TRANSFORMATION

There are two complementary points of view for describing object transformation.

1.Geometric Transformation: The object itself is transformed relative to the coordinate system or background. The mathematical statement of this viewpoint is defined by geometric transformations applied to each point of the object.

2.Coordinate Transformation: The object is held stationary while the coordinate system is transformed relative to the object. This effect is attained through the application of coordinate transformations.

An example that helps to distinguish these two viewpoints:

The movement of an automobile against a scenic background we can simulate this by

1. Moving the automobile while keeping the background fixed-(Geometric Transformation)

2.We can keep the car fixed while moving the background scenery- (Coordinate Transformation)

POINT REPRESENTATION

We can use a column vector (a 2x1 matrix) to represent a 2D point : $\begin{vmatrix} x \\ Y \end{vmatrix}$

General transformation of 2D points:

Representation method 1 : [B]=[T] [A]

- Pre-multiply transformation matrix.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Representation method 2 : [B]=[A] [T]

- Post multiply transformation matrix.

$$\begin{bmatrix} x' \\ y' \end{bmatrix}^T = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

X'=ax+cy
Y'=bx+dy

TYPES OF TRANSFORMATIONS…

Translation

Scaling

Rotating

Reflection

Shearing

2D TRANSLATION

It is the straight line movement of an object from one position to another is called Translation.

Here the object is positioned from one coordinate location to another.(Re-position a point along a straight line)

Translation of point:

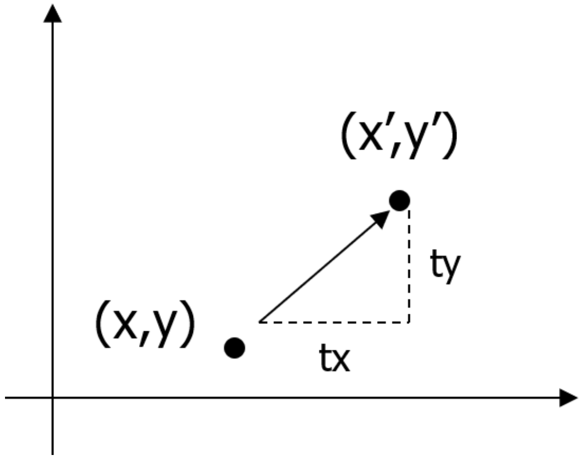
To translate a point from coordinate position (x, y) to another (x_1 y_1), we add algebraically the translation distances T_x and T_y to original coordinate.

The new point: (x', y')

$$x' = x + tx$$

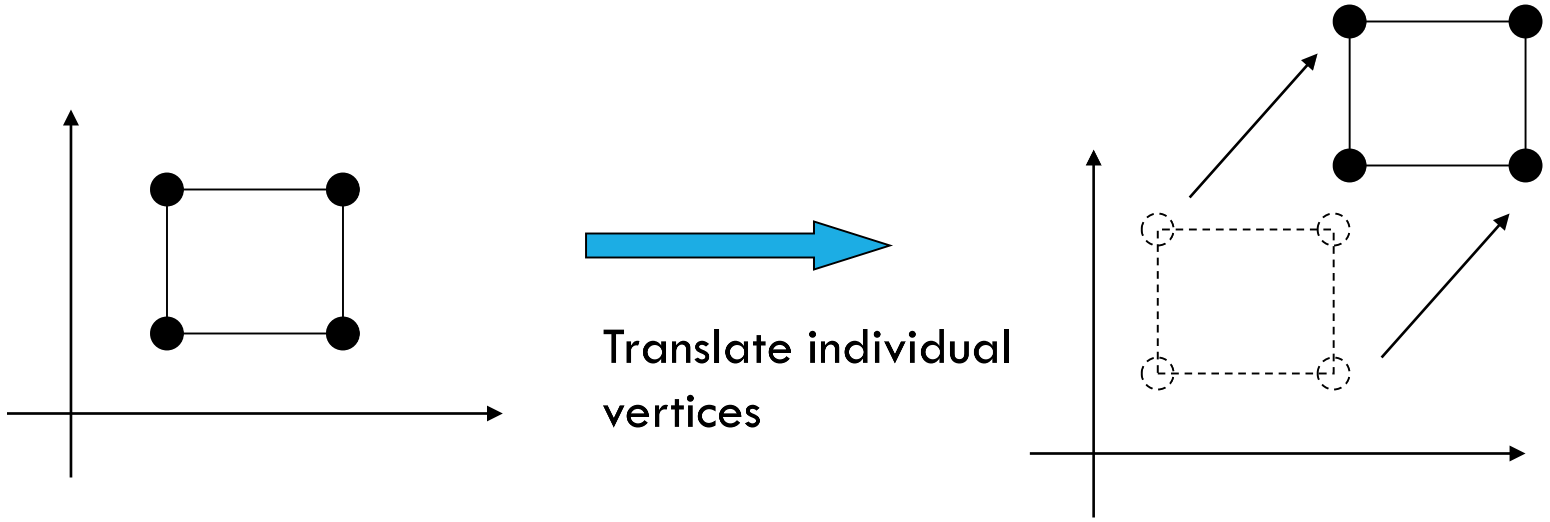
$$y' = y + ty$$

The translation pair (T_x,T_y) is called as shift vector.



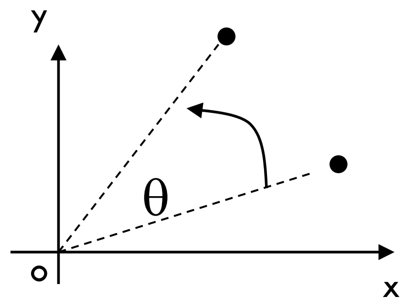
2D TRANSLATION

How to translate an object with multiple vertices?

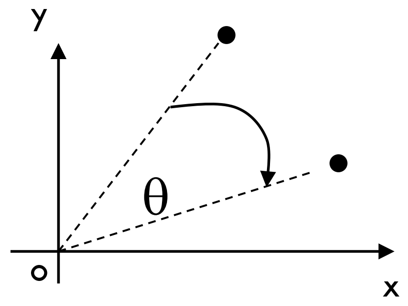


2D ROTATION

Default rotation center: Origin (0,0)



$\theta > 0$: Rotate counter clockwise



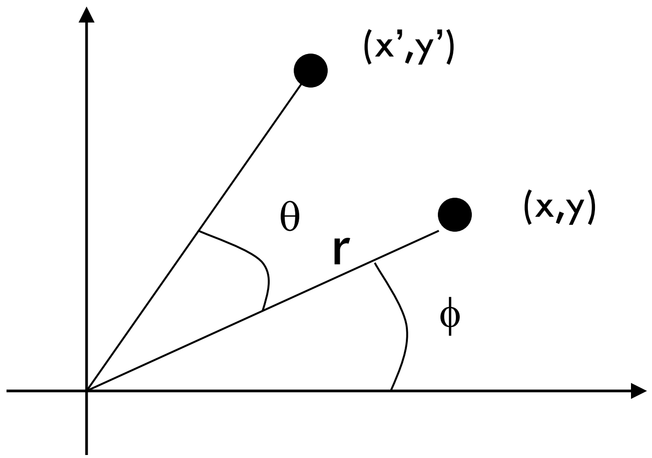
$\theta < 0$: Rotate clockwise

2D ROTATION

(x,y) -> Rotate *about the origin* by θ

$\longrightarrow$ (x', y')

How to compute (x', y') ?



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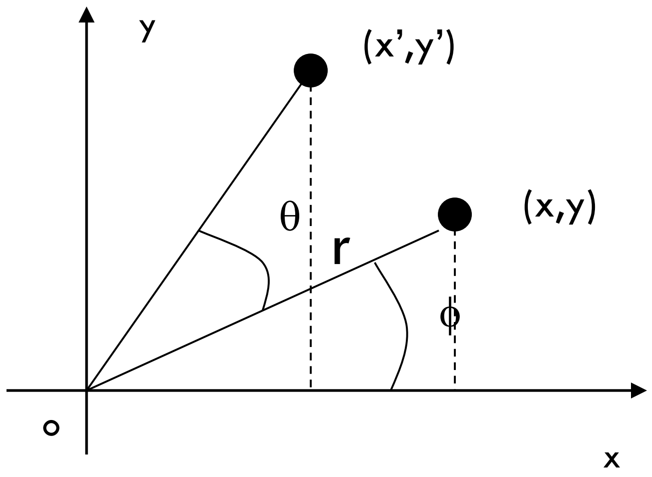


