Flight Dynamics and Control Revision

Setup

We assume:

- Aircraft behaves as a Rigid body.
- Mass is Constant.
- Symmetrical mass distribution.
- All rotations from centre of gravity.

Therefore, two equations are used -

Newton's Second law: $F = ma = m\frac{dv}{dt}$ (as Mass constant)

Euler's equation of Rotational Dynamics: $M = \frac{\partial}{dt} I(\omega)$

$$I = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

In addition, for this course we use a Body fixed axis system

x - Through nose

y - Through Right wing

z - Positive Downwards

origin - Centre of Gravity

Parameters:

Displacements = $[x, y, z]$

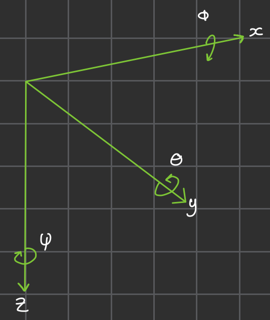
Rotation = $[\phi, \theta, \psi]$

Velocity = $[U, V, W]$

Roll Rate = $[p, q, r]$

Forces = $[X, Y, Z]$

Moment = $[L, M, N]$



However to go from Body fixed axis to Earth axis and vice versa we have to → Earth to Plane.

1. Align Heading (z)
2. Align pitch (y)
3. Align Roll (x)

Easy way to Remember PE = x, y, z

To align them we do following

Roll: $$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} U_e \\ V_e \\ W_e \end{bmatrix}$$

Pitch: $$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} U_e \\ V_e \\ W_e \end{bmatrix}$$

Yaw (Heading): $$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_e \\ V_e \\ W_e \end{bmatrix}$$

Giving the following equation

$$\begin{bmatrix} U_E \\ V_E \\ W_E \end{bmatrix} = R_{EB}(\psi, \theta, \phi) \begin{bmatrix} U_B \\ V_B \\ W_B \end{bmatrix}$$

Navigation:

Now the plane may have turned right but is tilted so for earth axis it is actually up.

So we use following Equation

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = R(\Phi)R(\theta)R(\psi)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + R(\Phi)R(\theta)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R(\Phi)\begin{bmatrix} \dot{\Phi} \\ 0 \\ 0 \end{bmatrix}$$

Remember $x - x$, $y - x - y$, $z - x - y - z$

Aircraft dynamics and State Space

This basically explains how an aircraft moves and its possible movements.

In this course we use the state space formulation to describe dynamics of aircraft.

$$\dot{x}(t) = f(x(t), \delta)$$

- $\dot{x}(t)$ refers to how state changes with time (eg. Speed, Distance)
- $x(t)$ refers to all things that aircraft doing currently 'State Vector'
- δ is everything which may influence the system

Next is the State Space which can be seen like this -

Imagine playing MSFS and the states are the velocity, heading, tilt, etc

The State Space is all the possible states, so the map your playing on.

From all of this we get something important which are Singular Points.

Firstly, trajectories are calculated using taylor expansion which gives us Regular points. However, some points cause the taylor expansion to not work. These points are called the 'Singular points (x_0)' which are also where

$\rightarrow f(x_0, \delta) = 0$.

$\therefore$ Signifying that $\dot{x} = 0$ so there is no movement unless we perturb it. Hence making it an equilibria and we want to know what type of equilibria and to do so we look at eigenvalues and vectors.

Eigenvalues:

Negative Real: System decays $\longrightarrow$ stable

Positive Real: System grows $\longrightarrow$ unstable

Negative Imaginary: Oscillates infinitely $\longrightarrow$ neutrally stable

Positive Imaginary: Spiral Behaviour $\longrightarrow$ Growth and decay

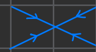
Negative Real: Spiral Inwards $\longrightarrow$ Stable

Positive Real: Spiral Outwards $\longrightarrow$ Unstable

Real = 0 : Oscillate in place $\longrightarrow$ Neutral

You can represent them in 2D-Space -

- Both Negative and Real



Attractor

- Both negative and Imaginary

Attractor

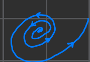
- Both Real but one negative and one positive

Saddle

- Both Real and Positive

Repellor

- Both Positive and Imaginary



Repellor

Bifurcation-

This where small changes can cause for a change in trajectory and State Spaces.

