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CS6363 DESIGN AND ANALYSIS OF ALGORITHMS

HOMEWORK-2

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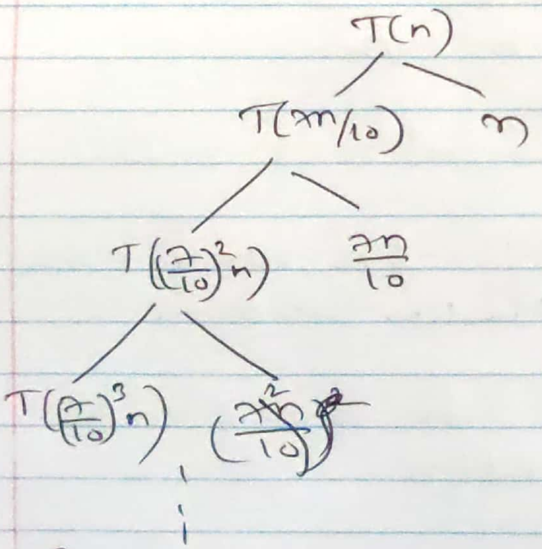
(1) Give asymptotic upper and lower bounds for the following recurrences

4.1b) $T(n) = T(7n/10) + n$.

Sol: Solving it using the recursion tree Method;



By continuing the tree, we arrive at the following step:

$$T(n) = n + \frac{7n}{10} + \left(\frac{7}{10}\right)^2 n + \left(\frac{7}{10}\right)^3 n + \cdots + \underbrace{\theta(1)}$$

since $T(n)$ is Constant for $n \leq 2$.

$$\Rightarrow T(n) = n\left(1 + \frac{7}{10} + \left(\frac{7}{10}\right)^2 + \cdots\right)$$

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Solving this geometric progression, gives

$$T(n) = n\left(\frac{1}{1-\frac{7}{10}}\right) + \Theta(1)$$

$$= n\left(\frac{10}{3}\right) + \Theta(1)$$

$$\Rightarrow T(n) = \Theta(n).$$

Also, ~~we~~ solving it using Master's Theorem

$$T(n) = aT(n/b) + f(n)$$

Here, given

$$T(n) = T(7n/10) + n$$

Considering $a=1$, $b=10/7$.

$$\Rightarrow n^{\log_b a} = n^{\log_{10/7} 1} = n^0 = 1 = \Theta(1).$$

$$\Rightarrow f(n) = n = \Omega(n^{0+1}) \text{ for some } \epsilon = 1$$

Hence, from case 3 of Master's Theorem

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ &

$$af(n/b) \leq cf(n)$$

Then $T(n) = \Theta(f(n))$.

$$1 * f\left(\frac{7n}{10}\right) \leq c \cdot f(n). \quad \text{for } c < 1 \left[\text{Here } c = \frac{7}{10}\right]$$

$\therefore$ We can deduce that $\boxed{T(n) = \Theta(n)}$

(4.1c) $T(n) = 16T(n/4) + n^2$

Sol: We can solve this using Master's Theorem

$$T(n) = aT(n/b) + f(n)$$

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(3)

Here $a = 16$, $b = 4$, $f(n) = n^2$.

$\Rightarrow n^{\log_b a} = n^{\log_4 16} \Rightarrow n^{\log_4 4^2} = n^{2(1)}$
$= \underline{\underline{n^2}}$

$\Rightarrow f(n) = \Theta(n^{\log_b a}) = \underline{\Theta(n^2)}$

Case(2) of Master's Theorem states that

If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Therefore;
$T(n) = \Theta(n^{\log_b a} \lg n) = \boxed{\Theta(n^2 \lg n)}$

4.1d) $T(n) = 7T(n/3) + n^2$

Sol: solving this using Master's Theorem,
$T(n) = aT(n/b) + f(n)$

Here, $a = 7$, $b = 3$, $f(n) = n^2$

$\Rightarrow n^{\log_b a} = n^{\log_3 7} = n^{1.77}$

$\Rightarrow f(n) = n^2 = \Omega(n^{1.77+\epsilon})$ for some $\epsilon > 0$ $(\epsilon = 0.33)$
$7 \cdot f(n/3) = 7 \cdot \frac{n^2}{9} \leq c \cdot n^2$ for $c < 1$ $[\because c = \frac{7}{9}]$.

