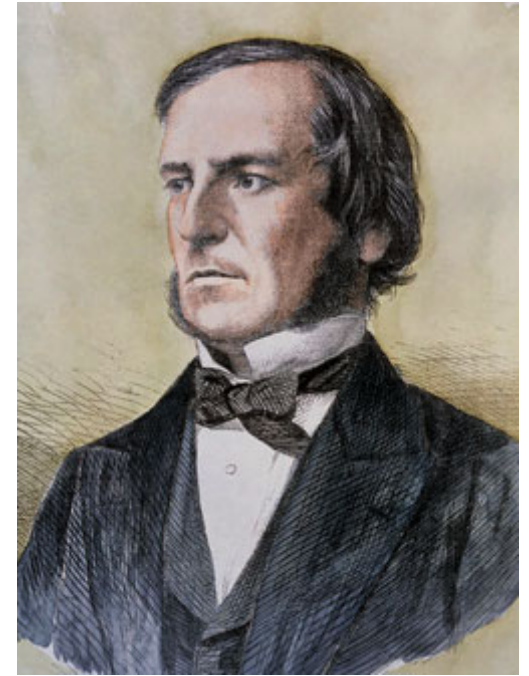


Topic 3

Boolean Algebra & Optimization

Boolean Algebra

- “Traditional” algebra
 - Variables represent real numbers
 - Operators operate on variables, and return real numbers
- *Boolean Algebra*
 - Developed mid-1800’s by George Boole to formalize human thought
 - Variables represent 0 or 1 only
 - Operators return 0 or 1 only
 - Basic operators
 - AND, OR, NOT



Boolean Algebra Terminology

- **Example equation:** $F(a,b,c) = a'bc + abc' + ab + c$
- *Variable*
 - Represents a value (0 or 1)
 - Three variables: a, b, and c
- *Literal*
 - Appearance of a variable, in true or complemented form
 - Nine literals: a', b, c, a, b, c', a, b, and c
- *Product term*
 - AND of literals
 - Four product terms: a'bc, abc', ab, c
- *Sum term*
 - OR of literals
 - No sum terms
- *Sum-of-products*
 - Equation written as OR of product terms only
 - Above equation is in sum-of-products form. “ $F = (a+b)c + d$ ” is not.

Basic Theorems of Boolean Algebra

- **(a) $x + 0 = x$;**
- **(a) $x + x' = 1$;**
- **(a) $x + x = x$;**
- **(a) $x + 1 = 1$;**
- **$(x')' = x$;**
- (b) $x \cdot 0 = 0$;**
- (b) $x \cdot x' = 0$;**
- (b) $x \cdot x = x$;**
- (b) $x \cdot 1 = x$;**
- (theorem 1)**
- (theorem 2)**
- (theorem 3)**
- (theorem 4)**
- (involution)**

Basic Theorems of Boolean Algebra

- (a) $x + y = y + x$; (b) $xy = yx$; (commutative)
- (a) $x + (y + z) = (x + y) + z$; (b) $x(yz) = (xy)z$; (associative)
- (a) $x(y + z) = xy + xz$; (b) $x + yz = (x+y)(x+z)$; (distributive)
- (a) $x + xy = x$; (b) $x(x + y) = x$; (absorption)
- (a) $xy + xy' = x$; (b) $(x + y)(x + y') = x$ (theorem 5)
- (a) $x + x'y = x + y$ (b) $x(x' + y) = xy$ (theorem 6)

Operator Precedence

- **The operator precedence for evaluating basic Boolean expressions is:**
 - Parenthesis
 - NOT
 - AND
 - OR
- **Example: $(x + y)'$**
 - Evaluate the parenthesized expression $(x + y)$ first and then the inversion
- **Example: $x + xy$**
 - Evaluate xy first and then OR it with the value of x

