Detecting misspecification in the distribution of random coefficients in demand models for differentiated goods

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Abstract

The differentiated products demand model initiated by Berry (1994) and Berry et al. (1995) is the workhorse model for demand estimation with market-level data. This model utilizes random coefficients to account for unobserved preference heterogeneity. The shape of the distribution of random coefficients matters greatly for many counterfactual quantities, such as the pass-through. In this paper, we develop new econometric tools to test this distribution and improve its estimation under a flexible parametrization. In particular, we construct new instruments that are designed to detect deviations from the underlying distribution of random coefficients. Then, we develop a formal moment-based specification test on the distribution of random coefficients. Next, we show that our instruments can strengthen the identifying power of the moment conditions used for estimation. Finally, we validate our approach with Monte Carlo simulations and an empirical application using data on car purchases in Germany. We show that these methods extend to the mixed logit demand model (with individual-level data).

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1 Introduction

The differentiated product demand model initiated by Berry (1994) and Berry, Levinsohn, and Pakes (1995) has been used in a wide array of empirical studies. It enables researchers to perform demand estimation in markets with differentiated products using either macro-level (market shares) or micro-level (individual purchases) data while allowing for unobserved heterogeneity in preferences as well as price endogeneity. This unobserved heterogeneity in preferences is modeled through the use of random coefficients (RCs) in the utility function. This framework allows researchers to estimate demand functions, price elasticities and counterfactual outcomes. Applications of the BLP model have notably studied the determinants of market power, the welfare effects resulting from a merger or the introduction of a new good and the economic impact of a tax or a subsidy[1].

The informativeness of the empirical analysis depends on how well the model can reproduce the underlying substitution patterns and approximate the shape of the demand curve, including its slope and curvature. A recent result in Miravete et al. (2022) shows that the commonly used Gaussian RC on price imposes strong restrictions on the demand's curvature and thus limits the range of the implied pass-through. The degree of pass through of taxes and costs is central to answering many questions in economics. However, estimating a more flexible demand system with a non-Gaussian distribution of random coefficients is challenging. First, there is a clear trade-off between the degree of flexibility and the precision of the estimates. It is thus important to be able to test the specification chosen by the researcher on the distribution of the RC (for instance, a Gaussian RC) and quantify the degree of misspecification before potentially moving to a more flexible specification. Second, to properly estimate a more flexible distribution of RC, the researcher must choose moment conditions (or equivalently instruments) that strongly identify this distribution. The current instruments work well with the standard Gaussian RC, but their performance appears to decline as the specification becomes more flexible.

[1]There are numerous applications of the BLP model including Berry et al. (1995)), Nevo (2000), Petrin (2002), Gentzkow and Shapiro (2006), Gallego and Hernando (2009), Add examples

In this paper, we provide a set of econometric tools to address these two challenges in practice. In particular, we construct a new set of instruments that are designed to detect deviations from the true distribution of random coefficients. Building on these instruments, we provide a formal moment-based specification test on the distribution of random coefficients, which allows researchers to test the chosen specification without having to re-estimate the model under a more flexible parametrization. Our instruments are designed to maximize the power of this test when the distribution of RC is misspecified. We also show how these instruments can strengthen the identifying power of the moment conditions used for estimation and thus be successful at estimating a flexibly parameterized distribution of RCs. Finally, building on our test procedure and our instruments, we develop a simple selection procedure to choose which product characteristics should be augmented with a random coefficient, which is an important question in practice.

The first contribution of our paper is to construct a new set of instruments that are designed to detect departures from the true distribution of RCs. The intuition we use is the following. Each distribution of RCs engenders a structural error, which, if correctly specified, is mean-independent with respect to a set of exogenous variables. This identifying condition can be transformed into unconditional moments, which can be used to test whether the distribution of RCs is correctly specified. We formally define this test and we construct instruments that maximize its power against a fixed alternative. We first assume that the econometrician knows the fixed alternative and derive an expression for the first-best instrument. We call this instrument the most powerful instrument (MPI) and show that this specific choice of instrument achieves consistency of the test. In a second stage, we provide two feasible approximations of the MPI that can be derived without knowledge of the fixed alternative. We call these feasible MPIs the interval instruments in reference to the way the approximation works.

The second contribution of our paper is to develop a formal moment-based specification test on the distribution of RCs. For instance, the researcher may be interested in testing whether a random coefficient is normally distributed. The general idea consists of testing unconditional moments that are implied by the model when the distribution of random coefficients is well specified. The researchers first choose a specification on the distribution of random coefficients and they estimate a pseudo-true value associated with this specification. They then choose a set of instruments (or equivalently moment conditions) to test the specification on the distribution of RCs. Our interval instruments represent a natural choice as they are designed to maximize the power of our test against misspecification in the distribution of RCs. Finally, we study the asymptotic properties of our test when the number of markets, T, goes to infinity and we prove the asymptotic validity of our test under mild assumptions. In particular, we consider parameter uncertainty (the pseudo-true value is estimated), and we control for the magnitude of the approximations that intervene in the estimation of the BLP model. Our asymptotic results complement previous work by Freyberger (2015) on the asymptotic properties of the BLP estimator when the number of markets grows to infinity.

Third, we show that our interval instruments can be effectively used to estimate the model, particularly when the distribution of RCs is flexibly parameterized. We do so by exploiting the duality between estimation and testing. In particular, we exhibit the connection between the MPI and the classical optimal instruments: the optimal instrument are an approximation of the MPI around the true parameter. Intuitively, the optimal instruments are designed to detect departures from the true distribution of RCs around the true parameter. The feasible approximations of the MPI that we provide can also achieve this objective and display some advantages over the usual optimal instruments. Overall, we believe these instruments will be useful for practitioners and complement the literature on BLP instruments, which have been shown to work well in the usual Gaussian case (Conlon and Gortmaker (2019), Reynaert and Verboven (2014), Gandhi and Houde (2019),Knittel and Metaxoglou (2008)).

To evaluate the performance of our test and instruments, we conduct two separate sets of simulation experiments. First, we compare the performance of our specification test using different sets of instruments including the instruments we construct in this paper and the instruments commonly used in the literature: Gandhi and Houde (2019), Reynaert and Verboven (2014). We show that the test has the correct empirical size and that the interval instruments significantly outperform the traditional instruments in terms of power under alternative distributions. Second, we evaluate the performance of the interval instruments in estimating the model when the distribution of

RC is flexibly parametrized; here we consider a Gaussian mixture. We show that our instruments outperform the traditional instruments in terms of the mean squared error. Finally, we apply the new tools developed in this paper to flexibly estimate the demand for cars in Germany from 2012 to 2018. The objective of the empirical exercise is first to see how well our instruments perform at estimating a flexible distribution of RC using a real dataset. Here, we will consider a flexible Gaussian mixture on price. Second, we are interested in using our test to see how the degree of misspecification decreases when we increase flexibility on the distribution of RC. Finally, in line with the findings in Miravete et al. (2022), we want to study how the shape of the distribution of RC on price can modify important counterfactual quantities such as the pass-through.

Related literature. First, our paper contributes to the literature on the flexible estimation of aggregate demand models for differentiated goods. The most effective way to introduce enough flexibility in these models while retaining tractability remains an open question. Compiani (2018) proposes a non-parametric estimator of the demand functions. If relaxing all the parametric assumptions makes this approach conceptually appealing, it also faces significant theoretical and practical difficulties (more stringent data requirements, large curse of dimensionality, limited scope for counterfactual analysis)[2]. Lu et al. (2019) and Wang (2021) propose semi-parametric estimators of the distribution of RC. These approaches are complementary to ours and the instruments and the test we develop in this paper can be useful to implement their non-parametric IV estimation procedures, which are known to be rather sensitive to the quality of the instruments (Chetverikov and Wilhelm (2017)). Finally, a recent strand of the literature suggests deriving bounds directly on the counterfactual quantities (Tebaldi et al. (2019), Ho and Pakes (2014)). Our paper also contributes to the literature on the non-parametric identification of the distribution of RC in demand models (Fox and Gandhi (2011), Fox et al. (2012), Dunker et al. (2017), Wang (2021), Berry and Haile (2014)). First, we slightly extend the identification result in Wang (2021) to link it directly

[2]It is not clear how restrictive the Type 1 Extreme Value assumption on the taste shock is. McFadden and Train (2000) shows that a mixed-logit model with flexibly distributed random coefficients can approximate any discrete choice model derived from random utility maximization. On the other hand, this assumption generates massive computational gains, which allows for studying sophisticated markets with many products and many characteristics. Thus, cost/benefit analysis seems to be largely in favor of the logit specification.

to the primitives of the model. Second, we provide a practical way of constructing moments that feature high identifying power with respect to the distribution of RCs.

Structure of the paper. In Section 2, we recall the baseline BLP model, define the structural error of the model, and provide conditions under which the distribution of random coefficients is non-parametrically identified. In Section 3, we derive the most powerful instrument and show how it relates to the classical optimal instruments. In Section 4, we construct two feasible approximations of the MPI. In Section 5, we present our specification test and show its asymptotic validity. In section 6, we conduct Monte Carlo simulations to first evaluate the consequences of misspecification on quantities of interest, and then evaluate the performance of our test and our instruments. In Section 7, we apply our new tools to flexibly estimate the demand for cars in Germany. In Section 8, we present our selection procedure to select the characteristics which must be augmented with RCs. We conclude the paper in section 9.

2 Model and identification

2.1 Indirect utility and moment restrictions

Indirect utility. We first describe the indirect utility function that induces the observed market shares. Our setting closely follows the one introduced in the seminal paper Berry et al. (1995). There are T markets indexed by $t = 1, ..., T$. There is a continuum of consumers indexed by i. There are J_t market-specific products in market t. Each consumer i chooses a product $j \in \{0, 1, ..., J_t\}$ where $j = 0$ corresponds to the outside option. For the sake of exposition and without loss of generality, we will assume throughout our analysis that the number of products is constant across markets ($\forall t,\ J_t = J$). Each product j is characterized by a vector of characteristics x_{jt}, which in most empirical settings includes price. Each consumer i derives an indirect utility u_{ijt} from purchasing good $j \in \{0, 1, \ldots, J\}$ in market t:

$$u_{ijt} = \underbrace{x'_{1jt}\beta + \xi_{jt}}_{\delta_{jt}} + x'_{2jt}v_i + \varepsilon_{ijt}. \tag{1}$$

with the following:

- x_{1jt} is a vector of product characteristics of size K_1 associated with product j and for which there is no preference heterogeneity; β represents preferences for x_{1jt}

- ξ_{jt} is an unobserved demand shock on product j in market t

- $\delta_{jt} \equiv x'_{1jt}\beta + \xi_{jt}$ denotes the mean utility for product good j, the part of the utility that is common to all consumers.

- x_{2jt} is a vector of product characteristics of size K_2 for which there is consumer preference heterogeneity; and v_i is the associated random coefficient that follows a distribution characterized by density f. The random coefficient is independent of all the other variables: $v_i \perp\!\!\!\perp (x_t, \xi_t, \varepsilon_{ijt})$.

- ε_{ijt} is a preference shock that follows an extreme value type I distribution independent of all other variables and across i, j, t

For individual i in market t, the indirect utility from purchasing the outside option is normalized to $u_{i0t} = \varepsilon_{i0t}$. From the random utility functions in (1), we can infer the demand functions for each good j in market t: $\rho_{jt}(f, \beta)$. Each consumer chooses the product that maximizes his or her utility. Let y_{ijt} equal 1 if individual i chooses good $j = 0, 1, \ldots, J$ in market $t = 1, \ldots, T$. We have the following:

$$\begin{aligned}
\forall j \neq 0, \quad \rho_{jt}(f, \beta) &\equiv \mathbb{P}_{f,\beta}(y_{ijt} = 1 | x_t, \xi_t) \\
&= \mathbb{P}_{f,\beta}(\text{good } j \text{ is chosen in market } t \text{ by individual } i | x_t, \xi_t) \\
&= \mathbb{P}_{f,\beta}(u_{ijt} > u_{ikt} \ \forall k \neq j | x_t, \xi_t) \\
&= \int_{\mathbb{R}^{K_2}} \frac{\exp\left\{x'_{1jt}\beta + \xi_{jt} + x'_{2jt}v\right\}}{1 + \sum_{k=1}^{J} \exp\left\{x'_{1kt}\beta + \xi_{kt} + x'_{2kt}v\right\}} f(v) dv. \qquad (2)
\end{aligned}$$

For the outside option, the demand function is simply written as follows:

$$\rho_{0t}(f, \beta) = \mathbb{P}_{f,\beta}(y_{i0t} = 1 | x_t, \xi_t) = \int_{\mathbb{R}^{K_2}} \frac{1}{1 + \sum_{k=1}^{J} \exp\left\{x'_{1kt}\beta + \xi_{kt} + x'_{2kt}v\right\}} f(v) dv.$$

Following the EV1 assumption on the idiosyncratic shock on utility, the demand functions take the usual logit form integrated over the distribution of preference heterogeneity. We assume in this paper that the observed market shares $(s_{jt})_{j,t} \in (0;1)^{J\times T}$ are equal to the shares generated by the model above at the true distribution f and the true preference parameter β.

$$\forall j, \quad \forall t, \qquad s_{jt} = \rho_{jt}(f, \beta). \tag{3}$$

Moment restrictions. Following the literature, we assume that the unobserved demand shock ξ_{jt} is mean independent of z_{jt}, a set of instrumental variables. Namely, $\mathbb{E}[\xi_{jt}|z_{jt}] = 0$ *a.s.*; The set z_{jt} traditionally consists of the exogenous characteristics of all the products on the market as well as cost shifters, which are meant to instrument for price. Indeed, the price of goods is usually considered to be an endogenous variable, as it is correlated with the unobserved demand shock ξ_{jt} through the profit maximization problem of firms[3]. To estimate the model, the researcher chooses transformations of the instruments z_{jt} to construct a set of unconditional moments. We refer to these transformations as estimation instruments and denote them $h_E(z_{jt})$. Likewise, in our analysis, we study transformations of the instruments that are designed to test the specification of the model. We refer to these instruments as testing instruments and we denote them $h_D(z_{jt})$ (D standing for detection).

2.2 Inverse demand function and structural error

Inverse demand function. For any given distribution of random coefficients $\tilde{f}$, we define the demand function $\rho \equiv (\rho_1(\cdot), ..., \rho_L(\cdot))$ as the function which maps the vector of mean utilities δ to the vector of market shares generated by the model under $\tilde{f}$:

$$\begin{aligned} \rho(\cdot, x_{2t}, \tilde{f}) : \ & \mathbb{R}^J \to [0,1]^J \\ & \delta \mapsto \int_{\mathbb{R}^{K_2}} \frac{\exp\left\{\delta + x'_{2jt}v\right\}}{1 + \sum_{k=1}^{J} \exp\left\{\delta_k + x'_{2kt}v\right\}} \tilde{f}(v)dv. \end{aligned}$$

[3] To deal with the endogeneity of prices, Berry et al. (1995) also suggests using exogenous own-product characteristics as well as exogenous characteristics from other products. The main idea behind the use of these instruments is to take advantage of the correlation between price and exogenous characteristics implied by profit-maximizing firms. To be precise, Berry et al. (1995) suggests using the sum of the characteristics from other products produced by the same firm and the sum of exogenous characteristics from rival firms' products as instruments

Berry (1994) shows by applying Brouwer's fixed point that for any (s_t, x_{2t}) and for any distribution of random coefficients $\tilde{f}$ (even when $\tilde{f}$ is not the true distribution), there exists a unique $\tilde{\delta} \in \mathbb{R}^J$ such that:

$$s_t = \rho(\tilde{\delta}, x_{2t}, \tilde{f}).$$

We define the solution to the previous system of equations as the inverse demand functions: $\rho^{-1}(s_t, x_{2t}, \tilde{f}) = \tilde{\delta}$. Unfortunately, there is no closed form expression for the inverse demand function, which must be recovered numerically.

Structural error. From what precedes, we can uniquely define the structural error $\xi_{jt}(\tilde{f}, \tilde{\beta})$ generated by a distribution of random coefficient $\tilde{f}$ and a homogeneous parameter $\tilde{\beta}$:

$$\xi_{jt}(\tilde{f}, \tilde{\beta}) = \rho_j^{-1}(s_t, x_{2t}, \tilde{f}) - x'_{1jt}\tilde{\beta}. \tag{4}$$

The non-linear nature of the model is captured by the inverse demand function which enters the expression of the structural error. The absence of an analytical formula for the inverse demand implies that there is no closed form expression for the structural error, which seriously complicates the estimation of the BLP demand model. If we consider a parametric family of distributions $\tilde{\mathcal{F}} = \{\tilde{f}(\cdot|\tilde{\lambda}) : \tilde{\lambda} \in \tilde{\Lambda}\}$, then the structural error generated by a specific element in $\tilde{f}(\cdot|\tilde{\lambda}) \in \tilde{\mathcal{F}}$ and $\tilde{\beta}$ is defined as follows:

$$\xi_{jt}(\tilde{f}(\cdot|\tilde{\lambda}), \tilde{\beta}) = \rho_j^{-1}(s_t, x_{2t}, \tilde{f}(\cdot|\tilde{\lambda})) - x'_{1jt}\tilde{\beta}.$$

2.3 Non-parametric identification

The main objective of this paper is to provide tools to test the specification on the distribution of random coefficients and to improve its estimation under a flexible specification. A natural first step is to study the conditions under which this distribution is non-parametrically identified. The identification of random coefficients in multinomial choice models has been studied extensively in the literature (Fox and Gandhi (2011), Fox et al. (2012), Dunker et al. (2017), Wang (2021), Berry and Haile (2014), Allen and Rehbeck (2020)). We summarize some of these findings in Appendix C.1. In this section, we build on an important identification result in Wang (2021) to recover a set of sufficient identifying conditions directly on the primitives of the model. We also show that the identification result holds with a less stringent exogeneity assumption than in Wang (2021).

In contrast to the rest of the literature, Wang (2021) adopts all the parametric assumptions in the standard BLP model and looks for a set of sufficient restrictions under which the identification of the demand functions implies the identification of the distribution of random coefficients. This approach allows him to obtain conditions that are much less stringent than the rest of the literature: no special regressor assumption, no full support assumption, and no continuity assumption on the set of covariates. Specifically, he shows that if the demand functions $\rho = (\rho_1, ..., \rho_J)$ are identified on an open set[4] of $\mathbb{R}^J$, then the distribution of random coefficients is identified. His proof astutely exploits the real analytic property of the demand functions[5]. In this paper, we build on this injectivity result to find sufficient identifying conditions directly on the primitives of the model (without assuming identification of the demand functions). We also show using a random permutation of the indices that we only require the demand shock ξ_{jt} to be mean independent of the instrumental variables z_{jt} across products, but we do not require this to hold for each product j taken separately. This is less restrictive, as demand shocks can now be on average non-zero for certain products and account for unobserved quality inherent to each product. Let us formally state the assumptions that we impose to recover the point identification of (f, β).

Assumption A.

(i) Strict exogeneity: $\mathbb{E}[\xi_{jt}|z_{jt}] = 0$ $a.s.$,

(ii) Completeness: for any measurable function g such that $\mathbb{E}[g(s_t, x_t)] < \infty$, if $\mathbb{E}[g(s_t, x_t)|z_{jt}] = 0$ $a.s.$, then $g(s_t, x_t) = 0$ $a.s.$,

(iii) $\mathbb{P}(s_t, x_{2t}, x_{1t}, z_t)$ is observed by the econometrician and market shares s_t are generated by the demand model defined in Section 2 by equations 1 and 3,

(iv) Detectable difference in distributions: we say f and $\tilde{f}$ differ (and write $f \neq \tilde{f}$) if there exists $\bar{v} \in \mathbb{R}^{K_2}$ such that $F(\bar{v}) \neq \tilde{F}(\bar{v})$,

[4]which can be achieved using theorem 1 in Berry and Haile (2014)

[5]In particular, the real analytic property yields that the local identification of ρ on $\mathcal{D} \subset \mathbb{R}^J$ implies the identification of ρ on $\mathbb{R}^J$. From the global identification of ρ, he is then able to show that the random coefficients' distribution is identified under a simple rank condition on x_{2t}

(v) Let $x_t = (x_{1t}, x_{2t})$ then x_t is such that $\mathbb{P}(x_t' x_t \text{ is positive definite}) > 0 \;\; \forall t$,

(vi) There exists $\bar{x}_t \in \mathcal{X}$ and an open set $\mathcal{D} \subset \mathbb{R}^J$ such that $\delta_t = \bar{x}_{1t}\beta_0 + \xi_t$ varies on $\mathcal{D}$ a.s.;

In A(i), we assume that the instruments are strictly exogenous. Assumption A(ii) is a completeness assumption that states that the instruments are strongly relevant with respect to (s_t, x_t). This assumption is typical of semiparametric or nonparametric IV models and is equivalent to a full rank assumption in a linear IV model. Intuitively, it tells us that if the inverse demands are different almost surely, then our instruments will be able to detect the difference. The completeness assumption is a strong assumption that has been widely used in this literature (Berry and Haile (2014), Dunker et al. (2017), Wang (2021)). Assumption A(v) is a standard rank condition. Assumption A(vi) is meant to ensure that there is enough variation in δ_t to utilize the injectivity result in Wang (2021). This assumption indicates that there needs to be sufficient variation in product characteristics across markets in the data in order for the econometrician to identify f. In practice, product characteristics are very similar from one market to the other and may not yield sufficient variation. A judicious solution is to create inter-market variation by interacting product characteristics with demographic variables characterizing each market. This is how we will proceed in the empirical application. Let us now state our formal identification result.

Proposition 1. Under Assumption A the distribution of random coefficients f and the homogeneous preference parameters β are non-parametrically identified.

$$(\tilde{f}, \tilde{\beta}) = (f, \beta) \iff \qquad \mathbb{E}[\xi_{jt}(\tilde{f}, \tilde{\beta}) | z_{jt}] = \mathbb{E}\left[\rho_j^{-1}(s_t, x_{2t}, \tilde{f}) - x_{1jt}'\tilde{\beta} \middle| z_{jt}\right] = 0 \;\; a.s.$$

The proof is in Appendix B.1. The identification result above tells us that under some fairly weak conditions and in the presence of instruments which generate sufficient variation in the product characteristics, the observed data identifies the distribution of random coefficients non-parametrically. Formally, the model is at the true pair (f, β) if and only if the associated structural error is mean independent of the instrumental variables z_{jt}. We will use this identification result to show consistency of our test under a specific choice of instruments that we will characterize thereafter.

3 Detecting misspecification: the most powerful instrument

The aim of this section is to recover the instrument with the greatest ability to detect misspecification in the distribution of RC. To do so, we consider a setting in which the econometrician wants to test a simple hypothesis of the form $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$. The upper bar is used to stress the fact that $\overline{H}_0$ is a simple hypothesis, in contrast to the composite hypothesis $H_0 : f \in \mathcal{F}_0$ that we study in section 5. Our approach builds on a simple intuition: if the model under $\overline{H}_0$ is misspecified, then the structural error will depart from the true demand shock ξ_{jt} and our goal is to find the best instrument to pin down this deviation. We proceed as follows. First, we introduce a moment-based test for $\overline{H}_0$ and we show its asymptotic validity. Next, we derive an analytical expression for the instrument that maximizes the power of our test against a fixed alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a)$. We call this instrument the most powerful instrument (MPI) and we show how it relates to the classical optimal instruments. In section 4, we provide two feasible approximations of the MPI, which have the critical property of being invariant with respect to the alternative $\overline{H}_a$.

3.1 A moment-based test

We want to test $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$ against $H_a : (f, \beta) \neq (f_0, \beta_0)$. For any set of testing instruments $h_D(z_{jt})$, we have the following implication:

$$\overline{H}_0 : (f, \beta) = (f_0, \beta_0) \implies \overline{H}'_0 : \quad \mathbb{E}[h_D(z_{jt})\xi_{jt}(f_0, \beta_0)] = 0.$$

We propose to test $\overline{H}_0$ indirectly through its implication $\overline{H}'_0$, which is a set of unconditional moment conditions. We test $\overline{H}'_0$ with a moment-based test and our test statistic writes as follows:

$$S_T(h_D, f_0, \beta_0) = TJ \left(\frac{1}{TJ} \sum_{j,t} \xi_{jt}(f_0, \beta_0) h_D(z_{jt}) \right)' \hat{\Omega}_0^{-1} \left(\frac{1}{TJ} \sum_{j,t} \xi_{jt}(f_0, \beta_0) h_D(z_{jt}) \right). \tag{5}$$

with $\hat{\Omega}_0$ a consistent estimator of Ω_0 the asymptotic variance-covariance matrix. Namely, $\Omega_0 = \mathbb{E}[\xi_{jt}^2(f_0, \beta_0) h_D(z_{jt}) h_D(z_{jt})']$. The number of markets T is the dimension that we let grow to infinity and under which we study the asymptotic properties of our test. As the focus of this section is on the construction of the most powerful instrument, we deliberately postpone to section 5 the treatment of the specific challenges implied by parameter uncertainty (i.e. when β_0 and f_0 must be estimated beforehand) and by the numerical approximations involved in the derivation of the structural error (in practice, the researcher only observes a numerical approximation of $\xi_{jt}(f_0, \beta_0)$). Additionally, to keep the results as simple as possible while retaining the key intuitions, we assume independence of the demand shocks in a given market conditional on z_{jt}. This last assumption is relaxed in the proofs in Appendix B.2 and in section 5.

Proposition 2. Assume that (s_t, x_t, z_t) i.i.d. across markets and consistent with the probability model defined by equations (1), (2) and (3) evaluated at (f, β), $\mathbb{E}[\|\xi_{jt}(f_0, \beta_0)h_D(z_{jt})\|^2] < +\infty$, Ω_0 full rank, for $k \neq j$, $\xi_{jt} \perp\!\!\!\perp \xi_{kt} | z_t$. We have the following:

- Under $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$, $\qquad S_T(h_D, f_0, \beta_0) \xrightarrow[T \to +\infty]{d} \chi^2_{|h_D|_0}$.
- Under $H_a' :\ \mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0, \beta_0)\right] \neq 0, \qquad \forall q \in \mathbb{R}^+,\ \ \mathbb{P}(S_T(h_D, f_0, \beta_0) > q) \xrightarrow[T \to +\infty]{} 1.$

with $|\cdot|_0$ being the counting norm.

The previous proposition indicates that as long as the testing instruments are functions of z_{jt}, our test procedure is asymptotically valid for $\overline{H}_0$. We are testing $\overline{H}_0$ by virtue of its implication $\overline{H}_0' : \mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0, \beta_0)\right] = 0$ and, as a consequence, the power properties of our test hinge critically on the choice of the testing instruments $h_D(z_{jt})$. This is the focus of the next paragraph.

3.2 The most powerful instrument (MPI)

The choice of testing instruments $h_D(z_{jt})$ is key to maximizing the rejection of $\overline{H}_0$ under any alternative $H_a : (f, \beta) \neq (f_0, \beta_0)$. To guide our choice of instruments, we first derive the instrument that maximizes the power of our test when the econometrician tests $\overline{H}_0$ against a fixed alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a) \neq (f_0, \beta_0)$. We refer to this instrument as the most powerful instrument (MPI). In practice, the researcher is often reluctant to fix the alternative. However, the MPI represents a useful first-best solution for which we provide two feasible approximations in section 4.

Derivation of the most powerful instrument. To construct the MPI, we use the following decomposition of the structural error generated under $\overline{H}_0$:

$$\xi_{jt}(f_0, \beta_0) = \underbrace{\xi_{jt}(f_a, \beta_a)}_{\text{structural error under } \overline{H}_a} + \underbrace{\xi_{jt}(f_0, \beta_0) - \xi_{jt}(f_a, \beta_a)}_{\Delta_{0,a}^{\xi_{jt}}},$$

with $\Delta_{0,a}^{\xi_{jt}}$ being the correction term due to misspecification under the alternative $\overline{H}_a$. Our goal is to compare the ability of our test for different candidates $h_D(z_{jt})$, to reject $\overline{H}_0$ under $\overline{H}_a$. The literature offers many ways to compare the power of competing tests (see Gourieroux and Monfort (1995) for a comprehensive review). First, we distinguish between exact and approximate methods. Exact methods rely on the exact distribution of the test statistic (under $\overline{H}_0$) and allow for a comparison in finite sample while asymptotic methods exploit the asymptotic distribution of the test statistic and are informative in larger samples. In our case, the exact distribution of our test is unknown; thus we will rely on asymptotic methods, which is the most common case in the literature. Second, we divide the methods into local and non-local methods. In parametric tests, local strategies are based on the analysis of the power properties of competing tests under a sequence of local alternatives θ_T which converges to θ_0 at a given rate (usually $\frac{1}{\sqrt{T}}$). The econometrician can compare two competing tests by means of their power functions (or more precisely, the limits of these power functions when sample sizes go to $+\infty$). This is called the direct approach. The dual approach, which is known as Pitman's relative efficiency, consists of comparing the rates at which the minimal number of observations must increase to ensure a given level of power. The approach we favor in this paper is the non-local approach developed in Bahadur (1960). Here, the econometrician chooses the test with the smallest level α needed to attain a given power against a fixed alternative and for a given number of observations. In other words, the econometrician chooses the test that minimizes the risk of type I error ceteris paribus.

There are several reasons for favoring Bahadur's non-local approach. First, it is better suited for the testing problem we study in this paper. We will see that the main comparison criterion, known as the asymptotic slope of the test, is straightforward to derive, whereas it is not clear how one should derive Pitman's efficiency criterion when the test concerns non-parametric objects such as distributions. Moreover, we study the properties of our test against a fixed alternative

$\overline{H}_a : (f, \beta) = (f_a, \beta_a)$ as in Bahadur's case, which is not necessarily local. Finally, the literature has highlighted many limitations of the local approach. Local criteria are often unable to discriminate between tests even when these tests lead to different decisions (see Silvey (1959)). In addition, as shown in Dufour and King (1991), a locally optimal test in a neighborhood of H_0 may perform very poorly away from H_0.

Let us now present the intuition for Bahadur's comparison approach. From section 3.1, we have:

$$\text{Under } \overline{H}_0: \qquad S_T \equiv S_T(h_D, f_0, \beta_0) \xrightarrow{d} S \qquad \text{with } \; S = \chi^2_{|h_D|_0}$$

Following the same notations as in Gourieroux and Monfort (1995), we denote:

$$\Lambda(s) = \mathbb{P}_{\overline{H}_0}(S \geq s)$$

The critical value is usually derived using the asymptotic distribution of the test statistic under the H_0. The approximate critical region at a given level α is then given by:

$$CR_\alpha = \{S_T \geq \Lambda^{-1}(\alpha)\} = \{\Lambda(S_T) \leq \alpha\}$$

The main idea in Bahadur's approach entails deriving the level of the test, if one takes the value of the test statistic as the critical value (this is also known as the p-value). Namely:

$$\alpha_T = \Lambda(S_T)$$

Bahadur suggests preferring the test that displays the lowest level α_T at least asymptotically. A formal analysis of the asymptotic behavior of α_T shows that it's better to consider the limit of a transformation of α_T than the limit of α_T directly. This gives rise to the concept of the approximate slope of the test.

Definition 1 (Asymptotic slope of the test)**.**

(i) $K_T = -\frac{2}{T}\log(\Lambda(S_T))$ is the approximate slope of the test

(ii) Under $\overline{H}_a$: $plim\ K_T = c(f_a, \beta_a)$ is the asymptotic slope of the test

Under the alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a)$, let us consider two sequences of tests based on S^1_T and S^2_T with asymptotic slopes $c^1(f_a, \beta_a)$ and $c^2(f_a, \beta_a)$ respectively. The test based on S^1_T is asymptotically preferred to the test based on S^2_T in Bahadur's sense if and only if $c^1(f_a, \beta_a) > c^2(f_a, \beta_a)$. To derive the asymptotic slopes or our test, we apply an important result in Geweke (1981), which states that if under H_0: $S_T \xrightarrow[T\to+\infty]{d} \chi^2_q$ (with any $q \in \mathbb{N}^*$), then $\frac{1}{T} S_T \xrightarrow{a.s.} c(f_a, \beta_a)$ (when the limit exists). In our test, the limiting distribution is chi-squared and thus, the asymptotic slope of our test with instrument $h_D(z_{jt})$ writes:

$$c_{h_D}(f_a, \beta_a) = plim\ \frac{1}{T} S_T(h_D, f_0, \beta_0) = J\mathbb{E}\left[\xi_{jt}(f_0, \beta_0)h_D(z_{jt})\right]' \Omega_0^{-1}\mathbb{E}\left[\xi_{jt}(f_0, \beta_0)h_D(z_{jt})\right]$$

Let us note that the asymptotic slope can also be interpreted as a measure of the speed of divergence of the test statistic in terms of population moments, i.e- Speed of divergence $\approx T \times c_{h_D}(f_a, \beta_a)$. The next proposition shows that the asymptotic slope of our test is maximized by the following instrument:

Proposition 3 (Most powerful instrument)**.**
Let $\mathcal{H}$ the set of measurable vectorial functions of z_{jt}, we have:

$$\mathbb{E}\left[\xi_{jt}(f_0, \beta_0)^2 | z_t\right]^{-1} \mathbb{E}[\Delta^{\xi_{jt}}_{0,a} | z_{jt}] \in \underset{h_D \in \mathcal{H}}{\text{argmax}}\ \ c_{h_D}(f_a, \beta_a)$$

The MPI is thus equal to the conditional expectation of the correction term $\mathbb{E}[\Delta^{\xi_{jt}}_{0,a} | z_{jt}]$ divided by a conditional variance term. One may notice that the MPI is almost identical to the classical optimal instruments but the second term differs. We explore the relationship between these two instruments in detail in section 3.3. It is well known that conditional variance term $\mathbb{E}\left[\xi_{jt}(f_0, \beta_0)^2 | z_{jt}\right]$, which also appears in the formulation of the optimal instruments, is difficult to model and estimate in practice. Researchers typically ignore this term or impose a restrictive and ad-hoc structure on the form that it can take (for instance, Reynaert and Verboven (2014)'s approximation of the optimal instruments in the BLP model ignores the variance term, add references here). In our case, the large dimension of z_{jt} makes the exercise even more difficult. Thus, we will disregard this term in the remainder of our analysis and take $h^*_D(z_{jt}) = \mathbb{E}[\Delta^{\xi_{jt}}_{0,a} | z_{jt}]$ as the reference MPI. This last expression corresponds to the exact formulation of the MPI under homoskedasticity. It is also worth mentioning that beyond maximizing power, $h_D^*(z_{jt})$ features other appealing properties including (i) consistency of the associated test and (ii) maximizing correlation with the structural error under the alternative.

(i) Consistency. By setting h_D equal to h_D^*, our moment-based test becomes consistent against any fixed alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a) \neq (f_0, \beta_0)$. Namely, we have:

Proposition 4 (Consistency of the test with the MPI)**.** Under assumption A and the same assumptions as in Proposition 2, we have:

$$\overline{H}_a : (f, \beta) = (f_a, \beta_a) \neq (f_0, \beta_0) \implies \forall q \in \mathbb{R}^+, \quad \mathbb{P}(S_T(h_D^*, f_0, \beta_0) > q) \underset{T \to +\infty}{\longrightarrow} 0$$

The proof of this result is in Appendix B.2.

(ii) Correlation with the structural error. An another interesting property of the MPI is to be the function of z_{jt} which maximizes the correlation with the structural error.

Proposition 5 (Correlation between the MPI and the structural error)**.**
Let $\mathcal{H}$ be the set of measurable functions of z_{jt}, we have under $\overline{H}_a$:

$$\forall \alpha \in \mathbb{R}^*, \quad \alpha\ \mathbb{E}[\Delta_{0,a}^{\xi_{jt}} | z_{jt}] \in \underset{h \in \mathcal{H}}{\arg\max}\ |\text{corr}(\xi_{jt}(f_0, \beta_0), h(z_{jt}))|$$

The proof is in Appendix B.2. Intuitively, the MPI $h_D^*(z_{jt})$ is designed to fully capture the exogenous variation contained in the correction term $\Delta_{0,a}^{\xi_{jt}}$ implied by the misspecification, which yields the result above.

