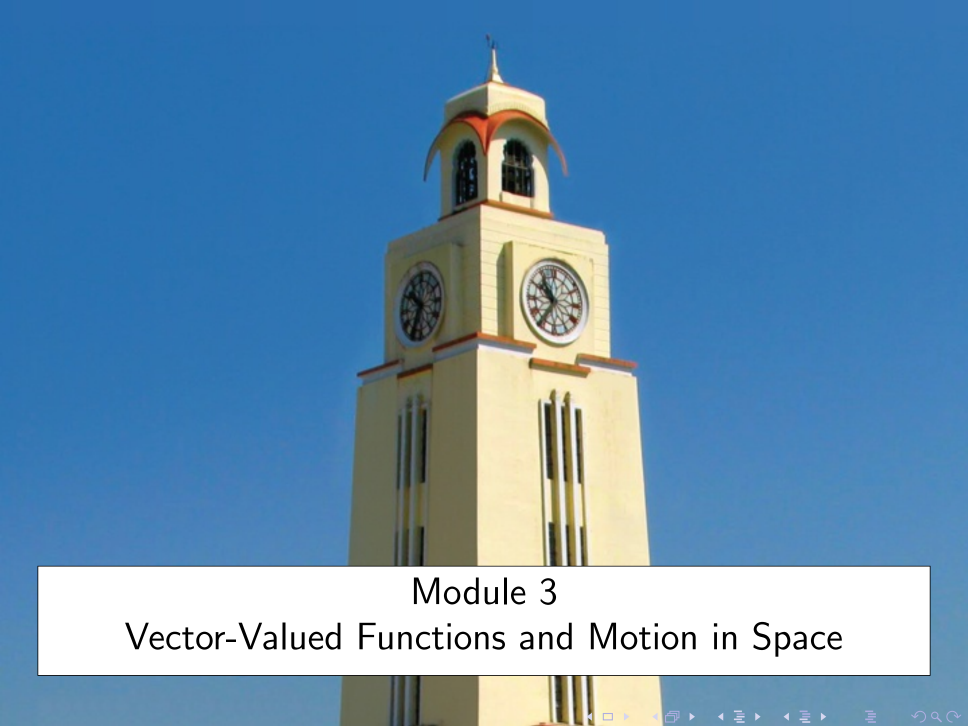

Module 3
Vector-Valued Functions and Motion in Space

Outline

In this module, we'll discuss the following topics:

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1. Curves in space and their tangents

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3. Arc length in space

Section 13.1

Curves in Space and Their Tangents

Curve in space

https://www.youtube.com/shorts/pLllc6pBv70?feature=share

Curve in space

When a particle moves through space during a time interval I, we can treat the particle coordinates as a function of time as

$$x = f(t),\ y = g(t),\ z = h(t), \qquad t \in I,$$

where f, g and h are real valued functions of the parameter t.

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$$x = f(t),\ y = g(t),\ z = h(t), \qquad t \in I,$$

where $f,\ g$ and h are real valued functions of the parameter t. Then, the locus of the points (x, y, z) is called the particle's path.

The curve traced by the particles's path is called curve in space or space curve.

Curve in space

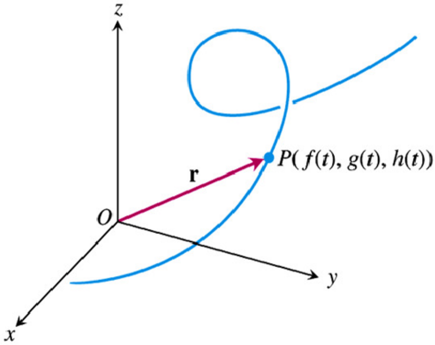

Figure: The position vector $\mathbf{r(t)} = \overrightarrow{OP}$ $= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ of a particle moving through space is a function of time.

A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$
$$= (f(t), g(t), h(t)),$$

from origin to the particle's position at time t is the particle's **position vector**.