Such as:

- Adding or Removing Singular Points
- Transform or Merge or Split

Forces and Moments on Aircraft

Cast your mind back to the two equations of motion:

$F = m\frac{dv}{dt}$ and $M = \frac{d}{dt}I(\omega)$

$v = U i + V j + W k$

$\omega = p i + q j + r k$

$F = X i + Y j + Z k$

$M = L i + M j + N k$

Brief example of how we use equation ↴

$F = m\frac{dv}{dt} = m\frac{d(Ui + Vj + Wk)}{dt}$

$= m\left(\frac{du}{dt}i + \frac{dv}{dt}j + \frac{dw}{dt}k + u\frac{di}{dt} + v\frac{dj}{dt} + w\frac{dk}{dt}\right)$

and $F = Xi + Yj + Zk$

$\therefore X = \frac{du}{dt}i + u\frac{di}{dt}$ $\quad Y = \frac{dv}{dt}i + v\frac{di}{dt}$ and so on

Same can be done for Euler Equation -

$M = \frac{d}{dt}I(\omega)$

$\frac{d}{dt}I(\omega) = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} \frac{dp}{dt} \\ \frac{dq}{dt} \\ \frac{dr}{dt} \end{bmatrix} =$

$= \left(I_{xx}\frac{dp}{dt} - I_{xz}\frac{dr}{dt}\right)i + \ldots\ldots$

Important to note that $\frac{di}{dt}$, $\frac{dj}{dt}$ and $\frac{dz}{dt}$ can be rewritten as

$\frac{di}{dt} = \omega \times i = \begin{bmatrix} i & j & k \\ p & q & r \\ 1 & 0 & 0 \end{bmatrix} = -rj + qk$

$\frac{dj}{dt} = \omega \times j = \begin{bmatrix} i & j & k \\ p & q & r \\ 0 & 1 & 0 \end{bmatrix} = \ldots$

This gives us following 6 equations:

- $m(\dot{u} - Vr + Wq) = X^a + mg_x$
- $m(\dot{v} + Ur - Wp) = Y^a + mg_y$
- $m(\dot{w} - Uq + Vp) = Z^a + mg_z$

- $I_{xx}\dot{p} - I_{xz}\dot{r} + (I_{zz} - I_{yy})qr - I_{xz}\cdot pq = L^a$
- $I_{yy}\dot{q} + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) = M^a$
- $I_{zz}\dot{r} - I_{xz}\cdot\dot{p} + (I_{yy} - I_{xx})pq + I_{xz}qr = N^a$

The values of g_x, g_y and g_z are found using:

$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} -g\cos(\theta) \\ g\cos(\theta)\sin(\phi) \\ g\cos(\theta)\cos(\phi) \end{bmatrix}$

The complete state space representation formula is

$\rightarrow \quad M\dot{x} = N(x) + F(\dot{x}, x) + G(x)$

$M \rightarrow$ Inertial Matrix

$N \rightarrow$ Non-Linear terms

$F(\dot{x}, x) \rightarrow$ Aerodynamic and propulsive forces

$G \rightarrow$ Gravitational forces

Now, when singular point is present $\dot{x} = 0$ meaning the rest of the equation equals zero. However if perturbed conditions are also present $\rightarrow x_0 + x'$

It gives following equation: $M\dot{x}' = (N_{x_0} + G_{x_0} + F_{x_0})x' + F_{\dot{x}_0}\dot{x}'$

Wings Level Trimmed flight

Firstly, when looking at the state vector of an aircraft we seperate it in Symmetrical and a symmetrical terms

$$x = (\underbrace{u, w, q, \theta}_{\text{Symmetrical}}, \overbrace{v, p, r, \phi, \psi}^{\text{A symmetrical}})$$

Now trimmed flight is where the aircraft is at a steady-state and where there are no unbalanced forces or moments

∴ Everything other than U_e, θ_e are equal to zero.

