Kinetics of Systems of Particles

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1 Linear Momentum of a System

Linear momentum of a system of particles is defined as the sum of the momentum of individual particles.

$$\vec{P}_{sys} = \sum_{j=1}^{n} \vec{P}_j$$

1.1 Center of Mass

Position vector to the center of mass from an origin O is defined as,

$$\vec{R}_{CM} = \frac{\sum_{j=1}^{n} m_j \vec{r}_i}{\sum_{j=1}^{n} m_j}$$

Differentiating w.r.t. time,

$$M\vec{V}_{CM} = \sum_{j=1}^{n} m_j \vec{v}_i = \vec{P}_{sys}$$

So the momentum of a system of particles with respect to a frame of reference O can be written as,

$$\vec{P}_{sys} = M\vec{V}_{CM}$$

For the above expression, it doesn't matter if the origin is accelerating or not. Differentiating again we get,

$$M\vec{a}_{CM} = \sum_{j=1}^{n} m_j \vec{a}_i$$

If the origin is an intertial frame of reference we can write,

$$m_j \vec{a}_j = \vec{F}_j$$

where $\vec{F}_j$ represents the force on j^{th} particle, including the forces from particles in the system, and particles outside of the system (external forces).

$$M\vec{a}_{CM} = \sum_{j=1}^{n} \vec{F}_j$$

We can write $\vec{F}_j$ as,

$$\vec{F}_j = \vec{F}_j^{\text{ext}} + \vec{F}_j^{\text{int}}$$

where $\vec{F}_j^{\text{ext}}$ represents the sum of the external forces applied on the particle, and $\vec{F}_j^{\text{int}}$ represents the sum of the internal forces applied on the particle. We can write $\vec{F}j^{\text{int}}$ as,

$$\vec{F}_j^{\text{int}} = \sum_{i \neq j}^{n} \vec{F}_{j,i}$$

Substituting this to the equation,

$$\begin{aligned}
M\vec{a}_{CM} &= \sum_{j=1}^{n} \left(\vec{F}_j^{\text{ext}} + \sum_{i \neq j}^{n} \vec{F}_{j,i} \right) \\
&= \sum_{j=1}^{n} \vec{F}_j^{\text{ext}} + \sum_{j=1}^{n} \sum_{i \neq j}^{n} \vec{F}_{j,i} \\
&= \sum_{j=1}^{n} \vec{F}_j^{\text{ext}} + \frac{1}{2} \sum_{j=1}^{n} \sum_{i \neq j}^{n} (\vec{F}_{j,i} + \vec{F}_{i,j})
\end{aligned}$$

By newton's third law, $\vec{F}_{j,i} + \vec{F}_{i,j} = 0$, hence the sum $\sum_{j=1}^{n} \sum_{i \neq j}^{n} (\vec{F}_{j,i} + \vec{F}_{i,j}) = 0$. Therefore,

$$M\vec{a}_{CM} = \sum_{j=1}^{n} \vec{F}_j^{\text{ext}}$$

Let $\vec{F}_{ext}$ be the sum of all external forces on the system of particles, it follows that,

$$\vec{F}_{ext} = M\vec{a}_{CM}$$

We can show that for a system of particles,

$$\frac{d\vec{P}_{sys}}{dt} = \vec{F}_{ext}$$

1.2 Conservation of Momentum in a System of Particles

If the net external force is zero, then it follows that,

$$\frac{d\vec{P}_{sys}}{dt} = 0$$

So the linear momentum is constant when the sum of external forces is zero. (Momentum in a specific direction is conserved if there's no net external force in that direction)

1.3 Linear Momentum in Center of Mass Frame

It can be shown that the linear momentum of a system is always zero in the COM frame. It doesn't matter whether the COM Frame is intertial or not. Let O be the origin (might be non-intertial)

$$\vec{v}_{j,cm} = \vec{v}_{j,O} - \vec{v}_{cm,O}$$

Let $\vec{v'}_j = \vec{v}_{j,cm}$. We can write the momentum of the system of particles as,

$$
\begin{aligned}
\vec{P}_{sys,cm} = \sum_{j=1}^{n} m_j \vec{v'}_j &= \sum_{j=1}^{n} m_j(\vec{v}_{j,O} - \vec{v}_{cm,O}) \\
&= \sum_{j=1}^{n} \vec{P}_{j,O} - \left(\sum_{j=1}^{n} m_j\right) \vec{v}_{cm,O} \\
&= \sum_{j=1}^{n} \vec{P}_{j,O} - M\vec{v}_{cm,O} \\
&= \vec{P}_{sys,O} - M\vec{v}_{cm,O} \\
&= 0
\end{aligned}
$$

Hence,

$$\vec{P}_{sys,cm} = 0$$

It can be derived in another way, since $\vec{P}_{sys,cm} = M\vec{v}_{cm,cm} = 0$ (COM has a zero velocity in COM Frame) the momentum is zero.