Vector valued function

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A vector function is of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \qquad \text{plane curve,}$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \qquad \text{space curve.}$$

Components

A vector function $\mathbf{r}$ can be written in component form as

$$\mathbf{r}(t) = (f(t), g(t), h(t)),$$

where f, g, h are real-valued functions called the component functions (or components) of $\mathbf{r}$.

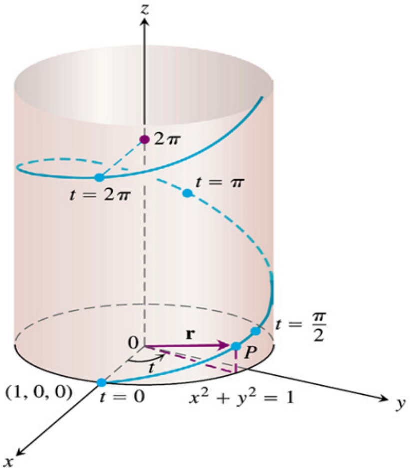

Figure: The upper half of a helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$.

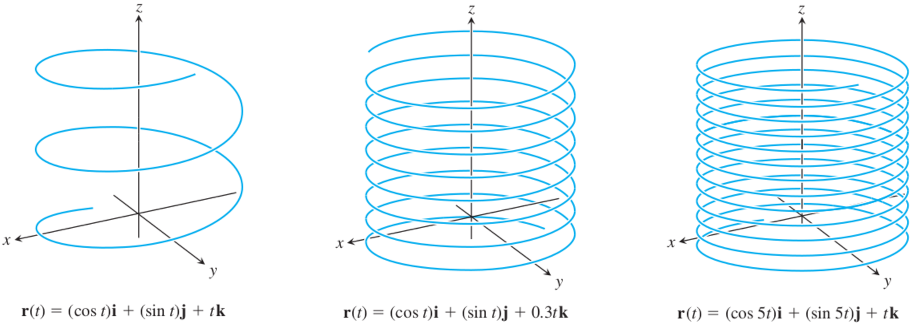

Figure: Helices spiral upward around a cylinder, like coiled springs.

Remark

Whether you use parametric equations for a curve, or a vector equation to represent the curve, is totally up to you. They are equivalent ways of describing the same thing.

Unless stated otherwise, the domain of a vector-valued function $\mathbf{r}$ is considered to be the intersection of the domains of the component functions f, g, and h.

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Example

Find the domain of $\mathbf{r}(t) = (\ln t)\mathbf{i} + (\sqrt{1-t})\mathbf{j} + t\mathbf{k}$.

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Example

Find the domain of $\mathbf{r}(t) = (\ln t)\mathbf{i} + (\sqrt{1-t})\mathbf{j} + t\mathbf{k}$.

Sol. Note that domain of $\ln t$ is $(0, \infty)$; domain of $\sqrt{1-t}$ is $(-\infty, 1]$; and the domain of t is $(-\infty, \infty)$. Thus their intersection , i.e. $(0, 1]$ is the domain of $\mathbf{r}(t)$.

Two different curves can have the same graph

For instance, each of the curves given by

$$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j},$$

and

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},$$

has the unit circle as its graph, but these equations do not represent the same curve, because the circle is traced out in different ways.

Limit of a vector function

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has a limit L as $t \to t_0$ and write $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to t_0} \mathbf{r}(t) = \lim_{t\to t_0} f(t)\mathbf{i} + \lim_{t\to t_0} g(t)\mathbf{j} + \lim_{t\to t_0} h(t)\mathbf{k},$$

provided all the limits of component functions exist.

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provided all the limits of component functions exist.

Thus $\lim_{t\to t_0} \mathbf{r}(t)$ exists if and only if each component limit exists.

