EMA 405 – Finite Element Analysis

Homework 2 – Link180 Elements and Symmetry

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Overview:

The objective of homework two is to learn how linear varying axial loads affect nodal displacements and axial stress and understanding how varying the numbers of elements for the same conditions changes the answers. Additionally, this homework assignment explores how to use symmetry for a roof truss problem. Both problems are also being modeled in the simulation software Ansys. The accuracy of the FE solutions for different numbers of elements are going to be compared to analytical answers derived from a finite element analysis approach.

Description of FE Model:

This homework assignment specified using Link180 elements to model the two problems. For the first problem, that creates some difficulty since load as a function of distance does not exist on Link180 nodes. To counter that, the increasing distributed load can be equivalently represented with appropriate point loads on the nodes. These force values were calculated by hand in an analytical approach (see appendix). The leftmost nodes for both problems are fully fixed (Ux=Uy=Uz=0). For the first problem, we will focus on displacements at each node or element as a function of the number of elements. For the second problem, we will focus on determining stresses and displacements of nodes using symmetry.

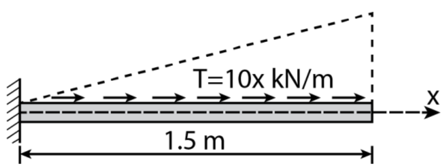

Fig. 1 Model of the bar subject to linear axial load.

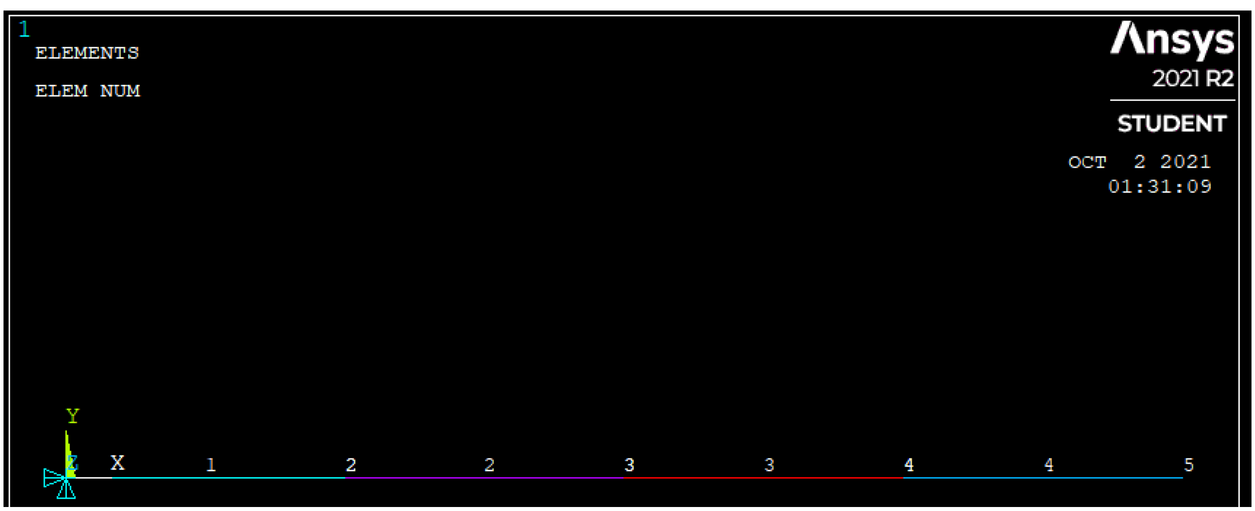

Fig. 2 Model of the four equal-length Link180 Elements in Ansys

The model in Fig. 2 shows how the bar can be modeled using four equal-length elements in Ansys. As clearly shown, the bar is constrained on the leftmost load, and the addition of equivalent forces on each node can accurately model a linear varying axial load.

FE Results

```
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

  NODE       UX
     1    0.0000
     2   0.99925E-005
     3   0.19671E-004
     4   0.28721E-004
     5   0.36830E-004
     6   0.43683E-004
     7   0.48966E-004
     8   0.52366E-004
     9   0.53569E-004
```

Fig 3. Example of nodal displacement results from Ansys

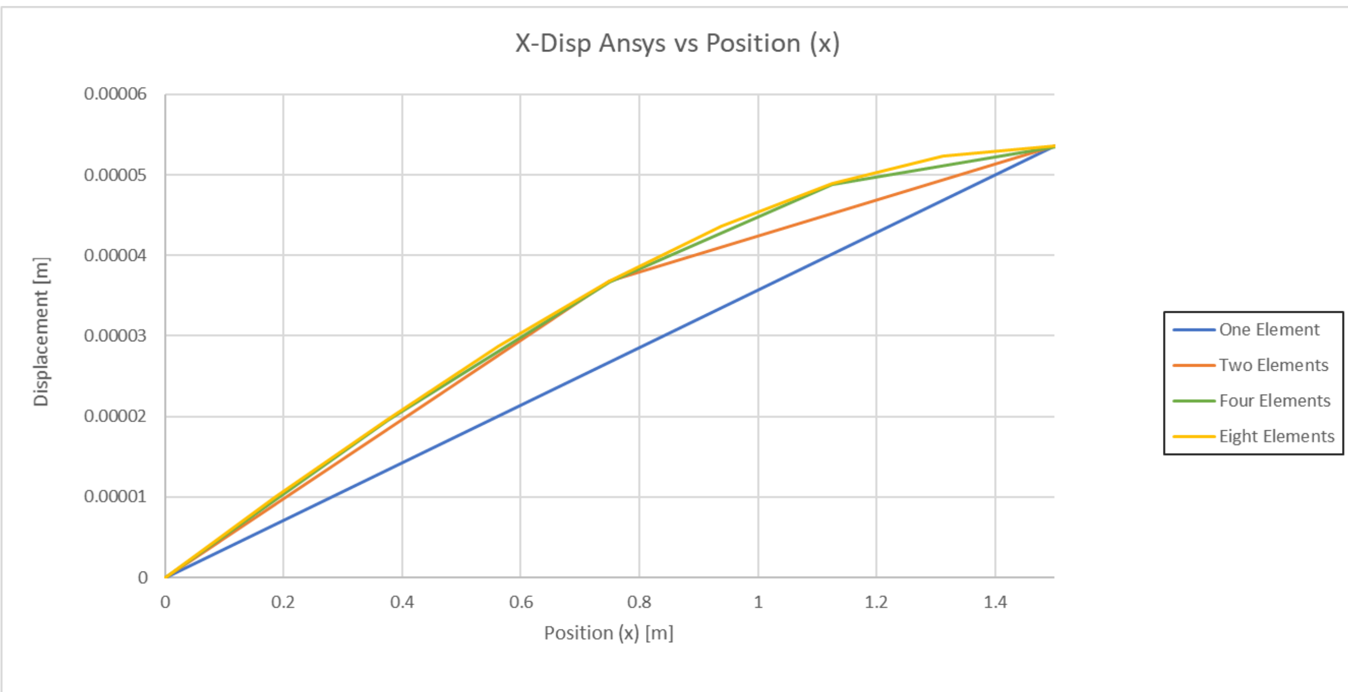

Fig 4. Displacement as a function of x for FE model

The image shown in Fig. 3 is an example of the output for nodal displacement in Ansys. By taking all the data points for each case of element number, we can model the x-displacement of the Link180 bars as a function of x such as in Fig. 4. For a linear varying axial load, we would expect a non-linear function of displacement vs length. It is clear that as the number of elements that are modeling the bar increases, the accuracy of the FE solutions through the model increases. The reasoning for that being that the FE model is trying to place a linear interpolation of the displacement vs position between each node, when, in reality, displacement vs position is non-linear. By increasing the number of elements, the distance between each node decreases, and as such, the interpolation is more accurate in the smaller distance.

