MATHEMATICS OF CRYPTOGRAPHY
PART II
ALGEBRAIC STRUCTURES

Objectives

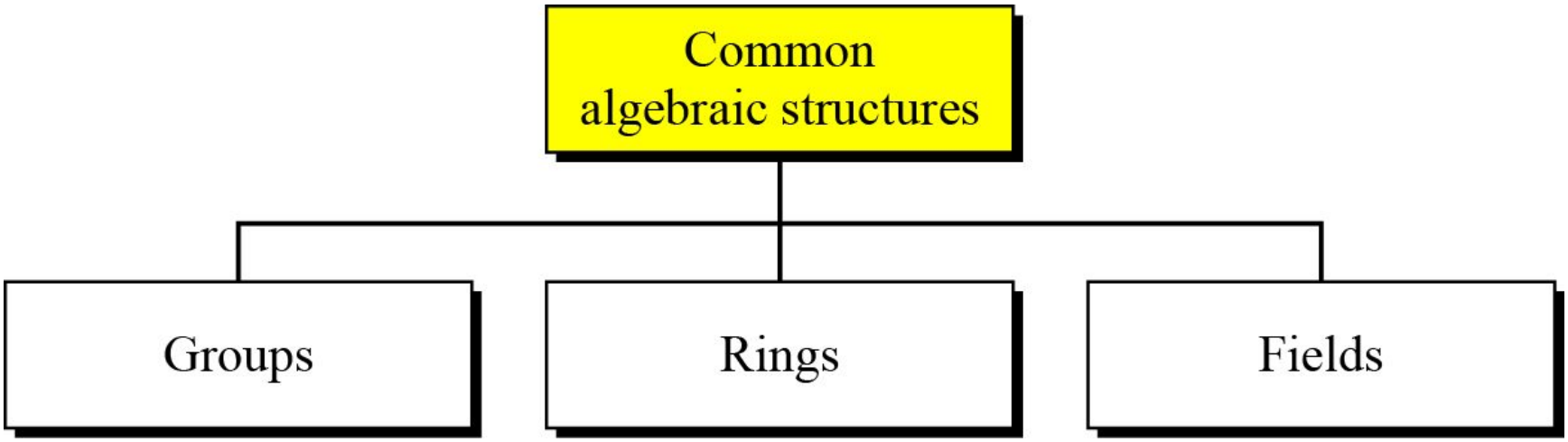
- To review the concept of algebraic structures
- To define and give some examples of groups
- To define and give some examples of rings
- To define and give some examples of fields
- To emphasize the finite fields of type GF (2^n) that make it possible to perform operations such as addition, subtraction, multiplication, and division on n-bit words in modern block ciphers

<modern symmetric-key ciphers are based on algebraic structure>

ALGEBRAIC STRUCTURES

- Cryptography requires sets of integers and specific operations that are defined for those sets.
- The combination of the set and the operations that are applied to the elements of the set is called an algebraic structure.
- Three common algebraic structures: groups, rings, and fields.

ALGEBRAIC STRUCTURES(cont.)



Common algebraic structure

Groups

- A group (G) is a set of elements with a binary operation (•) that satisfies four properties.
  - Closure
  - Associativity
  - Existence of identity
  - Existence of inverse

Groups(cont.)

- Closure
  - If a and b are elements of G, then c = a•b is also an element of G.
- Associativity
  - If a, b and c are elements of G, then (a•b) •c=a•(b•c)
- Existence of identity
  - For all *a* in G, there exist an element e, called the identity element, such that e•a=a•e=a
- Existence of inverse
  - For each *a* in G, there exists an element a', called the inverse of a, such that a•a'=a'•a=e

Groups(cont.)

- A Commutative group (Abelian group), is a group in which the operator satisfies four properties plus an extra property that is commutativity.
  - For all a and b in G, we have a • b = b • a

Groups(cont.)

- Although a group involves a single operation, the properties imposed on the operation allow the use of a pair of operations as log as they are inverses of each other.
- If the defined operation is addition, the group supports both addition and subtraction, because subtraction is addition using the additive inverse.
- This is also true for multiplication and division.
- However, a group can support only addition/subtraction or multiplication/division operations, but not the both at the same time.

Groups(cont.)

- Example

  The set of residue integers with the addition operator,

  $$G = < Zn , + >,$$

  is a commutative group.

  Check the properties…..

  *"*We can perform addition and subtraction on the elements of this set without moving out of the set."

  Assume- a=2, b=3, c=4, and identity element is 0

Example (cont..)

1. Closure is satisfied. The result of adding two integers in Zn is another integer in Zn.
2. Associativity is satisfied. The result of 4+(3+2) is same as (4+3)+2.
3. Commutativity is satisfied. We have 3+5=5+3.
4. The identity element is 0. we have 3+0=0+3=3.
5. Every element has an addititive inverse. The inverse of a element is its complement. For example, the inverse of 3 is -3 (n-3 in Zn) and the inverse of -3 is 3. the inverse allows us to perform subtraction on the set.