Example

If $\mathbf{r}(t) = \frac{\sin t}{t}\mathbf{i} + \frac{\tan^2 t}{\sin 2t}\mathbf{j} - \frac{t^3-8}{t+2}\mathbf{k}$, then

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$$\lim_{t\to 0}\mathbf{r}(t) = \left(\lim_{t\to 0}\frac{\sin t}{t}\right)\mathbf{i} + \left(\lim_{t\to 0}\frac{\tan^2 t}{\sin 2t}\right)\mathbf{j} + \left(\lim_{t\to 0}\frac{t^3-8}{t+2}\right)\mathbf{k}$$

Thus

$$\lim_{t\to 0}\mathbf{r}(t) = \mathbf{i} + 4\mathbf{k}.$$

Continuity

A vector function $\mathbf{r}(t)$ is continuous at a point $t = t_0$ in its domain if

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A vector function $\mathbf{r}(t)$ is continuous if it is continuous at every point in its domain.

Example

Discuss the continuity of the vector-valued function given by

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k},$$

where a is a constant, at $t = 0$.

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Example

The vector function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ is continuous in its domain as all its component functions $\cos t$, $\sin t$ and t are continuous.

Derivative of a vector function

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a derivative (is differentiable) at t if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

$$\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$$
$$= \left(f(t+\Delta t) - f(t)\right)\mathbf{i} + \left(g(t+\Delta t) - g(t)\right)\mathbf{j} + \left(h(t+\Delta t) - h(t)\right)\mathbf{k}.$$

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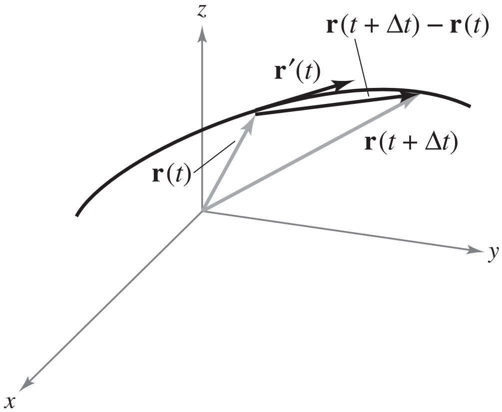
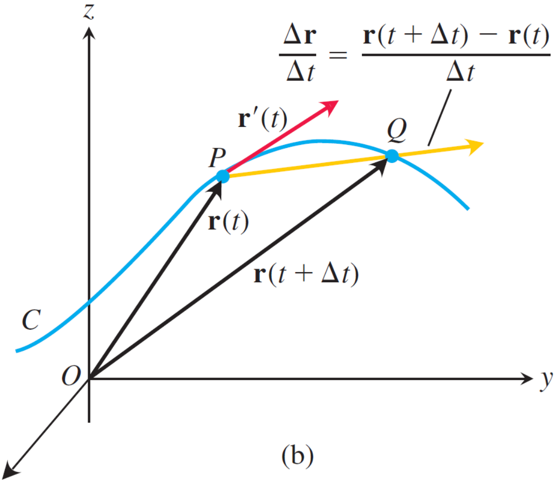
$$= (f(t+\Delta t) - f(t))\,\mathbf{i} + (g(t+\Delta t) - g(t))\,\mathbf{j} + (h(t+\Delta t) - h(t))\,\mathbf{k}.$$

$$\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = \left(\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\right)\mathbf{i} + \left(\lim_{\Delta t\to 0}\frac{g(t+\Delta t)-g(t)}{\Delta t}\right)\mathbf{j}$$
$$+\left(\lim_{\Delta t\to 0}\frac{h(t+\Delta t)-h(t)}{\Delta t}\right)\mathbf{k}$$
$$= \left(\frac{df}{dt}\right)\mathbf{i} + \left(\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{dh}{dt}\right)\mathbf{k}.$$


z
$\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$
$\mathbf{r}'(t)$
$\mathbf{r}(t+\Delta t)$
$\mathbf{r}(t)$
y
x

Derivative



(b)

Example

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

Note

The derivative $\mathbf{r}'(t)$ of the vector-valued function $\mathbf{r}$ is itself a vector-valued function.