Furthermore at this condition the steady state nature makes it a Singular point

$$\therefore \quad x_0 = (U_e, 0, 0, \theta_e, 0, 0, 0, 0, 0)$$

With this in mind we get the trim equation as.

$$\begin{bmatrix} X_e^a - mg\sin(\theta_e) \\ Z_e^a + mg\cos(\theta_e) \\ M_e^a \\ 0 \\ Y_e^a \\ L_e^a \\ N_e^a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Meaning that resultant moments and forces equal to zero and the aerodynamic and propulsive forces equal the gravitational forces

With all of this we can also make massive matrices for all the terms: N_{x_0}, G_{x_0}, F_{x_0}, $F_{\dot{x}_0}$ with each being a 9x9 matrix

Now the terms for aerodynamic and propulsive forces → F_{x_0}, $F_{\dot{x}_0}$ have the following structure

$$\begin{bmatrix} X_a \\ Z_a \\ M_a \\ 0 \\ Y_a \\ L_a \\ N_a \\ 0 \\ 0 \end{bmatrix}$$

This would be a 9x9 matrix

where a is one of the state vector terms

$F_{\dot{x}_0}$ would be the same but just $\dot{a}$ of all terms.

Every single term is a stability Derivative and follows the definitions

$$X_u = \frac{dx}{du}\bigg|_{x=0,\ \dot{x}=0}$$

and

$$X_{\dot{u}} = \frac{dx}{d\dot{u}}\bigg|_{x=0,\ \dot{x}=0}$$

physically interpreted –

F_{x_0} – Shows how forces and or moments change with state variables

$F_{\dot{x}_0}$ – Shows how forces and or moments change with rates of state Variables.

Simplifications –

Ok so far we have 9x9 matrix however not all terms are need as:

- The $\dot{\phi}, \dot{\theta}, \dot{\psi}$ derivates equate to zero as forces are not affected by the rate of change of Orientation.

* It is also important to note that q and $\dot{\theta}$ both describe different phenomena as q affects the flow field of the aircraft ∴ loading yet $\dot{\theta}$ only affects orientation in space *

- The $X_{\dot{w}}$, $Z_{\dot{w}}$, $M_{\dot{w}}$ Stability derivatives can be ignored as we will be mostly looking at subsonic aircraft

- The θ stability derivatives can also be ignored as it only has an affect near the ground.

· Curie Condition

A symmetrical cause doesn't produce an asymmetrical affect

Therefore we get the following matrix

$$
F_{x_0} = \begin{bmatrix}
X_u & X_w & X_q & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_u & Z_w & Z_q & 0 & 0 & 0 & 0 & 0 & 0 \\
M_u & M_w & M_q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Y_v & Y_p & Y_r & Y_\phi & Y_\varphi \\
0 & 0 & 0 & 0 & L_v & L_p & L_r & L_\phi & L_\varphi \\
0 & 0 & 0 & 0 & N_v & N_p & N_r & N_\phi & N_\varphi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Taking the Curie principle further we can rewrite the system as :

Symmetrical :

$$
\begin{bmatrix}
m & -X_{\dot{w}} & 0 & 0 \\
0 & m - Z_{\dot{w}} & 0 & 0 \\
0 & -M_{\dot{w}} & I_{yy} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \dot{U} \\ \dot{W} \\ \dot{q} \\ \dot{\Theta} \end{bmatrix}
=
\begin{bmatrix}
X_u & X_w & X_q & -mg\cos(\Theta_e) \\
Z_u & Z_w & Z_q + mU_e & mg\sin(\Theta_e) \\
W_u & W_w & W_q & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} U \\ W \\ q \\ \Theta \end{bmatrix}
$$

