Computing and Numerical Methods Report:

Case Study of Numerical Schemes for solving second-order ODE's

Q1(a)

$$\frac{d^2y}{dt^2} + \omega^2 y(t) = 0 \qquad (1)$$

The Analytical solution to the equation above can be found by considering harmonic functions as solutions, using the complex exponential:

$$y(t) = Ae^{(\omega t)i} \qquad (2)$$

$$\frac{d^2y}{dt^2} = -A\omega^2 e^{(\omega t)i} \qquad (3)$$

Evaluating our equation by substituting our complex exponential,

$$-A\omega^2 e^{(\omega t)i} + A\omega^2 e^{(\omega t)i} = 0 \qquad (4)$$

So, our complex exponential satisfies the second order ODE. Rewriting using Eulers identity:

$$y(t) = Ae^{(\omega t)i} = Bcos(\omega t + \varphi)$$

$$= Bcos\left(\sqrt{\frac{\rho_l g}{\rho L}}t + \varphi\right) \qquad (5)$$

Q1(b)

We arrive at the constants for our model:

$$\omega^2 = 19.62 \text{ rad}^2/\text{s}^2, \quad B = 0 \text{ m}$$

Hence giving the equation and graph below for the interval t: $[0,10]$,

$$y(t) = 0.1cos(4.4294t) \qquad (7)$$

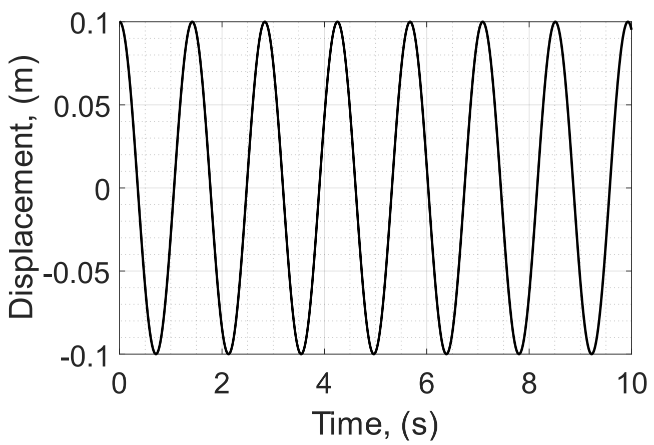

Figure 1: Analytical Solution

Q1(c)

$$\dot{\mathbf{Y}} = \mathbf{AY} \qquad (8)$$

$$\begin{bmatrix} \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} \qquad (9)$$

Q1(d) Eigenvalues:

$$\lambda_1 = \pm 4.4294i$$

Q2(a)

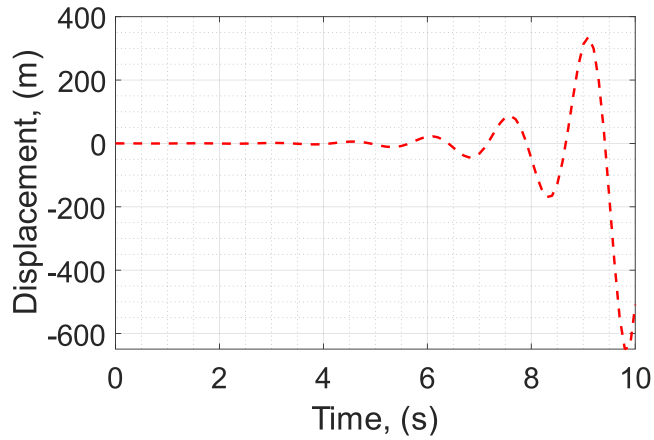

Figure 2: Explicit Euler Scheme

Q2(b)

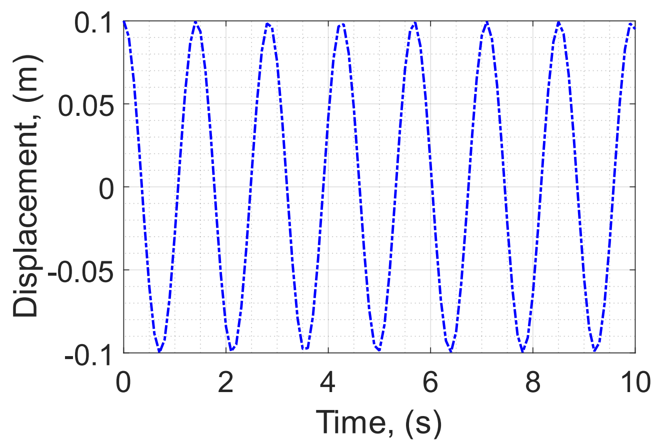

Figure 3: Runge-Kutta, 4th Order Scheme

Q2(c)