Smooth Curve

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1. The curve given by $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}$ is smooth as $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$ is continuous and nonzero for all t.

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2. Find all the value of t such that the curve $r(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is smooth.

Piecewise Smooth Curve

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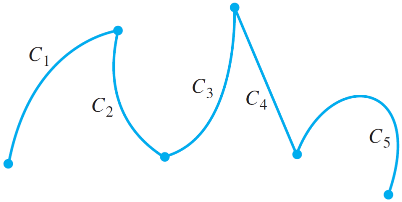

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be the position vector of a particle at time t moving along a smooth curve in space. Then

$$\begin{aligned}
\text{Velocity:} \quad & \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \\
\text{Speed:} \quad & v(t) = |\mathbf{v}(t)| \\
\text{Acceleration:} \quad & \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \\
\text{Direction of motion:} \quad & \frac{\mathbf{v}}{|\mathbf{v}|}
\end{aligned}$$

Remark

For a smooth curve speed is always positive (can never be zero).

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That is

$$\text{Velocity} = (\text{speed})(\text{direction}).$$

Question

The position vector of a moving particle moving in xy-plane along the cycloid $x = t - \sin t$, $y = 1 - \cos t$ is $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos(t))\mathbf{j}$. Find the particle's velocity and acceleration vectors at times $t = \pi$ and $t = 3\pi/2$.

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Sol. Given $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos(t))\mathbf{j}$. So

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + \sin(t)\mathbf{j}$$

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$$\text{and } \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2} = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

$$\begin{array}{lcll} \text{At } t = \pi & : & \mathbf{v}(\pi) = 2\mathbf{i}, & \mathbf{a}(\pi) = -\mathbf{j} \\ \text{At } t = 3\pi/2 & : & \mathbf{v}(3\pi/2) = \mathbf{i} - \mathbf{j}, & \mathbf{a}(3\pi/2) = -\mathbf{i}. \end{array}$$

Question

Given that $\mathbf{r}(t) = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$ is the position of a particle in space at any time t. Find the particle's speed and direction of motion at $t = 1$. Write the particle's velocity at that time as the product of its speed and direction.

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- $v(1) = |\mathbf{v}(1)| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$.

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- $v(1) = |\mathbf{v}(1)| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$.
- Direction of motion $= \frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$.

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- Particle's velocity at that time as the product of its speed and direction $\mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$.

Differentiation rules for vector functions

Let $\mathbf{u} = \mathbf{u}(t)$ and $\mathbf{v} = \mathbf{v}(t)$ be differentiable vector functions of t, **C** a constant vector and c any real number and f any real-valued function defined on an interval. Then

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3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$

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4. $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t).\mathbf{v}(t)] = \mathbf{u}(t).\mathbf{v}'(t) + \mathbf{u}'(t).\mathbf{v}(t)$

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1. $\frac{d}{dt}\mathbf{C} = 0$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t).\mathbf{v}(t)] = \mathbf{u}(t).\mathbf{v}'(t) + \mathbf{u}'(t).\mathbf{v}(t)$
6. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$

Chain rule for differentiation

If $\mathbf{u}(w)$ is a differentiable vector function of w and $w = f(t)$ is a differentiable scalar function of t, then

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{dw}\frac{dw}{dt}$$

Vector functions of constant lengt

When we track a particle moving on a sphere with center at origin, the position vector $\mathbf{r}(t)$ of point has a constant length equal to radius of sphere, then $\frac{d\mathbf{r}}{dt}$ is tangent to path of motion, and hence perpendicular to $\mathbf{r}(t)$, *i.e.*, if $\mathbf{u}(t)$ is a differentiable function of constant length, then vector and its first derivative are orthogonal *i.e.*, $\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0$.