1.4 Summary

Summary

- Momentum P of a system of particles can be written as,

$$\vec{P} = M\vec{v}_{CM}$$

- Rate of change of momentum,

$$\vec{F}_{ext} = M\vec{a}_{CM} = \frac{d\vec{P}_{sys}}{dt}$$

  If a component of $\vec{F}_{ext}$ is zero, then the momentum is conserved in that direction.

- Linear momentum $\vec{P}_{CM}$ is zero in COM Frame.

2 Angular Momentum of a System of Particles

2.1 Angular Momentum about a Fixed Origin

Angular momentum $\vec{L}_{sys}$ of a system is defined as,

$$\vec{L}_{sys} = \sum_{j=1}^{n} \vec{r_j} \times \vec{p_j}$$

Differentiating this,

$$\frac{d\vec{L}_{sys}}{dt} = \sum_{j=1}^{n} \left(\vec{r_j} \times \frac{d\vec{p_j}}{dt} + \frac{d\vec{r_j}}{dt} \times \vec{p_j} \right)$$

Since $\frac{d\vec{r_j}}{dt} \times \vec{p_j} = 0$ and $\vec{r_j} \times \frac{d\vec{p_j}}{dt} = \vec{r}_j \times \vec{F}_j$ it follows that,

$$\frac{d\vec{L}_{sys}}{dt} = \sum_{j=1}^{n} (\vec{r}_j \times \vec{F}_j)$$

It can be written as,

$$\begin{aligned}
\frac{d\vec{L}_{sys}}{dt} &= \sum_{j=1}^{n}(\vec{r}_j \times \vec{F}_j) \\
&= \sum_{j=1}^{n}\left(\vec{r}_j \times \vec{F}_j^{ext} + \sum_{i\neq j}^{n} \vec{r}_j \times \vec{F}_{j,i}\right) \\
&= \sum_{j=1}^{n}\left(\vec{r}_j \times \vec{F}_j^{ext} + \frac{1}{2}\sum_{i\neq j}^{n} (\vec{r}_j \times \vec{F}_{j,i} + \vec{r}_i \times \vec{F}_{i,j})\right)
\end{aligned}$$

By newton's third law,

$$\begin{aligned}
&= \sum_{j=1}^{n}\left(\vec{r}_j \times \vec{F}_j^{ext} + \frac{1}{2}\sum_{i\neq j}^{n} (\vec{r}_j \times \vec{F}_{j,i} - \vec{r}_i \times \vec{F}_{j,i})\right) \\
&= \sum_{j=1}^{n}\left(\vec{r}_j \times \vec{F}_j^{ext} + \frac{1}{2}\sum_{i\neq j}^{n} (\vec{r}_j - \vec{r}_i) \times \vec{F}_{j,i})\right) \\
&= \sum_{j=1}^{n}\left(\vec{r}_j \times \vec{F}_j^{ext} + \frac{1}{2}\sum_{i\neq j}^{n} (\vec{r}_{j,i} \times \vec{F}_{j,i})\right)
\end{aligned}$$

Since $\vec{r}_{j,i}$ and $\vec{F}_{j,i}$ are parallel and *act through the line joining particles*, their cross product is zero. Hence it follows that,

$$\frac{d\vec{L}_{sys}}{dt} = \sum_{j=1}^{n} \vec{r}_j \times \vec{F}_j^{ext}$$

The rate of change of angular momentum is equal to the sum of external torques on each particle. (*Note that this is only valid if the origin is inertial, but it's later shown that this is also valid in the COM Frame, even when the COM frame is non-inertial*)

$$\frac{d\vec{L}_{sys}}{dt} = \sum_{j=1}^{n} \vec{\tau}_j$$

We can define τ_{ext} as,

$$\vec{\tau}_{ext} = \sum_{j=1}^{n} \vec{\tau}_j$$

Therefore,

$$\vec{\tau}_{ext} = \frac{d\vec{L}_{sys}}{dt}$$

2.2 Angular Momentum and COM

Let $\vec{R}$ be the position vector of the COM, $\vec{v'}_j$ be the velocity of the j'th particle w.r.t. COM.