Asymmetrical

$$
\begin{bmatrix}
m & 0 & 0 & 0 & 0 \\
0 & I_{xx} & -I_{xz} & 0 & 0 \\
0 & -I_{xz} & I_{zz} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \dot{V} \\ \dot{p} \\ \dot{r} \\ \dot{\Phi} \\ \dot{\varphi} \end{bmatrix}
=
\begin{bmatrix}
Y_v & Y_p & Y_r - mU_e & mg\cos(\theta_e) & 0 \\
L_v & L_p & L_r & 0 & 0 \\
N_v & N_p & N_r & 0 & 0 \\
0 & 1 & \tan(\theta_e) & 0 & 0 \\
0 & 0 & \sec(\theta_e) & 0 & 0
\end{bmatrix}
\begin{bmatrix} V \\ p \\ q \\ \Phi \\ \varphi \end{bmatrix}
$$

Stability Derivatives

This Section Requires a lot of memorisation so get ready to learn everything ☺

All of the one we will look at are Longitudinal Stab. Derivs.

Firstly, lets look at the X^a, Z^a derivatives

At equilibrium Condition —

$$X^a = T_x - \tfrac{1}{2}\rho S_{ref} U_e^2 C_D$$
$$Z^a = -T_z - \tfrac{1}{2}\rho S_{ref} U_e^2 C_L$$

Not at equilibrium —

$$X^a = T_x - \tfrac{1}{2}\rho S_{ref} V_\infty^2 \left(C_D\cos(\alpha-\alpha_e) - C_L\sin(\alpha-\alpha_e)\right)$$
$$Z^a = -T_z - \tfrac{1}{2}\rho S_{ref} V_\infty^2 \left(C_L\cos(\alpha-\alpha_e) + C_D\sin(\alpha-\alpha_e)\right)$$

Now, from Intro To Aero last year we looked at the following (Recap):

- Aerodynamic Centre is where $\frac{\partial C_M}{\partial \alpha} = 0$
- $a_w = \frac{\partial C_L}{\partial \alpha}$
- $C_{M_{0w}} = \frac{X_{ac} - X_{ap}}{c} C_{L_{0w}}$
- X_{cp} is where Pitching Moment $= 0$
- $C_L = C_{L_{0w}} + a_w \cdot \alpha + \frac{S_H}{S_{ref}}\left[C_{L_{0H}} + a_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\alpha + a_\varepsilon \delta_\varepsilon + a_T \delta_T\right]$
- $k_n = -\frac{\partial C_M}{\partial C_L} = \bar{X}_{cp} - \bar{X}_{cg}$
- $C_M = C_{M_0} - k_n C_L$
- Trimmable

1. $C_M > 0 \rightarrow C_{M_0} = k_n C_L$ 2. $k_n > 0$

From that little recap we can now tackle the M^a derivative ⇒

$$M^a = T l_T + \frac{1}{2}\rho V_\infty^2 S_{ref}\, \bar{c}\, (C_{M_0} - k_n C_L)$$

Now you may be wondering about an odd term in there.

L_T is the moment arm of the aircraft CG to thrust line

$$\therefore\ L_T = (x_{cg} - x_T)\sin(\epsilon + \alpha_e) + (z_{cg} - z_T)\cos(\epsilon + \alpha_e)$$

Basically difference in distance in respective axis, factoring in angle

Now let us get straight into the meat of it all...