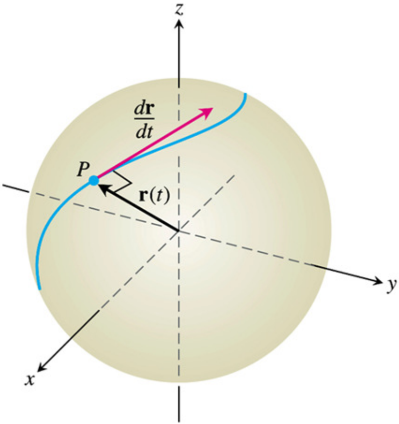

Figure: If a particle moves on a sphere in such a way that its position vector $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

Proof.

$$|\mathbf{u}(t)| = c \Rightarrow |\mathbf{u}(t)|^2 = c^2 \Rightarrow \mathbf{u}(t)\cdot\mathbf{u}(t) = c^2$$

$$\frac{d}{dt}(\mathbf{u}.\mathbf{u}) = 0$$

$$\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{u} = 0$$

$$2\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0$$

$$\mathbf{u}\cdot\frac{d\mathbf{u}}{dt} = 0$$

Hence we conclude that if $\mathbf{r}$ is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Hence we conclude that if $\mathbf{r}$ is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Example

For the vector function $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sqrt{3}\mathbf{k}$, we have $|\mathbf{r}(t)| = \sqrt{\sin^2 t + \cos^2 t + 3} = 2$ (a constant), so (without calculating $\frac{d\mathbf{r}}{dt}$) we must have $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0$.

Example

Given that $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}$ is the position of a particle in space at any time t. Find the time(s) in the interval $0 \leqslant t \leqslant 2\pi$ when the velocity and acceleration are orthogonal.

Example

Given that $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}$ is the position of a particle in space at any time t. Find the time(s) in the interval $0 \leqslant t \leqslant 2\pi$ when the velocity and acceleration are orthogonal.

Sol. We have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 2\mathbf{j}.$$

Thus

$$\mathbf{v} \cdot \mathbf{a} = 2t.$$

For $\mathbf{v}$ and $\mathbf{a}$ to be orthogonal, we need

$$\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= 0 \\ \Rightarrow 2t &= 0 \\ \Rightarrow t &= 0. \end{aligned}$$

Question

A particle moves along the top of the parabola $y^2 = 3x$ from left to right at a constant speed of 6 units per second. Find the velocity of the particle as it moves through the point $(3, 3)$.

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Sol. A possible parametrization of the parabola is

$$\mathbf{r}(t) = \frac{t^2}{3}\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}.$$

Question

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Sol. A possible parametrization of the parabola is

$$\mathbf{r}(t) = \frac{t^2}{3}\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}.$$

Thus

$$\frac{d\mathbf{r}}{dt} = \frac{2t}{3}\mathbf{i} + \mathbf{j}.$$

The point $(3,3)$ corresponds to $t=3$.

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=3} = 2\mathbf{i}+\mathbf{j}.$$

The velocity vector at $t=3$ is in the direction of the vector

$$\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}.$$

Therefore, the velocity vector is

$$\begin{aligned}\mathbf{v}(t)|_{t=3} &= (\text{speed})(\text{direction})\\ &= (6)\left(\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{j}\right)\end{aligned}$$

Section 13.2

Integrals of Vector Functions

Indefinite Integral

The indefinite integral of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, written as $\int \mathbf{r}(t)\ dt$.
If $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$, then

$$\int \mathbf{r}(t)\ dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is a constant vector of integration.

The vector $\mathbf{r}(t)$ is integrable over $[a, b]$ if and only if each of its components are integrable over $[a, b]$.

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That is if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is integrable over $[a, b]$, then

The vector $\mathbf{r}(t)$ is integrable over $[a, b]$ if and only if each of its components are integrable over $[a, b]$.