3.3 Connection with the optimal instruments

On the one hand, the MPI maximizes the power of the moment-based test for $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$. On the other hand, the optimal instruments minimize the asymptotic variance-covariance of the GMM estimator when the parameter of interest is identified by conditional moment restrictions. These two problems are seemingly unrelated. However, we show that the optimal instruments are a local approximation of the MPI around the true value of the parameter of interest. This connection between the MPI and the optimal instruments is critical to understand how the feasible approximations of the MPI we construct in section 4 can significantly improve the efficiency of the BLP estimator.

The estimation of the model works as follows: the researcher stipulates that f belongs to a parametric family $\mathcal{F}_0 = \{f_0(\cdot|\tilde{\lambda}) : \tilde{\lambda} \in \Lambda_0\}$ and wants to estimate the true parameter $\theta = (\beta', \lambda')'$ under this parametric restriction. For now let us assume that the model is correctly specified: $f \in \mathcal{F}_0$ and we shorten the notations by removing the dependence of the structural error in $f_0(\cdot|\tilde{\lambda})$, which becomes implicit in this context. Namely $\xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta})$ becomes $\xi_{jt}(\tilde{\theta})$. We further assume that θ is point identified by the following moment restriction[6]: $\mathbb{E}[\xi_{jt}(\theta)|z_{jt}] = 0$ $a.s.$; The econometrician must choose the set of instruments $h_E(z_{jt})$ (or equivalently moments) to include in the GMM objective function:

$$\hat{\theta} = \underset{\tilde{\theta}}{\text{Argmin}} \left(\sum_{j,t} \hat{\xi}_{jt}(\tilde{\theta}) h_E(z_{jt}) \right)' \hat{W} \left(\sum_{jt} \hat{\xi}_{jt}(\tilde{\theta}) h_E(z_{jt}) \right).$$

Optimal instruments in the BLP demand model. Traditionally, the instruments $h_E(z_{jt})$ are chosen to minimize the asymptotic variance-covariance of the estimator $\hat{\theta}$. The instruments that reach this objective are called the optimal instruments. The resulting estimator is said to be efficient in the sense that its asymptotic variance cannot be reduced by using additional moment conditions. There is a large body of literature on the derivation of optimal instruments in econometric models (Amemiya (1974), Chamberlain (1987), Newey (1990), Newey (2004)). The BLP estimator $\hat{\theta}$ is a non-linear GMM estimator and classical results in Chamberlain (1987) and Amemiya (1974) show that the optimal instruments in this case write:

$$h_E^*(z_{jt}) = \mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]^{-1} \mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}} \middle| z_{jt} \right]$$

and the corresponding efficiency bound (obtained by setting $h_E = h_E^*$) writes:

$$V^* = \mathbb{E}\left[\mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}} \middle| z_{jt} \right] \mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}} \middle| z_{jt} \right]' \mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]^{-1} \right]^{-1}$$

[6]The identification conditions in the parametric case are less stringent than the non-parametric identification conditions established in assumption A

For the sake of exhaustivity, we show this result in Appendix B.2.1. As for the MPI, the formulation of the optimal instruments above is obtained under the assumption of conditional independence of demand shocks ξ_{jt} in the same market: $k \neq j$, $\xi_{jt} \perp\!\!\!\perp \xi_{kt}|z_t$. In Appendix B.2.1, we derive the expression for the optimal instruments under weaker assumptions on the demand shock[7]. The conditional variance term $\mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]$ that appears in the formula of the optimal instruments is difficult to estimate and the common practice is to ignore it. We follow the literature and also disregard this term in our analysis.

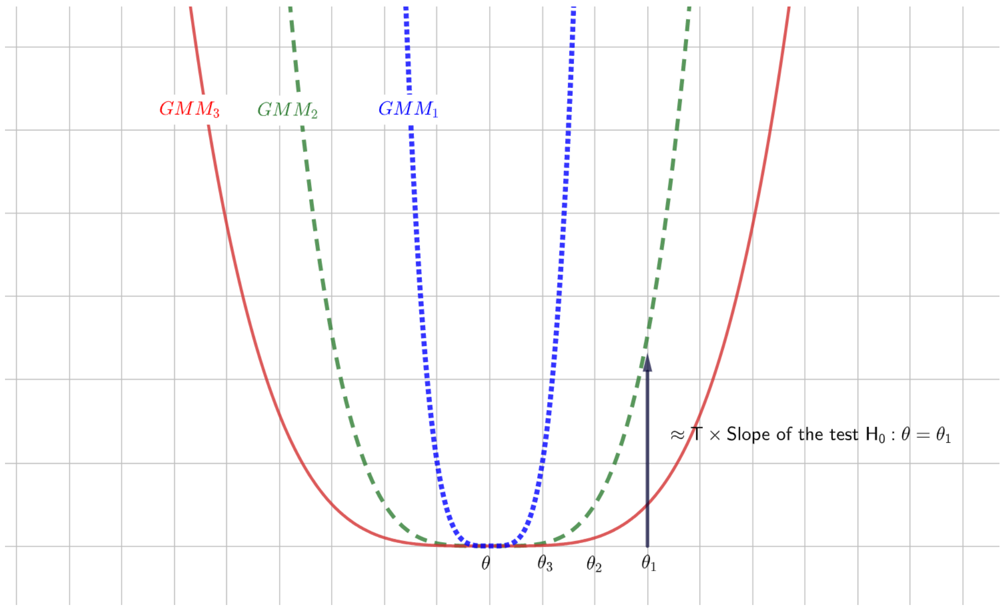
Interpretation of the optimal instruments as a local approximation of the MPI. The connection between the MPI and the optimal instruments stems from the duality between tests and estimation. Under the parametric assumption $f \in \mathcal{F}_0$, the simple hypothesis $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$ we studied previously becomes $\overline{H}_0 : \theta = \theta_0$. From what precedes, we know that the MPI maximizes the power of the moment-based test of $\overline{H}_0$ against a fixed alternative. Here, the fixed alternative is simply the true parameter θ, that the researcher seeks to estimate.

The GMM objective function used to estimate θ is based on the same test statistic as the one used to test $\overline{H}_0 : \theta = \theta_0$. Specifically, the researcher looks for the parameter $\hat{\theta}$ that minimizes this test statistic. Thus, contrary to the test we studied before, the null is no longer fixed, as different candidates $\overline{H}_0 : \theta = \theta_0$ are considered throughout the estimation procedure. A good set of estimation instruments $h_E(z_{jt})$ is such that the associated test largely rejects $\overline{H}_0 : \theta = \theta_0$ uniformly over the set of candidates $\theta_0 \neq \theta$. Equivalently, the slope of the GMM objective function should be as large as possible for any $\theta_0 \neq \theta$. We illustrate this in figure 1. Building on proposition 3, we can show that the MPI maximizes the asymptotic slope of the GMM objective function pointwise[8] and constitutes a power envelop on the GMM objective function. Unfortunately, the MPI depends on the tested candidate θ_0, which precludes from using it directly for estimation.

[7]We allow for unrestricted forms of correlation between demand shocks within a given market

[8]As in Proposition 3, we can apply Geweke (1981) to derive the asymptotic slope. Indeed, as long as we have a consistent estimator $\hat{\theta}$ to derive the 2-step optimal weighting matrix, we know that under H_0, our test statistic (which is also the GMM objective function) converges to a chi-square distribution.

Figure 1: Relation between the GMM objective function and the slopes of the tests $H_0 : \theta = \theta_i$ ($i = 1, 2, 3$)



GM1, GM2, and GM3 correspond to different sets of instruments. The set n^o1 is more effective because the slopes associated with $H_0 : \theta = \theta_i$ (i=1,2,3) are larger when $\theta \neq \theta_i$

A way to circumvent this issue is to derive an approximation of the MPI that is invariant with respect to the tested candidate $\overline{H}_0 : \theta = \theta_0$. This is exactly what the optimal instruments achieve locally around the true parameter θ. Take any candidate θ_0. It is straightforward to show that, in the parametric case, the associated MPI writes: $h^*_D(z_{jt}) = \mathbb{E}\left[\Delta^{\xi_{jt}}_{\theta_0,\theta}|z_{jt}\right]$ with $\Delta^{\xi_{jt}}_{\theta_0,\theta} = \xi_{jt}(\theta_0) - \xi_{jt}(\theta)$. By taking a Taylor expansion of $\xi_{jt}(\theta_0)$ around θ, we obtain the following:

$$\Delta^{\xi_{jt}}_{\theta_0,\theta} = \frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}}(\theta_0 - \theta) + o(||\theta_0 - \theta||_2)\,.$$

We immediately see that when θ_0 is in a neighborhood of θ, the MPI $h^*_D(z_{jt})$ is a linear combination of the optimal instruments $h^*_E(z_{jt})$:

$$h^*_D(z_{jt}) = \mathbb{E}\left[\Delta^{\xi_{jt}}_{\theta_0,\theta}|z_{jt}\right] \approx \underbrace{\mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}}\middle| z_{jt}\right]'}_{h^*_E(z_{jt})}(\theta_0 - \theta).$$

The previous equation highlights that, in a neighborhood of θ, the optimal instruments can be seen as a linear approximation of the MPI associated with $\overline{H}_0 : \theta = \theta_0$. This allows us to better understand how the optimal instruments work in practice: they reach efficiency by maximizing the slope of the GMM objective function around the true parameter. We can also infer from this analysis that good estimation instruments approximate the MPI sufficiently well over candidates $\theta_0 \neq \theta$ and particularly around θ. The optimal instruments are one particular way of achieving this outcome but we will see that the feasible approximations of the MPI that we provide in section 4 can also reach this objective.

Limitations of the optimal instruments for estimation. The optimal instruments are considered in the literature as the first-best instruments to estimate the model as they yield an efficient estimator. However, they also display certain limitations. First, the optimal instruments are infeasible as they depend on the true parameter θ, which is unknown to the researcher. The literature has proposed several approximations of the optimal instruments in the BLP model that we review them in section 4 (Berry (1994), Reynaert and Verboven (2014), Gandhi and Houde (2019), Conlon and Gortmaker (2019)). The performance of these approximations depend on how well they can approach the optimal instruments. Second, the optimal instruments are not robust to misspecification: they are only a valid approximation of the MPI when the distribution is well specified ($f \in \mathcal{F}_0$) and thus, there is no clear interpretation of the estimator obtained with the optimal instruments under misspecification[9]. On the other hand, if the approximation of the MPI doesn't depend on the specification, as it is the case for the approximations we propose in section 4, then the estimator can be interpreted as the one minimizing the misspecification with respect to the true distribution of RCs f within the parametric family $\mathcal{F}_0$. Finally, it isn't clear how the optimal instruments perform when θ_0 is away from θ. This is a key consideration in practice, especially when the dimension of θ_0 increases and minimization problem becomes more challenging. This calls for approximations of the MPI that are valid more globally than the optimal instruments.

[9]Interpretability and estimation under misspecification have been at the center of a fastly growing literature, which attempts to minimize the sensitivity of estimators to misspecification (Bonhomme and Weidner (2018)).

4 A feasible most powerful instrument

The MPI is the most powerful instrument to reject $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$ against a fixed alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a)$. Its derivation requires the knowledge of the alternative while in practice the econometrician typically wants to remain agnostic about the alternative. Moreover, the MPI is defined as a conditional expectation of a non-linear function with respect to a large dimension vector z_{jt}, and thus, even if the alternative $\overline{H}_a$ is known, the MPI can be difficult to estimate. In this section, we circumvent these two difficulties and provide two approximations of the MPI, which do not depend on $\overline{H}_a$ and that can be computed straightforwardly. We remain in the same configuration where the econometrician wants to test $\overline{H}_0 : (f, \beta) = (f_0, \beta_0)$ against a fixed alternative $\overline{H}_a : (f, \beta) = (f_a, \beta_a)$ but we now assume this alternative is unknown to the econometrician.

By construction, in the BLP demand model, the correction term writes:

$$\begin{aligned}\Delta_{0,a}^{\xi_{jt}} &= x_{1jt}'(\beta_a - \beta_0) + \rho_j^{-1}(s_t, x_{2t}, f_0) - \rho_j^{-1}(s_t, x_{2t}, f_a) \\ &= x_{1jt}'(\beta_a - \beta_0) + \Delta_j(s_t, x_{2t}, f_0, f_a).\end{aligned}$$

The previous equation shows that the correction term is the sum of a linear part, which is standard and a non-linear part which is specific to the BLP demand model.

Linear part. The linear part of the MPI writes: $\mathbb{E}[x_{1jt}|z_{jt}]'(\beta_a - \beta_0) = \mathbb{E}[x_{1jt}|z_{jt}]'\gamma$. Thus, for its linear part, the MPI is a linear combination of the conditional expectation of x_{1jt} with respect to the exogenous variables with unknown weights. If one is interested in pecifically testing that $\beta = \beta_0$, informative instruments simply consist of the variables in $\mathbb{E}[x_{1jt}|z_{jt}]$.

Non-linear part. The non-linear part $\Delta_j(s_t, x_{2t}, f_0, f_a)$ is the part which is implied by the misspecification on the distribution of RCs and for which we need to recover a feasible approximation. Equation (6) indicates that the non-linear part is the difference between the inverse demand functions generated by f_0 and f_a. We now go one step further and derive two analytical approximations of $\Delta_j(s_t, x_{2t}, f_0, f_a)$ which we then use as building blocks to construct our feasible approximations of the MPI's.

4.1 Local approximation

First, we consider a local approximation of $\Delta_j(s_t, x_{2t}, f_0, f_a)$. This approximation corresponds to the first order term in the expansion of $\Delta(s_t, x_{2t}, f_0, f_a)$ "around f_0" , which is recovered by exploiting the properties of the inverse demand function, which is both $\mathcal{C}^\infty$ and bijective in s_t.

Proposition 6.
A first order expansion of $\Delta(s_t, x_{2t}, f_0, f_a)$ around f_0 writes:

$$\Delta(s_t, x_{2t}, f_0, f_a) = \left(\frac{\partial \rho(\delta_t^0, x_{2t}, f_0)}{\partial \delta}\right)^{-1} \int_{\mathbb{R}^{K_2}} \left[\frac{\exp(\delta_t^0 + x_{2t}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt}^0 + x_{2kt}'v\right\}} - \rho(\delta_t^0, x_{2t}, f_0)\right] f_a(v) + \mathcal{R}_0.$$

with $\delta_t^0 = \rho^{-1}(s_t, x_{2t}, f_0)$ and $\mathcal{R}_0 = o\left(\int_{\mathbb{R}^{K_2}} |f_a(v) - f_0(v)| dv\right)$.

The proof is in Appendix B.3.1. We first observe that for any density f_0, we can construct artificial market shares s_t^0 such that $\rho^{-1}(s_t, x_{2t}, f_a) = \rho^{-1}(s_t^0, x_{2t}, f_0)$. We then recover the final result by taking a Taylor expansion of $\rho^{-1}(s_t^0, x_{2t}, f_0)$ around s_t and showing that the remainder is bounded. This approximation is local by design: it works best when f_a is a local deviation from f_0, even if it can be used more generally. To make this expression useful in practice, we must still overcome two difficulties. The distribution f_a is unknown to the econometrician. In addition, some variables such as δ_{jt}^0 are endogenous. Let us, however, notice that the previous expression may be particularly useful if the econometrician is interested in testing $\overline{H}_0$ against a fixed and known alternative as we did in the previous section.

Discretizing the integral. To solve for the fact f_a is unknown to the econometrician, we replace the integral in which f_a appears by a finite Riemann approximation. Namely,

$$\int_{\mathbb{R}} \frac{\exp\left\{x_{2jt}'v\right\}}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt}^0 + x_{2kt}'v\right\}} f_a(v) dv \approx \sum_{l=1}^L \omega_l(f_a) \frac{\exp(x_{2jt}'v_l)}{1 + \sum_{k=1}^J \exp(\delta_{kt}^0 + x_{2kt}'v_l)}.$$

with $\{v_l\}_{l=1,\ldots,L}$ L points chosen in the domain of definition of f_a, and $\omega_l(f_a)$ the weights associated with each point[10]. We provide more details on how to choose the points in Appendix C.4. It is

[10]in the usual Riemann sum, the weights correspond to density evaluated at point v_l : $f_a(v_l)$ times the width of the interval around v_l

important to observe that in the Riemann approximation, only the weights depend on the alternative f_a. This approximation can also be interpreted as approaching a continuous distribution with a discrete one: each point in $\{v_l\}_{l=1,...,L}$ represents a specific consumer type with an associated probability $w_l(f_a)$. The non-linear part of the MPI can thus be approximated as follows:

$$\mathbb{E}[\Delta_j(s_t, x_{2t}, f_0, f_a)|z_{jt}] \approx \sum_{l=1}^{L} \omega_l(f_a)\ \mathbb{E}[\pi_{j,l}(s_t, x_t)|z_{jt}],$$

$$\text{with } \pi_{j,l}(s_t, x_t) = \left(\frac{\partial \rho(\delta_t^0, x_{2t}, f_0)}{\partial \delta}\right)^{-1} \left[\frac{\exp(\delta_t^0 + x_{2t} v_l)}{1 + \sum_{k=1}^{J} \exp\left\{\delta_{kt}^0 + x_{2kt}' v_l\right\}} - \rho(\delta_t^0, x_{2t}, f_0)\right]_j.$$

Approximating the conditional expectation. Ideally, we would like to estimate the conditional expectation of $\pi_{j,l}(s_t, x_t)$ with respect to z_{jt}. In practice, this is often challenging because the dimension of z_{jt} can be very large and the functions $\pi_{j,l}(\cdot)$ are highly non-linear and non-separable in the endogenous variables: $\delta_t^0 = \rho^{-1}(s_t, x_{2t}, f_0)$ and the ones contained x_{2jt}. This makes it unappealing to use standard non-parametric estimation methods[11]. In the same spirit as Reynaert and Verboven (2014), we first project the endogenous variables on the space spanned by a relevant subset of z_{jt} and we plug them into our functions $\pi_{j,l}(\cdot)$. As we show in Appendix C.2, this strategy yields an estimator that converges faster to a first order approximation of the conditional expectation. Let us denote $\hat{\pi}_{j,l}(z_{jt})$ the resulting estimators of $\mathbb{E}[\Delta_j(s_t, x_{2t}, f_0, f_a)|z_{jt}]$.

Test procedure. From what precedes, the MPI (for its non-linear part) can be approximated as follows: $h_D^*(z_{jt}) \approx \sum_{l=1}^{L} \omega_l(f_a)\ \hat{\pi}_{j,l}(z_{jt})$. As we don't know the weights $\omega_l(f_a)$, we propose to take the vector $\hat{\pi}_j(z_{jt}) = (\hat{\pi}_{j,1}(z_{jt}), ..., \hat{\pi}_{j,L}(z_{jt}))'$ as our testing instruments. We call them interval instruments in reference to the way we divide the support in several intervals to construct our approximation. Following the test procedure presented in section 3.1, we perform a moment based test for $\overline{H}_0 : \mathbb{E}\left[\hat{\pi}_j(z_{jt})\xi_{jt}(f_0, \beta_0)\right] = 0$. Under the same assumptions in Proposition 2 and setting $h_D(z_{jt}) = \hat{\pi}_j(z_{jt})$, we have the following:

[11] For instance, a Sieve nonparametric estimator of the conditional mean. The dimension of z_{jt} makes this approach of little relevance in practice

$$\text{Under } H_0 : S_T(h_D, f_0, \beta_0) \xrightarrow[T\to+\infty]{d} \chi^2_L.$$

This approach has the advantage of remaining completely agnostic about f_a. One drawback is that the asymptotic distribution is a χ^2_L. In comparison, the infeasible MPI $h^*_D(z_{jt})$ is of dimension one and thus the associated test statistic is distributed as χ^2_L asymptotically. This increase in the number of degrees of freedom may lead to some loss of power. An alternative approach would consist in letting the researcher choose the weights $\{\hat{\omega}_l\}_{l=1,\ldots,L}$ and recover an instrument of dimension one. However, for this approach to work well and retain good power properties, the econometrician must choose the weights so that they approximately match the real weights $\{w_l(f_a)\}_{l=1,\ldots,L}$. This requires a good a apriori knowledge of the cumulative distribution function of the alternative distribution f_a.

4.2 Global approximation

Second, we consider a global approximation that is based on an identity which is valid everywhere and not only when f is close to f_a. Simple algebraic operations (see appendix B.3.2) allow us to derive the following expression for $\Delta_j(s_t, x_{2t}, f_0, f_a)$. Let $\delta^0_{jt} = \rho_j^{-1}(s_t, x_{2t}, f_0)$ and $\delta^a_{jt} = \rho_j^{-1}(s_t, x_{2t}, f_a)$. We have:

$$\Delta_j(s_t, x_{2t}, f_0, f_a) = \log\left(\frac{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^{J}\exp\left\{\delta^a_{kt}+x'_{2kt}v\right\}} f_a(v)dv}{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^{J}\exp\left\{\delta^0_{jt}+x'_{2kt}v\right\}} f_0(v)dv}\right).$$

As for the local approximation, we cannot directly exploit this approximation as some quantities such as f_a and δ^a_{jt} are unknown and some variables such as δ^0_{jt} are endogenous. To remedy these two difficulties, we apply the same methods as previously described: we discretize the integral and we exogenize the endogenous variables by projecting them onto the space spanned by a relevant subset of z_{jt}. To solve for the fact that the mean utility δ^a_{jt} under the alternative is unknown, we replace it with the mean utility under the null δ^0_{jt}. This should not alter the approximation too much given that δ^a_{jt} only enters the expression at the denominator within a sum, which averages out the differences between δ^a_{jt} and δ^0_{jt} across products. In fine, we are able to provide the following approximation for the non-linear part of the MPI:

$$\Delta_j(s_t, x_{2t}, f_0, f_a) \approx \log\left(\sum_{l=1}^{L} \bar{\omega}_l(f_a)\ \hat{\hat{\pi}}_{j,l}(z_{jt})\right) \text{ with } \hat{\hat{\pi}}_{j,l}(z_{jt}) = \frac{\frac{\exp(x'_{2jt}v_l)}{1+\sum_{k=1}^{J}\exp\left\{\hat{\delta}^0_{kt}+x'_{2kt}v_l\right\}}}{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^{J}\exp\left\{\hat{\delta}^0_{jt}+x'_{2kt}v\right\}} f_0(v)dv}.$$

$\bar{\omega}_l(f_a)$ correspond to the unknown weights and the $\hat{\hat{\pi}}_{j,l}(z_{jt})$ to our set of global interval instruments. The MPI can thus be approximated by the logarithm of a weighted sum of known functions of z_{jt}. As we did previously, we use $\hat{\hat{\pi}}_j(z_{jt}) = (\hat{\hat{\pi}}_{j,1}(z_{jt}), ..., \hat{\hat{\pi}}_{j,L}(z_{jt}))'$ as instruments to test $\overline{H}_0$. All the weights are positive and sum to 1, which entails that the correction term of interest is approximately an increasing function of our instruments. This approximation is said to be global as contrary to the second approximation we study, it does not require f_0 to be close to f_a. Nevertheless, if f_a close to f_0, then fraction κ inside the log is close to 1 and the well known approximation $\log(\kappa) \approx \kappa - 1$ allows us to rewrite the correction as a linear combination of our instruments.

Overall, the feasible MPIs that we derive in this section allow us to approximate the most powerful instrument against a fixed alternative while remaining agnostic about this alternative. The price to pay is that the properties that we establish for the MPI do not carry over to the feasible MPI. Nevertheless, our Monte Carlo simulations show that the interval instruments perform very well in practice. We further discuss the properties of the feasible MPI in C.6.

4.3 Feasible MPIs for estimation

In estimation, the researcher stipulates that f belongs to a parametric family $\mathcal{F}_0 = \{f_0(\cdot|\tilde{\lambda}) : \tilde{\lambda} \in \Lambda_0\}$ and wants to estimate the true parameter $\theta = (\beta', \lambda')'$ under this parametric restriction. If the parametric specification on f is incorrect, then the researcher wants to estimate the parameter θ that minimizes misspecification in the family $\mathcal{F}_0$. We showed in section 3.3 that a good set of estimation instruments $h_E(z_{jt})$ should approximate sufficiently well the MPI associated with the test $\overline{H}_0 : \theta = \theta_0$ uniformly over the set of candidates $\theta_0 \neq \theta$. Thus, we now seek to provide an approximation of the MPI that doesn't depend on the null and that can be computed without knowledge of the alternative. We position ourselves under an unknown fixed alternative $\overline{H}_a$: $(f, \beta) = (f_a, \beta_a)$ without necessarily assuming that $f_a \in \mathcal{F}_0$ (unlike the optimal instruments, we don't use the specification of the model to construct our approximations of the MPI). While in a testing procedure, the null $\overline{H}_0 : \theta = \theta_0$ is fixed, in an estimating procedure $\overline{H}_0$ varies from one iteration to the other, which prevents us from directly exploiting the interval instruments derived previously. We now slightly modify the interval instruments to make them suitable for estimation.

Starting with the local approximation constructed in section 4.2, it is sufficient to assume that we have a reference distribution $\tilde{f}_{0,a}$, which is close to both $f_0(\cdot|\lambda_0)$ and f_a. Then, we can show the following:

$$\mathbb{E}[\Delta_j(s_t, x_{2t}, f_0(\cdot|\lambda_0), f_a)|z_{jt}] \approx \sum_{l=1}^{L} \tilde{\omega}_l(f_0, f_a)\ \hat{\tilde{\pi}}_{j,l}(z_{jt})$$

$$\text{with } \hat{\tilde{\pi}}_{j,l}(z_{jt}) = \left(\frac{\partial \rho(\hat{\tilde{\delta}}_t, x_{2t}, \tilde{f}_{0,a})}{\partial \delta}\right)^{-1} \left[\frac{\exp(\hat{\tilde{\delta}}_t + x_{2t}v_l)}{1 + \sum_{k=1}^{J} \exp\left\{\hat{\tilde{\delta}}_{kt} + x_{2jk}v_l\right\}} - \rho(\hat{\tilde{\delta}}_t, x_{2t}, \tilde{f}_{0,a})\right]_j$$

As for the global approximation we derived in Section 4.2, it is straightforward to show that for any candidate $f_0(\cdot|\lambda_0)$, we can rewrite this approximation of the non-linear part of the MPI as follows:

$$\mathbb{E}[\Delta_j(s_t, x_{2t}, f_0(\cdot|\lambda_0), f_a)|z_{jt}] \approx \log\left(\sum_{l=1}^{L} \bar{\bar{\omega}}_l(f_0(\cdot|\lambda_0), f_a)\ \hat{\bar{\bar{\pi}}}_{j,l}(z_{jt})\right) \text{ with } \hat{\bar{\bar{\pi}}}_{j,l}(z_{jt}) = \frac{\exp(x'_{2jt}v_l)}{1 + \sum_{k=1}^{J} \exp\left\{\hat{\bar{\delta}}^a_{jt} + x_{2jk}v_l\right\}}$$

$$\textit{and } \bar{\bar{\omega}}_l(f_0, f_a) = \frac{\bar{\omega}_l(f_a)}{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^{J}\exp\left\{\delta^0_{jt}+x'_{2jk}v\right\}} f_0(\cdot|\lambda_0)(v)dv}$$

with $\hat{\bar{\delta}}^a_{jt}$ projected first stage estimates of δ^a_{jt}, which can be obtained, for example, under the logit specification. $\hat{\bar{\bar{\pi}}}_{j,l}(z_{jt})$ don't depend on f_0 and can be used for estimation.

Comparison with the usual approximations of the optimal instruments

5 Composite hypothesis

In the traditional estimation procedure, which encompasses almost all the applications of the BLP model, the econometrician is required to make a parametric assumption on the distribution of random coefficients to estimate the model. Formally, the econometrician assumes $f \in \mathcal{F}_0$ where $\mathcal{F}_0$ is a parametric family parametrized by $\tilde{\lambda}$, which is a parameter that must be estimated. In applied work, researchers typically assume that f is normally distributed. This parametric choice is rarely grounded in economic theory and, if too restrictive, is likely to impose arbitrary restrictions on some key counterfactual quantities such as the pass-through. However, adopting a more flexible parametrization on the distribution of RC leads to noisier estimates and can substantially increase the computational cost. Thus, the econometrician may be interested in testing the chosen parametric specification $H_0 : f \in \mathcal{F}_0$ before estimating a more complex model. This is the purpose of this section. We first define the pseudo-true value associated with a given specification and see how it enables us to come back to the simple hypothesis testing problem studied in section 3. Second, we define our test procedure and its implementation in practice. Finally, we study the asymptotic properties of our test.

5.1 The pseudo-true value under the parametric restriction $f \in \mathcal{F}_0$

To estimate the BLP model, researchers must make three choices. As mentioned previously, they must assume that f belongs to a parametric family $\mathcal{F}_0 = \{f_0(\cdot|\tilde{\lambda}) : \tilde{\lambda} \in \Lambda_0\}$. Second, they must choose a set of instruments $h_E(z_{jt})$ to estimate the model. Finally, they must choose a weighting matrix W, which weights the different moments included in the objective function. Given these three choices, we can define the BLP pseudo-true value[12] $\theta_0 = (\beta_0', \lambda_0')'$ as follows:

$$\theta(\mathcal{F}_0, h_E, W) \equiv \theta(\mathcal{F}_0) \equiv \theta_0 \in \underset{\tilde{\theta}}{\text{Argmin}}\ \mathbb{E}\left[\xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta}) h_E(z_{jt})\right]' W \mathbb{E}\left[h_E(z_{jt}) \xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta})\right].$$

Unless specified otherwise, we refer to θ_0 as the pseudo-true-value associated with the parametric family $\mathcal{F}_0$.

5.2 A moment-based test

The BLP pseudo-true value defined in the previous section allows us to map the non-parametric identification of f in Proposition 1 to the parametric assumption $H_0 : f \in \mathcal{F}_0$ that we want to test.

[12] Our definition of a pseudo-true value is closely related to the approach in White (1982) in the context of maximum likelihood. In his case, the pseudo true value minimizes the Kullback-Leibler distance between the assumed likelihood and the true likelihood, whereas in our case, the pseudo true value minimizes a weighted sum of population moments

Corollary 5.1. Under Assumption A, and assuming $h_E(z_{jt})$ and W are such that the pseudo-true value θ_0 is unique, then we have the following:

$$\begin{aligned} H_0 : f \in \mathcal{F}_0 &\iff \mathbb{E}[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)|z_{jt}] = \mathbb{E}\left[\rho_j^{-1}(s_t, x_{2t}, f_0(\cdot|\lambda_0)) - x'_{1jt}\beta_0 \middle| z_{jt}\right] = 0 \quad a.s. \\ &\iff \overline{H}_0 : (f, \beta) = (f_0(\cdot|\lambda_0), \beta_0). \end{aligned}$$

Corollary 5.1 is a direct implication of Proposition 1. Thus, under H_0, the structural error induced by the pseudo true value $\theta_0 = (\beta_0, \lambda_0)$ is mean independent of the instrumental variables z_{jt}. The pseudo true value reduces the dimensionality of the problem by allowing us to move from a composite hypothesis $H_0 : f \in \mathcal{F}_0$ to the simple hypothesis $\overline{H}_0 : (f, \beta) = (f_0(\cdot|\lambda_0), \beta_0)$ studied previously. In theory, one could directly use Corollary 5.1 to test H_0 via an integrated conditional moment test. We do not follow this route for at least two reasons. First, this test will contain no information on the nature of the misspecification (it could be completely unrelated to the distribution of RC). Second, in practice the dimension of z_{jt} is often very large, which substantially reduces the power of this kind of test. As we did in section 2, we propose a moment-based test of H_0[13]. Under H_0, for every set of testing instruments $h_D(z_{jt})$, the following moment conditions must hold:

$$H_0 : f \in \mathcal{F}_0 \iff \overline{H}_0 : (f, \beta) = (f_0(\cdot|\lambda_0), \beta_0) \implies \overline{H}'_0 : \ \mathbb{E}\left[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) h_D(z_{jt})\right] = 0.$$

We now develop a test procedure to test $\overline{H}'_0$. In comparison to the test in section 3.1, we must now account for the fact that the pseudo-true value needs to be estimated to derive the test statistic, which generates parameter uncertainty. Moreover, we propose a rigorous treatment of the numerical approximations involved in the derivation of the structural error.

Choice of instruments. As previously indicated, the power properties of our test hinge critically on the choice of testing instruments $h_D(z_{jt})$. We established the MPI and its feasible counterparts,

[13] Another testing approach would have entailed testing $H_0 : f \in \mathcal{F}_0$ against a larger class of densities that encompasses $\mathcal{F}_0$. For instance, if $\mathcal{F}_0$ is the family of normal distributions, encompassing families are mixtures of Gaussians with a larger number of components. We do not follow this route for two reasons. First, it is not desirable to restrict the alternative to a class of distributions which encompass the null as the econometrician does not know a priori the misspecification. Second, estimating the BLP model with a more flexible parametrization is challenging. An advantage of our test procedure is that it doesn't require estimating the model with a more flexible parametrization

the interval instruments, feature attractive properties in testing $\overline{H}_0 : (f, \beta) = (f_0(\cdot|\lambda_0), \beta_0)$ against any fixed alternative. Thus, it is natural to use these instruments for our specification test. In particular, we show that the consistency and optimal power properties of the test with the MPI (under certain choices of weighting matrix) carry over to the general specification test in Appendix B.5. Next, we present our test procedure and study the asymptotic properties of our test. We perform the entire analysis for an arbitrary choice of testing instruments $h_D(z_{jt})$.

5.3 Test procedure

Test statistic. For any choice of testing instruments $h_D(z_{jt})$, our objective is to test $\overline{H}_0'$: $\mathbb{E}[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)h_D(z_{jt})] = 0$[14] where $\theta_0 = (\beta_0, \lambda_0)$ is the pseudo-true value associated with the parametric family $\mathcal{F}_0$[15]. In order to test $\overline{H}_0$, we consider the following Wald test statistic:

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) = TJ \left(\frac{1}{TJ} \sum_{j,t} \widehat{\xi}_{jt}(f_0(\cdot|\hat{\lambda}), \hat{\beta})h_D(z_{jt}) \right)' \hat{\Sigma} \left(\frac{1}{TJ} \sum_{j,t} \widehat{\xi}_{jt}(f_0(\cdot|\hat{\lambda}), \hat{\beta})h_D(z_{jt}) \right).$$

where $\hat{\Sigma}$ is a weighting matrix chosen by the econometrician and $\hat{\theta} = (\hat{\beta}, \hat{\lambda})$ is a consistent estimator of θ_0. The number of markets T is the dimension that we let grow to infinity to the asymptotic properties of our test. We motivate this choice in Appendix C.3. Under some regularity conditions that we make explicit in the following section, the asymptotic distribution of the test statistic under $\overline{H}_0'$ is as follows:

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} Z'\Sigma Z, \tag{6}$$

$$\text{with } \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{j=1}^{J} \widehat{\xi}_{jt}(f_0(\cdot|\hat{\lambda}), \hat{\beta})h_D(z_{jt}) \xrightarrow{d} Z \sim \mathcal{N}(0, \Omega_0). \tag{7}$$

Σ is the probability limit of $\hat{\Sigma}$. We make Ω_0 explicit in the next subsection (in particular, the derivation of Ω_0 must take into account parameter uncertainty related to the estimation of the

[14]To derive formal results, we consider the moment condition $\mathbb{E}\left[\sum_{j=1}^{J} \xi(f_0(\cdot|\lambda_0), \beta_0)h_D(z_{jt})\right] = 0$ and not $\mathbb{E}[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)h_D(z_{jt})] = 0$. We show in the appendix that these two moment conditions are equivalent. Moreover, the formulation substantially simplifies the asymptotic analysis (as moments in the same market are "connected" through the contraction) and has the advantage of smoothly allowing for any form of correlation between ξ_{jt} and $\xi_{j't}$

[15]Under an alternative specification, the pseudo true value also depends on the estimation instruments $h_E(z_{jt})$ and the weighting matrix

pseudo-true value θ_0). Given that $\hat{\Sigma}$ is chosen by the econometrician and it is possible to derive a consistent estimator of Ω_0, the econometrician can always simulate the asymptotic distribution of our test statistic. In some polar cases, which we present hereafter, the asymptotic distribution of our test statistic are pivotal chi-square distributions which do not require to be simulated. Before we delve into the asymptotic properties of our test, let us briefly describe the procedure to follow in practice.

Practical implementation of the test.

1 The researcher chooses a parametric specification for the distribution of random coefficients $H_0 : f \in \mathcal{F}_0$ as well as instruments $h_E(z_{jt})$ and a weighting matrix $\hat{W}$ to estimate the BLP pseudo-true value θ_0.