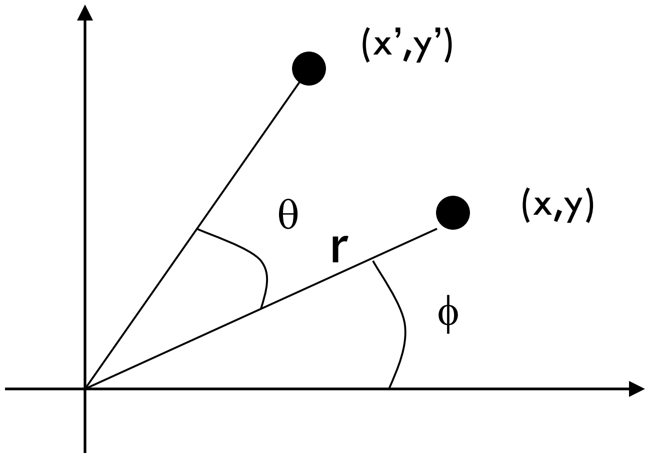
$x = r\cos(\phi) \quad y = r\sin(\phi)$

$x' = r\cos(\phi + \theta) \quad y' = r\sin(\phi + \theta)$

2D ROTATION

$$x = r\cos(\phi) \quad y = r\sin(\phi)$$

$$x' = r\cos(\phi + \theta) \quad y = r\sin(\phi + \theta)$$



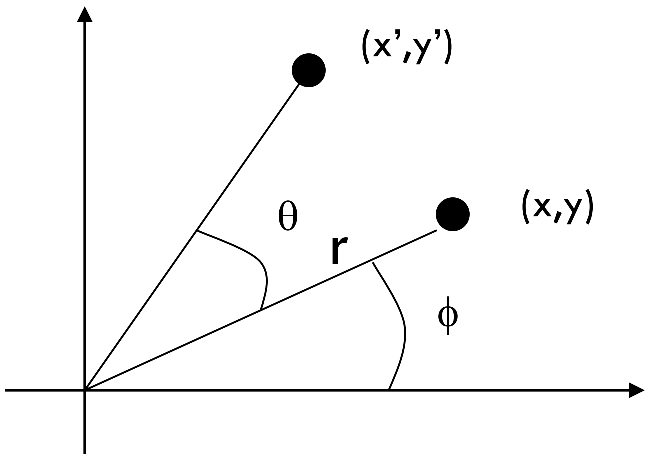
$$\begin{aligned} x' &= r\cos(\phi + \theta) \\ &= r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta) \\ &= x\cos(\theta) - y\sin(\theta) \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\phi + \theta) \\ &= r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta) \\ &= y\cos(\theta) + x\sin(\theta) \end{aligned}$$

2D ROTATION

$$x' = x\cos(\theta) - y\sin(\theta)$$

$$y' = y\cos(\theta) + x\sin(\theta)$$

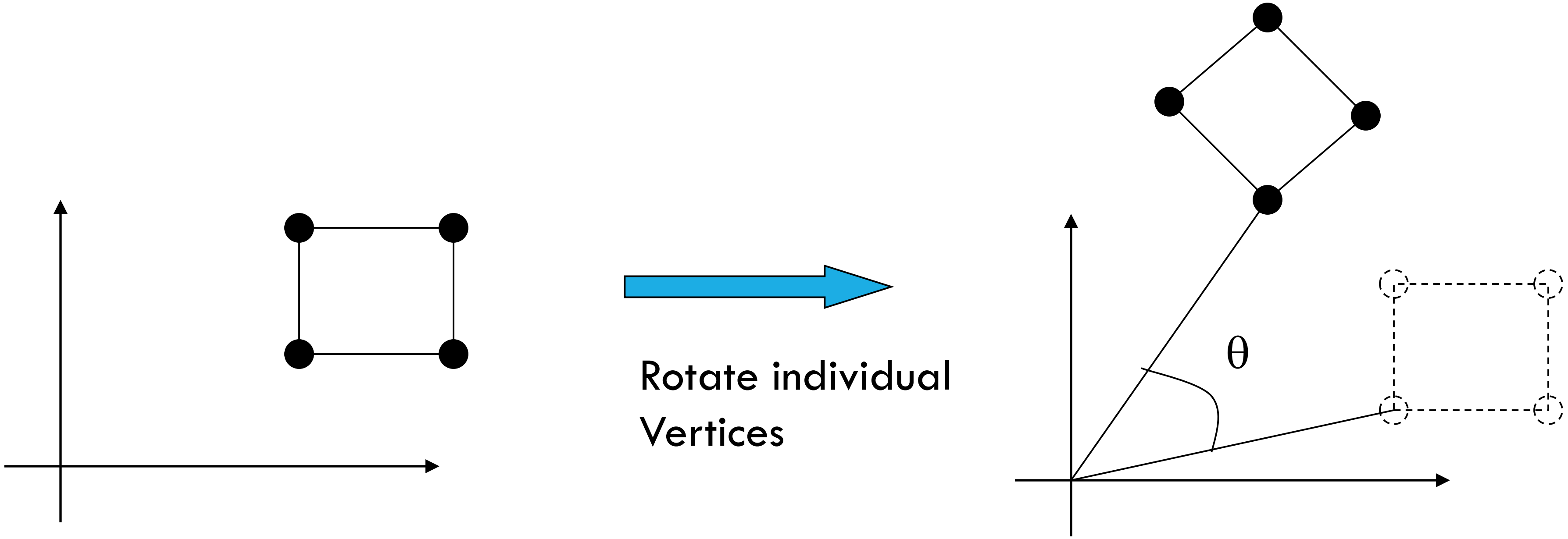


Matrix form?

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$

2D Rotation

- How to rotate an object with multiple vertices?

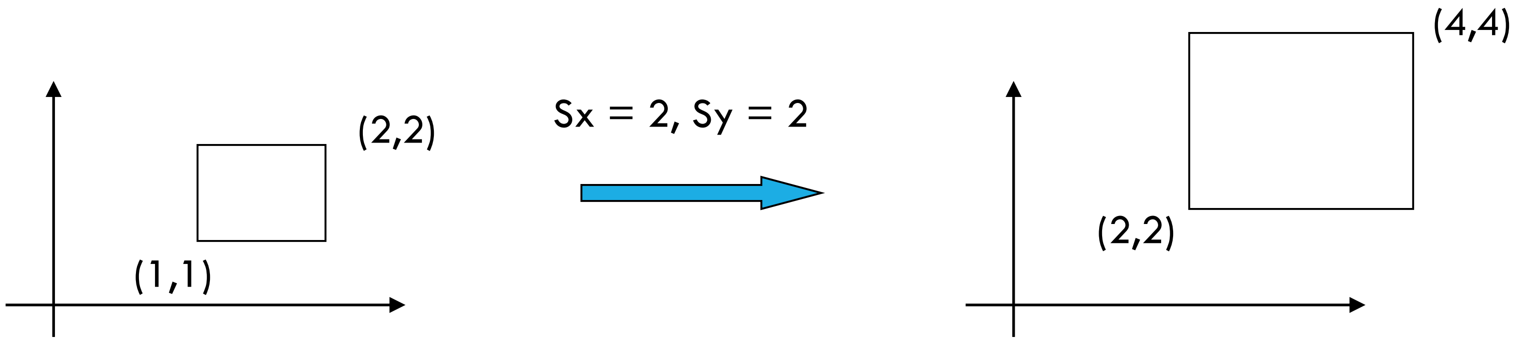


2D SCALING

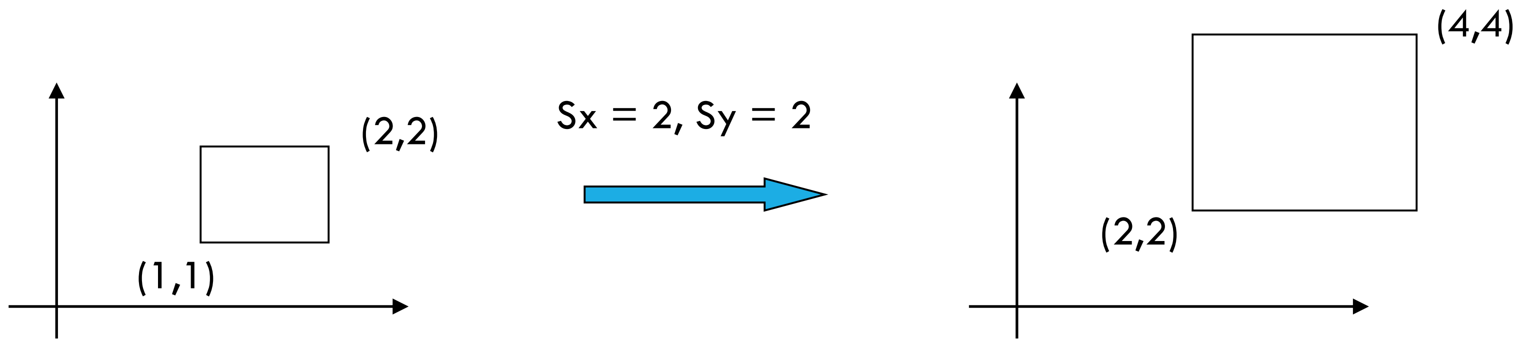
Scale: Alter the size of an object by a scaling factor (Sx, Sy), i.e.

$$x' = x \cdot Sx$$
$$y' = y \cdot Sy$$

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} Sx & 0 \\ 0 & Sy \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$$



2D SCALING



- Not only the object size is changed, it also moved!!
- Usually this is an undesirable effect
- We will discuss later (soon) how to fix it

SCALE FACTORS AFFECT SIZE AS FOLLOWING:

if $sx = sy$ uniform scaling

If $sx \neq sy$ nonuniform scaling

If $sx, sy < 1$, size is reduced, object moves closer to origin

If $sx, sy > 1$, size is increased, object moves further from origin

If $sx = sy = 1$, size does not change

SIMPLE NUMERICAL BASED ON AFFINE TRANSFORMATION 1…

1. Given a square with coordinate points A(0, 3), B(3, 3), C(3, 0), D(0, 0). Apply the translation with distance 1 towards X axis and 1 towards Y axis. Obtain the new coordinates of the square.

2. Given a circle C with radius 10 and center coordinates (1, 4). Apply the translation with distance 5 towards X axis and 1 towards Y axis. Obtain the new coordinates of C without changing its radius.

3. Given a line segment with starting point as (0, 0) and ending point as (4, 4). Apply 30 degree rotation anticlockwise direction on the line segment and find out the new coordinates of the line.

4. Given a triangle with corner coordinates (0, 0), (1, 0) and (1, 1). Rotate the triangle by 90 degree anticlockwise direction and find out the new coordinates.

5. Given a square object with coordinate points A(0, 3), B(3, 3), C(3, 0), D(0, 0). Apply the scaling parameter 2 towards X axis and 3 towards Y axis and obtain the new coordinates of the object.