That is if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is integrable over $[a, b]$, then

$$\int_a^b \mathbf{r}(t)\ dt = \left(\int_a^b f(t)\ dt\right)\mathbf{i} + \left(\int_a^b g(t)\ dt\right)\mathbf{j} + \left(\int_a^b h(t)\ dt\right)\mathbf{k}.$$

Question

Evaluate the integral

$$\int_0^1 \left[t e^{t^2} \mathbf{i} + e^{-t} \mathbf{j} + \mathbf{k} \right] \, dt.$$

Question

Evaluate the integral

$$\int_0^1 \left[te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right]\, dt.$$

Sol.

$$\begin{aligned}\int_0^1 \left[te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right]\, dt &= \left(\int_0^1 te^{t^2}\, dt\right)\mathbf{i} + \left(\int_0^1 e^{-t}\, dt\right)\mathbf{j} + \left(\int_0^1 dt\right)\mathbf{k} \\ &= \left(\frac{e^{t^2}}{2}\right)\Bigg|_{t=0}^{t=1} \mathbf{i} + \left(-e^{-t}\right)\Big|_{t=0}^{t=1}\mathbf{j} + (t)\Big|_{t=0}^{t=1}\mathbf{k} \\ &= \left(\frac{e-1}{2}\right)\mathbf{i} + (1-e^{-1})\mathbf{j} + \mathbf{k}.\end{aligned}$$

Question

Evaluate the integral

$$\int_0^1 \left[te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right]\, dt.$$

Sol.

$$\begin{aligned}
\int_0^1 \left[te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right]\, dt &= \left(\int_0^1 te^{t^2}\, dt\right)\mathbf{i} + \left(\int_0^1 e^{-t}\, dt\right)\mathbf{j} + \left(\int_0^1 dt\right)\mathbf{k} \\
&= \left(\frac{e^{t^2}}{2}\right)\Bigg|_{t=0}^{t=1} \mathbf{i} + \left(-e^{-t}\right)\Big|_{t=0}^{t=1}\mathbf{j} + (t)\Big|_{t=0}^{t=1}\mathbf{k} \\
&= \left(\frac{e-1}{2}\right)\mathbf{i} + (1-e^{-1})\mathbf{j} + \mathbf{k}.
\end{aligned}$$

Question

Solve the initial value problem for $\mathbf{r}(t)$ as a vector function of t.

$$\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$$

$$\mathbf{r}(0) = 100\mathbf{k}$$

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}.$$

Question

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$$\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$$
$$\mathbf{r}(0) = 100\mathbf{k}$$
$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}.$$

Sol. On integrating $\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$ gives

$$\frac{d\mathbf{r}}{dt} = -32t\mathbf{k} + \mathbf{C}_1 \tag{1}$$

On putting $t = 0$: $\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{C}_1$, but it is given to be $8\mathbf{i} + 8\mathbf{j}$. Thus we have $\mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$. Replacing the value of $\mathbf{C}_1$ in (1):

$$\frac{d\mathbf{r}}{dt} = -32t\mathbf{k} + 8\mathbf{i} + 8\mathbf{j}. \tag{2}$$

Again on integrating, (2) gives:

$$\mathbf{r}(t) = -16t^2\mathbf{k} + (8\mathbf{i} + 8\mathbf{j})t + \mathbf{C}_2 \tag{3}$$

On putting $t = 0$: $\mathbf{r}(0) = \mathbf{C}_2$, but it is given to be $100\mathbf{k}$. Thus we have $\mathbf{C}_2 = 100\mathbf{k}$. Replacing the value of $\mathbf{C}_2$ in (3):

$$\mathbf{r}(t) = (8t)\mathbf{i} + (8t)\mathbf{j} + (100 - 16t^2)\mathbf{k}.$$

Question

At time $t = 0$, a particle is located at the point $(1, 2, 3)$. It travels in a straight line to the point $(4, 1, 4)$, has speed 2 at $(1, 2, 3)$ and constant acceleration $3\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find an equation for the position vector $\mathbf{r}(t)$ of the particle at time t.