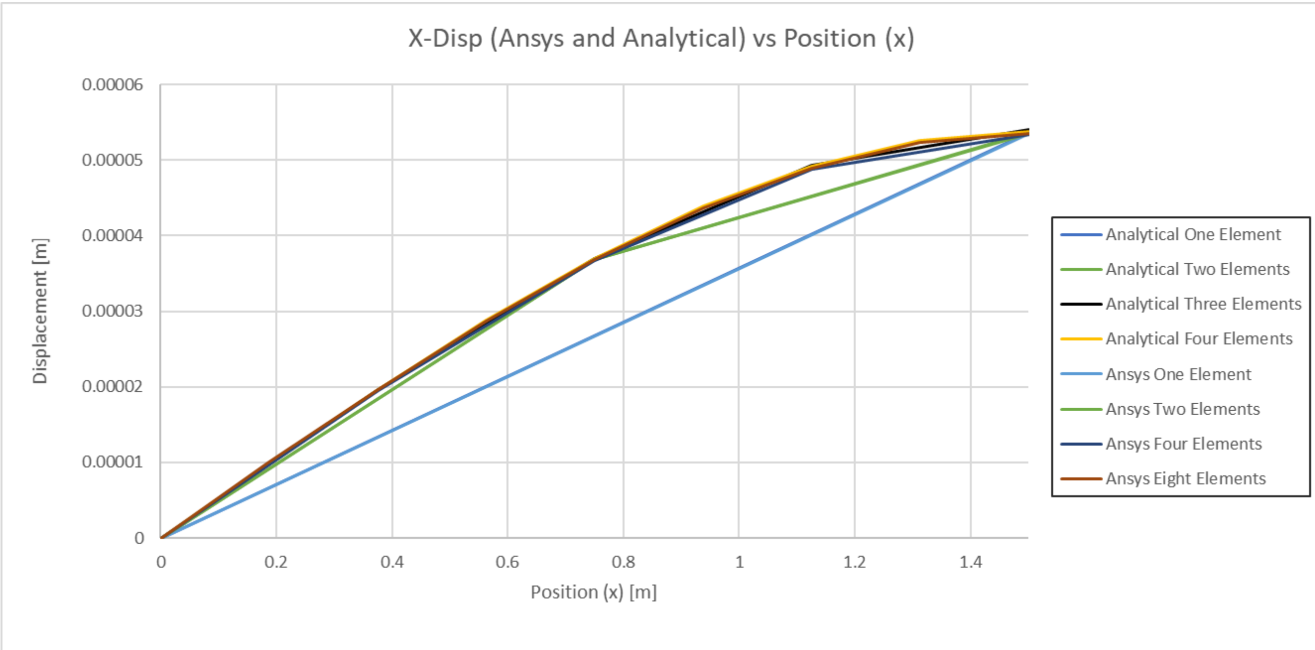

Fig 5. Analytical and FE model displacement as a function of x

In Fig. 5 we have compared the results of our FE model and plotted them with the analytical answers for displacement that we derive in the appendix. The result is as expected. The analytical answers practically fully overlap with the FE model answers for displacement vs position for each corresponding number of elements. Because of this, we can confidentially state that the FE model accurately solves the solutions for displacement vs time for the appropriate number of elements.

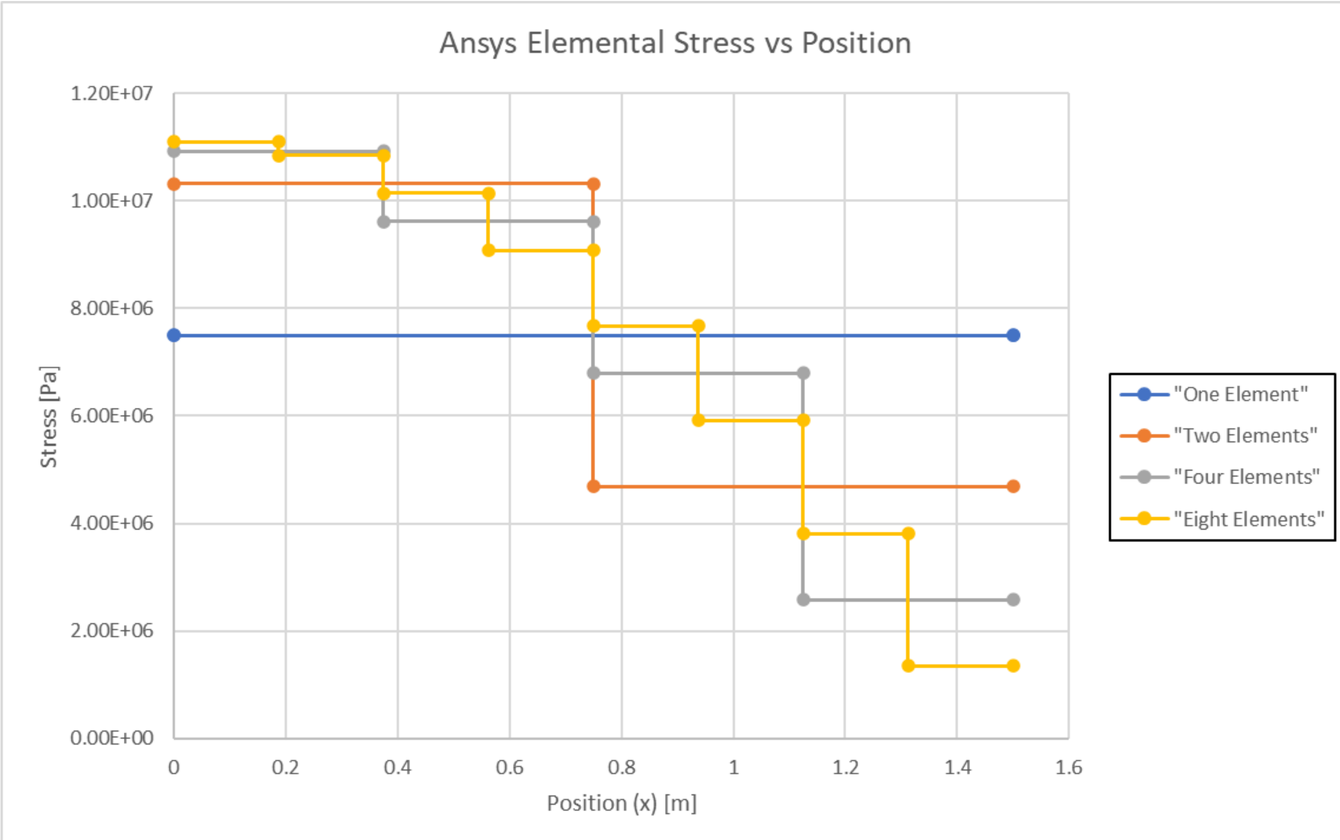

Fig 6. Axial Stress as a function of x for FE model.

Fig 6. plots the axial stress of each element as a matter of position. The reason for the horizontal lines is that the stresses are distributed through the element. The real stress distribution follows a quadratic function, so as the number of elements increases, the model for the stress distribution becomes more accurate.