6. find transformation matrix for

1. Rotation of 90° counter clockwise about the origin.
2. Rotation of 180° counter clockwise about the origin .
3. Rotation of 270° counter clockwise about the origin.

7. A unit square is transformed by a 2×2 transformation matrix. The resulting position vectors are:

$$[x'] = \begin{bmatrix} 0 & 0 \\ 2 & 3 \\ 8 & 4 \\ 6 & 1 \end{bmatrix}$$

Determine the transformation matrix used.

***Some answers are on next slide**

2. $$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

7. A A unit square is transformed by a 2×2 transformation matrix. The resulting position vectors are:

$$[x'] = \begin{bmatrix} 0 & 0 \\ 2 & 3 \\ 8 & 4 \\ 6 & 1 \end{bmatrix}$$

Determine the transformation matrix used.

Ans:

$$[x'] = [X][T] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \\ a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 3 \\ 8 & 4 \\ 6 & 1 \end{bmatrix}; \therefore [T] = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$$

6.

Some Common Cases of Rotation

Rotation of 90° counter clockwise about the origin

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Rotation of 180° counter clockwise about the origin

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotation of 270° counter clockwise about the origin

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In all the above cases det[T]=1

3. $$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} \cos30 & -\sin30 \\ \sin30 & \cos30 \end{bmatrix} \times \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 4 \times \cos30 - 4 \times \sin30 \\ 4 \times \sin30 + 4 \times \cos30 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 4 \times \cos30 - 4 \times \sin30 \\ 4 \times \sin30 + 4 \times \cos30 \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1.46 \\ 5.46 \end{bmatrix}$$

PERFORMING ROTATION ABOUT AN ARBITRARY POINT

$$P_x' = Q_x + (P_x - Q_x)\cos\alpha - (P_y - Q_y)\sin\alpha$$

$$P_y' = Q_y + (P_x - Q_x)\sin\alpha + (P_y - Q_y)\cos\alpha$$

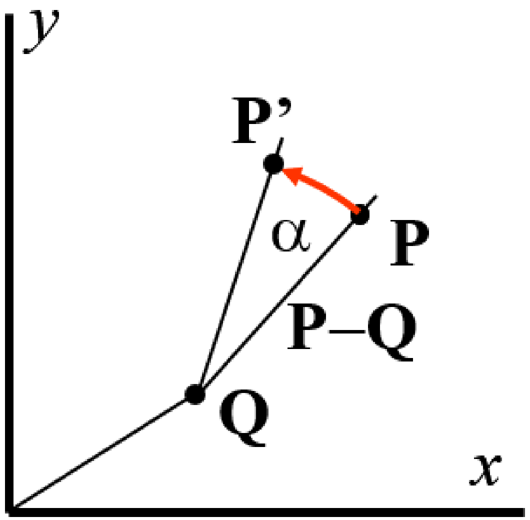


Fig a.

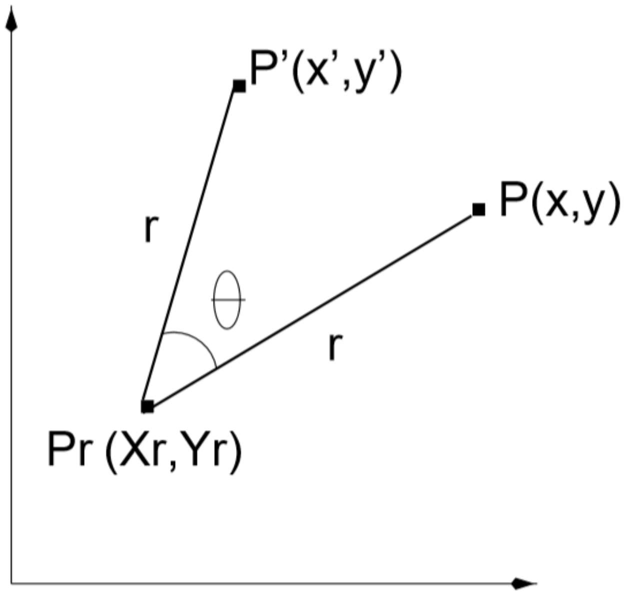
P'x Px Qx

$$x' = x_r + (x - x_r)\cos(\theta) - (y - y_r)\sin(\theta)$$

$$y' = y_r + (x - x_r)\sin(\theta) + (y - y_r)\cos(\theta)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\underline{P' = R(P - P_r) + P_r}$$



*If comparing with fig. a

X=Px Y=Py

X'=P'x Y'=P'y

Xr=Qx Yr=Qy

SCALING WITH RESPECT TO FIXED POINT

Scale with factors s_x and s_y:

$$P_x' = s_x P_x, \quad P_y' = s_y P_y$$

With respect to **F**:

$$P_x' - F_x = s_x (P_x - F_x),$$

$$P_y' - F_y = s_y (P_y - F_y)$$

or

$$P_x' = F_x + s_x (P_x - F_x),$$

$$P_y' = F_y + s_y (P_y - F_y)$$

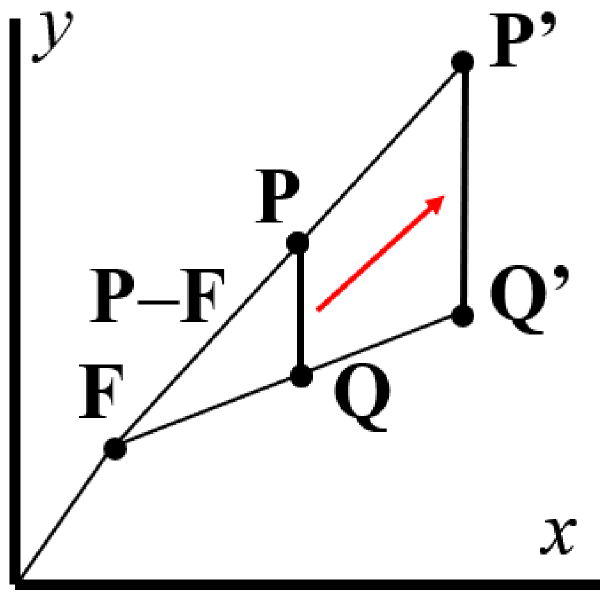


Fig b.

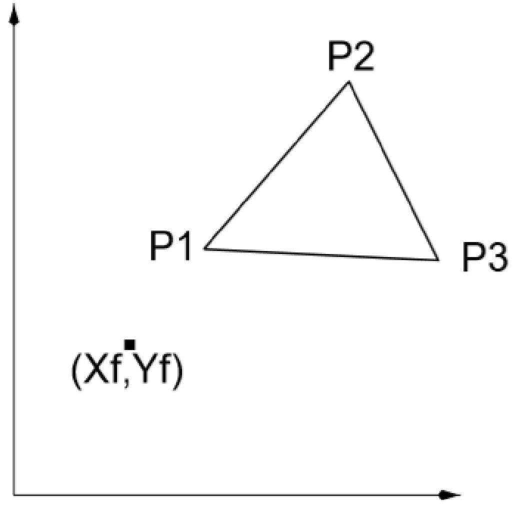
-Control the location of a scaled object by choosing the location of a point (fixed point)with respect to which the scaling is performed

$$x' = x_f + (x - x_f)s_x \quad \text{or} \quad x' = xs_x + x_f(1 - s_x)$$

$$y' = y_f + (y - y_f)s_y \quad \text{or} \quad y' = ys_y + y_f(1 - s_y)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_f(1 - s_x) \\ y_f(1 - s_y) \end{bmatrix}$$

$$\underline{P' = S\ P + T_f}$$



*If comparing with fig. b

X=Px Y=Py

X'=P'x Y'=P'y

Xf=Fx Yf=Fy

SIMPLE NUMERICAL BASED ON AFFINE TRANSFORMATION 2…

8. Prove that 2D rotation and scaling are commutative if $S_x = S_y$ and $\theta = n\pi$.

9. Rotate a triangle with vertices (10,20), (10,10), (20,10) about the origin by 30 degrees in counter clockwise direction and then translate it by $t_x=5$, $t_y=10$.

10.rotate a triangle ABC counter clockwise by 45^0 whose vertices are A(2 3) B(5 5) C(4 3) about point (1 1).

11. Find a coordinates of triangle ABC A(1 3) B(-2 4) C(3 -1) after it is scaled double size keeping coordinate (-2 4) fixed.

***Some answers are on next slide**

ANSWERS..

8. **Solution :** The matrix notation for scaling along S_x and S_y is as given below

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \text{ and}$$

The matrix notation for rotation is as given below.

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$S \cdot R = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & S_x\sin\theta \\ -S_y\sin\theta & S_y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} S_x\cos\theta & S_x\sin\theta \\ -S_x\sin\theta & S_x\cos\theta \end{bmatrix} \qquad \because S_x = S_y \quad \text{...I}$$

or

$$= \begin{bmatrix} -S_x & 0 \\ 0 & -S_y \end{bmatrix} \qquad \because \theta = n\pi \text{ where n is integer ...II}$$

$$R \cdot S = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

$$= \begin{bmatrix} S_x\cos\theta & S_y\sin\theta \\ -S_x\sin\theta & S_y\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} S_x\cos\theta & S_x\sin\theta \\ -S_x\sin\theta & S_x\cos\theta \end{bmatrix} \qquad \because S_x = S_y \quad \text{...III}$$

or

$$= \begin{bmatrix} -S_x & 0 \\ 0 & -S_y \end{bmatrix} \qquad \because \theta = n\pi \text{ where n is integer ...IV}$$

From equations I and III, and equations II and IV it is proved that 2D rotation and scaling commute if $S_x = S_y$ or $\theta = n\pi$ for integral n and that otherwise they do not.

