Let S be the implicit surface defined by $f(X)=0$.
Formally prove that the normal of S at point $p \in S$ is proportional to $\nabla f(p)$.

Proof: For every $p \in S$ there exists a curve $c = c(t)$ where $c \in S$ and $c(0) = p$.
Since $c(t) \in S$ we know $f(c(t)) = 0 \quad \forall t \in [0,1]$
Using the derivative

$$\frac{\partial f}{\partial c}\frac{\partial c}{\partial t} = \nabla f(c(t)) \cdot c'(t) = 0$$

For $t=0$ we thus know $\nabla f(c(0)) \cdot c'(0) = 0$

$$\Rightarrow \nabla f(p) \cdot c'(0) = 0$$

Since the derivative of c is a vector this dot product gives us $\nabla f(p) \perp c'(0)$ and since $c'(0)$ is defined on the plane, $\nabla f(p)$ is thus orthogonal to the plane and thus parallel to the normal of S.