Groups(cont.)

- Example:
  - The set Zn* with the multiplication operator, G = <Zn*, ×>, is also an abelian group.
  - We can perform multiplication and division on the elements of this set without moving out of the set.
- Example:
  - Let us define a set G = < {a, b, c, d}, •> and the operation as shown in Table.

•	*a*	*b*	*c*	*d*
a	*a*	*b*	*c*	*d*
b	*b*	*c*	*d*	*a*
c	*c*	*d*	*a*	*b*
d	*d*	*a*	*b*	*c*

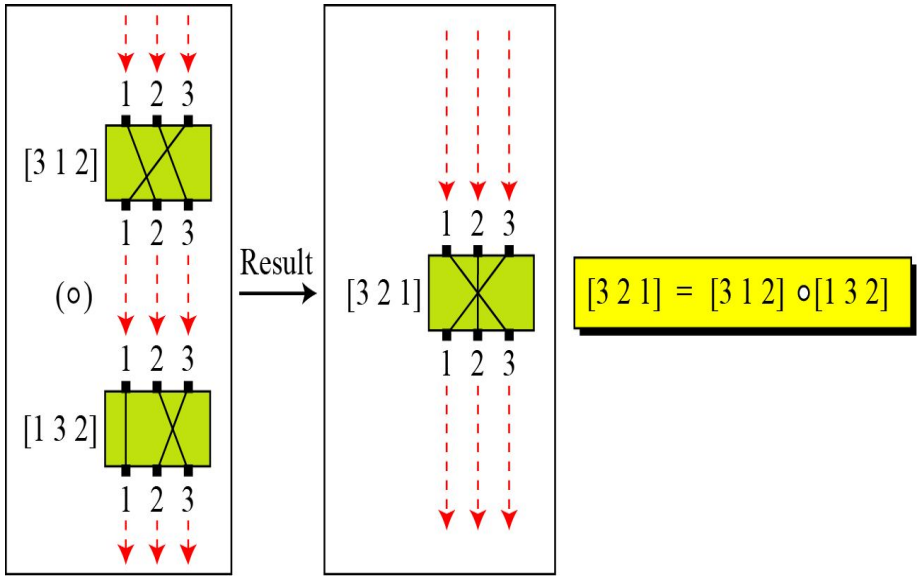
•	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

- The group is abelian????
- Closure ?
- Associativity?
- Commutativity?
- The group has an identity element, which is a.
- Each element has an inverse. The inverse pairs can be found by finding the identity in each row (shaded). The pairs are (a,a), (b,d), (c,c).

Groups(cont.)

- Example:
  - A very interesting group is the permutation group.
  - The set is the set of all permutations, and the operation is composition: applying one permutation after another.
  - Check for properties….
    - Is the group abelian???

Composition of two permutations



Groups(cont.)

- Example(cont.):

°	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 2 3]	[1 2 3]	[1 3 2]	[2 1 3]	[2 3 1]	[3 1 2]	[3 2 1]
[1 3 2]	[1 3 2]	[1 2 3]	[2 3 1]	[2 1 3]	[3 2 1]	[3 1 2]
[2 1 3]	[2 1 3]	[3 1 2]	[1 2 3]	[3 2 1]	[1 3 2]	[2 3 1]
[2 3 1]	[2 3 1]	[3 2 1]	[1 3 2]	[3 1 2]	[1 2 3]	[2 1 3]
[3 1 2]	[3 1 2]	[2 1 3]	[3 2 1]	[1 2 3]	[2 3 1]	[1 3 2]
[3 2 1]	[3 2 1]	[2 3 1]	[3 1 2]	[1 3 2]	[2 1 3]	[1 2 3]

Operation table for permutation group

With three inputs and three outputs, there can be 3! or 6 different permutations.
Only four properties are satisfied.

1. *Closure*
2. *Associativity.*
3. *Commutative?*
4. *The set has an identity element, which is [1 2 3].*
5. *Each element has inverse.*

Groups(cont.)

- In the previous example, we showed that a set of permutations with the composition operation is a group.
- This implies that using two permutations one after another cannot strengthen the security of a cipher.
- Because we can always find a permutation that can do the same job because of the closure property.

Groups(cont.)

- Finite Group
  - If the set has a finite number of elements; otherwise, it is an infinite group.
- Order of a Group |G|
  - The number of elements in the group.
  - If the group is finite, its order is finite
- Subgroups
  - A subset **H** of a group **G** is a subgroup of **G** if **H** itself is a group with respect to the operation on **G**

Groups(cont.)

- Subgroups(cont.)
  - If G=<S, •> is a group, H=<T, •> is a group under the same operation, and T is a nonempty subset of S, then H is a subgroup of G
  - The above definition implies that:
    - If a and b are members of both groups, then c=a•b is also member of both groups
    - The group share the same identity element
    - If a is a member of both groups, the inverse of a is also a member of both groups
    - Each group is a subgroup of itself

Groups(cont.)