9. The resultant coordinates of the triangle vertices are (3.66,32.32), (8.66,23.66), and (17.32,28.66) respectively.

11. (4 2) (-2 4) (8 -6)

TRANSFORMATIONS…

*Summary of transformations…

Translation: $P'=P + T$ (addition causes problems !!)

Scale: $P'=SP$

Rotation: $P'=RP$

Messy!

Transformations with respect to points: even more messy!

How to combine transformations?

* use homogeneous coordinates to express translation as matrix multiplication

HOMOGENOUS COORDINATES

- Uniform representation of translation, rotation, scaling
- Uniform representation of points and vectors
- Can represent points at infinity
- Compact representation of sequence of transformations

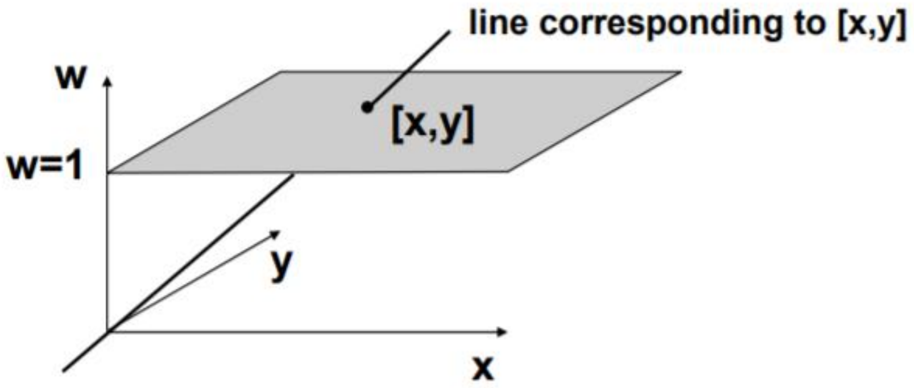
HOMOGENEOUS COORDINATES

Add extra coordinate:

$\mathbf{x} = (x, y, h)$

Cartesian coordinates: divide by h

$\mathbf{x} = (x/h, y/h)$



From homogeneous to 2d: [x,y,w] becomes [x/w,y/w]
From 2d to homogeneous: [x,y] becomes [kx,ky,k]
(can pick any nonzero k!)

- (x, y) can be represented by an infinite number of points in homogeneous coordinates

If $w = 6$, $(1/3, 1/2) \dashrightarrow (2, 3, 6)$

If $w = 12$, $(1/3, 1/2) \dashrightarrow (4, 6, 12)$

TRANSLATION MATRIX

Translation :

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

or

$$\mathbf{P}' = \mathbf{T}(t_x, t_y)\mathbf{P}$$

ROTATION MATRIX

Rotation :

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

or

$$\mathbf{P}' = \mathbf{R}(\theta)\mathbf{P}$$

SCALING MATRIX

$$\text{Scaling}:$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

or

$$\mathbf{P}' = \mathbf{S}(s_x, s_y)\mathbf{P}$$

SIMPLE NUMERICAL BASED ON HOMOGENEOUS TRANSFORMATION…

12. The unit square with vertices (1 1) , (2 1) , (2 2) , and (1 2) is scaled about the origin by factors of 4 and 2 in the x- and y- directions, respectively. Find the vertices of the resultant square.

13. Rotate a triangle with vertices A(2 2) B(4 2) C(4 4) by 90° about origin in counter clockwise direction.

Answer 12. The unit square with vertices $\binom{1}{1}$, $\binom{2}{1}$, $\binom{2}{2}$, and $\binom{1}{2}$ is scaled about the origin by factors of 4 and 2 in the x- and y- directions, respectively. We have

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 8 & 4 \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So the image is a square with vertices $\binom{4}{2}$, $\binom{8}{2}$, $\binom{8}{4}$, and $\binom{4}{4}$.

Answer 13. New coordinates A'(2 2) B'(-2 4) and C'(-4 4)

COMBINING TRANSFORMATIONS…

$$\mathbf{P}^{'} = \mathbf{M}_1\mathbf{P}$$ first transformation...

$$\mathbf{P}^{''} = \mathbf{M}_2\mathbf{P}^{'}$$ second transformation...

Combined:

$$\begin{aligned}\mathbf{P}^{''} &= \mathbf{M}_2(\mathbf{M}_1\mathbf{P}) \\ &= \mathbf{M}_2\mathbf{M}_1\mathbf{P} \\ &= \mathbf{MP} \quad \text{with } \mathbf{M} = \mathbf{M}_2\mathbf{M}_1\end{aligned}$$

Order of multiplication ←

SUCCESSIVE TRANSLATIONS...

$\mathbf{P}' = \mathbf{T}(t_{1x}, t_{1y})\mathbf{P}$ first translation

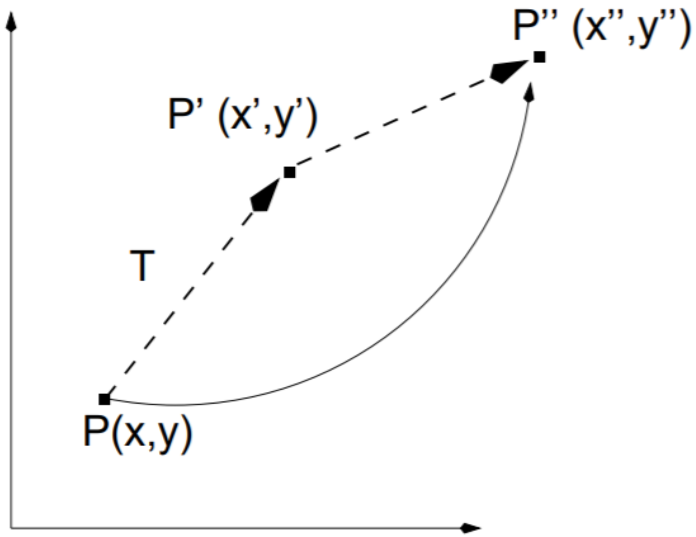
$\mathbf{P}'' = \mathbf{T}(t_{2x}, t_{2y})\mathbf{P}'$ second translation

Combined:

$$\mathbf{P}'' = \mathbf{T}(t_{2x}, t_{2y})\mathbf{T}(t_{1x}, t_{1y})\mathbf{P}$$

$$= \begin{pmatrix} 1 & 0 & t_{2x} \\ 0 & 1 & t_{2y} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & t_{1x} \\ 0 & 1 & t_{1y} \\ 0 & 0 & 1 \end{pmatrix}\mathbf{P} = \begin{pmatrix} 1 & 0 & t_{1x} + t_{2x} \\ 0 & 1 & t_{1y} + t_{2y} \\ 0 & 0 & 1 \end{pmatrix}\mathbf{P}$$

$$= \mathbf{T}(t_{1x} + t_{2x}, t_{1x} + t_{2y})\mathbf{P}$$



SUCCESSIVE SCALING…

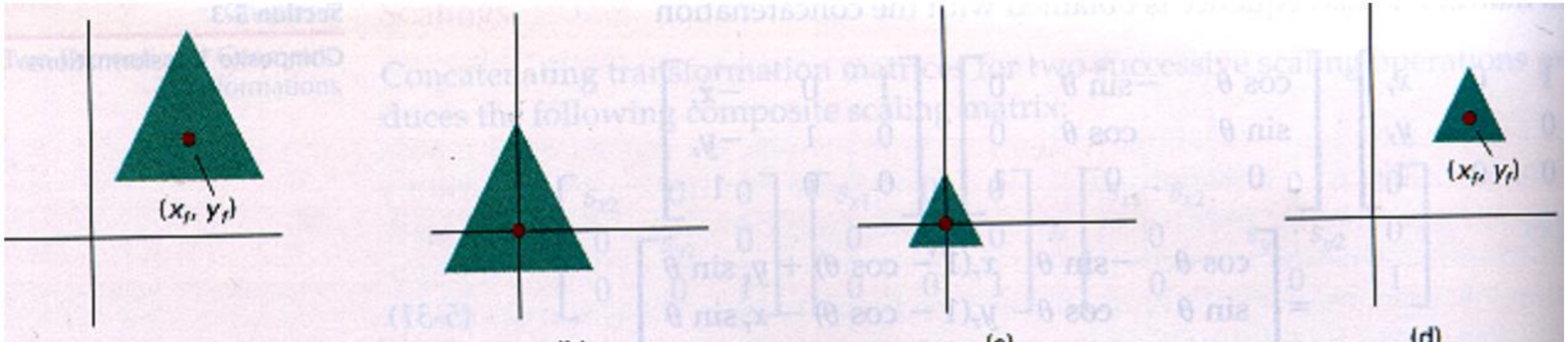
$$P' = S(s_{x_1}, s_{y_1})\ P, \quad P'' = S(s_{x_2}, s_{y_2})\ P'$$

$$\text{Thus, } P'' = S(s_{x_2}, s_{y_2})S(s_{x_1}, s_{y_1})\ P = S(s_{x_1}s_{x_2}, s_{y_1}s_{y_2})\ P$$

$$\begin{bmatrix} s_{x_2} & 0 & 0 \\ 0 & s_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x_1} & 0 & 0 \\ 0 & s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x_2}s_{x_1} & 0 & 0 \\ 0 & s_{y_2}s_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SCALING ABOUT FIXED POINT

[T]=
$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix}$$



(a) Original Position of Object and Fixed Point

(b) Translate Object so that Fixed Point (x_f, y_f) Is at Origin

(c) Scale Object with Respect to Origin

(d) Translate Object so that the Fixed Point Is Returned to Position (x_f, y_f)

