Mathematical Methods

Lecture 1

Vectors in 2D

Dr. Tony Lyons

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Contents

1 Vectors in 2-dimensions

1.1 Constructing vectors in the plane

- A vector in the plane is a **directed line segment** connecting two points in the plane, and is represented by an arrow pointing from the tail to the head.
- The two characteristics needed to define a vector are **length** and **direction**.
- Every vector in the plane can be represented in terms of the **basis vectors** $\hat{\imath}$ and $\hat{\imath}$.

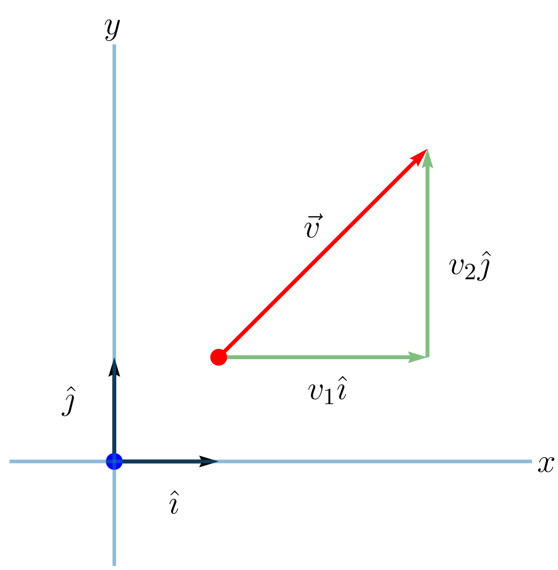

Figure 1: The **unit vectors** $\hat{\imath}$ and $\hat{\jmath}$ are used to write an arbitrary vector $\vec{v}$ in terms of components as $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

- The **unit vector** $\hat{\imath}$ is defined as the vector of length one unit pointing along the positive x-axis. Likewise, the **unit vector** $\hat{\jmath}$ is the vector of length one unit pointing along the positive y-axis, see Figure 1.

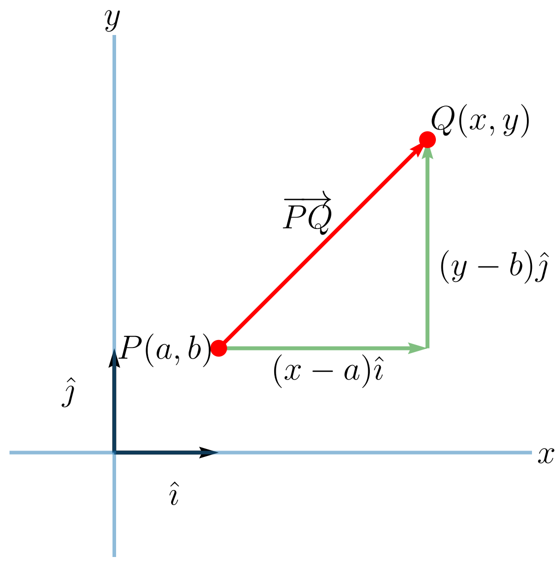

Figure 2: The vector connecting the point $P(a,b)$ to the point $Q(x,y)$, written in terms of the coordinates of those points.

Definition 1: The vector connecting two points

Given the points $P(a,b)$ and $Q(x,y)$ in the plane, then the vector connecting these two points is written in component form as

$$\overrightarrow{PQ} = (x-a)\hat{\imath} + (y-b)\hat{\imath}. \qquad \text{(Vector to points)}$$

That is to say, we start at the tip and subtract the tail, coordinate-by-coordinate, see 2

1.2 Addition of vectors

Definition 2: Vector addition

Given the vectors

$$\vec{u} = u_1\hat{\imath} + u2\hat{\jmath} \qquad \vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$$

then the sum of the vectors is given by

$$\vec{u} + \vec{v} = (u_1 + v_1)\hat{\imath} + (u_2 + v_2)\hat{\jmath}. \tag{Vector addition}$$

That is to say we add the vectors component-by-component.

Example 1: Vector addition

Given the vector connecting the points

$$O(0,0) \quad A(1,1) \quad C(3,8),$$

then the vector connecting O to C is equivalent to the sum of the vectors connecting O to A and connecting A to C.

Solution. The vector connecting O and A is given in component form by

$$\overrightarrow{OA} = (1-0)\hat{\imath} + (1-0)\hat{\jmath} = \hat{\imath} + \hat{\jmath}.$$

The vector connecting A to C is given in component form by

$$\overrightarrow{AC} = (3-1)\hat{\imath} + (8-1)\hat{\jmath} = 2\hat{\imath} + 7\hat{\jmath}.$$

The vector connecting the points O and C is given in component form by

$$\overrightarrow{OC} = (3-0)\hat{\imath} + (8-0)\hat{\jmath} = 3\hat{\imath} + 8\hat{\jmath}.$$

Adding the vectors $\overrightarrow{OA}$ and $\overrightarrow{AC}$ we have

$$\overrightarrow{OA} + \overrightarrow{AC} = \hat{\imath} + \hat{\jmath} + 2\hat{\imath} + 7\hat{\jmath} = 3\hat{\imath} + 8\hat{\jmath} = \overrightarrow{OC},$$

as required.

