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Bell's Original Paper

Determinism

$p \cdot \dot{x} = F \cdot \dot{x}$

AJ Leggett

Chapter 14

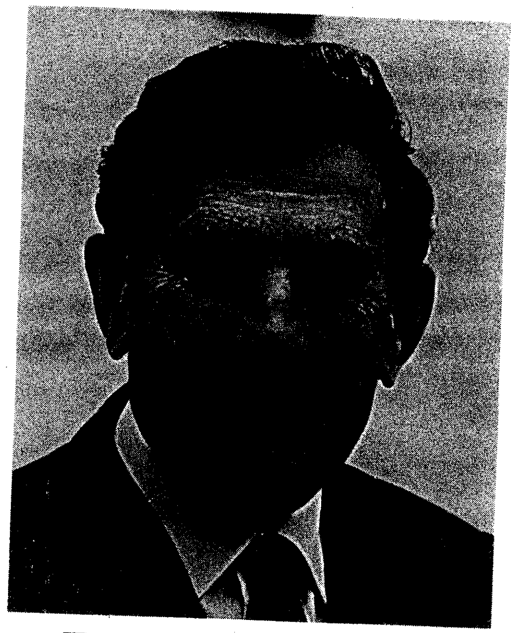
QUANTUM MECHANICS

SHELDON GOLDSTEIN

JOEL L. LEBOWITZ

Introduction

Papers Reprinted in Book



David Bohm (1917-1992). (Courtesy of Still Pictures, London.)

Quantum mechanics is undoubtedly the most successful theory yet devised by the human mind. Not one of the multitude of its calculated predictions has ever been found wanting, even in the last measured decimal place—nor is there any reason to believe that this will change in the foreseeable future. All the same, it is a bizarre theory. Let us quote Feynman,[1] one of the deepest scientist–thinkers of our century and one not known for his intellectual (or any other) modesty, on the subject: "There was a time when the newspapers said that only twelve men understood the theory of relativity. I do not believe there ever was such a time. There might have been a time when only one man did, because he was the only guy who caught on, before he wrote his

Sheldon Goldstein is a Professor of Mathematics at Rutgers University in New Brunswick, New Jersey. He worked for many years on probability theory and the rigorous foundations of statistical mechanics. In particular, he has investigated the ergodic properties of large systems, the existence of steady-state nonequilibrium ensembles, derivations of Brownian motion for interacting particles and of diffusion and subdiffusion limits for random motions in random environments. In recent years he has been concerned with the foundations of quantum mechanics.

Joel L. Lebowitz—see biographical note in Chapter 6

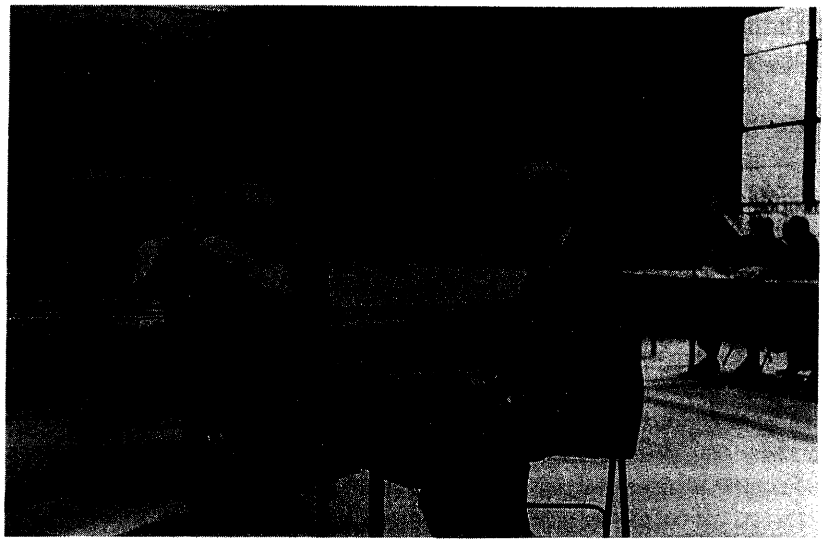
paper. But after people read the paper a lot of people understood the theory of relativity in some way or other, certainly more than twelve. On the other hand, I think I can safely say that nobody understands quantum mechanics. ... I am going to tell you what nature behaves like. If you will simply admit that maybe she does behave like this, you will find her a delightful, entrancing thing. Do not keep saying to yourself, if you can possibly avoid it, 'but how can it be like that?' because you will get 'down the drain,' into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that."

Feynman's point of view, expressed as usual with great vigor and clarity, characterizes the attitude of most physicists towards the foundations of quantum mechanics, a subject concerned with the meaning and interpretation of quantum theory—at least it did so before the work of Bell[2] and the experiments of Aspect *et al.*[3] (which came after the cited Feynman lecture). Even today, the subject is treated like a poor stepchild of the physics family. It is pretty much ignored in most standard graduate texts, and what is conveyed there is often so mired in misconception and confusion that it usually does more harm than good. To many physicists, it appears that not only does foundational research not lead to genuine scientific progress, but that it is in fact dangerous, with the potential for getting people "down the drain." Even to the more tolerant ones, it often seems that what is achieved merely supports what every good physicist should have known already.

While these perceptions are partly true, they are also in part misapprehensions arising from the, to our taste, much too practical approach taken by many physicists. Basic questions concerning the physical meaning of quantities such as the wave function which we manipulate in our computations are too important to be left to philosophers. One such question, whether the description of a physical system provided by its wave function is complete, is central to the articles reprinted in this chapter.

Albert Einstein (1879-1955) and **Niels Bohr (1885-1962).** (Courtesy of Ehrenfest Collection, AIP Emilio Segrè Visual Archives.)

Einstein, Podolsky, and Rosen (EPR)[4] argue that quantum mechanics provides at best an incomplete description of physical reality. Indeed, they claim that there are situations in which the very predictions of quantum theory demand that there be elements of physical reality—i.e., predetermined, preexisting values for physical quantities, which are *revealed* rather than *created* if and when we measure those quantities—that are not incorporated within the orthodox quantum framework. In the original version of the argument, these elements of reality are the (simultaneous) values of the position and momentum of a particle belonging to an EPR pair—a pair of particles whose quantum state, given by the EPR wave function, involves such strong quantum pair correlations that the position or momentum of one of the particles can be inferred from the measurement of that of the other. By the uncertainty principle, however, the position and momentum of one particle cannot simultaneously be part of the

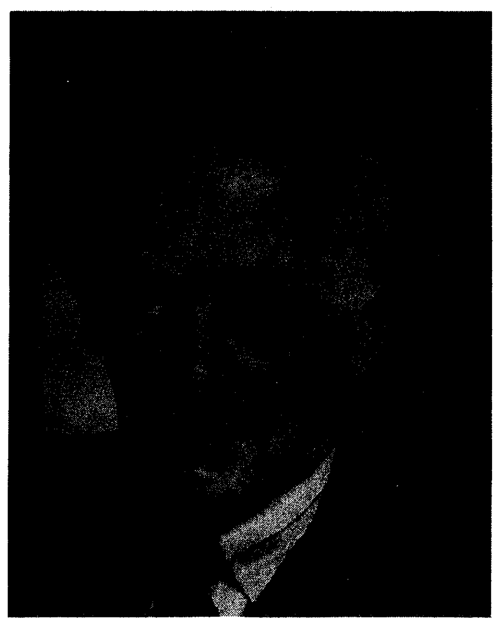


P. A. M. Dirac (1902-1984) and **Boris Podolsky (1896-1966)** in a corridor session, January 1964 [in background: **Donald A. Glaser (1926-), Oldweg Von Roos (1925-),** and **E. P. Wigner (1902-1995)**]. (Courtesy of *Physics Today* Collection, AIP Emilio Segrè Visual Archives.)

quantum description. In the later version of the EPR analysis due to Bohm,[5] which provides the framework for most of the experimental tests of quantum theory that were stimulated by the celebrated Bell's inequality paper,[2] these elements of reality are the values of the (simultaneous) components, in all possible directions, of the spins of the particles belonging to a Bohm-EPR pair—a pair of spin 1/2 particles prepared in the singlet $S = 0$ state—or, in another version, the simultaneous components of photon polarization in a suitable photon pair. We shall call these the Bohm-EPR elements of reality. (They again cannot simultaneously be part of the quantum description, because spin components in different directions do not commute.)

The EPR analysis begins with a criterion of reality: *"If, without in any way disturbing a system, we can predict with certainty ... the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."* EPR continue, "It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way ... . Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality." They then deduce the existence of the relevant elements of reality for an EPR pair from the predictions of quantum theory for the pair. In so doing, however, they crucially require a locality assumption that "the process of measurement carried out on the first system ... does not disturb the second system in any way." EPR conclude as follows: "While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible."

We wish to emphasize that in arguing here for the incompleteness of the quantum description, EPR were not questioning the



Max Born (1882–1970). (Courtesy of AIP Emilio Segrè Visual Archives.)

Erwin Schrödinger (1887–1961).
(Courtesy of Weber Collection, AIP Emilio Segrè Visual Archives.)

Eugene P. Wigner (1902–1995).
Photograph of charcoal drawing by Peter Geoffrey Cook. (Courtesy of Weber Collection, AIP Emilio Segrè Visual Archives.)

validity of the experimental predictions of quantum theory. On the contrary, they were claiming that these predictions were not only compatible with a more complete description—in particular, one involving their elements of reality—but also demanded one. Elsewhere, Einstein[6] asserts that in "a complete physical description, the statistical quantum theory would...take an approximately analogous position to the statistical mechanics within the framework of classical mechanics."

Niels Bohr,[7] in what is perhaps the definitive statement of his principle of complementarity, disagreed with the EPR conclusion, though he did not take the EPR analysis lightly. The central objection in Bohr's reply is that the EPR reality criterion "contains an ambiguity as regards the meaning of the expression 'without in any way disturbing a system.' Of course, there is...no question of a mechanical disturbance... . But...there is essentially the question of an influence on the very conditions which...constitute an inherent element of the description of any phenomenon to which the term 'physical reality' can be properly attached... ." While, with Bell,[8] we "have very little idea what this means," it does perhaps suggest "the feature of wholeness typical of proper quantum phenomena" elsewhere stressed by Bohr.[9]

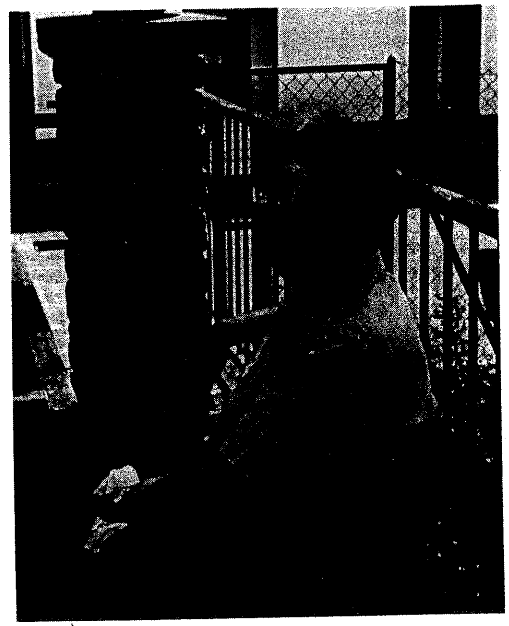
Bohm,[10, 11] on the other hand, not only agreed with EPR that the quantum description is incomplete, but showed explicitly how to extend the incomplete quantum description—by the introduction of "hidden variables"—into a complete one, in such a way that the indeterminism of quantum theory is completely eliminated. We shall call Bohm's deterministic completion of nonrelativistic quantum theory Bohmian mechanics. In Bohmian mechanics the hidden variables are simply the positions of the particles, which move, under an evolution governed by the wave function, in what is in effect the simplest possible manner.[12] We should emphasize that Bohmian mechanics is indeed an extension of quantum theory, in the sense that in this theory, as in quantum theory, the wave function evolves autonomously according to Schrödinger's equation. Moreover, it can be shown[12] that the statistical description in quantum theory, given by $\rho = |\psi|^2$, indeed takes, as Einstein wanted, "an approximately analogous position to the statistical mechanics within the framework of classical mechanics."

