HIPERWIND
PhD Summer School

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PhD school on uncertainty quantification and reliability assessment of offshore wind turbines

Methods for structural reliability analysis and related topics

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Overview

1) Concepts of reliability
2) The frequentist approach
3) The probabilistic approach
4) Methods for structural reliability analysis

1 Concepts of structural reliability

Concepts of reliability

What does “reliable structure” mean?

- Does not fail frequently
- The likelihood of failure is low
- Etc.

Are these the same?

Concepts of reliability

The “frequentist” approach:

- The reliability is a measure of failure frequencies, i.e., time between failures or the expected number of failure events per time period.

- The time to failure, t_F, is a random variable with a probability distribution $f_{t_F}(t)$

- Probability of failure is a function of time:

$$P(E|T) = P(t_F \leq T) = \int_0^T f_{t_F}(t)dt$$

where

E is a failure event;

T is a reference time period

- Well suited for interpretation of observations

Concepts of reliability

In case of rare events:

- Time between failures has less practical meaning
- There may be insufficient observations, or none at all

The “probabilistic” approach:

- The reliability is a measure of the probability of experiencing failure.
- The probability of failure is calculated based on a reliability model

- Note: the reliability is still defined with respect to a reference time period!

2 Reliability as measure of failure frequency

Reliability as measure of failure frequency

Mean time between failures (MTBF):

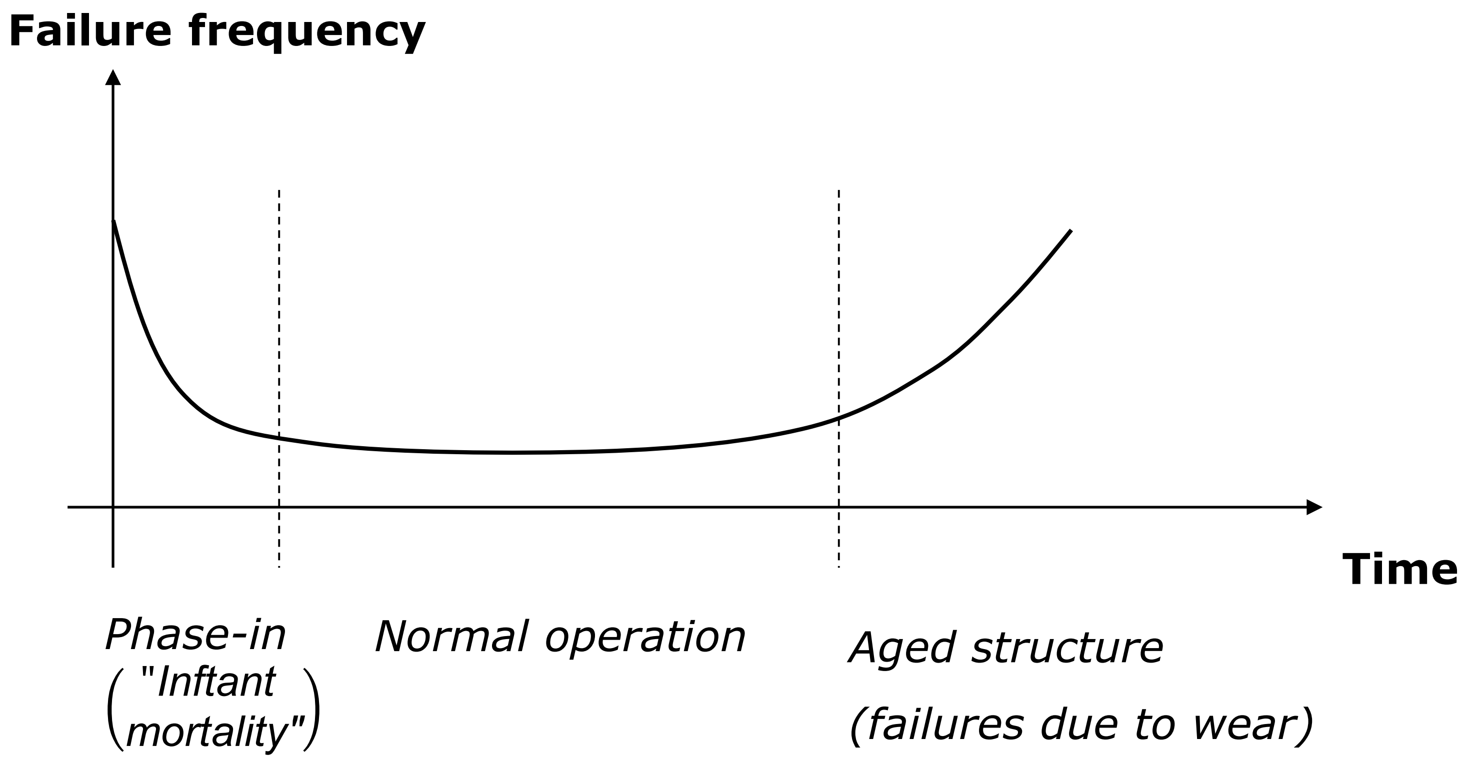
$$MTBF = \bar{t_F} = \lim_{N_T \to \infty} \frac{T.N_T}{N_E}$$

Reliability:

$$R(T) = 1 - P(E|t \leq T) = 1 - \int_0^T f_{t_F}(t)dt$$

Reliability as measure of failure frequency

- The famous “bathtub curve”:



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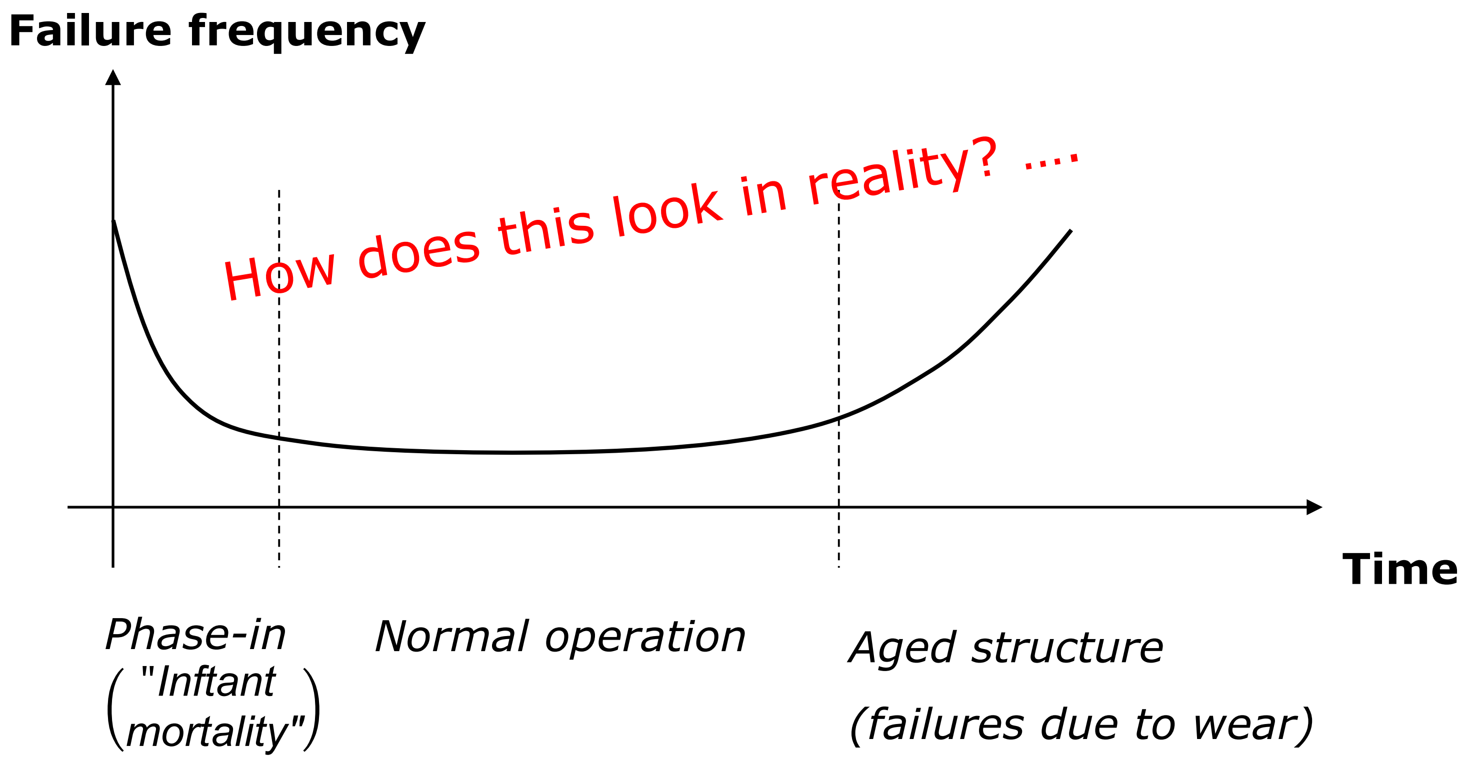
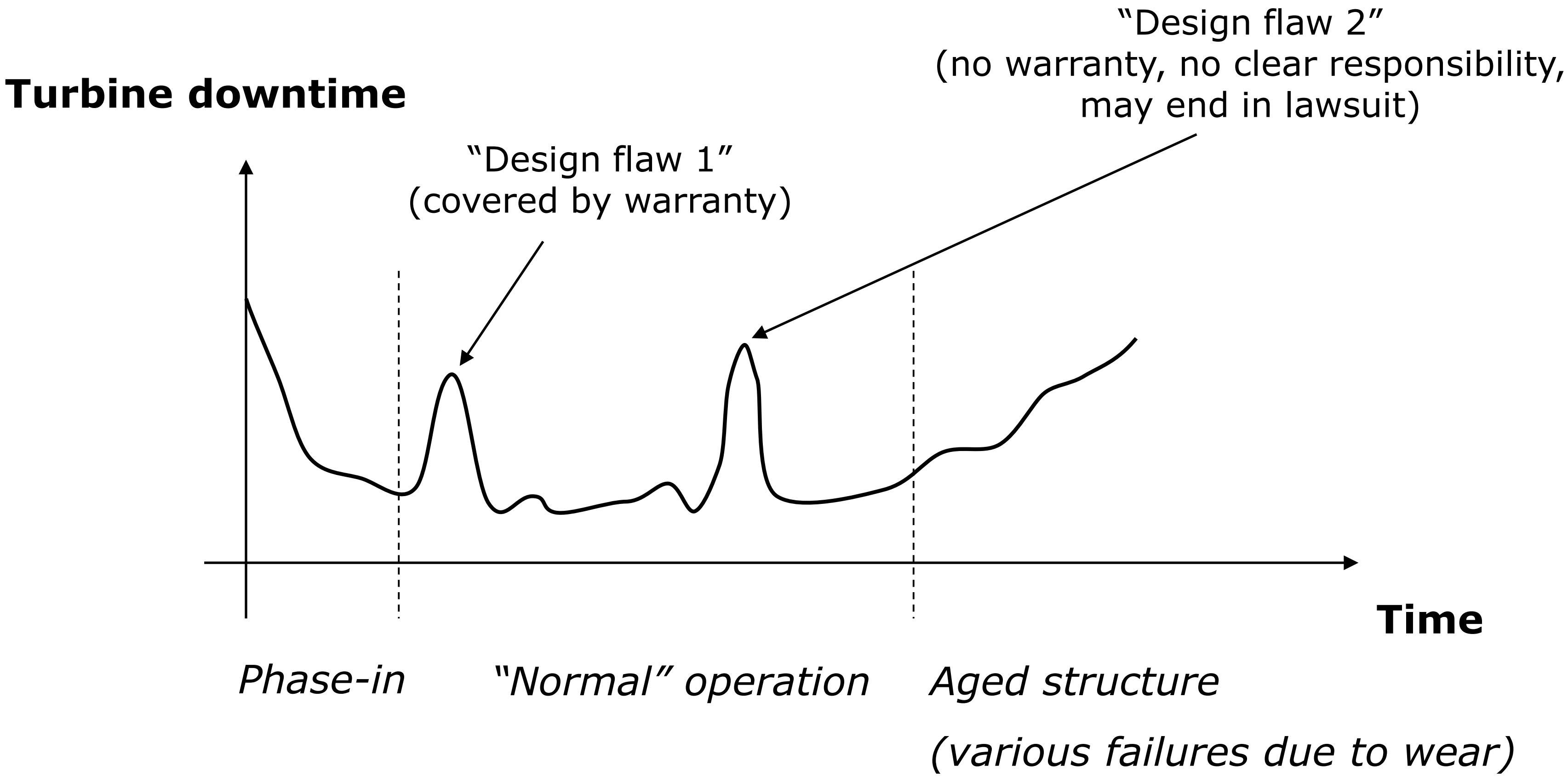


Reliability as measure of failure frequency

- The famous “bathtub curve”:

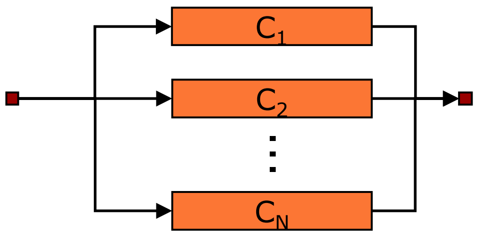


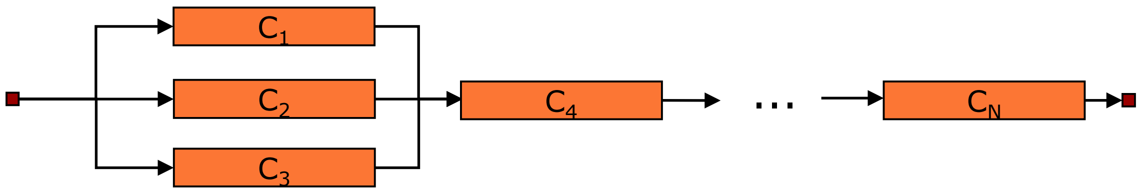
Reliability systems

Concept of reliability of a system of components

- An engineering system consisting of components

- Typically, components are connected in series or in parallel





Reliability systems

Assuming independent components:

- *for a series system*

$$R_{system}(t) = R_1(t) \cap R_2(t) \cap \cdots \cap R_N(t) = \prod_{i=1}^{N} R_i = \prod_{i=1}^{N} (1 - P_i)$$

$$P_{system}(t) = 1 - \prod_{i=1}^{N} (1 - P_i)$$

- *for a parallel system*

$$R_{system}(t) = R_1(t) \cup R_2(t) \cup \cdots \cup R_{N(t)} = \prod_{i=1}^{N} P_i = \prod_{i=1}^{N} (1 - R_i)$$

$$P_{system}(t) = 1 - \prod_{i}^{N} P_i$$

3 Structural reliability analysis

Structural reliability: the limit state equation

- We use an underlying deterministic model

$$g(\mathbf{X})$$

Where $\mathbf{X}$ is a vector of random variables with distribution $F_{\mathbf{X}}(\mathbf{X})$

- The limit state equation identifies the state of the structure:

$$g(\mathbf{X}) = \begin{cases} < 0 & \text{failure set} \\ = 0 & \text{failure surface} \\ > 0 & \text{safe set} \end{cases}$$

- Common forms:

$$g(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X}) \quad ; \quad g(\mathbf{X}) = \frac{R(\mathbf{X})}{S(\mathbf{X})} - 1 \quad ; \quad g(\mathbf{X}) = \log\left(\frac{R(\mathbf{X})}{S(\mathbf{X})}\right)$$

Structural reliability: the limit state equation

The reliability problem amounts to solving the integral:

$$p_f = \int \mathbb{I}_{g(\mathbf{X})\leq 0} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{X}$$

where $\mathbb{I}[]$ is the indicator function, $\mathbb{I} = 1$ for $g(\mathbf{X}) \leq 0$ and $\mathbb{I} = 0$ for $g(\mathbf{X}) > 0$

Structural reliability: the R-S model

Safety margin:

$$M = g(R, S) = R - S$$

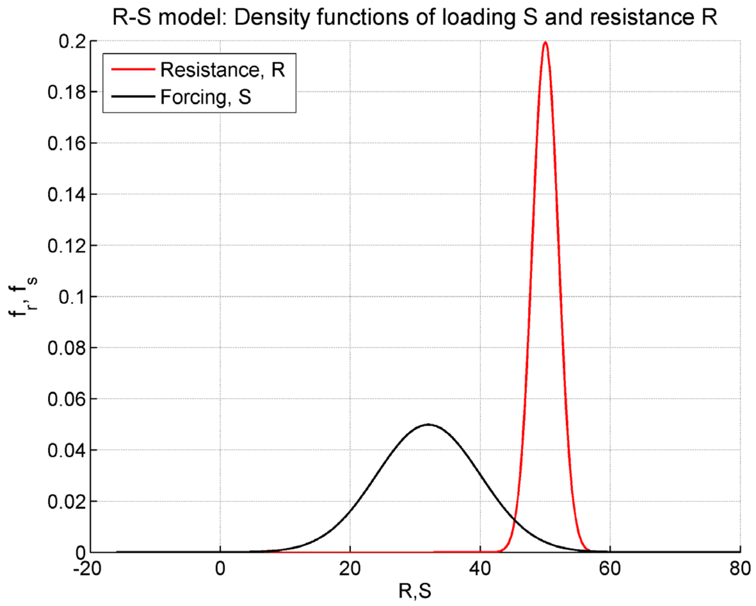
For normally distributed R and S,
M is normally distributed

$$\beta = \frac{E[M]}{D[M]}$$

$E[M]$: expected value; $D[M]$: standard deviation

$$p_f = P[M \le 0] = P\left[\frac{M - E[M]}{D[M]} \le -\frac{E[M]}{D[M]}\right] = P[U \le -\beta] = \Phi(-\beta)$$

β: elementary reliability index



Structural reliability: General nonlinear safety margin

- In the general case, R and S are not normal, and are nonlinear functions of $\mathbf{X}$, i.e.,

$$R = R(\mathbf{X}) \quad ; \quad S = S(\mathbf{X})$$

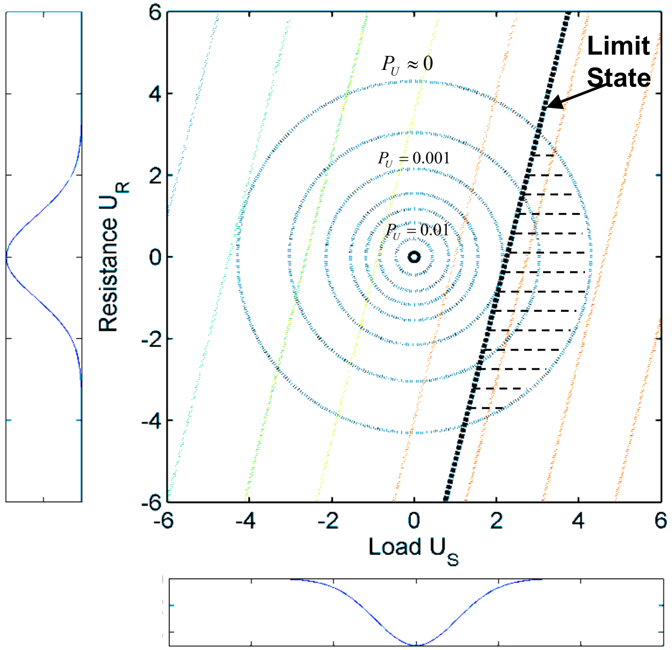
- The probability of failure given a general nonlinear safety margin $g(\mathbf{X})$ is found by solving the integral

$$p_f = \int \mathbb{I}_{g(\mathbf{X})\leq 0} f_{\mathbf{X}}(\mathbf{X}) d\mathbf{X}$$

where $\mathbb{I}[]$ is the indicator function, $\mathbb{I} = 1$ for $g(\mathbf{X}) \leq 0$ and $\mathbb{I} = 0$ for $g(\mathbf{X}) > 0$