Question

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Sol. Given

$$\mathbf{r}(0) = (1, 2, 3)$$
$$|\mathbf{v}(0)| = 2$$
$$\mathbf{a}(t) = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Integrating $\mathbf{a}(t)$:

$$\mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1 \tag{4}$$

Now the particle travels in the direction of the vector $(4-1)\mathbf{i}+(1-2)\mathbf{j}+(4-3)\mathbf{k}=3\mathbf{i}-\mathbf{j}+\mathbf{k}$. Its length is

$$\sqrt{3^2+(-1)^2+1^2}=\sqrt{11}.$$

So the direction is

$$\left(\frac{3}{\sqrt{11}}\right)\mathbf{i}-\left(\frac{1}{\sqrt{11}}\right)\mathbf{j}+\left(\frac{1}{\sqrt{11}}\right)\mathbf{k}.$$

Thus

$$\mathbf{v}(0) = \text{(speed)(direction)} = \left(\frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(\frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(\frac{2}{\sqrt{11}}\right)\mathbf{k}.$$

Now equation (4) gives $\mathbf{v}(0) = \mathbf{C}_1$ and so

$$\mathbf{C}_1 = \left(\frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(\frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(\frac{2}{\sqrt{11}}\right)\mathbf{k}.$$

Thus on replacing the value of $\mathbf{C}_1$ in (4)

$$\mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}.$$

Now integrate $\mathbf{v}(t)$ to obtain $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \left(3\frac{t^2}{2} + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{t^2}{2} + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2. \tag{5}$$

This gives $\mathbf{r}(0) = \mathbf{C}_2$ and since given $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, so we have $\mathbf{C}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and finally replace $\mathbf{C}_2$ in (5):

$$\mathbf{r}(t) = \left(\frac{3t^2}{2} + \frac{6t}{\sqrt{11}} + 1\right)\mathbf{i} - \left(\frac{t^2}{2} + \frac{2t}{\sqrt{11}} - 2\right)\mathbf{j} + \left(\frac{t^2}{2} + \frac{2t}{\sqrt{11}} + 3\right)\mathbf{k}.$$

Section 13.3

Arc Length in Space

- One of the special features of smooth space curves is that they have a measurable length.

- One of the special features of smooth space curves is that they have a measurable length.
- That enables us to find points along these curves by giving their directed distances along the curve from some base point.

Arc Length Along a Plane Curve

1. If a curve C is defined parametrically by $x = x(t),\ y = y(t),\ a \leqslant t \leqslant b$, where $x'(t),\ y'(t)$ are continuous and not simultaneously zero on $[a, b]$ and C traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Arc Length Along a Space Curve

1. The length of smooth curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \;\; a \leqslant t \leqslant b,$$

that is traced exactly once as t increases from $t = a$ to $t = b$, is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Arc Length Along a Space Curve

Since $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ is nothing but the speed of the moving particle so in terms of speed the arc length formula can be written as

$$L = \int_a^b |\mathbf{v}(t)| \ dt.$$

Arc Length parameter with base point at t_0

If we fix a base point t_0, then for each t,

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \; d\tau$$

becomes a function of t, called as ‘arc length function’ and s is called as arc length parameter of the curve.

1. If $t > t_0$, then $s(t) > 0$.

Arc Length parameter with base point at t_0

If we fix a base point t_0, then for each t,

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| \; d\tau$$

becomes a function of t, called as 'arc length function' and s is called as arc length parameter of the curve.

1. If $t > t_0$, then $s(t) > 0$.
2. If $t < t_0$, then $s(t) < 0$.

Remark

Using the fundamental theorem of calculus, we have

$$\frac{ds}{dt} = |\mathbf{v}(t)|.$$

Since $|\mathbf{v}(t)|$ is never zero for a smooth curve so $\frac{ds}{dt} > 0$. Thus $s(t)$ is strictly increasing function of t.

Unit Tangent Vector

We know that the velocity vector $\mathbf{v}(t)$ is tangent to the curve and so the vector

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|},$$

is the unit vector tangent to the (smooth) curve, called the **unit tangent vector**.