Application of Basic Theorems

- **Prove theorem 5(a): $xy + xy' = x$**

$$xy + xy'$$

$$= x(y + y') \quad \text{(distributive (a))}$$

$$= x \cdot 1 \quad \text{(theorem 2(a))}$$

$$= x \quad \text{(theorem 4(b))}$$

Application of Basic Theorems

- **Prove theorem 5(b): $(x + y)(x + y') = x$**

$$(x + y)(x + y')$$

$$= x + yy' \quad \text{(distributive (b))}$$

$$= x + 0 \quad \text{(theorem 2(b))}$$

$$= x \quad \text{(theorem 1(a))}$$

Application of Basic Theorems

- **Prove theorem 5(b): $(x + y)(x + y') = x$, alternatively**

$$(x + y)(x + y')$$

$$= (x + y)x + (x + y)y' \quad (\text{distributive (a)})$$

$$= xx + xy + xy' + yy' \quad (\text{distributive (a)})$$

$$= x + xy + xy' + 0 \quad (\text{theorem 2(b), 3(b)})$$

$$= x + x(y + y') \quad (\text{theorem 1(a), distributive (a)})$$

$$= x + x \quad (\text{theorem 2(a), 4(b)})$$

$$= x \quad (\text{theorem 3(a)})$$

Application of Basic Theorems

- **Prove theorem 6(a): $x + x'y = x + y$**

$$x + x'y$$

$$= (x + x')(x + y) \quad \text{(distributive (a))}$$

$$= 1 \cdot (x + y) \quad \text{(theorem 2(a))}$$

$$= x + y \quad \text{(theorem 4(b))}$$

Application of Basic Theorems

- **Exercises**

1. $x'y + x'$

2. $a'bc + a'$

3. $a'b'c + (a'b'c)'$

4. $(a + b)(c + b)(d' + b)(acd' + e)$

5. $wx'y' + wxz' + wx'yz'$

DeMorgan's Law

(a) $(x + y)' = x'y'$

(b) $(xy)' = x' + y'$

- **Very Useful**

Applications of DeMorgan's Law

- Find the complement of $F = x(y'z' + yz)$
- $F' = (x(y'z' + yz))'$ (All steps by DeMorgan's law)
 $= x' + (y'z' + yz)'$
 $= x' + (y'z')' \cdot (yz)'$
 $= x' + (y + z)(y' + z')$
- Exercise
 $((AB' + C)D' + E)'$

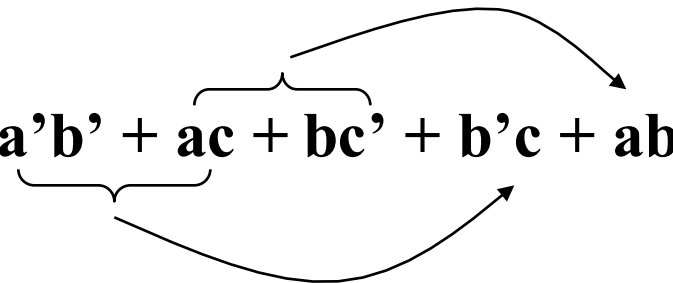
Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z$
- **Proof:**

$$\begin{aligned} & XY + X'Z + YZ \\ &= XY + X'Z + (X + X')YZ \\ &= XY + XYZ + X'Z + X'YZ \\ &= X(Y + YZ) + X'(Z + YZ) \\ &= XY(1 + Z) + X'Z(1 + Y) \\ &= XY + X'Z \end{aligned}$$

Application of Consensus Theorem

- $a'b' + \overbrace{ac + bc'} + b'c + ab = a'b' + ac + bc'$



XOR Properties

$$x \oplus 0 = x \text{ (a)} \qquad x \oplus 1 = x' \text{ (b)} \qquad \text{(theorem 1)}$$

$$x \oplus x = 0 \text{ (a)} \qquad x \oplus x' = 1 \text{ (b)} \qquad \text{(theorem 2)}$$

$$x \oplus y' = x' \oplus y = (x \oplus y)' \qquad \text{(theorem 3)}$$

$$x \oplus y = y \oplus x \qquad \text{(commutative)}$$

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) = x \oplus y \oplus z \qquad \text{(associative)}$$

Boolean Representation: Minterm and Maxterm

- A binary literal may be in the unprimed (true) form and primed (false) forms, representing true and false conditions respectively
 - E.g. a vs. a'
- **Minterm** is a product of n literals in which each literal appears exactly once in either true or complemented form, but not both
 - Minterm is represented by m_i
- **Maxterm** is a sum of n literals in which each literal appears exactly once in either true or complemented form, but not both
 - Maxterm is represented by M_i

Minterm and Maxterm

			Minterms		Maxterms	
x	y	z	Term	Designation	Term	Designation
0	0	0	$x'y'z'$	m_0	$x+y+z$	M_0
0	0	1	$x'y'z$	m_1	$x+y+z'$	M_1
0	1	0	$x'yz'$	m_2	$x+y'+z$	M_2
0	1	1	$x'yz$	m_3	$x+y'+z'$	M_3
1	0	0	$xy'z'$	m_4	$x'+y+z$	M_4
1	0	1	$xy'z$	m_5	$x'+y+z'$	M_5
1	1	0	xyz'	m_6	$x'+y'+z$	M_6
1	1	1	xyz	m_7	$x'+y'+z'$	M_7