- The reliability index is a function of p_f:

$$\beta = \Phi^{-1}(1 - p_f)$$

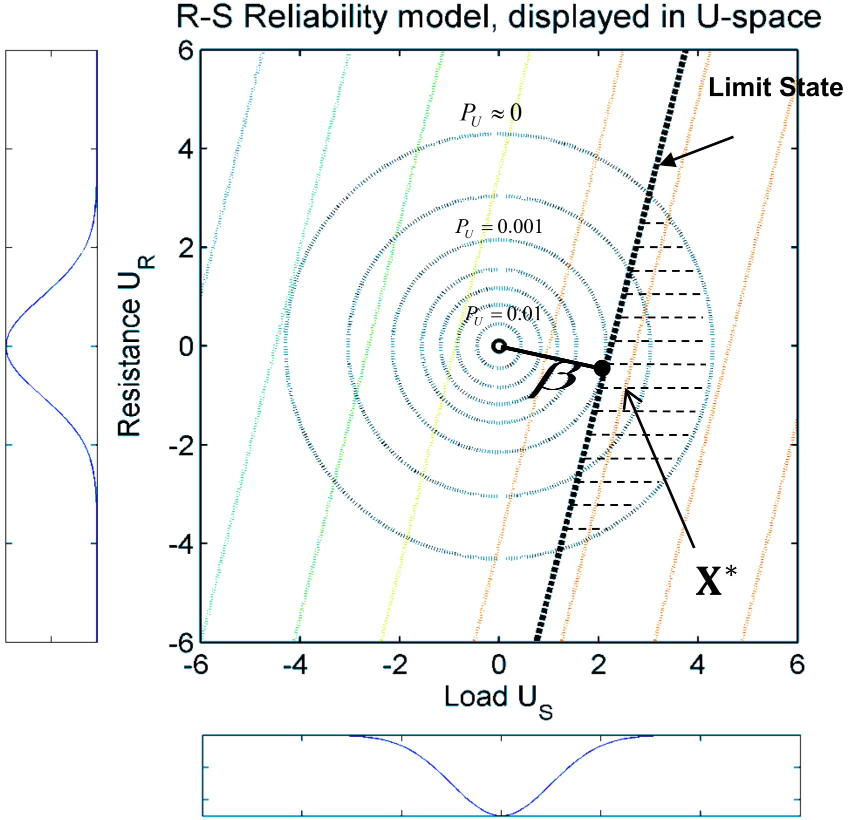


Structural reliability: geometric reliability index

- Geometric reliability index:
  the shortest distance between the limit state surface and the origin

- The design point $\mathbf{X}^*$: the point on the limit state surface closest to the origin

- At the design point $\mathbf{X} = \mathbf{x}^*$, the probability density $f_{\mathbf{X}}(\mathbf{X})$ attains its highest value within the failure set



Structural reliability: standard normal space

- Variables in $\mathbf{X}$ can have different distributions and numerical scales (i.e., orders of magnitude)

- $\mathbf{X}$ is transformed to the uniform, standard normal variable space using the exact transformation:

$$U_i = \Phi^{-1}\big(F(X_i)\big) \quad \text{and} \quad X_i = F_{X_i}^{-1}\big(\Phi(U_i)\big)$$

- In U-space, we can calculate the Hasofer-Lind reliability index:

$$\beta = \min_{g(\mathbf{U})=0} \sqrt{\sum_{i=1}^{n} u_i^2} = \mathbf{u}^{*T}\mathbf{u}^*$$

- The Hasofer-Lind reliability index represents the geometric reliability index in U-space. It satisfies the relation $\beta = \Phi^{-1}(1 - p_f)$ for linear safety margins.

4 Reliability methods

Reliability methods

- FORM / SORM
- Response surface
- Monte Carlo Simulations
- Variance reduction techniques
  - Importance sampling
  - Search-based importance sampling
  - Asymptotic sampling
- Other
  - Directional simulation
  - Model Correction Factor method

Reliability methods

- Example: let us calculate the reliability for a problem defined by the simple limit state function:

$$g(u) = -\frac{4}{25}(u_1 - 1)^2 - u_2 + 3.5$$

$$\nabla_g(u) = \begin{bmatrix} -8(u_1-1)/25 \\ -1 \end{bmatrix}$$

$$E\big(g(u)\big) = -\frac{4}{25}\cdot 2 + 3.5 = 3.18 \qquad ; \qquad D\big(g(u)\big) = \frac{16}{625}\cdot 2 + 1 = 1.0256$$

$$\frac{E\big(g(u)\big)}{D\big(g(u)\big)} = \frac{3.18}{1.0512} = 3.025 \ \neq \beta$$

- The safety margin is not linear $\Rightarrow E(M)/D(M)$ does not give the accurate reliability index
- We try to solve the problem using FORM and Monte Carlo

Exercise

We are given the simple limit state function

$$g(u) = -\frac{4}{25}(u_1 - 1)^2 - u_2 + 3.5$$

where u_1 and u_2 are independent, standard Normal random variables.

Your task is to determine the probability of failure and the reliability index associated with the limit state $g(u)$, using

a) a crude Monte Carlo simulation

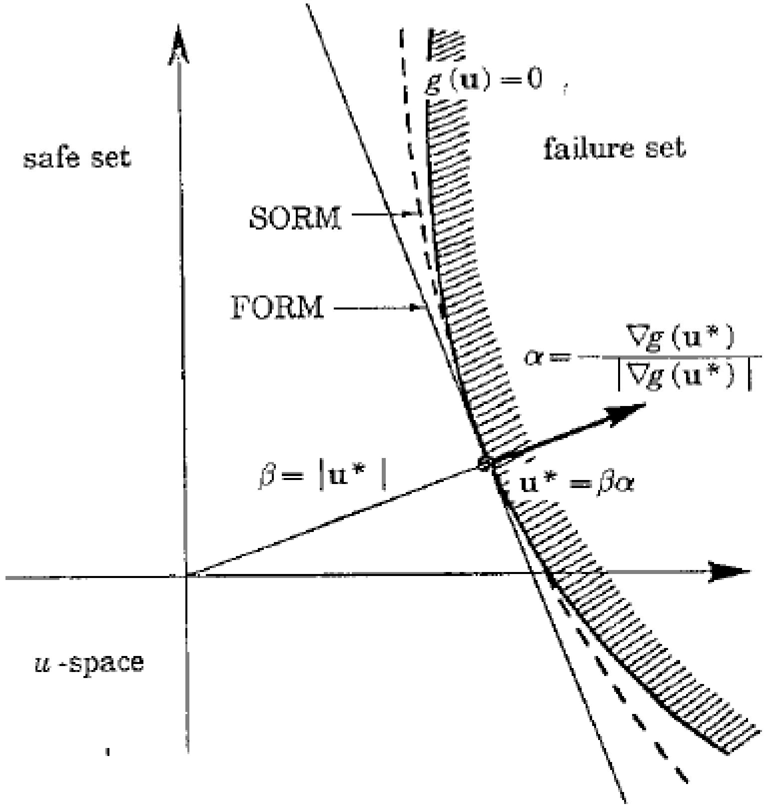
b) the First-Order Reliability Method (FORM)

Can we also use quasi-random sequences for the MC simulation and what would the result be?