Bohmian mechanics was ignored by most physicists, but it was taken very seriously by Bell, who declared[13] that "in 1952 I saw the impossible done." Bell quite naturally asked how Bohm had managed to do what von Neumann[14] had proclaimed to be—and almost all authorities agreed was—impossible. (It is perhaps worth noting that despite the almost universal acceptance among physicists of the soundness of von Neumann's proof of the impossibility of hidden variables, undoubtedly based in part on von Neumann's well-deserved reputation as one of the greatest mathematicians of the 20th century, Bell[15] felt that the assumptions made by von Neumann about the requirements for a hidden-variable theory are so unreasonable that "the proof of von Neumann is not merely false but *foolish*!" See also Ref. 16.) His ensuing hidden-variables analysis led to Bell's inequality, which must be satisfied by certain correlations between Bohm-EPR elements of reality—and, of course, by correlations between their measured values. He observed also that quantum theory predicts a sharp violation of the inequality when the quantities in question are measured.

Thus the specific elements of reality to which the EPR analysis would lead (if applied to the Bohm-EPR version) must satisfy correlations that are incompatible with those given by quantum theory. That is, these elements of reality, whatever else they may be, are demonstrably incompatible with the predictions of quantum theory and hence are certainly not part of any completion of it. It follows that there is definitely something wrong with the EPR analysis, since quantum mechanics cannot be (even partially) completed in the manner demanded by this analysis. In other words, had EPR been aware of the work of Bell, they might well have predicted that quantum theory is wrong and proposed an experimental test of Bell's inequality to settle the issue once and for all.

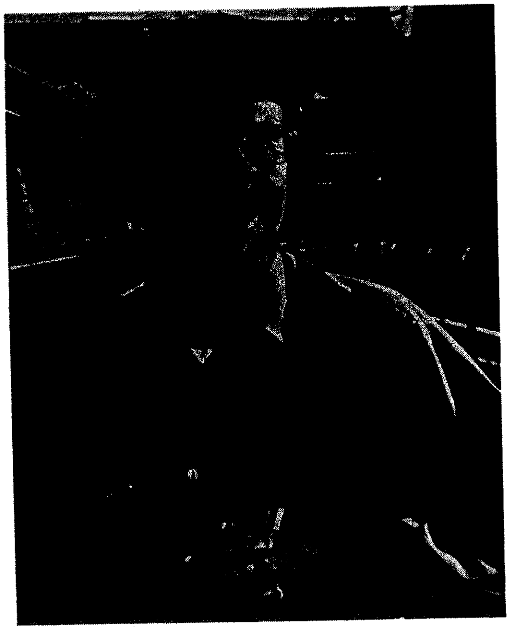
Of course, EPR were not aware of Bell's analysis, but Clauser, Horne, Shimony, and Holt were.[17] Their proposal for an experimental test has led to an enormous proliferation of experiments, the most conclusive of which was perhaps that of Aspect *et al.*[3] included here. The result: Quantum mechanics is right.

We note, however, that the predictions of (nonrelativistic) quantum mechanics—in particular, those for the experimental tests of Bell's inequality—are in complete agreement with the predictions of Bohmian mechanics. Thus the Bohm-EPR elements of reality are not part of Bohmian mechanics! This is because in Bohmian mechanics the result of what we speak of as measuring

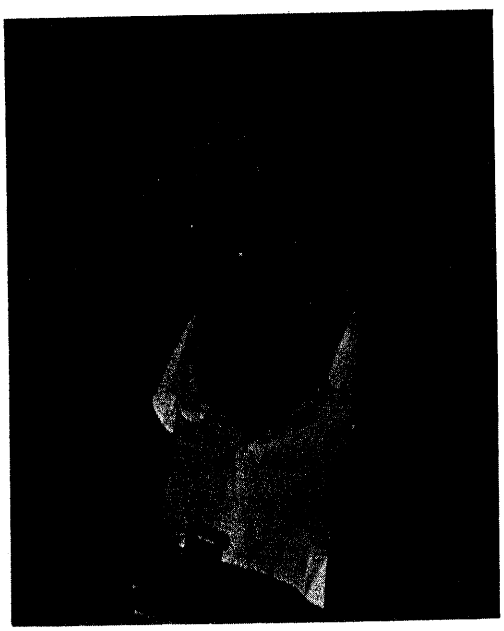


Abner E. Shimony (1928–). (Courtesy of Abner E. Shimony, Boston University.)

Yves A. Rocard (1903–1992), Louis duc de Broglie (1892–1987), Maurice duc de Broglie (1875–1960), and Francis Perrin (1901–). (Courtesy of AIP Emilio Segrè Visual Archives.)



John Bell (1928–1990). (Courtesy of Tatiana Fabergé and M. N. Fontaine, CERN.)



Alain Aspect (1947–). (Courtesy of Philip Pearle, Hamilton College.)

a spin component depends as much upon the detailed experimental arrangement for performing the measurement as it does upon anything existing prior to and independent of the measurement. This dependence is an example of what the experts in the hidden-variables field call contextuality[18] (see also Ref. 15), i.e., of the critical importance of not overlooking "the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."[19]

In fact, just before he arrived at his inequality, Bell noticed that "in this (Bohm's) theory an explicit causal mechanism exists whereby the disposition of one piece of apparatus affects the results obtained with a distant piece. ... Bohm of course was well aware of these features of his scheme, and has given them much attention. However, it must be stressed that, to the present writer's knowledge, there is no *proof* that *any* hidden variable account of quantum mechanics *must* have this extraordinary character. It would therefore be interesting, perhaps, to pursue some further 'impossibility proofs,' replacing the arbitrary axioms objected to above by some condition of locality, or of separability of distant systems."[18] Almost immediately, Bell found his inequality. Thus did Bohmian mechanics lead to Bell's refutation of the EPR claim to have "shown that the wave function does not provide a complete description." At the same time it showed, by explicit example, the correctness of the EPR belief "that such a theory is possible"!

While Bell's analysis, together with the results of experiments such as Aspect's, implies that the EPR analysis was faulty, where in fact did EPR go wrong? Since their only genuine assumption was that of locality quoted above, and since their subsequent reasoning is valid, it is this assumption that must fail, both for quantum theory and for nature herself. Aspect's experiment thus establishes perhaps the most striking implication of quantum theory: Nature is nonlocal! This conclusion is of course implicit in the very structure of quantum theory itself, based as it is on a field—the wave function—which for a many-body system lives not on physical space but on a $3n$-dimensional configuration space, a structure that allows for the entanglement of states of distant systems—as most dramatically realized in the EPR state itself. But while quantum mechanics may someday be replaced by a theory of an entirely different character, we may nonetheless conclude—though there are some who disagree[15]—from Bell and Aspect that the nonlocality it implies is here to stay.

One of the great foundational mysteries that remains very much unsolved is how nonlocality can be rendered compatible with special relativity, i.e., with Lorentz invariance. Here Bohmian mechanics is of no direct help, since it manifestly and fundamentally is not Lorentz invariant. But there is no reason to believe that a more appropriate completion of quantum theory, one that is Lorentz invariant and perhaps even generally covariant, cannot be found. However, one should not expect finding it to be easy.

One lesson of this story is perhaps that we would be wise to place greater trust in the mathematical structure of quantum theory, and less in the philosophy with which quantum theory is so often encumbered. For the EPR problem, the mathematical structure correctly suggests nonlocality, while the philosophy makes the questionable demand that the wave function provide a complete description, at least on the microscopic level. The paper by Aharonov and Bohm[20] included here supports this lesson. Aharonov and Bohm dramatically demonstrate that the electromagnetic vector potential has a reality in quantum theory far beyond what it has classically: A nonvanishing vector potential may generate a shift in an interference pattern for an electron confined to a region in which the magnetic field itself vanishes. The Aharonov-Bohm effect, while rather clear from the role played by the vector potential in Schrödinger's equation, is rather surprising from the perspective of the usual quantum philosophy, which, in attempting to explain quantum deviations from classical behavior, appeals to limitations on what can be measured or known arising from disturbances occurring during the act of measurement that are due to the finiteness of the quantum of action.

Philip Pearle (1936–) and **Yakir Aharonov (1932-).** (Courtesy of Philip Pearle, Hamilton College.)

It is appropriate to mention at this time—even though it is not the focus of any of the five papers included in this chapter—one of the strongest arguments for the conclusion that the quantum mechanical description is incomplete: the notorious measurement problem—or, what amounts to the same thing, the paradox of Schrödinger's cat. The problem is that the after-measurement wave function for system and apparatus arising from Schrödinger's equation for the composite system typically involves a superposition over terms corresponding to what we would like to regard as the various possible results of the measurement—e.g., different pointer orientations. Since it seems rather important that the actual result of the measurement be a part of the description of the after-measurement situation, it is difficult to see how this wave function could be the complete description of this situation. By contrast, with a theory or interpretation in which the description of the after-measurement situation includes, in addition to the wave function, at least the values of the variables that register the result, the measurement problem vanishes. (The remaining problem of then justifying the use of the "collapsed" wave function—corresponding to the actual result—in place of the original one is often confused with the measurement problem. The justification for this replacement is nowadays frequently expressed in terms of decoherence. One of the best descriptions of the mechanisms of decoherence, though not the word itself, can be found in the Bohm article reprinted here; see also Ref. 5. We wish to emphasize, however, as did Bell in his article "Against Measure-

Murray Gell-Mann (1929–) and Richard P. Feynman (1918–1988). (Courtesy of Marshak Collection, AIP Emilio Segrè Visual Archives.)

Stephen Hawking (1942–). (Courtesy of Ann Benedeuce.)

ment,"[21] that decoherence *per se* in no way comes to grips with the measurement problem itself.)

The orthodox response to the measurement problem is that we must distinguish between closed systems and open systems—those upon which an external "observer" intervenes. While we do not want to delve into the merits of this response here—nor is this the place to discuss the sundry proposals for alternate interpretations of quantum theory, such as those of Schulman,[22] Pearle,[23, 24] and of Ghirardi, Rimini, and Weber[25, 26] —we do wish to note one particular difficulty, much emphasized of late. This concerns the now-popular subject of quantum cosmology, concerned with the physics of the universe as a whole, certainly a closed system! A formulation of quantum mechanics that makes sense for closed systems seems to be demanded. Bohmian mechanics is one such formulation. Others also now generating a good deal of excitement are due to Griffiths,[27] Omnès,[28] and Gell-Mann and Hartle.[29] All of these exemplify the EPR conclusion "that the wave function does not provide a complete description of the physical reality."

