CS6363.004 Design and Analysis of Computer Algorithms

Homework 1 Solutions

Assigned: Wednesday 2/2/2022 **Due:** 11:59PM, Thursday, 2/10/2022

Please write your solutions in detail and submit them in ".pdf", ".docx" or "doc" format.

1. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for all positive integers n. **(25 pt)**

   Proof: By induction (you must always specify what your proof method).

   - **Base cases:**
     - For $n = 1$, we have $a_1 = 2$ which is less than 3^1.
     - For $n = 2$, we have $a_2 = 9 \leq {}^2$.
   - **Inductive hypothesis:** assume that $a_k \leq 3^k$ for $1 \leq k < n$.
   - **Show:** $a_n \leq 3^n$.

$$
\begin{aligned}
a_n = a_n &= 2a_{n-1} + 3a_{n-2} \text{ (by definition)} \\
&\leq 3^{n-1} + 3^{n-2} \text{ (by inductive hypothesis)} \\
&= 3^{n-2} \cdot (3 + 1) \\
&= 3^{n-2} \cdot 4 \\
&< 3^{n-2} \cdot 3^2 \\
&= 3^n
\end{aligned}
$$

2. Solve the Exercise 3.1-8 on page 53 from CLRS. **(25 pt)**

$$
\begin{aligned}
\Omega(g(n,m)) = \{f(n,m) : \ & \text{there exist positive constants } c, n_0, \text{ and } m_0 \\
& \text{such that } 0 \leq cg(n,m) \leq f(n,m) \\
& \text{for all } n \geq n_0 \text{ or } m \geq m_0\} .
\end{aligned}
$$

$$
\begin{aligned}
\Theta(g(n,m)) = \{f(n,m) : \ & \text{there exist positive constants } c_1, c_2, n_0, \text{ and } m_0 \\
& \text{such that } 0 \leq c_1 g(n,m) \leq f(n,m) \leq c_2 g(n,m) \\
& \text{for all } n \geq n_0 \text{ or } m \geq m_0\} .
\end{aligned}
$$

3. Exercise 4.3-3 on page 87 from CLRS. **(25 pt)**

   Guess: $T(n) \geq cnlgn$

$$
\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\geq 2c\lfloor n/2 \rfloor lg(\lfloor n/2 \rfloor) + n \\
&\geq 2c((n/2)-1)lg((n/2)-1)+n \\
&= 2c((n-2)/2)lg((n-2)/2)+n \\
&= c(n-2)(lg(n-2)-lg2)+n \\
&= c(n-2)lg(n-2)-c(n-2)+1 \\
&\leq c(n-2)lg(n-2)(\textbf{Mistake}!)
\end{aligned}
$$

Again, we didn't prove the exact claim which is $T(n) \geq cnlgn$. We fix this by making another guess $T(n) \geq c(n+2)lg(n+2)$.

$$
\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\geq 2c\lfloor (n/2)+2 \rfloor lg(\lfloor (n/2)+2 \rfloor) + n \\
&\geq 2c((n/2)-1+2)lg((n/2)-1+2)+n \\
&= 2c((n+2)/2)lg((n+2)/2)+n \\
&= c(n+2)(lg(n+2)-lg2)+n \\
&= c(n+2)lg(n+2)-c(n+2)+n \\
&= c(n+2)lg(n+2)-cn-2c+n \\
&= c(n+2)lg(n+2)+(1-c)n-2c \\
&\geq c(n+2)lg(n+2)
\end{aligned}
$$

if $(1-c)n-2c \geq 0$ which holds for $n \geq 2c/(1-c)$

Solve the Exercise 4.5-1 on page 96 from CLRS. **(25 pt)**

In all parts of this problem, we have $a = 2$ and $b = 4$, and thus $n^{\log_b a} = n^{\log_4 2} = n^{1/2} = \sqrt{n}$.

a. $T(n) = \Theta(\sqrt{n})$. Here, $f(n) = O(n^{1/2-\epsilon})$ for $\epsilon = 1/2$. Case 1 applies, and $T(n) = \Theta(n^{1/2}) = \Theta(\sqrt{n})$.

b. $T(n) = \Theta(\sqrt{n}\lg n)$. Now $f(n) = \sqrt{n} = \Theta(n^{\log_b a})$. Case 2 applies.

c. $T(n) = \Theta(n)$. This time, $f(n) = n^1$, and so $f(n) = \Omega(n^{\log_b a+\epsilon})$ for $\epsilon = 1/2$. In order for case 3 to apply, we have to check the regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$. Here, $af(n/b) = n/2$, and so the regularity condition holds for $c = 1/2$. Therefore, case 3 applies.

d. $T(n) = \Theta(n^2)$. Now, $f(n) = n^2$, and so $f(n) = \Omega(n^{\log_b a+\epsilon})$ for $\epsilon = 3/2$. In order for case 3 to apply, we again have to check the regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$. Here, $af(n/b) = n^2/8$, and so the regularity condition holds for $c = 1/8$. Therefore, case 3 applies.