2 Given a first-stage estimate $\hat{\theta}$, the researcher chooses testing instruments $h_D(z_{jt})$ and constructs the test statistic $S_T(h_D, \mathcal{F}_0, \hat{\theta})$.

3 Decision rule: for a test of level α, we reject $\overline{H}'_0$ and thus H_0 if:

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) > q_{1-\alpha},$$

with $q_{1-\alpha}$ is the $1-\alpha$ quantile of the asymptotic distribution of our test statistic stated above.

Two polar cases. For the sake of exposition, let us now describe two polar cases where the asymptotic distributions are pivotal chi-square distributions, which do not require to be simulated. Denote by $|\cdot|_0$ the counting norm.

1. **Sargan-Hansen J test:** If the set of estimation instruments and the set of testing instruments are the same ($h_E = h_D$), if $\hat{W}$ is the 2-step GMM optimal weighting matrix and if $\hat{\Sigma} = \hat{W}^{-1}$, then our test boils down to the usual over-identification specification test and we have under $\overline{H}'_0$:

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} \chi^2_{|h_E|_0 - |\theta|_0}.$$

2. **Non-overlapping h_D and h_E:** If Ω_0 is full rank[16] and if the econometrician sets $\hat{\Sigma} = \hat{\Omega}_0^{-1}$, then our test statistic has the following asymptotic distribution under $\overline{H}_0'$:

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} \chi^2_{|h_D|_0}.$$

5.4 Asymptotic validity

We now study the asymptotic properties of our test when the number of markets T goes to infinity. To establish the asymptotic validity and consistency of our test, we exploit classical results on the asymptotic normality of the non-linear GMM estimator (Hansen (1982), Newey (1990)) as well on the large-T asymptotics of the BLP estimator (Freyberger (2015)). The main challenge here is to control the magnitude of the approximations that intervene in the derivation of the structural error so that they can be neglected asymptotically. Contrary to Freyberger (2015), we do not assume the convergence of any moments ex-ante and we allow for the approximation error between demand and observed market shares to be non-zero. Next, we describe the approximations involved in the estimation of the BLP pseudo-true value and then we state a set of sufficient conditions for our test to be asymptotically valid.

5.4.1 The BLP estimator

The BLP estimator is an empirical counterpart of the BLP pseudo-true value defined previously. The minimization is done with respect to sample analogs. Additionally, we know that there is no closed form expressions for the structural error $\xi_{jt}(f_0(.|\tilde{\lambda}), \tilde{\beta})$, and thus, a feasible counterpart $\hat{\xi}_{jt}(f_0(.|\tilde{\lambda}), \tilde{\beta})$ must be used instead. For a given parametric family $\mathcal{F}_0$, the BLP estimator writes as follows:

$$\hat{\theta}(\mathcal{F}_0, h_E, \hat{W}) \equiv \hat{\theta} = \underset{\tilde{\theta}}{\text{Argmin}} \left(\sum_{jt} \hat{\xi}_{jt}(f_0(.|\tilde{\lambda}), \tilde{\beta}) h_E(z_{jt}) \right)' \hat{W} \left(\sum_{jt} \hat{\xi}_{jt}(f_0(.|\tilde{\lambda}), \tilde{\beta}) h_E(z_{jt}) \right). \tag{8}$$

The construction of the feasible structural error $\hat{\xi}_{jt}(f_0(.|\tilde{\lambda}), \tilde{\beta})$ requires the following 3 numerical approximations:

[16] If Ω_0 is singular, one can always use directly the asymptotic distribution in 6 or apply the singularity-robust procedure proposed in Andrews and Guggenberger (2019)

1. The econometrician does not observe a continuum of consumers as in the theoretical model but only empirical averages $\hat{s}_{jt}$ over the n_t individuals in market t.

$$\hat{s}_{jt} = \frac{1}{n_t} \sum_{i=1}^{n_t} y_{ijt}. \tag{9}$$

   where $y_{ijt} \in \{0; 1\}$ are iid choices over the $i = 1, \ldots, n_t$.

2. There is no closed form for $\rho_j(., x_{2t}, f_0(\cdot|\tilde{\lambda}))$, the integral has to be computed through numerical integration. A prominent example is Monte Carlo integration:

$$\hat{\rho}_j(\delta, x_{2t}, f_0(|\tilde{\lambda})) = \frac{1}{R} \sum_{r=1}^{R} \frac{\exp(\delta_j + x_{2jt} v_r)}{1 + \sum_{k=1}^{J_t} \exp(\delta_k + x'_{2kt} v_r)}. \tag{10}$$

   with v_r iid draws from $f_0(\cdot|\tilde{\lambda})$.

3. There is no analytical way to recover the inverse of the demand functions $\rho^{-1}(s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))$. The most popular way to derive the inverse demand is by solving the following contraction:

$$C : (\cdot, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) : \delta \mapsto \delta + \log(s_t) - \log(\rho(\delta, x_{2t}, f_0(\cdot|\tilde{\lambda}))).$$

   This solution has given rise to the popular nested fixed point GMM procedure. Another solution that has gained traction in the literature is the MPEC procedure (Dubé et al. (2012)) that replaces the BLP inversion at each step of the minimization by imposing equilibrium constraints on the minimization program.

Regardless of the method chosen[17], we can only recover an approximation of the true inverse demand and consequently of the structural error. In the next section, we explicitly state the assumptions that allow us to neglect these approximations asymptotically.

[17]there exist many estimation procedures that can be used to estimate the BLP pseudo true value: NFP GMM, MPEC, GEL (generalized empirical likelihood) , CUE (continuously updated GMM). For the sake of exposition, we will focus on the NFP GMM estimator as it is the procedure that is used in practice. We want to point out that the test and the methods that we develop in this paper will also work with other estimation procedures under some modifications in the regularity conditions

5.4.2 Regularity assumptions

Assumption B.

(i) $(s_t, x_t, z_t)_{t=1}^T$ are i.i.d. across markets and are consistent with the probability model defined by equations (1), (2) and (3) evaluated at (f, β),

(ii) Strong Exogeneity: $\mathbb{E}[\rho_j^{-1}(s_t, x_{2t}, f) - x'_{1jt}\beta | z_{jt}] = 0$ a.s.,

(iii) Finite moment conditions: x_{2t} has bounded support and x_{1t} has finite 4th moments.

In B(i), we assume that the data are i.i.d. across markets, an assumption which we could relax slightly (technically, only certain moments need to be identical across markets), and that the data is generated by the BLP demand model at a given pair (f, β). In B(ii), we assume exogeneity of our instrumental variables. Note that to show the asymptotic validity of our specification test, we do not require that (f, β) to be point identified as we just test the validity of certain moment conditions[18]. B(iii) is standard and is a necessary condition to recover asymptotic normality of the BLP estimator .

Assumption C.

$\mathcal{F}_0$ is such that

(i) λ_0 belongs to the interior of Λ_0 with Λ_0 compact,

(ii) $\tilde{\lambda} \mapsto \rho(\delta, x_{2t}, f_0(\cdot|\tilde{\lambda}))$ is well defined and continuously differentiable on Λ_0,

(iii) $\forall(\lambda, \lambda')$ such that $\lambda \neq \lambda'$, $\exists v^* \in Supp(\mathcal{F}_0)$ such that $f_0(v^*|\lambda) \neq f_0(v^*|\lambda')$.

In C(i), we assume that, given for any DGP, the associated pseudo-true-value λ_0 associated with the family $\mathcal{F}_0$ lies in a compact set Λ_0. This condition is standard in establishing the consistency and asymptotic normality of M-estimators. Second in C(ii), we impose that the demand function and its derivative with respect to λ should both be well defined and continuous. Finally, in C(iii), we restrict our attention to families that cannot have two parameters yielding the same density (to avoid issues of non-uniqueness of the pseudo-true value). assumption 3 is implied by identification

Next, we impose conditions on the instruments that are used for estimation $h_E(z_{jt})$ and for testing $h_D(z_{jt})$ given a distributional assumption $\mathcal{F}_0$ and on the BLP estimator itself:

[18] We do not need to assume completeness of z_{jt}, which is a strong assumption

Assumption D.

For a given $\mathcal{F}_0$ which satisfies Assumption C and for some weighting matrix W and Σ:

(i) Finite moments for instruments: $h_E(z_{jt})$ and $h_D(z_{jt})$ are not perfectly colinear and have finite 4th moments,

(ii) Local identification: $\Gamma(\mathcal{F}_0, \theta_0, h_E) = \mathbb{E}\left[\sum_j h_E(z_{jt})\frac{\partial \xi_{jt}(f_0(\cdot|\lambda_0),\beta_0)}{\partial \theta'}\right]$ and $\Gamma(\mathcal{F}_0, \theta_0, h_D)$ are full rank (i.e of rank $|\theta_0|$),

(iii) Global identification of θ_0: $\forall \tilde{\theta} \neq \theta_0$:

$$\mathbb{E}\left[\sum_j \xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta})h_E(z_{jt})'\right] W\mathbb{E}\left[\sum_j h_E(z_{jt})\xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta})\right] > \mathbb{E}\left[\sum_j \xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)h_E(z_{jt})'\right] W\mathbb{E}\left[\sum_j h_E(z_{jt})\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)\right]$$

, (iv) W and Σ are symmetric positive definite and $\hat{W} \xrightarrow{P} W$, $\hat{\Sigma} \xrightarrow{P} \Sigma$,

(v) $\hat{\theta}$ minimizes objective function 8 and satisfies the FOC of the minimization problem:

$$\left(\sum_{jt} \frac{\partial \hat{\xi}_{jt}(f(\cdot|\hat{\lambda}), \hat{\beta})}{\partial \theta} h_E(z_{jt})\right)' \hat{W} \left(\sum_{jt} \hat{\xi}_{jt}(f(\cdot|\hat{\lambda}), \hat{\beta})h_E(z_{jt})\right) = 0.$$

Assumption D restricts the class of instruments which can be used for estimation and/or for testing. More specifically D(i) and D(ii) are common regularity conditions necessary to establish asymptotic results whereas D(iii) is an identification condition which ensures that the pseudo true value θ_0 is uniquely defined; a condition that is critical to show consistency of the BLP estimator. Finally, Assumptions D(iv) and D(v) impose regularity conditions on the weighting matrix as well as on the BLP estimator itself.

The next assumptions ensure that the numerical approximations defined in subsection 5.4.1 do not interfere with the asymptotic theory.

Assumption E.

(i) Let n_t be the number of individuals in market t, $(n_t)_{t=1}^T$ is i.i.d. and independent from all other variables. First it must be that $\forall t$ $\sqrt{T}\mathbb{E}(n_t^{-1/2}) \underset{T\to+\infty}{\to} 0$. Second observed market share $\hat{s}_t$ in market t must write

$$\hat{s}_{jt} = \frac{1}{n_t}\sum_{i=1}^{n_t} y_{ijt}$$

with $(y_{ijt})_{i=1}^{n_t}$ iid draws conditional on (x_t, ξ_t) by drawing $\varepsilon_{ijt} \overset{iid}{\sim} EV(1)$

(ii) Let R be the number of simulations, then the simulated demand writes:

$$\hat{\rho}_t(\delta, x_{2t}, f_0(\cdot|\tilde{\lambda})) = \frac{1}{R}\sum_r \frac{\exp(\delta_j + x'_{2jt}v_r)}{1 + \sum_k \exp(\delta_k + x'_{2kt}v_r)}$$

where $v_r \overset{iid}{\sim} f_0(\cdot|\tilde{\lambda})$ and $\frac{T}{R} \underset{T\to+\infty}{\to} 0$

(iii) Let H be the stopping time for the contraction (which depends on T) and ϵ the fixed Lipschitz constant of the contraction mapping used to invert the demand function, then it must be that $\sqrt{T}\epsilon^H \underset{T\to+\infty}{\to} 0$

A sufficient condition for E(i) to hold is that the minimum number of individuals observed in any market is of higher order than the total number of markets. This condition can be checked in practice. Note that by making stronger assumptions on the higher moments and the support of the observed characteristics, it is possible to find milder conditions on the number of individuals relative to the number of markets. E(ii) and E(iii) can also be checked in practice and are more manageable because R and H are chosen by the researcher and can always be increased so that these assumptions hold.

Given our assumptions, we derive the asymptotic distribution of our test statistic under the null as well as its speed of divergence under the alternative, which ensures that the test is consistent.

Theorem 5.2. Let $\hat{\theta} = \hat{\theta}(\mathcal{F}_0, \hat{W}, h_E)$ be the BLP estimator associated with distributional assumption $\mathcal{F}_0$, weighting matrix $\hat{W}$, estimating instruments h_E. Under assumptions B-E,

- Under $\overline{H}'_0:\ \mathbb{E}\left[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)h_D(z_{jt})\right] = 0$

$$S_T(h_D, \mathcal{F}_0, \hat{\theta}) \underset{T\to+\infty}{\overset{d}{\to}} Z'\Sigma Z, \qquad\qquad Z \sim \mathcal{N}(0, \Omega_0)$$

  where

$$\Omega_0 = \begin{pmatrix} I_{|h_D|0} & G \end{pmatrix} \begin{pmatrix} \Omega(\mathcal{F}_0, h_D) & \Omega(\mathcal{F}_0, h_D, h_E) \\ \Omega(\mathcal{F}_0, h_D, h_E)' & \Omega(\mathcal{F}_0, h_E) \end{pmatrix} \begin{pmatrix} I_{|h_D|0} \\ G' \end{pmatrix}$$

with

$$\Omega(\mathcal{F}_0, h_D, h_E) = cov\bigg(\sum_j \xi_{jt}(f(.|\lambda_0), \beta_0)h_D(z_{jt}), \sum_j \xi_{jt}(f(.|\lambda_0), \beta_0)h_E(z_{jt})\bigg)$$

$$G = -\Gamma(\mathcal{F}_0, \theta_0, h_D)\left[\Gamma(\mathcal{F}_0, \theta_0, h_E)'W\Gamma(\mathcal{F}_0, \theta_0, h_E)\right]^{-1}\Gamma(\mathcal{F}_0, \theta_0, h_E)'W$$

- Under H_a' : $\mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0(.|\lambda_0), \beta_0)\right] \neq 0$

$$\forall q \in \mathbb{R}^+, \quad \mathbb{P}(S_T(h_D, \mathcal{F}_0, \hat{\theta}) > q) \underset{T\to+\infty}{\to} 1$$

The proof of the Theorem 5.2 is in Appendix B.4 and consists of 3 main steps. First, we show that under the assumptions in E, the numerical approximation becomes negligible asymptotically. Second, we show consistency and asymptotic normality of the BLP estimator. Finally, we derive the asymptotic distribution of the test statistic, taking into account parameter uncertainty (θ_0 is estimated and not observed). The apparent complexity of the asymptotic variance covariance matrix Ω_0 is a direct consequence of parameter uncertainty.

6 Monte Carlo experiments

In this section, we conduct three distinct sets of Monte Carlo experiments. First, we implement a simple simulation exercise to assess the effects of incorrectly specifying the distribution of random coefficients on quantities of interest such as price elasticities or cross-price elasticities, which are known to play a key role in shaping the counterfactuals. In a second set of Monte Carlo experiments, we study the finite sample performances of the specification test designed in section 5 with different sets of testing instruments. We first examine the size of our test in finite sample. Then, we investigate the power properties of our test under alternative specifications (with alternatives including a mixture of Gaussians, gamma distribution and local alternatives). We show that our test with the interval instruments significantly outperforms the traditional J-test with the usual instruments. In a third set of Monte Carlo simulations, we assess the finite sample performance of the selection procedures introduced in section 8. Finally, in the last Monte Carlo exercise, we study the performance of the interval instruments to estimate the parameters of the model by means of comparison with the commonly used instruments in the literature.

6.1 Simulation design

For the sake of exposition, we will keep the same simulation design for all the simulation experiments. The simulation design closely follows the simulation design used in Dubé et al. (2012), Reynaert and Verboven (2014). The market includes $J = 12$ products, which are characterized by 3 exogenous product attributes x_a, x_b and x_c which follow a joint normal distribution. The price p is endogenous and depends on the observed and unobserved characteristics and on some cost shifters c_1 and c_2. Consumer heterogeneity is present only in x_c only and the random coefficient v_i associated with x_c will follow various distributions depending on the simulation exercise. The sample size T varies between 50, 100 and 200 markets. We can summarize the DGP as follows.

$$u_{ijt} = 2 + x_{ajt} + 1.5x_{bjt} - 2p_{jt} + x_{cjt}v_i + \xi_{jt} + \varepsilon_{ijt} \qquad \xi \sim \mathcal{N}(0,1),\ \varepsilon \sim EV1$$

$$\text{and} \begin{bmatrix} x_{a,j} \\ x_{b,j} \\ x_{c,j} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -0.8 & 0.3 \\ -0.8 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{bmatrix} \right)$$

$$p_{jt} = 1 + \xi_{jt} + u_{jt} + \sum_{k=a}^{c} x_{kjt} + c_{1jt} + c_{2jt} \quad \text{with } u_{j,t} \sim U[-4,-2],\ c_{1jt} \sim U[2,4] \ and\ c_{2jt} \sim U[3,5]$$

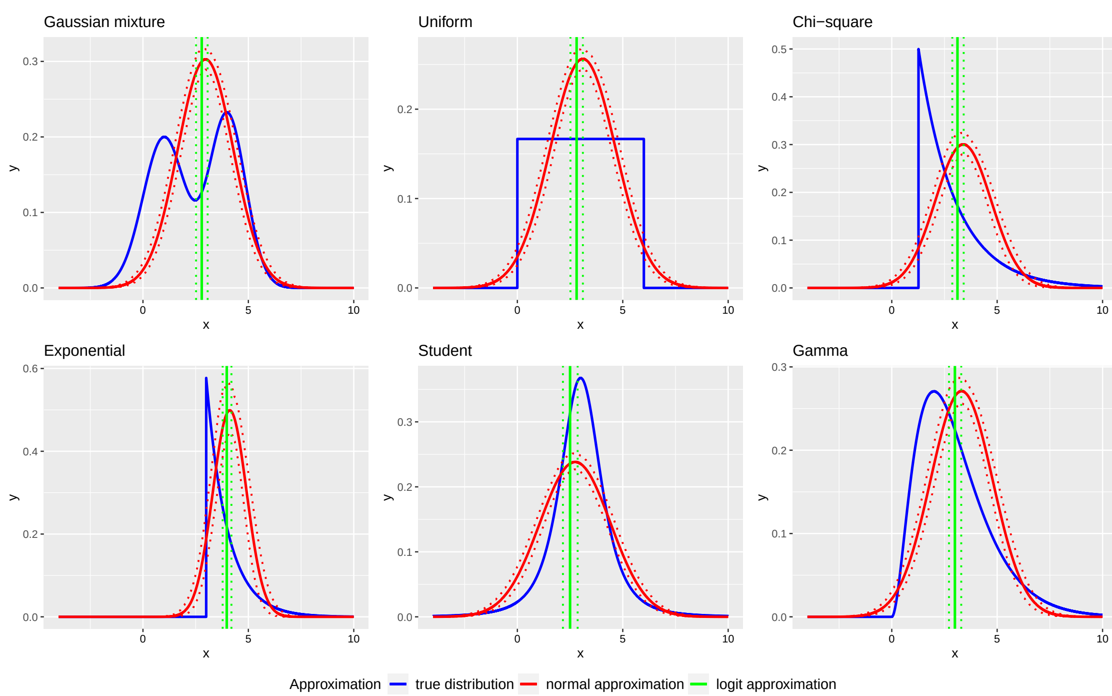
The market shares are generated by integrating over 20000 consumers (assuming the logit form). This allows us to essentially remove the approximation error between the observed market shares and the theoretical ones.

6.2 Counterfactuals under an alternative distribution

We now present a simple exercise to illustrate how the misspecification of random coefficients can affect the estimation of quantities of interest such as price elasticities and cross-price elasticities.

To do so, we simulate data using the simulation design introduced above and we take various distributions for the random coefficient v_i (respectively: Gaussian mixture, Uniform, Chi-square, Exponential, Student, Gamma). We ensure that all the distributions have the same mean and variance (respectively 3 and 3). For each distribution, we simulate $T = 100$ markets of data and we estimate the model either assuming no heterogeneity (simple logit) or assuming that v_i is normally distributed. We replicate the same exercise 500 times for each distributions. This allows us to recover the mean estimate for the parameters as well as to construct 95% "confidence bins" (by trimming the observations below the 2.5% quantile and above the 97.5% quantile). We plot the true densities and their approximations in Figure 2.

Figure 2: Approximation of the true density



In a second stage, we simulate $N = 5000$ draws from the true distributions as well as from the closest logit and normal approximations to compute the demand, the price-elasticity and the cross-price elasticity[19] for the product j^* with the highest value for x_c. The cross-price elasticity is arbitrarily taken for product $j = 1$ with respect to p_{j^*}. We derive the quantities of interest for 100

[19]The expressions for both price-elasticities and the cross-price elasticities are in the appendix

equally spaced values of p_{j^*} ranging in $]0, 10[$. We plot the elasticities (Figure 3) and cross-price elasticities (Figure 4) generated by the true distribution as well as those generated by the logit and normal approximations, respectively. We proceed similarly with the demand functions (see in Figure 11 in the appendix). One can observe that, as expected, the logit specification poorly replicates the substitution patterns. In particular, it consistently overstates the true elasticities and cross-elasticities. The absence of consumer heterogeneity on characteristic c implies that consumers can "renounce' more easily to product j^* when its price increases. By introducing some heterogeneity, the normal approximation can somewhat attenuate this issue. However, significant discrepancies in the shape of elasticities and cross price elasticities remain. As most counterfactual analyzes rely on the substitution patterns generated by the model, these differences will inevitably create significant biases.

Figure 3: Price elasticities

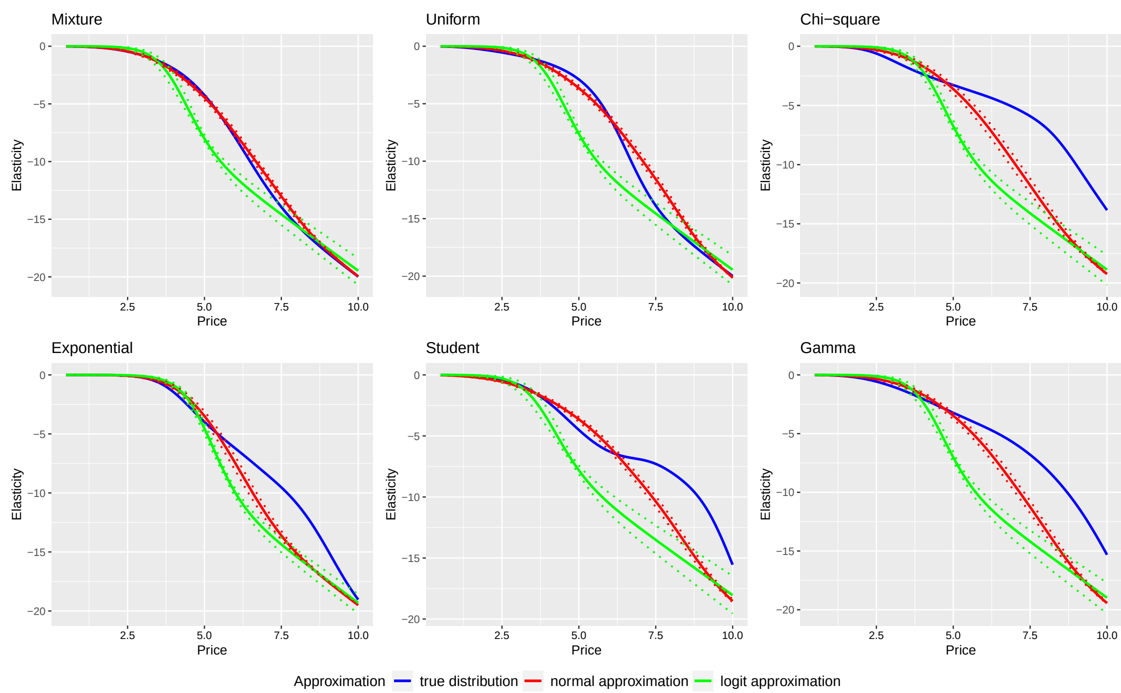
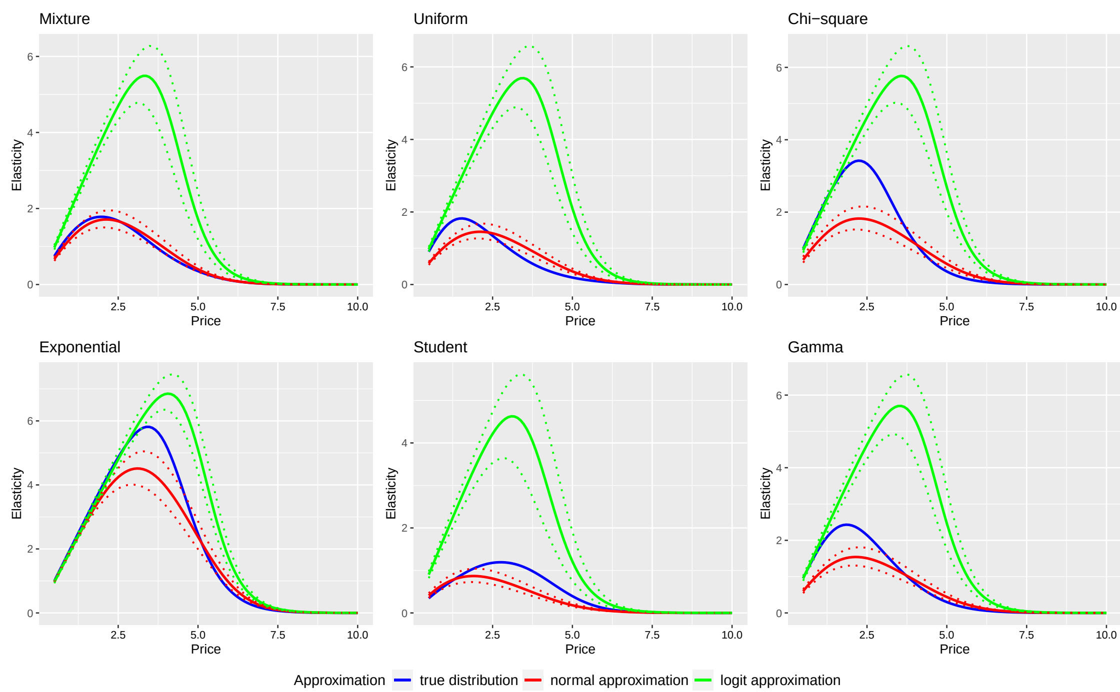


Figure 4: Cross-price elasticities



6.3 Finite sample performance of the specification test

We now study the empirical size and power of our test under different sample sizes and for different sets of testing and estimating instruments. Once again the data is generated according the simulation design exhibited previously for various distributions of v_i. The assumption made throughout the simulations is $H_0 : f \in \mathcal{F}_0$ where $\mathcal{F}_0$ is the family of normal distributions. In words, we always assumes that the random coefficient is normally distributed and we test this hypothesis. We set the nominal size to 5%. We study the finite sample performances of the specification test that we presented in section 5 using different sets of estimation and testing instruments. For the estimation instruments $h_E(z_{jt})$, we consider the two sets of instruments the most commonly used in the literature, the differentiation instruments of Gandhi and Houde (2019) and the "optimal" instruments of Reynaert and Verboven (2014) (both of these sets are approximations of the optimal instruments). For testing instruments, we use the instruments used for estimation (in this case our test boils down to the Sargan J-test). We also use the global and local approximations of the MPI that we construct in Sections 4.2 and 4.1. We denote these instruments respectively I Local and I Global.

The BLP estimator is estimated following the NFP GMM procedure described in Section 5.4.1. For the optimization, only an analytic Jacobian is provided. We make sure that the number of tested restrictions is of the same magnitude across the different sets of instruments. More details on the exact sets of instruments and on the estimation procedure for this specific set of simulations are given in Appendix D.2.

6.3.1 Empirical size

The size is the probability of rejecting the null hypothesis when the null is true, so we compute the empirical size by counting and averaging the number of times we reject the null for nominal size 5% over the 1000 simulations when the random coefficient v_i is normally distributed. Below in Table 1 are the empirical sizes of the test with the different sets of instruments described above for the different sample sizes and different true distributions of the random coefficient $v_i \sim f \in \mathcal{F}_0$.

Table 1: Empirical size for nominal size 5% (1000 replications)

Number of markets	T=50						T=100						T=200					
Estimation instruments	"Differentiation"			"Optimal'			"Differentiation"			"Optimal'			"Differentiation"			"Optimal'		
Test type	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local
$v_i \sim \mathcal{N}(-1, 0.5^2)$	0.294	0.083	0.091	0.145	0.078	0.063	0.138	0.078	0.058	0.094	0.084	0.047	0.08	0.052	0.053	0.064	0.05	0.04
$v_i \sim \mathcal{N}(0, 0.75^2)$	0.293	0.084	0.085	0.148	0.081	0.071	0.137	0.061	0.06	0.1	0.059	0.05	0.074	0.053	0.045	0.062	0.048	0.036
$v_i \sim \mathcal{N}(1, 1^2)$	0.287	0.084	0.083	0.142	0.084	0.073	0.142	0.055	0.054	0.098	0.053	0.047	0.079	0.042	0.03	0.058	0.035	0.025
$v_i \sim \mathcal{N}(2, 2^2)$	0.288	0.087	0.077	0.145	0.071	0.072	0.138	0.069	0.051	0.099	0.053	0.056	0.077	0.044	0.041	0.069	0.037	0.044
$v_i \sim \mathcal{N}(3, 3^2)$	0.287	0.089	0.071	0.137	0.075	0.066	0.145	0.074	0.06	0.098	0.06	0.061	0.076	0.044	0.037	0.061	0.046	0.046

We observe that with a moderate sample size ($T = 50, J = 12$), all the tests are over-sized. This is within expectations and due to the approximations inherent to estimation of the BLP models as described in Section 5.4.1 and the relatively large number of instruments [20] used for estimation and testing[21]. Increasing the sample size improves the tests' empirical levels and shifts them towards nominal level, which is a good indication of the validity of our test. Even with a relatively large number of markets ($T = 200$), the Sargan Hansen tests remain slightly oversized (rejection rate is

[20] number of over-identifying restrictions lies between 6 and 8

[21] Sargan Hansen tests are known to suffer from size and power distortions as the number of instruments increases

still slightly above 5%). On the other hand, for the test with interval instruments, the empirical size appears to match the nominal level for all but two configurations, where it seems to be slightly undersized.

6.3.2 Empirical power

Power is the probability of rejecting the null hypothesis when the null is false, so we compute empirical power by counting and averaging the number of times we reject the null for nominal size 5% over the 1000 simulations when the distribution of random coefficients is misspecified. The simulation setup remains the same as previously with the only modification being that the true distribution of v_i is now either a mixture of normals, or a Gamma. We report the power against the different alternatives in the subsequent tables. The main takeaway from our results is that the test with the interval instruments as testing instruments (I global and I local) largely outperforms the traditional Sargan J-test against all the alternative distributions considered in our simulations.

Power against Gaussian mixture alternatives. We simulate data with the random coefficients distributed according to the Gaussian mixtures described below. We plot the true distributions in Figure 5. We report the results in Table 2. We observe that the test with the interval instruments has great power against all the mixtures tested. The rejection rates go to 1 very quickly in comparison to the Sargan J-tests.

$$v = Dv_1 + (1-D)v_2, \quad \mathbb{P}(D=1) = p, \quad \mathbb{P}(D=0) = 1-p$$
$$v_1 \sim \mathcal{N}(-\sqrt{\frac{3p}{1-p}} + 2, 1) \quad v_2 \sim \mathcal{N}(\sqrt{\frac{3(1-p)}{p}} + 2, 1)$$

With $p \in \{0.1; 0.2; 0.3; 0.4; 0.5\}$

Figure 5: Densities of the alternative distributions, Gaussian mixtures

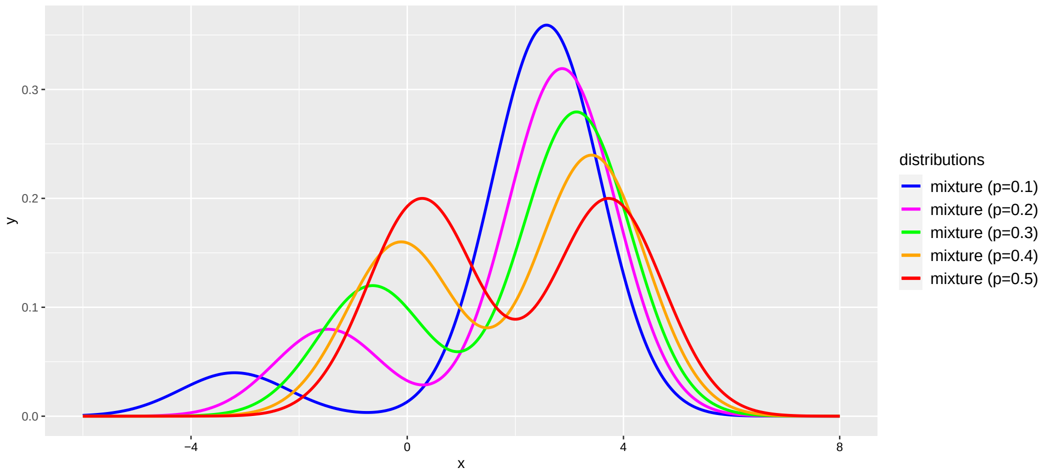


Table 2: Empirical power, Gaussian mixture alternatives (1000 replications)

Number of markets	T=50						T=100						T=200					
Estimation instruments	"Differentiation"			"Optimal'			"Differentiation"			"Optimal'			"Differentiation"			"Optimal'		
Test type	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local
Mixture 1	0.533	0.991	0.987	0.719	0.989	0.989	0.604	1	1	0.967	1	1	0.829	1	1	1	1	1
Mixture 2	0.626	0.996	0.998	0.613	0.997	0.998	0.723	1	1	0.905	1	1	0.933	1	1	1	1	1
Mixture 3	0.629	0.992	0.995	0.43	0.996	0.997	0.741	1	1	0.7	1	1	0.941	1	1	0.977	1	1
Mixture 4	0.601	0.983	0.982	0.275	0.981	0.981	0.713	1	0.999	0.368	1	1	0.921	1	1	0.672	1	1
Mixture 5	0.56	0.907	0.904	0.157	0.9	0.906	0.635	0.993	0.995	0.124	0.995	0.996	0.855	1	1	0.146	1	1

Power against Gamma alternatives. We simulate data with the random coefficients distributed according to the Gamma distribution described bellow. We plot the true distributions in Figure 6. We report the results in table 3. We observe that the test with interval instruments has great power against all the gamma distributions tested except for the first one, which we can see on the plot has a distribution that is relatively close to a normal. Even for the 1st gamma distribution, it still outperforms the traditional sets of instruments. For all the other gamma distributions, the rejection rates go to 1 very quickly in comparison to the Sargan J-tests. This confirms the superiority of the interval instruments in detecting misspecification in the distribution of random coefficients. In Appendix D.2, we also study the power properties of our test against local alternatives.

$$v \sim \Gamma(2, k) \quad with\ k \in \{0.25, 0.5, 0.75, 1, 1.5\}$$

Figure 6: Densities of the alternative distributions, Gamma distributions

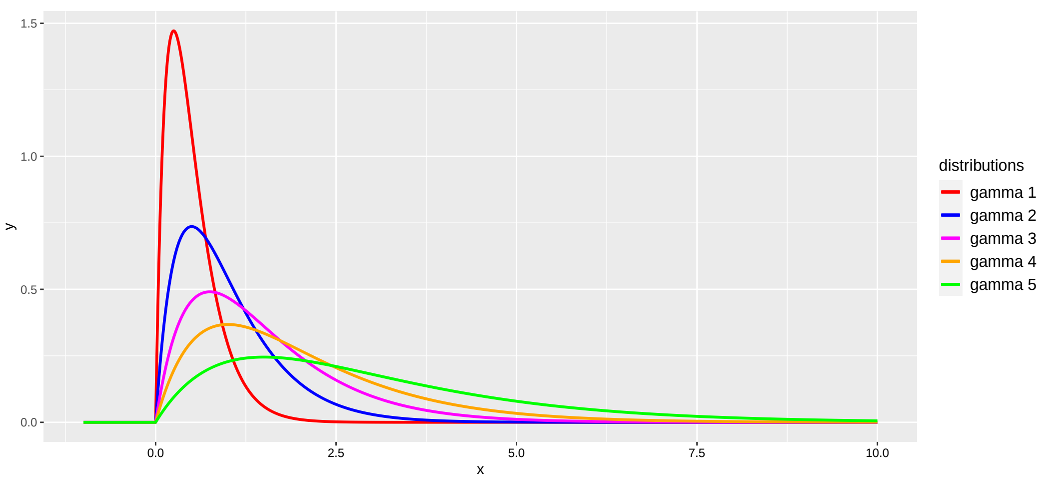


Table 3: Empirical power, Gamma alternatives (1000 replications)

Number of markets	T=50						T=100						T=200					
Estimation instruments	“Differentiation”			“Optimal’			“Differentiation”			“Optimal’			“Differentiation”			“Optimal’		
Test type	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local	J	I Global	I local
Gamma 1	0.293	0.106	0.093	0.142	0.082	0.074	0.154	0.083	0.073	0.094	0.092	0.08	0.118	0.155	0.139	0.066	0.156	0.138
Gamma 2	0.516	0.747	0.752	0.14	0.781	0.77	0.562	0.983	0.978	0.095	0.982	0.98	0.492	1	1	0.08	1	1
Gamma 3	0.607	0.96	0.962	0.157	0.963	0.969	0.693	0.998	1	0.156	1	1	0.922	1	1	0.161	1	1
Gamma 4	0.622	0.97	0.99	0.207	0.962	0.995	0.748	0.999	1	0.263	1	1	0.933	1	1	0.412	1	1
Gamma 5	0.687	0.991	0.999	0.371	0.988	0.999	0.812	1	1	0.585	1	1	0.976	1	1	0.865	1	1

6.4 Finite sample performance of interval instruments for estimation

In our last simulation exercise, we evaluate the performance of our interval instruments in estimating the non-linear parameters when the distribution of random coefficients is flexibly parametrized. To do so we simulate data with a distribution of random coefficients following a mixture of Gaussians and we estimate the parameters of this mixture. We provide a method to estimate the preference parameters when the distribution of the RC is a mixture in Section C.5 of the appendix. In particular, we provide a new parametrization of the model, which yields substantial practical gains and may be of interest for researchers independent of the rest of the paper. The simulation design remains the same as previously. We assume that the random coefficient v_i is distributed according to the following mixture: $v_i \sim 0.25\mathcal{N}(-2, 0.5) + 0.75\mathcal{N}(4, 0.5)$. Thus, there are 5 non-linear parameters to estimate: the means and variances of each component of the mixture and the mixing probability. Our objective is to compare the performance of the global and local interval instruments with the most commonly used instruments: the differentiation instruments from Gandhi and Houde (2019) and the "optimal instruments" from Reynaert and Verboven (2014). In Table 30, we report the empirical biases and the square root of the MSE for the estimators of the non-linear parameters for each set of instruments and for different sample sizes. In the appendix, we report the same information for the linear parameters (see Tables 26, 27, and 29) as well as the distribution of the empirical distribution of the non-linear estimates. Table 30 allows us to directly compare the performances of the three sets of instruments in estimating the non-linear parameters. We first observe that for all the sets of instruments, the empirical biases and $\sqrt{MSE}$ of the estimators decrease when the sample size increases, which is reassuring. Furthermore, it appears clearly that the differentiation instruments perform worse than the "optimal instruments" and the interval instruments. The empirical $\sqrt{MSE}$ of the estimators with the differentiation instruments is up to 12 times larger than with the interval instruments and up to 6 times larger than with the "optimal instruments". We reach the same conclusions when we study the empirical biases. The interval instruments appear to perform better than the "optimal instruments" even if the difference is less significant than with the differentiation instruments. For the sake of conciseness, we do not report the results obtained with a mixture of 3 components but the observations we make with two components are even more exacerbated. In the appendix, as a means of comparison, we perform the same exercise when the distribution of random coefficients is a simple Gaussian and here, we do not observe any significant differences between the different sets of instruments, which confirms that the interval instruments make a difference when the distribution of RC is flexible.