For solving the exercise using the FORM method, the gradient of $g(u)$ will be needed. Given the simple form of the limit state function, we can determine the gradient analytically as

$$\nabla_g(u) = \begin{bmatrix} -8(u_1 - 1)/25 \\ -1 \end{bmatrix}$$

First and second-order reliability methods



$$p_f \approx p^{FORM} = \Phi(-\beta)$$

$$p_f \approx p_{asymptotic}^{SORM} = \Phi(-\beta) \prod_{j=1}^{q-1} \left(1 - \beta\kappa_j\right)^{-1/2}$$

$$p_f \approx p_{parabolic}^{SORM} = \phi(\beta) \cdot Re\left[i \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_{i=0}^{i\infty} \frac{\exp\{(t+\beta)^2/2\}}{t} \left\{ \prod_{j=1}^{q-1} \left(1 - \beta\kappa_j\right)^{-1/2} \right\} dt \right]$$

where k_j are the principal curvatures of the limit state surface at $\mathbf{u}^*$

First-order parameter sensitivity

$$\frac{d\beta}{d\theta} = \boldsymbol{\alpha}^T \frac{d\mathbf{u}^*}{d\theta} = -\frac{1}{|\nabla g(\mathbf{u}^*)|}\frac{\partial g(\mathbf{u}^*)}{\partial \theta}$$

- $\boldsymbol{\alpha}$: importance factors

$$\boldsymbol{\alpha} = -\frac{\nabla g(\mathbf{u}^*)}{|\nabla g(\mathbf{u}^*)|}$$

- At the design point u^*, the failure surface is linearly approximated by

$$M = \beta - \boldsymbol{\alpha}^T \mathbf{U}$$

- The squared sum of the importance factors equals unity:

$$\sum_{i=1}^{n} \alpha_i^2 = 1$$

FORM: finding $\mathbf{u}^*$ by optimization

- At the design point:

$$\mathbf{u}^* = \lambda \nabla g(\mathbf{u}^*)$$

$$g(\mathbf{u}^*) = 0$$

- Linearization at a point $\mathbf{u}^0$ “close to $\mathbf{u}^*$”:

$$\begin{aligned} g(\mathbf{u}^*) &\approx g(\mathbf{u}^0) + \nabla g(\mathbf{u})^T (\mathbf{u}^* - \mathbf{u}^0) \\ &= g(\mathbf{u}^0) + \nabla g(\mathbf{u}^0)^T (\lambda \nabla g(\mathbf{u}^0) - \mathbf{u}^0) \end{aligned}$$

- The condition $g(\mathbf{u}^*) = 0$ gives:

$$\lambda = \frac{\nabla g(\mathbf{u}^0)^T \mathbf{u}^0 - g(\mathbf{u}^0)}{\nabla g(\mathbf{u}^0)^T \nabla g(\mathbf{u}^0)}$$

FORM: finding $\mathbf{u}^*$ by optimization

Iterative procedure to find $\mathbf{u}^*$ and β:

1) Define a starting point $\mathbf{u}^0$, set iteration $i = 0$

2) Calculate the limit state function value, $g(\mathbf{u}^i)$

3) Calculate the gradient $\nabla g(\mathbf{u}^i)$

4) Estimate an updated value $\mathbf{u}^{i+1}$:

$$u^{i+1} = \lambda^i \nabla g(u^i) = \frac{\nabla g(\mathbf{u}^i)^T \mathbf{u}^i - g(\mathbf{u}^i)}{\nabla g(\mathbf{u}^\mathrm{i})^\mathrm{T} \nabla g(\mathbf{u}^i)} \nabla g(\mathbf{u}^i)$$

5) Update the reliability index: $\beta^{i+1} = \sqrt{(\mathbf{u}^{i+1})^T \mathbf{u}^{i+1}}$

6) Stop if $\left|\beta^{i+1} - \beta^i\right| < \varepsilon$

Monte Carlo simulation

- Estimating probability of failure:

$$p_f = \int_\Omega \mathbb{I}(g(\mathbf{X}) \leq 0) f_{\mathbf{x}}(\mathbf{X}) d\mathbf{x} \Rightarrow$$

$$p_f \approx \frac{1}{N} \sum_{i=1}^{n} \mathbb{I}(g(\mathbf{X}) \leq 0)$$

- Pros:

Very robust: not sensitive to limit state surface shape, can solve system reliability problems, number of dimensions does not influence convergence speed

- Cons:

Very computationally expensive for small probabilities ($N \sim 200 \cdot p_f^{-1}$ for convergence)

Monte Carlo simulation

- A Monte Carlo simulation does not require evaluation of gradients or fitting of surfaces
- Due to this simplicity, importance factors are not part of the solution
- It is however possible to find approximate importance factors by fitting a hyperplane to the observations close to the failure surface:

1) Choose the samples which are close to the limit state surface ($g(\mathbf{X}) \approx 0$). If too few samples are within that category, the set ($g(\mathbf{X}) \leq 0$) may be more appropriate choice.

2) Fit a linearized limit state function:

$$g_L(\mathbf{X}) = a_0 + \sum_{i=1}^{n} a_i X_i = 0$$

Where (a_0, a_i) are the coefficients of the hyperplane fit to $g(\hat{\mathbf{x}}) \approx 0$