- Exercise:
  - Is the group H = $<Z_{10}, +>$ a subgroup of the group G = $<Z_{12}, +>$?

Groups(cont.)

- Exercise:
  - Is the group H = $<Z_{10}, +>$ a subgroup of the group G = $<Z_{12}, +>$?
- Solution:
  - The answer is no. Although H is a subset of G, the operations defined for these two groups are different. The operation in H is addition modulo 10; the operation in G is addition modulo 12.

Groups(cont.)

- Cyclic subgroups
  - If a subgroup of a group can be generated using the power of an element, the subgroup is called the cyclic subgroup.
  - The term *power* here means repeatedly applying the group operation to the element.

$$a^n \rightarrow a \bullet a \bullet \ldots \bullet a \quad (n \text{ times})$$

Groups(cont.)

- Four cyclic subgroups can be made from the group G = $<Z_6, +>$.
- They are $H_1 = <\{0\}, +>$, $H_2 = <\{0, 2, 4\}, +>$, $H_3 = <\{0, 3\}, +>$, and $H_4 = G$.

$$0^0 \bmod 6 = 0$$

$$1^0 \bmod 6 = 0$$
$$1^1 \bmod 6 = 1$$
$$1^2 \bmod 6 = (1 + 1) \bmod 6 = 2$$
$$1^3 \bmod 6 = (1 + 1 + 1) \bmod 6 = 3$$
$$1^4 \bmod 6 = (1 + 1 + 1 + 1) \bmod 6 = 4$$
$$1^5 \bmod 6 = (1 + 1 + 1 + 1 + 1) \bmod 6 = 5$$

$$2^0 \bmod 6 = 0$$
$$2^1 \bmod 6 = 2$$
$$2^2 \bmod 6 = (2 + 2) \bmod 6 = 4$$

$$3^0 \bmod 6 = 0$$
$$3^1 \bmod 6 = 3$$

$$4^0 \bmod 6 = 0$$
$$4^1 \bmod 6 = 4$$
$$4^2 \bmod 6 = (4 + 4) \bmod 6 = 2$$

$$5^0 \bmod 6 = 0$$
$$5^1 \bmod 6 = 5$$
$$5^2 \bmod 6 = 4$$
$$5^3 \bmod 6 = 3$$
$$5^4 \bmod 6 = 2$$
$$5^5 \bmod 6 = 1$$

Groups(cont.)

- Exercise:
  - Find out the cyclic subgroups for group G = $<Z_{10}^*, \times>$.

Groups(cont.)

- Three cyclic subgroups can be made from the group $G = \langle Z_{10}^*, \times \rangle$. G has only four elements: 1, 3, 7, and 9. The cyclic subgroups are $H_1 = \langle \{1\}, \times \rangle$, $H_2 = \langle \{1, 9\}, \times \rangle$, and $H_3 = G$.

$$1^0 \bmod 10 = 1$$

$$3^0 \bmod 10 = 1$$
$$3^1 \bmod 10 = 3$$
$$3^2 \bmod 10 = 9$$
$$3^3 \bmod 10 = 7$$

$$7^0 \bmod 10 = 1$$
$$7^1 \bmod 10 = 7$$
$$7^2 \bmod 10 = 9$$
$$7^3 \bmod 10 = 3$$

$$9^0 \bmod 10 = 1$$
$$9^1 \bmod 10 = 9$$

Groups(cont.)

- Cyclic group
  - The group **G** has a cyclic subgroup $\mathbf{H_4 = G}$. This means that the group **G** is a cyclic group.
  - In this case, the elements that generates the cyclic subgroup can also generate the group itself.
  - This element is referred to as a *generator.*

Groups(cont.)

- Cyclic group(cont.)
- Example:
  - Three cyclic subgroups can be made from the group G = < Z10*, ×>.
  - The cyclic subgroups are H1 = <{1}, ×>, H2 = <{1, 9}, ×>, and H3 = G.
  - The group G = <Z_{10}*, ×> is a cyclic group with two generators, g = 3 and g = 7.
  - The group G = <Z_6, +> is a cyclic group with two generators, g = 1 and g = 5.

Groups(cont.)

- Lagrange's Theorem
  - Assume that G is a group, and H is a subgroup of G. If the order of G and H are |G| and |H|, respectively, then, based on this theorem, |H| divides |G|.
    - Application of Lagrange's Theorem: Given a group G of order |G|, the orders of the potential subgroups can be easily determined if the divisors of |G| can be found.

- Order of an Element
  - The order of an element is the order of the cyclic group it generates.

Groups(cont.)

- Example:
  - In the group G = $<Z_6, +>$, the orders of the elements are:

    ord(0) = 1, ord(1) = 6, ord(2) = 3, ord(3) = 2, ord(4) = 3, ord(5) = 6.