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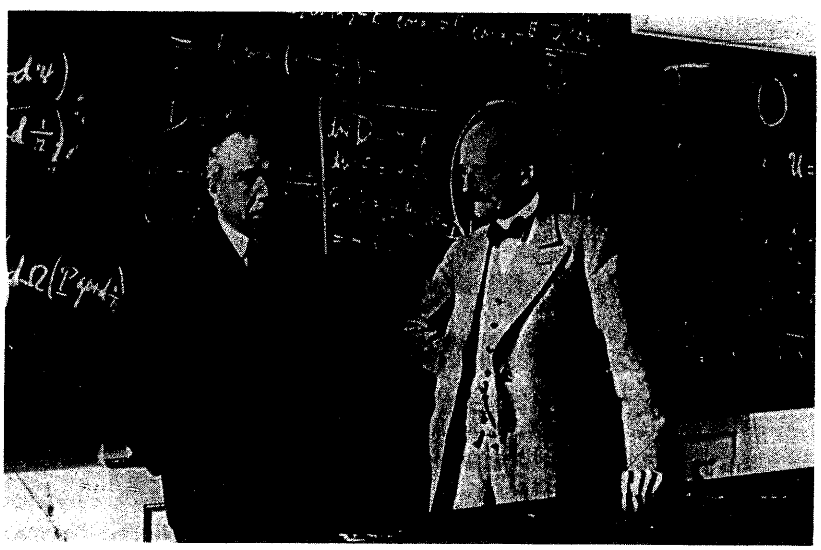
N. David Mermin (1935–) and **Victor F. Weisskopf (1908–).** (Courtesy of Abner E. Shimony, Boston University.)



Niels Bohr (1885–1962) and **Max Planck (1858–1947).** (Courtesy of Margrethe Bohr Collection, AIP Emilio Segrè Visual Archives.)

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way, whenever the conditions set down in it occur. Regarded not as a necessary, but merely as a sufficient, condition of reality, this criterion is in agreement with classical as well as quantum-mechanical ideas of reality.

To illustrate the ideas involved let us consider the quantum-mechanical description of the behavior of a particle having a single degree of freedom. The fundamental concept of the theory is the concept of *state*, which is supposed to be completely characterized by the wave function ψ, which is a function of the variables chosen to describe the particle's behavior. Corresponding to each physically observable quantity A there is an operator, which may be designated by the same letter.

If ψ is an eigenfunction of the operator A, that is, if

$$\psi' \equiv A\psi = a\psi, \qquad (1)$$

where a is a number, then the physical quantity A has with certainty the value a whenever the particle is in the state given by ψ. In accordance with our criterion of reality, for a particle in the state given by ψ for which Eq. (1) holds, there is an element of physical reality corresponding to the physical quantity A. Let, for example,

$$\psi = e^{(2\pi i/h)p_0x}, \qquad (2)$$

where h is Planck's constant, p_0 is some constant number, and x the independent variable. Since the operator corresponding to the momentum of the particle is

$$p = (h/2\pi i)\partial/\partial x, \qquad (3)$$

we obtain

$$\psi' = p\psi = (h/2\pi i)\partial\psi/\partial x = p_0\psi. \qquad (4)$$

Thus, in the state given by Eq. (2), the momentum has certainly the value p_0. It thus has meaning to say that the momentum of the particle in the state given by Eq. (2) is real.

On the other hand if Eq. (1) does not hold, we can no longer speak of the physical quantity A having a particular value. This is the case, for example, with the coordinate of the particle. The operator corresponding to it, say q, is the operator of multiplication by the independent variable. Thus,

$$q\psi = x\psi \neq a\psi. \qquad (5)$$

In accordance with quantum mechanics we can only say that the relative probability that a measurement of the coordinate will give a result lying between a and b is

$$P(a,b) = \int_a^b \bar{\psi}\psi dx = \int_a^b dx = b - a. \qquad (6)$$

Since this probability is independent of a, but depends only upon the difference $b-a$, we see that all values of the coordinate are equally probable.

A definite value of the coordinate, for a particle in the state given by Eq. (2), is thus not predictable, but may be obtained only by a direct measurement. Such a measurement however disturbs the particle and thus alters its state. After the coordinate is determined, the particle will no longer be in the state given by Eq. (2). The usual conclusion from this in quantum mechanics is that *when the momentum of a particle is known, its coordinate has no physical reality.*

More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first.

From this follows that either (1) *the quantum-mechanical description of reality given by the wave function is not complete* or (2) *when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.* For if both of them had simultaneous reality—and thus definite values—these values would enter into the complete description, according to the condition of completeness. If then the wave function provided such a complete description of reality, it would contain these values; these would then be predictable. This not being the case, we are left with the alternatives stated.

In quantum mechanics it is usually assumed that the wave function *does* contain a complete description of the physical reality of the system in the state to which it corresponds. At first weird

sight this assumption is entirely reasonable, for the information obtainable from a wave function seems to correspond exactly to what can be measured without altering the state of the system. We shall show, however, that this assumption, together with the criterion of reality given above, leads to a contradiction.

2.

For this purpose let us suppose that we have two systems, I and II, which we permit to interact from the time $t=0$ to $t=T$, after which time we suppose that there is no longer any interaction between the two parts. We suppose further that the states of the two systems before $t=0$ were known. We can then calculate with the help of Schrödinger's equation the state of the combined system I+II at any subsequent time; in particular, for any $t>T$. Let us designate the corresponding wave function by Ψ. We cannot, however, calculate the state in which either one of the two systems is left after the interaction. This, according to quantum mechanics, can be done only with the help of further measurements, by a process known as the *reduction of the wave packet*. Let us consider the essentials of this process.

Let $a_1, a_2, a_3, \cdots$ be the eigenvalues of some physical quantity A pertaining to system I and $u_1(x_1), u_2(x_1), u_3(x_1), \cdots$ the corresponding eigenfunctions, where x_1 stands for the variables used to describe the first system. Then Ψ, considered as a function of x_1, can be expressed as

$$\Psi(x_1, x_2)=\sum_{n=1}^{\infty} \psi_n(x_2)u_n(x_1), \tag{7}$$

where x_2 stands for the variables used to describe the second system. Here $\psi_n(x_2)$ are to be regarded merely as the coefficients of the expansion of Ψ into a series of orthogonal functions $u_n(x_1)$. Suppose now that the quantity A is measured and it is found that it has the value a_k. It is then concluded that after the measurement the first system is left in the state given by the wave function $u_k(x_1)$, and that the second system is left in the state given by the wave function $\psi_k(x_2)$. This is the process of reduction of the wave packet; the wave packet given by the infinite series (7) is reduced to a single term $\psi_k(x_2)u_k(x_1)$.

The set of functions $u_n(x_1)$ is determined by the choice of the physical quantity A. If, instead of this, we had chosen another quantity, say B, having the eigenvalues $b_1, b_2, b_3, \cdots$ and eigenfunctions $v_1(x_1), v_2(x_1), v_3(x_1), \cdots$ we should have obtained, instead of Eq. (7), the expansion

$$\Psi(x_1, x_2)=\sum_{s=1}^{\infty} \varphi_s(x_2)v_s(x_1), \tag{8}$$

where φ_s's are the new coefficients. If now the quantity B is measured and is found to have the value b_r, we conclude that after the measurement the first system is left in the state given by $v_r(x_1)$ and the second system is left in the state given by $\varphi_r(x_2)$.

We see therefore that, as a consequence of two different measurements performed upon the first system, the second system may be left in states with two different wave functions. On the other hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system. This is, of course, merely a statement of what is meant by the absence of an interaction between the two systems. Thus, *it is possible to assign two different wave functions* (in our example ψ_k and φ_r) *to the same reality* (the second system after the interaction with the first).

Now, it may happen that the two wave functions, ψ_k and φ_r, are eigenfunctions of two noncommuting operators corresponding to some physical quantities P and Q, respectively. That this may actually be the case can best be shown by an example. Let us suppose that the two systems are two particles, and that

$$\Psi(x_1, x_2)=\int_{-\infty}^{\infty} e^{(2\pi i/h)(x_1-x_2+x_0)p}dp, \tag{9}$$

where x_0 is some constant. Let A be the momentum of the first particle; then, as we have seen in Eq. (4), its eigenfunctions will be

$$u_p(x_1)=e^{(2\pi i/h)px_1} \tag{10}$$

corresponding to the eigenvalue p. Since we have here the case of a continuous spectrum, Eq. (7) will now be written

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \psi_p(x_2) u_p(x_1) dp, \tag{11}$$

where

$$\psi_p(x_2) = e^{-(2\pi i/h)(x_2 - x_0)p}. \tag{12}$$

This ψ_p however is the eigenfunction of the operator

$$P = (h/2\pi i)\partial/\partial x_2, \tag{13}$$

corresponding to the eigenvalue $-p$ of the momentum of the second particle. On the other hand, if B is the coordinate of the first particle, it has for eigenfunctions

$$v_x(x_1) = \delta(x_1 - x), \tag{14}$$

corresponding to the eigenvalue x, where $\delta(x_1 - x)$ is the well-known Dirac delta-function. Eq. (8) in this case becomes

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \varphi_x(x_2) v_x(x_1) dx, \tag{15}$$

where

$$\varphi_x(x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/h)(x - x_2 + x_0)p} dp$$
$$= h\delta(x - x_2 + x_0). \tag{16}$$

This φ_x, however, is the eigenfunction of the operator

$$Q = x_2 \tag{17}$$

corresponding to the eigenvalue $x + x_0$ of the coordinate of the second particle. Since

$$PQ - QP = h/2\pi i, \tag{18}$$

we have shown that it is in general possible for ψ_k and φ_r to be eigenfunctions of two noncommuting operators, corresponding to physical quantities.

Returning now to the general case contemplated in Eqs. (7) and (8), we assume that ψ_k and φ_r are indeed eigenfunctions of some noncommuting operators P and Q, corresponding to the eigenvalues p_k and q_r, respectively. Thus, by measuring either A or B we are in a position to predict with certainty, and without in any way disturbing the second system, either the value of the quantity P (that is p_k) or the value of the quantity Q (that is q_r). In accordance with our criterion of reality, in the first case we must consider the quantity P as being an element of reality, in the second case the quantity Q is an element of reality. But, as we have seen, both wave functions ψ_k and φ_r belong to the same reality.

Previously we proved that either (1) the quantum-mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (1) leads to the negation of the only other alternative (2). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.

One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality *only when they can be simultaneously measured or predicted.* On this point of view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this.

While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

f increases linearly with electron energy, one obtains the surprisingly good curve C shown in Fig. 6. This agreement is probably partially fortuitous, and a more rigorous development must eventually involve allowances for the effects of varying cross section, scattering angle, and inelasticity of impact in detail. The general character of the problem, however, seems to be indicated by the above considerations, and furnishes further evidence for the inelasticity of electron impact in molecular gases.

These experiments are being continued to determine electron mobilities in other gases. The authors desire to thank the Research Committee of Stanford University for a grant which has made these investigations possible.

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Note on the Quantum-Mechanical Theory of Measurement

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In recent notes by Einstein, Podolsky and Rosen and by Bohr, attention has been called to the fact that certain results of quantum mechanics are not to be reconciled with the assumption that a system has independently real properties as soon as it is free from mechanical interference. We here investigate in general, and in abstract terms, the extent of this disagreement. When suitably formulated, such an assumption gives to certain types of questions the same answers as does quantum mechanics; this is true of the formulas usually given in discussions of the theory of measurement. There exists, however, a general class of cases in which contradictions occur. That such contradictions are not restricted to the abstract mathematical theory, but can be realized in the commonest physical terms, is shown by the working out of an example.