Monte Carlo simulation

- The mean and standard deviation of g_L are given by

$$\mu_g = a_0 + a_1\mu_{X_1} + a_2\mu_{X_2} + \cdots$$

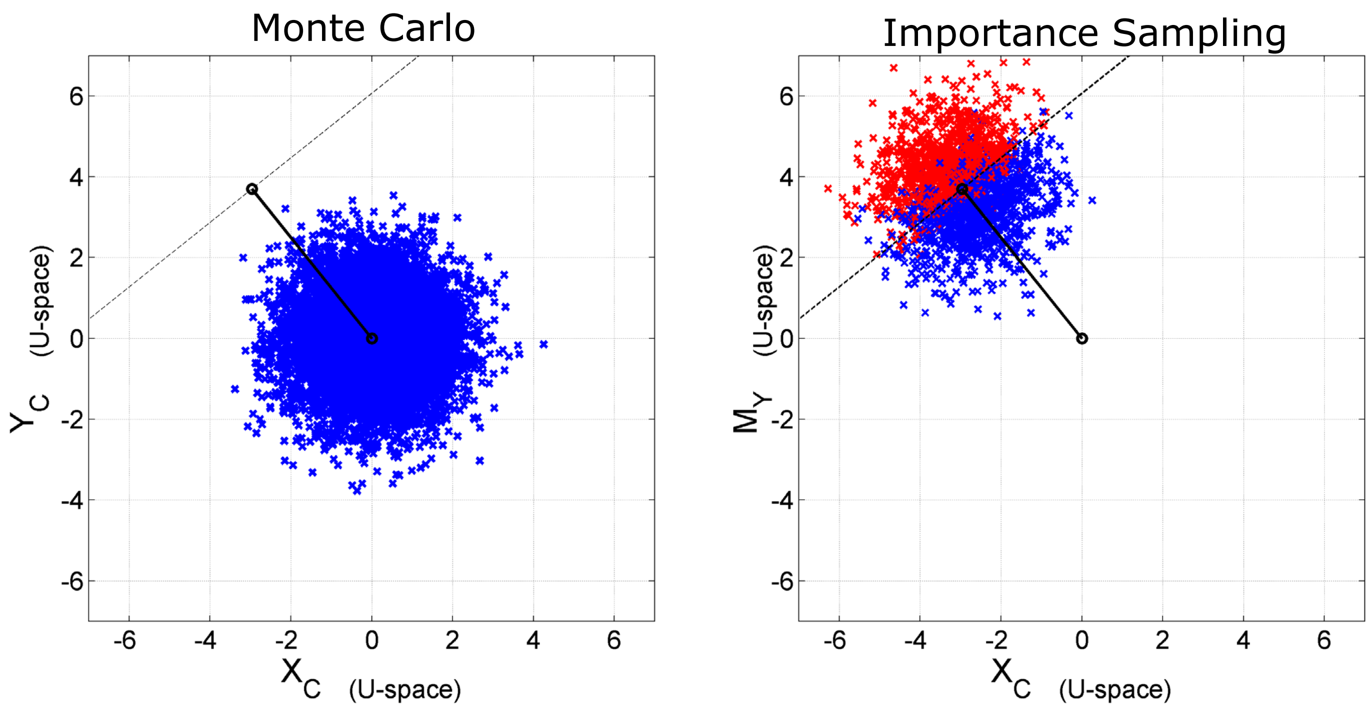
$$\sigma_g = \left[\sum_{i=1}^{n} \left(a_i \sigma_{X_i}\right)^2\right]^{\frac{1}{2}}$$

- When $\mathbf{X}$ is mapped into standard normal space $\mathbf{U}$, gradients of the reliability index follow from FORM theory:

$$\left.\frac{\partial\beta}{\partial U_i}\right|_{\mathbf{u}^*} = \alpha_i = \frac{a_i \sigma_{X_i}}{\sigma_g}$$

Importance sampling

- When P_F is small, a Crude Monte Carlo simulation requires large number of trials
- Moving the sampling density closer to the failure domain reduces the number of necessary trials



Importance sampling

- We introduce the importance sampling density, $h(\mathbf{X})$, where
$$h(\mathbf{X}) \neq 0 \text{ if } f(\mathbf{X}) \neq 0$$

- The probability integral can be represented as

$$p_f = \int \mathbb{I}_{g(\mathbf{X}) \leq 0} \frac{f_{\mathbf{X}}(\mathbf{X})}{h_{\mathbf{X}}(\mathbf{X})} h_{\mathbf{X}}(\mathbf{X}) d\mathbf{X}$$

$$p_f = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{g(\mathbf{X}) \leq 0} \frac{f_{\mathbf{X}}(\mathbf{X})}{h_{\mathbf{X}}(\mathbf{X})}$$

- Ideally, all samples from $h(X)$ would fall within the failure domain – this ensures the fastest possible convergence of the importance sampling – however this is difficult to achieve

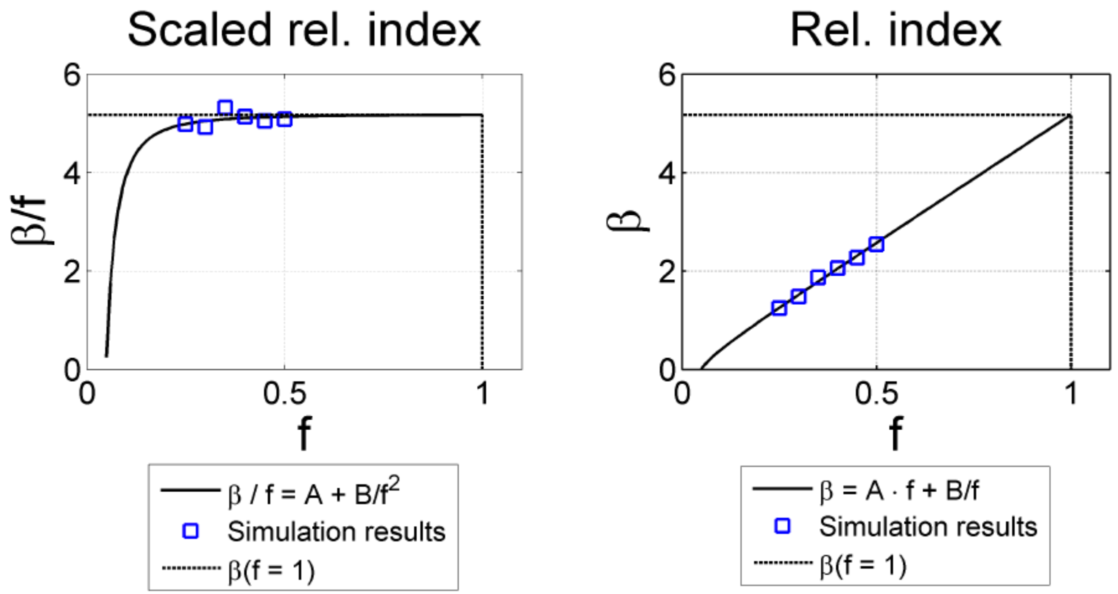
- In practice, the design point obtained by a FORM analysis is a reasonable choice – with approximately half of the points within the failure domain

Asymptotic sampling

The variance of input variables is increased in order to achieve more failure outcomes.