  - In the group G = $<Z_{10}^*, \times>$, the orders of the elements are: ord(1) = 1, ord(3) = 4, ord(7) = 4, ord(9) = 2.

Examples

Let us consider the following statements

(i) $(\mathbb{Z}_{10}, +)$ is a cyclic group.

(ii) $(\mathbb{Z}, +)$ is not a cyclic group.

Select the correct option from below.

A. (i) and (ii) both are true.

B. Only (ii) is true.

C. Only (i) is true.

D. (i) and (ii) both are false.

Examples

Order of 3 in the group $(\mathbb{Z}_5, +)$ is _____.

A. 2

B. 5

C. 1

D. 3

Examples

Let $G = \{a \in \mathbb{Z}_{10} \mid gcd(a, 10) = 1\}$. Let us consider the following statements

(i) G is a group under multiplication modulo 10.

(ii) The number of elements in set G is 5.

Select the correct option from below.

A. (i) and (ii) both are true.

B. Only (ii) is true.

C. Only (i) is true.

D. (i) and (ii) both are false.

Ring

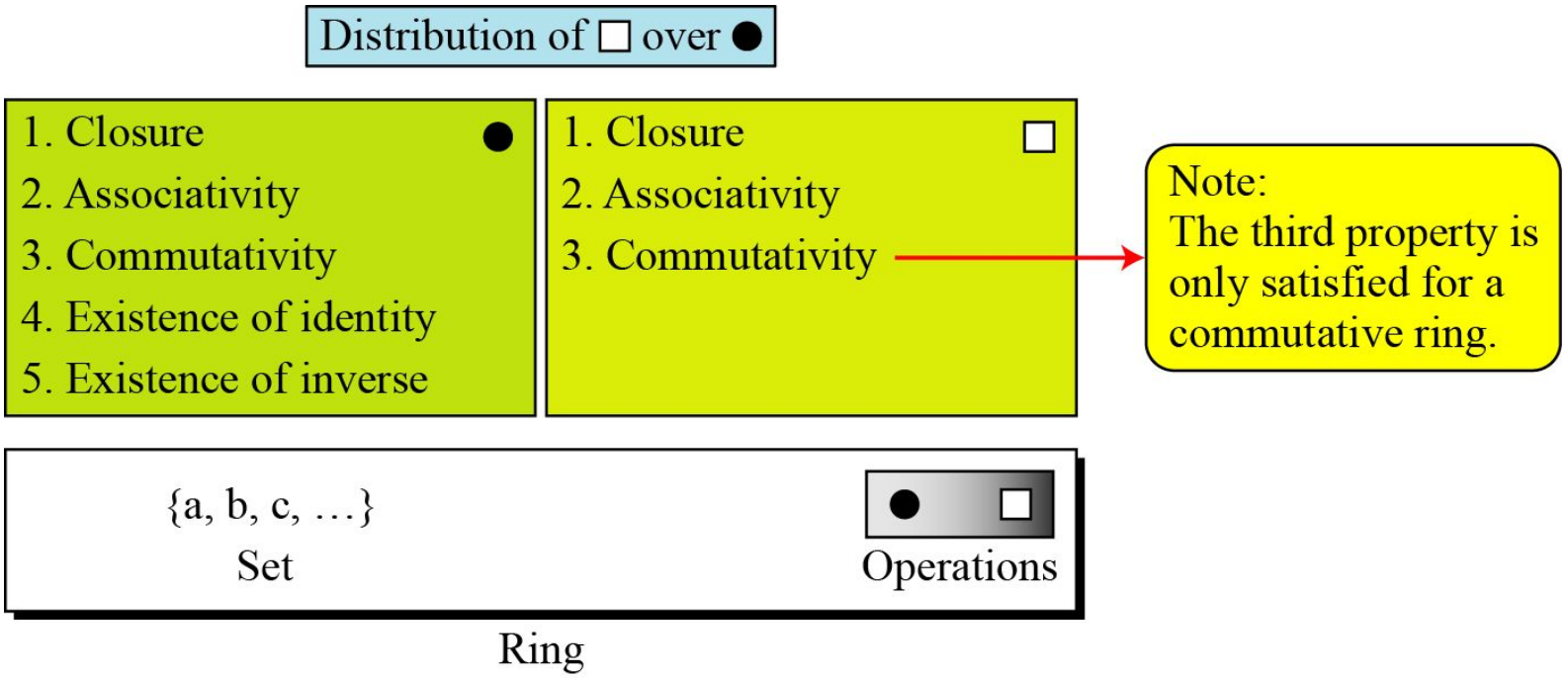
- A ring, R = <{…}, •, ■ >, is an algebraic structure with two operations.
- First operation must satisfy all five properties
- Second operation must satisfy only the first two
- In addition, second operation must be distributed over first
  - i.e. for all a, b, and c elements of R, we have,

    a ■ (b • c) = (a ■ b) • (a ■ c) and

    (a • b) ■ c = (a ■ c) • (a ■ c)

Ring(cont.)

