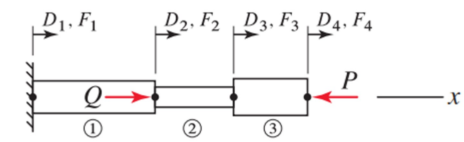
HW1 Q1

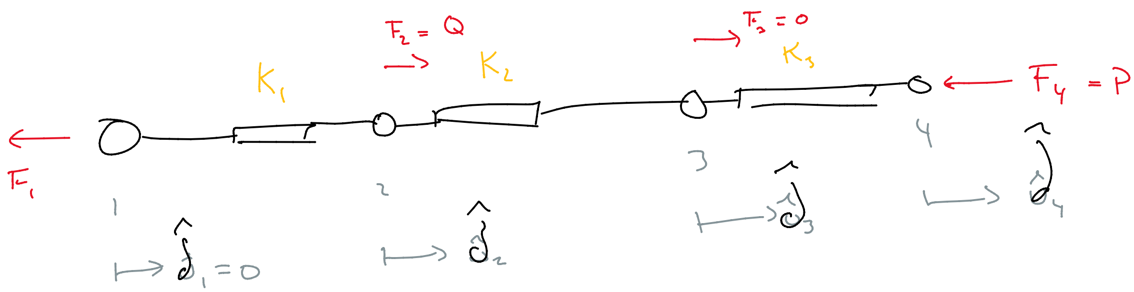
Wednesday, September 15, 2021 2:52 PM

For each of the following problem, formulate the global stiffness matrix K, and determine the global equilibrium equations [K]{D}={F}. For Problem 1, solve the system of linear equilibrium equations you have derived and determine the nodal displacements, reactions at the support, and nodal forces.

Problem 1 (60 pts)



$$\kappa_1 = \frac{A_1E_1}{L_1} \quad \kappa_2 = \frac{A_2E_2}{L_2} \quad \kappa_3 = \frac{A_3E_3}{L_3}$$



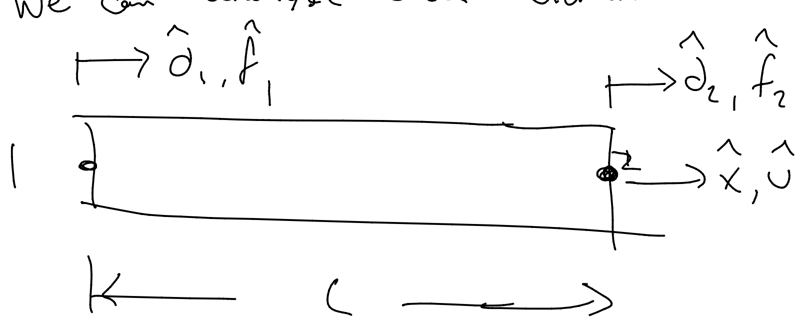
This problem has 3 elements, 4 nodes

Element	1	α	nodes	1,2
"	2	α	"	2,3
"	3	α	"	3,4

In Equilibrium:

$$\sum F = 0 \Rightarrow \sum_{\#\text{ nodes}} \hat{f} = F$$

We can analyze each element such that:



A Bar w/ length L, area A, Young's modulus E.

A bar w/ length L, area A, Young's modulus E.

Displacement function

$$\hat{U} = a_1 + a_2 \hat{x} \Rightarrow$$ Linear interpolation function for element.