The reliability index is expressed as function of the scaling variable $f = \frac{1}{\sigma}$

$$\beta = A \cdot f + \frac{B}{f}$$

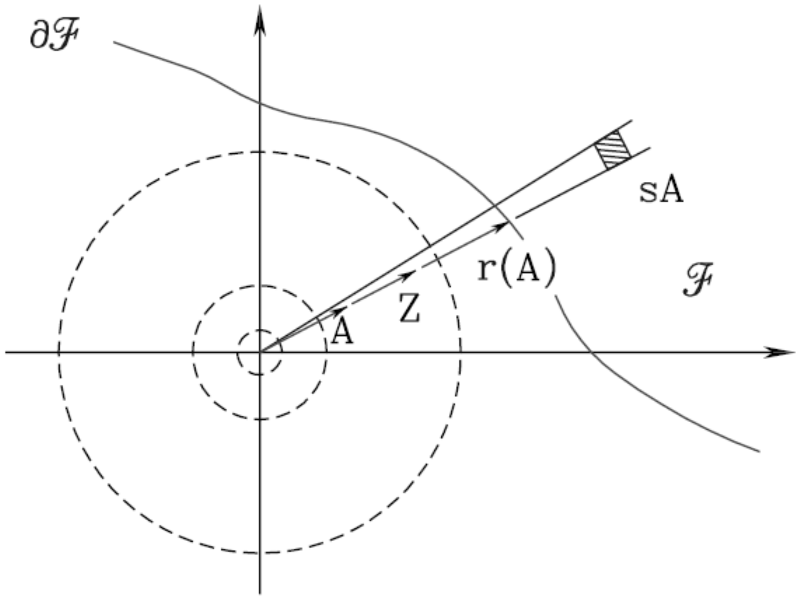


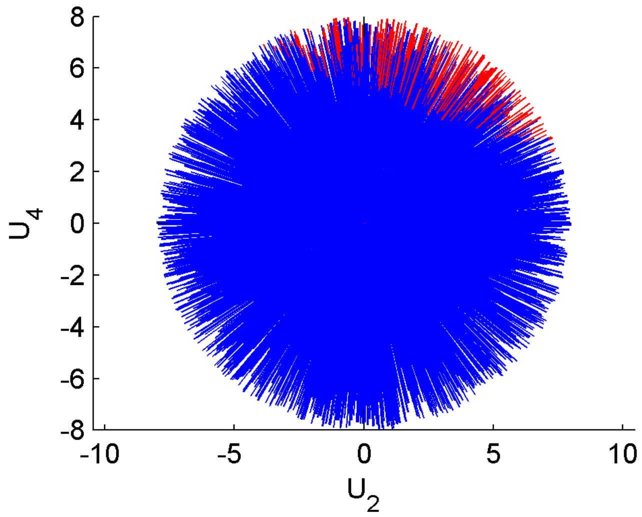
Directional Simulation

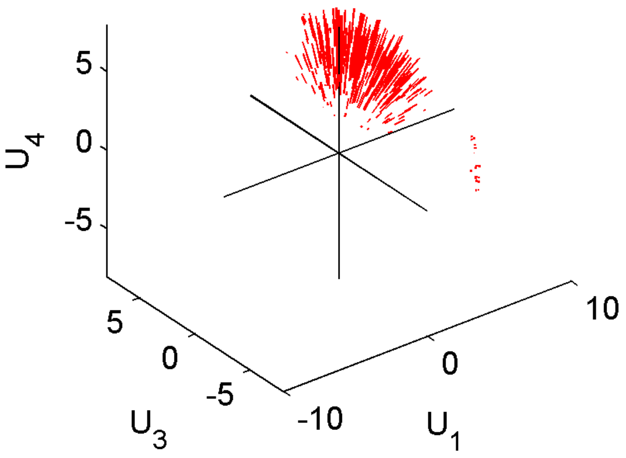
- Define “direction vector”, taking a random direction in the solution domain

- Find distance to the failure surface in the chosen direction

- Integrate the failure probability over the part of the line that falls within the failure domain

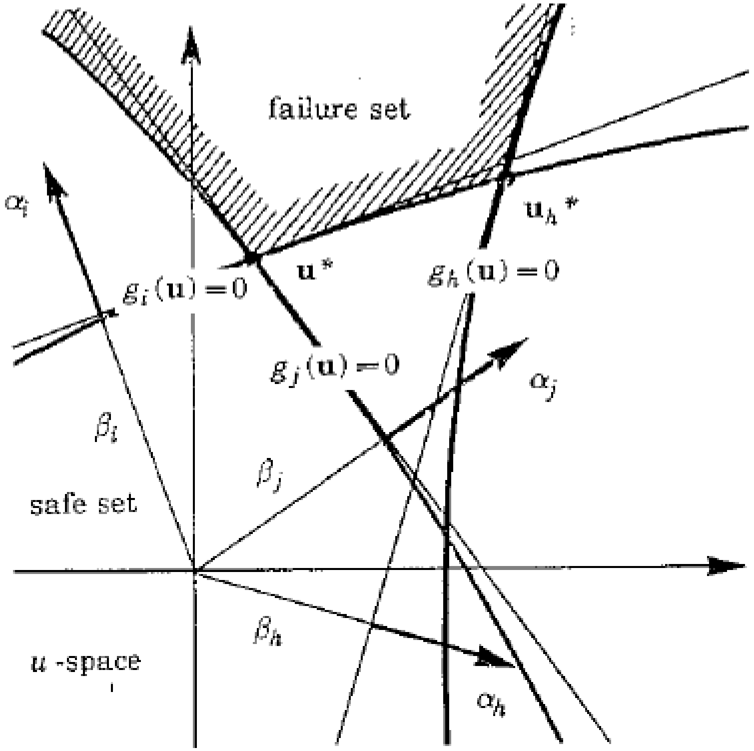






Multiple safety margins (system problem)

Small intersection (parallel system)



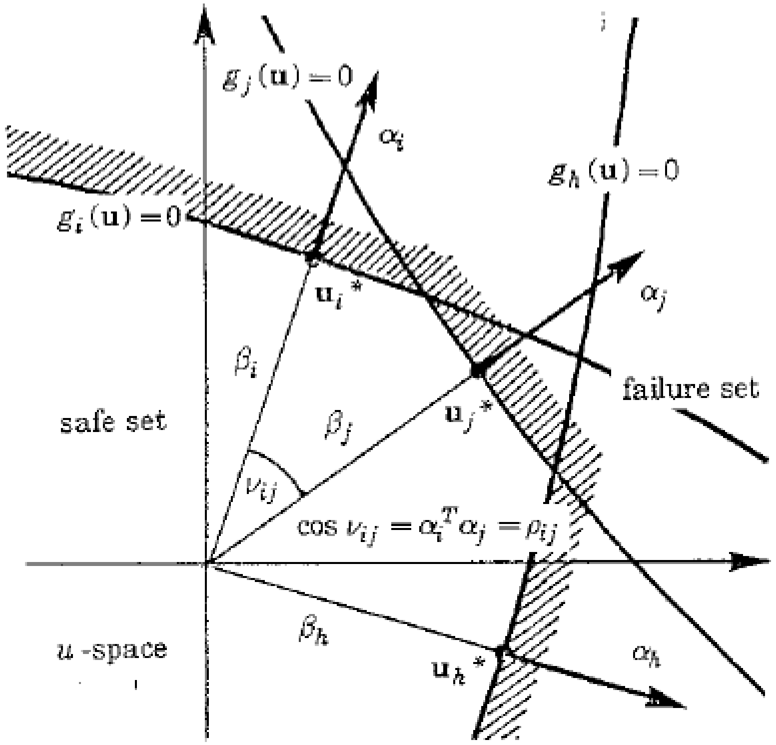
$$p_f \approx p^{FORM} = \Phi_k(-\boldsymbol{\beta}, \mathbf{R})$$