- Commutative Ring- is a ring in which the commutative property is also satisfied for the second operation.

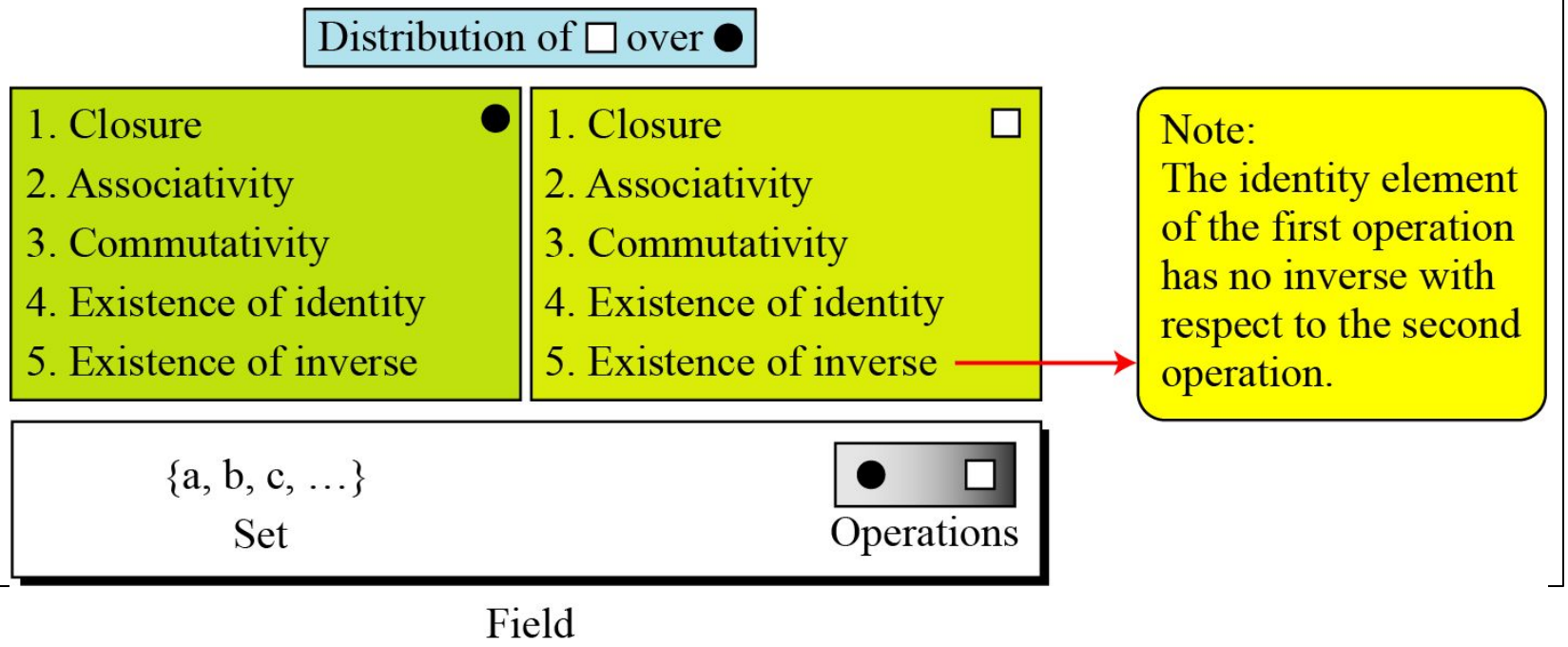


Ring(cont.)

- The set Z with two operations, addition and multiplication, is a commutative ring.
- We show it by R = <Z, +, ×>.
- Addition satisfies all of the five properties; multiplication satisfies only three properties.

Field

- A field, denoted by F = <{…}, •, ■ > is a commutative ring in which the second operation satisfies all five properties defined for the first operation except that the identity of the first operation has no inverse.



Field(cont.)

- Finite Fields
  - Only finite fields are extensively used in cryptography.
  - It is a field with finite number of elements, are very important structures in cryptography.
  - Galois showed that for a field to be finite, the number of elements should be p^n, where p is a prime and n is a positive integer.

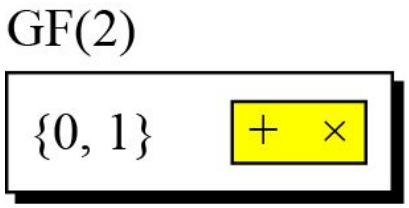
A Galois field, GF(p^n), is a finite field with p^n elements.

Field(cont.)