Matrix form

$$\hat{U} = \begin{bmatrix} 1, & \hat{x} \end{bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

As a function of nodal displacements:

$$\hat{u}(0) = \hat{d}_1 = a_1$$

$$\hat{u}(L) = \hat{d}_2 = a_1 + a_2 L \Rightarrow a_2 = \frac{\hat{d}_2 - \hat{d}_1}{L}$$

$$\hat{U} = \hat{d}_1 + \left(\frac{\hat{d}_2 - \hat{d}_1}{L}\right) \hat{x}$$

$$\hat{U} = \left(1 - \frac{\hat{x}}{L}\right) \hat{d}_1 + \left(\frac{\hat{y}}{L}\right) \hat{d}_2$$

In matrix form:

$$\hat{U} = \begin{bmatrix} 1 - \frac{\hat{x}}{L}, & \frac{\hat{x}}{L} \end{bmatrix} \begin{Bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{Bmatrix}$$

$$\hat{U} = \begin{bmatrix} N_1, & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{Bmatrix} = N \hat{d}$$

$$N_1 = 1 - \frac{\hat{x}}{L} \qquad N = \frac{\hat{x}}{L}$$

This is our shape function.

This is our shape function.

We then use the constitutive model

$$\sigma = E \varepsilon$$

Strain - stress relationship:

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{d}{d\hat{x}} \left(N \hat{d} \right)$$

$\hat{d} \neq$ function of $\hat{x}$

Therefore, $\varepsilon = \underbrace{\frac{dN}{d\hat{x}}}_{\text{Strain-disp.} = B} \hat{d} = B \hat{d}$

$$\varepsilon = \frac{d}{dx} \left[\frac{L - \hat{x}}{L} , \frac{\hat{x}}{L} \right] \hat{d}$$

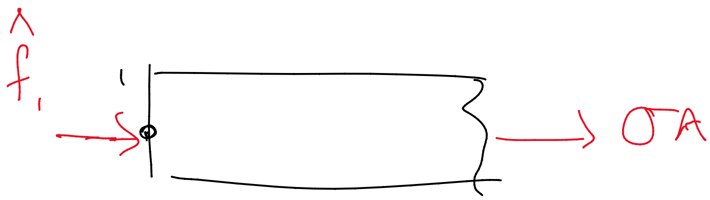
$$\varepsilon = B \hat{d}$$

Then, $\sigma = E B \hat{d}$

$$\sigma = E \left[\frac{-1}{L} , \frac{1}{L} \right] \begin{Bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{Bmatrix}$$

$$\sigma = E \, \frac{d_2 - d_1}{L}$$

For equilibrium,



$$\sum F = 0 = \hat{f}_1 + \hat{f}_2$$

$$\sum F = 0 = \hat{f}_1 + \sigma A$$

$$\Rightarrow \hat{f}_1 = -\sigma A = -\frac{AE}{L}\left(\hat{d}_2 - \hat{d}_1\right)$$

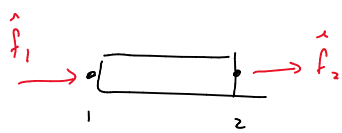
$$\hat{f}_1 = \frac{AE}{L}\left(\hat{d}_1 - \hat{d}_2\right) = k\left(\hat{d}_1 - \hat{d}_2\right)$$

$\hookrightarrow k = \frac{AE}{L}$

$$\hat{f}_2 = -\hat{f}_1 = k\left(\hat{d}_2 - \hat{d}_1\right)$$

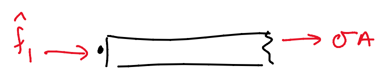
Then, we can apply this to our nodes 1, 2, 3, 4, elements ①, ②, ③

In our case, that means we can link $\hat{\delta}_1, \hat{\delta}_2$ to D_1, D_2 when we consider elements 1, 2, 3 continuously using stress-strain relationship



$$\sigma = E B \hat{\delta}$$

In matrix form w/ some workaround:



$$k_1 \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} \Rightarrow \Sigma F = 0 = \hat{f}_1 + \hat{f}_2 = \hat{f}_1 + \sigma A$$

$k_1 = \frac{A_1 E_1}{L}$

$\hookrightarrow \hat{f}_1 = \sigma A = k_1(\hat{\delta}_1 - \hat{\delta}_2)$

For each node, $\hookrightarrow \hat{f}_2 = k_1(\hat{\delta}_2 - \hat{\delta}_1)$

Derivation of this for elements 1, 2, 3

Equivalent to (in global eq.)

$$K_1 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{bmatrix} ①$$

for element 1.