1. Introduction

SOME time ago there appeared a paper by Einstein, Podolsky and Rosen[1] entitled "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" The writers concluded that the answer must be negative, on the ground that quantum mechanics forbids simultaneous measurement of two noncommuting variables even when both variables simultaneously possess "physical reality," in the sense that either might be measured "without in any way disturbing the system." Recently Bohr[2] has upheld the view that quantum-mechanical description of nature can be considered complete, by demonstrating how the restrictions on simultaneous measurement which it imposes are inherent in the character and use of the measuring instruments. These measuring instruments must always be included as part of the physical situation from which our experience is obtained, and by doing this one sees that quantum mechanics provides a complete and peculiarly apt interpretation of experience.

Bohr has again clearly called attention to this circumstance, and has remarked that one must be careful not to suppose that a system is an independent seat of "real" attributes simply because it has ceased to interact dynamically with other systems. The paper of Einstein, Podolsky and Rosen has shown the sort of situations in which this characteristic of quantum mechanics may become especially prominent. This indicates an extension of the usual discussions of the theory of measurement.[3] In the present note a discussion more comprehensive in this respect will be summarized, and some further consideration will be given to the possibility of illustrating the point in question in concrete physical terms.

We shall have to make use of the concepts and results presented in von Neumann's rigorous and

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935). Referred to as EPR.

[2] N. Bohr, Phys. Rev. **48**, 696 (1935).

[3] Cf. W. Heisenberg, *The Physical Principles of the Quantum Theory*, particularly pp. 55ff.; J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Chap. VI; W. Pauli, *Handbuch der Physik*, Vol. 24, No. 1, pp. 143ff., p. 165.

detailed discussion[3] of the theory of measurement by means of an instrument. Since the mathematical language of von Neumann's work is not that most current among physicists, it is desirable first to explain the meaning of these concepts in more usual terms. A similar rephrasing of the proofs of important results is omitted in the interests of brevity, and the results are simply stated.

2. Possible Types of Statistical Information about a System

Our statistical information about a system may always be expressed by giving the expectation values of all observables.[4] Now the expectation value of an arbitrary observable F, for a state whose wave function is φ, is

$$\bar{F}=(\varphi,\ F\varphi). \tag{1}$$

If we do not know the state of the system, but know that w_i (with $\sum_i w_i=1$) are the respective probabilities of its being in states whose wave functions are φ_i, then we must assign as the expectation value of F the weighted average of its expectation values for the states[5] φ_i. Thus

$$\bar{F}=\sum_i w_i(\varphi_i,\ F\varphi_i). \tag{2}$$

This formula for $\bar{F}$ is the appropriate one when our system is one of an ensemble[6] of systems of which numbers proportional to w_i are in the states φ_i. It must not be confused with any such formula as

$$\bar{F}=(\sum_i (w_i)^{\frac{1}{2}}\varphi_i,\ F\sum_i (w_i)^{\frac{1}{2}}\varphi_i),$$

which corresponds to the system's having a wave function which is a linear combination of the φ_i. This last formula is of the type of (1), while (2) is an altogether different type.

An alternative way of expressing our statistical information is to give the probability that measurement of an arbitrary observable F will give as result an arbitrary one of its eigenvalues, say δ. When the system is in the state φ, this probability is

$$|(\varphi,\ \chi_\delta)|^2, \tag{1'}$$

where χ_δ is the eigenfunction of F corresponding to the eigenvalue δ. When we know only that w_i are the probabilities of the system's being in the states φ_i, the probability in question is

$$\sum_i w_i|(\varphi_i,\ \chi_\delta)|^2. \tag{2'}$$

Formula (2′) is not the same as any special case of (1′) such as

$$|(\sum_i (w_i)^{\frac{1}{2}}\varphi_i,\ \chi_\delta)|^2.$$

It differs generically from (1′) as (2) does from (1).

When such equations as (1), (1′) hold, we say that the system is in the "pure state" whose wave function is φ. The situation represented by Eqs. (2), (2′) is called a "mixture" of the states φ_i with the weights w_i. It can be shown[7] that the most general type of statistical information about a system is represented by a mixture. A pure state is a special case, with only one nonvanishing w_i. The term "mixture" is usually reserved for cases in which there is more than one nonvanishing w_i. It must again be emphasized that a mixture in this sense is essentially different from any pure state whatever.

3. Reduction of Wave Packets

Let $\Psi(x_1, x_2)$ be the wave function for two systems I and II which have at some previous time interacted and have now ceased to interact. One can show[8] that there always exists an expansion, which is in general unique, in the form

$$\Psi(x_1,\ x_2)=\sum_k (w_k)^{\frac{1}{2}}\varphi_{\lambda_k}(x_1)\xi_{\rho_k}(x_2), \tag{3}$$

where the φ_{λ_k} are eigenfunctions of an observable L corresponding to eigenvalues λ_k, and the ξ_{ρ_k} are eigenfunctions of an observable R corre-

[4] v. Neumann, reference 3, p. 163. We confine ourselves throughout to the case of discrete spectra. (φ, ψ) means the Hermitian inner product, $\int\varphi^*\psi d\tau$. The same letters, $A, B \cdots$, are used to denote observables and the corresponding operators.

[5] For convenience we shall often refer to "the state ψ" instead of "the state whose wave function is ψ." All functions are normalized unless otherwise stated.

[6] The usefulness of this concept has recently been remarked upon by Kemble (Phys. Rev. **47**, 973 (1935)).

[7] v. Neumann, reference 3, pp. 167–168.

[8] v. Neumann, reference 3, pp. 225 ff. In the following arguments we use the word "observable" to mean "complete set of commuting observables," in the sense of Dirac, *Principles of Quantum Mechanics*, §17 (1st edition). In like fashion a set of eigenvalues of such a set of observables is referred to as an "eigenvalue" of the "observable."

sponding to eigenvalues ρ_k. The λ_k are all distinct, and so are the ρ_k. It can be shown[9] that, so far as system II alone is concerned, the statistical information available when (3) is the wave function of the combined systems is represented by a mixture of the states ξ_{ρ_k} with the weights w_k. A similar result holds, of course, for system I. A measurement of L and R on the total system can never give any other value than ρ_i for R to correspond to the value λ_i of L. Thus a measurement of L suffices to predict the value of R and the state of system II, which, *after*[9] such a measurement, is always in one of the pure states ξ_{ρ_k}. Eq. (3) shows that the coupling between the systems has been such as to make system I a suitable instrument for measuring the observable R on system II, the quantity L serving as a "pointer reading."

Now the conclusions to be drawn on the basis of these developments are just those we should expect if we ascribed "real" characteristics to system II as soon as it ceased to interact with system I; this will be shown explicitly later. The contradictions we wish to investigate can be brought out only by going beyond the considerations given in connection with Eq. (3). We may either look for particular cases in which the expansion (3) is not unique—e.g., the example given by EPR and the one we shall give in Section 5—or develop a way of interpreting expansions of a less special type.

The second alternative brings us directly to the general method of "reducing the wave packet." This procedure is commonly known and accepted among physicists, and is applied by EPR; but the writer has been unable to find in the literature an explicit description of its application to the present case. Such a description is briefly as follows: If M is any observable of system I, and ψ_μ its eigenfunction corresponding to the eigenvalue μ, then we can express $\Psi(x_1, x_2)$ as a series in the orthogonal functions $\psi_\mu(x_1)$, with coefficients which are functions[10] of x_2:

$$\Psi(x_1, x_2) = \sum_\mu \psi_\mu(x_1)\zeta_\mu(x_2), \tag{4}$$

where $$\zeta_{\mu'}(x_2) = \int \psi_{\mu'}^*(x_1)\Psi(x_1, x_2)dx_1. \tag{5}$$

The statistical information which one will have about system II after[9] a measurement of M on system I has given the value μ' may be obtained by the following process: We suppose a large number of measurements made on combined systems I+II prepared so that their wave functions are given by (4), each of these measurements consisting in a determination of the values of M for system I and some observable F for system II. We then obtain the relative numbers of times the different values δ are found for F, counting only those measurements in which the value μ' is found for M. These relative numbers are by definition proportional to the quantities $|(\Psi(x_1, x_2), \psi_{\mu'}(x_1)\chi_\delta(x_2))|^2$ (cf. Eq. (1′)), and these quantities are, by (5), just equal to $|(\zeta_{\mu'}, \chi_\delta)|^2$. Since this is true for all observables F, we see that after a measurement on system I has given the value μ' for M, system II is in the pure state with wave function given—apart from normalization—by (5).

4. Probability Calculations and Their Results

We are now ready to discuss in detail the degrees of agreement and disagreement between the results of quantum-mechanical calculations and those to be expected on the assumption that a system once freed from dynamical interference can be regarded as possessing independently real properties. For we can give a definite form to this assumption, and base on it a method for answering all questions which can be asked about the probabilities of finding different results by measurements on system II. This we call Method A:

Assumption and method A. We assume that during the interaction of the two systems each system made a transition to a definite state, in which it now is, system I being in one of the states φ_{λ_k} and system II in one of the states ξ_{ρ_k}. These transitions are not causally determined, and there is no way of finding out *which* transitions occurred, except by making a suitable measurement. In the absence of measurements we know

[9] i.e., *immediately* after. The wave functions used are in general not stationary solutions of the wave equation, but in our discussion we can abstract from the time, because our statistical information about a system at one time can be calculated from that at another time according to a definite differential equation (v. Neumann, reference 3, p. 186).

[10] Unnormalized, and in general not orthogonal to each other.

only that the probabilities of the different transitions are respectively w_k, and that if system I is in the state φ_{λ_i} system II is in the state ξ_{ρ_i}. This provides a sufficient basis for making all needed calculations of probabilities, the methods being those of ordinary probability theory.

Method B. We shall compare with the results of Method A those obtained by quantum-mechanical calculations, using the facts explained in connection with Eqs. (3) and (4), (5).

There are four types of questions for which answers may be required. The notation used in discussing them is the same as that previously described; in particular, the reader is reminded that the observables L and R have a special significance through their connection with the expansion (3), while M and S are arbitrary observables. The questions and their answers are as follows:

(a) If S, having eigenvalues σ and eigenfunctions η_σ is measured on system II without any measurements having been made on system I, what is the probability of obtaining the result σ'? Both methods give the same result

$$\sum_k w_k |(\xi_{\rho_k}, \eta_{\sigma'})|^2, \tag{6}$$

as is at once evident. (Cf. remarks following Eq. (3).)

(b) If L has been measured on system I and the value λ_i obtained, what is the probability of finding the value σ' for S in II? Both methods at once give the answer

$$|(\xi_{\rho i}, \eta_{\sigma'})|^2. \tag{7}$$

When $S=R$ and $\sigma'=\rho'$, we get in particular the value

$$|(\xi_{\rho i}, \xi_{\rho'})|^2 = \delta_{\rho_i \rho'}, \tag{7'}$$

so that a definite result is predicted. The possibility of such definite predictions was taken by EPR as a "criterion of the physical reality" of the observable R; it is, *par excellence*, the bit of evidence which might incline one to believe Assumption A to be true.

(c) If M has been measured on system I and the value μ' obtained, what is the probability of finding the value ρ_i for R in II? Some calculations are required for this case, but are omitted, being analogous to those which will be given in detail for (d). Both methods give the result

$$w_i |(\varphi_{\lambda_i}, \psi_{\mu'})|^2 / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2]. \tag{8}$$

(d) If M has been measured on system I and the value μ' obtained, what is the probability of finding the value σ' for S in II?