U- Derivative

As we stated Earlier

$$X_u^a = \left.\frac{\partial X}{\partial U}\right|_{u = U_e,\ \alpha = \alpha_e}$$

The effect of perturbation in U can also be expressed as the following:

$$\frac{\partial X^a}{\partial U} = \frac{\partial X^a}{\partial \alpha}\frac{\partial \alpha}{U_\infty} + \frac{\partial X^a}{\partial V_\infty}\frac{\partial V_\infty}{\partial U} \approx \frac{\partial X^a}{\partial V_\infty}$$

Same for M^a and Z^a

∴ Only the V_∞ has an effect

So we get the U derivative by differentiating X^a by V_∞

$$\text{Also}\quad \frac{\partial}{\partial V_\infty} = \frac{\partial M_\infty}{\partial V_\infty}\cdot\frac{\partial}{\partial M_\infty} = \frac{1}{C_s}\frac{\partial}{\partial M_\infty}$$

∴

$$\frac{\partial X^a}{\partial U} = \frac{1}{C_s}\left(\frac{\partial T_x}{\partial M_\infty}\right)_e - \rho U_e S_{ref}(C_D)_e - \frac{1}{2}\rho U_e^2 S_{ref}\frac{1}{C_s}\left(\frac{\partial C_D}{\partial M_\infty}\right)_e$$

$$\frac{\partial Z^a}{\partial U} = -\frac{1}{C_s}\left(\frac{\partial T_z}{\partial M_\infty}\right)_e - \rho U_e S_{ref} C_{L_e} - \frac{1}{2}\rho U_e^2 S_{ref}\frac{1}{C_s}\left(\frac{\partial C_L}{\partial M_\infty}\right)_e$$

$$\frac{\partial M^a}{\partial U} = \frac{1}{C_s}\left(\frac{\partial T}{\partial M_\infty}\right)_e l_T + \frac{1}{2}\rho U_e^2 S_{ref}\bar{c}\frac{1}{C_s}\left[\frac{\partial C_{M_0}}{\partial M_\infty} - \frac{\partial k_n}{\partial M_\infty}C_{L_e} - k_n\left(\frac{\partial C_L}{\partial M_\infty}\right)_e\right]$$

W - Derivatives

The perturbation:

$$\frac{\partial X}{\partial W} = \frac{\partial X}{\partial \alpha}\frac{\partial \alpha}{\partial W} + \frac{\partial X}{V_\infty}\frac{\partial V_\infty}{\partial W} \approx \frac{1}{U_e}\frac{\partial X^a}{\partial \alpha}$$

Therefore, W derivatives depend on α and also a factor $\frac{1}{U_e}$

∴

$$\frac{\partial X^a}{\partial W} = \frac{1}{U_e}\left(\frac{\partial T_x}{\partial \alpha}\right)_e + \frac{1}{2}\rho U_e^2 S_{ref}\left[C_{L_e} - \left(\frac{\partial C_D}{\partial \alpha}\right)_e\right]$$

$$\frac{\partial Z^a}{\partial W} = -\frac{1}{U_e}\left(\frac{\partial T_z}{\partial \alpha}\right)_e - \frac{1}{2}\rho U_e^2 S_{ref}\left[C_{D_e} + \left(\frac{\partial C_L}{\partial \alpha}\right)_e\right]$$

Note that if the whole aircraft is said to be Parabolic:

$$\frac{\partial C_D}{\partial \alpha} = \frac{2a}{\pi A\!R e} C_L$$

Now the $\frac{\partial M^a}{\partial W}$ derivative, just note that the $\frac{\partial k_n}{\partial \alpha} = 0$ as we are not operating near Stalled Conditions

$$\therefore\ \frac{\partial M^a}{\partial W} = \frac{1}{U_e}\left(\frac{\partial T}{\partial \alpha}\right)_e l_T - \frac{1}{2}\rho U_e S_{ref} k_n \bar{c}\left(\frac{\partial C_L}{\partial \alpha}\right)_e$$

q- Derivatives

This has a bit more to it...