$\hat{f}_1 = k_1(D_1 - D_2)$

$\hat{f}_2 = k_1(D_2 - D_1)$

w/ respect to element 1

& so on, which is where the matrix comes from.

Element 2 equivalently models:

$$K_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{bmatrix} ② \quad \left\{ \begin{aligned} \hat{f}_2 &= k_2(D_2 - D_3) \\ \hat{f}_3 &= K_2(D_3 - D_2) \end{aligned} \right.$$

Matrix form on left

And so on for elements 3

$$K_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \end{bmatrix} ③ \quad \left\{ \begin{aligned} \hat{f}_3 &= k_3(D_3 - D_4) \\ \hat{f}_4 &= k_3(D_4 - D_3) \end{aligned} \right.$$

however, this maps to: $\underset{\sim}{F} = \underset{\sim}{f}_{①} + \underset{\sim}{f}_{②} + \underset{\sim}{f}_{③}$

③

In global terms, however, this maps to: $\underset{\sim}{F} = \underset{\sim}{f}_{①} + \underset{\sim}{f}_{②} + \underset{\sim}{f}_{③}$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix}$$

$$k_1 = \frac{A_1 E_1}{L_1} \qquad k_2 = \frac{A_2 E_2}{L_2} \qquad k_3 = \frac{A_3 E_3}{L_3}$$

From the info. we know from our problem:

$$\begin{bmatrix} F_1 \\ Q \\ 0 \\ -P \end{bmatrix} = \overbrace{\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}}^{\text{Global Stiffness Matrix}} \begin{bmatrix} 0 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix}$$

Multiplying out rhs:

$$\begin{bmatrix} F_1 \\ Q \\ 0 \\ -P \end{bmatrix} = \begin{bmatrix} k_1 \cdot 0 - k_1 \cdot D_2 + 0 \cdot D_3 + 0 \cdot D_4 \\ -k_1 \cdot 0 + (k_1+k_2)(D_2) - k_2 \cdot D_3 + 0 \cdot D_4 \\ 0 \cdot 0 - k_2 \cdot D_2 + (k_2+k_3)(D_3) - k_3 \cdot D_4 \\ 0 \cdot 0 + 0 \cdot D_2 - k_3 \cdot D_3 + k_3 \cdot D_4 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ Q \\ 0 \\ -P \end{bmatrix} = \begin{bmatrix} -k_1 D_2 \\ k_1 D_2 + k_2 D_2 - k_2 D_3 \\ -k_2 D_2 + k_2 D_3 + k_3 D_3 - k_3 D_4 \\ -k_3 D_3 + k_3 D_4 \end{bmatrix}$$

From here, Solve the system of equations:

$$\boxed{F_1 = -k_1 D_2}$$

① $\boxed{Q = k_1 D_2 + k_2 D_2 - k_2 D_3}$

$0 = -k_2 D_2 + k_2 D_3 + k_3 D_3 - k_3 D_4$

ADD

⑤ $Q = k_1 D_2 + k_3 D_3 - k_3 D_4$

ADD

$$⑥\ Q = k_1D_2 + k_3D_3 - k_3D_4$$

$$-P = -k_3D_3 + k_3D_4$$

ADD

$$Q - P = k_1D_2 \Rightarrow \boxed{F_1 = P - Q}$$

$$Q - P = k_1D_1 + k_1D_1 - k_2D_3 - k_3D_3 + k_3D_4$$

$$\cancel{k_1D_1} = \cancel{k_1D_1} + k_2D_2 - k_2D_3 - k_3D_3 + k_3D_4$$

$$-k_2D_2 + k_2D_3 = -k_3D_3 + k_3D_4$$

$$F_1 + Q = k_2D_2 - k_2D_3$$

$$O = -k_2D_2 + k_2D_3 + k_3D_3 - k_3D_4$$

$$-P = -k_3D_3 + k_3D_4$$

ADD

$$-P = -k_2D_2 + k_2D_3$$

Determining Displacements

$$\boxed{D_2 = \frac{F_1}{-k_1}} \qquad \boxed{D_1 = 0}$$

$$\begin{bmatrix} F_1 \\ Q \\ O \\ -P \end{bmatrix} = \begin{bmatrix} -k_1D_2 \\ k_1D_2 + k_2D_2 - k_2D_3 \\ -k_2D_2 + k_2D_3 + k_3D_3 - k_3D_4 \\ -k_3D_3 + k_3D_4 \end{bmatrix}$$

$$Q = D_3(-k_2) + \frac{F_1}{-k_2}(k_1 + k_2)$$

subbing $D_2 = \frac{F_1}{-k_1}$

$$\boxed{D_3 = \frac{Q + \frac{F_1}{k_1}(k_1 + k_2)}{-k_2}}$$

$$O = D_2(-k_2) + D_3(k_2 + k_3) + D_4(-k_3)$$

$$D_2 \cdot k_2 - D_3(k_2 + k_3) = D_4(-k_3)$$

$$(-k_3)D_4 = \frac{F_1}{\cancel{-k_2}}\cancel{(k_2)} + \frac{\left[Q + \frac{F_1}{k_1}(k_1 + k_2)\right](k_2 + k_3)}{k_2}$$

$$\boxed{D_4 = \frac{F_1}{-k_3} + \frac{\left[Q + \frac{F_1}{k_1}(k_1 + k_2)\right](k_2 + k_3)}{-k_2k_3}}$$

HW1 Q2

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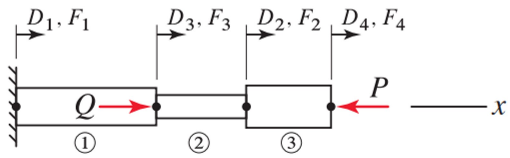
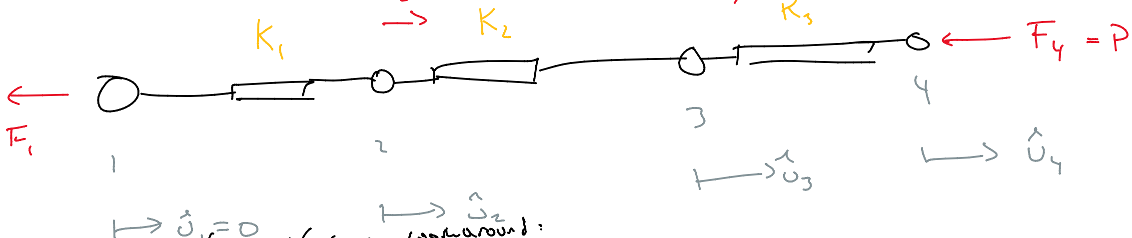
Problem 2 (20 pts)



$\kappa_1 = \dfrac{A_1E_1}{L_1}$ $\kappa_2 = \dfrac{A_2E_2}{L_2}$ $\kappa_3 = \dfrac{A_3E_3}{L_3}$

$F_2 = Q$ $F_3 = 0$



In matrix form w/ some workaround:

$$k_1 \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} \hat d_1 \\ \hat d_2 \end{bmatrix} = \begin{bmatrix} \hat f_1 \\ \hat f_2 \end{bmatrix}$$

$k_1 = \dfrac{A_1E_1}{L}$

Equivalent to

$$k_1 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat f_1 \\ \hat f_2 \\ \hat f_3 \\ \hat f_4 \end{bmatrix}$$

for element 1.

For each node,

$\hat f_1 = k_1(D_1 - D_2)$

$\hat f_2 = k_1(D_2 - D_1)$

w/ respect to element 1

& so on, which is where the matrix comes from.