Case (3) of MT states that
If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ &
$af(n/b) \leq c \cdot f(n)$
Then $T(n) = \Theta(f(n))$

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Hence, here $T(n) = \Theta(f(n))$
$= \Theta(n^2)$

$\therefore \boxed{T(n) = \Theta(n^2)}$

4.1 e) $T(n) = 7T(n/2) + n^2$

Sol: Solving it using Master's Theorem, $T(n) = aT(n/b) + f(n)$
$a = 7,\ b = 2,\ f(n) = n^2$

$n^{\log_b a} = n^{\log_2 7} = n^{2.807}$

$f(n) = n^2 = O(n^{\log_b a - \epsilon}) = O(n^{2.807 - \epsilon})$ for $\epsilon > 0$.
[Here $\epsilon = 0.807$]

Case 1 of MT states that
If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then
$T(n) = \Theta(n^{\log_b a})$

Therefore, $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_2 7})$.

$\therefore \boxed{T(n) = \Theta(n^{\log_2 7})}$

2 Ex 8.1-4 on Page 194 from CLRS.

Sol: Given that
A sequence of n elements to sort.
n/k subsequences consisting k elements

elements in a given subsequence are smaller than elements in succeeding subsequence and greater than elements in preceding subsequence

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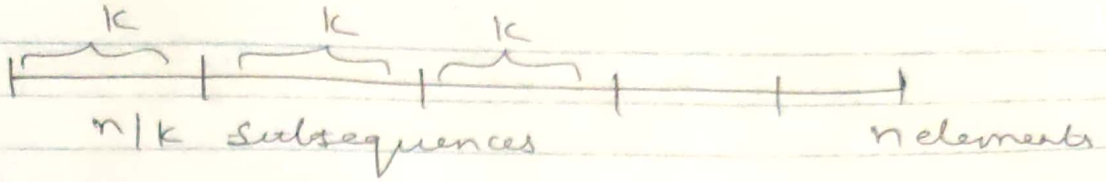

Hence, sorting each subsequence gives the entire sorted array.

From the decision tree method.
Worst case no. of comparisions
= length of longest path from root to reachable leaf
= height (depth) of decision tree.

For each subsequence, there are K elements.
$\therefore$ There are $k!$ permutations that are associated to leaves. — (2)

For n/k subsequences,
Total no. of permutations $= (k!)^{n/k}$ — (1)

In a tree with height h, there are atmost 2^h leaves — (1)

From (1) & (2);

$$(k!)^{n/k} \leq 2^h$$

$$2^h \geq (k!)^{n/k}$$

Applying log on both sides;

$$\log 2^h \geq \log (k!)^{n/k}$$

$$h \geq \frac{n}{k} \log(k!)$$

(6)

$$h \geq \frac{n}{k} \log k!$$

$$\geq \frac{n}{k} \log(k*(k-1)*(k-2)* \cdots 1)$$

$$\geq \frac{n}{k}[\log k + \log(k-1) + \cdots + \log 1]$$

$$[\because \log(A*B) = \log A + \log B]$$

$$\geq \frac{n}{k}\left[\sum_{i=1}^{k} \log i\right]$$

$$\geq \frac{n}{k}\left[\sum_{i=k/2}^{k} \log i\right]$$

$$\geq \frac{n}{\cancel{k}}\left[\frac{\cancel{k}}{2} \log k/2\right]$$

$$\geq \frac{n}{2} \log k/2$$

We can show that $\boxed{1/2 \log k/2 \geq c \log k}$ (1) for some $c > 0$ & $n \geq n_0$

$$\frac{\frac{1}{2}\log k/2}{\log k} \geq c$$

$$\frac{1}{2}\frac{[\log k - \overset{1}{\log 2}]}{\log k} \geq c$$

$$\Rightarrow c \leq \frac{1}{2}\left[1 - \frac{1}{\log k}\right] \quad \left[\text{if } k \to \infty, \frac{1}{\log k} \to 0\right]$$

$$\Rightarrow c \leq 1/2.$$

Lets consider $c = 1/4$; then

$$\frac{1/2 \log k/2}{\log k} \geq 1/4.$$

$$\Rightarrow 2 \log k/2 \geq \log k.$$

$$2 \log k - 2 \geq \log k$$

$$\log k \geq 2 \quad \Rightarrow k \geq 4.$$

$\Rightarrow h \geq n * c \log k$ for $c \leq 1/2$ & $k \geq 4$. from (1).

