Mathematical Methods

Lecture 2

Dot products & Angles

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Contents

1 Vector Algebra

1.1 Representations of vectors

- Given a vector in 2D we can represent it n several different ways.

- We can use the standard algebraic definition

$$\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} = \left[\begin{array}{c} u_1 \\ u_2 \end{array} \right]. \tag{1.1}$$

- We can also represent it as vertical **or** horizontal **arrays**:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ or } \vec{u} = \begin{bmatrix} u_1, u_2 \end{bmatrix} \tag{1.2}$$

- These vertical and horizontal arrays are equivalently valid representations, and we usually use one or the other depending on the context.

- In a similar way, we can represent a 3D vector algebraically as

$$\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}.$$

- Likewise, we can represent this 3D vector as a vertical or horizontal array like

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ or } \vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}. \tag{1.3}$$

- These representations using arrays are especially useful when doing vector algebra using a computer program such as Python, Matlab or Mathematica etc.

Definition 1: Symbolic vector algebra

Symbolically we can define vector addition, subtraction and scalar multiplication in 2D and 3D in the expected ways

Addition

$$\begin{aligned}\vec{u}+\vec{v} &= u_1\hat{\imath}+u_2\hat{\jmath}+v_1\hat{\imath}+v_2\hat{\jmath} \\ &= (u_1+v_1)\hat{\imath}+(u_2+v_2)\hat{\jmath} \end{aligned} \tag{2D}$$

$$\begin{aligned}\vec{u}+\vec{v} &= u_1\hat{\imath}+u_2\hat{\jmath}+u_3\hat{k}+v_1\hat{\imath}+v_2\hat{\jmath}+v_3\hat{k} \\ &= (u_1+v_1)\hat{\imath}+(u_2+v_2)\hat{\jmath}+(u_3+v_3)\hat{k} \end{aligned} \tag{3D}$$

Subtraction

$$\begin{aligned}\vec{u}-\vec{v} &= u_1\hat{\imath}+u_2\hat{\jmath}-v_1\hat{\imath}-v_2\hat{\jmath} \\ &= (u_1-v_1)\hat{\imath}+(u_2-v_2)\hat{\jmath} \end{aligned} \tag{2D}$$

$$\begin{aligned}\vec{u}-\vec{v} &= u_1\hat{\imath}+u_2\hat{\jmath}+u_3\hat{k}-v_1\hat{\imath}-v_2\hat{\jmath}-v_3\hat{k} \\ &= (u_1-v_1)\hat{\imath}+(u_2-v_2)\hat{\jmath}+(u_3-v_3)\hat{k} \end{aligned} \tag{3D}$$

Scalar Multiplication

$$\begin{aligned}\alpha\vec{u} &= \alpha(u_1\hat{\imath}+u_2\hat{\jmath}) \\ &= \alpha u_1\hat{\imath}+\alpha u_2\hat{\jmath} \end{aligned} \tag{2D}$$

$$\alpha\vec{u}+\vec{v} = \alpha u_1\hat{\imath}+\alpha u_2\hat{\jmath}+\alpha u_3\hat{k} \tag{3D}$$

Definition 2: Numerical vector algebra

Numerically we define vector addition, subtraction and scalar multiplication in 2D and 3D in a similar way:

Addition

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Subtraction

$$\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_2 - v_3 \end{bmatrix}$$

Scalar Multiplication

$$\alpha\vec{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{bmatrix}$$

- Here we give the definition in 3D only, it should be obvious how things work in 2D from this.
- We use the vertical representation just to keep everything inside the borders of this box. Using the horizontal representation is equally valid.

Definition 3: Re-scaling of vectors

In a similar manner, we define scalar multiplication by $\alpha \in \mathbb{R}$ of the vectors $\vec{u} = u_1\hat{\imath} + u - 2\hat{\jmath} \in \mathbb{R}^2$ and $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k} \in \mathbb{R}^3$ in the following way

$$\begin{aligned} \alpha\vec{u} &= (\alpha u_1)\hat{\imath} + (\alpha u_2)\hat{\jmath} \in \mathbb{R}^2 \\ \alpha\vec{v} &= (\alpha v_1)\hat{\imath} + (\alpha v_2)\hat{\jmath} + (\alpha v_3)\hat{k} \in \mathbb{R}^3 \end{aligned} \tag{1.4}$$

i.e. we multiply each component of the vectors by the scalar α.

2 The dot product

IMPORTANT! 1: The dot product

- Given the vectors in the plane $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$ and $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath}$, the **dot product** of the pair is given by:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 \qquad \text{(Dot product)}$$

  i.e. we multiply component-by-component.