$$\vec{r}_j = \vec{R} + \vec{r'}_j$$
$$\vec{v}_j = \vec{v}_{CM} + \vec{v'}_j$$

Let L_O be the angular momentum around the origin O.

$$\begin{aligned}\vec{L}_O &= \sum_{j=1}^{n} \vec{r}_j \times m_j \vec{v}_j \\ &= \sum_{j=1}^{n} (\vec{R} + \vec{r'}_j) \times m_j(\vec{v}_{CM} + \vec{v'}_j) \\ &= \vec{R} \times \sum_{j=1}^{n} m_j \vec{v}_{CM} + \vec{R} \times \sum_{j=1}^{n} m_j \vec{v'}_j + \left(\sum_{j=1}^{n} m_j \vec{r'}_j\right) \times \vec{v}_{CM} + \sum_{j=1}^{n} m_j \vec{r'}_j \times \vec{v'}_j\end{aligned}$$

In the 2nd term, $\sum_{j=1}^{n} m_j \vec{v'}_j$ is the momentum of the system in the COM frame, which is zero. Therefore the 2nd term is zero. In the 3rd term, $\sum_{j=1}^{n} m_j \vec{r'}_j$ gives the $M\vec{R}_{CM,CM}$ which is zero. So middle two terms are zero. Therefore,

$$\begin{aligned}\vec{L}_O &= \vec{R} \times \sum_{j=1}^{n} m_j \vec{v}_{CM} + \sum_{j=1}^{n} m_j \vec{r'}_j \times \vec{v'}_j \\ &= \vec{L}_{CM} + \vec{R} \times \vec{P}_{sys,O}\end{aligned}$$

Differentiating this equation we get,

$$\frac{d\vec{L}_O}{dt} = \frac{d\vec{L}_{CM}}{dt} + \vec{R} \times \frac{d\vec{P}_{sys,O}}{dt} + \vec{V}_{CM} \times \vec{P}_{sys,O}$$

Since $\vec{V}_{CM} \times \vec{P}_{sys,O}$ is zero (vectors are parallel) it follows that,

$$\frac{d\vec{L}_O}{dt} = \frac{d\vec{L}_{CM}}{dt} + \vec{R} \times \frac{d\vec{P}_{sys,O}}{dt}$$
$$\frac{d\vec{L}_O}{dt} = \frac{d\vec{L}_{CM}}{dt} + \vec{R} \times \vec{F}_{ext}$$
$$\frac{d\vec{L}_O}{dt} = \frac{d\vec{L}_{CM}}{dt} + \vec{R} \times \vec{F}_{ext}$$

$$\begin{aligned} \vec{\tau}_O &= \sum_{j=1}^{n}(r_j \times \vec{F}_j) \\ &= \sum_{j=1}^{n}(\vec{r'}_j + \vec{R}) \times \vec{F}_j \\ &= \sum_{j=1}^{n}(\vec{r'}_j \times \vec{F}_j) + \vec{R} \times \sum_{j=1}^{n} \vec{F}_j \end{aligned}$$

In the above expression, $\sum_{j=1}^{n}(\vec{r'}_j \times \vec{F}_j)$ represents the *torque about COM*. The acceleration of the COM didn't matter in deriving this equation, hence it's valid even when the COM accelerates. Therefore,

$$\vec{\tau}_O = \vec{\tau}_{CM} + \vec{R} \times \vec{F}_{ext}$$

Since $\vec{\tau}_O = \frac{d\vec{L}_O}{dt}$. It follows that,

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_{CM} + \vec{R} \times \vec{F}_{ext}$$

(*Usually* $\frac{d\vec{L}}{dt} = \vec{\tau}$ *is valid only when the frame is inertial, but the above equations suggest that* $\frac{d\vec{L}_CM}{dt} = \vec{\tau}_{CM}$ *is valid even when the COM frame is non-inertial*).