- GF(*p*) Fields
  - When *n* = 1, we have GF(*p*) field.
  - This field can be the set Z_p, {0, 1, …, p − 1}, with two arithmetic operations (addition and multiplication).
  - In this set each element has an additive inverse and that nonzero elements have a multiplicative inverse (no multiplicative inverse for 0).

Field(cont.)

- A very common field in this category is GF(2) with the set {0, 1} and two operations, addition and multiplication.



+	0	1
0	0	1
1	1	0

Addition

×	0	1
0	0	0
1	0	1

Multiplication

a	0	1
−a	1	0

a	0	1
a^{-1}	—	1

Inverses

GF(2) field

Field(cont.)

- We can define GF(5) on the set Z_5 (5 is a prime) with addition and multiplication operators.

GF(5)

{0, 1, 2, 3, 4} + ×

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Addition

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Multiplication

Additive inverse

a	0	1	2	3	4
−a	0	4	3	2	1

a	0	1	2	3	4
a^{-1}	—	1	3	2	4

Multiplicative inverse

GF(5) field

- Summary:

Algebraic Structure	*Supported Typical Operations*	*Supported Typical Sets of Integers*
Group	$(+ \ -)$ or $(\times \ \div)$	$\mathbf{Z}_n$ or $\mathbf{Z}_n^*$
Ring	$(+ \ -)$ and $(\times)$	$\mathbf{Z}$
Field	$(+ \ -)$ and $(\times \ \div)$	$\mathbf{Z}_p$

GF(2^n) FIELDS

- In cryptography, we often need to use four operations(addition,subtraction,multiplication,and division).
- In other words, we need to use fields.
- However, when we work with computers, the positive integers are stored in the computers as n-bit words in which n is usually 8,16,32 and so on.
- Range of integers is 0 to $2^n - 1$
- Hence modulus is 2^n
- What if we want to use field???? <there are two choices>

GF(2^n) FIELDS (cont.)

- Solution 1
  - Use GF(p), with the set Zp, where p is the largest prime number less than 2^n
  - But the problem ???
    - It is inefficient because we can not use the integers from p to 2^n -1.
    - Example- if n=4,the largest prime less than 2^4 is 13. this means we cannot use integers 13,14, and 15.
    - If n=8 ????????
- Solution 2
  - Use GF(2^n)
  - Use a set of 2^n elements
  - The elements in this set are n-bit words
  - E.g. for n=3, the set is {000,001,010,011,100,101,110,111}

GF(2^n) FIELDS (cont.)

- Let us define a GF(2^2) field in which the set has four 2-bit words: {00, 01, 10, 11}.
- We can redefine addition and multiplication for this field in such a way that all properties of these operations are satisfied. Addition and multiplication are defined in terms of polynomials.

Addition

⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

Identity: 00

Multiplication

⊗	00	01	10	11
00	00	00	00	00
01	00	01	10	11
10	00	10	11	01
11	00	11	01	10

Identity: 01

An example of GF(2^2) field

Polynomials

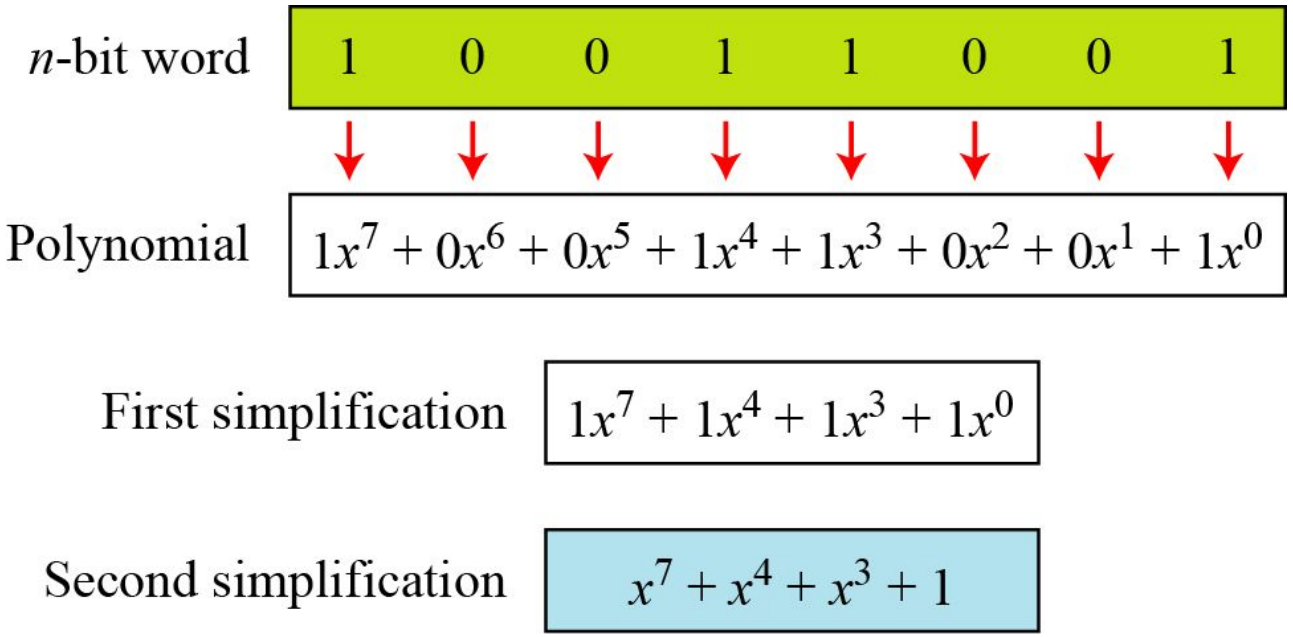
- We can directly define the rules for addition and multiplication operations on *n*-bit words that satisfy the properties in GF(2^n).
- A polynomial of degree $n - 1$ is an expression of the form