Exercise 1: Vectors Between Points

Given the points

$$A(-1,4),\ B(1,-3),\ C(-2,-5),\ D(3,4) \tag{1.1}$$

find the following vectors in the form $x\hat{\imath} + y\hat{\jmath}$:

(i) $\overrightarrow{AB}$

(ii) $\overrightarrow{BC}$

(iii) $\overrightarrow{CD}$

(iv) $\overrightarrow{DA}$.

Draw each of these vectors.

2 Re-scaling vectors

Definition 3: Scalar multiplication of vectors

Given the scalar (i.e. number) α and the vector $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$ then the re-scaled $\alpha\vec{v}$ is given by

$$\alpha\vec{v} = \alpha v_1\hat{\imath} + \alpha v_2\hat{\jmath} \qquad \text{(Vector re-scaling)}$$

- Besides addition of vectors, it is also possible to re-scale a vector simply by multiplying each component by a fixed number called a **scalar**.
- In Figure 3 the vector $\vec{v} = 42\hat{\imath} + 2\hat{\jmath}$ is re-scaled by factors of -1, 0.5 and 2.
- The effect of multiplying a vector by a **negative number** is to **reverse direction** of the vector.
- Multiplying by a **number greater than 1** always **stretches** the vector, and preserves its direction.
- Multiplying by a **number less than 1** always **shrinks** the vector, see Figure 3.

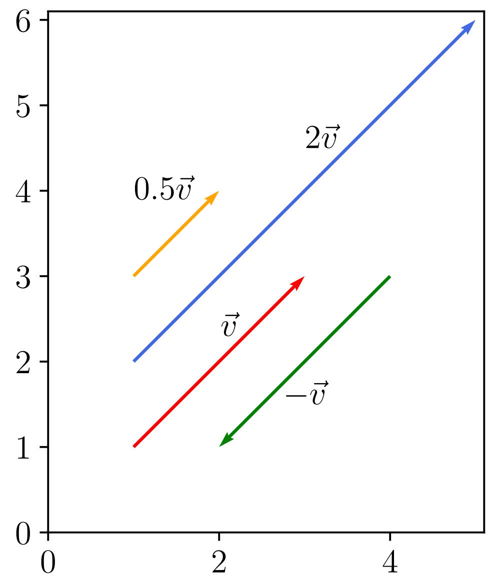

Figure 3: The vector $\vec{v} = 2\hat{\imath} + 2\hat{\jmath}$ along with its scalar multiples $-\vec{v}$, $0.5\vec{v}$ and $2\vec{v}$.

Exercise 2: Re-scaling vectors

Given the vector $\vec{v} = 2\hat{\imath} - 4\hat{\jmath}$ answer the following:

(i) Calculate the vector $3\vec{v}$ in the form $x\hat{\imath} + y\hat{\jmath}$.

(ii) Calculate the vector -$2\vec{v}$ in the form $x\hat{\imath} + y\hat{\jmath}$.

(iii) Calculate the vector $\frac{1}{2}\vec{v}$ in the form $x\hat{\imath} + y\hat{\jmath}$.

(iv) Calculate the vector $-\frac{1}{4}\vec{v}$ in the form $x\hat{\imath} + y\hat{\jmath}$.

(v) Draw all 5 vectors.

2.1 The norm of a vector

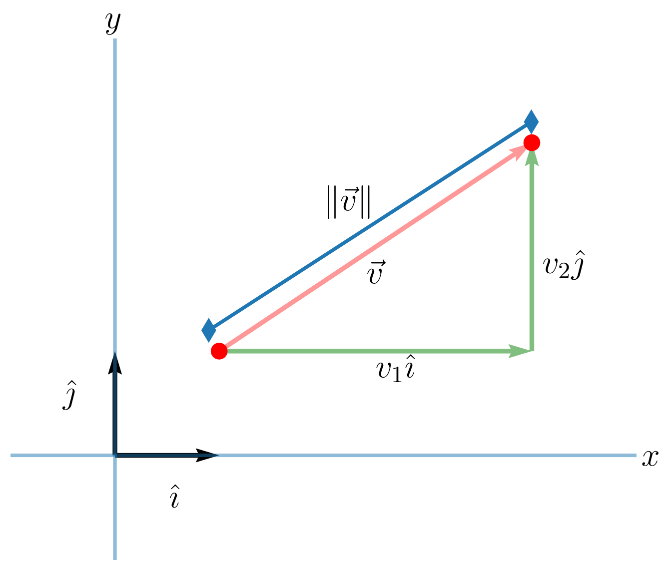

Figure 4: The vector $\vec{v}$ resolved into its components v_1 and v_2 and into its norm $\|\vec{v}\|$.