The X_q, Z_q and M_q derivatives show the effect of pitch rate about CG of aircraft. Therefore we need to get radial velocity distribution

$$U_i = -(z - z_{cg})q$$

$$W_i = (x - x_{cg})q$$

With this in mind lets look at perturbation

$$\frac{dx}{dq} = \frac{\partial x}{\partial U} \cdot \frac{\partial U_i}{\partial q} + \frac{\partial x}{\partial W} \cdot \frac{\partial W_i}{\partial q}$$

$$-(z - z_{cg})\frac{\partial x}{\partial U} + (x - x_{cg})\frac{\partial x}{\partial W}$$

Same for $\frac{\partial z}{\partial q}$

$\frac{\partial M}{\partial q}$ differs as you need to take moment from CG

$$\therefore \frac{\partial M}{\partial q} = -(z - z_{cg}) \cdot \left(\frac{\partial M}{\partial u} + \frac{\partial x}{\partial q}\right) + (x - x_{cg}) \cdot \left(\frac{\partial M}{\partial w} + \frac{\partial z}{\partial q}\right)$$

Now in a conventional aircraft the affect of U_i is negligble to that of W_i $\therefore$ only the tailplane's derivative of W used.

So we get

$$l_H' = x_H - x_{cg}$$

$$h_H' = z_H - z_{cg}$$

$$(X_q)_H = \frac{1}{2}\rho U_e l_H' S_H \left[C_{L_{e_H}} - \left(\frac{\partial C_{D_H}}{\partial \alpha}\right)_e\right]$$

$$(Z_q)_H = -\frac{1}{2}\rho U_e l_H' S_H \left[C_{D_{e_H}} + \left(\frac{\partial C_{L_H}}{\partial \alpha}\right)_e\right]$$

$$(M_q)_H = l_H'(Z_q)_H - h_H'(X_q)_H + \left(\frac{\partial M_H}{\partial q}\right)_H$$

So you can essentially say that Z_q and X_q derivs same as W derivatives just that it is for tail planes.

$\dot{W}$ - Derivatives

$\dot{W}$ derivatives occur due to lag effects which occure due to vortices creating a downwash on the aircraft.

So we use the following:

- $l_H = x_H - x_W$
- W (downwash velocity at wing ac) $= U_e \cdot \varepsilon(t)$
- W_H (Delayed downwash velocity) $= U_e \cdot \varepsilon_H(t)$

- Downwash angle is ε and is proportional to the angle of attack or α

By doing a taylor Expansion and chain rule we get the following

$$\frac{\partial \varepsilon_H}{\partial \dot{w}} = -\frac{l_H}{U_e}\left(\frac{\partial \varepsilon}{\partial \alpha}\right)_e$$ Illustrating how downwash changes with α.

With this we get...

- $X_{\dot{w}} = \frac{l_H}{U_e}\left(\frac{\partial \varepsilon}{\partial \alpha}\right)_e (X_w)_H$
- $Z_{\dot{w}} = \frac{l_H}{U_e}\left(\frac{\partial \varepsilon}{\partial \alpha}\right)_e (Z_w)_H$
- $M_{\dot{w}} = l_H' Z_{\dot{w}} - h_H' X_{\dot{w}} + \frac{l_H}{U_e}\left(\frac{\partial \varepsilon}{\partial \alpha}\right)_e (M_w)_H$

That is it :)

Now Remember them all

1. Perturbations
2. Equations
3. Physical Meaning

State Space Structures

To analyse the dynamic response of an aircraft near equilibrium at steady, trimmed flight we use the following

Symmetrical: $M_S \dot{x}_S' = R_S x_S' \qquad x_S' = [U', W', q', \theta']$

Asymmetrical: $M_A \dot{x}_A' = R_A x_A' \qquad x_A' = [V', p', r', \phi', \psi']$

R is the System Matrix: $N_{x_0} + F_{x_0} + G_{x_0}$

Both equations can be rewritten in terms of the jacobian

$$\dot{x}_S' = (J_S)_e x_S'$$

As we know, with the jacobian we can get eigenvalues /vectors

$$|(J_S)_e - I\lambda| = 0$$

With those eigenvalues λ we get two equations:

Symmetrical: $A_S\lambda^4 + B_S\lambda^3 + C_S\lambda^2 + D_S\lambda + E_S = 0$

↑ This is called: Longitudinal Quartic Equation

Asymmetrical: $\lambda(A_A\lambda^4 + B_A\lambda^3 + C_A\lambda^2 + D_A\lambda + E_A\lambda) = 0$

Called: Lateral quintic Equation

From these equations we Apply Raths Descrimiant which states that for stability we need the following:

- $A, B, C, D, E > 0$ always
- $R = BCD - B^2E - AD^2$

Remember using:
Boys Cant Dance but
Big Boys Eat but
All Dads Dance

If $R=0 \rightarrow$ Roots opposite Sign $\rightarrow$ Instability

If $R<0 \rightarrow$ Stable

- $E \geq 0$ as if $E < 0$ we get instability

Approximating quartic and quintic Equations

After using Rauth's Descriminant to know if stable or not we have to approximate them.

Quartic:

The equation: $A_S\lambda^4 + B_S\lambda^3 + C_S\lambda^2 + D_S\lambda + E = 0$

Factorises into 2 brackets:

$$\left(A_S\lambda^2 + B_S\lambda + C_S\right)\left(\lambda^2 + \frac{C_S D_S - B_S E_S}{C_S^2}\lambda + \frac{E_S}{C_S}\right) = 0$$

Remember using: First is Basic But Second has CDs takeaway the BEEs and split it with the Seas. Finally everything is split by Cans.

Quintic:

The equation: $\lambda(A_A\lambda^4 + B_A\lambda^3 + C_A\lambda^2 + D_A\lambda + E) = 0$

Factorised into 3: $\lambda\left(\lambda + \frac{B_A}{A_A}\right)\left(\lambda + \frac{E_A}{D_A}\right)\left(\lambda^2 + \frac{C_A}{D_A}\lambda + \frac{D_A}{B_A}\right) = 0$

Typical Aerodynamic Modes

SPPO (Short Period Pitch Oscillations —

- Longitudinal Stability
- A_s, B_s, C_s Large
- It is dependent on Lift and pitching moment

SPPO is effected by k_n ∴ if

- k_n Large $\omega_d = \sqrt{\frac{C_s}{A_s} - \left(\frac{B_s}{2A_s}\right)^2}$ and is Stable.

- k_n decreasing ⟶ Oscillation decrease but still stable

- Negative k_n ⟶ Un Stable

∴ k_n is proportional to stability

Phugoid Mode

- Longitudinal Stability
- $\lambda^2 + \frac{CD - BE}{C^2}\lambda + \frac{E_s}{C_s}$ has 2nd and 3rd term Smaller
- Governed by X force opposite to SPPO
- Pitching Moment ignored
- Phugoid damping inversely proportional to aerodynamic Efficiency or L/D

Roll Subsidence

- Lateral Stability
- $\lambda + \frac{B_A}{A_A} = 0$ has a large λ
- Governed by Rolling moment damping effect.
- Occurs due to Roll rate perturbation (p) inducing opposing Rolling moment which causes damping to occur
- This mode is always stable and highly damped

Note Pilot would need to maintain Aeileon input to keep rolling motion.

Spiral Mode

- Lateral stability
- $\lambda + \frac{E_A}{D_A} = 0$ has a Small λ
- This is a coupled yawing and Rolling motion
- Since λ is Small the mode changes slowly
- Neglect Small accelerations and lateral forces
- Governed by Rolling and Yawing Moments

$$L_v v' + L_p p' + L_r r' = 0$$
$$N_v v' + N_p p' + N_r r' = 0$$

Dutch Roll

- Coupled Yawing, Rolling and Slide slipping oscillation
- Primary oscillations are Yawing and Slide Slip and Secondary is Rolling
- Neglect Roll moment, p' and ϕ in calculations.
- frequency and Damping influenced by N_v and Y_v.