Method A: If the measurements are carried out on a large number of similarly prepared pairs of systems, the fraction giving the value μ' for M is $\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2$. The fraction giving this value and having system II in state $\xi_{\rho k}$ is $w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2$. Then the fraction giving the values μ' for M and σ' for S is $\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2 \times |(\xi_{\rho_k}, \eta_{\sigma'})|^2$. Dividing this by the fraction giving the value μ' for M, we find as the required *a posteriori* probability,

$$[\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2 |(\xi_{\rho_k}, \eta_{\sigma'})|^2] / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2]. \tag{9A}$$

Method B: The wave function from which we must calculate this probability is, by (5), (3):

$$\int \psi_{\mu'}^*(x_1)\Psi(x_1, x_2)dx_1 = \sum_k (w_k)^{\frac{1}{2}}(\psi_{\mu'}, \varphi_{\lambda_k})\xi_{\rho_k}(x_2).$$

On normalizing this function and taking the square of its inner product with $\eta_{\sigma'}$ one gets for the required probability

$$[|\sum_k (w_k)^{\frac{1}{2}}(\varphi_{\lambda_k}, \psi_{\mu'})(\xi_{\rho_k}, \eta_{\sigma'})|^2] / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2], \tag{9B}$$

where the denominator comes from normalization.

The difference between (9A) and (9B) comes from the well-known phenomenon of "interference" between probability amplitudes. The absence of such an effect in case (a) is usually stressed in discussions of the theory, since it shows plainly the effect which the mere attaching of an instrument must in general have on the behavior of a system. Since case (d) is not mentioned, it is possible for a reader to form the impression that the theory is consistent with Assumption A.[11]

The formal discrepancy between (9A) and (9B) is a consequence of the fact that, according to the remarks following (4) and (5), after a measurement of M on system I has been made system II is in a pure state, which is in general not one of the $\xi_{\rho k}$. Now no possible manipulation of the w_k will produce from the statistics of the mixture those of any pure state other than one of the $\xi_{\rho k}$. Thus not only is Method A inconsistent with Method B, but also there is no conceivable modification of Method A which could produce consistency between Assumption A and Method B.

The contradiction here, like that between quantum mechanics and the classical doctrine of causality, indicates a radical change in concept rather than a mere change in the details of a mechanism. The idea which is found to be untenable may, roughly, be said to be that of the independent existence of two entities, the state of system II and one's knowledge of its state, only the latter being affected by measurements made on system I. Quantum theory shows that this is not an adequate concept of the relation between subject and object.

5. A Physical Example

The inconsistency of quantum mechanics with the point of view which finds its definite formulation in Assumption A has been demonstrated mathematically. One may still wish to inquire whether it can be realized in a concrete physical example; for we may certainly suppose that not all mathematical operators, even though subject to the proper formal requirements,[12] correspond to experimentally measurable quantities.

An example has been outlined in mathematical form by EPR. As Bohr has remarked, and as is evident from the mathematics, the physical realization of this example involves certain difficulties, in particular the necessity of abstracting altogether from the time in circumstances in which this is not really permissible. We shall now outline an example in physical terms, in which full account is taken of the time-dependence of the quantities involved.

In order to give a physical example of the point in question, it would suffice to describe any case in which one system (I) is used as a measuring instrument in observing another system (II), and in which, after interaction has ceased, some other observable besides the one suited to serve as a "pointer reading" can be measured on system I. The resulting inferences about system II would in the main fall under case (d), in which the contradiction between quantum mechanics and Assumption A is evident. A variety of such examples could doubtless be given.[13] There is, however, a particularly neat and striking special type of example, to which EPR have directed attention; and the one we shall describe is of this type. Before describing it in detail, we shall indicate the nature of this sort of example and its connection with the argument of the preceding section.

The characteristic feature of such an example is obtained by choosing a case in which the expansion (3) is not unique.[14] An assumption in the form of Assumption A can be stated corresponding to each of two expansions of type (3). Each of these assumptions is consistent with a number of quantum-mechanical results of the forms (6), (7), (8), and particularly of the form (7′); thus both of them are to be accepted as

[11] Cf. Pauli, reference 3, p. 89. The remarks there given are entirely correct, but liable to be misleading to an unwary reader with a predilection for Assumption A. The same is to some extent true of the remarks in v. Neumann, reference 3, p. 232, and Heisenberg, reference 3, pp. 59–62.

[12] v. Neumann, reference 3, pp. 75 ff.

[13] e.g., the example we shall discuss could be stated in this way. In the construction and discussion of such examples a difficulty arises owing to the fact that in experimental practice the observables used are almost always incomplete (cf. note 8), and that the theory of measurements of incomplete observables is not altogether free from ambiguity (cf. v. Neumann, reference 3, pp. 184–185). For this reason it is much easier and more satisfactory to treat our example—which itself involves the use of incomplete observables—in the way we have here used.

[14] The condition for this is that the w_k be not all distinct (cf. v. Neumann, reference 3, p. 232 and p. 175). In the example of EPR, the w_k are all equal and *all* states of system II are included. This means that there are an infinite number of different expansions of the form (3), and that any measurement made is as likely to give any one result as any other. Measurement in quantum mechanics has in general a twofold aspect: it gives information (of a statistical nature) about the properties of the state of the system *before* the measurement, and it enables us to predict the state of the system *after* the measurement. In order that it may serve the first purpose, care must be used in choosing a suitable coupling of object and instrument. In the example of EPR, the coupling has been so violent that all trace of the original state of system II is lost, so that the word "prediction" is the only correct one to apply to one's conclusions about system II.

true, according to the general attitude underlying Assumption A. Now each assumption asserts that system II has made a transition to a definite (though unknown) state, and that it is a state for which a certain observable has a well-defined value. In the case chosen for the example the two requirements thus imposed on the state of system II cannot be fulfilled simultaneously by any quantum-mechanical state, because of the limitations imposed by the uncertainty principle. The point of view expressed in Assumption A is accordingly found to conflict with quantum mechanics.

With this outline of the argument in mind, we proceed to the detailed discussion of the example. In this we shall use the language of experimental physics rather than that of the mathematical formalism.

We consider the determination of the position of a heavy particle of mass M, say a proton, by the use of a "microscope." This microscope is designed for use, not with γ-rays, but with light charged particles of mass m, say electrons. If we shine soft light in from the side of the "barrel" and let it be scattered from an electron which is on its way to the "lens," a transverse component[15] of the electron's momentum can be calculated from the Doppler effect on the scattered light. Under suitable circumstances this will enable us to infer the momentum[15] of the proton quite accurately. On the other hand, we have only to let the electron travel undisturbed to the photographic plate in order to be able to infer with considerable accuracy the corresponding coordinate[15] of the proton. The "Assumption A" which corresponds to the first experiment is: During the collision between the electron and the proton, the proton made a transition into some state with well-defined momentum; *which* state this is can be determined by a measurement made on the electron. Corresponding to the second experiment one has an "Assumption A" which reads exactly the same except that the word "momentum" is replaced by the word "coordinate." Since one is still free to choose which experiment is to be performed *after* the electron has been scattered into the microscope[16] and ceased to interact with the proton, these assumptions must both be true at once, if one accepts the point of view on which they are based. But we shall see that their simultaneous truth can be in conflict with the uncertainty principle.

Before deriving the actual expressions for the uncertainties in the two alternative predictions, let us consider briefly their physical origin. In principle the Doppler effect experiment can be made with arbitrary precision (cf. latter part of note 16). The uncertainty Δp may accordingly be regarded as fixed by the original uncertainties in the momenta of the particles; to make it small, we have only to prepare them properly beforehand. Now the use of particles so prepared will in some degree limit the accuracy of our prediction of x. This comes about through the fact that the predictions we want must refer to a definite time, and that the prediction of x read directly from the photographic plate refers to the moment of the collision: this is not precisely known, because our long wave trains take a finite time to pass over each other. In making a prediction for a definite moment, which we choose to be that at which the wave trains cease to overlap, we must allow for the distance the proton may have moved since the collision. This leads to the existence of a lower limit for the product $\Delta p \Delta x$; but since the electron, with its small mass, moves rapidly across the region where the proton may be found, whereas the proton's motion after the collision is comparatively sluggish, this limit will be found to be not h, but about $(m/M)h$.

[15] i.e., transverse to the axis of the microscope. This component of the momentum will be called simply the "momentum," and the corresponding coordinate will be called the "coordinate" x.

[16] Of course the electron need not have been scattered into the microscope; in any given case we can only go on with the experiment in good faith, hoping that it will have been so scattered. The experiment can fail; and from the fact that it has succeeded when one choice was made, quantum mechanics of course offers no such inference as that it "would have" succeeded if the other choice had been made. But under the proper circumstances, such an inference does follow from the point of view of Assumption A. The circumstances required are, that in the Doppler effect experiment enough quanta must be sent in so that "if the electron is in the barrel," several are sure to be scattered back into a suitably small solid angle. The quanta must be extremely soft, so that scattering a great many of them will not change the electron's momentum too much; and their energy must be measured very accurately. This means that the electron is in the barrel a long time; and it turns out that, to secure the accuracy represented by Eq. (14), the length of the microscope must be much greater than $(M/m)^{11/3}\Delta x$, while the lateral distance to the source and analyzer of the light must be much greater than $(M/m)^{13/3}\Delta x$. These requirements make in principle no difficulty.

To show this, we shall discuss the possible sources of uncertainty in the prediction of x. These are: (1) The finite resolving power of the microscope; (2) inaccuracy of focus; (3) allowance for the proton's motion between the moment of the collision and the moment to which the predictions refer.

(*1*) *Finite resolving power*. This gives an uncertainty

$$\Delta_1 x \sim (\lambda/\epsilon) \sim (h/p\epsilon), \tag{10}$$

where ϵ is the numerical aperture and λ, p are wave-length and momentum of the electron.

(*2*) *Inaccuracy of focus*. Let the electron[17] be sent in through a slit of width s. At the far side of the field of view the half-width of the beam will have become, through diffraction, about $(s/2)+(L\lambda/2s)$, where L is the breadth of the field of view. By proper choice of s, this expression takes on its minimum value, $(\lambda L)^{\frac{1}{2}}$. Now L must be equal to $(h/\Delta p)$, the whole length of the wave train; otherwise the restriction of the field of view would cause enough diffraction of the scattered waves to spoil the significance of the contemplated Doppler effect experiment. Thus we get for the uncertainty in x from this source

$$\Delta_2 x \sim \epsilon(\lambda h/\Delta p)^{\frac{1}{2}} = \epsilon h/(p\Delta p)^{\frac{1}{2}}. \tag{11}$$

(*3*) *Allowance for proton's motion*. The time available for this motion is of the order of magnitude

$$\Delta t \sim (h/\Delta p)/(p/m),$$

which is the time required for the electron to travel the length of such a wave train as must be used. To make the proton's velocity after the collision as small as possible, we can send the two particles into the field of view with equal and opposite momenta. Then if the electron were scattered exactly at a right angle, the resulting x component of the proton's velocity would be zero. But on account of the finite aperture we must take it to be roughly

$$v_x' \sim \epsilon(p/M).$$

We then get

$$\Delta_3 x \sim v_x'\Delta t \sim \epsilon(m/M)(h/\Delta p). \tag{12}$$

Thus we have finally

$$\begin{aligned}\Delta p\Delta x &\sim \Delta p(\Delta_1 x+\Delta_2 x+\Delta_3 x)\\ &\sim h\{(\Delta p/\epsilon p)+\epsilon(\Delta p/p)^{\frac{1}{2}}+\epsilon(m/M)\}.\end{aligned} \tag{13}$$

By making Δp extremely small compared to p, we can make

$$\Delta p\Delta x \sim (m/M)h. \tag{14}$$

By taking ϵ also to be small, we can in principle make $\Delta p\Delta x$ *arbitrarily* small; also we could in principle dispense with the advantage we obtained by using the disparity in mass of two known particles.