- Alternatively, the dot product may also be written as

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos(\theta), \qquad \text{(Dot product)}$$

  where θ is the angle of opening between the two vectors.

- To find $\cos(\theta)$ we can divide both sides of the formula by $\|\vec{v}\| \, \|\vec{w}\|$, to give

$$\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{\|\vec{v}\| \, \|\vec{w}\| \cos(\theta)}{\|\vec{v}\| \, \|\vec{w}\|},$$

  and so cancelling gives

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}.$$

- Taking $\cos^{-1}$ gives

$$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$$

2.1 Angle of opening

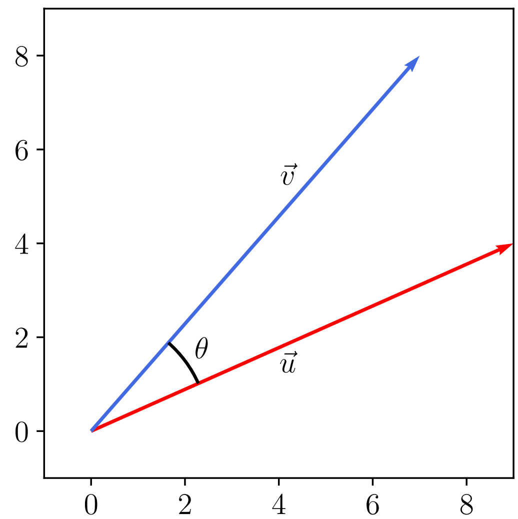

Figure 1: The angle of opening between the vectors $\vec{u}$ and $\vec{v}$.

- Figure 1 shows two vectors $\vec{u}$ and $\vec{v}$ in the plane.
- Since $\vec{u}$ and $\vec{v}$ do not point in the same direction the angle of opening between cannot not be zero, as can be seen from the figure.
- Using both forms of the dot product given in above, this angle of opening may be computed from the components and norms of the vectors.

Example 1

Given the vectors shown in Figure 1 are given in component form by

$$\vec{u} = 9\hat{\imath} + 4\hat{\jmath} \qquad \vec{v} = 7\hat{\imath} + 8\hat{\jmath},$$

find the following

(i) $\|\vec{u}\|$ and $\|\vec{v}\|$

(ii) $\vec{u} \cdot \vec{v}$

(iii) The angle of opening between $\vec{u}$ and $\vec{v}$

Solution.

(i) The norms of the vectors are given by

$$\begin{aligned} \|\vec{u}\| &= \sqrt{9^2 + 4^2} = \sqrt{97} \\ \|\vec{v}\| &= \sqrt{7^2 + 8^2} = \sqrt{113}. \end{aligned}$$

(ii) The dot-product of the vectors is given by

$$\vec{u} \cdot \vec{v} = (9\hat{\imath} + 4\hat{\jmath}) \cdot (7\hat{\imath} + 8\hat{\jmath}) = 9(7) + 4(8) = 95$$

(iii) The angle between the vectors is given by

$$\theta = \cos^{-1}\left(\frac{95}{(\sqrt{97})(\sqrt{113})}\right) = 24.852^\circ$$

IMPORTANT! 2: $\cos^{-1}(x)$ and $\arccos(x)$

Another name for the function $\cos^{-1}(x)$ is $\arccos(x)$. Most computer programs, such as Python, Mathematica, Matlab etc., use some version of the name **arccos(x)** to represent $\cos^{-1}(x)$.

Exercise 1

Given the vectors

$$\vec{p} = 3.1\hat{\imath} - 6.5\hat{\jmath} \text{ and } \vec{q} = 1.2\hat{\imath} + 4.3\hat{\jmath},$$

(i) $\|\vec{p}\|$ and $\|\vec{q}\|$.

(ii) $\vec{p} \cdot \vec{q}$.

(iii) The angle of opening between $\vec{p}$ and $\vec{q}$.

IMPORTANT! 3: 3D dot products and angles

The formulae for the dot product and the angle of opening between two vectors in 3D are the exact same. The only difference is there is an extra component when calculating the norms and dot product.

Example 2

Given the vectors

$$\vec{u} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k} \qquad \vec{v} = -\hat{\imath} + 4\hat{\jmath} + \hat{k},$$

find the following

(i) $\|\vec{u}\|$ and $\|\vec{v}\|$

(ii) $\vec{u} \cdot \vec{v}$

(iii) The angle of opening between $\vec{u}$ and $\vec{v}$

Solution.