Table 4: Estimation non-linear parameters of the mixture (1000 replications)

	Instruments	Differentiation					"Optimal"					Interval Global					Interval Local				
	Parameter	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L
Sample size	true	-2	0.5	4	0.5	0.25	-2	0.5	4	0.5	0.25	-2	0.5	4	0.5	0.25	-2	0.5	4	0.5	0.25
T=50, J=12	bias	0.214	0.184	-0.022	-0.045	0.027	0.076	0.059	0.026	-0.111	0.01	0.017	0	-0.045	0.004	0.005	-0.006	-0.005	-0.039	-0.001	0.003
	$\sqrt{MSE}$	0.633	0.734	0.281	0.35	0.075	0.361	0.483	0.212	0.281	0.036	0.277	0.391	0.227	0.259	0.024	0.251	0.34	0.214	0.244	0.019
T=50, J=20	bias	0.189	0.347	0.022	-0.081	0.025	0.074	0.11	0.028	-0.089	0.01	0.013	0.042	-0.018	-0.003	0.004	0.019	0.033	-0.023	0.01	0.003
	$\sqrt{MSE}$	0.566	0.887	0.184	0.291	0.059	0.328	0.563	0.163	0.228	0.033	0.248	0.415	0.166	0.22	0.021	0.228	0.38	0.15	0.184	0.018
T=100, J=12	bias	0.233	0.226	0.02	-0.066	0.027	0.054	0.037	0.019	-0.066	0.007	0.004	-0.012	-0.027	0.005	0.002	0	0	-0.028	0.007	0.001
	$\sqrt{MSE}$	0.592	0.703	0.256	0.305	0.072	0.279	0.4	0.154	0.211	0.028	0.167	0.282	0.157	0.201	0.013	0.127	0.225	0.143	0.164	0.005
T=100, J=20	bias	0.198	0.423	0.047	-0.101	0.025	0.074	0.107	0.033	-0.074	0.01	-0.009	-0.005	-0.008	-0.009	0.001	-0.003	0.004	-0.01	0.004	0.001
	$\sqrt{MSE}$	0.552	0.89	0.164	0.27	0.055	0.311	0.52	0.129	0.194	0.034	0.115	0.264	0.115	0.169	0.005	0.104	0.226	0.103	0.125	0.004
T=200, J=12	bias	0.184	0.167	0.011	-0.049	0.019	0.026	0.011	0.021	-0.061	0.004	-0.006	-0.027	-0.015	-0.001	0.001	0.002	-0.007	-0.016	0.006	0.001
	$\sqrt{MSE}$	0.466	0.601	0.176	0.262	0.053	0.184	0.313	0.113	0.172	0.018	0.088	0.219	0.108	0.164	0.003	0.091	0.174	0.099	0.123	0.003

7 Empirical application

In this section, we apply the tools we developed previously to investigate the importance of flexibility on counterfactual quantities such as price elasticities, curvature and pass-through. To do so, we use data on new car registrations in Germany from 2012 to 2018[22]. Cars are highly differentiated products which create room for rich substitution patterns. Moreover, the role of road transport in air pollution is significant. In 2017, road transport was responsible of approximately 19% of total greenhouse has emissions in EU-28[23]. There have been several tax policies implemented across the world to encourage consumers to switch more environmental friendly cars. Random coefficient logit model has been utilized in the recent research on the car market to analyze different tax schemes Grigolon et al. (2018), and role of import tariffs Miravete et al. (2018a). In these studies, as in much of the literature using random coefficient logit models with aggregate data, the flexibility in the heterogeneity of preferences for price has been either not taken into account (i.e. no random coefficient on price (normalized with respect to income)) or assumed normal. We start by the commonly used approach in the literature and assume normally distributed random coefficient.

[22]The dataset is kindly provided by Kevin Remmy https://kevinremmy.com/research/

[23]Retrieved from https://www.eea.europa.eu/data-and-maps/indicators/transport-emissions-of-greenhouse-gas transport-emissions-of-greenhouse-gases-12 on October 21, 2022.

Then using our test, we increase flexibility by using mixtures of higher degrees and estimating corresponding parameters. Finally, we compare the counterfactual quantities with flexibly estimated model and the one that assumes normal distribution.

7.1 Data

The data includes state-level new car registrations, publicly available by the German Federal Motor Transport Authority (KBA) from 2012 to 2018. This gives us 112 markets defined by state-year pairs. Data on car characteristics and price are scraped from General German Automobile Club and include horsepower, engine type, size, weight, fuel costs, CO2 emission, number of doors, segment and body type. We define a product using brand-model-engine type-body type combination, e.g. BMW-1 Series-Diesel-Hatchback [24]. To construct market shares, we use the share of new car registrations of a given product in a market where market size is determined by the number of households in the market following the literature. The data is complemented by information on demographics such as the number of households at the state-year level, average income per household at the state-year level [25] and yearly average gas price data from ADAC and electricity cost data from the German Economics Ministry.

Summary statistics Shares of products sold by engine type are presented in Table 5. We focus our analysis on the combustion engine vehicles as in our sample period electric-vehicle cars constitute a small market share (less than 5% among the cars sold/registered) as shown in. Between the diesel and gasoline cars, we observe that the market share for diesel decreases starting from 2016. The timing is in line with the emissions scandal (also called Dieselgate) started in September 2015, which we control for in our empirical analysis.

[24]Each car is identified by its manufacturer and type key code (HSN/TSN) which pins down car model, body type, engine type, engine size. Following the literature, we aggregate products with the same brand-model-engine type-body type combination and we use the characteristics of the car with the highest sales.

[25]State level income https://ec.europa.eu/eurostat/web/products-datasets/-/nama_10r_2hhinc

Table 5: Shares (%) of new registrations by engine type

	Year						
Fuel Type	2012	2013	2014	2015	2016	2017	2018
Diesel	46.8	46.1	46.3	46.4	43.9	36.2	30.0
Gasoline	52.6	52.9	52.6	52.3	54.4	60.8	66.5
Battery EV	0.1	0.2	0.3	0.4	0.3	0.7	1.1
Hybrid EV	0.5	0.8	0.7	0.6	1	1.4	1.6
Plug-in hybrid EV	0	0	0.1	0.3	0.4	0.9	0.9

Table 6 provides sales-weighted averages for prices as a share of average household income and characteristics observed. We observe that the difference in fuel consumption and resulting fuel costs steadily ranks diesel more efficient than gasoline. On the other hand, the average price of diesel cars sold is higher compared the gasoline cars. This implies a potential trade-off in terms of costs of car ownership at the time of purchase. With a fixed mileage in mind, a consumer with high sensitivity to fuel costs might be willing to pay higher for a more fuel-efficient car. We also observe higher horsepower that for both types of cars, and a slight increase in the size of the cars sold.

Table 6: Summary Statistics (Sales weighted)

	Year						
	2012	2013	2014	2015	2016	2017	2018
Diesel							
Price/income	0.74	0.72	0.73	0.72	0.71	0.69	0.68
Size (m2)	8.31	8.31	8.32	8.36	8.42	8.48	8.53
Horsepower (kW/100)	1.09	1.07	1.11	1.11	1.14	1.16	1.21
Fuel cost (euros/100km)	7.90	7.18	6.63	5.53	4.94	5.25	5.83
Fuel cons. (Lt./100km)	5.19	4.98	4.89	4.73	4.61	4.61	4.71
CO2 emission (g/km)	136.19	130.50	127.69	123.58	120.42	120.49	123.27
Nb. of products/market	133	138	146	150	151	149	143
Gasoline							
Price/income	0.46	0.46	0.46	0.46	0.46	0.45	0.43
Size (m2)	7.23	7.27	7.28	7.30	7.36	7.46	7.53
Horsepower (kW/100)	0.79	0.78	0.80	0.82	0.85	0.88	0.91
Fuel cost (euros/100km)	9.48	8.61	8.11	7.27	6.69	7.06	7.40
Fuel cons. (Lt./100km)	5.76	5.47	5.40	5.31	5.25	5.34	5.38
CO2 emission (g/km)	135.80	128.18	125.27	122.89	121.22	122.86	123.26
Nb. of products/market	157	171	179	185	186	193	188

Note: Provided statistics are sales weighted averages across products. Total number of markets (State*Year) is 112

Inter-market variation. As we define a market by a year and state combination in Germany, we need to account for the inherent differences in market characteristics. This is important as such variation in market specific characteristic would bias preferences estimates if not not accounted for. Our data displays both inter-temporal variation, which leads to differences in the choice sets. Our data also features geographical variation. In Germany, the states present considerable variation in income per inhabitant, population density and average distance driven[26]. To understand which characteristics are affected by geographical differences, we do the following exercise. We first create sales-weighted characteristics that we include in our model, namely price, fuel cost, size, horsepower,

[26] For the area in km^2: 419 (Bremen) to 70522 (Bavaria). For population per km^2 in 2020: 0.6 million (Bremen) to 13 million (Bavaria), GDP per capita 2020: 28k (Mecklenburg-Western Pomerania) to 64k (Hamburg). For average driving distance in 2019: 13079 km (Mecklenburg-Vorpommern) to 9531 (Berlin) https://de.statista.com/statistik/daten/studie/644381/umfrage/fahrleistung-privater-pkw-in-deutschland-nach-bundesland/

height, gasoline dummy, and foreign dummy by market. Then we regress these quantities on the demographics of interest which are average income, population density and we control for the time trend. The results of these regressions are presented in Table 7. The results suggest that income plays a role in preferences for price[27], size, horsepower positively (i.e. higher income is associated with larger cars, and higher horsepower) and for foreign status, height, and gasoline states negatively. Although weaker, a similar pattern is observed for the effect of population density.

Table 7: Linear regressions of sales-weighted car characteristics on demographics

	Income(/1000)	Population density (/100)	Time trend
Price(×1000)	0.138**	0.069*	0.286*
	(0.013)	(0.011)	(0.059)
Fuel cost (euros/100km)	-0.0069	-0.0036	0.3587**
	(0.0063)	(0.0056)	(0.0296)
Size(mˆ2)	0.0058**	0.0018*	0.0176*
	(0.00079)	(0.00070)	(0.00371)
Horsepower (KW/100)	0.0028**	0.0012*	0.0129**
	(0.00028)	(0.00025)	(0.00132)
Foreign	−0.0050**	−0.0014*	0.0295**
	(0.00052)	(0.00046)	(0.00246)
height(m)	−0.00051**	−0.00043**	0.00181*
	(0.000061)	(0.000054)	(0.000286)
Gazoline	−0.0067**	−0.0024*	0.0131*
	(0.00059)	(0.00053)	(0.00280)

Note: * p-value lower than 0.01, ** p-value lower than $1e^{-10}$

Instruments for price We use a combination of cost shifters together with number of competing products in a given market, class and engine type and number of competing products in a given market and engine type. There are three complementary data sets to construct the cost shifters: mean hourly labor cost, price of steel (interacted with weight of the car), exchange rates between Germany and the country of assembly.

1. Labour cost: We use mean nominal hourly labour cost per employee in the manufacturing sector of the country of assembly of the models. The data on labour costs come from Inter-

[27]In the main analysis, we use price/income to capture the income effect

national Labor Organization Statistics (ILOSTAT) [28].

2. Price of steel: We also collect the price of steel futures at the January of each year.

3. Exchange rates: We construct the exchange rates between Germany and the country of assembly of each car model using exchange rate data from OECD [29].

7.2 Empirical specification

Indirect utility of consumer i, purchasing product j in market t (defined as a state, year pair) is given by

$$u_{ijt} = \underbrace{x'_{1jt}\beta + \xi^*_{jt}}_{\delta_{jt}} + x'_{2jt}\alpha_i + \varepsilon_{ijt}$$

The mean utility $\delta_{jt} = x'_{1jt}\beta + \xi^*_{jt}$ captures homogeneous preferences. The individual specific part of the utility is captured by $x'_{2jt}\alpha_i$. The demand shock on product j is decomposed as follows

$$\xi^*_{jt} = Brand_j + State_t + Year_t + \xi_{jt}$$

where $Brand_j$ is a brand fixed effects that capture unobserved heterogeneity attributable to brands and fixed across states and time, $State_t$ captures state specific preferences that are fixed across time and brands and $Year_t$ captures year specific preferences that are fixed across states and brands and . Therefore $State_t$ and $Year_t$ play a role in explaining the outside good shares that is not purchasing.

We first estimate the model assuming that consumer have heterogenous preferences for price, size and fuel cost (i.e. variables in x_2) and v_i follows a standard normal distribution. Then we question the normality assumption on price and test the normality assumption against a more flexible alternative. If our specification test rejects the normality assumption for some variables, we will we make use of gaussian mixtures to increase flexibility on price coefficient. To increase the flexibility sequentially, we consider gaussian mixtures with a degree higher than the assumed distribution in the null. For instance, in the first step we test the normality against a bimodal distribution.

[28] https://www.ilo.org/ilostat-files/Documents/Excel/INDICATOR/LAC_4HRL_ECO_CUR_NB_A_EN.xlsx
[29] https://data.oecd.org/conversion/exchange-rates.htm

7.3 Estimation

In the appendix, we provide results with baseline specifications including the simple conditional logit and the nested logit (with and without state and year fixed effects).

Estimation conditional logit (no heterogeneity)

Table 8: Logit estimation

	Baseline		**× Income (/1000)**		**× Pop. density(/100)**		**× Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-3.1	7.4e-03	-	-	-	-	-	-
Fuel Cost	-0.37	1.8e-03	-	-	-	-	-0.054	3.7e-04
Size(m^2)	0.11	3.2e-03	-0.0047	6.3e-05	-	-	-	-
Horsepower(KW/100)	3.7	1.1e-02	-0.039	1.7e-04	-	-	-0.027	5.1e-04
Foreign	0.14	5.1e-03	-0.017	1e-04	-	-	-	-
Height(m)	3.6	1.7e-02	-0.006	3.4e-04	-0.039	3.5e-04	-	-
Gasoline	1.1	4.7e-03	-0.014	8.5e-05	-	-	-	-
Constant	-12	1.1e-02	-	-	-	-	-	-

Note: Brand, Year and State FE's are included

Estimation with Gaussian random coefficients

Table 9: Traditional BLP (normal RC)

	Baseline		**× Income (/1000)**		**× Pop. density(/100)**		**× Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-	-	-	-	-	-	-	-
Fuel Cost	-	-	-	-	-	-	-0.071	3.9e-03
Size(m^2)	-	-	0.0018	4.1e-04	-	-	-	-
Horsepower(KW/100)	0.67	7.2e-02	0.0037	1.3e-03	-	-	-0.033	5.8e-03
Foreign	0.096	6.3e-02	-0.016	1.3e-03	-	-	-	-
Height(m)	3.3	1.4e-02	-0.0067	1.4e-03	-0.031	5.2e-04	-	-
Gasoline	0.84	5.9e-02	-0.0083	1.1e-03	-	-	-	-
Constant	-6.7	6e-02	-	-	-	-	-	-
Gaussian RC	$\hat{\beta}$	S.E	$\hat{\sigma}$	S.E				
Price/income	-3.2	3.5e-02	1.3	7.4e-02				
Fuel Cost	-0.36	2.8e-02	0.36	1.4e-02				
Size(m^2)	-2.6	1.9e-02	1.4	1.9e-02				

Note: Brand, Year and State FE's are included

Table 9 presents the parameter estimates obtained under the assumption that the preferences for price, fuel cost and size follow a normal distribution. The estimates are in line with the literature and with the closest paper to our setting Grigolon et al. (2018). The homogeneous preferences show that consumers value cars with higher horsepower, greater height and gasoline engine more on average. For the characteristics with heterogenous preferences, we see that the mean valuation for price, fuel cost and size. However, the variation in the preferences are quite significant in terms of the comparison to the mean effect and are precisely identified.

Estimation Following the selection procedure developed in section 8

- **Stage 0**: reference model simple logit

Table 10: Selection Stage 0

characteristic	R^2
Price	0.0559
Fuel cost	0.0355
Size	0.0494
Horsepower	0.0324
Foreign	0.0068
Height	0.0393
gazoline	0.0050

Table 11: Test Gaussian specification

test	value statistic	critical value
J test	1423.7	28.9
I test (local)	619.2	15.5

- **Stage 1:** Gaussian RC on price

Table 12: Estimation stage 1

	Baseline		$\times$ **Income (/1000)**		$\times$ **Pop. density(/100)**		$\times$ **Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-	-	-	-	-	-	-	-
Fuel Cost	-0.21	5.4e-03	-	-	-	-	0.0081	9.8e-04
Size(m^2)	0.7	1.8e-02	-0.011	3.6e-04	-	-	-	-
Horsepower(KW/100)	0.75	8.2e-02	0.014	1.3e-03	-	-	-0.08	6.1e-03
Foreign	0.011	5.9e-02	-0.012	1.2e-03	-	-	-	-
Height(m)	3.9	1.6e-02	-0.01	1.3e-03	-0.036	3.8e-04	-	-
Gasoline	0.88	5.3e-02	-0.0085	1.1e-03	-	-	-	-
Constant	-12	7.1e-02	-	-	-	-	-	-
Gaussian RC	$\hat{\beta}$	S.E	$\hat{\sigma}$	S.E				
Price/income	-5.6	3e-02	1.7	8.9e-02				

Note: Brand, Year and State FE's are included

Table 13: Selection Stage 1

characteristic	R^2
Price	0.0266
Fuel cost	0.0401
Size	0.0252
Horsepower	0.0152
Foreign	0.0059
Height	0.0304
Gasoline	0.0496

Table 14: Test Stage 1

test	value statistic	critical value
J test	5082.8	30.1
Interval test	1646.6	11.1
p-value	$< 1e^{-16}$	

- **Stage 2:** Gaussian RC on price and gasoline

Table 15: Estimation stage 2

	Baseline		**× Income (/1000)**		**× Pop. density(/100)**		**× Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-	-	-	-	-	-	-	-
Fuel Cost	-0.22	5.6e-03	-	-	-	-	0.013	9.8e-04
Size(mˆ2)	0.83	2.7e-02	-0.012	3.6e-04	-	-	-	-
Horsepower(KW/100)	1.2	7.9e-02	0.011	1.3e-03	-	-	-0.084	6e-03
Foreign	-0.052	5.9e-02	-0.012	1.2e-03	-	-	-	-
Height(m)	3.6	1.6e-02	-0.0047	1.3e-03	-0.034	3.7e-04	-	-
Gasoline	-	-	-0.0031	9.4e-04	-	-	-	-
Constant	-13	7e-02	-	-	-	-	-	-
Gaussian RC	$\hat{\beta}$	S.E	$\hat{\sigma}$	S.E				
Price/income	-6.5	3.3e-02	1.9	7.1e-03				
Gasoline	-2.4	7.2e-03	3.1	9e-03				

Note: Brand, Year and State FE's are included

Table 16: Selection Stage 2

characteristic	R^2
Price	0.066
Fuel cost	0.033
Size	0.033
Horsepower	0.010
Foreign	0.019
Height	0.029
gazoline	0.065

Table 17: Test Stage 2

test	value statistic	critical value
J test	5017.6	32.7
Interval test	1936.1	15.5
p-value	$< 1e^{-16}$	

- **Stage 3:** Gaussian mixture on price and Gaussian RC on gasoline

Table 18: Estimation stage 3

	Baseline		**× Income (/1000)**		**× Pop. density(/100)**		**× Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-	-	-	-	-	-	-	-
Fuel Cost	-0.21	5.4e-03	-	-	-	-	0.012	9.7e-04
Size(m^2)	0.67	1.9e-02	-0.0069	3.5e-04	-	-	-	-
Horsepower(KW/100)	1.5	7.8e-02	0.0057	1.3e-03	-	-	-0.087	6.2e-03
Foreign	0.072	5.9e-02	-0.015	1.2e-03	-	-	-	-
Height(m)	4.3	1.7e-02	-0.018	1.3e-03	-0.034	3.7e-04	-	-
Gasoline	-	-	-0.014	1.1e-03	-	-	-	-
Constant	-12	5.7e-02	-	-	-	-	-	-
Gaussian RC	$\hat{\beta}$	S.E	$\hat{\sigma}$	S.E				
Gasoline	0.32	9.4e-03	1.6	4e-02				
Gaussian Mixture	$\hat{\beta}_1$	S.E	$\hat{\sigma}_1$	S.E	$\hat{\beta}_2$	S.E	$\hat{\sigma}_2$	S.E
Price/income	-13	2.6e-02	0.00000051	8.3e-15	-5.1	2.6e-02	1.5	1.4e-11
Probability	0.82	6.5e-04						

Note: Brand, Year and State FE's are included

Table 19: Selection Stage 3

characteristic	R^2
Price	0.039
Fuel cost	0.019
Size	0.037
Horsepower	0.047
Foreign	0.020
Height	0.023
Gasoline	0.057

Table 20: Test Stage 3

test	value statistic	critical value
J test	4600.17	30.14
Interval test	1074.22	9.49
p-value	$< 1e^{-16}$	

- Stage 4: Gaussian mixture on price and on gasoline

Table 21: Estimation stage 4

	Baseline		**× Income (/1000)**		**× Pop. density(/100)**		**× Time trend**	
Homogeneous Preferences	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E	$\hat{\beta}$	S.E
Price/income	-	-	-	-	-	-	-	-
Fuel Cost	-0.22	5.4e-03	-	-	-	-	0.018	9.5e-04
Size(m^2)	0.87	1.7e-02	-0.013	3.4e-04	-	-	-	-
Horsepower(KW/100)	1.8	4.6e-02	0.0018	1.1e-03	-	-	-0.072	6.1e-03
Foreign	-0.093	5.9e-02	-0.012	1.2e-03	-	-	-	-
Height(m)	2.9	1.1e-02	0.0041	1.2e-03	-0.035	3.6e-04	-	-
Gasoline	-	-	-0.033	1.2e-03	-	-	-	-
Constant	-13	6.6e-02	-	-	-	-	-	-
Gaussian Mixture	$\hat{\beta}_1$	S.E	$\hat{\sigma}_1$	S.E	$\hat{\beta}_2$	S.E	$\hat{\sigma}_2$	S.E
Price/income	-7.6	3.2e-02	0.0000000071	4.9e-18	-4	3.2e-02	1	5.8e-14
Gasoline	5.6	3.3e-02	0.000000019	5.1e-17	0.64	3.3e-02	0.00000014	1.4e-16
Probability	0.18	7.4e-04						

Note: Brand, Year and State FE's are included

Table 22: Selection Stage 4

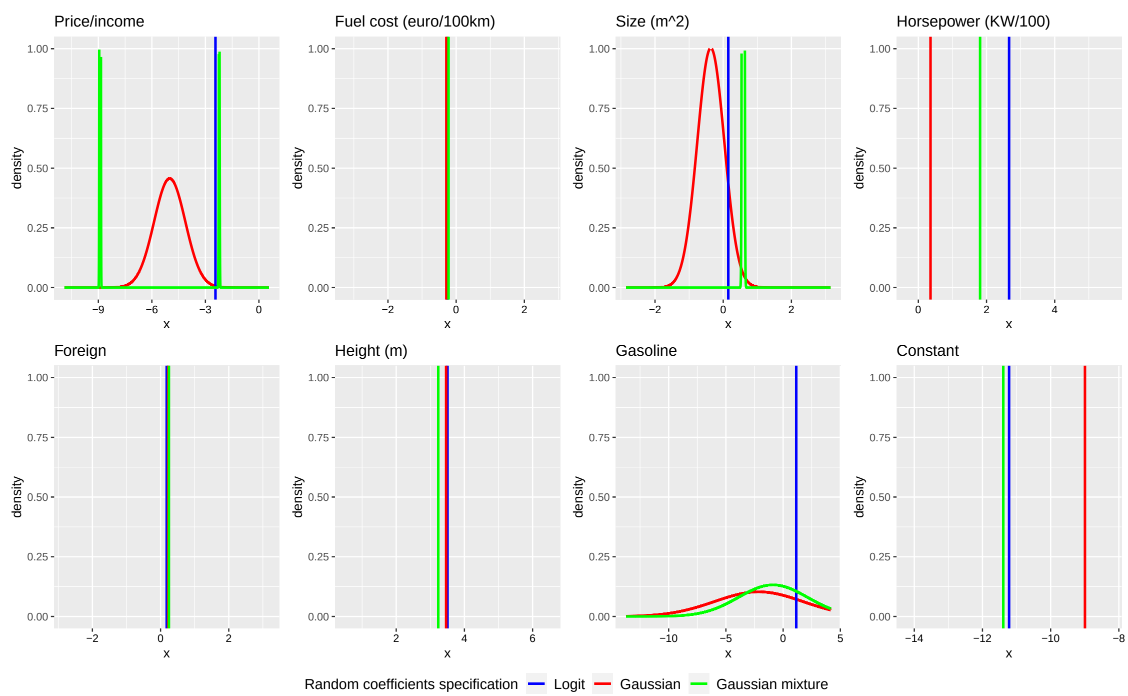
characteristic	R^2
Price	0.013
Fuel cost	0.015
Size	0.015
Horsepower	0.013
Foreign	0.016
Height	0.015
Gasoline	0.013

Table 23: Test Stage 4

test	value statistic	critical value
J test	4559.10	27.59
Interval test	265.61	5.99
p-value	$< 1e^{-16}$	

Densities

Figure 7: Estimated distributions of RCs



7.4 Counterfactual quantities

Counterfactual quantities of interest. The goal of this section is to present the impact of added flexibility in the quantities of interest that are central to assessing results of many tax policies. We first present demand elasticities under the traditional approach of normally distributed random coefficient of price, fuel cost and size and provide a comparison with the elasticities obtained that allows flexible distribution in random coefficient for price. Then, we move on to the pass-through which is affected not only by the subtitution patterns (first derivative of demand) but also by the curvature (function of second price derivative of demand) as highlighted by recent work of Miravete et al. (2018b).

- Price elasticity
- Demand curvature

  Recent work of Miravete et al. (2022) emphasizes the importance of the second derivative of demand due to the link between the demand curvature and elasticities are determined by the demand specification. As a result, they argue that demand curvature plays an important role in the pass-through rates. In this section, we show how added flexiblity changes the demand curvature.

  Demand curvature of the inverse demand function is given by

$$\rho(p) \equiv s(p) \cdot \frac{s_{pp}(p)}{[s_p(p)]^2}$$

- Marginal cost and Mark-ups

  The price elasticity of demand is important for counterfactual analysis given the role it plays in finding the equilibrium prices under multi-product Bertrand competition Berry et al. (1995). The profit of a multi-product firm $f \in \mathcal{F}$ with the price vector $\mathbf{p}$ writes[30]:

$$\Pi_f(\mathbf{p}) = \sum_{j \in J_f} (p_j - c_j)\, M s_j(\mathbf{p})$$

  where J_f is the set of products produced by firm f, c_j is the marginal cost for product j, M is the market size and $s_j(p)$ is the market share of product j. The first-order condition with respect to product j's price p_j, therefore, writes

$$s_j(\mathbf{p}) + \sum_{j' \in J_f} (p_{j'} - c_{j'}) \frac{\partial s_{j'}(\mathbf{p})}{\partial p_j} = 0$$

  and in matrix form as:

$$\mathbf{s}(\mathbf{p}) + (\boldsymbol{\Delta}(\mathbf{p}))\,(\mathbf{p} - \mathbf{c}) = 0$$

[30]For the sake of representation we omit x, ξ, and f, and simple write $s_j(\mathbf{p})$ instead of $s_j(\mathbf{p}, x, \xi; f)$

where $\mathbf{\Delta}(\mathbf{p}) = \frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}$ if j' and j are produced by the same firm and equals to zero otherwise. One can obtain equilibrium prices by solving for the equilibrium using the following equation.

$$\mathbf{p} = \mathbf{c} - (\Delta(\mathbf{p}))^{-1}\,\mathbf{s}(\mathbf{p})$$

- Pass-through

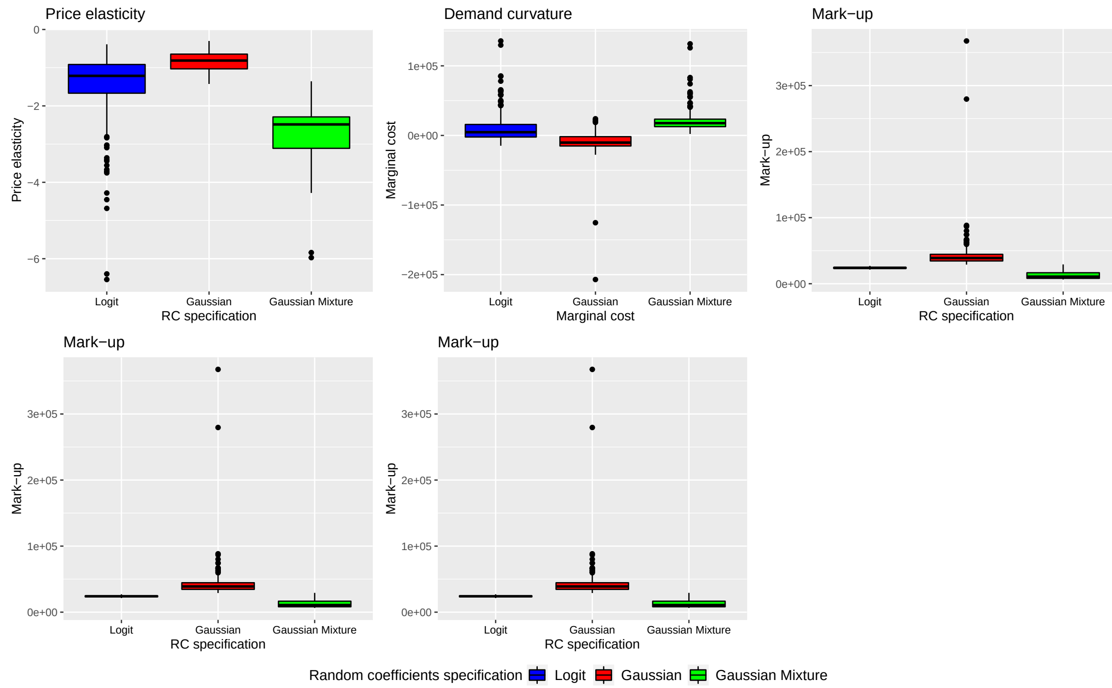
Table 24: Mean counterfactual quantities under different specifications on RCs

RC specification on price	Logit	Gaussian	Gaussian Mixture
mean own price-elasticity	-1.40	-0.84	-2.64
Demand curvature	1.00	1.40	1.28
Marginal cost	9366	-8939	20315
Mark-up	24029	42334	13080
Pass-through	9366	19877	20315

Note:

Summary of results.