As explained in the preliminary discussion of the example, the comparison of a result such as (14) with the uncertainty principle shows that Assumption A is inconsistent with quantum mechanics.

Concluding Remark

Both by mathematical arguments and by discussion of a conceptual experiment we have seen that the assumption that a system when free from mechanical interference necessarily has independently real properties is contradicted by quantum mechanics. This conclusion means that a system and the means used to observe it are to be regarded as related in a more subtle and intimate way than was assumed in classical theory. It does not mean that quantum mechanics is not to be regarded as a satisfactory way of correlating and describing experience; it does illustrate the difficulty, often remarked upon by Bohr, which is inherent in the problem of the distinction between subject and object.

[17] Only one beam need be narrow. Under (3) we shall see that it is expedient to make the wave-lengths of electron and proton originally equal, so that $\Delta_2 x$ will be the same whichever beam is limited. By admitting the proton through a slit much broader than s, we can assure that the electron escapes unless it is scattered through a fairly large angle, thus avoiding the prevalence of spurious effects.

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Measurement of Quantum Mechanical Operators

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The limitation on the measurement of an operator imposed by the presence of a conservation law is studied. It is shown that an operator which does not commute with a conserved (additive) quantity cannot be measured exactly (in the sense of von Neumann). It is also shown for a simple case that an approximate measurement of such an operator is possible to any desired accuracy.

1. INTRODUCTION

IT was pointed out by Wigner[1] that the presence of a conservation law puts a limitation on the measurement of an operator which does not commute with the conserved quantity. The limitation is such that the measurement of such an operator is only approximately possible. An approximate measurement can be done by a measuring apparatus which is large enough in the sense that the apparatus should be a superposition of sufficiently many states with different quantum numbers of the conserved quantity. He has proved these statements for a simple case where the x component of the spin of a spin one-half particle is measured, the z component of the angular momentum being the conserved quantity. The aim of this paper is to present a proof of the above statement for the general case.

In Sec. 2, we will prove that an exact measurement of an operator M which does not commute with a conserved operator L_1 is impossible. In Sec. 3, we will prove that an *approximate* measurement of the operator M is possible if L_1 has discrete eigenvalues and is bounded in the Hilbert space of the measured object.

2. IMPOSSIBILITY OF AN EXACT MEASUREMENT OF AN OPERATOR WHICH DOES NOT COMMUTE WITH A CONSERVED QUANTITY

Suppose we measure a self-adjoint operator M for a system represented by a Hilbert space $\mathfrak{H}_1$. Assume that M has discrete eigenvalues μ and corresponding eigenvectors $\phi_{\mu\rho}$ which are orthonormal and complete in $\mathfrak{H}_1$,

$$M\phi_{\mu\rho}=\mu\phi_{\mu\rho}, \tag{2.1}$$

$$(\phi_{\mu\rho},\phi_{\mu'\rho'})=\delta_{\mu\mu'}\delta_{\rho\rho'}. \tag{2.2}$$

The measuring apparatus is represented by a Hilbert space $\mathfrak{H}_2$. Then a state of the combined system of the measured object and the measuring apparatus is represented by a unit vector in $\mathfrak{H}_1\otimes\mathfrak{H}_2$.

According to von Neumann,[2] the measurement of the operator M in a state ϕ is accomplished by choosing an apparatus in a state ξ (fixed normalized state independent of ϕ) in $\mathfrak{H}_2$ such that the combined system, if it is in the state $\phi_{\mu\rho}\otimes\xi$ before the measurement, goes over after a finite time t into

$$U(t)(\phi_{\mu\rho}\otimes\xi)=\textstyle\sum_{\rho'}\phi_{\mu\rho'}\otimes X_{\mu\rho\rho'}, \tag{2.3}$$

where $U(t)$ is a unitary operator describing the time-development of the combined system. In order to be able to distinguish the different measured values of the operator M in terms of states of measuring apparatus after the measurement, we require

$$(X_{\mu\rho\rho''},X_{\mu'\rho'\rho'''})=0,\quad\text{if}\quad \mu\neq\mu'. \tag{2.4}$$

We note that, because we are not measuring the degeneracy parameter ρ, we have to allow the possibility that the measuring object remains in any linear combination of $\phi_{\mu\rho'}$, with fixed μ but with arbitrary ρ.[3]

We now assume the existence of a universal conservation law for a self-adjoint operator L which is additive in the sense that

$$L=L_1\otimes 1+1\otimes L_2, \tag{2.5}$$

where L_1 and L_2 are self-adjoint operators in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Actually this additivity will be used only before and after the measurement, when the two systems are separated. By universal, we mean that, whatever measuring apparatus we take, $U(t)$ commutes with L,

$$[U(t),L]=0. \tag{2.6}$$

Our claim is that (2.3) is impossible unless

$$[L_1,M]=0. \tag{2.7}$$

For the proof, we first note that, because of the unitarity of $U(t)$ and the conservation law (2.6), we

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[1] E. P. Wigner, Z. Physik **131**, 101 (1952).

[2] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932; English ed.: Princeton University Press, Princeton, 1955).

[3] For any state of the measured object, we can write

$$U(t)(\phi\otimes\xi)=\Sigma_{\mu\rho}\,\phi_{\mu\rho}\otimes X_{\mu\rho}(\phi),$$

where $X_{\mu\rho}(\phi)$ depends on ϕ. The Eqs. (2.3) and (2.4) give the most general form of the above equation satisfying (1) the distinguishability of the measured result,

$$(X_{\mu\rho}(\phi),X_{\mu'\rho'}(\phi))=0,\quad\text{if}\quad\mu\neq\mu',$$

and (2) the requirement that the probability of an eigenvalue μ in the state ϕ, as measured by the state of the measuring apparatus after the measurement, should give the conventionally postulated value,

$$\Sigma_\rho\|X_{\mu\rho}(\phi)\|^2=\Sigma_\rho|(\phi_{\mu\rho},\phi)|^2.$$

have

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ L(\phi_{\mu\rho}\otimes\xi))\\ &=(U(t)(\phi_{\mu'\rho'}\otimes\xi),\ U(t)L(\phi_{\mu\rho}\otimes\xi))\\ &=(U(t)(\phi_{\mu'\rho'}\otimes\xi),\ LU(t)(\phi_{\mu\rho}\otimes\xi))\\ &=(\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},L\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''}).\end{aligned}\tag{2.8a}$$

Hence, as the necessary condition for the conservation law for the operator L, we can write

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ L(\phi_{\mu\rho}\otimes\xi))\\ &=(\sum_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},L\sum_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''}).\end{aligned}\tag{2.8b}$$

Using the additivity of L, (2.5), we obtain

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ L(\phi_{\mu\rho}\otimes\xi))\\ &=(\phi_{\mu'\rho'},L_1\phi_{\mu\rho})(\xi,\xi)+(\phi_{\mu'\rho'},\phi_{\mu\rho})(\xi,L_2\xi)\\ &=\sum_{\rho''\rho'''}[(\phi_{\mu'\rho'''},L_1\phi_{\mu\rho''})(X_{\mu'\rho'\rho'''},X_{\mu\rho\rho''})\\ &\qquad+(\phi_{\mu'\rho'''},\phi_{\mu\rho''})(X_{\mu'\rho'\rho'''},L_2X_{\mu\rho\rho''})].\end{aligned}\tag{2.8c}$$

Because of the orthogonalities, (2.2) and (2.4), we finally obtain

$$(\phi_{\mu'\rho'},L_1\phi_{\mu\rho})=0,\quad\text{if}\quad \mu\neq\mu',\tag{2.9a}$$

or

$$(\phi_{\mu'\rho'},L_1\phi_{\mu\rho})=\delta_{\mu\mu'}(\phi_{\mu'\rho'},L_1\phi_{\mu\rho}).\tag{2.9b}$$

We are now ready to prove that L_1 commutes with M. For this purpose we decompose M into projection operators

$$M=\sum_\mu \mu P_\mu;\quad P_\mu\phi_{\mu'\rho'}=\delta_{\mu\mu'}\phi_{\mu'\rho'}.\tag{2.10}$$

To prove the commutativity of L_1 and M, (2.7), it is sufficient to prove the commutativity of L_1 and P_μ,

$$P_\mu L_1-L_1P_\mu=0.\tag{2.11}$$

From the self-adjoint nature of P_μ, (2.9b) and (2.10), we see that

$$(\phi_{\mu'\rho'},P_\mu L_1\phi_{\mu''\rho''})=\delta_{\mu\mu'}\delta_{\mu'\mu''}(\phi_{\mu'\rho'},L_1\phi_{\mu''\rho''}),$$
$$(\phi_{\mu'\rho'},L_1P_\mu\phi_{\mu''\rho''})=\delta_{\mu\mu''}\delta_{\mu'\mu''}(\phi_{\mu'\rho'},L_1\phi_{\mu''\rho''}),$$

which manifestly demonstrate (2.11). Thus we have succeeded in proving that (2.3)–(2.6) imply (2.7).[4]

Although we have assumed in the above proof that M has a discrete spectrum, the conclusion holds for any self-adjoint operator M. Namely, suppose

$$M=\int\mu dP(\mu)$$

is a spectral decomposition of M. If M can be measured exactly, $P(\mu)$ for each μ can obviously be measured exactly. Since the projection operator $P(\mu)$ has a discrete eigenvalue 1 or 0, the above proof tells us that $P(\mu)$ for each μ commutes with L_1, which in turn implies (2.7).

3. POSSIBILITY OF AN APPROXIMATE MEASUREMENT

In this section we will discuss the problem of whether the operator M, which does not commute with the conserved operator L_1 of the preceding section, can be measured approximately. We will prove that this is possible if L has a discrete spectrum and L_1 has only a finite number of eigenvalues.

We may assume that the eigenvalues of L_1 are[5] $0, \pm1, \pm2, \cdots\pm l$. We decompose L's into projection operators

$$L=\sum_\lambda\lambda P(\lambda),\tag{3.1a}$$

$$L_i=\sum_\lambda\lambda P_i(\lambda),\quad i=1,2.\tag{3.1b}$$

The additivity, (2.5), implies

$$P(\lambda)=\sum_{|\lambda'|\leq l}P_1(\lambda')P_2(\lambda-\lambda').\tag{3.2}$$

As a first step of our proof, we state the following Lemma which will be proved at the end of this section.

Lemma. Given two sets of vectors $\Psi_\alpha{}^i$ and $\Psi_\alpha{}^f$ in a Hilbert space $\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$ satisfying

$$(\Psi_\alpha{}^i,P(\lambda)\Psi_\beta{}^i)=(\Psi_\alpha{}^f,P(\lambda)\Psi_\beta{}^f),\tag{3.3}$$

for every λ, then there exists a Hilbert space $\mathfrak{H}_2'$ con-

[4] If L_2 is unbounded, the above proof does not exclude the possibility that one can measure M, even if M does not commute with L_1, by a measuring apparatus (ξ or $X_{\mu\rho\rho'}$) in a state which is outside the domain of L, because (2.8) would then be meaningless.