(i) The norms of the vectors are given by

$$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$
$$\|\vec{v}\| = \sqrt{(-1)^2 + 4^2 + 1^2} = \sqrt{18}.$$

(ii) The dot-product of the vectors is given by

$$\vec{u} \cdot \vec{v} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) \cdot (-\hat{\imath} + 4\hat{\jmath} + vk) = 2(-1) - 1(4) + 3(1) = -3$$

(iii) The angle between the vectors is given by

$$\theta = \arccos\left(-\frac{3}{(\sqrt{14})(\sqrt{18})}\right) = 100.89^\circ$$

Example 3

Given the vectors

$$\vec{p} = 3.4\hat{\imath} + 4.2\hat{\jmath} - 4.5\hat{k} \qquad \vec{q} = 3.8\hat{\imath} + 7.4\hat{\jmath} + 3.4\hat{k},$$

find the following

(i) $\|\vec{p}\|$ and $\|\vec{q}\|$

(ii) $\vec{p} \cdot \vec{q}$

(iii) The angle of opening between $\vec{p}$ and $\vec{q}$

3 Perpendicular vectors

- Given the dot product of some vectors $\vec{v}$ and $\vec{w}$, the angle of opening between the vectors may be calculated using Definition **??** to give

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|\,\|\vec{w}\|}\right).$$

- When the dot product of the vectors is zero (i.e. $\vec{v}\cdot\vec{w} = 0$), then we find the angle of opening is

$$\theta = \cos^{-1}(0) = 90^\circ,$$

and so perpendicular vectors always have an angle of opening of 90° and a dot product zero.

3.1 Constructing perpendicular vectors

- Given an arbitrary vector

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$$

we can use the fact the perpendicular vectors have dot product zero to find the components of a vector perpendicular to $v = \vec{v}$.

- Specifically, **either** of the vectors

$$\vec{v}^{\perp} = -v_2\hat{\imath} + v_1\hat{\jmath} \qquad \vec{v}^{\perp} = v_2\hat{\imath} - v_1\hat{\jmath}$$

will be perpendicular to the original vector $\vec{v}$.

- It is easy to see that

$$\begin{cases} \vec{v}\cdot\vec{v}^{\perp} = (v_1\hat{\imath} + v_2\hat{\jmath})\cdot(-v_2\hat{\imath} + v_1\hat{\jmath}) = -v_1v_2 + v_2v_1 = 0 \\ \vec{v}\cdot\vec{v}^{\perp} = (v_1\hat{\imath} + v_2\hat{\jmath})\cdot(v_2\hat{\imath} - v_1\hat{\jmath}) = v_1v_2 - v_2v_1 = 0 \end{cases}$$

and so $\vec{v}$ is perpendicular to $\vec{v}^{\perp}$ in each case.

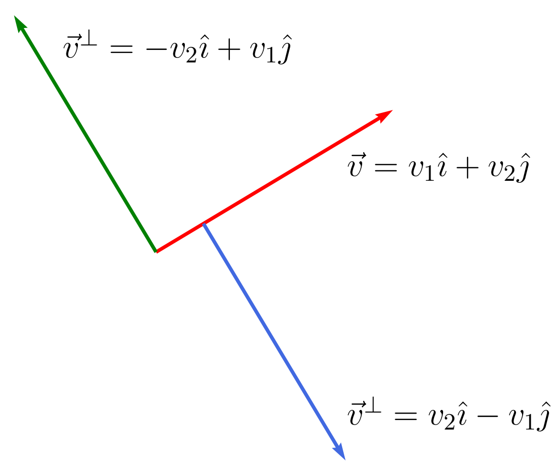

Figure 2: The vectors $\vec{v}^{\perp} = -v_2\hat{\imath} + v_1\hat{\jmath}$ and $\vec{v}^{\perp} = v_2\hat{\imath} - v_1\hat{\jmath}$ perpendicular to $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

3.2 The unit vectors

- We should note that the unit vectors $\hat{\imath}$, $\hat{\jmath}$ are defined so that they point along perpendicular axes.
- They are also defined so they both have length one. Given these conditions it follows that

$$\hat{\imath} \cdot \hat{\imath} = 1 \qquad \hat{\jmath} \cdot \hat{\jmath} = 1 \qquad \hat{\imath} \cdot \hat{\jmath} = 0.$$

- Hence, each unit vector dotted with itself gives one (this is because the vectors have length 1), and different unit vectors dotted together gives zero (this is because the vectors are $\perp$).