SUCCESSIVE ROTATION…

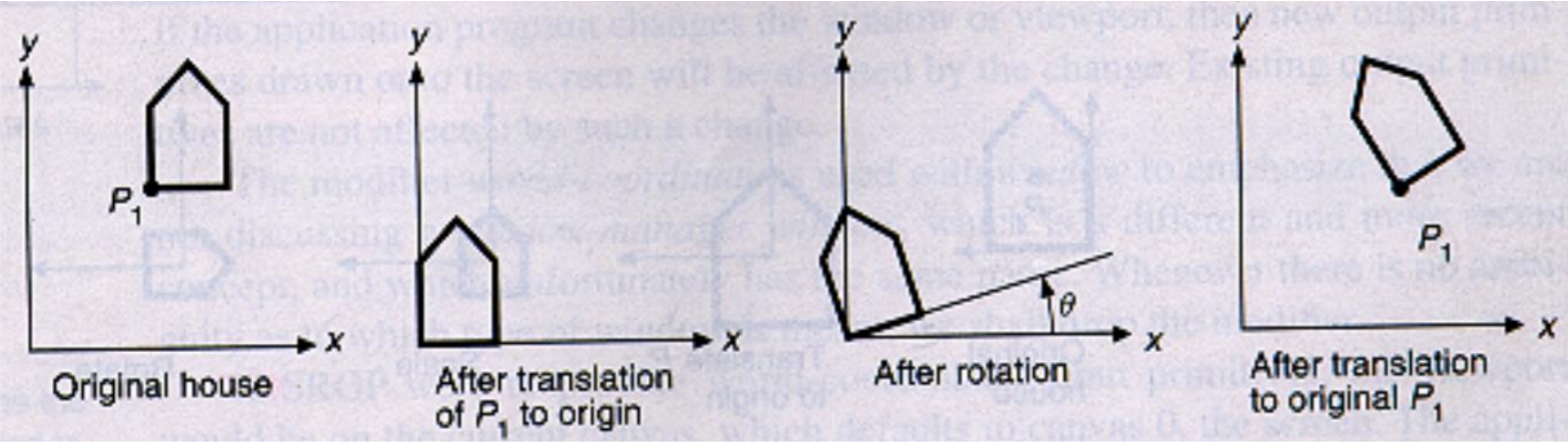
- Successive rotations

$$P' = R(\theta_1)\ P, \quad P'' = R(\theta_2)\ P'$$

$$\text{Thus, } P'' = R(\theta_1)R(\theta_2)\ P = R(\theta_1 + \theta_2)\ P$$

ROTATION ABOUT AN ARBITRARY POINT

$$[T]= \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$



COMBINING TRANSFORMATION (SAME KIND)…

Composite translations :

$$\mathbf{T}(t_{2x}, t_{2y})\mathbf{T}(t_{1x}, t_{1y}) = \mathbf{T}(t_{1x} + t_{2x}, t_{1x} + t_{2y})$$

Composite rotations :

$$\mathbf{R}(\theta_2)R(\theta_1) = \mathbf{R}(\theta_1 + \theta_2)$$

Composite scaling :

$$\mathbf{S}(s_{2x}, s_{2y})\mathbf{S}(s_{1x}, s_{1y}) = \mathbf{S}(s_{1x}s_{2x}, s_{1y}s_{2y})$$

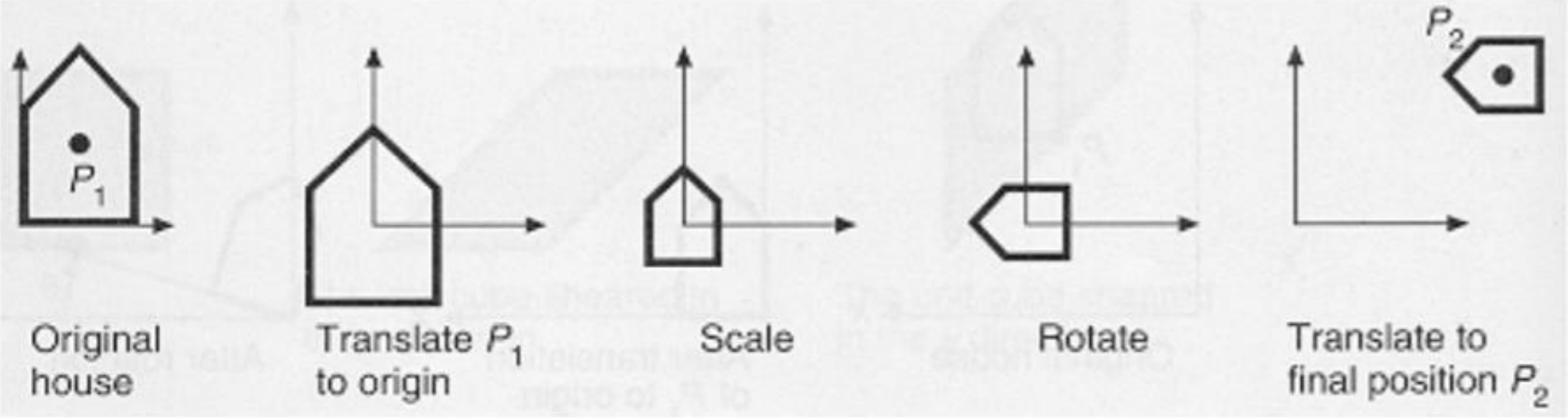
COMBINING TRANSFORMATION (DIFFERENT)...

The transformation matrices of a series of transformations can be concatenated into a single transformation matrix

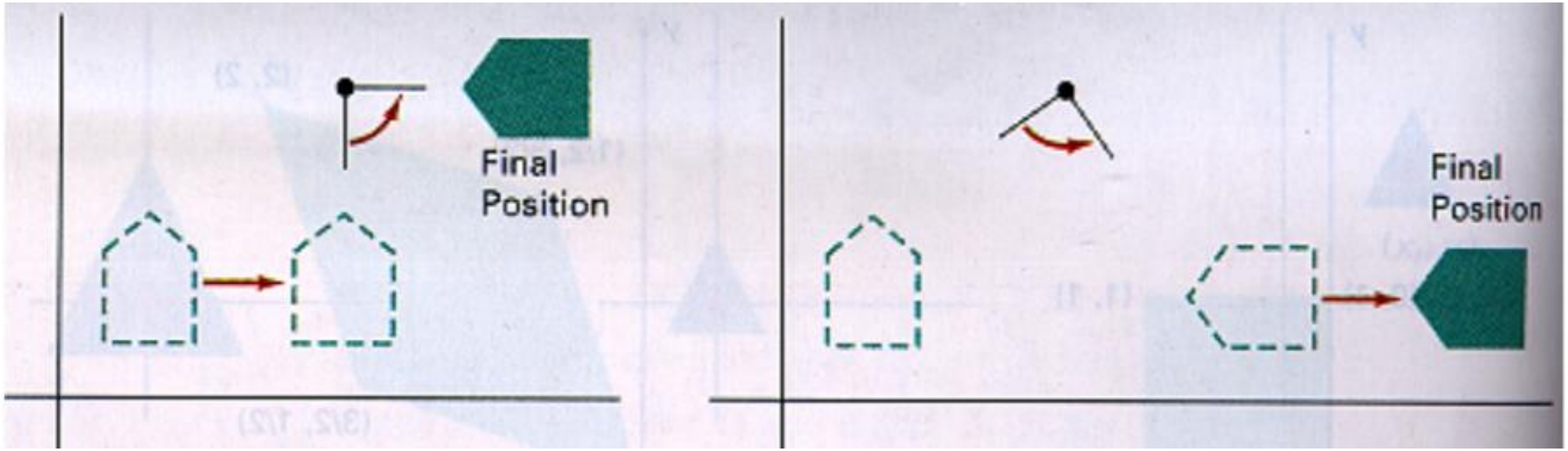
Example (Steps):

1. Translate to origin
2. Perform scaling and rotation
3. Translate to fixed point

* It is important to reserve the order in which a sequence of transformations is performed !!



It is important to reserve the order in which a sequence of transformations is performed !!



SIMPLE NUMERICAL BASED ON HOMOGENEOUS TRANSFORMATION (COMBINATION)...

12.Conduct a combination of transformations in sequence for a triangle whose vertices are (1,1) ,(1,3) and (5,1).

Translate the right-angle vertex to the origin ($Tx = -1$, $Ty = -1$) and then rotate it counter clockwise direction by 45°

Also plot graph for the same.

13. translate the square ABCD with co ordinates (0 0) (5 0) (5 5) (0 5) respectively by 5 units in both directions and then scale it by 2 times in x direction and 1.5 times in y direction.

14. A square has vertices p1 = (1 1) , p2 = (2 1) , p3 = (2 2) , and p4 = (1 2) . Determine the new vertices of the square after a rotation about p2 through an angle of $\pi/4$.

ANSWERS…

Answer 12.

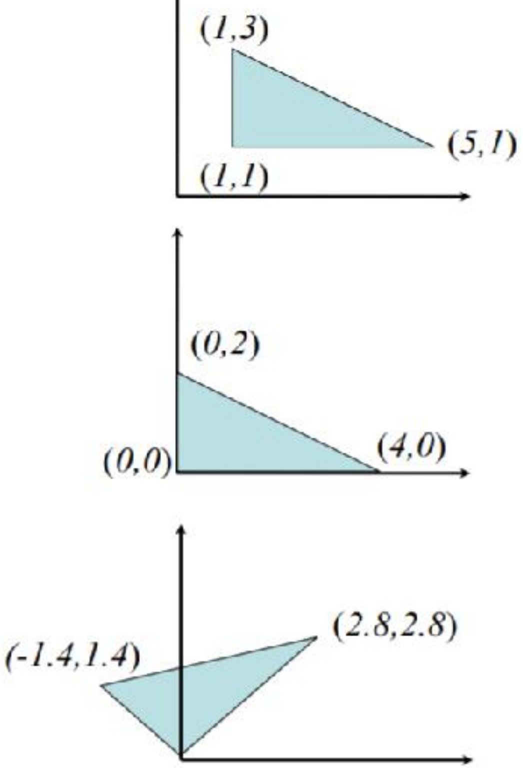
- Conduct a combination of transformations in sequence
  - Translate the right-angle vertex to the origin ($Tx = -1$, $Ty = -1$)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}.$$

  - Rotate 45º ($^{\pi}/_{4}$ radian)

    $\sin {}^{\pi}/_{4} = \cos {}^{\pi}/_{4} = 0.7071$

$$\begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$



OTHER 2D TRANSFORMATIONS…

Reflection:

It is a transformation which produces a mirror image of an object. The mirror image can be either about x-axis or y-axis. The object is rotated by180°.