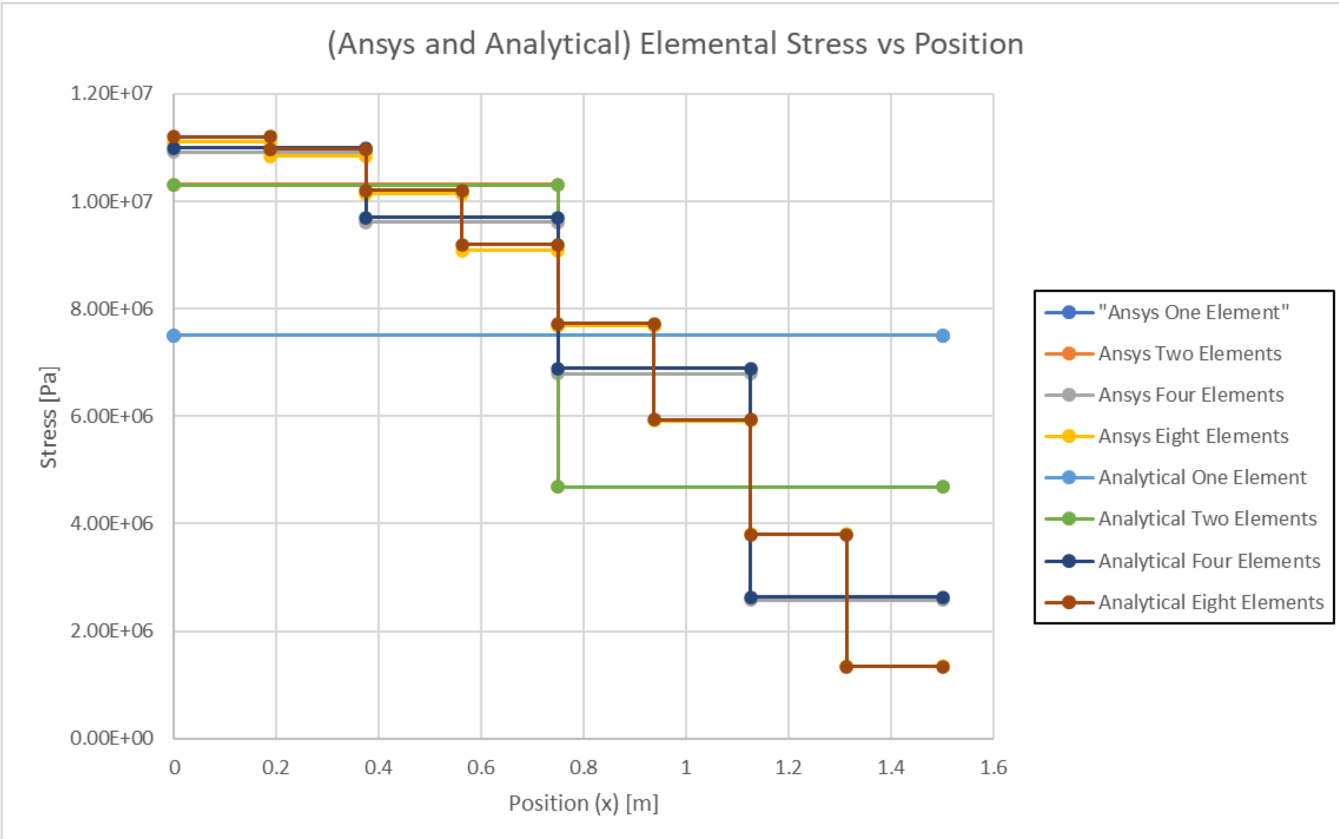

Fig 7. Analytical and FE model axial stress as a function of x.

Figs. 6 and 7 model the FE and analytical answers for elemental stress versus position. Similar to displacement, the results are as expected. As the number of elements increases, the model better represents that the internal stress decreases with position. Additionally, with just a few elements, the internal stress versus position of the bar is poorly represented. Knowing that the axial load is linearly varying, the axial stress would be estimated to be quadratically decreasing until it reaches zero stress at the end. By increasing the number of elements, it is clear that the accuracy increases as it will more increment in smaller gaps and not lose as much accuracy between nodes. The analytical and FE model solutions align almost perfectly in respect to the number of elements. Each answer is also accurate with respect to the elemental stress. However, accuracy of the bar stress on a nodal position perspective will depend on the number of elements and increase as the number of elements increases.

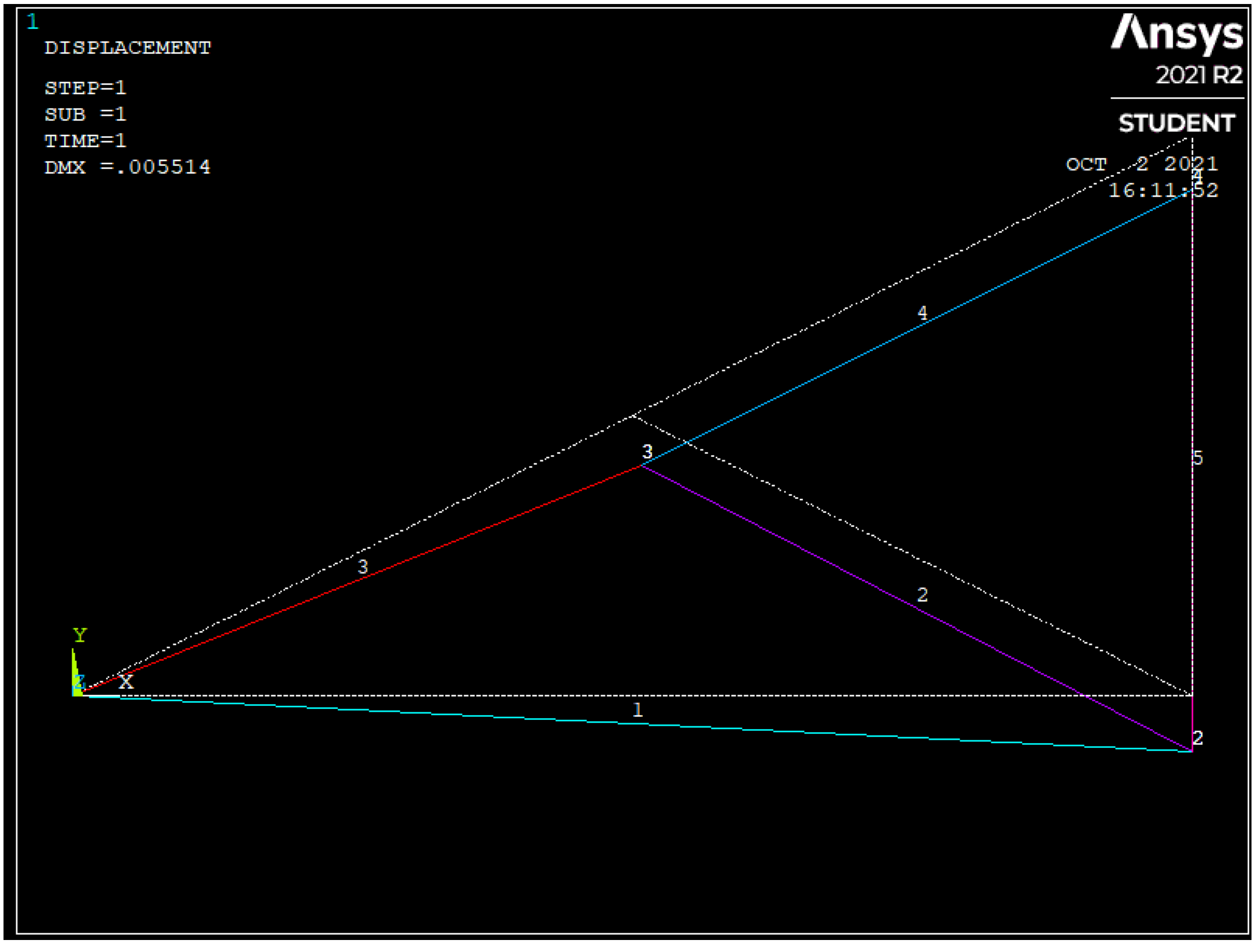

Fig. 8 Ansys model of Problem 2.

Fig. 8 helps identify how the nodes and elements were labeled. Additionally, the white outline is an outline of the original truss, while the outlined colored elements is the deformed object. It represents what the truss will look like when the forces are applied.

```
THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN THE GLOBAL COORDINATE SYSTEM

  NODE       UX            UY            UZ           USUM
     1   0.0000        0.0000        0.0000        0.0000
     2   0.0000      -0.55144E-002   0.0000       0.55144E-002
     3  0.84621E-003-0.48868E-002   0.0000       0.49595E-002
     4   0.0000      -0.53240E-002   0.0000       0.53240E-002

ELEMENT=      1        LINK180
  NODE     SX          SY          SZ          SXY         SYZ         SXZ
     1  0.0000       0.0000      0.0000      0.0000      0.0000      0.0000
     2  0.0000       0.0000      0.0000      0.0000      0.0000      0.0000

ELEMENT=      2        LINK180
  NODE     SX          SY          SZ          SXY         SYZ         SXZ
     2 -0.22361E+008 0.0000      0.0000      0.0000      0.0000      0.0000
     3 -0.22361E+008 0.0000      0.0000      0.0000      0.0000      0.0000

ELEMENT=      3        LINK180
  NODE     SX          SY          SZ          SXY         SYZ         SXZ
     1 -0.67082E+008 0.0000      0.0000      0.0000      0.0000      0.0000
     3 -0.67082E+008 0.0000      0.0000      0.0000      0.0000      0.0000

ELEMENT=      4        LINK180
  NODE     SX          SY          SZ          SXY         SYZ         SXZ
     3 -0.44721E+008 0.0000      0.0000      0.0000      0.0000      0.0000
     4 -0.44721E+008 0.0000      0.0000      0.0000      0.0000      0.0000

ELEMENT=      5        LINK180
  NODE     SX          SY          SZ          SXY         SYZ         SXZ
     2  0.10000E+008 0.0000      0.0000      0.0000      0.0000      0.0000
     4  0.10000E+008 0.0000      0.0000      0.0000      0.0000      0.0000
```

Fig. 9 Displacements of nodes and elemental stresses

Fig. 9 displays the displacement of the nodes, and stresses of the elements that were modeled in Fig. 8.