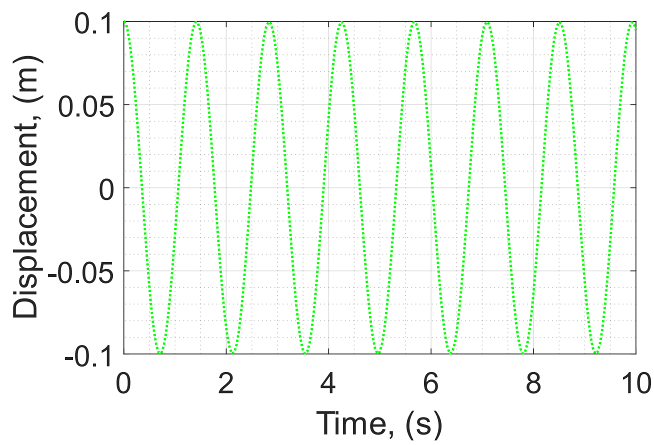

Figure 4: MATLAB™ ode45 in-built function

Q2(d) Comparison + plot of all on one

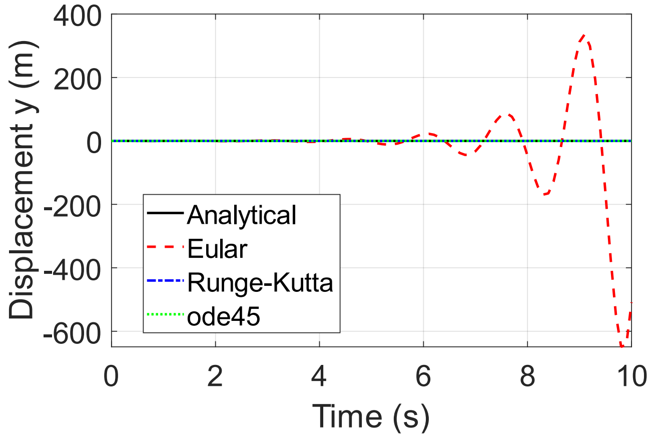

Figure 5: Comparison of Solutions

Q2(e)

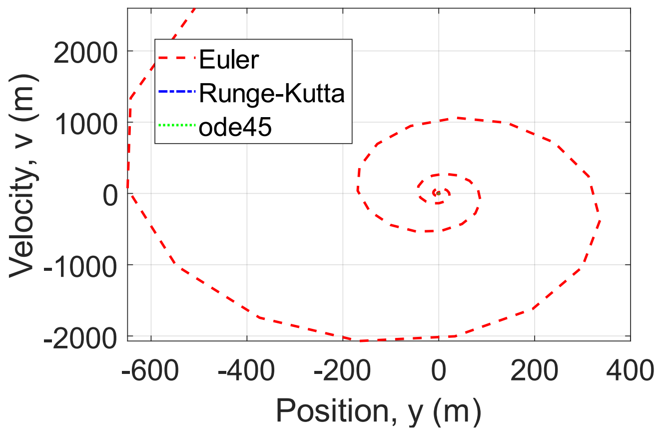

Figure 6: Velocity vs Displacement for all Schemes

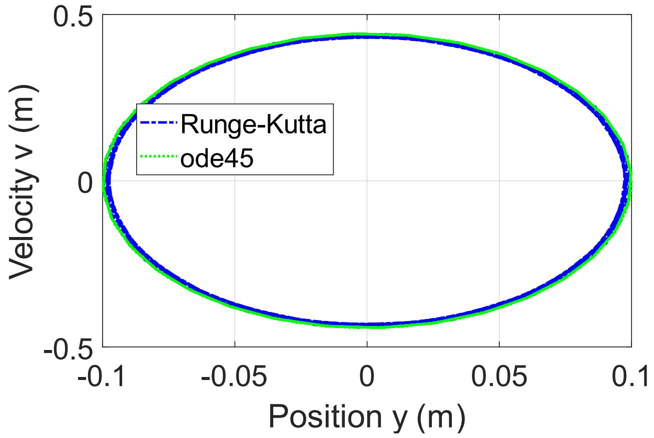

Figure 7: Velocity vs Displacement for Bounded Stable Schemes

Comparison

Q2(f)