Element 2 equivalently modeled:

$$k_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat f_1 \\ \hat f_2 \\ \hat f_3 \\ \hat f_3 \end{bmatrix} \left\{ \begin{aligned} \hat f_2 &= k_2(D_2 - D_3) \\ \hat f_3 &= k_2(D_3 - D_2) \end{aligned} \right\}$$

Matrix form on left

And so on for elements 3

$$K_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} \hat f_1 \\ \hat f_2 \\ \hat f_3 \\ \hat f_4 \end{bmatrix} \left\{ \begin{aligned} \hat f_3 &= k_3(D_3 - D_4) \\ \hat f_4 &= k_3(D_4 - D_3) \end{aligned} \right.$$

Then, the global stiffness matrix can be formed by adding elements 1, 2, 3:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}$$

In global Eq, we can formulate:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

$$k_1 = \frac{A_1 E_1}{L_1}$$

$$k_2 = \frac{A_2 E_2}{L_2}$$

$$k_3 = \frac{A_3 E_3}{L_3}$$

We can plug in our knows from the problem:

$F_2 = Q$, $F_3 = 0$, $F_4 = -P$, $D_1 = 0$ to get:

$$\begin{Bmatrix} F_1 \\ Q \\ 0 \\ -P \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

$$\{F\} = [K]\{D\}$$

HW1 Q3

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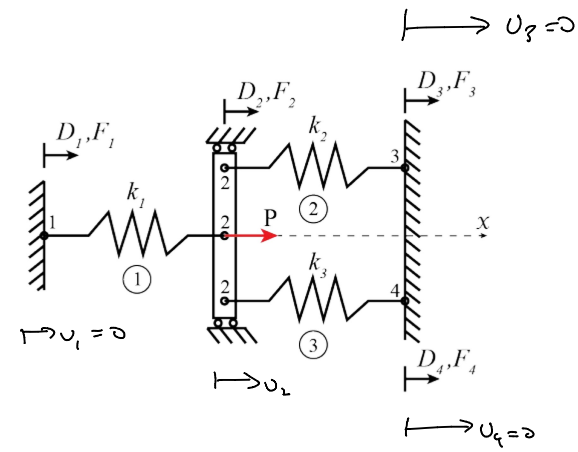
Problem 3 (20 pts)



- Three elements, 4 nodes
- Stiffness given 1 : k_1 2 : k_2 3 : k_3
- Displacements

node 1	node 2	node 3 & 4
$U_1 = 0$	$U_2 = U_2$	$U_3 = U_4 = 0$

- Forces

Node 1	Node 2	node 3	node 4
F_1	P	F_3	F_4

FE Analysis of element 1

$$\hat{f}_1 = k_1 (U_1 - U_2)$$

$$\hat{f}_2 = k_1 (U_2 - U_1)$$

In matrix form

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

For element 2

$$\hat{f}_2 = k_2 (U_2 - U_3)$$

$$f_2 = k_2(U_2 - U_3)$$

$$\hat{f}_3 = k_2(U_3 - U_2)$$

In matrix form:

$$\begin{bmatrix} \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix}$$

For element ③

$$\hat{f}_2 = k_3(U_2 - U_4)$$

$$\hat{f}_4 = k_3(U_4 - U_2)$$

In matrix form:

$$\begin{bmatrix} \hat{f}_2 \\ \hat{f}_4 \end{bmatrix} = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} U_2 \\ U_4 \end{bmatrix}$$

Note: node 2 $\propto$ 3, 4 then in global eq we can add elements 1, 2, 3 to get the global stiffness matrix:

$$\boxed{\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & \underline{k_1 + k_2 + k_3} & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}}$$

Assembling $[k]\{U\} = \{F\}$

$$\{U\} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 0 \\ U_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\{F\} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ P \\ F_3 \\ F_4 \end{pmatrix}$$

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} \quad \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

Then $[k]\{U\} = \{F\}$

$$\boxed{\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \\ F_4 \end{Bmatrix}}$$