Figure 8: Counterfactual quantities under different specifications



- Demand curves
- Price elasticities

Figure 9: Estimated demand functions under different specifications

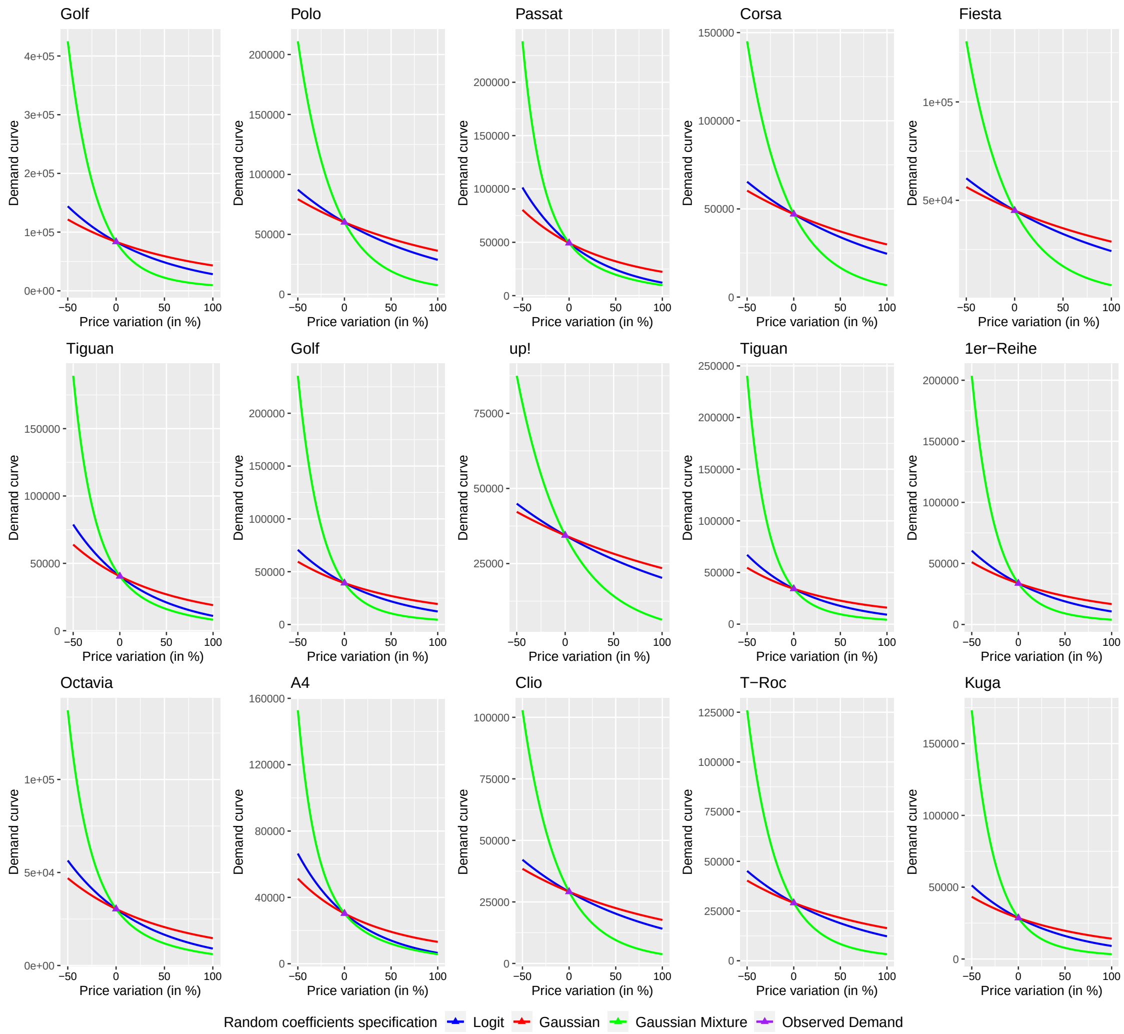
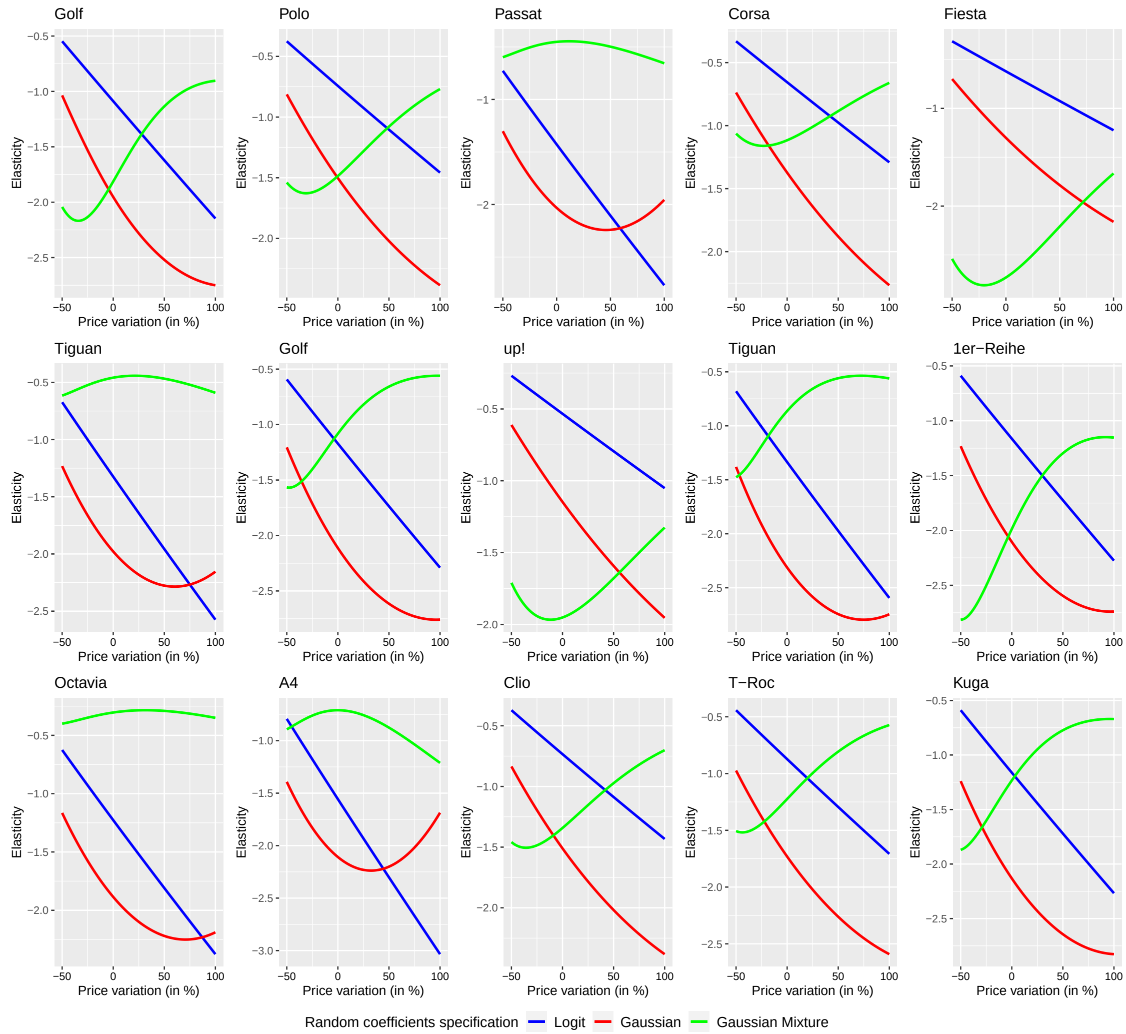


Figure 10: Estimated elasticities under different specifications



8 Model selection

In practice, the researcher does not know a priori which characteristics display preference heterogeneity and must be augmented with a random coefficient, and which distribution of RC is specified in an overly restrictive way. In most empirical applications, this choice is made arbitrarily by the researcher. However, given how critical the choice of the specification is to recovering the right substitution patterns, we believe it is important to have a more data-driven approach. Furthermore, the model is generally costly to estimate - even more so with a large number of random coefficients - and thus, it is paramount to develop a procedure that doesn't require the econometrician to estimate the model with all the possible combinations of random coefficients. Consequently, we build on our test and our instruments to provide a procedure to sequentially select the characteristics that display omitted preference heterogeneity, that is heterogeneity that the current model does not account for. First, we describe the selection procedure. Then, we discuss the general idea behind this selection procedure.

Practical implementation of the selection procedure

1. **Estimation stage:** the researcher estimates model M_0 with parametric restriction $\mathcal{F}_0$ and recovers the structural error $\xi_{jt}(f_0(\cdot|\widehat{\lambda}), \widehat{\beta})$. A natural initial model M_0 is the conditional logit, which does not display any random coefficient (i.e. $v_i = v \ \forall i$). This model can be directly estimated via 2SLS.

2. **Selection stage:** the researcher decides which characteristic must be augmented with a random coefficient or must have its RC distribution made more flexible. Let I_1, the set of indices whose characteristic displays no RC in M_0 and I_2, the set of characteristics with a RC in M_0. For $k = 1, \ldots, K$, the econometrician performs the following regressions:

   - If $k \in I_1$: $\quad \widehat{\xi}_{jt} = \gamma_0 + \sum_{m \in I_1 \setminus \{k\}} \gamma_m x_{m,jt} + \sum_{l=1}^{L} \gamma_l \pi_{j,l}^{k}(z_{jt}) + v_{jt,k}$
   - If $k \in I_2$: $\quad \widehat{\xi}_{jt} = \gamma_0 + \sum_{m \in I_1} \gamma_m x_{m,jt} + \sum_{l=1}^{L} \gamma_l \pi_{j,l}^{k}(z_{jt}) + v_{jt,k}$

   with $\pi_{j,l}^{k}(z_{jt})$ the interval instruments (local or global) defined in section 4 associated with characteristic k. In practice, we construct these instruments by taking points in the domain of f_k. The researcher then derives the coefficient of determination for each regression $R_k^2 = 1 - \frac{\sum_{j,t}(\widehat{v}_{jt,k} - \bar{v}_k)^2}{\sum_{j,t}(\widehat{\xi}_{jt} - \bar{\xi})^2}$ and chooses the characteristic whose associated regression best explains the residual variation of the structural error:

$$k^* = \underset{k \in \{1, \ldots, K_x\}}{Argmax} \ R_k^2$$

3. **Validation stage:** in this stage, we want to test the validity of the reference model M_0 against the alternative specification selected in the previous stage. Namely, we test $H_0 : f \in \mathcal{F}_0$ against $H_1 : f_{k^*} \notin \mathcal{F}_{0k}$ with f_{k^*} being the component of f associated with characteristic k^*.

We conduct this test using the specification test developed in section 5. If H_0 is not rejected or if the model has become too difficult to estimate, then the researcher stops here. Otherwise, the researcher continues to the next stage.

4. We define the new reference model M_1, which corresponds to M_0 augmented with a random coefficient on k^* if $k^* \in I_1$ or an RC with a more flexible distribution if $k^* \in I_2$. Repeat stages 1-3 by setting $M_0 = M_1$.

Our selection procedure alternates between selecting the alternative specification that explains the best the remaining variation in the residuals[31] and a testing stage in which we test the validity of the current model against the selected alternative. In this sense, our procedure resembles forward selection via residuals with a model validity check at each step. The test ensures the validity of the selected model and precludes overfitting, i.e making the distribution of RC too flexible[32]. In addition, our model selection procedure significantly reduces computational time in comparison with alternative selection procedures that require computing risks for all possible models (for instance AIC, BIC, mean squared error of prediction, Mallows CP, GMM-BIC from Andrews (1999), Davidson and MacKinnon (1981) non-nested comparison approach, etc). Our selection procedure only requires estimating the model as many times as the number of times the researcher decides to add flexibility to the model. For instance, if the objective is to know where to include random coefficients, in the worst case, the researcher needs to estimate the model as many times as there are product characteristics.

The selection stage builds on the same intuition as for the detection of misspecification: the structural error contains information about the true distribution of random coefficients and this information can be exploited to test the specification (as we did previously) or to know in which direction additional heterogeneity should be added. Using the same notations as previously, let

[31]Let us remark that in the selection stage, other measures of fit could be used to select the best alternative (e.g.: the adjusted coefficient of determination, the t-test statistic of relevance, the AIC). However, the coefficient of determination works well in practice and has a more straightforward interpretation in our context. See the next paragraph, which summarizes the general idea behind the selection procedure.

[32]This differs greatly from forward selection of control variables in linear models which cannot select the true model in general due to the lack of a specification test, see Smith (2018) for a recent review

M_0 our reference model, I_1 the set of indices with no RC and I_2 the set of indices with an RC. If the true model is M_1, which is M_0 with a misspecified distribution of RC on characteristic k, then following the development in section 3, the structural error writes as follows

- if $k \in I_1$, we have the following decomposition:

$$\begin{aligned}\xi_{jt}(f_0,\beta_0) &= \xi_{jt}(f,\beta) + \sum_{m\in I_1\setminus\{k\}} (\beta_{0m}-\beta_m)x_{m,jt} + \Delta_j(s_t, x_{2t}, (f_0, \delta^k_{\beta_0}), f) \\ &= \sum_{m\in I_1\setminus\{k\}} (\beta_{0m}-\beta_m)x_{m,jt} + \mathbb{E}[\Delta_j(s_t, x_{2t}, (f_0, \delta^k_{\beta_0}), f)|z_{jt}] + u^k_{jt} \quad \text{with} \quad \mathbb{E}[u^k_{jt}|z_{jt}] = 0\end{aligned}$$

  δ_{β_0} is the Dirac probability measure which puts weights 1 on β_0.In section 3, we derived 2 approximations for $\mathbb{E}[\Delta_j(s_t, x_{2t}, (f_0, \delta^k_{\beta_0}), f)|z_{jt}] \approx \sum_{l=1}^{L} \omega_l \pi^k_{j,l}(z_{jt})$. Consequently, under model M_1 a linear regression which explains well $\xi_{jt}(f_0,\beta_0)$ simply writes:

$$\xi_{jt}(f_0,\beta_0) = \sum_{m\in I_1\setminus\{k\}} \gamma_m x_{m,jt} + \sum_{l=1}^{L} \gamma_l \pi^k_{j,l}(z_{jt}) + \tilde{u}^k_{jt}$$

- if $k \in I_2$, we have the following decomposition:

$$\begin{aligned}\xi_{jt}(f_0,\beta_0) &= \xi_{jt}(f,\beta) + \sum_{m\in I_1} (\beta_{0m}-\beta_m)x_{m,jt} + \Delta_j(s_t, x_{2t}, f_0, f) \\ &= \sum_{m\in I_1} (\beta_{0m}-\beta_m)x_{m,jt} + \mathbb{E}[\Delta_j(s_t, x_{2t}, f_0, f)|z_{jt}] + u^k_{jt} \quad \text{with} \quad \mathbb{E}[u^k_{jt}|z_{jt}] = 0\end{aligned}$$

  Consequently, under model M_1 a linear regression which explains well $\xi_{jt}(f_0,\beta_0)$ simply writes:

$$\xi_{jt}(f_0,\beta_0) = \sum_{m\in I_1} \gamma_m x_{m,jt} + \sum_{l=1}^{L} \gamma_l \pi^k_{j,l}(z_{jt}) + \tilde{u}^k_{jt}$$

9 Conclusion

In this paper, we develop a set of tools to gradually and parsimoniously increase the flexibility of the distribution of random coefficients in the demand model for differentiated demand model initiated by Berry et al. (1995). In particular, we construct a new set of instruments that are designed to detect deviations from the true distribution of random coefficients. Building on these instruments, we provide a formal moment-based specification test on the distribution of random coefficients, which allows researchers to test the chosen specification without having to re-estimate the model under a more flexible parametrization. Our instruments are designed to maximize the power of this test when the distribution of RC is misspecified. By exploiting the duality between estimation and testing, we show that these instruments can also improve the estimation of the BLP model under a flexible parametrization. Then, building on our test and our instruments, we develop a simple selection procedure to choose which product characteristics should be augmented with a RC. Our Monte Carlo simulations confirm that the interval instruments we develop in this paper outperform the traditional instruments both for testing and estimation purposes. Finally, we apply these new tools to flexibly estimate the demand for cars in Germany. We show that these tools can be applied to the equally popular mixed logit demand model (with individual level data).

In future work, we plan to see if we can generalize these instruments to other non-linear moment-based models, as well as to the general problem of testing distributional assumptions in structural models. From a broader perspective, our paper is part of an existent discussion on the most effective way to model unobserved preference heterogeneity in structural models. We believe that misspecification is a concern if and only if it affects the counterfactual quantities related to the question the researcher seeks to address. In the case of the BLP demand model, our paper and others show that misspecification in the distribution of random coefficients can substantially distort substitution patterns and, thus, is likely to alter the counterfactual quantities.

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Appendices

A Extension to the mixed logit demand model

The main difference between the BLP demand model and the mixed logit model is that the latter one assumes that the econometrician observes individual data. Let us consider the baseline mixed logit model with no endogeneity and consumer level data[33]. Indirect utility function of consumer i making choice $j \in \{0, 1, \dots, J\}$ is given by:

$$u_{ij} = x'_{1ij}\beta_0 + x'_{2ij}v_i + \varepsilon_{ij}. \tag{11}$$

Where

- ε_{ij} is a preference shock that follows a type I extreme value distribution independent of all other variables and across i, j
- x_{1ij} is a vector of product characteristics interacted with consumer characteristics of size K_1 which display no preference heterogeneity
- x_{2ij} is a vector of product characteristics interacted with consumer characteristics of size K_2 which display preference heterogeneity
- v_i is a vector of random coefficients of size K_2 which jointly follows a joint distribution characterized by a density f

Each consumer chooses the product which maximizes his/her utility in each market. For any couple $(\tilde{f}, \tilde{\beta})$, demand for product j from consumer i writes:

$$\forall j \neq 0, \quad \rho_j(x_i, \tilde{\beta}, \tilde{f}) = \int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{1ij}\tilde{\beta} + x'_{2ij}v)}{1 + \sum_{k=1}^{J} \exp\left\{x'_{1ik}\tilde{\beta} + x'_{2ik}v\right\}} \tilde{f}(v)dv.$$

[33]In the mixed logit case, the absence of endogenous variables here is not an unrealistic assumption as the econometrician can always model unobserved product quality by incorporating product fixed effects into the utility function

For the outside option, we have:

$$\text{for } j=0, \quad \rho_j(x_i,\tilde{\beta},\tilde{f}) = \int_{\mathbb{R}^{K_2}} \frac{1}{1+\sum_{k=1}^J \exp\left\{x'_{1ik}\tilde{\beta}+x'_{2ik}v\right\}}\tilde{f}(v)dv.$$

Structural error. As we did in the case of the BLP demand model, we can define the structural error generated by $(\tilde{\beta},\tilde{f})$ as follows. Let y_{ij} equal to 1 if individual i chooses good $j=0,1,\ldots,J$.

$$\xi_{ij}(\tilde{\beta},\tilde{f}) = y_{ij} - \rho_j(x_i,\tilde{\beta},\tilde{f})$$

By construction, at the true (f,β), we have $\mathbb{E}[\xi_{ij}(\beta,f)|x_i] = \mathbb{E}[y_{ij}|x_i] - \rho_j(x_i,\beta,f) = 0$a.s.;

most powerful instrument and approximations. As in the aggregate demand model, let us see how we can construct instruments to detect misspecification in the distribution of RC. Given that the model displays no endogeneity, the set of exogenous variables is simply x_i. We now want to find the transformation of x_i which provides the most detection power against a wrong distribution. With this objective in mind we consider a situation where the econometrician has a candidate (f_0,β_0) and wants to test that the model is well specified, namely: $H_0:(f,\beta)=(f_0,\beta_0)$. Under an alternative $H_1:(f,\beta)=(f_a,\beta_a)$, the expression for the Most Powerful Instrument (i.e the instrument which maximizes the correlation between the Structural Error and any instrument in the class of measurable functions of x_i) is the same as previously:

$$\begin{aligned}
\mathbb{E}[\Delta_{0,a}^{\xi_j}|x_{ij}] &= \Delta_j(x_i,f_0,\beta_0,f_a,\beta_a) \\
&= \rho_j(x_i,\beta_0,f_0) - \rho_j(x_i,\beta_a,f_a) \\
&= \int_{\mathbb{R}} \rho_j(x_i,\beta_0,f_0) - \frac{\exp(x'_{1ij}\beta_a + x'_{2ij}v)}{1+\sum_{k=1}^J \exp\{x'_{1ik}\beta_a + x'_{2ik}v\}} f_a(v)
\end{aligned}$$

Several remarks are in order. First, contrary to the BLP case, the correction term $\Delta_{0,a}^{\xi_j}$ is a function of the exogenous variables x_i and thus we don't need to estimate its conditional expectation.Second, β_a and f_a are usually unknown to the econometrician and thus we cannot exploit directly this expression. As did for the BLP case, we propose 2 feasible approximations of the MPI.

- **Global approximation:** we replace the unknown β_a by a known substitute β_0[34]. As for the unknown distribution of RC f_a, we proceed as in the BLP case and we replace the integral with a finite sum. Namely, we have:

$$\mathbb{E}[\Delta_j(x_i, f_0, f_a)|x_i] \approx \sum_{l=1}^{L} \omega_l \underbrace{\left[\rho_j(x_i, \beta_0, f_0) - \frac{\exp(x'_{1ij}\beta_0 + x'_{2ij}v_l)}{1 + \sum_{k=1}^{J} \exp\{x'_{1ik}\beta_0 + x'_{2ik}v_l\}}\right]}_{\pi_{j,l}(x_i)}$$

  with $\{v_l\}_{l=1,\dots,L}$ L points chosen in the support of f_a, and ω_l the unknown weights associated with each point

- **Local approximation:** we provide a local approximation which is accurate when f_0 is close to the true density f_a. To derive this local approximation, we need to impose additional restrictions on β_0 and β_a so that $\|\beta_a - \beta_0\| = O\left(\int_{\mathbb{R}^{K_2}} |f_0(v) - f_a(v)|dv\right)$

  Assumption 1. We assume that $\beta_0 = \beta_0^*$ and $\beta_a = \beta_a^*$ where (β_0^*, β_a^*) are both pseudo true values which maximize the conditional expectation of their respective population log-likelihoods. Namely,

$$\beta_0^* = \underset{\beta \in \mathbb{R}^{K_1}}{\operatorname{argmax}}\ \mathbb{E}\big[L(x_i, y_i, \beta, f_0)\big|x_i\big] \text{ with } \ L(x_i, y_i, \beta, f_0) = \sum_{j=0}^{J} \mathbf{1}\{y_{ij} = 1\} \log(\rho_j(x_i, \beta, f_0))$$

$$\beta_a^* = \underset{\beta \in \mathbb{R}^{K_1}}{\operatorname{argmax}}\ \mathbb{E}\big[L(x_i, y_i, \beta, f_a)\big|x_i\big] \text{ with } \ L(x_i, y_i, \beta, f_a) = \sum_{j=0}^{J} \mathbf{1}\{y_{ij} = 1\} \log(\rho_j(x_i, \beta, f_a))$$

  Now we can derive the following first order approximation of the $\Delta_j(x_i, f_0, \beta_0, f_a, \beta_a)$

  Proposition 7.
  Under Assumption1, a first order expansion of $\Delta_j(x_i, f_0, \beta_0, f_a, \beta_a)$ around f_0 writes:

$$\begin{aligned}\Delta_j(x_i, f_0, \beta_0, f_a, \beta_a) &= g_j(x_i, \beta_0, f_0) - g_j(x_i, \beta_a, f_a)\\ &= \int_{\mathbb{R}^{K_2}} \frac{exp(x'_{1ij}\beta_0 + x'_{2ij}v)}{1 + \sum_{k=1}^{J} \exp\{x'_{1ik}\beta_0 + x'_{2ik}v\}}(f_0(v) - f_a(v))dv + \frac{\partial \rho_j(x_i, \beta, f_a)}{\partial \beta}\bigg|_{\beta=\beta_0}(\beta_1 - \beta_0) + \mathcal{R}_0\end{aligned}$$

  with $\mathcal{R}_0 = \int_{\mathbb{R}^{K_2}} |f_0(v) - f_a(v)|dv$

[34] in simulations, we find that the homogeneous parameters are usually close to each other even when the distributions are somewhat remote from each other

The proof is in section B. Building on this approximation, we can construct the following local feasible approximation of the MPI:

$$\mathbb{E}[\Delta_j(x_i, f_0, f_a)|x_i] \approx \sum_{l=1}^{L} \bar{\omega}_{1l} \underbrace{\left[\rho_j(x_i, \beta_0, f_0) - \frac{\exp(x'_{1ij}\beta_0 + x'_{2ij}v_l)}{1 + \sum_{k=1}^{J} \exp\{x'_{1ik}\beta_0 + x'_{2ik}v_l\}}\right]}_{\bar{\pi}_{1,j,l}(x_i)}$$

$$+ \sum_{l=1}^{L} \bar{\omega}_{2l} \underbrace{\frac{\partial}{\partial \beta}\left\{\frac{exp(x'_{1ij}\beta_0 + x'_{2ij}v_l)}{1 + \sum_{k=1}^{J} \exp\{x'_{1ik}\beta_0 + x'_{2ik}v_l\}}\right\}}_{\bar{\pi}_{2,j,l}(x_i)}$$

with $\{v_l\}_{l=1,\ldots,L}$ L points chosen in the support of f_a, and $\bar{\omega}_l$ the unknown weights associated with each point. The interval instruments are simply the set $(\bar{\pi}_{1,j,l}(x_i), \bar{\pi}_{2,j,l}(x_i))$.

Specification test. TBD

B Proofs

B.1 Identification

In this subsection we prove that under assumptions in A, the distribution of random coefficients f is point identified.

B.1.1 Proof of Proposition 1

For purposes of identification, we assume that the true distribution of the data is fully observed and is generated by the demand model introduced in section 2.

$$P(s_t, x_{2t}, x_{1t}, z_t)$$

Assume that the data is generated by the couple (f, β) and by distribution of random variables $P(\xi_t, x_{2t}, x_{1t}, z_t)$ which satisfy the assumptions in A. We want to show the following implication:

$$\begin{aligned}(\tilde{f}, \tilde{\beta}) = (f, \beta) \iff \quad & \mathbb{E}[\xi_{jt}(\tilde{f}, \tilde{\beta})|z_{jt}] = 0 \ a.s. \\ \iff \quad & \mathbb{E}\left[\rho_j^{-1}(s_t, x_{2t}, \tilde{f}) - x'_{1jt}\tilde{\beta}\middle| z_{jt}\right] = 0 \ a.s.;\end{aligned}$$

First let us show that the standard exogeneity condition assumed in Berry and Haile (2014) and in Wang (2021) implies the moment condition we utilize in this paper:

By construction, we can rewrite the exogeneity condition A (i) as follows:

$$\mathbb{E}[\xi_{jt}|z_{jt}] = \sum_{k=1}^{J} \Pr(j=k)\mathbb{E}\left[\xi_{jt}\middle|z_{jt}, j=k\right] = \frac{1}{J}\sum_{k=1}^{J} \mathbb{E}\left[\xi_{jt}\middle|z_{jt}, j=k\right]$$

The exogeneity condition in Wang (2021) assumes: $\forall k, \quad \mathbb{E}\left[\xi_{jt}\middle|z_{jt}, j=k\right] = 0$. From what precedes, this condition implies the exogeneity condition $\mathbb{E}[\xi_{jt}|z_{jt}] = 0\ a.s.$ in A (i). This assumption is required for non-parametric identification of the demand functions but not for the non-parametric identification of the distribution of RC.

Now let us prove the identification result. As an artifact for our proof, let us consider a new indexation, which is done exogenously across markets. We denote j' the exogenous indices. Consequently, a same product j doesn't necessarily have the same indices across markets. As the new indexation is done exogenously, we have for any j':

$$\mathbb{E}[\xi_{jt}(\tilde{f},\tilde{\beta})|z_{jt}] = \mathbb{E}[\xi_{jt}(\tilde{f},\tilde{\beta})|z_{jt}, j \to j'] \equiv \mathbb{E}_{j'}[\xi_{j't}(\tilde{f},\tilde{\beta})|z_{j't}]a.s.$$

$j \to j'$ indicates index j has been changed into j'. Consequently, we have:

$$\mathbb{E}[\xi_{jt}(\tilde{f},\tilde{\beta})|z_{jt}] = 0\ a.s. \iff \forall j' \ \ \mathbb{E}_{j'}[\xi_{j't}(\tilde{f},\tilde{\beta})|z_{j't}] = 0\ a.s.$$

As a consequence, we can rewrite the initial equivalence as follows:

$$(\tilde{f},\tilde{\beta}) = (f,\beta) \iff \quad \forall j', \ \ \mathbb{E}_{j'}[\xi_{j't}(\tilde{f},\tilde{\beta})|z_{j't}] = 0\ a.s.$$

Given the random permutation $j \to j'$, which is market dependent, we must redefine our matrices and vectors as follows: $\hat{x}_t = M_t x_t$ with $(M_t)_{i,k} = \mathbf{1}\{i = j_t, k = j'_t\}$. Likewise $\hat{s}_t = M_t s_t$. M_t is a random matrix. It is straight forward to show the direct implication.

$$(\tilde{f},\tilde{\beta}) = (f,\beta) \implies \quad \forall j', \qquad \mathbb{E}_{j'}\left[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - x'_{1j't}\tilde{\beta}\middle|z_{j't}\right] = \mathbb{E}_{j'}[\xi_{j't}(f,\beta)|z_{j't}] = 0\ a.s.$$

The reverse implication is much more intricate to prove and we will exploit other results in the literature. We want to show:

$$(\tilde{f},\tilde{\beta}) \neq (f,\beta) \implies \quad \exists j' \quad \Big| \quad \mathop{\mathbb{E}}_{j'}\left[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - \hat{x}'_{1jt}\tilde{\beta}\middle| z_{j't}\right] = 0 \text{ a.s. does not hold}$$

First, let us assume that $\tilde{f} = f$ and $\tilde{\beta} \neq \beta$, then we have:

$$\rho^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - \hat{x}_{1t}\tilde{\beta} = \underbrace{\rho^{-1}(\hat{s}_t,\hat{x}_{2t},f) - x_{1t}\beta}_{\hat{\xi}_t(f,\beta)} + \hat{x}_{1t}(\beta - \tilde{\beta})$$

By assumption, we have: $P(x'_{1t}x_{1t} \;\; dp) > 0$. M_t is symmetric, idempotent and full rank. As a consequence,

$$P(\hat{x}'_{1t}\hat{x}_{1t} \;\; dp) = P(x'_{1t}M_t x_{1t} \;\; dp) = P(x'_{1t}x_{1t} \;\; dp) > 0$$

Therefore, we have $\forall \; \gamma \neq 0 \in \mathbb{R}^K$,

$$\begin{aligned} P(\gamma'\hat{x}'_{1t}\hat{x}_{1t}\gamma > 0) > P(\hat{x}'_{1t}\hat{x}_{1t} \;\; dp) > 0 &\iff P(\|\hat{x}_{1t}\gamma\|^2 > 0) > 0 \\ &\iff P(\hat{x}_{1t}\gamma \neq 0) > 0 \end{aligned}$$

Thus, $\exists j' \mid x'_{1j't}(\beta - \tilde{\beta}) = 0 \; a.s$ does not hold. To conclude, there exists j' such that:

$$\mathop{\mathbb{E}}_{j'}[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},f) - x'_{1j't}\tilde{\beta}|z_{j't}] = \underbrace{\mathop{\mathbb{E}}_{j'}[\xi_{j't}(f,\beta)|z_{j't}]}_{=0} + \underbrace{\mathop{\mathbb{E}}_{j'}[x'_{1j't}(\beta - \tilde{\beta})|z_{j't}]}_{=\,0\;a.s\text{ does not hold from the completeness}}$$

Now let us assume that $\tilde{f} \neq f$ and we want to show that $\forall \tilde{\beta} \in \mathbb{R}^k$, $\exists j'$ such that:

$$\mathop{\mathbb{E}}_{j'}\left[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - x'_{1j't}\tilde{\beta}\middle| z_{jt}\right] = 0 \; a.s \;\; \text{does not hold}$$

First note that $\forall j'$,

$$\mathop{\mathbb{E}}_{j'}\big[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - x'_{1j't}\tilde{\beta}\big|z_{j't}\big] = \underbrace{\mathop{\mathbb{E}}_{j'}\big[\xi_{j't}(f,\beta)\big|z_{j't}\big]}_{=0} + \mathop{\mathbb{E}}_{j'}\big[\rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},\tilde{f}) - \rho_{j'}^{-1}(\hat{s}_t,\hat{x}_{2t},f) - x'_{1j't}(\tilde{\beta}-\beta)\big|z_{j't}\big]$$

Thus, we need to show that $\exists j' \left| \underset{j'}{\mathbb{E}}\left[\rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) - \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, f) - x'_{1j't}(\tilde{\beta} - \beta)\right] = 0 \right.$ a.s doesn't hold. From the completeness condition, a sufficient condition is: $\exists j' \left| \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) - \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, f) - x'_{1j't}(\tilde{\beta} - \beta) = 0 \right.$ $a.s$ does not hold. Let $\gamma = (\tilde{\beta} - \beta)$.

By contradiction, it can be easily be shown that $\rho(\hat{\delta}_t, \hat{x}_{2t}, f) - \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, \tilde{f}) \neq 0 \implies \exists j' \ \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) \neq \rho_j^{-1}(\hat{s}_t, \hat{x}_{2t}, f) + \gamma' x_{1j't}$. Indeed, assume that $\rho(\hat{\delta}_t, \hat{x}_{2t}, f) - \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, \tilde{f}) \neq 0$ and $\forall j' \ \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) = \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, f) + \gamma' x_{1j't}$. Then, we have: $\rho(\rho^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}), \hat{x}_{2t}, \tilde{f}) = \rho(\rho^{-1}(\hat{s}_t, \hat{x}_{2t}, f) + \hat{x}_{1t}\gamma, \hat{x}_{2t}, \tilde{f}) = \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, \tilde{f}) \neq \rho(\hat{\delta}_t, \hat{x}_{2t}, f) = \hat{s}_t$. Therefore, we have a contradiction.

Thus, the next step is to show that $\forall \gamma, \ \tilde{f} \neq f \implies \rho(\hat{\delta}_t, \hat{x}_{2t}, f_0) - \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, f) = 0$ a.s does not hold.

To this end, we are going to exploit the identification result shown by Wang (2021). Following the notations in this paper, we define $\mu_i = \hat{x}_{1t}\Gamma + \hat{x}_{2t}v_i = \hat{x}_t\mathbf{v}$ with $\mathbf{v}_i = (\Gamma, v_i)$. Here Γ is a degenerate random variable characterized by constant c such that $P(\Gamma = c) = 1$. Let $G_{\mu|\hat{x}_t}$ the distribution of $\mu_i|\hat{x}_t$ under $f^\dagger = (c = 0, f)$ and $G_{\tilde{\mu}|\hat{x}_t}$ the distribution of $\mu_i|\hat{x}_t$ under $\tilde{f}^\dagger = (c = \gamma, \tilde{f})$. The following result is shown in Wang (2021): for any $\hat{\bar{x}}_t \in Supp(\hat{x}_t)$

$$\exists j' \mid \rho_{j'}(\hat{\delta}_t, G_{\mu|\hat{\bar{x}}_t}) - \rho_{j'}(\hat{\delta}_t, G_{\tilde{\mu}|\hat{\bar{x}}_t}) = 0 \text{ on open set } \mathcal{D} \subset \mathbb{R}^J \implies G_{\mu|\hat{\bar{x}}_t} = G_{\tilde{\mu}|\hat{\bar{x}}_t}$$

Note that thanks to the real analytic property of the demand functions ρ, Wang (2021) does not require a full support assumption on $\hat{\delta}_t$

Fix the value of $\hat{x}_t$ as follows: $\hat{x}_t = \bar{M}_t\bar{x}_t = \hat{\bar{x}}_t$.By assumption, there exists $\bar{x}_t \in Supp(x_t)$ such that $\bar{x}_t'\bar{x}_t$ is dp and $\delta_t = \bar{x}_{1t}\beta + \xi_t$ varies on an open set $\bar{\mathcal{D}}$ almost surely.These properties naturally transmit to $\hat{\bar{x}}_t$. The chosen permutation $\bar{M}_t$ doesn't matter. Given the result in Wang (2021), in order to prove that $\rho(\hat{\delta}_t, \hat{x}_{2t}, f_0) - \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, f) = 0$ $a.s$ does not hold, we just need to prove that $\forall \gamma, \quad \tilde{f} \neq f \implies G_{\tilde{\mu}|\hat{\bar{x}}_t} \neq G_{\mu|\hat{\bar{x}}_t}$. As the density functions are assumed to be continuous, $\tilde{f} \neq f \implies \exists v^* \in \mathbb{R}^{K_2} \ \ \tilde{F}(v^*) \neq F(v^*)$. Take $x^* = (0_{K_1}, \hat{\bar{x}}_{2t}v^*)' = \hat{\bar{x}}_t(0_{K_1}, v^*)'$:

$$G_{\mu|\hat{\hat{x}}_t}(x^*) = P(x_t \mathbf{v}_i \leq x^* | x_t = \hat{\hat{x}}_t) = P((x_t'x_t)^{-1}x_t'x_t\mathbf{v}_i \leq (x_t'x_t)^{-1}x_t'\bar{x}_t(0_{K_1}, v^*)' | x_t = \hat{\hat{x}}_t)$$
$$= (1_{K_1}, P(v_i \leq v^* | x_t = \hat{\hat{x}}_t))' = (1_{K_1}, F(v^*))'$$

The last equality comes from independence of v_i and x_t. Likewise, $G_{\tilde{\mu}|\hat{\hat{x}}_t}(x^*) = (1\{\gamma > 0\}, \tilde{F}(v^*))'$

Therefore, $\exists x^*,\ \forall \gamma\ \ G_{\tilde{\mu}|\hat{\hat{x}}_t}(x^*) \neq G_{\mu|\hat{\hat{x}}_t}(x^*)$. Following the result in Wang (2021), we have that for all $\gamma \in \mathbb{R}^{K_1}$, $\rho(\hat{\delta}_t, \hat{x}_{2t}, f) - \rho(\hat{\delta}_t + \hat{x}_{1t}\gamma, \hat{x}_{2t}, \tilde{f}) = 0\ a.s$ does not hold which in turn implies that for all $\gamma \in \mathbb{R}^{K_1}$, $\exists j'\ \ \rho_j^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) - \rho_j^{-1}(\hat{s}_t, \hat{x}_{2t}, f) + \hat{x}_{1jt}'\gamma = 0\ a.s$ does not hold.