Conclusions

This homework assignment was useful to study stress and displacement distributions of two different types of problems with nodes and number of elements in consideration. The results from this assignment were obtained using FE models in Ansys Student and accuracy checks are reported in the appendix. Using more elements to model a link was shown to have better accuracy in modeling throughout the objects, however the endpoints were always precise. The answers were shown to be exact when compared to the analytical answers.

HW2 Analytical 1a

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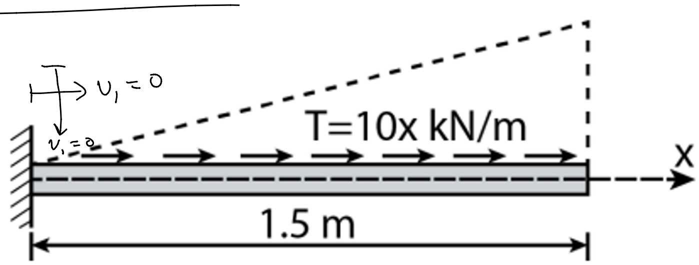
APPENDIX



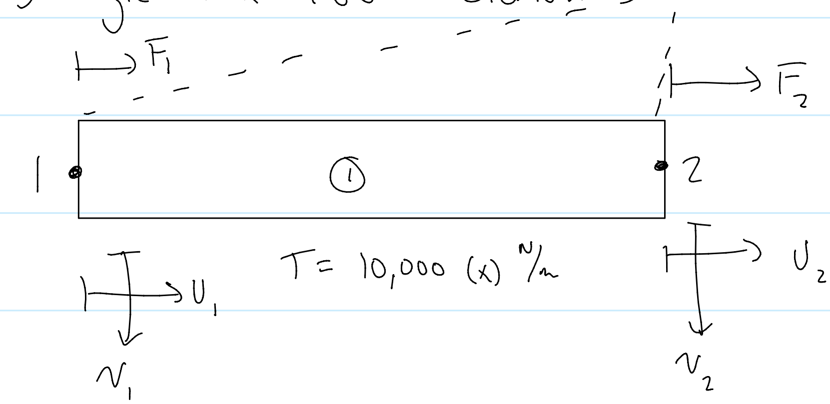
Firstly, Conversion of Units to provide accurate results:

$$A = \frac{10\ cm^2}{} \left| \frac{(1m)^2}{(100\ cm)^2} \right. = \underline{0.001\, m^2 = A}$$

$$E = 210\ GPa \cdot \frac{10^9\ Pa}{GPa} = \underline{210 \cdot 10^9\ Pa = E}$$

$$T = 10 \cdot x\ \frac{kN}{m} \cdot \frac{10^3\ N}{kN} = \underline{10{,}000\ N/m = T}$$

a) Single-link 180 element



Since weight is not a factor, & no forces in the y-direction

Our global Setup for D_y will look like:

$$[K]\{D_y\} = \underset{\sim}{0}$$

The zero vector.
The Stiffness matrix will try to link D to 0 thus

$$[K]\{D_y\} = \underset{\sim}{0}$$

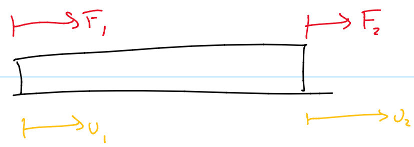
→ The zero vector. The stiffness matrix will try to link D to $\underset{\sim}{0}$, thus

$$\boxed{\underset{\sim}{D_y} = \underset{\sim}{0}}$$

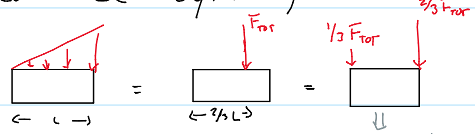
The same is true for parts b, c, d & for any number of elements for this bar.

The displacement in x-dir. can be set up by finding the stiffness matrix & applying $[K]\{D_x\} = \{F_x\}$

Element ①:

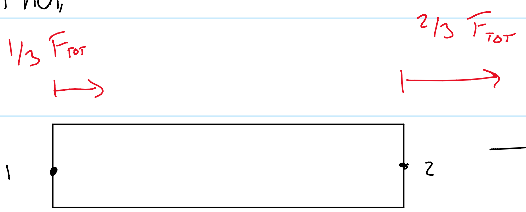


For distributed loads, we can use symmetry to apply FEA ... such that:



This is what we will use.

Then, we can model element 1 such that:



$$F_{end} = 10{,}000\,\tfrac{N}{m}(1.5) = 15{,}000\,\tfrac{N}{m}$$

$$\rightarrow F_{TOT} = F_{Triangle} = \frac{1}{2}(1.5m)\left(15{,}000\,\frac{N}{m}\right)$$

$$F_{TOT} = 11{,}250\,N$$

Then $\hat{f}_1 = \frac{1}{3}(F_{TOT}) = \frac{1}{3}(11250\,N) = 3750\,N = \hat{f}_1$

$\hat{f}_2 = \frac{2}{3}(F_{TOT}) = \frac{2}{3}(11250\,N) = 7500\,N = \hat{f}_2$

Then, $\hat{f}_1 = k_1(\hat{u}_1 - \hat{u}_2)$; $\hat{f}_2 = k_1(\hat{u}_2 - \hat{u}_1)$

$$k_1 = \frac{A_1 E_1}{L_1}$$

$$k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}$$

Constraints

$U_1 = 0$, $U_2 = ?$ $\Rightarrow$ $\hat{f}_1 = 3750\,N$

$\hat{f}_2 = 7500\,N$

$U_1 = 0$, $U_2 = ?$ $\Rightarrow$ $\hat{f}_1 = 3750\ N$

$\hat{f}_2 = 7500\ N$

In global eq. we have to note that node 1 also has a reactive force R_1. Then, we can map this to global eq. such that:

$$k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ D_2 \end{bmatrix} = \begin{bmatrix} R_{1x} + 3750 \\ 7500 \end{bmatrix}$$

$$k_1 \begin{bmatrix} -D_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} R_{1x} + 3750 \\ 7500 \end{bmatrix}$$

$-k_1 D_2 = R_{1x} + 37500$

$\underline{k_1 D_2 = 7500}$

$-7500 = R_{1x} + 37500$

$\underline{R_{1x} = -11250\ \hat{x}}$

Also, $K_1 = \frac{A_1 E_1}{L_1} = \frac{(0.001 m^2)(210e9\ Pa)}{1.5m}$

$\underline{K_1 = 1.4e8}$ $\rightarrow$ $k_1 D_2 = 7500$

$D_2 = \frac{7500}{1.4e8}$ $\Rightarrow$ $\boxed{D_2 = 5.36e^{-5}}$

Stress analysis

General stress: $\sigma = E\varepsilon = E\frac{d}{L}$

in terms of our elements,

$$\sigma = \sigma^1 + \sigma^2 + \sigma^3 + \ldots +$$

Where $\sigma^1 = \sigma_{element\ 1} = \sigma' = \frac{E}{L}\begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$

In general for our problem $\sigma' = \frac{E}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \end{bmatrix}$