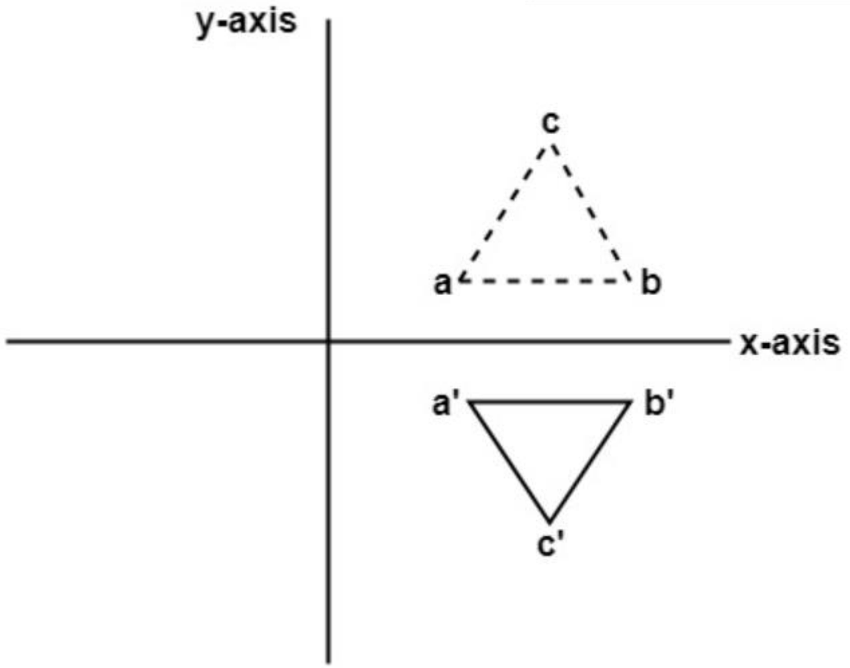
Types of Reflection:

- Reflection about the x-axis
- Reflection about the y-axis
- Reflection about an axis perpendicular to xy plane and passing through the origin
- Reflection about line y=x

REFLECTION ABOUT X-AXIS:

The object can be reflected about x-axis with the help of the following matrix

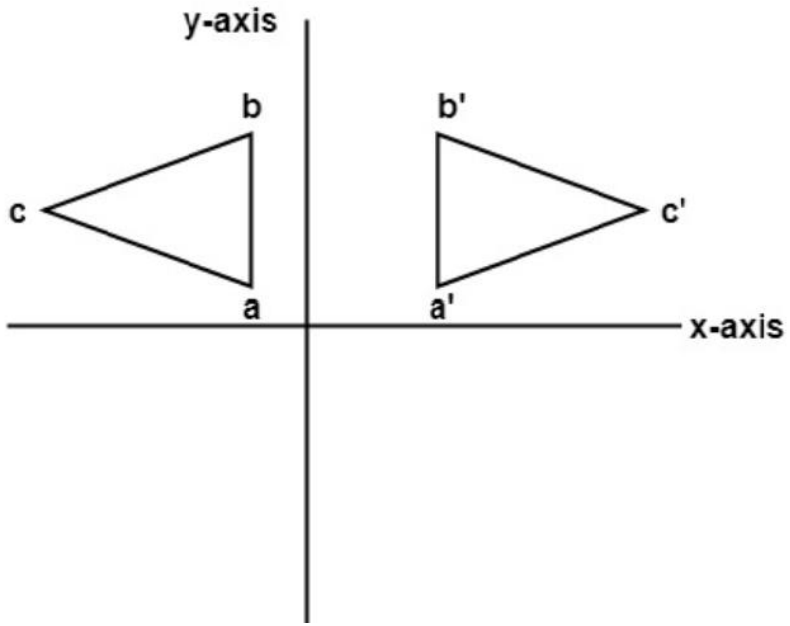
$$\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



REFLECTION ABOUT Y-AXIS

The object can be reflected about y-axis with the help of following transformation matrix

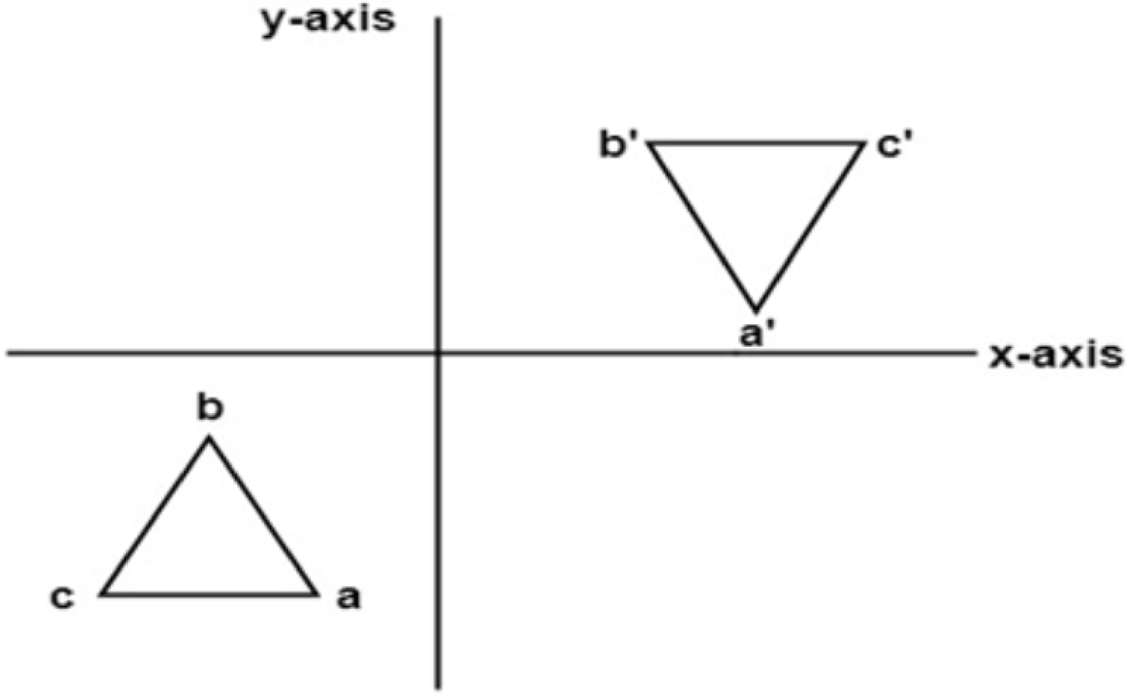
$$\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



REFLECTION ABOUT AN AXIS PERPENDICULAR TO XY PLANE AND PASSING THROUGH ORIGIN:

Reflection about an axis perpendicular to xy plane and passing through origin:
In the matrix of this transformation is given below

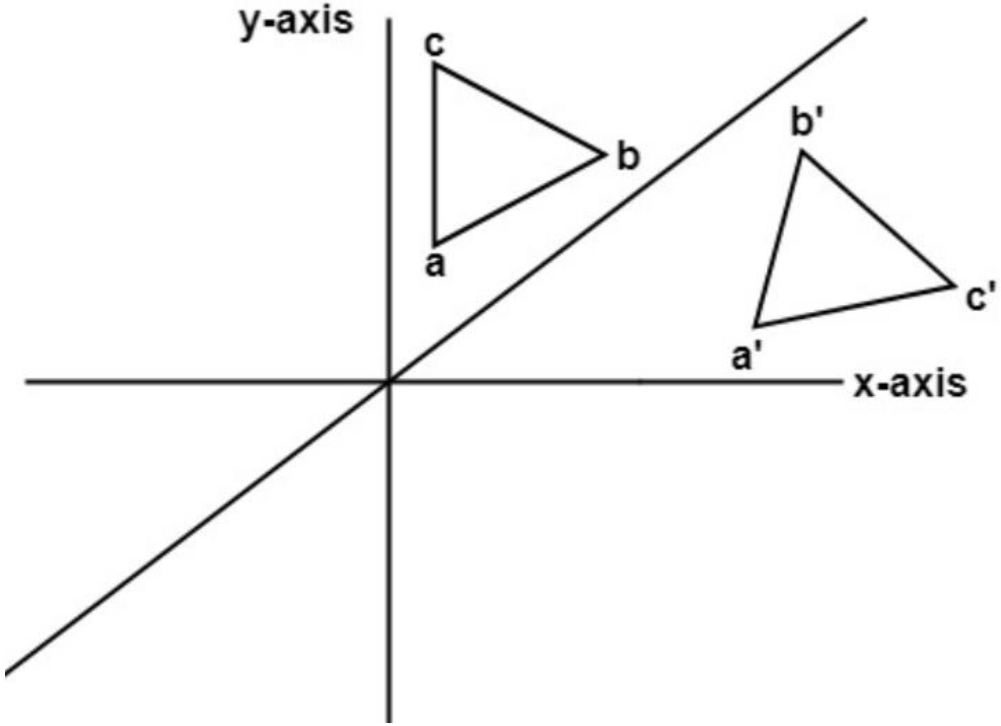
$$\begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