To conclude: $\forall \beta \in \mathbb{R}^k$, there exists j' such that:

$$\rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, \tilde{f}) - \rho_{j'}^{-1}(\hat{s}_t, \hat{x}_{2t}, f) - x_{1j't}'(\tilde{\beta} - \beta) = 0\ a.s\ \text{ does not hold}$$

which is what we wanted to show.

B.1.2 Proof of corollary 5.1

Let us assume that specification $\mathcal{F}_0$, instruments $h_E(z_{jt})$ and weighting matrix yields a unique pseudo true value θ_0.

$$\theta_0 = \underset{\tilde{\theta}}{Argmin}\ \mathbb{E}[\xi_{jt}(f_0(.|\tilde{\lambda}, \tilde{\theta})h_E(z_{jt})']W\mathbb{E}[h_E(z_{jt})\xi_{jt}(f_0(.|\tilde{\lambda}, \tilde{\theta})]$$

Under $H_0 : f \in \mathcal{F}_0$ and $f = f_0(\cdot|\lambda)$. By the mean independence assumption on the unobserved quality ξ_{jt}, we have at the true $\theta = (\beta, \lambda)$:

$$\xi_{jt}(f_0(.|\lambda), \beta) = \rho_j^{-1}(s_t, x_{2t}, f_0(.|\lambda)) - x_{1jt}'\beta = \xi_{jt} \implies \mathbb{E}[\big(\xi_{jt}(f_0(.|\lambda), \beta)h_E(z_{jt})] = 0$$

Thus, θ is solution to the previous minimization problem and as the solution is unique: $\theta_0 = \theta$. As a consequence, $\xi_{jt}(f_0(.|\lambda_0), \beta_0)) = \xi_{jt}$ and $\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0), \beta_0)|z_{jt}] = 0\ as$

Under an alternative specification: $f \notin \mathcal{F}_0$, we know from the identification proof that $\forall \tilde{\theta} = (\tilde{\beta}, \tilde{\lambda})$,

$$\mathbb{E}\left[\rho_j^{-1}(s_t, x_{2t}, f_0(.|\tilde{\lambda})) - x_{1jt}'\tilde{\beta}\middle| z_{jt}\right] = 0\ a.s\ \text{ does not hold}$$

In particular, the last equation holds for $\tilde{\theta} = \theta_0$

B.2 Detecting misspecification: the most powerful instrument

Proof of Proposition 2.

- Under $\overline{H}_0 : (f,\beta) = (f_0,\beta_0)$. By assumption, the data is i.i.d. across markets, $\mathbb{E}[\|\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\|^2] = \frac{1}{J}\mathbb{E}[\sum_j \|\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\|^2] < +\infty$, the CLT applies:

$$\frac{\sqrt{T}}{TJ}\sum_{j,t} h_D(z_{jt})\xi_{jt}(f_0,\beta_0) = \frac{\sqrt{T}}{TJ}\sum_{j,t} h_D(z_{jt})\xi_{jt} \underset{T\to+\infty}{\longrightarrow} \mathcal{N}(0,\tilde{\Omega}_0).$$

with:

$$\begin{aligned}
\tilde{\Omega}_0 &= \mathbb{E}\left[\left(\frac{1}{J}\sum_{j=1}^{J} h_D(z_{jt})\xi_{jt}\right)\left(\frac{1}{J}\sum_{j=1}^{J} h_D(z_{jt})\xi_{jt}\right)'\right] \\
&= \frac{1}{J^2}\mathbb{E}\left[\sum_{j=1}^{J} h_D(z_{jt})h_D(z_{jt})'\xi_{jt}^2 + \sum_{j=1}^{J}\sum_{k\neq j} h_D(z_{jt})h_D(z_{kt})'\xi_{jt}\xi_{kt}\right] \\
&= \frac{1}{J^2}\mathbb{E}\left[\sum_{j=1}^{J} h_D(z_{jt})h_D(z_{jt})'\xi_{jt}^2\right] + \frac{1}{J^2}\sum_{j=1}^{J}\sum_{k\neq j}\mathbb{E}\left[h_D(z_{jt})h_D(z_{kt})'\underbrace{\mathbb{E}[\xi_{jt}\xi_{kt}|z_{jt},z_{kt}]}_{=0}\right] \\
&= \frac{1}{J}\mathbb{E}\left[h_D(z_{jt})h_D(z_{jt})'\xi_{jt}^2\right] \\
&= \frac{1}{J}\Omega_0.
\end{aligned}$$

Third line comes from $\xi_{jt} \perp\!\!\!\perp \xi_{kt}|z_t$. By assumption, Ω_0 is full rank. Thus, we have:

$$\begin{aligned}
S_T(h_D,f_0,\beta_0) &= TJ\left(\frac{1}{TJ}\sum_{j,t}\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\right)'\hat{\Omega}_0^{-1}\left(\frac{1}{TJ}\sum_{j,t}\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\right) \\
&= T\left(\frac{1}{TJ}\sum_{j,t}\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\right)'\hat{\tilde{\Omega}}_0^{-1}\left(\frac{1}{TJ}\sum_{j,t}\xi_{jt}(f_0,\beta_0)h_D(z_{jt})\right) \underset{T\to+\infty}{\overset{d}{\longrightarrow}} \chi^2_{|h_D|_0}.
\end{aligned}$$

- Under H'_a : $\mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right] \neq 0$. The data is i.i.d. across markets, by the law of large numbers: $\frac{1}{TJ}\sum_{j,t} h_D(z_{jt})\xi_{jt}(f_0,\beta_0) \overset{P}{\to} \mathbb{E}\left[\frac{1}{J}\sum_j h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right]$. It follows by the continuous mapping theorem:

$$
\begin{aligned}
\frac{S_T(h_D,f_0,\beta_0)}{T} &\xrightarrow{P} J\mathbb{E}\left[\frac{1}{J}\sum_j h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right]'\Omega_0^{-1}\mathbb{E}\left[\frac{1}{J}\sum_j h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right]\\
&= J\underbrace{\mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right]'\Omega_0^{-1}\mathbb{E}\left[h_D(z_{jt})\xi_{jt}(f_0,\beta_0)\right]}_{\kappa(h_D,f_0,\beta_0)}
\end{aligned}
$$

Under H_a', $\kappa(h_D,f_0,\beta_0)$ is strictly positive because Ω_0 is positive definite. Thence,

$$
\begin{aligned}
\forall q\in\mathbb{R}\quad \underset{T\to\infty}{lim}\ \mathbb{P}(S_T(h_D,f_0,\beta_0)>q) &= \underset{T\to\infty}{lim}\ \mathbb{P}\left(\frac{S(h_D,f_0,\beta_0)-q}{T}>0\right)\\
&= \mathbb{P}(J\kappa(h_D,f_0,\beta_0)>0)\\
&= 1.
\end{aligned}
$$

where the 2nd equality holds because convergence in probability implies convergence in distribution.

□

Proof of Proposition 3. To shorten notations, let $\xi_{jt0}\equiv\xi_{jt}(f_0(\cdot|\lambda_0),\beta_0)$, $\xi_{jta}\equiv\xi_{jt}(f_a,\beta_a)$ and ξ_{t0} and ξ_{ta} their stacked versions over j. Likewise, we define $h_D(z_t)=(h_D(z_{1t}),...,h_D(z_{Jt}))'$ The asymptotic slope of the test writes:

$$
\begin{aligned}
c_{h_D}(f_a,\beta_a) &= \mathbb{E}\left(\sum_j \xi_{jt0}h_D(z_{jt})\right)'\mathbb{E}\left(\left(\sum_j \xi_{jt0}h_D(z_{jt})\right)\left(\sum_{j'} \xi_{j't0}h_D(z_{j't})\right)'\right)^{-1}\mathbb{E}\left(\sum_j \xi_{jt0}h_D(z_{jt})\right)\\
&= \mathbb{E}(\xi_{t0}'h_D(z_t))\mathbb{E}(h_D(z_t)'\xi_{t0}\xi_{t0}'h_D(z_t))^{-1}\mathbb{E}(h_D(z_t)'\xi_{t0})\\
&= \mathbb{E}({\Delta_{0,a}^{\xi_t}}'h_D(z_t))\mathbb{E}(h_D(z_t)'\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t)h_D(z_t))^{-1}\mathbb{E}(h_D(z_t)'\Delta_{0,a}^{\xi_t})
\end{aligned}
$$

Third line comes from $\mathbb{E}({\Delta_{0,a}^{\xi_t}}'h_D(z_t))=\mathbb{E}((\xi_{t0}-\xi_{ta})'h_D(z_t))=\mathbb{E}(\xi_{t0}'h_D(z_t))$ because ξ_{ta} is the true structural error. Then the slope of the test taking $h_D^*=\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t)^{-1}\mathbb{E}(\Delta_{0,a}^{\xi_t}|z_t)$ is equal to:

$$
c_{h_D^*}(f_a,\beta_a)=\mathbb{E}\left(\mathbb{E}(\Delta_{0,a}^{\xi_t}|z_t)'\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t)^{-1}\mathbb{E}(\Delta_{0,a}^{\xi_t}|z_t)\right)
$$

To finish the proof, we must show that for any set of instruments h_D, we have: $c_{h_D^*}(f_a,\beta_a)\geq c_{h_D}(f_a,\beta_a)$.

Denote $\tilde{h}_D(z_t)=\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t)^{1/2}h_D(z_t)$ and $\tilde{h}_D^*(z_t)=\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t)^{1/2}h_D^*(z_t)$.With these new notations, we have:

$$
\begin{aligned}
c_{h_D^*}(f_a,\beta_a)-c_{h_D}(f_a,\beta_a) &= \mathbb{E}\left(\tilde{h}_D^*(z_t)'\tilde{h}_D^*(z_t)\right)-\mathbb{E}\left(\tilde{h}_D^*(z_t)'\tilde{h}_D(z_t)\right)\mathbb{E}\left(\tilde{h}_D(z_t)'\tilde{h}_D(z_t)\right)^{-1}\mathbb{E}\left(\tilde{h}_D(z_t)'h_D^*(z_t)\right)\\
&= G'\begin{pmatrix}\mathbb{E}\left(\tilde{h}_D^*(z_t)'\tilde{h}_D^*(z_t)\right) & \mathbb{E}\left(\tilde{h}_D^*(z_t)'\tilde{h}_D(z_t)\right)\\ \mathbb{E}\left(\tilde{h}_D(z_t)'\tilde{h}_D^*(z_t)\right) & \mathbb{E}\left(\tilde{h}_D(z_t)'\tilde{h}_D(z_t)\right)\end{pmatrix}G\\
&= G'\mathbb{E}\left(\tilde{H}\tilde{H}'\right)G\geq 0
\end{aligned}
$$

with $\tilde{H} = (\tilde{h}^*_D(z_t), \tilde{h}_D(z_t))'$ and $G = \left(1, -\mathbb{E}\left(\tilde{h}^*_D(z_t)'\tilde{h}_D(z_t)\right)\mathbb{E}\left(\tilde{h}_D(z_t)'\tilde{h}_D(z_t)\right)^{-1}\right)'$

$\square$

Proof of Proposition 4.

From corollary 5.1. Under Assumption A,

$$\begin{aligned}
H_a :\ f \notin \mathcal{F}_0 \implies & \quad \mathbb{E}[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)|z_{jt}] \neq 0 \ \ a.s \\
\implies & \quad \mathbb{E}[\xi_{jt}(f_0(.|\lambda_0), \beta_0)|z_{jt}]^2 > 0 \ a.s \\
\implies & \quad \mathbb{E}\big[\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0), \beta_0)|z_{jt}]^2\big] > 0 \\
\implies & \quad \mathbb{E}\big[\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0), \beta_0)\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0)|z_{jt}]|z_{jt}]\big] > 0 \\
\implies & \quad \mathbb{E}\big[\xi_{jt}(f_0(.|\lambda_0), \beta_0)\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0)|z_{jt}]\big] > 0 \\
\implies & \ \forall \alpha \neq 0 \ \ H_1' :\ \mathbb{E}\big[\xi_{jt}(f_0(.|\lambda_0), \beta_0)\underbrace{\alpha\mathbb{E}[\Delta^{\xi_{jt}}_{0,a}|z_{jt}]}_{h^*_D(z_{jt})}\big] \neq 0
\end{aligned}$$

From theorem 5.2, under assumptions B-E,

$$H_a \implies \forall q \in \mathbb{R}^+, \ \ \mathbb{P}(S(h^*_D, \mathcal{F}_0, \hat{\theta}) > q) \to 1$$

$\square$

Proof of Proposition 5: Correlation between the MPI and the structural error.

Let $\mathcal{H}$ the set of measurable functions of z_{jt}, we want to show under $\bar{H}_a$:

$$\forall \alpha \in \mathbb{R}^*, \ \ \alpha\mathbb{E}[\Delta^{\xi_{jt}}_{0,a}|z_{jt}] \in \arg\max_{h \in \mathcal{H}} \text{corr}(\xi_{jt}(f_0, \beta_0), h(z_{jt}))$$

We proceed in 2 steps. First, we derive the upper bound by showing that for any $h \in \mathcal{H}$, we have:

$$\text{corr}\left(\xi_{jt}(f_0, \beta_0), h(z_{jt})\right) \leq \sqrt{\frac{\text{var}\left(\mathbb{E}[\Delta^{\xi_{jt}}_{0,a}|z_{jt}]\right)}{\text{var}(\xi_{jt}(f_0, \beta_0))}}$$

To do so, we use the definition of the conditional expectation and the Cauchy Schwarz inequality. First notice that we have: $\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}] = \mathbb{E}[\xi_{jt}(f_0, \beta_0)|z_{jt}]$. By definition of the conditional expectation, we have for any $h \in \mathcal{H}$,

$$\mathbb{E}[h(z_{jt})\xi_{jt}(f_0, \beta_0)] = \mathbb{E}[h(z_{jt})\mathbb{E}[\xi_{jt}(f_0, \beta_0)|z_{jt}]]$$

It follows that:

$$|\text{cov}\,(h(z_{jt}), \xi_{jt}(f_0, \beta_0))| = \text{cov}\,(h(z_{jt}), \mathbb{E}[\xi_{jt}(f_0, \beta_0)|z_{jt}]) \leq \sqrt{\text{var}(h(z_{jt}))\text{var}\,(\mathbb{E}[\xi_{jt}(f_0, \beta_0)|z_{jt}])}$$

The inequality comes from the Cauchy Schwarz inequality. The result follows by using the definition of the correlation coefficient.

Second, we show that the upper bound is reached by taking for any $\alpha \in \mathbb{R}^*$, $h^*(z_{jt}) = \alpha\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]$.

$$\begin{aligned} \text{cov}\left(\xi_{jt}(f_0, \beta_0), \alpha\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]\right) &= \alpha\ \text{cov}\left(\Delta_{0,a}^{\xi_{jt}}, \mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]\right) \\ &= \alpha\ \text{var}\left(\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]\right) \end{aligned}$$

Consequently,

$$\text{corr}\,(\xi_{jt}(f_0, \beta_0), h^*(z_{jt})) = \frac{\alpha}{\sqrt{\alpha^2}}\sqrt{\frac{\text{var}\left(\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]\right)}{\text{var}(\xi_{jt}(f_0, \beta_0))}} \implies |\text{corr}\,(\xi_{jt}(f_0, \beta_0), h^*(z_{jt}))| = \sqrt{\frac{\text{var}\left(\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]\right)}{\text{var}(\xi_{jt}(f_0, \beta_0))}}$$

□

B.2.1 Connection with optimal instruments

In the parametric case, the BLP parameter θ is identified by the following non-linear conditional moment restriction $\mathbb{E}[\xi_{jt}(\theta)|z_{jt}] = 0$. The derivation of the optimal instruments in this context has been studied by Amemiya (1974). For an arbitrary choice of $h_E(z_{jt})$, the GMM estimator with the 2-step efficient weighting matrix has the following asymptotic distribution:

$$\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}\left(0, (\Gamma(\mathcal{F}_0, \theta, h_E)'\Omega(\mathcal{F}_0, h_E)^{-1}\Gamma(\mathcal{F}_0, \theta, h_E))^{-1}\right)$$

with the same notations as previously:

$$\Omega(\mathcal{F}_0, h_E) = \mathbb{E}\Big[\Big(\sum_j \xi_{jt}(\theta) h_E(z_{jt})\Big)\Big(\sum_j h_E(z_{jt})\xi_{jt}(\theta)\Big)'\Big]$$

$$\Gamma(\mathcal{F}_0, \theta_0, h_E) = \mathbb{E}\left[\sum_j h_E(z_{jt})\frac{\partial \xi_{jt}(\theta)}{\partial \tilde{\theta}'}\right]$$

For the sake of exposition, we will assume that unobserved demand shock ξ_{jt} is independent across observations, namely: $\mathbb{E}\left[\xi_{jt}(\theta)\xi_{j't}(\theta)|z_t\right] = 0$ for $j \neq j'$. The general case extends naturally. The optimal instrument $h_E^*(z_{jt})$ are chosen to minimize the asymptotic variance covariance matrix. We derive the form of the optimal instruments in the context of BLP by adapting well known results in Chamberlain (1987) and Amemiya (1974)

Lemma 2.1. Optimal instruments in the BLP model

In our setting and assuming $f \in \mathcal{F}_0$, the optimal instruments $h_E^*(z_{jt})$ write:

$$h_E^*(z_{jt}) = \mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]^{-1}\mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \theta}\bigg| z_{jt}\right]$$

and the corresponding efficiency bound (obtained by setting $h_E = h_E^*$) writes:

$$V^* = \mathbb{E}\left[\sum_j \mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \theta}\bigg| z_{jt}\right]\mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \theta}\bigg| z_{jt}\right]'\mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]^{-1}\right]^{-1}$$

Proof. To shorten the notations, we denote: $\sigma^2(z_{jt}) = \mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]$ and $d(z_{jt}) = \mathbb{E}\left[\frac{\partial \xi_{jt}(\theta)}{\partial \theta}\Big| z_{jt}\right]$. Likewise, we define

$$\Omega_0(h_E) = \mathbb{E}\Big[\sum_j \mathbb{E}[\xi_{jt}(\theta)^2|z_{jt}]h_E(z_{jt})h_E(z_{jt})'\Big]$$

We want to prove that for any set of instruments $h_E(z_{jt})$ that $V^*(z_{jt}) - \Gamma_0(h_E)'\Omega_0(h_E)^{-1}\Gamma_0(h_E)$ matrix is semi definite positive.

$$V^*(z_{jt}) - \Gamma_0(h_E)\Omega_0(h_E)^{-1}\Gamma_0(h_E)' =$$
$$= \mathbb{E}\Big[\sum_j d(z_{jt})d(z_{jt})'\sigma^2(z_{jt})\Big] - \mathbb{E}\left[\sum_j \frac{\partial \xi_{jt}(\theta)}{\partial \theta}h_E(z_{jt})'\right]\Omega_0(h_E)^{-1}\mathbb{E}\left[\sum_j \frac{h_E(z_{jt})\partial \xi_{jt}(\theta)'}{\partial \theta}\right]$$

$$= \mathbb{E}\Big[\sum_j d(z_{jt})d(z_{jt})'\sigma^{-2}(z_{jt})\Big] - \mathbb{E}\left[\sum_j d(z_{jt})h_E(z_{jt})'\right]\mathbb{E}\Big[\sum_j \sigma^2(z_{jt})h_E(z_{jt})h_E(z_{jt})'\Big]\mathbb{E}\left[\sum_j h_E(z_{jt})d(z_{jt})'\right]$$

$$= \mathbb{E}\Big[\tilde{\mathbf{D}}(\mathbf{z_{jt}})'\tilde{\mathbf{D}}(\mathbf{z_{jt}})\Big] - \mathbb{E}\Big[\tilde{\mathbf{D}}(\mathbf{z_{jt}})'\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})\Big]\mathbb{E}\Big[\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})'\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})\Big]^{-1}\mathbb{E}\left[\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})'\tilde{\mathbf{D}}(\mathbf{z_{jt}})\right]$$

Second line comes from law of iterated expectations. Third line is a matricial way to rewrite the second line. $\tilde{\mathbf{D}}(\mathbf{z_{jt}})$ a matrix which stacks $d(z_{jt})/\sigma(z_{jt})$ over the set of products (each line corresponds to one product j). Likewise, let $\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})$ a matrix which stacks $h_E(z_{jt})\sigma(z_{jt})$ over the set of products (each line corresponds to one product j). Now let us define the following matrices.

$$\tilde{\mathbf{X}} = \begin{pmatrix}\tilde{\mathbf{D}}(\mathbf{z_{jt}}) & \tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})\end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{M}} = \begin{pmatrix}\mathbf{I}_{|\theta_0|} & -\mathbb{E}\Big[\tilde{\mathbf{D}}(\mathbf{z_{jt}})'\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})\Big]\mathbb{E}\Big[\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})'\tilde{\mathbf{H}}_\mathbf{E}(\mathbf{z_{jt}})\Big]^{-1}\end{pmatrix}'$$

We have: $V^*(z_{jt}) - \Gamma_0(h_E)\Omega_0(h_E)^{-1}\Gamma_0(h_E) = \tilde{\mathbf{M}}'\mathbb{E}[\tilde{\mathbf{X}}'\tilde{\mathbf{X}}]\tilde{\mathbf{M}}$

The matrix above is clearly semi definite positive. □

B.3 Feasible most powerful instrument

B.3.1 Local approximation of the MPI

Proof of Proposition 6

Proof. First, we define $s_t^0 = \rho(\delta_t, x_{2t}, f_0(.|\lambda_0))$ with δ_t the true mean utility. From lemma 2.2 ρ^{-1} is $\mathcal{C}^\infty$ and in particular, ρ^{-1} is $\mathcal{C}^1$. Thus, the Taylor expansion of $\rho^{-1}(s_t^0, x_{2t}, f_0(.|\lambda_0))$ around s_t writes:

$$\rho^{-1}(s_t^0, x_{2t}, f_0(.|\lambda_0)) = \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)) + \frac{\partial \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))}{\partial s}\Bigg|_{s=s_t}(s_t^0 - s_t) + o\big(||s_t^0 - s_t||\big)$$

$$\delta_t = \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)) + \frac{\partial \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))}{\partial s}\Bigg|_{s=s_t}(s_t^0 - s_t) + o\big(||s_t^0 - s_t||\big)$$

We now derive an expression for the first derivative of the inverse function. We make use of lemma 2.3: for any $\delta \in \mathbb{R}^J$, $\frac{\partial \rho(\delta, x_{2t}, f)}{\partial \delta}$ is invertible.

$$\frac{\partial \rho(\rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)), x_{2t}, f_0(.|\lambda_0))}{\partial s} = I_J \iff \frac{\partial \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))}{\partial s} \left(\frac{\partial \rho(\rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)), x_{2t}, f_0(.|\lambda_0))}{\partial \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))} \right) = I_J$$
$$\iff \frac{\partial \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))}{\partial s} = \left(\frac{\partial \rho(\delta_t^0, x_{2t}, f_0(.|\lambda_0))}{\partial \delta} \right)^{-1}$$

with $\delta_t^0 = \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))$.Consequently,

$$\underbrace{\rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)) - \delta_t}_{\Delta(s_t, x_{2t}, f_0, f_a)} = -\left(\frac{\partial \rho(\delta_t^0, x_{2t}, f_0(.|\lambda_0))}{\partial \delta} \right)^{-1} (s_t^0 - s_t) + o\big(||s_t^0 - s_t||\big) \tag{12}$$

with $\delta_t^0 = \rho_j^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))$

Now let us show that there exists a constant M such that $||s_t^0 - s_t|| \leq M\tau(f_0(.|\lambda_0) - f_a)$. with $\tau(f_0 - f_a) = \int_{\mathbb{R}^{K_2}} |f_0(v|\lambda_0) - f_a(v)|dv$. Norms are equivalent in a finite vectorial space and without loss of generality, we will derive the results with the L_1 norm. By definition:

$$s_t^0 - s_t = \int_{\mathbb{R}^{K_2}} \frac{exp(\delta_t + x_{2t}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt} + x_{2jk}'v\right\}} (f_0(v|\lambda_0) - f_a(v))dv$$

Taking the L_1 norm of this vector:

$$\begin{aligned}
||s_t^0 - s_t||_1 &= \sum_{j=1}^J \left| \int_{\mathbb{R}^{K_2}} \frac{\exp(\delta_{jt} + x_{2jt}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt} + x_{2jk}'v\right\}} (f_0(v|\lambda_0) - f_a(v))dv \right| \\
&\leq \sum_{j=1}^J \int_{\mathbb{R}^{K_2}} \underbrace{\left| \frac{\exp(\delta_{jt} + x_{2jt}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt} + x_{2jk}'v\right\}} \right|}_{\leq 1} |f_0(v|\lambda_0) - f_a(v)|dv \\
&\leq J \int_{\mathbb{R}^{K_2}} |f_0(v|\lambda_0) - f_a(v)|dv = J\tau(f_0(.|\lambda_0) - f_a)
\end{aligned}$$

This proves the statement. As a consequence, we have: $||s_t^0 - s_t||_1 = O(\tau(f_0(.|\lambda_0) - f_a))$ and $o(||s_t^0 - s_t||) = o(\tau(f_0(.|\lambda_0) - f_a))$

The problem with the term $s_t^0 - s_t$ is that it is an expression of δ_t which we do not know under misspecification. As we want to be able to compute this approximation of the error term, it is not convenient in practice to have an expression which depends on δ_t. On the other hand, we know δ_t^0 and thus, the simple idea that we exploit is to take a Taylor expansion of the term above around δ_t^0. First, let us remark that from equation 12, we have that:

$$||\delta_t - \delta_t^0|| = ||\delta_t - \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0)|| = O(||s_t^0 - s_t||) = O(\tau(f_0(.|\lambda_0) - f_a))$$

Now let us take the Taylor expansion of $s_t^0 - s_t$ around δ_t^0:

$$\begin{aligned} s_t^0 - s_t = & \int_{\mathbb{R}^{K_2}} \frac{exp(\delta_t^0 + x_{2t}v)}{1 + \sum_{k=1}^J \exp\left\{\tilde{\delta}_{kt} + x'_{2jk}v\right\}}(f_0(v|\lambda_0) - f_a(v))dv \\ + & \underbrace{\int_{\mathbb{R}^{K_2}} \frac{\partial}{\partial \delta'}\left\{\frac{exp(\delta_t^0 + x_{2t}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt}^0 + x'_{2jk}v\right\}}\right\}(\delta_t - \delta_t^0)(f_0(v|\lambda_0) - f_a(v))dv}_{B} + o\big(||\delta_t - \delta_t^0||\big) \end{aligned}$$

From what precedes, we know that $o(||\delta_t - \delta_t^0||) = o(\tau(f_0(.|\lambda_0) - f_a))$. Now, let us show that term B in the previous expansion is also $o(\tau(f_0(.|\lambda_0) - f_a))$. Again taking the L_1 norm:

$$\begin{aligned} ||B||_1 = & \sum_{j=1}^J \left| \sum_{l=1}^J \int_{\mathbb{R}^{K_2}} \frac{\partial}{\partial \delta_l}\left\{\frac{\exp(\delta_{jt}^0 + x'_{2jt}v)}{1 + \sum_{k=1}^J \exp\left\{\tilde{\delta}_{kt} + x'_{2jk}v\right\}}\right\}(\delta_{lt} - \delta_{lt}^0)(f_0(v|\lambda_0) - f_a(v))dv \right| \\ \leq & \sum_{j=1}^J \sum_{l=1}^J \int_{\mathbb{R}^{K_2}} \underbrace{\left| \frac{\partial}{\partial \delta_l}\left\{\frac{\exp(\delta_{jt}^0 + x'_{2jt}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt}^0 + x'_{2jk}v\right\}}\right\}\right|}_{\leq 1} |\delta_{lt} - \tilde{\delta}_{lt}||f_0(v|\lambda_0) - f_a(v)|dv \\ \leq & J^2 \tau(f_0(.|\lambda_0) - f) O(\tau(f_0(.|\lambda_0) - f_a)) = O(\tau(f_0(.|\lambda_0) - f_a)^2) = o(\tau(f_0(.|\lambda_0) - f_a)) \end{aligned}$$

Thus, $||B||_1 = o(\tau(f_0(.|\lambda_0) - f_a))$ and by combining all the results together, we get the final result. When $f_0(.|\lambda_0)$ gets ‘’close “ to f_a, we have the following approximation:

$$\begin{aligned} \Delta(s_t, x_{2t}, f_0, f_a) = & \left(\frac{\partial \rho(\delta_t^0, x_{2t}, f_0(.|\lambda_0))}{\partial \delta}\right)^{-1} \int_{\mathbb{R}^{K_2}} \frac{\exp(\delta_t^0 + x_{2t}v)}{1 + \sum_{k=1}^J \exp\left\{\delta_{kt}^0 + x'_{2jk}v\right\}}(f_a(v) - f_0(v|\lambda_0))dv \\ & + o(\tau(f_a - f_0(.|\lambda_0))) \end{aligned}$$

$\delta_t^0 = \rho^{-1}(s_t, x_{2t}, f_0(.|\lambda_0))$ and $\tau(f_a - f_0(.|\lambda_0)) = \int_{\mathbb{R}^{K_2}} |f_a(v) - f_0(.|\lambda_0)(v)|dv$. □

B.3.2 Global approximation of the MPI

Derivation of $\Delta_j(s_t, x_{2t}, f_0, f_a)$

Proof.

$$1 = \frac{\rho(\delta_{jt}, x_{2t}, f_a)}{\rho(\delta_{jt}^0, x_{2t}, f_0)} = \frac{\int_{\mathbb{R}^{K_2}} \frac{\exp(\delta_{jt}+x'_{2jt}v)}{1+\sum_{k=1}^J \exp\{\delta_{kt}+x'_{2kt}v\}} f_a(v)dv}{\int_{\mathbb{R}^{K_2}} \frac{\exp(\delta_{jt}^0+x'_{2jt}v)}{1+\sum_{k=1}^J \exp\{\delta_{kt}^0+x'_{2kt}v\}} f_0(v)dv} \iff \frac{\exp(\delta_{jt}^0)}{\exp(\delta_{jt})} = \frac{\int_{\mathbb{R}^{K_2}} \frac{\exp(x_{2t}v)}{1+\sum_{k=1}^J \exp\{\delta_{kt}+x'_{2kt}v\}} f_a(v)dv}{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^J \exp\{\delta_{jt}^0+x'_{2kt}v\}} f_0(v)dv}$$

□

B.3.3 Approximation of the MPI in the mixed logit case

Proof of Proposition 7. By definition, we have:

$$\begin{aligned} g_j(x_i, ., f):\ & \mathbb{R}^{K_1} \to [0,1] \\ & \beta \mapsto \int_{\mathbb{R}^{K_2}} \frac{\exp\left\{x'_{ij1}\beta + x'_{2ij}v\right\}}{1+\sum_{k=1}^J \exp\left\{x'_{ik1}\beta + x'_{2ik}v\right\}} f(v)dv \end{aligned}$$

g is $\mathcal{C}^\infty$ on $\mathbb{R}^{K_1}$. Thus, we can take a first order Taylor expansion of $g_j(x_i, ., f_1)$ around β_0^*:

$$g_j(x_i, \beta_1, f_1) = g_j(x_i, \beta_0^*, f_1) + \left.\frac{\partial g(x_i, \beta, f_1)}{\partial \beta}\right|_{\beta=\beta_0^*} (\beta_1 - \beta_0^*) + o(||\beta_1 - \beta_0^*||)$$

This yields immediately,

$$\begin{aligned} g(x_i, \beta_0^*, f_0^*) - g(x_i, \beta_1, f_1) = & \int_{\mathbb{R}^{K_2}} \frac{exp(x'_{1ij}\beta_0^* + x'_{2ij}v)}{1+\sum_{k=1}^J \exp\left\{x'_{1ik}\beta_0^* + x'_{2ik}v\right\}} (f_0^*(v) - f_1(v))dv + \\ & \left.\frac{\partial g(x_i, \beta, f_1)}{\partial \beta}\right|_{\beta=\beta_0} (\beta_a - \beta_0) + o(||\beta_- \beta_0^*||) \end{aligned}$$

Now let us show that $||\beta_1 - \beta_0^*|| = ...$

By construction, the pseudo true values β_0^* and β_1^* maximize the conditional expectation of the log-likelihood:

$$\beta_0^* = \underset{\beta \in \mathbb{R}^{K_1}}{argmax}\ \mathbb{E}\big[L(x_i, y_i, \beta, f_0^*)\big|x_i\big] \text{ with } L(x_i, y_i, \beta, f_0^*) = \sum_{j=0}^J \mathbf{1}\{y_{ij} = 1\} log(g_j(x_i, \beta, f_0^*))$$

The same goes for β_1^*:

$$\beta_1^* = \underset{\beta \in \mathbb{R}^{K_1}}{argmax}\ \mathbb{E}\big[L(x_i, y_i, \beta, f_1^*)\big|x_i\big] \text{ with } \ L(x_i, y_i, \beta, f_1^*) = \sum_{j=0}^{J} \mathbf{1}\{y_{ij} = 1\} log(g_j(x_i, \beta, f_1^*))$$

When the true distribution of RC is f_1, we have:

$$\mathbb{E}\big[L(x_i, y_i, \beta, f_0^*)\big|x_i\big] = \sum_{j=0}^{J} g_j(x_i, \beta_1^*, f_1) log(g_j(x_i, \beta, f_0^*))$$

$$\mathbb{E}\big[L(x_i, y_i, \beta, f_1)\big|x_i\big] = \sum_{j=0}^{J} g_j(x_i, \beta_1^*, f_1) log(g_j(x_i, \beta, f_1))$$

[TBC]

$\square$

B.4 Specification Test: composite hypothesis

In this section, we prove theorem 5.2, which is the main asymptotic result of the paper. The section is organized as follows. First, we establish the equivalence between the moment condition around which we build our test $\mathbb{E}\left[\sum_{jt} \xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) h_D(z_{jt})\right] = 0$ and the one characterizing $H_0' : \mathbb{E}\left[\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) h_D(z_{jt})\right] = 0$. Then, we introduce the notations used in the proofs and we decompose $\widehat{\xi}$ according to the BLP approximations. Second we provide technical lemmas which prove that under the assumptions in E, the BLP approximations vanish asymptotically. Third, we prove that the BLP estimator is consistent and asymptotically normal. Finally, we prove the main theorem and we show that under the null the test is pivotal in the 2 polar cases described in the main text.

B.4.1 Equivalence between moment conditions

Let $h_D(z_{jt})$ our detection instruments. For conciseness, we omit the dependence in f_0 and denote $\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) = \xi_{jt}(\theta_0)$. We want to prove that the following two moment conditions are equivalent:

$$\mathbb{E}\left[\xi_{jt}(\theta_0)h_D(z_{jt})\right] = 0 \iff \mathbb{E}\left[\sum_{j=1}^{J}\xi_{jt}(\theta_0)h_D(z_{jt})\right] = 0$$

Let R_t a categorial random variable which exogenously selects a product j with probability $\frac{1}{J}$. Formally, we have $(\xi_{jt}(\theta_0), z_{jt}) \perp R_{jt}$. By construction, we have:

$$\begin{aligned}
\mathbb{E}\left[\xi_{jt}(\theta_0)h_D(z_{jt})\right] &= \sum_{k=1}^{J}\mathbb{E}\left[\xi_{kt}(\theta_0)h_D(z_{kt})R_{kt}\right] = \sum_{k=1}^{J}\mathbb{E}\left[\xi_{kt}(\theta_0)h_D(z_{kt})\right]\mathbb{E}[R_{kt}] \\
&= \frac{1}{J}\mathbb{E}\left[\sum_{k=1}^{J}\xi_{kt}(\theta_0)h_D(z_{kt})\right]
\end{aligned}$$

Second line results from independence of $(\xi_{jt}(\theta_0), z_{jt})$ and R_{jt}. This proves the result.