So, for element 1

$$\sigma^{1} = \frac{210E9\ Pa}{1.5m}\left(-1\cancel{U_1}^{\to 0} + U_2\right)$$

$$\boxed{\sigma^{1} = 7.504\,E6\ Pa}$$

HW2 Analytical 1b

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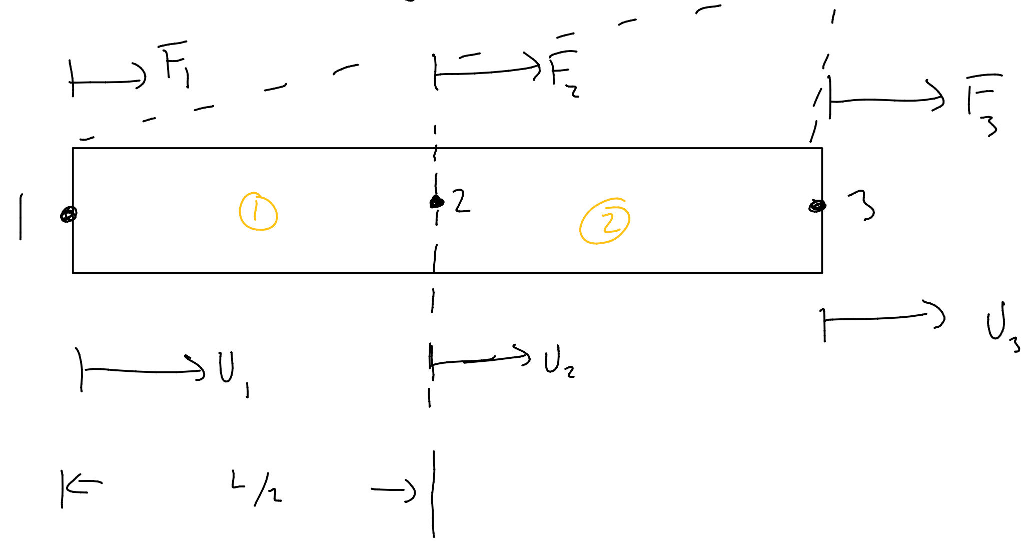
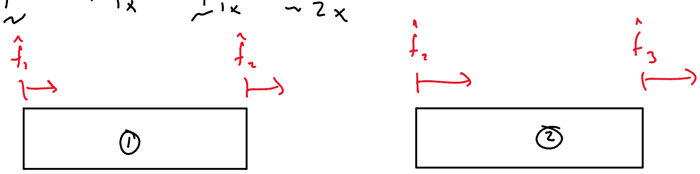
b) Two 1D element



From (a) we have proved $\underset{\sim}{D}_y = \underset{\sim}{0}$. Then, we will just be focusing on $\underset{\sim}{D}_x$.

$$\underset{\sim}{F} = R_{1x} + \underset{\sim}{f}_{1x} + \underset{\sim}{f}_{2x}$$



$F_{TOT} = \frac{1}{2}(L_1)(T(2))$

$F_{TOT} = \frac{1}{2}\left(\frac{1.5}{2}\right)\left(T\left(\frac{1.5}{2}\right)\right)$

$F_{TOT} = \frac{1}{2}(0.75\ m)(7500\ N/m)$

$F_{TOT} = 2813\ N$

Uniform load = $(L_2)(T(\text{node } 2))$

Triangle load = $\frac{1}{2}(L_2)(T(\text{node } 3) - T(\text{node } 2))$

Uniform: $(0.75\ m)(7500\ \frac{N}{m}) = 5625\ N$

Triangle: $\frac{1}{2}(0.75\ m)(15000 - 7500)$

$= 2812.5\ N$

Uniform load distributed $\frac{1}{2}$ on each node. Triangle $\frac{1}{3}, \frac{2}{3} \rightarrow$ to bigger load concentration side.

$\Rightarrow$ Element ①: $\hat{f}_1 = \frac{1}{3} F_{TOT} = 938\ N$

$\hat{f}_2 = \frac{2}{3} F_{TOT} = 1875\ N$

$\Rightarrow$ Element ①: $\hat{f}_1 = \frac{1}{3} F_{TOT}$ ...

$\hat{f}_2 = \frac{2}{3} F_{TOT} = 1875 N$

Element ②: $\hat{f}_2 = \frac{1}{2} F_{uni} + \frac{1}{3} F_{Tri} = 3750 N$

$\hat{f}_3 = \frac{1}{2} F_{Uni} + \frac{2}{3} F_{Tri} = 4688 N$

$$\underset{\sim}{F} = \begin{bmatrix} R_{1x} + \hat{f}_1^{①} \\ \hat{f}_2^{①} + \hat{f}_2^{②} \\ f_3^{②} \end{bmatrix} = \begin{bmatrix} R_{1x} + 938 \\ 1875 + 3750 \\ 4688 \end{bmatrix}$$

Using two elements, we can find $\underset{\sim}{D}$ such that:

Element ①: $\hat{f}_1 = k_1 (\hat{\delta}_1 - \hat{\delta}_2)$

$\hat{f}_2 = k_1 (\hat{\delta}_2 - \hat{\delta}_1)$

Element ②: $\hat{f}_2 = k_2 (\hat{\delta}_2 - \hat{\delta}_3)$

$\hat{f}_3 = k_2 (\hat{\delta}_3 - \hat{\delta}_2)$

Mapping to global equation $[K]\{D\} = \{F\}$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} \cancel{D_1} \, 0 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} R_1 + 937.5 \\ 5625 \\ 4688 \end{bmatrix}$$

$$\begin{bmatrix} -k_1 D_2 \\ D_2 (k_1 + k_2) - k_2 D_3 \\ -k_2 D_2 + k_2 D_3 \end{bmatrix} = \begin{bmatrix} R_1 + 937.5 \\ 5625 \\ 4688 \end{bmatrix}$$

By inspection, it is clear that

$k_1 = k_2 \quad (L_1 = L_2), (A_2 = A_1) (E_1 = E_1)$

$$= K = \frac{(0.001 \, m^2)(210e9 \, Pa)}{0.75 \, m}$$

$$\underline{K = 2.8e8 \, \frac{Pa}{m}}$$

Then, $-K D_2 = R_{1x} + 937.5$

$(K(2D_2 - D_3) = 5628)$

$+ (K(D_3 - D_2) = 4688)$

$\Rightarrow K(D_2) = 10316$

$$\boxed{D_2 = 3.684 E^{-5}}$$

$2.8e^8 (D_3 - 3.684e^{-5}) = 4688$

$$\boxed{D_3 = 5.36 E^{-5}}$$

$$2.8e^{8}\left(D_3 - 3.684e^{-5}\right) = 4688$$

$$\boxed{D_3 = 5.36\ E^{-5}}$$

$$-KD_2 = R_{1x} + 937.5$$

$$-2.8 \cdot 10^{8} \cdot 3.684 \cdot 10^{-5} = R_{1x} + 937.5$$

$$\boxed{R_{1,x} = -11250\ N}$$

Stress Analysis

Derived from Previous

Element ①

$$\sigma^{①} = \frac{E}{L}\left(-U_1 + U_2\right)$$

$$\sigma^{①} = \frac{210E9\,Pa}{0.75\,m}\left(3.684 \cdot 10^{-5}\,m\right)$$

$$\boxed{\sigma^{①} = 1.03\ E7\ Pa}$$

Element ②

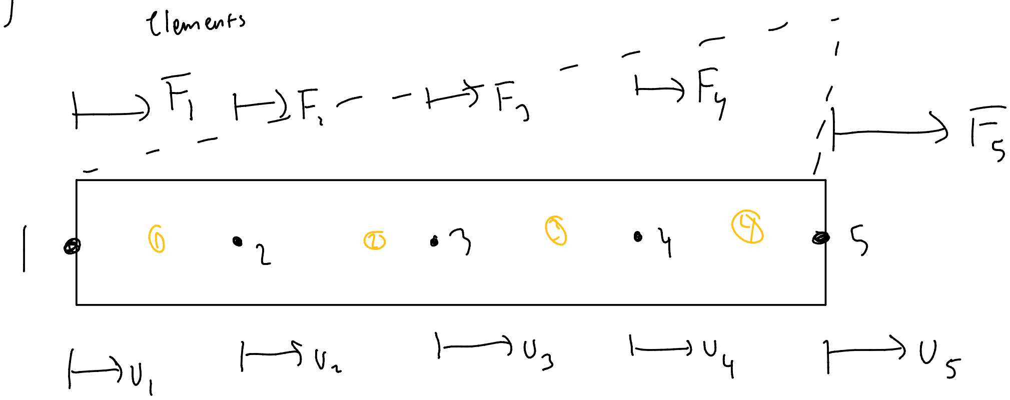
$$\sigma^{②} = \frac{210E9}{.75}\left(-3.684 \cdot 10^{-5} + 5.36E^{-5}\right)$$

$$\boxed{\sigma^{②} = 4.69\ E6\ Pa}$$

HW2 Analytical 1c

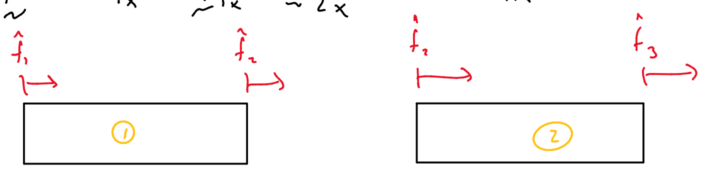
Monday, September 27, 2021 9:41 PM

C) Four Equal-length Link-180 elements



Using Same assumptions (a), (b),

$$\underset{\sim}{F} = R_{1x} + \underbrace{\underset{\sim}{f}_{1x} + \underset{\sim}{f}_{2x} + \underset{\sim}{f}_{3x} + \underset{\sim}{f}_{4x}}_{\text{Elements } 1,2,3,4}$$