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0x^0$$

- where x^i is called the ith term and a_i is called coefficient of the *i*th term.

Polynomials (cont.)

- We can represent the 8-bit word (10011001) using a polynomial.



Polynomials (cont.)

- Find the 8-bit word related to the polynomial $x^5 + x^2 + x$, we first supply the omitted terms.
- Since $n = 8$, it means the polynomial is of degree 7. The expanded polynomial is,

$$0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0$$

- This is related to the 8-bit word 00100110.

Polynomials (cont.)

- Operations on polynomials
  - Actually involves two operations
    - Operation on coefficients and operation on polynomials
  - Hence, need to define two fields
  - What for coefficient??
  - What for polynomials???

Polynomials (cont.)

- Operations on polynomials
  - Actually involves two operations
    - Operation on coefficients and operation on polynomials
  - Hence, need to define two fields
  - What for coefficient??
  - What for polynomials???
  - Coefficients are made of 0 or 1; we can use GF(2) and GF(2^n) for polynomials….

Polynomials (cont.)

- Modulus
  - Why Modulus?
  - Addition of two polynomials never creates a polynomial out of the set.
  - However, multiplication of two polynomials may create a polynomial with degree more than n-1.
  - This means we need to divide the result by a modulus and keep only the remainder, as we deed in modular arithmetic.
  - For the sets of polynomials in GF(2^n), a group of polynomials of degree *n* is defined as the modulus.

Polynomials (cont.)

- irreducible polynomials.
  - No polynomial in the set can divide this polynomial
  - Can not be factored into a polynomial with degree of less than n

Degree	*Irreducible Polynomials*
1	$(x + 1), (x)$
2	$(x^2 + x + 1)$
3	$(x^3 + x^2 + 1), (x^3 + x + 1)$
4	$(x^4 + x^3 + x^2 + x + 1), (x^4 + x^3 + 1), (x^4 + x + 1)$
5	$(x^5 + x^2 + 1), (x^5 + x^3 + x^2 + x + 1), (x^5 + x^4 + x^3 + x + 1),$ $(x^5 + x^4 + x^3 + x^2 + 1), (x^5 + x^4 + x^2 + x + 1)$

Polynomials (cont.)

- Polynomial addition

Addition and subtraction operations on polynomials are the same operation.

Polynomials (cont.)

- Example
- Let us do $(x^5 + x^2 + x) \oplus (x^3 + x^2 + 1)$ in GF(2^8). We use the symbol $\oplus$ to show that we mean polynomial addition. The following shows the procedure:
- <keep the uncommon term and delete the common term>

$$
\begin{array}{ll}
0x^7 + 0x^6 + 1x^5 + 0x^4 + 0x^3 + 1x^2 + 1x^1 + 0x^0 & \oplus \\
0x^7 + 0x^6 + 0x^5 + 0x^4 + 1x^3 + 1x^2 + 0x^1 + 1x^0 & \\
\hline
0x^7 + 0x^6 + 1x^5 + 0x^4 + 1x^3 + 0x^2 + 1x^1 + 1x^0 & \rightarrow \quad x^5 + x^3 + x + 1
\end{array}
$$

Polynomials (cont.)

- Short cut method
  - Addition in GF(2) means the exclusive-or (XOR) operation.
  - So we can exclusive-or the two words, bits by bits, to get the result.
  - In the previous example, $x^5 + x^2 + x$ is 00100110 and $x^3 + x^2 + 1$ is 00001101.
  - The result is 00101011 or in polynomial notation $x^5 + x^3 + x + 1$.

Polynomials (cont.)

- Multiplication
  - The coefficient multiplication is done in GF(2).
  - The multiplying x^i by x^j results in x^{i+j}.
  - The multiplication may create terms with degree more than $n - 1$, which means the result needs to be reduced using a modulus polynomial.

Polynomials (cont.)