REFLECTION ABOUT LINE Y=X

The object may be reflected about line y = x with the help of following transformation matrix

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$



SHEARING

Shearing in the X-direction: The homogeneous matrix for shearing in the x-direction is shown below:

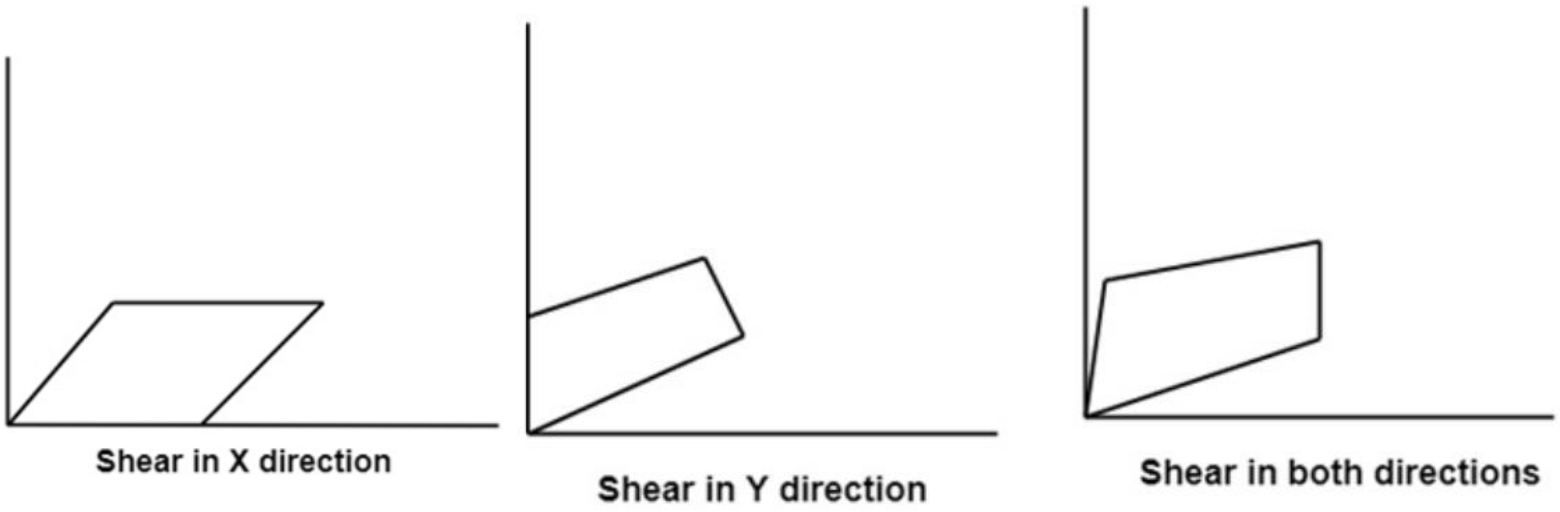
$$\begin{matrix} 1 & 0 & 0 \\ sh_x & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Shearing in the Y-direction: Here shearing is done by sliding along vertical or y-axis.

$$\begin{matrix} 1 & sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Shearing in X-Y directions:

$$\begin{matrix} 1 & sh_y & 0 \\ sh_x & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



SIMPLE NUMERICAL BASED ON HOMOGENEOUS TRANSFORMATION (COMBINATION)...

15. Rotate a triangle with vertices (10,20), (10,10), (20,10) about the origin by 30 degrees and then translate it by tx=5, ty=10.

16. Consider a square A(1 1) B(2 1) C(2 2) D(1 2) rotate it about the point B(2 1) through an angle of $\pi/4$ in clockwise direction.

17. The reflection along the line y=x is equivalent to the reflection along the X-axis followed by counter clockwise rotation by Θ degrees. Find the value of Θ.

18. Perform X-shear & Y-shear on a triangle having A(2,1), B(4,3),C (2,3).Consider the constant value b = c = 2.

19. A triangle is defined by A(2 2) B(4 2) c(4 4). Find the transformed coordinates after rotating it by 90° about origin and Reflecting it about line y = -x.

20. Give a 3x3 homogeneous matrix to rotate the image clockwise by 90°.Then shift the image to right by 10 units. Finally scale the image by twice as large.

ANSWERS..

14. A square has vertices $\boldsymbol{p}_1 = \binom{1}{1}$, $\boldsymbol{p}_2 = \binom{2}{1}$, $\boldsymbol{p}_3 = \binom{2}{2}$, and $\boldsymbol{p}_4 = \binom{1}{2}$. Determine the new vertices of the square after a rotation about $\boldsymbol{p}_2$ through an angle of $\pi/4$. The transformation matrix is

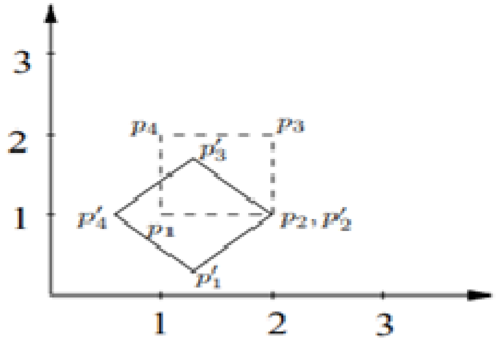


Figure 1: Rotation about p_2.

$$\mathrm{Rot}_{(2,1)}(\pi/4) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2-\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2}+1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Apply the transformation above to the homogeneous coordinates of the vertices:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2-\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2}+1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2-\frac{\sqrt{2}}{2} & 2 & 2-\frac{\sqrt{2}}{2} & 2-\sqrt{2} \\ 1-\frac{\sqrt{2}}{2} & 1 & 1+\frac{\sqrt{2}}{2} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1.2929 & 2 & 1.2929 & 0.5858 \\ 0.2929 & 1 & 1.7071 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Thus the new vertices are $\boldsymbol{p}_1' \approx \binom{1.2929}{0.2929}$, $\boldsymbol{p}_2' = \binom{2}{1}$, $\boldsymbol{p}_3' \approx \binom{1.2929}{1.7071}$, and $\boldsymbol{p}_4' \approx \binom{0.5858}{1.0}$, as illustrated in Figure 1.

REFLECTION ABOUT GIVEN LINE

Equation of given line :y=mx+c , where m=slope of line and c=y-intercept

Steps

Translate
- Subtract distance To bring y/x intercept to origin.

Rotate by angle Θ
- find using slop equation..i.e m=tan Θ.

Reflect
- Reflect using matrix for x/y axis.

Inverse rotate by Θ
- Inverse of previous rotation by Θ.

Inverse translate
- Inverse of previous translate

INVERSE TRANSFORMATIONS

Translation :

$$\mathbf{T}^{-1}(t_x, t_y) = \mathbf{T}(-t_x, -t_y)$$

Rotation :

$$\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$$

Scaling :

$$\mathbf{S}^{-1}(s_x, s_y) = \mathbf{S}\left(\frac{1}{s_x}, \frac{1}{s_y}\right)$$

NUMERICAL....

21. Triangle ABC with coordinates A(2 3) B(6 3) and C(4 8) is reflected along line $y=3x+4$. Find the coordinates of the resultant triangle.

22. reflect point (3 4) along straight line $y=-2x+6$.

TRANSFORMATIONS BETWEEN COORDINATE SYSTEMS

COORDINATE SYSTEMS

Screen Coordinates : The coordinate system used to address the screen (device coordinates)

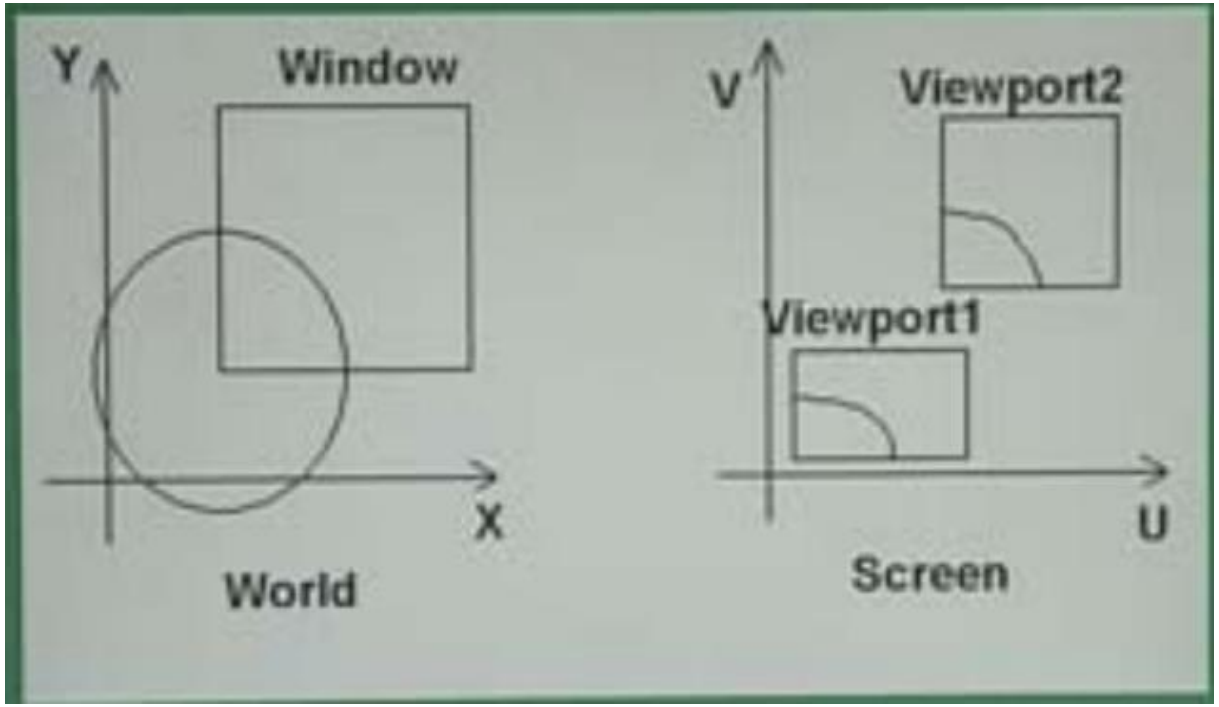
World Coordinates: A user-defined application specific coordinate system having its own units of measure , axis, origin, etc.