Subscription i of minterm is the decimal equivalent of the corresponding binary combination

Minterm in Truth Table

x con1	y con2	z con3	F result
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Result would happen if con1 is **false** AND con2 is **false** AND con3 is **true**, $x'y'z$

Result would happen if con1 is **false** AND con2 is **true** AND con3 is **true**, $x'yz$

Result would happen if con1 is **true** AND con2 is **false** AND con3 is **false**, $xy'z'$

Result would happen if con1 is **true** AND con2 is **false** AND con3 is **true**, xyz

Result would be true if any of these four conditions is true, implies OR logic,
This relationship is expressed by:
$$F = x'y'z + x'yz + xy'z' + xyz$$

Minterm Expression From Truth Table

- A Boolean Equation can be derived from a truth table and expressed as a sum-of-minterms (**standard-sum-of-products**)
- **The minterms chosen in the sum-of-minterms expression are those which produce a logic 1 for the corresponding output**
- Example:

$$\begin{aligned} F &= x'y'z + x'yz + xy'z' + xy'z \\ &= m_1 + m_3 + m_4 + m_5 \\ &= \Sigma m(1, 3, 4, 5) \end{aligned}$$

x con1	y con2	z con3	F result
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Exercise

- Find minterm logic equation from these truth table

x	y	z	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

W	X	Y	Z	F		
0	0	0	0	1	m0	$W'X'Y'Z'$
0	0	0	1	0	m1	$W'X'Y'Z$
0	0	1	0	0	m2	$W'X'YZ'$
0	0	1	1	1	m3	$W'X'YZ$
0	1	0	0	0	m4	$W'XY'Z'$
0	1	0	1	0	m5	$W'XY'Z$
0	1	1	0	0	m6	$W'XYZ'$
0	1	1	1	1	m7	$W'XYZ$
1	0	0	0	1	m8	$WX'Y'Z'$
1	0	0	1	0	m9	$WX'Y'Z$
1	0	1	0	0	m10	$WX'YZ'$
1	0	1	1	0	m11	$WX'YZ$
1	1	0	0	0	m12	$WXY'Z'$
1	1	0	1	0	m13	$WXY'Z$
1	1	1	0	0	m14	$WXYZ'$
1	1	1	1	1	m15	$WXYZ$

Minterms and Maxterms

- **The complement of Minterm is the corresponding Maxterm, vice versa**
 - $m_i' = M_i$
 - e.g.: $m_0 = x'y'z'$
 $m_0' = (x'y'z')' = x + y + z = M_0$ (DeMorgan's)
- **Conversion between Standard Forms**
 - the term numbers missing from one form will be the term numbers used in the other form
 - e.g.: if all the terms are indexed by $0 \sim 7$, then
 $F = \Sigma m(1, 2, 4, 7) = \Pi M(0, 3, 5, 6)$

Minterms and Maxterms

- Example: In the given truth table, F1 is output of a 3-input device**

Truth Table

x	y	z	F1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Sum-of-minterms

$$F1 = x'y'z + xy'z' + xy'z + xyz' + xyz$$

$$F1 = m_1 + m_4 + m_5 + m_6 + m_7$$

$$F1 = \Sigma (1, 4, 5, 6, 7)$$

Product-of-maxterms

$$F1 = (x+y+z) \cdot (x+y'+z) \cdot (x+y'+z')$$

$$F1 = M_0 \cdot M_2 \cdot M_3$$

$$F1 = \Pi (0, 2, 3)$$

Incompletely Specified Functions

- In a circuit, some input conditions may never happen, then the output is not completely specified
- The corresponding output is designated as “x”, called *don't care*
- A *don't care* output could be either 0 or 1
- $F = \Sigma m(1, 3, 4) + \Sigma d(2, 5)$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	X
0	1	1	1
1	0	0	1
1	0	1	X
1	1	0	0
1	1	1	0

Simplified Forms

- **The minterm and maxterm forms can be further simplified**
 - Boolean function may contain less number of terms
 - Each term may have less literals
 - e.g.:

Simplified SOP

$$F1 = x + y'z$$

Simplified POS

$$F1 = (x + y')(x + z)$$

Why to simplify? & How to?

- Why?
- How to? Boolean theorems. And more....