However, even if L_2 is unbounded, as long as L_1 is bounded, we can modify the above argument in the following way. We introduce unitary operators

$$V(S)=\exp(iLS);\quad V_j(S)=\exp(iL_jS),\quad j=1,2.\tag{i}$$

Because of the additivity, (2.5),

$$V(S)=V_1(S)\otimes V_2(S).\tag{ii}$$

Then, by the conservation law (2.6), we have

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ V(S)(\phi_{\mu\rho}\otimes\xi))\\ &=(\Sigma_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\ V(S)(\Sigma_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})).\end{aligned}\tag{iii}$$

By the orthogonality, (2.2),

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ V_2(S)(\phi_{\mu\rho}\otimes\xi))\\ &=(\Sigma_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\ V_2(S)(\Sigma_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''}))\\ &=0\quad(\text{for }\mu\neq\mu')\end{aligned}\tag{iv}$$

Combining the two equations above and using the additivity, (ii), we obtain for $\mu\neq\mu'$,

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ F(S)(\phi_{\mu\rho}\otimes\xi))\\ &=(\Sigma_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\ F(S)(\Sigma_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})),\end{aligned}\tag{v}$$

where

$$F(S)=(1/iS)[V_1(S)-1]\otimes V_2(S).\tag{vi}$$

Since $F(S)\to L_1$, as $S\to0$, we obtain

$$\begin{aligned}(\phi_{\mu'\rho'}\otimes\xi,\ L_1(\phi_{\mu\rho}\otimes\xi))\\ &=(\Sigma_{\rho'''}\phi_{\mu'\rho'''}\otimes X_{\mu'\rho'\rho'''},\ L_1(\Sigma_{\rho''}\phi_{\mu\rho''}\otimes X_{\mu\rho\rho''})).\end{aligned}\tag{vii}$$

Because of the orthogonality, (2.4), we finally obtain (2.9a) from which we conclude (2.11) as before.

[5] The proof holds without this specification but notations become complicated, especially in dividing various regions of values of λ.

taining $\mathfrak{H}_2$ and a unitary operator U on $\mathfrak{H}'=\mathfrak{H}_1\otimes\mathfrak{H}_2'$ such that (1) a self-adjoint operator L_2' (representing the conserved quantity in $\mathfrak{H}_2'$) is defined on $\mathfrak{H}_2'$ coinciding with L_2 on $\mathfrak{H}_2$, (2) U commutes with the conserved quantity L' on $\mathfrak{H}'$, $L'=L_1\otimes 1+1\otimes L_2'$, and

$$(3)\quad \Psi_\alpha{}^f=U\Psi_\alpha{}^i. \tag{3.4}$$

If the set of the indices α is finite, $\mathfrak{H}_2'$ can be taken to be $\mathfrak{H}_2$.

This Lemma is used in the following way. We will construct states ξ, $X_{\mu\rho}$, ψ, and $\eta_{\mu\rho}$ satisfying

$$(X_{\mu\rho},X_{\mu'\rho'})=0,\quad \text{if}\quad \mu\neq\mu', \tag{3.5}$$

$$(X_{\mu\rho},\eta_{\mu'\rho'})=0,\quad \text{for any } \mu,\ \rho,\ \mu',\ \rho', \tag{3.6}$$

$$\|\psi\otimes\eta_{\mu\rho}\|^2<\epsilon,$$

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=0,\quad \text{for } (\mu,\rho)\neq(\mu',\rho') \tag{3.7}$$

in such a way that the two sets of vectors

$$\Psi_\alpha{}^i=\phi_{\mu\rho}\otimes\xi,\quad \Psi_\alpha{}^f=\phi_{\mu\rho}\otimes X_{\mu\rho}+\psi\otimes\eta_{\mu\rho},\quad \alpha=(\mu,\rho), \tag{3.8}$$

fulfill (3.3). If we succeed in constructing such states, then by the Lemma, there exists a unitary operator U in $\mathfrak{H}'$ which conserves L' and for which (3.4) holds. This implies, for any normalized state ϕ in $\mathfrak{H}_1$,

$$U(\phi\otimes\xi)=\sum_{\mu\rho}(\phi_{\mu\rho},\phi)(\phi_{\mu\rho}\otimes X_{\mu\rho})+\eta(\phi), \tag{3.9}$$

$$\eta(\phi)=\sum_{\mu\rho}(\phi_{\mu\rho},\phi)(\psi\otimes\eta_{\mu\rho}), \tag{3.10}$$

and, due to (3.7),

$$\|\eta(\phi)\|^2<\epsilon. \tag{3.11}$$

Thus if we choose the setup of a measurement in such a way that the Hilbert space of the measuring instrument is $\mathfrak{H}_2'$, the initial state of the instrument is ξ, and the time development of the combined system of the measured object and the measuring apparatus in a certain time interval t is described by $U(t)=U$, then we can measure the operator M in terms of the states $X_{\mu\rho}$ of the measuring apparatus after the measurement within the inaccuracy representing by $\eta(\phi)$. This inaccuracy can be made as small as one desires by making ϵ small enough. Because we are only concerned with the effect of the conservation law of L, we have assumed in the above argument that, if U is a unitary operator commuting with the conserved quantity, then there always exists an experimental setup whose time development in a certain time period is described by U. There may be many other conditions on U in addition to that it commutes with L. Hence, our argument does not assure that a system exists whose Hamiltonian leads to U.

We now give an explicit construction of states ξ, $X_{\mu\rho}$, ψ and $\eta_{\mu\rho}$. For this purpose we denote by $\mathfrak{H}_{2\lambda}$ the subspace of $\mathfrak{H}_2$ which is spanned by eigenvectors of L_2 with an eigenvalue λ. ψ is taken to be a normalized eigenstate of L_1 with the eigenvalue 0,

$$L_1\psi=0,\quad (\psi,\psi)=1. \tag{3.12}$$

ξ, $X_{\mu\rho}$, and $\eta_{\mu\rho}$ are given by

$$\xi=\sum_\lambda\xi_\lambda,\quad X_{\mu\rho}=\sum_\lambda X_{\mu\rho\lambda},\quad \eta_{\mu\rho}=\sum_\lambda\eta_{\mu\rho\lambda}, \tag{3.13}$$

where ξ_λ, $X_{\mu\rho\lambda}$, and $\eta_{\mu\rho\lambda}$ are vectors in $\mathfrak{H}_{2\lambda}$ to be specified below.

ξ_λ is any state in $\mathfrak{H}_{2\lambda}$ with the norm given by

$$(\xi_\lambda,\xi_\lambda)=0,\qquad \text{for}\quad |\lambda|>N, \tag{3.14a}$$

$$=(2N+1)^{-1},\quad \text{for}\quad |\lambda|\leq N. \tag{3.14b}$$

N is any integer satisfying

$$N>2l/\epsilon-\tfrac{1}{2}. \tag{3.15}$$

The $X_{\mu\rho\lambda}$ are any states in $\mathfrak{H}_{2\lambda}$, orthogonal to each other and with the norm given by

$$(X_{\mu\rho\lambda},X_{\mu'\rho'\lambda})=0,\qquad \text{for}\quad |\lambda|>N-2l, \tag{3.16a}$$

$$=(2N+1)^{-1}\delta_{\mu\mu'}\delta_{\rho\rho'},\quad \text{for}\quad |\lambda|\leq N-2l. \tag{3.16b}$$

The orthogonal complement of the set $\{X_{\mu\rho\lambda}|\mu,\rho \text{ varying}\}$ in $\mathfrak{H}_{2\lambda}$ will be denoted by $\mathfrak{H}_{2\lambda}{}^\eta$.

$\eta_{\mu\rho\lambda}$ are taken from $\mathfrak{H}_{2\lambda}{}^\eta$ and defined in the following way

(I) For $|\lambda|>N+l$ or $|\lambda|\leq N-3l$,

$$\eta_{\mu\rho\lambda}=0. \tag{3.17a}$$

(II) For $N+l\geq|\lambda|>N-l$,[6]

$$(\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda})=(2N+1)^{-1}\sum_{\substack{|\lambda'|\leq l\\|\lambda-\lambda'|\leq N}}(\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}). \tag{3.17b}$$

(III) For $N-l\geq|\lambda|>N-3l$, $\eta_{\mu\rho\lambda}$ are any states in $\mathfrak{H}_{2\lambda}{}^\eta$ orthogonal to each other and with the norm given by

$$(\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda})=(2N+1)^{-1}(\phi_{\mu\rho},Q_\lambda\phi_{\mu\rho})\delta_{\mu\mu'}\delta_{\rho\rho'}, \tag{3.17c}$$

where Q_λ is a projection operator given by

$$Q_\lambda=\sum_{\substack{|\lambda'|\leq l\\|\lambda-\lambda'|>N-2l}}P_1(\lambda'). \tag{3.18}$$

Note that $(\phi_{\mu\rho},Q_\lambda\phi_{\mu\rho})$ is non-negative (between 0 and 1).

We now show that ξ, $X_{\mu\rho}$, ψ, and $\eta_{\mu\rho}$ thus constructed have the desired properties. ξ is normalized due to (3.14). (3.5) and (3.6) are trivially satisfied by our

[6] This means that $\eta_{\mu\rho\lambda}$ is defined by

$$\eta_{\mu\rho\lambda}=(2N+1)^{-\frac{1}{2}}\sum_{\substack{|\lambda'|\leq l\\|\lambda-\lambda'|\leq N}}P_1{}^{(\lambda)}(\lambda')\phi_{\mu\rho}{}^{(\lambda)}$$

where we have made an isometric linear mapping of $\mathfrak{H}_1$ into $\mathfrak{H}_{2\lambda}{}^\eta$ and $\phi_{\mu\rho}$ and $P_1(\lambda')$ thus mapped are called $\phi_{\mu\rho}{}^{(\lambda)}$ and $P_1{}^{(\lambda)}(\lambda')$.

choice. To prove (3.3), we rewrite (3.3) using (3.2):

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})(\xi, P_2(\lambda-\lambda')\xi)$$
$$= \sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})(X_{\mu\rho}, P_2(\lambda-\lambda')X_{\mu'\rho'}) + (\eta_{\mu\rho},P_2(\lambda)\eta_{\mu'\rho'}), \quad (3.19)$$

where we have also used (3.12). By (3.13), this is equivalent to

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}) \times[\|\xi_{\lambda-\lambda'}\|^2-(X_{\mu\rho,\lambda-\lambda'},X_{\mu'\rho',\lambda-\lambda'})] = (\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda}). \quad (3.20)$$

We divide the range of λ into 4 parts and prove (3.20) separately for λ in each of these 4 regions.

(I) If $|\lambda|>N+l$, then $|\lambda-\lambda'|>N$ and (3.20) is trivially satisfied because all terms vanish.

(II) If $N+l\geq|\lambda|>N-l$, then $|\lambda-\lambda'|>N-2l$, and hence the term containing X still vanishes. Due to (3.14b), the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})\|\xi_{\lambda-\lambda'}\|^2 = (2N+1)^{-1} \sum_{\substack{|\lambda'|\leq l \\ |\lambda-\lambda'|\leq N}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'}),$$

which is equal to the right-hand side of (3.20) calculated by (3.17b).