- Example
  - Find the result of $(x^5 + x^2 + x) \otimes (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$. <

$$P_1 \otimes P_2 = x^5(x^7 + x^4 + x^3 + x^2 + x) + x^2(x^7 + x^4 + x^3 + x^2 + x) + x(x^7 + x^4 + x^3 + x^2 + x)$$
$$P_1 \otimes P_2 = x^{12} + x^9 + x^8 + x^7 + x^6 + x^9 + x^6 + x^5 + x^4 + x^3 + x^8 + x^5 + x^4 + x^3 + x^2$$
$$P_1 \otimes P_2 = (x^{12} + x^7 + x^2) \bmod (x^8 + x^4 + x^3 + x + 1) = x^5 + x^3 + x^2 + x + 1$$

  - To find the final result, divide the polynomial of degree 12 by the polynomial of degree 8 (the modulus) and keep only the remainder.

Polynomials (cont.)

- Polynomial division with coefficients in GF(2)

$$
\begin{array}{r|l}
 & x^4 + 1 \\
\hline
x^8 + x^4 + x^3 + x + 1 & x^{12} + x^7 + x^2 \\
 & x^{12} + x^8 + x^7 + x^5 + x^4 \\
\hline
 & x^8 + x^5 + x^4 + x^2 \\
 & x^8 + x^4 + x^3 + x + 1 \\
\hline
\text{Remainder} & \boxed{x^5 + x^3 + x^2 + x + 1}
\end{array}
$$

Polynomials (cont.)

- Example:
  - In GF (2^4), find the inverse of ($x^2 + 1$) modulo ($x^4 + x + 1$).
  - Finding multiplicative inverse- use extended Euclidean Algorithm
- Solution
  - The answer is ($x^3 + x + 1$) <How to proof the answer>

q	r_1	r_2	r	t_1	t_2	t
$(x^2 + 1)$	$(x^4 + x + 1)$	$(x^2 + 1)$	(x)	(0)	(1)	$(x^2 + 1)$
(x)	$(x^2 + 1)$	(x)	(1)	(1)	$(x^2 + 1)$	$(x^3 + x + 1)$
(x)	(x)	(1)	(0)	$(x^2 + 1)$	$(x^3 + x + 1)$	(0)
	(1)	(0)		$(x^3 + x + 1)$	(0)	

Polynomials (cont.)

- Example:
  - In GF(2^8), find the inverse of (x^5) modulo ($x^8 + x^4 + x^3 + x + 1$).

- Solution

q	r_1	r_2	r	t_1	t_2	t
(x^3)	$(x^8 + x^4 + x^3 + x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	(0)	(1)	(x^3)
$(x + 1)$	(x^5)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$
(x)	$(x^4 + x^3 + x + 1)$	$(x^3 + x^2 + 1)$	(1)	(x^3)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$
$(x^3 + x^2 + 1)$	$(x^3 + x^2 + 1)$	(1)	(0)	$(x^4 + x^3 + 1)$	$(x^5 + x^4 + x^3 + x)$	(0)
	(1)	(0)		$(x^5 + x^4 + x^3 + x)$	(0)	

Polynomials (cont.)

- A better algorithm: Obtain the result by repeatedly multiplying a reduced polynomial by x.
- Example:
  - Find the result of multiplying $P_1 = (x^5 + x^2 + x)$ by $P_2 = (x^7 + x^4 + x^3 + x^2 + x)$ in GF(2^8) with irreducible polynomial $(x^8 + x^4 + x^3 + x + 1)$

Polynomials (cont.)

- Solution:
  - We first find the partial result of multiplying x^0, x^1, x^2, x^3, x^4, and x^5 by P_2. Note that although only three terms are needed, the product of $x^m \otimes P_2$ for m from 0 to 5 because each calculation depends on the previous result.

Powers	*Operation*	*New Result*	*Reduction*
$x^0 \otimes P_2$		$x^7 + x^4 + x^3 + x^2 + x$	No
$x^1 \otimes P_2$	$\boldsymbol{x} \otimes (x^7 + x^4 + x^3 + x^2 + x)$	$x^5 + x^2 + x + 1$	**Yes**
$x^2 \otimes P_2$	$\boldsymbol{x} \otimes (x^5 + x^2 + x + 1)$	$x^6 + x^3 + x^2 + x$	No
$x^3 \otimes P_2$	$\boldsymbol{x} \otimes (x^6 + x^3 + x^2 + x)$	$x^7 + x^4 + x^3 + x^2$	No
$x^4 \otimes P_2$	$\boldsymbol{x} \otimes (x^7 + x^4 + x^3 + x^2)$	$x^5 + x + 1$	**Yes**
$x^5 \otimes P_2$	$\boldsymbol{x} \otimes (x^5 + x + 1)$	$x^6 + x^2 + x$	No
$\mathbf{P_1 \times P_2} = (x^6 + x^2 + x) + (x^6 + x^3 + x^2 + x) + (x^5 + x^2 + x + 1) = \boldsymbol{x^5 + x^3 + x^2 + x + 1}$			