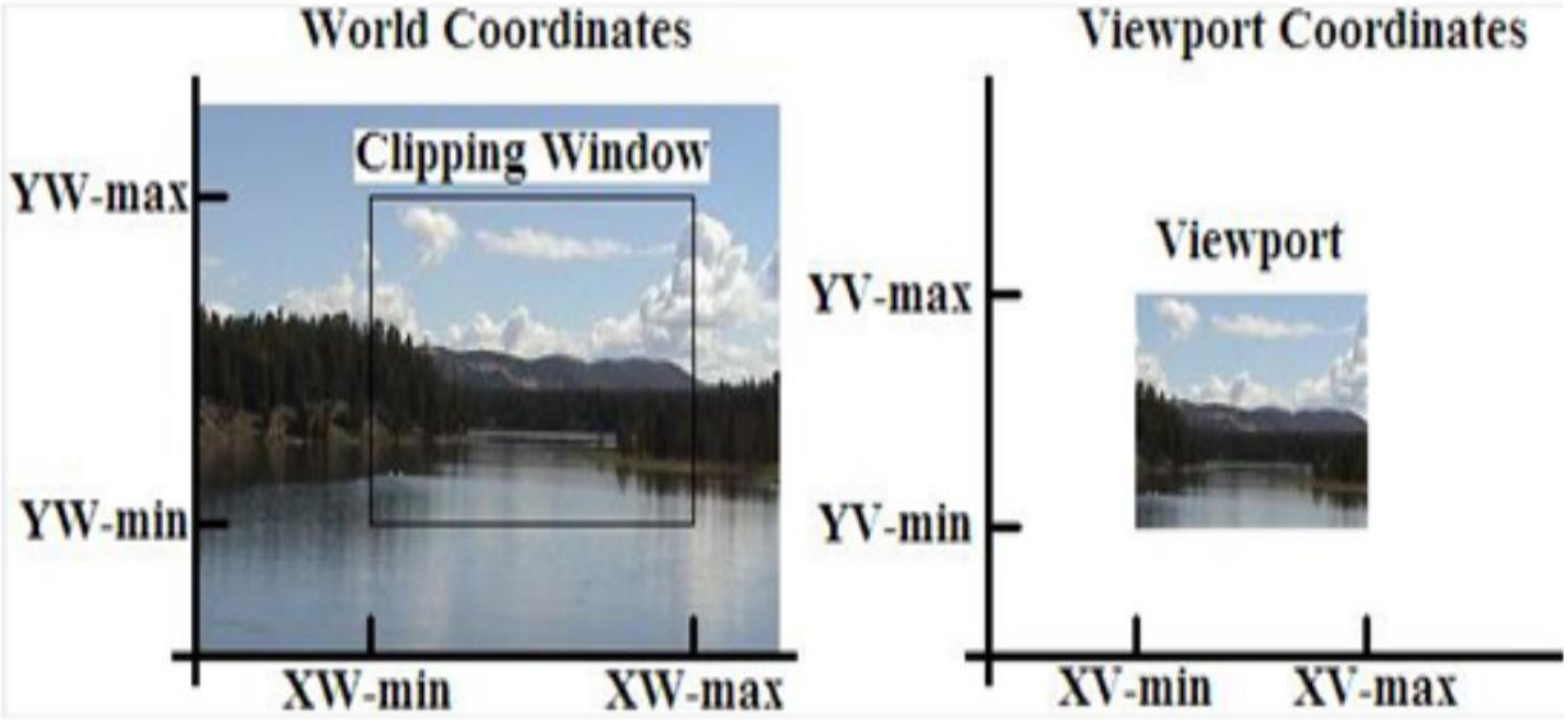
Window: The rectangular region of the world that is visible.

Viewport: The rectangular region of the screen space that is used to display the window.

WINDOW TO VIEWPORT TRANSFORMATION

Purpose is to find the transformation matrix that maps the window in world coordinates to the viewport in screen coordinates.




World Coordinates
Viewport Coordinates
Clipping Window
YW-max
YW-min
XW-min
XW-max
Viewport
YV-max
YV-min
XV-min
XV-max

THE OVERALL TRANSFORMATION:

- Translate the window to the origin
- Scale it to the size of the viewport
- Translate it to the viewport location
- Window: (x, y space) denoted by:

  $x_{min}, y_{min}, x_{max}, y_{max}$
- Viewport: (u, v space) denoted by:

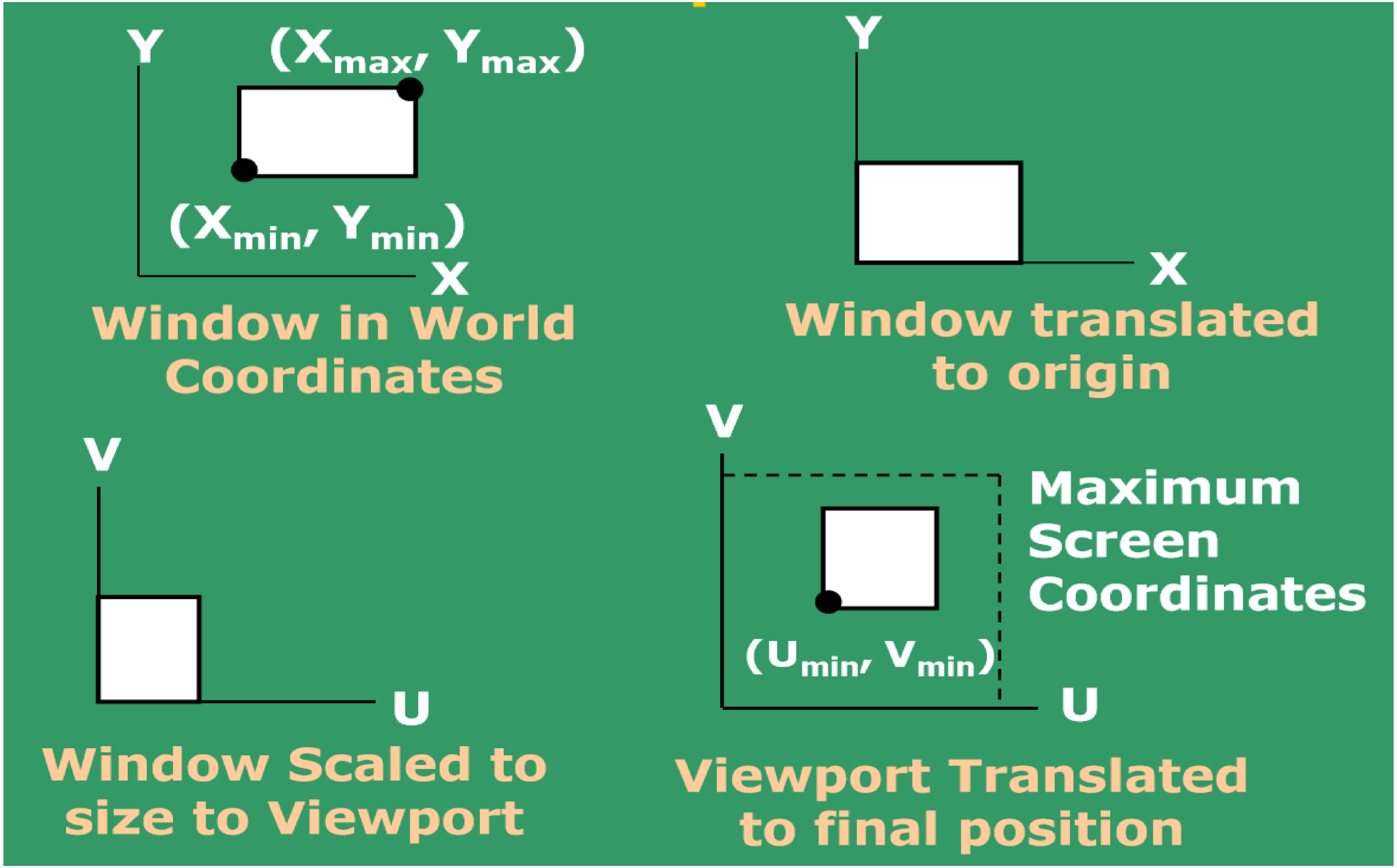
  $u_{min}, v_{min}, u_{max}, v_{max}$

$$M_{WV} = T(U_{\min}, V_{\min}) * S(S_x, S_y) * T(-x_{\min}, -y_{\min});$$

$$S_x = (U_{\max} - U_{\min}) / (x_{\max} - x_{\min});$$

$$S_y = (V_{\max} - V_{\min}) / (y_{\max} - y_{\min});$$

$$M_{WV} = \begin{bmatrix} S_x & 0 & (-x_{\min} * S_x + U_{\min}) \\ 0 & S_y & (-y_{\min} * S_y + V_{\min}) \\ 0 & 0 & 1 \end{bmatrix}$$


Y
(X_{max}, Y_{max})
(X_{min}, Y_{min})
X
Window in World Coordinates
Y
X
Window translated to origin
V
U
Window Scaled to size to Viewport
V
Maximum Screen Coordinates
(U_{min}, V_{min})
U
Viewport Translated to final position

THANK YOU !!!