(III) If $N-l\geq|\lambda|>N-3l$, then $|\lambda-\lambda'|\leq N$ and hence $\|\xi_{\lambda-\lambda'}\|^2$ is always $(2N+1)^{-1}$. By the orthogonality, (2.2), the definition (3.16b) and the equation

$$\sum_{|\lambda'|\leq l} P_1(\lambda)=1, \quad (3.21)$$

the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[(2N+1)^{-1}-(X_{\mu\rho,\lambda-\lambda'},X_{\mu'\rho',\lambda-\lambda'})]$$
$$=(2N+1)^{-1}\delta_{\mu\mu'}\delta_{\rho\rho'} \sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu\rho}) \times[1-(2N+1)(X_{\mu\rho,\lambda-\lambda'},X_{\mu'\rho',\lambda-\lambda'})]. \quad (3.22)$$

Because of (3.16b), the inside of the square bracket of (3.22) vanishes for $|\lambda-\lambda'|\leq N-2l$ and is unity for $|\lambda-\lambda'|>N-2l$. Thus, due to (3.17c) and (3.18), (3.22) is equal to the right-hand side of (3.20).

(IV) If $N-3l\geq|\lambda|$, then $|\lambda-\lambda'|\leq N-2l$ and the left-hand side of (3.20) becomes

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda)\phi_{\mu'\rho'})(1-\delta_{\mu\mu'}\delta_{\rho\rho'}).$$

Because of (3.21) and the orthogonality, (2.2), this expression vanishes and hence is equal to the right-hand side of (3.20) which also vanishes due to (3.17a). This completes the proof of (3.4).

Finally, we will prove (3.7). Since ψ is normalized, $\|\psi\otimes\eta_{\mu\rho}\|$ is equal to $\|\eta_{\mu\rho}\|$. By (3.13), we have

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=\sum\nolimits_\lambda (\eta_{\mu\rho\lambda},\eta_{\mu'\rho'\lambda}).$$

By (3.20), we get

$$\sum_\lambda(\eta_{\mu\rho\lambda},\eta_{\mu\rho'\lambda'})$$
$$= \sum_{\substack{\lambda\lambda' \\ |\lambda'|\leq l}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[\|\xi_{\lambda-\lambda'}\|^2-\|X_{\mu\rho,\lambda-\lambda'}\|^2\delta_{\mu\mu'}\delta_{\rho\rho'}]$$
$$= \sum_{\substack{\lambda\lambda' \\ |\lambda'|\leq l}} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})[\|\xi_\lambda\|^2-\|X_{\mu\rho,\lambda}\|^2\delta_{\mu\mu'}\delta_{\rho\rho'}].$$

By (3.21) and (2.2),

$$\sum_{|\lambda'|\leq l} (\phi_{\mu\rho},P_1(\lambda')\phi_{\mu'\rho'})=\delta_{\mu\mu'\rho\rho'}.$$

By (3.14) and (3.16)

$$\sum\nolimits_\lambda [\|\xi_\lambda\|^2-\|X_{\mu\rho\lambda}\|^2]=4l(2N+1)^{-1}.$$

Combining these, and using (3.15), we obtain

$$\|\psi\otimes\eta_{\lambda\rho}\|^2=\frac{4l}{2N+1}<\epsilon.$$

$$(\eta_{\mu\rho},\eta_{\mu'\rho'})=0, \quad \text{for } (\mu,\rho)\neq(\mu',\rho')$$

In the above construction, $\mathfrak{H}_{2\lambda}$ for $|\lambda|\leq N-3l$ should have at least the dimension of $\mathfrak{H}_1$. We need higher dimension for $\mathfrak{H}_{2\lambda}$ with $N-3l<|\lambda|\leq N$.[7]

Finally we will give a proof of our Lemma. For this purpose, we denote the subspace of $\mathfrak{H}$ spanned by eigenvectors of L with eigenvalue λ by $\mathfrak{H}_\lambda$, the subspace spanned by $P(\lambda)\Psi_\alpha{}^i$ with varying α by $\mathfrak{H}_\lambda{}^i$, the subspace spanned by $P(\lambda)\Psi_\alpha{}^f$ with varying α by $\mathfrak{H}_\lambda{}^f$, the orthogonal complement of $\mathfrak{H}_\lambda{}^i$ in $\mathfrak{H}_\lambda$ by $\mathfrak{H}_\lambda{}^i{}_\perp$, and the orthogonal complement of $\mathfrak{H}_\lambda{}^f$ in $\mathfrak{H}_\lambda$ by $\mathfrak{H}_\lambda{}^f{}_\perp$. Obviously

$$\mathfrak{H}=\oplus_\lambda(\mathfrak{H}_\lambda{}^i\oplus\mathfrak{H}_\lambda{}^i{}_\perp)=\oplus_\lambda(\mathfrak{H}_\lambda{}^f\oplus\mathfrak{H}_\lambda{}^f{}_\perp). \quad (3.23)$$

We will first show that

$$U_\lambda(\sum\nolimits_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^i)=\sum\nolimits_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^f, \quad (3.24)$$

defines a unitary mapping U_λ of $\mathfrak{H}_\lambda{}^i$ onto $\mathfrak{H}_\lambda{}^f$, where $\{C_\alpha\}$ is a set of arbitrary complex numbers. To see this, we note that, due to (3.3), $\sum_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^i$ and

[7] In the above construction, the measuring apparatus is a superposition of eigenstates of L_2 with different eigenvalues λ varying over the range of the order $1/\epsilon$. However, if one counts the number of equations to be satisfied, one finds a possibility of constructing a similar measuring apparatus which is a superposition of eigenstates of L_2 with eigenvalues, near a certain large value of the order $1/\epsilon$, but varying only over the range of the order of the dimension of $\mathfrak{H}_1$, provided that the latter is finite. Here we will not pursue the problem of such minimization, but we will only note that, if we do minimize the number of eigenvalues of L_2 to be used in the measuring apparatus, then $X_{\mu\rho}$ will be nearly strictly determined and if that is the case, there is a fair chance that $X_{\mu\rho}$ cannot be made macroscopically distinguishable any better than $\phi_{\mu\rho}$.

$\sum_\alpha C_\alpha P(\lambda)\Psi_\alpha{}^f$ converge, diverge, or vanish simultaneously. Hence, U_λ is a one-to-one mapping of $\mathfrak{H}_\lambda{}^i$ onto $\mathfrak{H}_\lambda{}^f$. Since this mapping is linear and, due to (3.3), isometric, U_λ is a unitary mapping of $\mathfrak{H}_\lambda{}^i$ onto $\mathfrak{H}_\lambda{}^f$ as was to be proved. This also proves that the dimensions of $\mathfrak{H}_\lambda{}^i$ and $\mathfrak{H}_\lambda{}^f$ are the same.

If this dimension is finite, the dimensions of $\mathfrak{H}_\lambda{}^i{}_\perp$ and $\mathfrak{H}_\lambda{}^f{}_\perp$ are the same. Then there always exists a unitary mapping $U_{\lambda\perp}$ of $\mathfrak{H}_\lambda{}^i{}_\perp$ onto $\mathfrak{H}_\lambda{}^f{}_\perp$. Now we define an operator U in $\mathfrak{H}$.

$$U=\oplus_\lambda(U_\lambda\oplus U_{\lambda\perp}). \tag{3.25}$$

Because of the unitarity of U_λ and $U_{\lambda\perp}$ and the decomposition, (3.23), U is obviously unitary. For any $\Psi\epsilon\mathfrak{H}$,

$$U\Psi=\sum_\lambda (U_\lambda\Psi_\lambda{}^i+U_{\lambda\perp}\Psi_\lambda{}^i{}_\perp), \tag{3.26}$$

where

$$\Psi=\sum_\lambda(\Psi_\lambda{}^i+\Psi_\lambda{}^i{}_\perp),\ \Psi_\lambda{}^i\epsilon\mathfrak{H}_\lambda{}^i,\ \Psi_\lambda{}^i{}_\perp\epsilon\mathfrak{H}_\lambda{}^i{}_\perp, \tag{3.27}$$

is a unique decomposition of Ψ according to the first equation of (3.23). Since the subspace $\mathfrak{H}_\lambda$ of $\mathfrak{H}$ spanned by eigenvectors of $L=L_1\otimes 1+1\otimes L_2$ with the eigenvalue λ is mapped onto itself by U, U commutes with L. This completes the proof for the case where the dimension of $\mathfrak{H}_\lambda{}^i$ and $\mathfrak{H}_\lambda{}^f$ is finite.

If this dimension is infinite, then the dimensions of $\mathfrak{H}_\lambda{}^i{}_\perp$ and $\mathfrak{H}_\lambda{}^f{}_\perp$ can be different. In such a case we introduce a new Hilbert space $\mathfrak{H}_2{}^r$ (on which the conserved quantity $L_2{}^r$ is defined) in such a way that the dimension of $\mathfrak{H}_\lambda{}^r$ is at least the number of indices α where $\mathfrak{H}_\lambda{}^r$ is the subspace of $\mathfrak{H}^r\equiv\mathfrak{H}_1\otimes\mathfrak{H}_2{}^r$ spanned by the eigenstates of $L^r=L_1\otimes 1+1\otimes L_2{}^r$ with eigenvalues λ. Then since the dimension of $\mathfrak{H}_\lambda{}^i$ and $\mathfrak{H}_\lambda{}^f$ does not exceed the cardinal number of the set of the indices α, $\mathfrak{H}_\lambda{}^{ir}\equiv\mathfrak{H}_\lambda{}^i{}_\perp\oplus\mathfrak{H}_\lambda{}^r$ and $\mathfrak{H}_\lambda{}^{fr}\equiv\mathfrak{H}_\lambda{}^f{}_\perp\oplus\mathfrak{H}_\lambda{}^r$ have the same dimension. Hence, there always exists a unitary mapping $U_{\lambda\perp}$ of $\mathfrak{H}_\lambda{}^{ir}$ onto $\mathfrak{H}_\lambda{}^{fr}$.

We are now in the position to construct the Hilbert space $\mathfrak{H}_2'$ and the unitary operator U for this case. $\mathfrak{H}_2'$ is taken to be $\mathfrak{H}_2\oplus\mathfrak{H}_2{}^r$. L_2' on $\mathfrak{H}_2'$ is taken to be $L_2\oplus L_2{}^r$. $\mathfrak{H}'$ can be decomposed as

$$\mathfrak{H}'=\oplus_\lambda(\mathfrak{H}_\lambda{}^i\oplus\mathfrak{H}_\lambda{}^{ir})=\oplus_\lambda(\mathfrak{H}_\lambda{}^f\oplus\mathfrak{H}_\lambda{}^{fr}). \tag{3.28}$$

U is defined as unitary mapping

$$U=\oplus_\lambda(U_\lambda\oplus U_{\lambda\perp}). \tag{3.29}$$

Instead of (3.26), (3.27), we have, for any $\Psi\epsilon\mathfrak{H}'$

$$U\Psi=\sum_\lambda (U_\lambda\Psi_\lambda{}^i+U_{\lambda\perp}\Psi_\lambda{}^{ir}), \tag{3.30}$$

$$\Psi=\sum_\lambda (\Psi_\lambda{}^i+\Psi_\lambda{}^{ir}),\ \Psi_\lambda{}^i\epsilon\mathfrak{H}_\lambda{}^i,\ \Psi_\lambda{}^{ir}\epsilon\mathfrak{H}_\lambda{}^{ir}. \tag{3.31}$$

Then by the same argument as in the previous case, we can show the unitarity of U, and commutativity with L', where L' is defined as $L'\equiv L_1\otimes 1+1\otimes L_2'$.

We note that in our application of the Lemma, the number of the indices α is the same as the dimension of $\mathfrak{H}_1$.

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