Response To Control Inputs

Everything we have looked at has excluded pilot inputs.

So let us get started

Pitching Moment: $M = M_e + M_q q' + M_u U' + M_w W'$

M_e = Equilibrium pitching moment (trimmed flight)

$M_u U', M_q q', M_w W'$ = Pitching Moment due to forward, vertical velocity and pitch rate

The elevator deflection is also usually added on top $\Rightarrow M_{\delta_E} \delta_E$

Now, for control inputs we use equation -

$$M\dot{x}' = Rx' + Bu'$$

Bu' = Effect of Control inputs

Control Derivatives

We have 4 control inputs

Symmetrical (Longitudinal) -

- Thrust
- Elevator

Asymmetrical (Lateral) -

- Aileron
- Rudder

Longitudinal Control Derivatives -

Since we only care about elevator and Throttle inputs we get the following -

Elevator -

$$X^a_{\delta_e} = -\frac{1}{2}\rho U_e^2 S_{ref}\left[\frac{\partial C_D}{\partial \delta_E}\cos(\alpha-\alpha_e) - \frac{\partial C_L}{\partial \delta_E}\sin(\alpha-\alpha_e)\right]$$

$$Z^a_{\delta_E} = -\frac{1}{2}\rho U_e^2 S_{ref}\left[\frac{\partial C_L}{\partial \delta_E}\cos(\alpha-\alpha_e) + \frac{\partial C_D}{\partial \delta_E}\sin(\alpha-\alpha_e)\right]$$

$$M^a_{\delta_E} = l_H' z_{\delta_E} - h_H' x_{\delta_E} + \frac{1}{2}\rho U_e^2 S_H \bar{C}_H \left(\frac{\partial C_{M_H}}{\partial \delta_E}\right)_e$$

Throttle Derivatives:

Basically Same thing as Elevator derivatives

$$X_{c_T} = \frac{1}{2}\rho U_e^2 S_{ref}\cos(\epsilon + \alpha_e)$$

$$Z_{c_T} = -\frac{1}{2}\rho U_e^2 S_{ref}\sin(\epsilon + \alpha_e)$$

$$M_{c_T} = \frac{1}{2}\rho U_e^2 S_{ref}\, C_T$$

Lateral Derivatives:

Rudder

S_{VS} = Surface Area of vertical Stabilliser

$$Y_{\delta_R} = \frac{1}{2}\rho U_e^2 S_{VS}\, a_R$$

$$L_{\delta_R} = \frac{1}{2}\rho U_e^2 S_{VS}(z_{VS} - z_{cg})\, a_R$$

$$N_{\delta_R} = -\frac{1}{2}\rho U_e^2 S_{VS}(X_{VS} - X_{cg})\, a_R$$

Transfer Functions

This is where transfer functions are used to simply describe how an input frequency affects stability and dynamic Response.

Essentially just giving us a clearer image of whats happening

We used earlier: $M\dot{x}' = Rx' + Bu'$

Laplace Transform

$$G(s) = (sM - R)^{-1}B$$

Very Waffly so don't know what is and isn't needed.

Feedback Control

Simple Explanation, just do loads of questions

Basically a system we tune using three controllers to improve stability.

Feedback Control adjust inputs based on errors from setpoint you set

Equation: $y(t) = C \cdot x(t)$

- $y(t)$ = Outputs
- C = Matrix that maps inputs to outputs
- $x(t)$ = Inputs

For errors we use: $e(t) = \overset{\text{Setpoint}}{r} - y(t)$

For this course we use a PID controller

	Rise Time	Overshoot	Settling time	SSE
P	Decrease	Increase	N/A	Decrease
I	Decrease	Increase	Increase	Decrease
D	Increase	Decrease	Decrease	N/A

∴ P and I are basically same and D opposite of I