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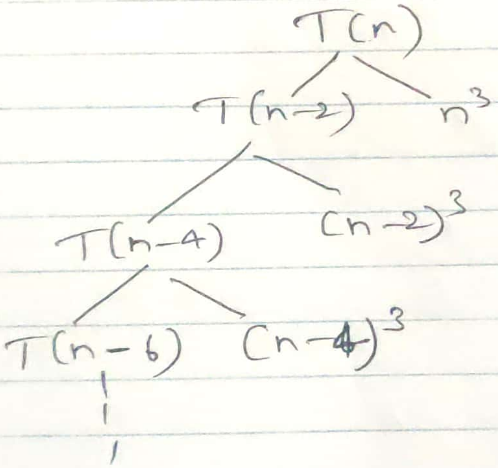
$\Rightarrow \quad h \geq cn \log k$

$\Rightarrow$ Worst No. of comparisions = $\boxed{\Omega(n \log k)}$

③ Solve the recurrence relation $T(n) = T(n-2) + n^3$ by using substitution method

Sol. To make a guess, let us check ~~the~~ using recurrence tree method.



$\Rightarrow T(n) = n^3 + (n-2)^3 + (n-4)^3 + \cdots + 2^3 + \underbrace{O(1)}_{\text{for 0 input}}$

$= 2^3 + 4^3 + \cdots + (n-4)^3 + (n-2)^3 + n^3$

↳ Sum of cubes of even numbers

$= 2(n(n+1))^2$

$= 2(n^2+n)^2 = 2(n^4+2n^3+n^2)$

$\Rightarrow T(n) = \cancel{\Theta(n^4)}\ O(n^4)$

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Hence, our guess is $T(n) = O(n^4)$

We need to prove that

$$T(n) \le c \cdot n^4 \text{ for all } n \geq n_0, \exists\, c > 0, n_0 > 0$$

We can prove this by Induction

Basis step: Assume that $T(n) = 8$ for $n = 2$ when $T(0) = 0$.

$8 \le c \cdot (16)$

Hence base is true. for $c > 1/2$

Inductive hypothesis:

Assume that

$T(n) \le cn^4$ for all $k < n$

i.e $T(k) \le ck^4$

Lets us find out

$$T(n) = T(n-2) + n^3$$

$$\le c(n-2)^4 + n^3 \quad \text{[From Inductive hypothesis]}$$

$$= c(n^4 - 8n^3 + 24n^2 - 32n + 16) + n^3$$

$$= cn^4 - n^3 \underbrace{(8c-1)}_{\geq 0 \text{ for } c \geq 1/8} - c\,(\underbrace{32n - 24n^2 - 16}_{> 0 \text{ always since no real roots}})$$

$$= cn^4 - (\text{some quantity} > 0)$$

$$T(n) \le cn^4 \quad [\text{for } c \geq 1/2,\ n > 0].$$

$\therefore$ $T(n) = O(n^4)$

(4) Design a divide-and-conquer algorithm to compute the factorial of a positive integer n. Set up and solve the recurrence relation for the number of multiplications made by your algorithm.

Sol. Factorial of a positive integer

$n! = n * (n-1) * (n-2) * \cdots 1.$

Divide and conquer algorithm for factorial:

Divide: Divide the calculating of n! as following

$n! = n * (n-1)!$ i.e $n! \begin{cases} n \\ (n-1)! \end{cases}$

Conquer: Recursively calculate $(n-1)!$

Combine: After obtaining $(n-1)!$ result multiply it to n.

$n! = n * (n-1)!$

(10)

Algo:

Fact(n):

i) If n=0 or n=1;
result = 1;

(ii) Else
result = n * Fact(n-1);

Recurrence relation for no. of multiplications:-

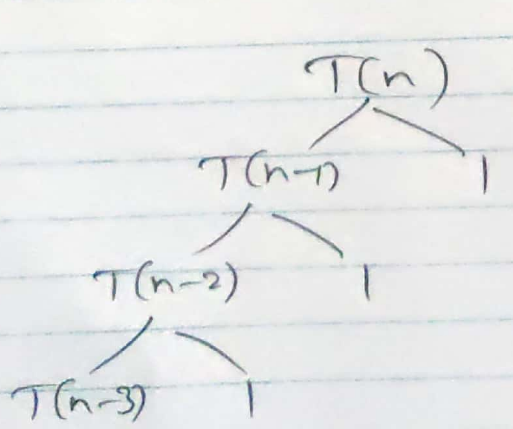
Let T(n): No. of multiplications for input n.

From the algo:

$$T(n) = \underbrace{T(n-1)}_{\text{for recursively calculating } (n-1)!} + 1 \rightarrow \text{multiplications in combine step.}$$

[Assuming comparing n to 0 or 1 & subtracting n-1 takes negligible time]

$$\Rightarrow T(n) = T(n-1) + 1$$



$$\Rightarrow T(n) = \underbrace{1 + 1 + 1 + \cdots}_{n-1 \text{ times}} + \underbrace{T(1)}_{1}$$

$$\Rightarrow T(n) = n$$

$$\therefore T(n) = O(n)$$

$\Rightarrow$ The no. of multiplications made $= \boxed{O(n)}$

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