- $\boldsymbol{\beta} = [\beta_1, \beta_2, \dots, \beta_k]^T$ is the vector of component reliability indexes with respect to safety margins $M = M_1, M_2, \dots, M_k$
- $\mathbf{R}$ is the correlation matrix for $\boldsymbol{\beta}$, with

$$\rho_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j, \qquad i = 1, \dots, k\,; j = 1, \dots, k$$

Multiple safety margins (system problem)

Large intersection (series system)



$$p_f \approx p^{FORM} = 1 - \Phi_k(\boldsymbol{\beta}, \mathbf{R})$$

Reliability systems

- When multiple components or multiple failure scenarios are present
- Reliability of parallel and series systems defined in U-space:

$$P_f = 1 - \Phi_k(\boldsymbol{\beta}, \mathbf{R}) \qquad \text{(Series system)}$$

$$P_f = \Phi_k(-\boldsymbol{\beta}, \mathbf{R}) \qquad \text{(Parallel system)}$$

- In principle, all general systems can be expressed as a combination of parallel and series components
- FORM/SORM and response surface methods can only find a single design point => the result corresponds to a component reliability
- Simulation-based methods are insensitive to number of system components, their convergence is not affected

Reliability systems

Simple bounds on the system reliability for a set of safety margins $M_1, M_2, \dots, M_m$

- Simple bounds for series system:

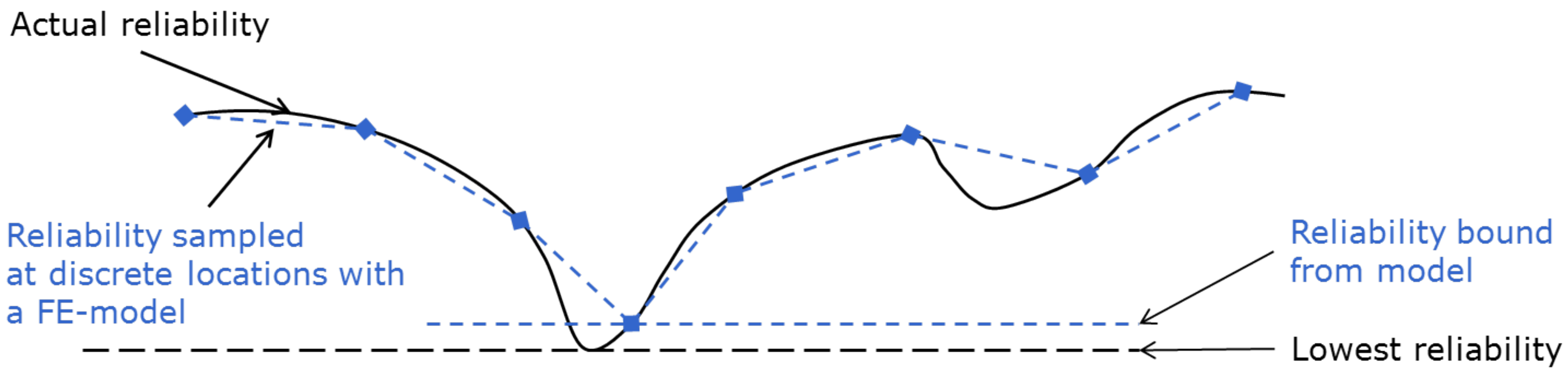
$$\max_{i=1\dots m} P(M_i \leq 0) \leq P_f^S \leq \sum_{i=1}^{m} P(M_i \leq 0)$$

- Simple bounds for parallel system:

$$0 \leq P_f^P \leq \min_{i=1\dots m} (P(M_i \leq 0)$$

Reliability systems

- Spatial variation of properties is also a system-reliability phenomenon
- When estimating the reliability over a spatial domain, the domain discretization has an effect on the results
- In discretized structures, in fact we are only determining reliability bounds, not the true reliability of the structure



Uncertainty modelling

Limit state function:

$$g = R \cdot X_{CAPACITY} - S \cdot X_{DEMAND}$$

R, *S*: resistance and load (capacity and demand) variables

$X_{CAPACITY}$ and X_{DEMAND}: model uncertainty variables

- Example from literature, concerning uncertainties in wind turbine load models (Tarp-Johansen et al., 2002):

$$X_{DEMAND} = X_L = X_{exp} X_{st} X_{aero} X_{dyn} X_{str} X_{sim} X_{ext}$$

- For unbiased models, model uncertainties will have a mean of 1

- Uncertainty types: epistemic or aleatory?

Calibration of partial safety factors

Given a limit state equation

$$g(\mathbf{X}) = R(\mathbf{X}) - S(\mathbf{X})$$

The corresponding design equation is

$$g = \frac{1}{\gamma_m \gamma_n} R_c - \gamma_f S_c$$

Where $\gamma_m, \gamma_f, \gamma_n$ are partial safety factors for material strength, load effects, and consequences of failure respectively.

R_c: characteristic (typically 5% quantile) strength of the material

S_c: characteristic (typically 98% quantile) load

Calibration of partial safety factors

- If we carry out reliability analysis using $g(\mathbf{X})$, we obtain a design point

$$\mathbf{X}^* = R^* - S^* = 0$$

- If we know the characteristic values R^* and S^*, we can calibrate partial safety factors:

$$\gamma_m \gamma_n = \frac{R_c}{R^*} \quad ; \quad \gamma_f = \frac{S^*}{S_c}$$

Calibration of partial safety factors

Calibration against target reliability levels:

- We use a modified limit state equation, including a "design parameter", z.

$$g(\mathbf{X}) = R(z\mathbf{X}_R) - S(\mathbf{X}_S)$$

- By changing the parameter z and carrying out repeated reliability analyses, we find a value of z for which the estimated reliability equals the target reliability

- Using the obtained value for z, we estimate partial safety factors calibrated to the target reliability level, by solving the equation below for either γ_m, γ_n or γ_f:

$$g = R\left(z\frac{1}{\gamma_m\gamma_n}X_{R,c}\right) - S(\gamma_f X_{S,c}) = 0$$

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