$F_{Tot} = \frac{1}{2}(L_1)(T(2))$

$F_{Tot} = \frac{1}{2}\left(\frac{1.5}{4}\right)\left(T\left(\frac{1.5}{4}\right)\right)$

$\underline{F_{Tot} = 703.125 N}$

Uniform load $= (L_2)(T(\frac{node}{2}))$

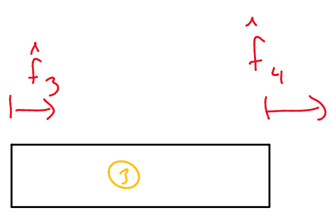
Triangle load $= \frac{1}{2}(L_2)(T(\frac{node}{3}) - T(\frac{node}{2}))$

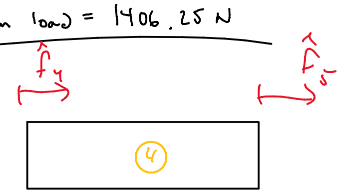
↳ Will be Same for all elements

$\underline{\text{Triangle load} = 703.125 N}$

Uniform load $= \frac{1.5}{4} m \cdot 10{,}000\ N/m \cdot \frac{1.5}{4}$

$\underline{\text{Uniform load} = 1406.25 N}$





$\underline{\text{Triangle load}: 703.125 N}$

Uniform Load: $L_3 \cdot T(N_3)$

$= \frac{1.5m}{4} \cdot T\left(\frac{1.5}{2} m\right)$

$= 0.375 m \cdot \frac{10{,}000 N}{m} \cdot \frac{1.5}{2}$

$\underline{\text{Uniform load } 3 = 2812.5 N}$

$\underline{\text{Triangle load}: 703.125 N}$

Uniform load: $(L_4)\ T(N_4)$

$= \frac{1.5m}{4} \cdot T\left(\frac{1.5m}{4} \cdot 3\right)$

$= 0.375 m \cdot \frac{10{,}000 N}{m} \cdot \frac{1.5}{4} \cdot 3$

$\underline{\text{Uniform Load } 4 = 4218.75 N}$

$$\underset{\sim}{F} = R + f^{①} + f_x^{②} + \underset{\sim}{f}_x^{③} + f_x^{④}$$

Uniform load

So... $F = R_{1x} + \underset{\sim}{f}_x^{(1)} + \underset{\sim}{f}_x^{(2)} + \underset{\sim}{f}_x^{(3)} + \underset{\sim}{f}_x^{(4)}$

From Analysis of beams:

Element ①: $\underset{\sim}{f}_x^{(1)} = \begin{Bmatrix} \frac{1}{2}F_{Uniform} + \frac{1}{3}F_{Tri} \\ \frac{1}{2}F_{Uniform} + \frac{2}{3}F_{Tri} \end{Bmatrix} \begin{matrix} \hat{f}_1 \\ \hat{f}_2 \end{matrix}$

This pattern for all elements.

$$\underset{\sim}{f}_x^{(1)} = \begin{Bmatrix} 0 + \frac{1}{3}(703.125) \\ 0 + \frac{2}{3}(703.125) \end{Bmatrix} \begin{matrix} \hat{f}_1 \\ \hat{f}_2 \end{matrix}$$

$$\boxed{\underset{\sim}{f}_x^{(1)} = \begin{Bmatrix} 234.4 \\ 468.8 \end{Bmatrix} \begin{matrix} \hat{f}_1 \\ \hat{f}_2 \end{matrix} [N]}$$

Element ②: $\underset{\sim}{f}_x^{(2)} = \begin{Bmatrix} \frac{1}{2} \cdot 1406.25 + 234.4 \\ \frac{1}{2} \cdot 1406.25 + 468.8 \end{Bmatrix} \begin{matrix} \hat{f}_2 \\ \hat{f}_3 \end{matrix} [N]$

$$\boxed{\underset{\sim}{f}_x^{(2)} = \begin{Bmatrix} 937.5 \\ 1171.88 \end{Bmatrix} \begin{matrix} \hat{f}_2 \\ \hat{f}_3 \end{matrix} [N]}$$

Element ③: $\underset{\sim}{f}_x^{(3)} = \begin{Bmatrix} \frac{1}{2}(2812.5) + 234.4 \\ \frac{1}{2}(2812.5) + 468.8 \end{Bmatrix} \begin{matrix} \hat{f}_3 \\ \hat{f}_4 \end{matrix} [N]$

$$\boxed{\underset{\sim}{f}_x^{(3)} = \begin{Bmatrix} 1640.7 \\ 1875.1 \end{Bmatrix} \begin{matrix} \hat{f}_3 \\ \hat{f}_4 \end{matrix} [N]}$$

Element ④: $\underset{\sim}{f}_x^{(4)} = \begin{Bmatrix} \frac{1}{2}(4218.75) + 234.4 \\ \frac{1}{2}(4218.75) + 468.8 \end{Bmatrix} \begin{matrix} \hat{f}_4 \\ \hat{f}_5 \end{matrix} [N]$

$$\boxed{\underset{\sim}{f}_x^{(4)} = \begin{Bmatrix} 2343.78 \\ 2578.18 \end{Bmatrix} \begin{matrix} \hat{f}_4 \\ \hat{f}_5 \end{matrix} [N]}$$

From previous we know $R_x := -11250$ N

Then,
$$F = \begin{bmatrix} R_{1x} + \hat{f}_1^{(1)} \\ \hat{f}_2^{(1)} + \hat{f}_2^{(2)} \\ \hat{f}_3^{(2)} + \hat{f}_3^{(3)} \\ \hat{f}_4^{(3)} + \hat{f}_4^{(4)} \\ \hat{f}_5^{(4)} \end{bmatrix} = \begin{bmatrix} R_{1x} + 234.4 \\ 468.8 + 937.5 \\ 1171.88 + 1640.7 \\ 1875.1 + 2343.75 \\ 2578.2 \end{bmatrix} = \begin{bmatrix} R_{1x} + 234.4 \\ 1312.58 \\ 2812.58 \\ 4218.85 \\ 2578.2 \end{bmatrix}$$

From previous analysis we know

Element ①: $k_1(D_1 - D_2) = \hat{f}_1$

$k_1(D_2 - D_1) = \hat{f}_2$

And so on for ②, ③, ④

Then

$$\underline{\underline{K}} = \begin{bmatrix} K_1 & -K_1 & & & \text{Sym} \\ -K_1 & K_1 + K_2 & -K_2 & & \\ 0 & -K_2 & K_2 + K_3 & -K_3 & \\ 0 & 0 & -K_3 & K_3 + K_4 & -K_4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -K_2 & K_2+K_3 & -K_3 & \\ 0 & 0 & -K_3 & K_3+K_4 & -K_4 \\ 0 & 0 & 0 & -K_4 & K_4 \end{bmatrix}$$

Global $\underline{\underline{K}}\{D\} = \{F\}$

$$\begin{bmatrix} K_1 & -K_1 & & \text{Sym} & \\ -K_1 & K_1+K_2 & -K_2 & & \\ 0 & -K_2 & K_2+K_3 & -K_3 & \\ 0 & 0 & -K_3 & K_3+K_4 & -K_4 \\ 0 & 0 & 0 & -K_4 & K_4 \end{bmatrix} \begin{bmatrix} 0 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} -11015.6 \\ 1312.6 \\ 2812.6 \\ 4218.9 \\ 2578 \end{bmatrix}$$