Let a_j be the acceleration of j'th particle w.r.t an inertial frame of reference O. Let a_{CM} represent the acceleration of the COM w.r.t. O. Let a_j' be the acceleration of the j'th particle w.r.t. COM frame.

$$a_j' = a_j - a_{CM}$$

$$\begin{aligned}
\frac{d\vec{L}_{CM}}{dt} &= \frac{d}{dt}\left(\sum_{j=1}^{n} \vec{r'}_j \times m_j \vec{v'}_j\right) \\
&= \sum_{j=1}^{n} \vec{r'}_j \times m_j \vec{a'}_j + \cancelto{0}{\sum_{j=1}^{n} \vec{v'}_j \times m_j \vec{v'}_j} \\
&= \sum_{j=1}^{n} \vec{r'}_j \times m_j (\vec{a_j} - \vec{a}_{CM}) \\
&= \cancelto{\sum_{j=1}^{n} \vec{r'}_j \times \vec{F}_j^{\,ext}}{\sum_{j=1}^{n} \vec{r'}_j \times \vec{F}_j} - \sum_{j=1}^{n} m_j \vec{r'}_j \times \vec{a}_{CM}
\end{aligned}$$

Here in the last term $\sum_{j=1}^{n} m_j \vec{r'}_j$ represents the *position vector to COM from the COM*, which is zero. Therefore the above expression reduces to,

$$\frac{d\vec{L}_{CM}}{dt} = \sum_{j=1}^{n} \vec{r'}_j \times \vec{F}_j^{\,ext} = \sum_{j=1}^{n} \tau_j = \vec{\tau}_{CM}$$

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2.3 Angular Momentum about an Arbitrary Point

Consider a system of n particles. Let O be a point with zero acceleration, let O' be an arbitrary point (it might be accelerating). Let $\vec{r}_j$ be the position vector of j'th particle, let $\vec{R}$ be the position of the COM w.r.t. O' frame. Let $\vec{r'}_j$ be the position vector of j'th particle w.r.t. O'.

Angular momentum about O' can be written as,

$$
\begin{aligned}
\vec{L}_{O'} &= \sum_{j=1}^{n} (\vec{r'}_j \times m_j \vec{v'}_j) \\
&= \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times (\vec{v}_j - \vec{v}_o)
\end{aligned}
$$

Differentiating this we get,

$$
\begin{aligned}
\frac{d\vec{L}_{O'}}{dt} &= \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times m_j(\vec{a}_j - \vec{a}_0) + \cancelto{0}{\sum_{j=1}^{n} (\vec{v}_j - \vec{v}_0) \times m_j(\vec{v}_j - \vec{v}_0)} \\
&= \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times \vec{F}_j^{ext} - \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times m_j \vec{a}_0 \\
&= \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times \vec{F}_j^{ext} - \left(\sum_{j=1}^{n} m_j (\vec{r}_j - \vec{r}_0) \right) \times \vec{a}_0 \\
&= \sum_{j=1}^{n} (\vec{r}_j - \vec{r}_0) \times \vec{F}_j^{ext} - M\vec{R} \times \vec{a}_0
\end{aligned}
$$

If one of the following condition is satisfied, 2nd term becomes zero.

- $\vec{a}_0 = 0$, that is if the O' frame is inertial.
- $\vec{R} = 0$, i.e. O' is the COM
- $\vec{R}$ and $\vec{a}_0$ are parallel, which means, the acceleration of O' is towards/away from the COM.

If one of those are satisfied,

$$
\frac{d\vec{L}_{O'}}{dt} = \vec{\tau}_{ext}
$$

2.4 Summary

Summary

- If O satisfies one of the following conditions,
  - O is an inertial frame
  - O is the COM
  - Acceleration of O is towards or away from the COM

  then the following holds

$$\frac{d\vec{L}_{sys}}{dt} = \sum_{j=1}^{n} \vec{r}_j \times \vec{F}_j^{ext} = \vec{\tau}_{ext}$$

  Note that this is not valid in non-inertial frames(except in 3rd case).

- If the total external torque on a system is zero, then the angular momentum of the system is conserved.

- Let O be *any* reference frame, and R be the position vector to COM, $\vec{P}$ be the linear momentum of the system,

$$\vec{L}_O = \vec{L}_{CM} + \vec{R} \times \vec{P} = \vec{L}_{CM} + \vec{R} \times M\vec{V}_{CM}$$

  where M is the total mass.

- Let O be an inertial frame of reference (*COM might be accelerating*), then

$$\frac{d\vec{L}_{sys}}{dt} = \vec{\tau}_O = \vec{\tau}_{CM} + \vec{R} \times \vec{F}_{ext}$$

3 Kinetics of Rigid Bodies