B.4.2 Notations

In the proofs, we will adopt the following notations. If the derivations are done under the parametric assumption $H_0 : f \in \mathcal{F}_0$ then we omit the dependence in f_0 and interchangeably use $\xi_{jt}(f_0(.|\lambda), \beta)$ and $\xi_{jt}(\theta)$. We also omit the dependence of the BLP pseudo true value in W and $h_E(z_{jt})$ [35]. Then define the following objectives of the GMM minimization

$$\begin{aligned}
\hat{\mathcal{Q}}_T(\tilde{\theta}) &= \left(\frac{1}{T}\sum_{j,t}\widehat{\xi}_{jt}(\tilde{\theta})h_E(z_{jt})\right)'\hat{W}\left(\frac{1}{T}\sum_{j,t}\widehat{\xi}_{jt}(\tilde{\theta})h_E(z_{jt})\right) \\
\mathcal{Q}_T(\tilde{\theta}) &= \left(\frac{1}{T}\sum_{j,t}\xi_{jt}(\tilde{\theta})h_E(z_{jt})\right)'\hat{W}\left(\frac{1}{T}\sum_{j,t}\xi_{jt}(\tilde{\theta})h_E(z_{jt})\right) \\
\mathcal{Q}(\tilde{\theta}) &= \mathbb{E}\left[\sum_{j}\xi_{jt}(\tilde{\theta})h_E(z_{jt})\right]'W\mathbb{E}\left[\sum_{j}\xi_{jt}(\tilde{\theta})h_E(z_{jt})\right]
\end{aligned}$$

We also define the following moments

$$\begin{aligned}
\hat{g}_T(\tilde{\theta}, h) &= \frac{1}{T}\sum_{jt}\hat{\xi}_{jt}(\tilde{\theta})h(z_{jt}) \\
g_T(\tilde{\theta}, h) &= \frac{1}{T}\sum_{jt}\xi_{jt}(\tilde{\theta})h(z_{jt})
\end{aligned}$$

[35] The BLP pseudo true value depends on W and $h_E(z_{jt})$ when the model is misspecified

$$g(\tilde{\theta}, h) = \mathbb{E}\left[\sum_j \xi_{jt}(\tilde{\theta}) h(z_{jt})\right]$$

And recall the definition of $\Gamma(\mathcal{F}_0, \tilde{\theta}, h)$ which is used interchangeably with $\Gamma(\tilde{\theta}, h)$

$$\hat{\Gamma}_T(\tilde{\theta}, h) = \frac{1}{T}\sum_{j,t} h(z_{jt}) \frac{\partial}{\partial \theta} \widehat{\xi}_{jt}(\tilde{\theta})'$$

$$\Gamma_T(\tilde{\theta}, h) = \frac{1}{T}\sum_{j,t} h(z_{jt}) \frac{\partial}{\partial \theta} \xi_{jt}(\tilde{\theta})'$$

$$\Gamma(\tilde{\theta}, h) = \mathbb{E}\left[\sum_j h(z_{jt}) \frac{\partial}{\partial \theta} \xi_{jt}(\tilde{\theta})'\right]$$

Furthermore, unless specified, all limits are taken with respect to T; Additionally, we denote by the expression $X = o_P(T^{\kappa})$ a random variable or statistic which is asymptotically degenerate of order T^a, ie $X = o_P(T^{\kappa}) \Leftrightarrow \forall e > 0 \; \mathbb{P}(|X|T^{-\kappa} > e) \underset{T\to\infty}{\rightarrow} 0$, and denote by $X = O_p(T^{\kappa})$ a random variable which is (bounded in probability) of order T^{κ}, ie $\forall e_1 > 0 \exists e_2 > 0, \exists T_N : \forall T \geqslant T_N \; \mathbb{P}(|X|T^{-\kappa} > e_2) < e_1$. Properties of $o_P(1)$ and $O_P(1)$ random variables are used throughout these proofs.

B.4.3 Feasible Structural Error and BLP approximations

We now decompose the difference between the true structural error $\xi_{jt}(\tilde{\theta})$ and the feasible structural error $\hat{\xi}_{jt}(\tilde{\theta})$ in terms of the different approximations involved in the derivation of the feasible structural error $\hat{\xi}_{jt}(\tilde{\theta})$. In market t given an assumption $\mathcal{F}_0$, a parameter $\tilde{\lambda}$, market shares s_t and product characteristics with preference heterogeneity x_{2t} there exists a unique $\delta_t \in \mathbb{R}^J$ such that $s_t = \rho(\delta_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))$ (Brouwer's fixed point theorem, see Berry (1994)) so that $\delta_t = \rho^{-1}(s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))$. There is no closed form for $\rho^{-1}(s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))$ so the NFP algorithm is used. Denote as C the contraction used to find the mean utilities which solve the demand equal market share constraint

$$C(\cdot, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) : \delta \in \mathbb{R}^J \mapsto \delta + log(s_t) - log(\rho(\delta, x_{2t}, f_0(\cdot|\tilde{\lambda})))$$

So that for some starting mean utility $\delta_0 \in \mathcal{B} \subset \mathbb{R}^J$ where $\mathcal{B}$ is bounded, the mean utility obtained via NFP at the limit is equal to the unique vector which solves the constraint

$$\delta_t(f_0(\cdot|\tilde{\lambda})) = \rho^{-1}(s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) = \lim_{H\to\infty} C^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))$$

Similarly the error generated by $(f_0(\cdot|\tilde{\lambda}, \tilde{\beta}))$ can be obtained from NFP at the limit

$$\xi_t(f_0(\cdot|\tilde{\lambda}), \tilde{\beta}) = \delta_t(f_0(\cdot|\tilde{\lambda})) - x_{1t}\tilde{\beta} = \lim_{H\to\infty} C^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) - x_{1t}\tilde{\beta}$$

This way we obtain a vector of mean utilities for each market t. There are 3 approximations to consider, market shares are not truly observed, the demand integral has to be simulated, and the contraction is never taken to its limit, so define $\widehat{\xi}(f_0, \tilde{\lambda}))$ $\widehat{\delta}(f_0, \tilde{\lambda}))$ and $\hat{C}$ for some starting value δ_0

$$\widehat{\xi}_t(f_0(\cdot|\tilde{\lambda}), \tilde{\beta}) = \hat{C}^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) - x_{1jt}\tilde{\beta}, \qquad \widehat{\delta}(f_0, \tilde{\lambda})) = \hat{C}^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0, \tilde{\lambda}))$$
$$\hat{C} : \delta \mapsto \delta + log(\hat{s}_t) - log(\hat{\rho}(\delta, x_{2t}, f_0(\cdot|\lambda_0)))$$

Consequently we decompose the difference between the error generated by $(f_0(\cdot|\tilde{\lambda}), \tilde{\beta})$ and its feasible approximation into 3 differences

$$\begin{aligned}
\xi_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta}) - \widehat{\xi}_{jt}(f_0(\cdot|\tilde{\lambda}), \tilde{\beta}) &= \delta_{jt}(f_0(\cdot|\tilde{\lambda})) - \widehat{\delta}_{jt}(f_0(\cdot|\tilde{\lambda})) \\
&= \lim_{H\to\infty} C_j^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))) - \hat{C}_j^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) \\
&= \lim_{H\to\infty} C_j^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))) - C_j^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) \\
&\quad + C_j^{(H)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))) - C_j^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) \\
&\quad + C_j^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) - \hat{C}_j^{(H)}(\delta_0, \hat{s}_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) \\
&\equiv \rho_j^{-1}(s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})) - D_j(\rho, s_t, \tilde{\lambda}) \\
&\quad + D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, \hat{s}_t, \tilde{\theta}) \\
&\quad + D_j(\rho, \hat{s}_t, \tilde{\theta}) - D_j(\hat{\rho}, \hat{s}_t, \tilde{\theta})
\end{aligned}$$

In the fourth line, we simply introduce shortened notations for the same objects.

B.4.4 Technical Lemmas

The 1st and 2nd lemma establish the smoothness of ρ^{-1} and the invertibility of the Jacobian matrix of ρ with respect to δ. In the 3rd lemma, we derive the Lipschitz constant of the contraction and we prove that it is bounded away from 0 and 1. The 4th lemma ensures that for key moments and quantities the BLP approximations can be ignored uniformly asymptotically.

Lemma 2.2. ρ^{-1} is $\mathcal{C}^{\infty}$

Proof. We know that the demand function ρ is $\mathcal{C}^\infty$ and invertible on $\mathbb{R}^J$. Moreover,$\forall\delta \in \mathbb{R}^J$, $\frac{\partial\rho(\delta,x_{2t},f)}{\partial\delta} \neq 0$. As a consequence, $\rho^{-1} : [0,1]^J \to \mathbb{R}^J$ the inverse demand function is also $\mathcal{C}^\infty$. $\square$

Lemma 2.3. For any $\delta \in \mathbb{R}^J$, $\frac{\partial\rho(\delta,x_{2t},f)}{\partial\delta}$ is invertible.

Proof. $\frac{\partial\rho}{\partial\delta}$ is a $J \times J$ matrix such that $\left(\frac{\partial\rho}{\partial\delta}\right)_{j,k}$ is:

$$\frac{\partial\rho_j\left(\delta_t, x_{2t}, f\right)}{\partial\delta_{kt}} = \begin{cases} \int \mathcal{T}_{jt}(v)\left(1 - \mathcal{T}_{kt}(v)\right) f(v)dv & if \quad j = k \\ -\int \mathcal{T}_{jt}(v)\mathcal{T}_{kt}(v)f(v)dv & if \quad j \neq k \end{cases}$$

with $\mathcal{T}_{jt}(v) \equiv \frac{exp\{\delta_{jt}+x'_{2jt}v\}}{1+\sum_{j'=1}^{J} exp\{\delta_{j't}+x'_{2j't}v\}}$

One can easily check that $\frac{\partial\rho}{\partial\delta}$ is strictly diagonally dominant. Indeed for each row j:

$$\left|\frac{\partial\rho_j\left(\delta_t, x_{2t}, f\right)}{\partial\delta_{kt}}\right| - \sum_{k\neq j}\left|\frac{\partial\rho_j\left(\delta_t, x_{2t}, f\right)}{\partial\delta_{kt}}\right| = \int \mathcal{T}_{jt}(v)\underbrace{\left(1 - \sum_{k=1}^{J}\mathcal{T}_{kt}(v))\right)}_{>0} f(v)dv > 0$$

$\square$

Lemma 2.4 (Contraction Mapping Lipschitz Constant)**.**
Given parametric assumption $\mathcal{F}_0$, under assumptions B-E, assume that starting mean utility δ_0 is in $\mathcal{B}$ where $\mathcal{B}$ is compact, then without loss of generality there exists some $(\underline{a}, \bar{a}) \in \mathbb{R}^2$ with $\bar{a} > \underline{a}$ such that for any $b \in \mathcal{B}$ for any $j = 1, \ldots, J$ $\underline{a} \leqslant b_j \leqslant \bar{a}$, furthermore denote by $\mathcal{X}$ the compact support of x_{2jt}. Then on $\mathcal{B}$ the map $C(\cdot, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}_0))$ is a contraction with Lipschitz constant

$$\epsilon = \max_{j=1,\ldots,J} \sup_{a\in\mathcal{B}, b\in[0;\bar{a}-\underline{a}]^J, x_2\in\mathcal{X}, \tilde{\lambda}\in\Lambda_0} 1 - \frac{\int \frac{exp(a_j+b_j+x'_{2j}v)}{\left(1+\sum_k \exp(a_k+b_k+x'_{2k}v)\right)^2} f_0(v|\tilde{\lambda})dv}{\int \frac{exp(a_j+b_j+x'_{2j}v)}{1+\sum_k \exp(a_k+b_k+x'_{2k}v)} f_0(v|\tilde{\lambda})dv}$$

which is in $(0;1)$

Proof. This proof is inspired by the proof of the Theorem in Appendix 1 of Berry et al. (1995). Let $C_j(\cdot) \equiv C(\cdot, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}_0))$, we first determine the partial derivative of $C_j(\cdot)$

$$\frac{\partial C_j(a)}{\partial a_j} = 1 - \frac{1}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))}\int \frac{exp(a_j + x'_{2kt}v)(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v)) - exp(2(a_j + x'_{2kt}v))}{(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv$$

$$= \frac{1}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} \int \frac{exp(2(a_j + x'_{2jt}v))}{(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv$$

$$\frac{\partial C_j(a)}{\partial a_{j'}} = \frac{1}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} \int \frac{exp(a_j + x'_{2jt}v)exp(a_{j'} + x'_{2j't}v)}{(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv$$

Note that for any $j = 1, \ldots, J$ all partial derivatives of $C_j(\cdot)$ are strictly positive and that the sum of its derivatives evaluated in a equals

$$\begin{aligned}
\sum_{k=1}^{J} \frac{\partial C_j(a)}{\partial a_k} &= \frac{1}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} \int \frac{exp(a_j + x'_{2jt}v) \sum_{k=1}^{J} exp(a_k + x'_{2kt}v)}{(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv \\
&= \frac{1}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} \int \frac{exp(a_j + x'_{2jt}v)(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v) - 1)}{(1 + \sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv \\
&= 1 - \frac{\int \frac{exp(a_j + x'_{2jt}v)}{(1+\sum_{k=1}^{J} exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))}
\end{aligned}$$

For any $(a_1, a_2) \in \mathcal{B}^2$ let $\tilde{a} = (||a_1 - a_2||_\infty, \ \ldots \ , ||a_1 - a_2||_\infty) \in \mathbb{R}^J$ then

$$\begin{aligned}
C_j(a_1) - C_j(a_2) &= C_j(a_2 + a_1 - a_2) - C_j(a_2) \leqslant C_j(a_2 + \tilde{a}) - C_j(a_2) \\
&\leqslant \int_{0^J}^{||a_1 - a_2||_\infty^J} \frac{\partial C_j(a_2 + b)}{\partial a} db \\
&\leqslant ||a_1 - a_2||_\infty \sup_{a \in \mathcal{B}, b \in [0; \bar{a} - \underline{a}]^J} \sum_{k=1}^{J} \frac{\partial C_j(a + b)}{\partial a_k} \\
&\leqslant ||a_1 - a_2||_2 \max_{j=1,..J} \sup_{a \in \mathcal{B}, b \in [0; \bar{a} - \underline{a}]^J, x_2 \in \mathcal{X}, \tilde{\lambda} \in \Lambda_0} \sum_{k=1}^{J} \frac{\partial C_j(a + b)}{\partial a_k} \\
&\equiv ||a_1 - a_2||_2 \ \epsilon
\end{aligned}$$

where the 1st inequality holds because $C_j(\cdot)$ is increasing in all its inputs, the 2nd inequality holds by the fundamental theorem of calculus and by the total derivative formula, the 3rd and 4th inequalities hold by properties of norms.

We now prove that $\sup_{a \in \mathcal{B}, b \in [0; \bar{a} - \underline{a}]^J, \tilde{\lambda} \in \Lambda_0} \sum_{k=1}^{J} \frac{\partial C_j(a+b)}{\partial a_k} \in (0; 1)$ which will imply that $\epsilon \in (0; 1)$. To do so we have to prove that $\sum_{k=1}^{J} \frac{\partial C_j(a, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))}{\partial a_k}$ is continuous in $(a, x_{2t}, \tilde{\lambda})$ and takes values in $(0; 1)$ almost surely, this way because $\mathcal{B}$, $\mathcal{X}$ and Λ_0 are compact by Weierstrass' Extreme Value Theorem the sum of partial derivatives will also take values in a compact which is inside $(0; 1)$, then the supremum will become a maximum which can be attained and which is inside $(0;1)$. The sum of partial derivatives is almost surely in $(0;1)$ because

$$
\begin{aligned}
&\int \frac{exp(a_j + x'_{2jt}v)}{(1+\sum_{k=1}^J exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv - \rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda})) \\
&= \int \frac{exp(a_j + x'_{2jt}v)}{(1+\sum_{k=1}^J exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv - \int \frac{exp(a_j + x'_{2jt}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2kt}v)} f_0(v|\tilde{\lambda})dv \\
&= -\int \frac{exp(a_j + x'_{2jt}v)\sum_{k=1}^J exp(a_k + x'_{2kt}v)}{(1+\sum_{k=1}^J exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv < 0 \\
&\Rightarrow \frac{\int \frac{exp(a_j + x'_{2jt}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2kt}v)} f_0(v|\tilde{\lambda})dv}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} < 1 \\
&\Rightarrow \sum_{k=1}^J \frac{\partial C_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))}{\partial a_k} = 1 - \frac{\int \frac{exp(a_j + x'_{2jt}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2kt}v)} f_0(v|\tilde{\lambda})dv}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} > 0 \\
&\quad - \frac{\int \frac{exp(a_j + x'_{2jt}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2kt}v)} f_0(v|\tilde{\lambda})dv}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} < 0 \\
&\Rightarrow \sum_{k=1}^J \frac{\partial C_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))}{\partial a_k} = 1 - \frac{\int \frac{exp(a_j + x'_{2jt}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2kt}v)} f_0(v|\tilde{\lambda})dv}{\rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))} < 1
\end{aligned}
$$

Continuity of the sum of the partial derivatives in (a, x_{2t}) is trivial, continuity in $\tilde{\lambda}$ also holds because $f_0(\cdot|\tilde{\lambda})$ must be continuously differentiable via Assumption D. $\forall e_1 > 0, \exists e_2 : \forall(\lambda_1, \lambda_2) : ||\lambda_1 - \lambda_2||_2 \leqslant e_2$ implies $|f_0(v|\lambda_1) - f_0(v|\lambda_2)| < e_1$ for all v which in turn implies

$$
\begin{aligned}
\forall x_2 \in \mathcal{X}, \forall a \in \mathcal{B} \quad &\left| \int \frac{exp(a_j + x'_{2j}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2k}v)} (f_0(v|\lambda_1) - f_0(v|\lambda_2))dv \right| \\
&\leqslant \int \frac{exp(a_j + x'_{2j}v)}{1+\sum_{k=1}^J exp(a_k + x'_{2k}v)} |f_0(v|\lambda_1) - f_0(v|\lambda_2)|dv \leqslant e_1 \\
&\left| \int \frac{exp(a_j + x'_{2j}v)}{(1+\sum_{k=1}^J exp(a_k + x'_{2k}v))^2} (f_0(v|\lambda_1) - f_0(v|\lambda_2))dv \right| \leqslant e_1
\end{aligned}
$$

thus both $\tilde{\lambda} \mapsto \rho_j(a, x_{2t}, f_0(\cdot|\tilde{\lambda}))$ and $\tilde{\lambda} \mapsto \int \frac{exp(a_j + x'_{2jt}v}{(1+\sum_{k=1}^J exp(a_k + x'_{2kt}v))^2} f_0(v|\tilde{\lambda})dv$ are continuous and so is their ratio. □

Lemma 2.5 (Uniform Convergence of Objective Function wrt BLP Approximations)**.**
Given parametric assumption $\mathcal{F}_0$, under assumptions B-E and $\forall h$ which satisfies D

$$\underset{\tilde{\theta} \in \Theta_0}{sup} \sqrt{T} ||\hat{g}_T(\tilde{\theta}, h) - g_T(\tilde{\theta}, h)||_2 \xrightarrow{P} 0$$

$$\underset{\tilde{\theta}\in\Theta_0}{sup}||\hat{\Gamma}_T(\tilde{\theta},h)-\Gamma_T(\tilde{\theta},h)||_2 \xrightarrow{P} 0$$

$$\underset{\tilde{\theta}\in\Theta_0}{sup}|\hat{\mathcal{Q}}_T(\tilde{\theta})-\mathcal{Q}(\tilde{\theta})| \xrightarrow{P} 0$$

Proof. Parts of this proof are inspired from Freyberger (2015). We prove the 3 statements of the Lemma in order

1. Using the properties of the sup, the fact that $\forall (A,B)$ rv, $\forall e>0$, $\forall \alpha \in (0,1)$, $\mathbb{P}(A+B>e) \leqslant \mathbb{P}(A>\alpha e)+\mathbb{P}(B>(1-\alpha)e)$ and the previous decomposition of the difference between ξ and $\widehat{\xi}$ we can find an upper bound on the probability that that the difference between $\hat{g}_T(\cdot)$ and $g_T(\cdot)$ is above a deviation: For any $e_1>0$

$$\begin{aligned}\mathbb{P}(\underset{\tilde{\theta}}{sup}\ \sqrt{T}||\hat{g}_T(\theta,h)-g_T(\theta,h)||_2>e_1) &= \mathbb{P}(\underset{\tilde{\theta}}{sup}\ \sqrt{T}\frac{1}{T}||\sum_{j,t}(\widehat{\xi}_t(f_0(\cdot|\tilde{\lambda}),\tilde{\beta})-\xi_t(f_0(\cdot|\tilde{\lambda}),\tilde{\beta}))h(z_{jt})||_2>e_1)\\ &\leqslant \mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \sqrt{T}||\frac{1}{T}\sum_{j,t}(\rho^{-1}(s_t,x_{2t},f_0(\cdot|\tilde{\lambda}_0))-D_j(\rho,s_t,\tilde{\lambda}))h(z_{jt})||_2>\frac{e_1}{3})\\ &+\mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \sqrt{T}||\frac{1}{T}\sum_{j,t}(D_j(\rho,s_t,\tilde{\lambda})-D_j(\rho,\hat{s}_t,\tilde{\lambda}))h(z_{jt})||_2>\frac{e_1}{3})\\ &+\mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \sqrt{T}||\frac{1}{T}\sum_{j,t}(D_j(\rho,\hat{s}_t,\tilde{\lambda})-D_j(\hat{\rho},\hat{s}_t,\tilde{\lambda}))h(z_{jt})||_2>\frac{e_1}{3})\end{aligned}$$

   Then we can prove that each element of the upper bound converges to 0

   (a) By properties of contractions and using Lemma 2.4 we have

$$|\rho^{-1}(s_t,x_{2t},f_0(\cdot|\tilde{\lambda}_0))-D_j(\rho,s_t,\tilde{\lambda})| \leqslant \epsilon^H|\rho^{-1}(s_t,x_{2t},f_0(\cdot|\tilde{\lambda}_0))-\delta_0| \leqslant \epsilon^H\kappa$$

   for some constant κ which exists due to the compactness of Λ_0, $\mathcal{X}$ and $\mathcal{B}$. Thus using the iid nature of the data **??**(i), the speed of the NFP algorithm Assumption E(iii), the triangle inequality, Markov inequality and Cauchy-Schwarz inequality the 1st element converges to 0

$$\begin{aligned}&\mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \sqrt{T}||\frac{1}{T}\sum_{j,t}(\rho^{-1}(s_t,x_{2t},f_0(\cdot|\tilde{\lambda}_0))-D_j(\rho,s_t,\tilde{\lambda}))h(z_{jt})||_2>\frac{e_1}{3})\\ &\leqslant \mathbb{P}(\sqrt{T}\epsilon^H\kappa||\frac{1}{T}\sum_{j,t}h(z_{jt})||_2>\frac{e_1}{3}) \leqslant \mathbb{P}(\sqrt{T}\epsilon^H\frac{1}{T}\sum_{j,t}||h(z_{jt})||_2>\frac{e_1}{3})\\ &\leqslant \frac{3\kappa}{e_1}\sqrt{T}\epsilon^H\sum_j\sqrt{\mathbb{E}(||h(z_{jt})||_2^2)} \underset{T\to\infty}{\to} 0\end{aligned}$$

(b) Note that D_j is continuously differentiable in $s \in (0;1)$ so that it is uniformly continuous in s. Indeed C is $\mathcal{C}^\infty$ in s so that

$$\frac{\partial D(\rho, s_t, \tilde{\lambda})}{\partial s} = \prod_{h=1}^{H} \frac{\partial C(C^{(h-1)}(\delta_0, s_t, x_{2t}, f_0(\cdot|\tilde{\lambda})), s_t, x_{2t}, f_0(\cdot|\tilde{\lambda}))}{\partial s}$$

Next because Λ_0 is compact it can be covered by some finite union of closed balls in $\mathbb{R}^{K_2}$, ie $\Lambda_0 \subset \cup_{c=1}^{N} \Lambda_{0,c}^{N}$ with $\forall c = 1, \ldots, N \quad \Lambda_{0,c}^{N} = \{\tilde{\lambda} : ||\tilde{\lambda} - \lambda_c||_2 \leqslant r_N\}$, $\lambda_c \in \Lambda_0$ and $r_N \underset{N\to\infty}{\to} 0$. Consequently

$$\begin{aligned}
&\mathbb{P}(\sup_{\tilde{\lambda}} \frac{1}{\sqrt{T}} || \sum_{j,t} (D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, \hat{s}_t, \tilde{\lambda})) h_E(z_{jt}) ||_2 > \frac{e_1}{3}) \\
&\leqslant \mathbb{P}(\max_{c=1,\ldots,N} \sup_{\tilde{\lambda} \in \Lambda_{0,c}^{N}} \frac{1}{\sqrt{T}} || \sum_{j,t} (D_j(\rho, s_t, \tilde{\theta}) - D_j(\rho, \hat{s}_t, \tilde{\theta})) h_E(z_{jt}) ||_2 > \frac{e_1}{3}) \\
&\leqslant \sum_{c=1}^{N} \mathbb{P}(\sup_{\tilde{\lambda} \in \Lambda_{0,c}^{N}} \frac{1}{\sqrt{T}} \sum_{j,t} |D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, \hat{s}_t, \tilde{\lambda})| \, ||h_E(z_{jt})||_2 > \frac{e_1}{3}) \\
&\leqslant \sum_{c=1}^{N} \mathbb{P}(\frac{1}{\sqrt{T}} || \sum_{j,t} (D_j(\rho, s_t, \lambda_c) - D_j(\rho, \hat{s}_t, \lambda_c)) h_E(z_{jt}) ||_2 > \frac{e_1}{9}) \\
&\quad + \sum_{c=1}^{N} \mathbb{P}(\sup_{\tilde{\lambda} \in \Lambda_{0,c}^{N}} \frac{1}{\sqrt{T}} \sum_{j,t} |D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, s_t, \lambda_c)| \, ||h_E(z_{jt})||_2 > \frac{e_1}{9}) \\
&\quad + \sum_{c=1}^{N} \mathbb{P}(\sup_{\tilde{\lambda} \in \Lambda_{0,c}^{N}} \frac{1}{\sqrt{T}} \sum_{j,t} |D_j(\rho, \hat{s}_t, \lambda_c) - D_j(\rho, \hat{s}_t, \tilde{\lambda})| \, ||h_E(z_{jt})||_2 > \frac{e_1}{9})
\end{aligned}$$

where the last inequality was obtained using the triangle inequality. Then by uniform continuity of D_j in s it follows that $\exists e_2 > 0$ such that $\forall c$ $\frac{1}{\sqrt{T}} || \sum_{j,t} (D_j(\rho, s_t, \lambda_c) - D_j(\rho, \hat{s}_t, \lambda_c)) h_E(z_{jt}) ||_2 > \frac{e_1}{9}$ implies $\frac{1}{\sqrt{T}} || \sum_{j,t} (s_t - \hat{s}_t) ||_2 > e_2$ thence letting $\mathbb{P}^* = \mathbb{P}(\cdot | n_t, x_t, \xi_t)$

$$\begin{aligned}
&\mathbb{P}^*(\frac{1}{\sqrt{T}} || \sum_{j,t} (D_j(\rho, s_t, \lambda_c) - D_j(\rho, \hat{s}_t, \lambda_c)) h_E(z_{jt}) ||_2 > \frac{e_1}{9}) \leqslant \mathbb{P}^*(\frac{1}{\sqrt{T}} || \sum_{j,t} (s_t - \hat{s}_t) ||_2 > e_2) \\
&\leqslant \frac{J \sum_t \mathbb{E}^*(||s_t - \hat{s}_t||_2)}{e_2 \sqrt{T}} = \frac{J \sum_t \mathbb{E}^* \left(\sqrt{\sum_j (s_{jt} - \hat{s}_{jt})^2} \right)}{e_2 \sqrt{T}} \leqslant \frac{J \sum_t \sqrt{\sum_j \mathbb{E}^* \left((s_{jt} - \hat{s}_{jt})^2 \right)}}{e_2 \sqrt{T}} \\
&\leqslant \frac{J \sum_t \sqrt{\sum_j \mathbb{E}^* \left((\frac{1}{n_t} \sum_{i=1}^{n_t} y_{ijt} - \mathbb{E}^*(y_{ijt}))^2 \right)}}{e_2 \sqrt{T}} = \frac{J \sum_t \sqrt{\sum_j Var^*(\frac{1}{n_t} \sum_{i=1}^{n_t} y_{ijt})}}{e_2 \sqrt{T}} \\
&\leqslant \frac{J \sum_t \sqrt{\sum_j \frac{1}{n_t} Var^*(y_{ijt})}}{e_2 \sqrt{T}} \leqslant \frac{J^{3/2}}{e_2} \frac{1}{\sqrt{T}} \sum_t \frac{1}{\sqrt{n_t}}
\end{aligned}$$

where Markov inequality, Jensen inequality, the fact that $y_{ijt} \in \{0;1\}$, that ε_{ijt} is iid extreme-value type 1 distributed across i, j and t, and the fact that n_t is iid and independent of all other variables have been used. Then taking the expectations and summing over N on both sides implies by Assumption E(i)

$$\sum_{c=1}^{N} \mathbb{P}(\frac{1}{\sqrt{T}}||\sum_{j,t}(D_j(\rho, s_t, \lambda_c) - D_j(\rho, \hat{s}_t, \lambda_c))h_E(z_{jt})||_2 > \frac{e_1}{9}) \leqslant \frac{J^{3/2}N}{e_2}\sqrt{T}\mathbb{E}(n_t^{-1/2}) \underset{T\to\infty}{\to} 0$$

Next using continuity of D_j in $\tilde{\lambda}$ it must be that for any $e_1 > 0$ there exists some N such that $\forall\tilde{\lambda} \in \Lambda_{0,c}^N$ such that $||\tilde{\lambda} - \lambda_c||_2 \leqslant r_N$ implies

$$\frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, s_t, \lambda_c)|\ ||h_E(z_{jt}||_2 \leqslant \frac{e_1}{9}$$

because $r_N \underset{N\to\infty}{\to} 0$. By definition of the supremum it also implies that

$$\underset{\tilde{\lambda}\in\Lambda_{0,c}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, s_t, \lambda_c)|\ ||h_E(z_{jt}||_2 \leqslant \frac{e_1}{9}$$

The contraposition is that

$$\underset{\tilde{\lambda}\in\Lambda_{0,c}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, s_t, \lambda_c)|\ ||h_E(z_{jt}||_2 > \frac{e_1}{9}$$

implies $\forall\tilde{\lambda} \in \Lambda_{0,c}^N \quad ||\tilde{\lambda}-\lambda_c||_2 > r_N$ which is impossible by definition of $\Lambda_{0,c}^N$. Consequently

$$\sum_{c=1}^{N}\mathbb{P}(\underset{\tilde{\lambda}\in\Lambda_{0,c}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho, s_t, \tilde{\lambda}) - D_j(\rho, s_t, \lambda_c)|\ ||h_E(z_{jt}||_2 > \frac{e_1}{9})$$
$$\leqslant \sum_{c=1}^{N}\mathbb{P}(\cap_{\tilde{\lambda}\in\Lambda_{0,c}^N}||\tilde{\lambda} - \lambda_c||_2 > r_N) = 0$$

Similarly

$$\sum_{c=1}^{N}\mathbb{P}(\underset{\tilde{\lambda}\in\Lambda_{0,c}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho, \hat{s}_t, \tilde{\lambda}) - D_j(\rho, \hat{s}_t, \lambda_c)|\ ||h_E(z_{jt}||_2 > \frac{e_1}{9}) = 0$$

(c) With the same arguments as in (b)

$$\mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \frac{1}{\sqrt{T}}||\sum_{j,t}(D_j(\rho, \hat{s}_t, \tilde{\lambda}) - D_j(\hat{\rho}, \hat{s}_t, \tilde{\lambda}))h_E(z_{jt})||_2 > \frac{e_1}{3})$$
$$\leqslant \sum_{c=1}^{N}\mathbb{P}(\frac{1}{\sqrt{T}}||\sum_{j,t}(D_j(\rho, \hat{s}_t, \lambda_c) - D_j(\hat{\rho}, \hat{s}_t, \lambda_c))h_E(z_{jt})||_2 > \frac{e_1}{9})$$

$$+\sum_{c=1}^{N}\mathbb{P}(\underset{\tilde{\lambda}\in\Lambda_{0,c}^{N}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho,\hat{s}_t,\tilde{\lambda})-D_j(\rho,\hat{s}_t,\lambda_c)|\ ||h_E(z_{jt})||_2>\frac{e_1}{9})$$

$$+\sum_{c=1}^{N}\mathbb{P}(\underset{\tilde{\lambda}\in\Lambda_{0,c}^{N}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\hat{\rho},\hat{s}_t,\lambda_c)-D_j(\hat{\rho},\hat{s}_t,\tilde{\lambda})|\ ||h_E(z_{jt})||_2>\frac{e_1}{9})$$

$$=\sum_{c=1}^{N}\mathbb{P}(\frac{1}{\sqrt{T}}||\sum_{j,t}(D_j(\rho,\hat{s}_t,\lambda_c)-D_j(\hat{\rho},\hat{s}_t,\lambda_c))h_E(z_{jt})||_2>\frac{e_1}{9})$$

where $D_j(\rho,s_t,\lambda_c)=C^{(H)}(\delta_0,s_t,x_{2t},f_0(\cdot|\lambda_c))$. D_j is $\mathcal{C}^\infty$ in $\rho\in(0;1)$, moreover $\rho_j(\delta_t,x_{2t},f_0(\cdot|\tilde{\lambda}))$ and $\hat{\rho}_j(\delta_t,x_{2t},f_0(\cdot|\tilde{\lambda}))$ are continuously differentiable in Λ_0. Therefore there exists some $e_2>0$ such that

$$\frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho,\hat{s}_t,\lambda_c)-D_j(\hat{\rho},\hat{s}_t,\lambda_c)|\ ||h_E(z_{jt})||_2>\frac{e_1}{9}$$

implies $\underset{a\in\mathcal{B}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}||\rho(a,x_{2t},f_0(\cdot|\lambda_c)-\hat{\rho}(a,x_{2t},f_0(\cdot|\lambda_c))||_2>e_2$, and as $\mathcal{B}$ is compact we can cover it by $\tilde{N}$ closed balls $\mathcal{B}_b^{\tilde{N}}=\{a\in\mathcal{B}:||a-a_b||\leqslant r_{\tilde{N}}\}$ with $a_b\in\mathcal{B}$ for any $b=1,\ldots,\tilde{N}$ so that

$$\begin{aligned}
&\sum_{c=1}^{N}\mathbb{P}(\frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho,\hat{s}_t,\lambda_c)-D_j(\hat{\rho},\hat{s}_t,\lambda_c)|\ ||h_E(z_{jt})||_2>\frac{e_1}{9})\\
&\leqslant\sum_{c=1}^{N}\mathbb{P}(\underset{a\in\mathcal{B}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}||\rho(a,x_{2t},f_0(\cdot|\lambda_c)-\hat{\rho}(a,x_{2t},f_0(\cdot|\lambda_c))||_2>e_2)\\
&\leqslant\sum_{c,b}\mathbb{P}(\underset{a\in\mathcal{B}_b^{\tilde{N}}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}||\rho(a,x_{2t},f_0(\cdot|\lambda_c)-\hat{\rho}(a,x_{2t},f_0(\cdot|\lambda_c))||_2>e_2)\\
&=\sum_{c,b}\mathbb{P}(\frac{1}{\sqrt{T}}\sum_{j,t}||\rho(a_b,x_{2t},f_0(\cdot|\lambda_c)-\hat{\rho}(a_b,x_{2t},f_0(\cdot|\lambda_c))||_2>e_2)
\end{aligned}$$

where the last equality was obtained reusing arguments from (b). As a consequence let $F_{jt}(v)=\frac{exp(a_{bj}+x_{2jt}'v)}{1+\sum_k exp(a_{bk}+x_{2kt}'v)}$ and $\mathbb{P}^*(\cdot)=\mathbb{P}(\cdot|x_t,\xi_t)$ then using Markov inequality and Cauchy-Schwarz inequality

$$\begin{aligned}
&\mathbb{P}^*(\frac{1}{\sqrt{T}}\sum_{j,t}||\rho(a_b,x_{2t},f_0(\cdot|\tilde{\lambda}))-\hat{\rho}(a_b,x_{2t},f_0(\cdot|\tilde{\lambda}))||_2>e_2)\\
&\leqslant\frac{J\sum_t\mathbb{E}^*(||\hat{\rho}(a_b,x_{2t},f_0(\cdot|\tilde{\lambda}))-\rho(a_b,x_{2t},f_0(\cdot|\tilde{\lambda}))||_2)}{e_2\sqrt{T}}\\
&\leqslant\frac{J\sum_t\sqrt{\sum_j\mathbb{E}^*\left((\frac{1}{R}\sum_{r=1}^{R}F_{jt}(v_R)-\mathbb{E}^*(F_{jt}(v_R)))^2\right)}}{e_2\sqrt{T}}=\frac{J\sum_t\sqrt{\sum_j Var^*(\frac{1}{R}\sum_{r=1}^{R}F_{jt}(v_r))}}{e_2\sqrt{T}}\\
&\leqslant\frac{J^{3/2}}{e_2}\sqrt{\frac{T}{R}}
\end{aligned}$$

where the fact that v_r are iid draws from $f_0(\cdot|\tilde{\lambda})$ independent from all other variables has been used. It follows by taking the expectation and summing over N and $\tilde{N}$ that

$$\mathbb{P}(\underset{\tilde{\lambda}}{sup}\ \frac{1}{\sqrt{T}}\sum_{j,t}|D_j(\rho,\hat{s}_t,\tilde{\lambda})-D_j(\hat{\rho},\hat{s}_t,\tilde{\lambda})|\ h_E(z_{jt})||_2 \underset{T\to\infty}{\to} 0$$

by Assumption E(i).