Comparison

Q3(a)

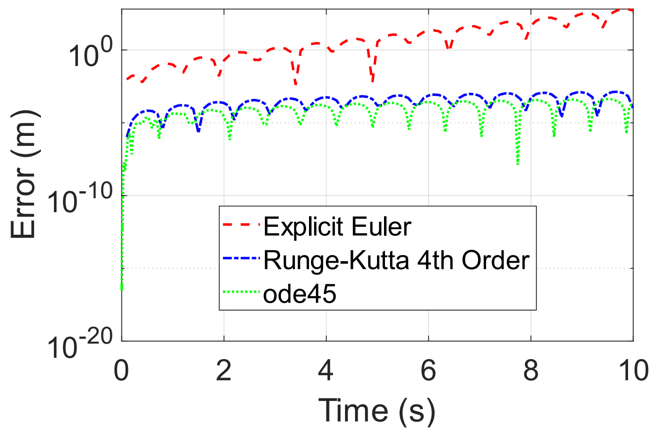

Figure 8: Absolute Relative Error Plot to Analytical Solution

Discussion

Q3(b)

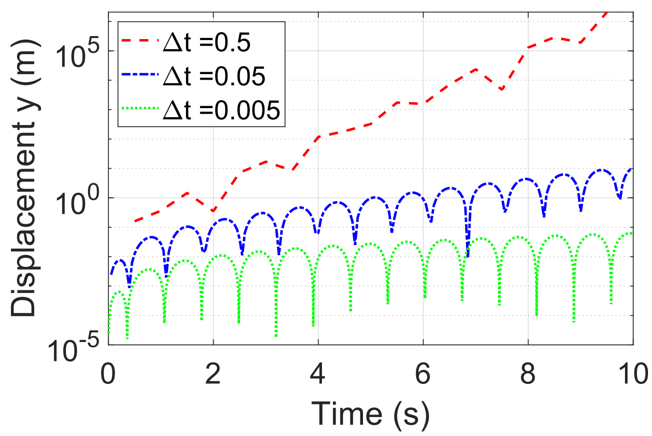

Figure 9: Absolute Relative Error Plot to Analytical Solution, FE

Q3(c)

