1/2$

Case 3 of Master Theorem applies.

Hence, $T(n) = \Theta(f(n)) = \boxed{\Theta(n)}$

d) $T(n) = 2T(n/4) + n^2$

Here $a=2$, $b=4$, $f(n) = n^2$.

$n^{\log_b a} = n^{\log_4 2} = n^{\log_{2^2} 2} = \sqrt{n}$.

$f(n) = n^2 = \Omega(n^{1/2+\epsilon})$ for $\epsilon = 3/2$

$2 \cdot f(n/4) = 2 * \frac{n^2}{16} = \frac{n^2}{8} \leq c \cdot n^2$ for $c = 1/2$

Case 3 of Master theorem applies.

Hence, $T(n) = \Theta(f(n)) = \boxed{\Theta(n^2)}$