$k_1 = k_2 = k_3 = k_4 = \frac{AE}{L} = \frac{0.001 \cdot 210E9}{1.5/4} = 5.6E8$

$-K_1 D_2 = -11015.6$

$D_2 = \frac{-11015.6}{-5.6E8} = \boxed{1.967\ E-5 = D_2}$

$D_2 (K_1 + K_2) - K_2 \cdot D_3 = 1312.6$

$D_3 = \frac{1312.6 - (1.97E^{-5})(2)(5.6E8)}{-5.6E8}$

$\boxed{D_3 = 3.7\ E-5}$

$-K_2 D_2 + D_3 (K_2 + K_3) - D_4 K_3 = 2812.6$

$K(-D_2 + 2D_3 - D_4) = 2812.6$

$-1.97E-5 + (3.7E-5)(2) - D_4 =$

$\boxed{D_4 = 4.93\ E-5}$

$-K_4 D_4 + K_4 D_5 = 2578$

$D_5 = \frac{2578 + K D_4}{K}$

$\boxed{D_5 = 5.4\ E-5}$

Stress Analysis

Element ① $= \sigma^{①} = \frac{E}{L}(-U_1 + U_2)$

$$\sigma^{①} = \frac{210E9}{.375\,m}\left(1.967 \cdot 10^{-5}\right)$$

$$\boxed{\sigma^{①} = 1.101\,E7\ Pa}$$

Element 2

$$\sigma^{②} = \frac{210E9}{.375}\left(-1.967E{-5} + 3.7E{-5}\right)$$

$$\boxed{\sigma^{②} = 9.7\,E6\ Pa}$$

Element ③

$$\sigma^{③} = \frac{210E9}{.375}\left(-3.7E{-5} + 4.93E{-5}\right)$$

$$\boxed{\sigma^{③} = 6.89\,E6\ Pa}$$

Element ④

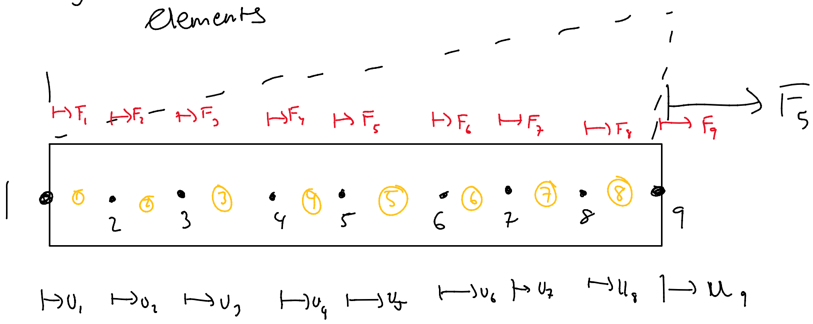
$$\sigma^{④} = \frac{210E9}{.375}\left(4.93E{-5} + 5.4E{-5}\right)$$

$$\boxed{\sigma^{④} = 2.632\,E6\ Pa}$$

HW2 Analytical 1d

Tuesday, September 28, 2021 5:34 PM

d) Eight - Equal - length elements



Using same Analysis as e),

Triangle Load $= \frac{1}{2}(L_{element})(T(L_{element}))$

↳ Length of each element

↳ The increase in load through element

Triangle Load $= \frac{1}{2}\left(\frac{1.5\,m}{8}\right)\left(10{,}000\,N/m \cdot 1.5/8\right)$

$\boxed{\text{Dist. Load} = 175.8\,N}$ ⇒ At each element $\frac{1}{3}$ on left node, $\frac{2}{3}$ on right node

Uniform Load $= (L_{element})(T(L_{node}))$ → Nodes 1–8 for elements 1–8

For elements ①–⑧ ↳ At each element $\frac{1}{2}$ on left node, $\frac{1}{2}$ on right.

① $UL = \left(\frac{1.5}{8}m\right)(10{,}000\,N/m \cdot 0)$ nodes 1, 2

$= 0\,N$

② $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5}{8}\right)$ nodes 2, 3

$= 352\,N$

③ $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5 \cdot 2}{8}\right)$ nodes 3, 4

$= 703.125\,N$

④ $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5 \cdot 3}{8}\right)$ nodes 4, 5

$= 1054.7\,N$

⑤ $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5 \cdot 4}{8}\right)$ nodes 5, 6

$= 1406.3\,N$

⑥ $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5}{8} \cdot 5\right)$ nodes 6, 7

$= 1757.8\,N$

⑦ $UL = \left(\frac{1.5}{8}m\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5}{8} \cdot 6\right)$ nodes 7, 8

$= 2109.4\,N$

⑧ $UL = \left(\frac{1.5}{8}\right)\left(10{,}000\,\frac{N}{m} \cdot \frac{1.5}{8} \cdot 7\right)$ nodes 8, 9

$= 2461\,N$

$F = R_x + \underset{\sim}{f}^{①} + \underset{\sim}{f}^{②} + \underset{\sim}{f}^{③} + \dots + \underset{\sim}{f}^{⑧}$

$= -11250\,N$ from Prev. ← Elements 2–8 have $\frac{1}{3} + \frac{2}{3}$ Tri-load $= 175.8$

$R_{1x} + \frac{1}{3}(175.8)$

$175.8 + \frac{1}{2}(352)$

$$F = \begin{bmatrix} R_{1x} + \frac{1}{3}(175.8) \\ 175.8 + \frac{1}{2}(352) \\ 175.8 + \frac{1}{2}(352 + 703.1) \\ 175.8 + \frac{1}{2}(703.1 + 1054.7) \\ 175.8 + \frac{1}{2}(1054.7 + 1406.3) \\ 175.8 + \frac{1}{2}(1406.3 + 1757.8) \\ 175.8 + \frac{1}{2}(1757.8 + 2109.4 \\ 175.8 + \frac{1}{2}(2109.4 + 2461) \\ \frac{2}{3}(175.8) + \frac{1}{2}(2461) \end{bmatrix} N$$

have $\frac{1}{3} + \frac{2}{3}$ Tri-load $= 175.8$

$$F = \begin{bmatrix} -1191.4 \\ 351.8 \\ 703.35 \\ 1054.7 \\ 1406.4 \\ 1757.8 \\ 2109.4 \\ 2461 \\ 1347.1 \end{bmatrix} N$$

$\Rightarrow$ We also know from part c) that

$k_1 = k_2 = \dots = k_8 = K.$

Also, the force-disp modeling follows a pattern such that we can skip to global eq.

$$[K]\{D\} = \{F\}$$

$$\begin{bmatrix} k_1 & -k_1 & & & & & & & \\ -k_1 & k_1+k_2 & -k_2 & & & \text{Sym} & & & \\ 0 & -k_2 & k_2+k_3 & -k_3 & & & & & \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 & & & & \\ 0 & 0 & 0 & -k_4 & k_4+k_5 & -k_5 & & & \\ 0 & 0 & 0 & 0 & -k_5 & K_5+K_6 & -K_6 & & \\ 0 & 0 & 0 & 0 & 0 & -k_6 & K_6+K_7 & -k_7 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_7 & K_7+K_8 & -K_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_8 & K_8 \end{bmatrix} \begin{Bmatrix} 0 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix} = \begin{bmatrix} -1191.4 \\ 351.8 \\ 703.35 \\ 1054.7 \\ 1406.4 \\ 1757.8 \\ 2109.4 \\ 2461 \\ 1347.1 \end{bmatrix}$$

$$k_1 = k_2 = \dots = k_8 = \frac{A_e E_c}{L_c} = \frac{0.001\,m^2 \cdot 210E9\,Pa}{(1.5m/8)}$$