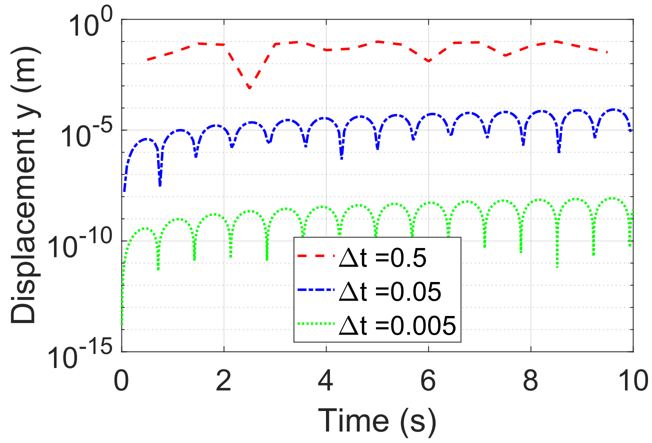

Figure 10: Absolute Relative Error Plot to Analytical Solution, RK4

Q3(d) Investigate the effects of initial conditions

Euler

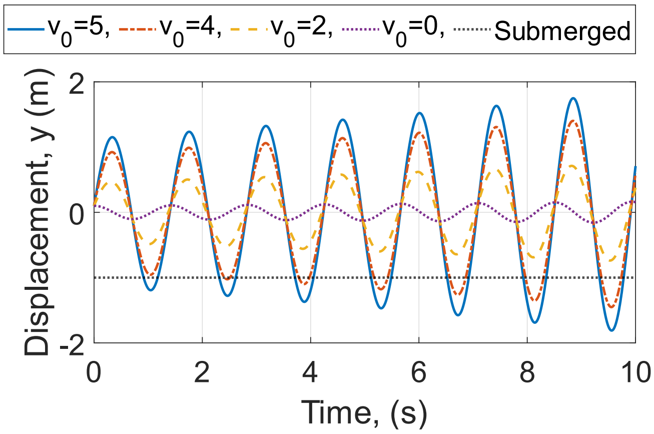

Figure 11: Varying Initial Conditions, FE

RK4

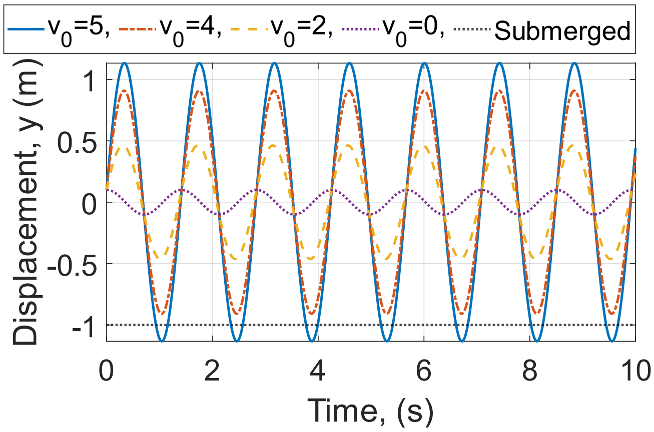

Figure 12: Varying Initial Conditions, RK4

Analytical Solution

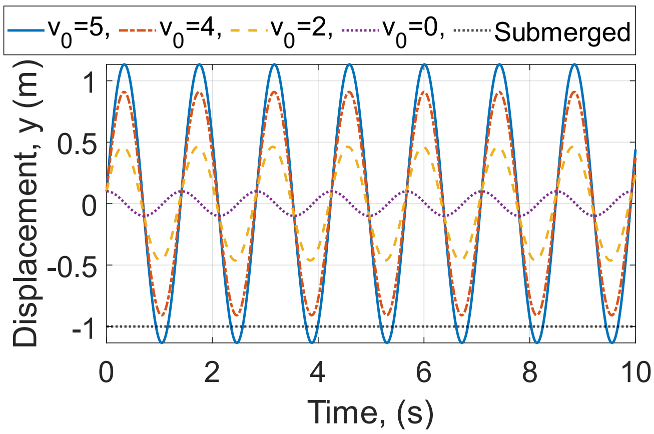

Figure 13: Varying Initial Conditions, Analytical Solution

Q4(a) Compared to the mass spring model the buoyancy model acts in the vertical direction so have some effects of gravity which have not been included into the modelled equation. the spring mass system does not include these effects.

The mathematical formulation of the system is equivalent as both model have a restoring force promoting oscillations with resistive forces causing damping

Difficulties arose when making the comparison of the Explicit Euler and the other schemes due to instability of the time step, we were using.

Q4(b) Including Damping, with c being our damping coefficient, changes our equation of motion to:

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega^2 y(t) = 0 \qquad (10)$$

$$\begin{bmatrix}\ddot{y}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & -\omega^2\\ 1 & -c\end{bmatrix}\begin{bmatrix}\dot{y}\\ y\end{bmatrix} \qquad (11)$$

Q4(b)(i) The damped system has an increasing period and decreasing amplitude over time. However, the undamped case shows a constant time period and amplitude.

Q4(b)(ii) The effects of damping on numerical stability include increasing the stability of solutions due to inclusion of a dissipative mechanism which prevents runaway solutions. Additionally, it ensures solutions remain bounded as energy is dissipated over time and the scheme is unlikely to diverge.

Q4(b)(iii) Damped case solution:

$$y(t) = 0.1e^{-0.1t}cos(4.4283t) \qquad (12)$$

Q4(b)(iv)

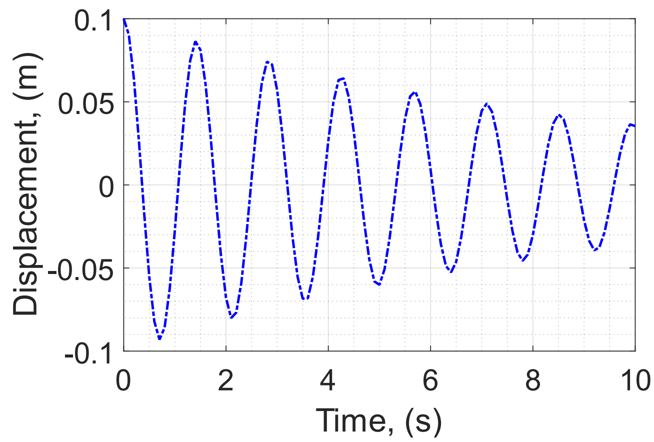

Figure 14: Damped Solution, RK4