Given a vector $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$ (see Figure 5), then Pythagoras's Theorem gives the length of the vector as

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}, \tag{2.1}$$

which is obvious given the decomposition of $\vec{v}$ into its components along the x and y-axes. It is clear that we may apply the norm to any vector in the plane.

Exercise 3: Vector Norms

Given the vector $\vec{u} = \hat{\imath} + 3\hat{\jmath}$ and the points $A(1, 4)$ anf $B(-2, 3)$ answer the following:

(i) Find the length of $\vec{v}$

(ii) Find the length of $\overrightarrow{AB}$

(iii) Draw the vector $\vec{v}$ and its individual components (i.e. $\hat{\imath}$ & $3\hat{\jmath}$).

(iv) Do the same for $\overrightarrow{AB}$.

(v) Apply Pythagoras' Theorem to get the hypotenuse of each of these triangles to show that you get the norm in each case.

2.2 The direction of a vector

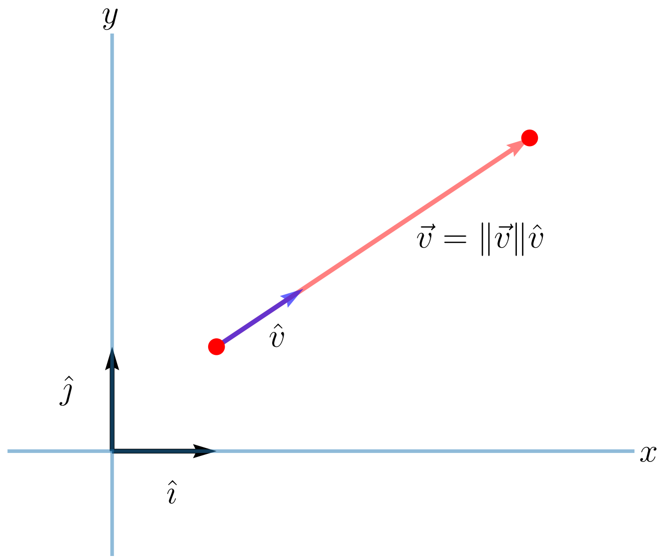

Figure 5: The vector $\vec{v}$ resolved into its components v_1 and v_2 and into its norm $\|\vec{v}\|$.

Definition 4: Unit vectors

The direction of a vector $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$ is given by its associated **unit vector** $\hat{u}$, which is defined as

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{v_1}{\|\vec{v}\|}\hat{\imath} + \frac{v_2}{\|\vec{v}\|}\hat{\jmath}. \tag{2.2}$$

- A vector is characterised by a length and direction, and we have seen that the length is given by the norm.

- Any vector may also be written as

$$\vec{v} = \|\vec{v}\|\,\hat{v}. \tag{2.3}$$

- Equation (2.3) informs us that a vector $\vec{v}$ can be interpreted as the re-scaling of the unit vector $\hat{v}$ by the vector length $\|\vec{v}\|$, to give the vector $\vec{v}$.

- In this sense a vector has both **length** and **direction**.

Example 2

Given the vector

$$\vec{u} = 4\hat{\imath} - 3\hat{\jmath} \tag{2.4}$$

answer the following:

(i) Find the norm $\|\vec{u}\|$

(ii) Find the unit vector $\hat{u}$

(iii) Show that $\|\hat{u}\| = 1$

(iv) Show that $\vec{u} = \|\vec{u}\|\,\hat{u}$

(v) Use GeoGebra Calculator to draw the vectors $\vec{u}$ and $\hat{u}$.

Solution.

(i) The norm $\|\vec{u}\|$

$$\|\vec{u}\| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

(ii) The unit vector $\hat{u}$

$$\hat{u} = \frac{4\hat{\imath} - 4\hat{\jmath}}{5} = \frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath}$$

(iii) If $\|\hat{u}\| = 1$

$$\|\hat{u}\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{1} = 1$$

(iv) If $\vec{u} = \|\vec{u}\|\, \hat{u}$

$$\|\vec{u}\|\, \hat{u} = 5\left(\frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath}\right) = 4\hat{\imath} - 3\hat{\jmath} = \vec{u}.$$

Exercise 4: Unit vectors

Given the vector

$$\vec{w} = -2\hat{\imath} + 5\hat{\jmath} \tag{2.5}$$

answer the following:

(i) Find the norm $\|\vec{w}\|$

(ii) Find the unit vector $\hat{w}$

(iii) Show that $\|\hat{w}\| = 1$

(iv) Show that $\vec{w} = \|\vec{w}\|\, \hat{w}$

(v) Draw the vectors $\vec{w}$ and $\hat{w}$.