$$K = 1.12\ E9$$

From Matrix: $\boxed{D_1 = 0}$

$-k_1 D_2 = -1191.4\ N$

$$\boxed{D_2 = \frac{-1191.4}{-1.12E9} = 9.99\,E{-6}}$$

$K(2D_2 - D_3) = 351.8\ N$

$$D_3 = -\frac{351.8N - 2D_2K}{K}$$

$$\boxed{D_3 = 1.97\ E{-5}}$$

$K(-D_2 + 2D_3 - D_4) = 703.35\ N$

$$K(-D_2 + 2D_3 - D_4) = 703.35\ N$$

$$\boxed{D_4 = 2.88\ E-5}$$

$$K(-D_3 + 2D_4 - D_5) = 1054.7\ N$$

$$\boxed{D_5 = 3.70\ E-5}$$

$$K(-D_4 + 2D_5 - D_6) = 1406.4N$$

$$\boxed{D_6 = 4.39\ E-5}$$

$$K(-D_5 + 2D_6 - D_7) = 1752.8N$$

$$\boxed{D_7 = 4.92\ E-5}$$

$$K(-D_6 + 2D_7 - D_8) = 2109.4N$$

$$\boxed{D_8 = 5.26\ E-5}$$

$$K(-D_7 + 2D_8 - D_9) = 2461$$

$$\boxed{D_9 = 5.38\ E-5}$$

Stress Analysis

$$\sigma^{1} = \frac{210E9}{.1875}(9.99E-6) = 1.12\ E7\ Pa$$

$$\sigma^{2} = \frac{210E9}{.1875}(-9.99E-6 + 1.97E-5) = 1.097\ E7\ Pa$$

$$\sigma^{3} = \frac{210E9}{.1875}(-1.97E-5 + 2.88E-5) = 1.02\ E7\ Pa$$

$$\sigma^{4} = \frac{210E9}{.1875}(-2.88\ E-5 + 3.7\ E-5) = 9.19\ E6\ Pa$$

$$\sigma^{5} = \frac{210E9}{.1875}(-3.7E-5 + 4.39E-5) = 7.73E6\ Pa$$

$$\sigma^{6} = \frac{210E9}{.1878}(-4.39\ E-5 + 4.92E-5) = 5.94\ E6\ Pa$$

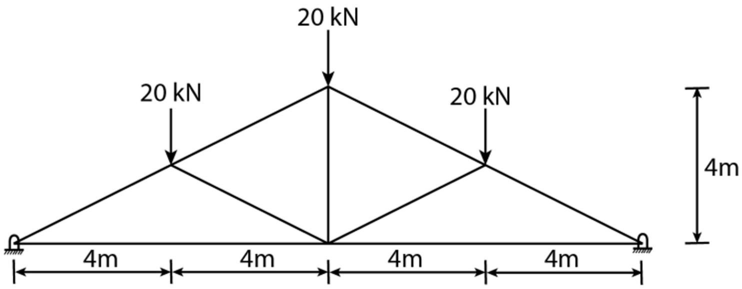
$$\sigma^{7} = \frac{210E9}{.1875}(-4.92\ E-5 + 5.26\ E-5) = 3.8\ E6\ Pa$$

$$\sigma^{8} = \frac{210E9}{.1825}(-5.26E--5 + 5.38E-5) = 1.34\ E6\ Pa$$

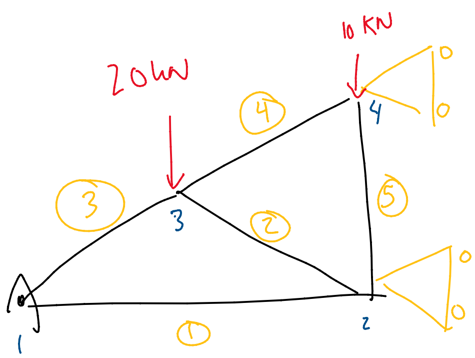
HW2 Analytical 2

Monday, September 27, 2021 6:10 PM

All elements have $E = 210\text{E}9$ Pa, $A = 1.10^{-3}\ m^2$



Using symmetry, we can model the following truss such that it is equivalently:



Now, we only need to solve for 5 elements.

A truss can be modeled such that you can use a shape matrix

$$N = \begin{array}{cccc} U_i & V_i & U_{i+1} & V_{i+1} \\ \end{array}$$
$$\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Where $C = \cos(\text{element})$
$S = \sin(\text{element})$

For each element to determine the nodal stiffness matrix such that: $\frac{AE}{L} N = k$

$$[K] = k^{(1)} + k^{(2)} + \dots + k^{(5)}$$

Element 1

$\sin\theta = 0$

$\cos\theta = 1$

$$\frac{AE}{L_1}\begin{array}{c c} & \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ & \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \end{array}$$

Element 2

$$\sin\theta = \frac{2}{\sqrt{2^2+4^2}} \approx 0.4472$$

$$\cos\theta = \frac{-4}{\sqrt{2^2+4^2}} = -0.8944$$

$$k^{(2)} = \frac{AE}{L_2}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cccc} .8 & -.4 & -.8 & .4 \\ -.4 & .2 & .4 & -.2 \\ -.8 & .4 & .8 & -.4 \\ .4 & -.2 & -.4 & .2 \end{array}\right] \end{array}$$

Element 3 $\longleftrightarrow$ Same as 4 except for Start/Stop nodes.

$\sin\theta = 0.4472$

$\cos\theta = 0.8944$

$$k^{(3)} = \frac{AE}{L_3}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cccc} .8 & .4 & -.8 & .4 \\ .4 & .2 & -.4 & -.2 \\ -.8 & -.4 & .8 & .4 \\ -.4 & -.2 & .4 & .4 \end{array}\right] \end{array}$$

Element 4

$$k^{(4)} = \frac{AE}{L_4}\begin{array}{c} \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} \\ \left[\begin{array}{cccc} .8 & .4 & -.8 & .4 \\ .4 & .2 & -.4 & -.2 \\ -.8 & -.4 & .8 & .4 \\ -.4 & -.2 & .4 & .4 \end{array}\right] \end{array}$$

$$k^{④} = \frac{AE}{L_4}\begin{bmatrix} .4 & .2 & -.4 & .2 \\ -.8 & -.4 & .8 & .4 \\ -.4 & -.2 & .4 & .4 \end{bmatrix}$$

Element 5

$\cos\theta = 0$

$\sin\theta = 1$

$$k^{⑤} = \frac{AE}{L_5}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$K = k^{①} + k^{②} + \ldots + k^{⑤}$$

$AE = 210e9 \cdot 1 \cdot 10^{-3}$

$AE = 210 \cdot 10^{6}\ m^2 \cdot Pa$

$L_1 = L_5 = 4m$

$L_2 = L_3 = L_4 = \sqrt{2^2+4^2} = 4.47\ m$

$$K = \frac{AE}{L_1}\begin{array}{cc} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cancel{0} \end{bmatrix} & \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \end{array} + \frac{AE}{L_2}\begin{array}{cc} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} & \\ \begin{bmatrix} .8 & -.4 & -.8 & .4 \\ -.4 & \cancel{.2} & \cancel{.4} & \cancel{-.2} \\ -.8 & .4 & \cancel{.8} & \cancel{-.4} \\ .4 & -.2 & -.4 & \cancel{.2} \end{bmatrix} & \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array} +$$

$$+ \frac{AE}{L_3}\begin{array}{cc} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} & \\ \begin{bmatrix} .8 & .4 & -.8 & .4 \\ .4 & .2 & -.4 & -.2 \\ -.8 & -.4 & \cancel{.8} & \cancel{.4} \\ -.4 & -.2 & .4 & \cancel{.4} \end{bmatrix} & \begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array} \end{array} + \frac{AE}{L_4}\begin{array}{cc} \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} & \\ \begin{bmatrix} \cancel{.8} & \cancel{.4} & -.8 & \cancel{.4} \\ .4 & \cancel{.2} & -.4 & \cancel{-.2} \\ -.8 & -.4 & .8 & .4 \\ -.4 & -.2 & .4 & \cancel{.4} \end{bmatrix} & \begin{array}{c} u_3 \\ v_3 \\ u_4 \\ v_4 \end{array} \end{array} + \frac{AE}{L_5}\begin{array}{cc} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cancel{1} & 0 & \cancel{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cancel{1} \end{bmatrix} & \begin{array}{c} u_2 \\ v_2 \\ u_4 \\ v_4 \end{array} \end{array}$$

$U_1 = 0 \quad v_1 = 0 \quad u_4 = 0 \quad u_2 = 0$

$$\begin{array}{c} \begin{array}{cccc} v_2 & u_3 & v_3 & v_4 \end{array} \\ K = AE \begin{bmatrix} 0.295 & .0894 & -.0447 & -.25 \\ & .5369 & .0894 & .0894 \\ & & .1789 & -.0447 \\ \text{Sym} & & & .3132 \end{bmatrix} \end{array} \begin{Bmatrix} v_2 \\ u_3 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -20{,}000 \\ -10{,}0000 \end{Bmatrix}$$

Using matrix solver,

$u_3 = 8.5E-4 \quad v_3 = -4.88E-3$

$v_4 = -5.32E-3 \quad v_2 = -5.514E-3$

Stress Analysis of Elements

$$\sigma^{①} = \frac{E}{L_1} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ U_2 \\ v_3 \end{Bmatrix}$$

By doing this w/ respective angles,

$\sigma^{①} = 0 \text{ Pa}$

$\sigma^{②} = -22.4E6 \text{ Pa}$

$\sigma^{(2)} = -22.4\ E6\ Pa$

$\sigma^{(3)} = -67.1\ E6\ Pa$

$\sigma^{(4)} = 16\ E6\ Pa$