For a smooth curve, we can write

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{d\mathbf{r}}{ds}\left|\mathbf{v}(t)\right|.$$

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Thus we have

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{d\mathbf{r}}{ds}.$$

Thus $\dfrac{d\mathbf{r}}{ds}$ is the unit tangent vector in the direction of the velocity vector $\mathbf{v}$.

Question

Find the following curve's unit tangent vector and length of the indicated portion of the curve

$$\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \ \sqrt{2} \leqslant t \leqslant 2.$$

Question

Find the following curve's unit tangent vector and length of the indicated portion of the curve

$$\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \;\; \sqrt{2} \leqslant t \leqslant 2.$$

Sol. Given $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}$, so

$$\mathbf{v}(t) = (t\cos t)\mathbf{i} - (t\sin t)\mathbf{j}$$
$$|\mathbf{v}(t)| = \sqrt{(t\cos t)^2 + (-t\sin t)^2} = |t| = t.$$

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

Now

$$L = \int_{\sqrt{2}}^{2} |\mathbf{v}(t)| \; dt = \int_{\sqrt{2}}^{2} t \; dt = 1.$$

Question

Find the point on the curve

$$\mathbf{r}(t) = (12\sin t)\mathbf{i} - (12\cos t)\mathbf{j} + 5t\mathbf{k},$$

at a distance 13π units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.

Question

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$$\mathbf{r}(t) = (12\sin t)\mathbf{i} - (12\cos t)\mathbf{j} + 5t\mathbf{k},$$

at a distance 13π units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.

Sol. Given $\mathbf{r}(t) = (12\sin t)\mathbf{i} - (12\cos t)\mathbf{j} + 5t\mathbf{k}$, so

$$\mathbf{v}(t) = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{(12\cos t)^2 + (12\sin t)^2 + (5)^2} = 13.$$

Let the point be $P(t_0) = (12\sin t_0, -12\cos t_0, 5t_0)$ [note that the initial point $(0, -12, 0)$ is associated with $t = 0$]. Then

$$s(t_0) = \int_0^{t_0} |\mathbf{v}(\tau)|\, d\tau = 13t_0.$$

But it is given to be -13π, so we must have $13t_0 = -13\pi$ and so $t_0 = -\pi$. Then the point is

$$(12\sin(-\pi), -12\cos(-\pi), 5(-\pi)) = (0, 12, -5\pi).$$

Question

Find the arc length parameter along the curve from the point where $t = 0$

$$\mathbf{r}(t) = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k}, \ -1 \leqslant t \leqslant 0.$$

Also find the length of the portion of the curve.

Question

Find the arc length parameter along the curve from the point where $t = 0$

$$\mathbf{r}(t) = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k}, \ -1 \leqslant t \leqslant 0.$$

Also find the length of the portion of the curve.

Sol. Given $\mathbf{r}(t) = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k}$, so

$$\mathbf{v}(t) = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$
$$|\mathbf{v}(t)| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7.$$

Thus

$$s(t) = \int_0^t |\mathbf{v}(\tau)| \, d\tau = \int_0^t 7 \, d\tau = 7t.$$

Thus

$$s(t) = \int_0^t |\mathbf{v}(\tau)|\, d\tau = \int_0^t 7\, d\tau = 7t.$$

Now we have

$$s(-1) = 7(-1) = -7.$$

Thus the required arc length is 7.

Question

Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k},$$

from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, , 0)$.

Question

Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k},$$

from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, , 0)$.

Sol. First note that the point $(0, 0, 1)$ is associated with $t = 0$ and the point $(\sqrt{2}, \sqrt{2}, , 0)$ is associated with $t = 1$. Now we have

$$\mathbf{v}(t) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k}$$
$$|\mathbf{v}(t)| = 2\sqrt{1 + t^2}.$$

Thus the required length is

$$
\begin{aligned}
L &= 2\int_0^1 \sqrt{1+t^2}\;dt \\
&= 2\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\right]_0^1 \\
&= \sqrt{2} + \ln(1+\sqrt{2}).
\end{aligned}
$$