2. The 2nd statement is not formally proven as it largely builds on the proof of the 1st statement. To see why recall that

$$\hat{\Gamma}_T(\tilde{\theta},h)-\Gamma_T(\tilde{\theta},h)=\frac{1}{T}\sum_{jt}h(z_{jt})\frac{\partial}{\partial\theta}(\widehat{\xi}(\tilde{\theta})-\xi_{jt}(\tilde{\theta}))'$$

More precisely let $e_j'=(0\ \ldots\ 0\ \underbrace{1}_{\text{j-th coordinate}}\ 0\ \ldots\ 0)$ then

$$\frac{\partial\xi_{jt}(\tilde{\theta})}{\partial\beta}=-x_{1jt},\quad \frac{\partial}{\partial\lambda}\xi_{jt}(\tilde{\theta})=-e_j'\left(\frac{\partial\rho(\delta_t(\tilde{\lambda}),x_{2t},f_0(\cdot|\tilde{\lambda}))}{\partial\delta}\right)^{-1}\int\frac{exp(\delta_{jt}(\tilde{\lambda})+x_{2jt}'v)}{1+\sum_{k=1}^{J}exp(\delta_{kt}(\tilde{\lambda})+x_{2kt}'v)}\frac{\partial}{\partial\lambda}f_0(v|\tilde{\lambda})dv$$

Thus the columns of the matrix $\hat{\Gamma}_T(\tilde{\theta},h)-\Gamma_T(\tilde{\theta},h)$ associated to the derivative in β are equal to 0. Furthermore using an uniform continuity argument $\left|\frac{\partial\hat{\xi}_{jt}(\tilde{\theta})}{\partial\lambda}-\frac{\partial\xi_{jt}(\tilde{\theta})}{\partial\lambda}\right|>e_1$ is implied by $||\widehat{\delta}_t(\tilde{\lambda})-\delta_t(\tilde{\lambda})||_2>e_2$ for some $e_2>0$. Using the compactness of Λ_0 and Assumption E it is straightforward that $\underset{\tilde{\lambda}}{sup}\ ||\hat{\Gamma}_T(\tilde{\theta},h)-\Gamma_T(\tilde{\theta},h)||_2\xrightarrow{P}0$ for any h which satisfies the conditions in Assumption D.

3. The 3rd statement follows from the 1st. Indeed using Cauchy-Schwarz and properties of the supremum

$$\begin{aligned}\underset{\tilde{\theta}\in\Theta_0}{sup}\ |\hat{\mathcal{Q}}_T(\tilde{\theta})-\mathcal{Q}_T(\tilde{\theta})|=&|(\hat{g}_T(\tilde{\theta},h_E)-g_T(\tilde{\theta},h_E))'\hat{W}(\hat{g}_T(\tilde{\theta},h_E)-g_T(\tilde{\theta},h_E))\\&-2(\hat{g}_T(\tilde{\theta},h_E)-g_T(\tilde{\theta},h_E))'\hat{W}g_T(\tilde{\theta},h_E))|\\\leqslant&\underset{\tilde{\theta}\in\Theta_0}{sup}\ ||(\hat{g}_T(\tilde{\theta},h_E)-g_T(\tilde{\theta},h_E))||_2^2\bar{\mu}(\hat{W})\\&+2\underset{\tilde{\theta}\in\Theta_0}{sup}\ ||(\hat{g}_T(\tilde{\theta},h_E)-g_T(\tilde{\theta},h_E))||_2\underset{\tilde{\theta}\in\Theta_0}{sup}\ ||g_T(\tilde{\theta},h_E))||_2\bar{\mu}(\hat{W})\end{aligned}$$

where $\bar{\mu}(\cdot)$ maps a square matrix towards its maximum eigenvalue. By D(iv) and definition of the L_2 matrix norm, $\bar{\mu}(\hat{W})\xrightarrow{P}\bar{\mu}(W)$. Then we apply Jennrich's ULLN: the data is iid, Θ_0 is compact, and $g_T(\tilde{\theta},h_E)=\sum_j\xi_{jt}(f_0(\cdot|\tilde{\lambda}),\tilde{\beta})h_E(z_{jt})$ has an enveloppe with finite absolute 1st moment because $\xi_{jt}(f_0(\cdot|\tilde{\lambda}),\tilde{\beta})=\rho^{-1}(s_t,x_{2t},\tilde{\lambda})-x_{1jt}'\tilde{\beta}$ and $\rho^{-1}(\cdot)$ has a maximum because it is continuous and its input are in a compact and because $\tilde{\beta}$ is in a compact and x_{1jt} has finite 4th moments, see Assumption B; Thus by the CMT $\underset{\tilde{\theta}\in\Theta_0}{sup}\ ||g_T(\tilde{\theta}, h_E))||_2 \xrightarrow{P} \underset{\tilde{\theta}\in\Theta_0}{sup}\ ||g(\tilde{\theta}, h_E)||_2$; Finally using the 1st statement we have $||(\hat{g}_T(\tilde{\theta}, h_E) - g_T(\tilde{\theta}, h_E))||_2 \xrightarrow{P} 0$ therefore by the CMT

$$\underset{\tilde{\theta}\in\Theta_0}{sup}\ |\hat{\mathcal{Q}}_T(\tilde{\theta}) - \mathcal{Q}_T(\tilde{\theta})| \xrightarrow{P} 0$$

□

B.4.5 Asymptotic Properties of the BLP estimator

Lemma 2.6 (Consistency of BLP Estimator)**.**
Given parametric assumption $\mathcal{F}_0$ and under assumptions B-E

$$\hat{\theta} \xrightarrow{P} \theta_0$$

Proof. We prove consistency using arguments for the consistency of M-estimators. For any $e_1 > 0$ such that $|\hat{\theta} - \theta_0| > e_1$ then by Assumption D(iii) there exists some $e_2 > 0$ such that $\mathcal{Q}(\hat{\theta}) - \mathcal{Q}(\theta_0) > e_2$ as θ_0 is the unique minimizer of the objective. Thence for any $e_1 > 0$, $\exists e_2 > 0$ such that

$$\begin{aligned}
\mathbb{P}(|\hat{\theta} - \theta_0| > e_1) &\leqslant \mathbb{P}(\mathcal{Q}(\hat{\theta}) - \mathcal{Q}(\theta_0) > e_2) \\
&= \mathbb{P}(\hat{\mathcal{Q}}_T(\theta_0) - \mathcal{Q}(\theta_0) + \mathcal{Q}(\hat{\theta}) - \hat{\mathcal{Q}}_T(\hat{\theta}) + \hat{\mathcal{Q}}_T(\hat{\theta}) - \hat{\mathcal{Q}}_T(\theta_0) > e_2) \\
&\leqslant \mathbb{P}(\hat{\mathcal{Q}}_T(\theta_0) - \mathcal{Q}(\theta_0) + \mathcal{Q}(\hat{\theta}) - \hat{\mathcal{Q}}_T(\hat{\theta}) > e_2) \\
&\leqslant \mathbb{P}(\hat{\mathcal{Q}}_T(\theta_0) - \mathcal{Q}(\theta_0) > (1-\alpha)e_2) + \mathbb{P}(\mathcal{Q}(\hat{\theta}) - \hat{\mathcal{Q}}_T(\hat{\theta}) > \alpha e_2)
\end{aligned}$$

where $\alpha \in (0; 1)$, the 2nd inequality comes from the fact that $\hat{\mathcal{Q}}_T(\hat{\theta}) - \hat{\mathcal{Q}}_T(\theta_0)$ is almost surely negative by definition of $\hat{\theta}$, and the 3rd inequality is obtained by utilizing properties of indicator functions. Then by a direct implication of Lemma 2.5 the right-hand-side converges to 0.

□

Lemma 2.7 (Asymptotic Normality of BLP Estimator)**.**
Given parametric assumption $\mathcal{F}_0$, under assumptions B-E and under $H_0 : f \in \mathcal{F}_0$

$$\sqrt{T}(\hat{\theta} - \theta_0) = (\Gamma'(\theta_0, h_E)W\Gamma(\theta_0, h_E))^{-1}\sqrt{T}\Gamma'(\theta_0, h_E)Wg_T(\theta_0, h_E) + o_P(1)$$

Furthermore under $H_0; f \in \mathcal{F}_0$

$$\sqrt{T}(\hat{\theta}-\theta_0) \xrightarrow{d} \mathcal{N}(0, (\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E))^{-1}\Gamma'(\theta_0,h_E)W\Omega(\mathcal{F}_0,h_E)W\Gamma(\theta_0,h_E)$$
$$(\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E))^{-1})$$

Proof. We prove asymptotic normality using arguments from M-estimators asymptotics. From Taylor's theorem there exists some $\tilde{\theta}$ such that $||\tilde{\theta}-\theta_0||_2 \leqslant ||\hat{\theta}-\theta_0||_2$ and

$$\begin{aligned}
&\hat{g}_T(\hat{\theta},h_E) = \hat{g}_T(\theta_0,h_E) + \hat{\Gamma}_T(\tilde{\theta},h_E)(\hat{\theta}-\theta_0) \\
&\Rightarrow \sqrt{T}\hat{\Gamma}'_T(\hat{\theta},h_E)\hat{W}\hat{g}_T(\hat{\theta},h_E) = \sqrt{T}\hat{\Gamma}'_T(\hat{\theta},h_E)\hat{W}\hat{g}_T(\theta_0,h_E) + \hat{\Gamma}'_T(\hat{\theta},h_E)\hat{W}\hat{\Gamma}_T(\tilde{\theta},h_E)\sqrt{T}(\hat{\theta}-\theta_0) = 0 \\
&\Leftrightarrow \sqrt{T}(\hat{\theta}-\theta_0) = -\left(\hat{\Gamma}'_T(\hat{\theta},h_E)\hat{W}\hat{\Gamma}_T(\tilde{\theta},h_E)\right)^{-1}\sqrt{T}\hat{\Gamma}'_T(\hat{\theta},h_E)\hat{W}\hat{g}_T(\theta_0,h_E)
\end{aligned}$$

where the 1st implication is due to the FOC Assumption D(v). Then, we apply the CMT to $(A,B) \mapsto (A'BA)^{-1}A'B$ which is a continuous mapping if A and B are full rank so that when taking $A = \hat{\Gamma}_T(\hat{\theta},h_E)$ and $B = \hat{W}$ we obtain:

$$\sqrt{T}(\hat{\theta}-\theta_0) = -\left(\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E)\right)^{-1}\sqrt{T}\Gamma'(\theta_0,h_E)Wg_T(\theta_0,h_E) + o_P(1)$$

To prove that $plim\ \hat{\Gamma}_T(\hat{\theta},h_E) = plim\ \hat{\Gamma}_T(\tilde{\theta},h_E) = \Gamma(\theta_0,h_E)$ we make the following decomposition

$$\hat{\Gamma}_T(\hat{\theta},h_E) - \Gamma(\theta_0,h_E) = \hat{\Gamma}_T(\hat{\theta},h_E) - \Gamma_T(\hat{\theta},h_E) + \Gamma_T(\hat{\theta},h_E) - \Gamma(\hat{\theta},h_E) + \Gamma(\hat{\theta},h_E) - \Gamma(\theta_0,h_E)$$

where the 1st difference is $o_P(1)$ by Lemma 2.5, the 3rd difference is $o_P(1)$ by the CMT and the consistency of $\hat{\theta}$, see Lemma 2.6, and the 2nd difference is $o_P(1)$ by Jennrich's ULLN. The ULLN can be applied if and only if $\sum_j h_E(z_{jt})\frac{\partial \xi_{jt}(\theta)}{\partial \theta}$ has an enveloppe with finite 1st absolute moments: $\xi_{jt}(\theta) = \rho^{-1}(s_t, x_{2t}, f_0(\cdot|\lambda)) - x'_{1jt}\beta$ and $\frac{\partial \xi_{jt}(\theta)}{\partial \beta} = x_{1jt}$ with x_{1jt} has finite moments of order 4 by Assumption B(iv), whereas $\frac{\partial \xi_{jt}(\theta)}{\partial \lambda} = \frac{\partial \rho^{-1}(s_t,x_{2t},f_0(\cdot|\tilde{\lambda}))}{\partial \lambda}$ and ρ^{-1} is C^∞ with arguments (s_t, x_{2t}, λ) which take values in a compact thus $\frac{\partial \rho^{-1}}{\partial \lambda}$ has bounds.

Thence $plim\ \hat{\Gamma}_T(\hat{\theta},h_E) = plim\ \hat{\Gamma}_T(\tilde{\theta},h_E) = \Gamma(\theta_0,h_E)$ which is full rank by Assumption D(ii), $plim\ \hat{W} = W$ which is full rank by Assumption D(iv), and by Lemma 2.5 $plim\ \sqrt{T}(\hat{g}_T(\theta_0,h_E) - g_T(\theta_0,h_E)) = 0$ so we can apply the aforementioned CMT and by the CLT which can be applied because $g(\theta_0,h_E) = 0$ under the null

$$\sqrt{T}(\hat{\theta}-\theta_0) = -\left(\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E)\right)^{-1}\sqrt{T}\Gamma'(\theta_0,h_E)Wg_T(\theta_0,h_E) + o_P(1)$$

$$\xrightarrow{d}\mathcal{N}(0,(\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E))^{-1}\Gamma'(\theta_0,h_E)W\Omega(\mathcal{F}_0,h_E)W\Gamma(\theta_0,h_E)$$
$$(\Gamma'(\theta_0,h_E)W\Gamma(\theta_0,h_E))^{-1})$$

□

B.4.6 Asymptotic distribution of the test statistic

Proof of Theorem 5.2

Proof. This proof leans heavily on the proof of Lemma 2.7. By Taylor's theorem there exists $\tilde{\theta}$ such that $||\tilde{\theta}-\theta_0||_2 \leqslant ||\hat{\theta}-\theta_0||_2$

$$\begin{aligned}
\sqrt{T}\hat{g}_T(\hat{\theta},h_D) &= \sqrt{T}\hat{g}_T(\theta_0,h_D)+\hat{\Gamma}_T(\tilde{\theta},h_D)\sqrt{T}(\hat{\theta}-\theta_0)\\
&= (I_{|h_D|_0} \quad -\Gamma(\theta_0,h_D)(\Gamma'(\theta_0,h_D)W\Gamma(\theta_0,h_D))^{-1}\Gamma'(\theta_0,h_D)W)\sqrt{T}\begin{pmatrix} g_T(\theta_0,h_D)\\ g_T(\theta_0,h_E)\end{pmatrix}+o_P(1)\\
&\equiv (I_{|h_D|_0} \quad G)\sqrt{T}\begin{pmatrix} g_T(\theta_0,h_D)\\ g_T(\theta_0,h_E)\end{pmatrix}+o_P(1)
\end{aligned}$$

The second equality is obtained by relying on the proof of Lemma 2.7 to express $\sqrt{T}(\hat{\theta}-\theta_0)$ as a function of moments, by relying on Lemma 2.5 so that *plim* $\sqrt{T}\hat{g}_T(\theta_0,h_D) =$ *plim* $\sqrt{T}g_T(\theta_0,h_D)$ and *plim* $\hat{\Gamma}_T(\tilde{\theta},h_D) =$ *plim* $\Gamma_T(\theta_0,h_D)$, and by using the CMT.

- Under $H_0 : f \in \mathcal{F}_0$ then $\mathbb{E}\left[\sum_j h_D(z_{jt})\xi_{jt}(\theta_0)\right] = 0$ by LIE. So using the CLT and Slutsky's Lemma we obtain

  $$\sqrt{T}\hat{g}_T(\hat{\theta},h_D) \xrightarrow{d} Z \sim \mathcal{N}(0,\Omega_0)$$

  where

  $$\Omega_0 = \begin{pmatrix} I_{|h_D|_0} & G\end{pmatrix}\begin{pmatrix}\Omega(\mathcal{F}_0,h_D) & \Omega(\mathcal{F}_0,h_D,h_E)\\ \Omega(\mathcal{F}_0,h_D,h_E)' & \Omega(\mathcal{F}_0,h_E)\end{pmatrix}\begin{pmatrix} I_{|h_D|_0}\\ G'\end{pmatrix}$$

  with

  $$\Omega(\mathcal{F}_0,h_D) = \mathbb{E}\Bigg[\Bigg(\sum_j \xi_{jt}(f(.|\lambda_0),\beta_0)h_D(z_{jt})\Bigg)\Bigg(\sum_j h_D(z_{jt})\xi_{jt}(f_0(.|\lambda_0),\beta_0)\Bigg)'\Bigg]$$

$$\Omega(\mathcal{F}_0, h_D, h_E) = \mathbb{E}\Bigg[\Bigg(\sum_j \xi_{jt}(f(.|\lambda_0), \beta_0) h_D(z_{jt})\Bigg)\Bigg(\sum_j h_E(z_{jt})\xi_{jt}(f_0(.|\lambda_0), \beta_0)\Bigg)'\Bigg]$$

$$G = -\Gamma(\theta_0, h_D)\left[\Gamma(\theta_0, h_E)' W \Gamma(\theta_0, h_E)\right]^{-1} \Gamma(\theta_0, h_E)' W$$

Thence by the continuous mapping theorem:

$$S(h_D, \mathcal{F}_0, \hat{\theta}) = \hat{g}_T(\hat{\theta}, h_D)' \hat{\Sigma} \hat{g}_T(\hat{\theta}, h_D) \xrightarrow{d} Z' \Sigma Z$$

□

- Under H_1' : $\mathbb{E}\left[\sum_j h_D(z_{jt})\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)\right] \neq 0$, we have by Lemma 2.5, by consistency of $\hat{\theta} \xrightarrow{P} \theta_0$ and the CMT:

$$\hat{g}_T(\hat{\theta}, h_D) = g_T(\theta_0, h_D) + o_P(1)$$

Thus by Assumption D(iv) and the CMT

$$\frac{S(h_D, \mathcal{F}_0, \hat{\theta})}{T} \xrightarrow{P} \underbrace{\mathbb{E}\left[\sum_j h_D(z_{jt})\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)\right]' \Sigma \mathbb{E}\left[\sum_j h_D(z_{jt})\xi_{jt}(f_0(\cdot|\lambda_0), \beta_0)\right]}_{\kappa(h_D, \mathcal{F}_0, \theta_0)}$$

Under H_1', $\kappa(h_D, \mathcal{F}_0, \theta_0)$ is strictly positive because Σ is positive definite. Thence,

$$\begin{aligned} \forall q \in \mathbb{R} \quad \underset{T\to\infty}{lim}\ \mathbb{P}(S(h_D, \mathcal{F}_0, \hat{\theta}) > q) &= \underset{T\to\infty}{lim}\ \mathbb{P}\left(\frac{S(h_D, \mathcal{F}_0, \hat{\theta}) - q}{T} > 0\right) \\ &= \mathbb{P}(\kappa(h_D, \mathcal{F}_0, \theta_0) > 0) \\ &= 1 \end{aligned}$$

where the 2nd equality holds because convergence in probability implies convergence in distribution.

□

B.4.7 Application of Theorem 5.2 to the 2 polar cases

1. **Sargan-Hansen J test**

   If $h_D = h_E$, with W and Σ are set to be equal to the GMM 2-step optimal weighting matrix

$$\Sigma = W = \mathbb{E}\Bigg[\Bigg(\sum_j \xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) h_E(z_{jt})\Bigg)\Bigg(\sum_j \xi_{jt}(f_0(\cdot|\lambda_0), \beta_0) h_E(z_{jt})\Bigg)'\Bigg]^{-1} = \Omega(\mathcal{F}_0, h_E)^{-1}$$

Then under H_0:

$$S(h_D, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} \chi^2_{|h_E|_0 - |\theta|_0}$$

Proof. By applying theorem 5.2, we have:

$$S(h_D, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} Z'\Sigma Z \quad \text{with} \quad Z \sim \mathcal{N}(0, \Omega_0)$$

If $h_D = h_E$ and $W = \Omega(\mathcal{F}_0, h_E)^{-1}$ then Ω_0 simplifies to

$$\begin{aligned}
\Omega_0 &= \Omega(\mathcal{F}_0, h_E) - \Gamma(\theta_0, h_E)\left[\Gamma(\theta_0, h_E)'\Omega(\mathcal{F}_0, h_E)^{-1}\Gamma(\theta_0, h_E)\right]^{-1}\Gamma(\theta_0, h_E)' \\
&= \Omega(\mathcal{F}_0, h_E)^{1/2} M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)} \Omega(\mathcal{F}_0, h_E)^{1/2}
\end{aligned}$$

with $M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)} \equiv I_{|h_E|_0} - P_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)}$ is the orthogonal projection on the space orthogonal to $\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)$. Let $\tilde{Z} \sim \mathcal{N}(0, I_{|h_E|_0})$, we have by definition:

$$\begin{aligned}
Z = \Omega(\mathcal{F}_0, h_E)^{1/2} M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)} \tilde{Z} &\implies \Sigma^{1/2} Z = M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)} \tilde{Z} \\
&\implies Z'\Sigma Z = \tilde{Z}' M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)} \tilde{Z}
\end{aligned}$$

Second line comes from symmetry and idempotence of $M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)}$. Orthogonal projections have eigenvalues equal to either 0 or 1 with the number of eigenvalues equal to one corresponding to the rank of the space it projects into, which in our case is $|h_E| - |\theta|_0$. If we denote by V the matrix of eigenvectors of $M_{\Omega(\mathcal{F}_0, h_E)^{-1/2}\Gamma(\theta_0, h_E)}$ then note that $V'\tilde{Z} \sim \mathcal{N}(0, I_{|h_E|_0})$ so that

$$Z'\Sigma Z = \sum_{k=1}^{|h_E|_0 - |\theta|_0} (V'\tilde{Z})_k^2 \sim \chi^2_{|h_E|_0 - |\theta|_0}$$

□

2. **Non-overlapping h_D and h_E**

   If Ω_0 is full rank and if the econometrician sets $\Sigma = \Omega_0^{-1}$, then our test statistic has the following asymptotic distribution under H_0:

$$S(h_T, \mathcal{F}_0, \hat{\theta}) \xrightarrow{d} \chi^2_{|h_D|_0}$$

   One sufficient condition for Ω_0 being full rank is $(\xi_{jt}(f(\cdot|\lambda_0), \beta_0))_{j=1}^J$ is independent across j and $(h_E(z_{jt}), h_D(z_{jt}))$ not being perfectly colinear.

Proof. The asymptotic result is direct; $(\xi_{jt}(f_0(\cdot|\lambda_0),\beta_0))_{j=1}^J$ being independent across j and $(h_E(z_{jt}),h_D(z_{jt}))$ not being perfectly colinear implies that

$$\Omega(\mathcal{F}_0,h_E,h_D)=\sum_j \mathbb{E}\left[\xi_{jt}(f_0(\cdot|\lambda_0),\beta_0)^2 h_E(z_{jt})h_D(z_{jt})'\right]$$

$$\Rightarrow \Omega_0=\sum_j (I_{|h_D|0}\quad G)Var\left(\xi_{jt}(f_0(\cdot|\lambda_0),\beta_0)\begin{pmatrix}h_D(z_{jt})\\ h_E(z_{jt})\end{pmatrix}\right)\begin{pmatrix}I_{|h_D|0}\\ G'\end{pmatrix}$$

Thus Ω_0 is positive definite because it is the sum of positive definite matrices. □

B.5 Properties of the MPI in the composite specification test: $f\in\mathcal{F}_0$

Proof of Proposition 4.

From corollary 5.1. Under Assumption A,

$$\begin{aligned}
H_a:\ f\notin\mathcal{F}_0 &\implies \quad \mathbb{E}[\xi_{jt}(f_0(\cdot|\lambda_0),\beta_0)|z_{jt}]\neq 0\ \ a.s\\
&\implies \mathbb{E}[\xi_{jt}(f_0(.|\lambda_0),\beta_0)|z_{jt}]^2>0\ a.s\\
&\implies \mathbb{E}\big[\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0),\beta_0)|z_{jt}]^2\big]>0\\
&\implies \mathbb{E}\big[\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0),\beta_0)\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0)|z_{jt}]|z_{jt}]\big]>0\\
&\implies \mathbb{E}\big[\xi_{jt}(f_0(.|\lambda_0),\beta_0)\mathbb{E}[\xi_{jt}(f_0(.|\lambda_0)|z_{jt}]\big]>0\\
&\implies \forall\alpha\neq 0\ \ H_1':\ \mathbb{E}\big[\xi_{jt}(f_0(.|\lambda_0),\beta_0)\underbrace{\alpha\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]}_{h_D^*(z_{jt})}\big]\neq 0
\end{aligned}$$

From theorem 5.2, under assumptions B-E,

$$H_a\implies \forall q\in\mathbb{R}^+,\ \ \mathbb{P}(S(h_D^*,\mathcal{F}_0,\hat{\theta})>q)\to 1$$

□

C Various Elements

C.1 Literature on the identification of the distribution of RC

In this section, we briefly summarize some recent findings on the identification of random coefficients in multinomial choice models has been extensively studied in the literature. In their seminal paper, Berry and Haile (2014) shows the identification of the demand functions ρ in a framework that encompasses the BLP model but their result does not entail identification of the random coefficients' distribution per se. To achieve their identification result, they require a completeness condition on the instruments as well as additional conditions (eg: connected substitutes) to ensure invertibility of the demand functions. They also need to impose that at least one of the product characteristic has a coefficient that is not random and that is equal to 1. Notice that in BLP model, the structure implied by the logit shock guarantees invertibility of the demand functions.

Fox et al. (2012) provides conditions under which the distribution of random coefficients is identified in a mixed logit model with micro-level data and no endogeneity. Their identification result requires continuous characteristics in x_{2t} and rules out interaction terms (eg polynomial terms of x_{2jt}). Moreover, their result is restricted to distributions of random coefficients with a compact support - excluding for instance a normally distributed random coefficient.

Fox and Gandhi (2011) investigates the identification of the joint distribution of random coefficients v_i and idiosyncratic shocks ε_{ijt} in aggregate demand models without endogeneity. They also consider a setting where endogeneity is introduced in a very restrictive way. They show identification under a special regressor assumption and finite support of the unobserved heterogeneity. The special regressor assumption assumes that a variable in x_{1t} has full support and has an associated coefficient that is either 1 or -1. This special regressor assumption is very common in the literature on the identification of random coefficients (see Ichimura and Thompson (1998), Berry and Haile (2009), Matzkin (2007) and Lewbel (2000)). Their framework does not nest the standard BLP model as ϵ_{ijt} and v_i are both assumed to have a finite support but it is more general in other dimensions. They do not exploit the logit distributional assumption on ε_{ijt}, they do not impose independence between v_i and ε_{ijt}, their identification argument can be extended to the case where multiple goods are purchased.

In a setting much closer to ours, Dunker et al. (2017) studies the identification of the distribution of random coefficients in endogenous aggregate demand models which includes the BLP model as a special case (in particular, no parametric assumption is made on the idiosyncratic shock ε_{ijt}). They make a clever use of the Radon transform to identify f. The price they have to incur for flexibility is that they need to make stringent assumptions on the product characteristics: variables in x_t are required to be continuous and to satisfy a joint full support assumption. The idea is to exploit the variation in the covariates in order to trace out the distribution of rc f. Unfortunately, these requirements are rarely met in real data sets.

In contrast to the rest of the literature, Wang (2021) adopts all the parametric assumptions in the standard BLP model and looks for the set of minimal assumptions under which the distribution of random coefficients is identified. This approach allows him to obtain sufficient conditions which are much less stringent than the rest of the literature (no special regressor assumption, no full support assumption, no continuity assumption). To be more specific, he shows that if the demand functions are identified on an open set of $\mathbb{R}^J$[36], then the distribution of random coefficients is identified. His proof astutely exploits the real analytic property of the demand functions[37].

C.2 Feasible MPI: conditional expectation

C.3 Choice of the large-T asymptotics

In this paper, we decide to study the asymptotics of our test when the number of markets T grows to infinity. First, the markets are considered to be independent from each other, which allows us to recover some independence between moments. Unless strong conditions are imposed on the demand shocks within a market, the general moments are connected through the contraction. What's more, from an economic stand point, it's hard to conceptually think of a market with an unlimited number of products.

C.4 Construction of the interval instruments in practice

We now provide more details on how to construct the interval instruments in practice. The procedure to construct the interval instruments is as follows:

1. Given $(\mathcal{F}_0, \hat{W}, h_E)$, the researcher derives the BLP estimator $\hat{\theta}$

[36] which can be achieved using theorem 1 in Berry and Haile (2014)

[37] In particular, the real analytic property yields that the local identification of ρ on $\mathcal{D} \subset \mathbb{R}^J$ implies identification of ρ on $\mathbb{R}^J$. From global identification of ρ, he is then able to show that the random coefficients' distribution is identified under a simple rank condition on x_{2t}

2. Then the researcher chooses L points $(v_l)_{l=1}^L \in \mathbb{R}^L$ in the presumed support of $f_0(\cdot|\hat{\lambda})$.

3. Finally, the researcher can construct a set of L interval instruments based on the approximations of the MPI that we develop in sections 4.2 and 4.1.

   - Global approximation: $\{\pi_{j,l}(z_{jt})\}_{l=1,\dots,L}$ interval instruments which are such that:

$$\mathbb{E}\left[\Delta_j(s_t, x_{2t}, f_0, f_a)|z_{jt}\right] \approx \log\left(\sum_{l=1}^{L} \omega_l \ \pi_{j,l}(z_{jt})\right) \text{ with } \ \pi_{j,l}(z_{jt}) = \frac{\frac{\exp(x'_{2jt}v_l)}{1+\sum_{k=1}^{J}\exp\left\{\hat{\delta}_{kt}^0 + x'_{2kt}v_l\right\}}}{\int_{\mathbb{R}^{K_2}} \frac{\exp(x'_{2jt}v)}{1+\sum_{k=1}^{J}\exp\left\{\hat{\delta}_{jt}^0 + x'_{2kt}v\right\}} f_0(v)dv}$$

     with $\hat{\delta}_t^0$ the linear projection of δ_t^0 on z_{jt} (or a carefully chosen subset of z_{jt}).

   - Local approximation: $\{\bar{\pi}_{j,l}(z_{jt})\}_{l=1,\dots,L}$ interval instruments such that

$$\mathbb{E}[\Delta_j(s_t, x_{2t}, f_0, f_a)|z_{jt}] \approx \sum_{l=1}^{L} \bar{\omega}_l \ \bar{\pi}_{j,l}(z_{jt})$$

$$\text{with } \bar{\pi}_{j,l}(z_{jt}) = \left(\frac{\partial \rho(\hat{\delta}_t^0, x_{2t}, f_0)}{\partial \delta}\right)^{-1} \left[\frac{\exp(\hat{\delta}_t^0 + x_{2t}v_l)}{1 + \sum_{k=1}^{J}\exp\left\{\hat{\delta}_{kt}^0 + x'_{2kt}v_l\right\}} - \rho_j(\hat{\delta}_t^0, x_{2t}, f_0)\right]$$

     with $\hat{\delta}_t^0$ the linear projection of δ_t^0 on z_{jt} (or a carefully chosen subset of z_{jt}).

Choice of the L points in the domain of f_a The researcher doesn't know a priori the support of the true density f_a. Thus, he/she must choose points in the domain of definition of f_a. If this choice coincides with points of the support where $|f_0(\cdot|\lambda_0) - f_a|$ is large, then this choice generates more informative instruments. In practice, one can take points in the high density regions of $f_0(\cdot|\lambda_0)$ (eg if $\mathcal{F}_0$ is the Gaussian family, then one can take points around the mean λ_0). The choice of of the number of instruments N obeys a usual bias variance tradeoff. On the one hand, a large L allows to better approximate the MPI and thus increases the detection ability of the instruments. On the other hand, it is well-known that a larger number of instruments can induce finite sample bias and can distort asymptotic distributions of estimators and tests such as the over-identification test[38]; For these reasons we advise not to use too few or too many interval instruments, in our simulations and application we use between 10 and 20 instruments. We leave a formal analysis of the optimal choice of L and of the general approximations properties of the interval instruments for future work.

[38]see Roodman (2009) for a review on the effect of many possibly weak moments on estimation and testing

C.5 Estimation procedure when the distribution of RC is a mixture

In this section, we present a procedure to estimate the BLP model when the distribution of RC is parametrized as a mixture. Namely, we perform the estimation under $H_0 : f \in \mathcal{F}_0$ with $\mathcal{F}_0$ the family of Gaussian mixtures with L components. The pdf of a Gaussian mixture writes as follows:

$$\forall x \in \mathbb{R}\ ,\ \ f_0(x|\lambda_0) = \sum_{l=1}^{L} p_{l0} f_l(x|\lambda_{l0}) \qquad \sum_{l=1}^{L} p_{l0} = 1 \qquad L \geqslant 1$$

where $f_{l0}(\cdot|\lambda_{l0})$ is the pdf of a $\mathcal{N}(\mu_{l0}, \sigma_{l0}^2)$.

As long as the means are different ($\mu_{l0} \neq \mu_{l'0}\ \forall l \neq l'$), the gaussian mixture is uniquely characterized by the vector $\lambda_0 = (p_{10}, \ldots, p_{L0}, \mu_{10}, \ldots, \mu_{L0}, \sigma_{10}^2, \ldots, \sigma_{L0}^2)$ up to permutations of indexes[39]. The objective of our procedure is to estimate the parameters of the model $\theta_0 = (\beta_0, \lambda_0)$ where λ_0 characterizes the mixture. In general, the problem of estimating a density by a mixture is solved through the use of the well-known Expectation-Maximization (EM) algorithm. In our case, the application of this algorithm is made difficult by two main obstacles. First, we do not observe directly the random coefficients. Second, we do not have individual choice data which would have enabled us to construct a likelihood as in Train (2008). As an alternative, we propose to adapt the BLP estimation procedure to estimate the parameters of a mixture of gaussians instead of the single normal distribution. The mixture affects the derivation of the market shares. The random coefficient v_i is now a gaussian mixture. Hence, $v_i = \sum_{l=1}^{L} \mathbf{1}\{D_i = l\} v_{il}$ where $(v_{il})_{i=1}^n$ are iid and have density $f_{l0}(\cdot|\lambda_{l0})$ known up to λ_{l0} for $l = 1, \ldots, L$, and where $(D_i)_{i=1}^n$ are iid categorically distributed with pmf $\mathbb{P}(D_i = l) = p_{l0}$. For all market t and product j, the demand functions are as follows:

$$\begin{aligned}
\rho_j(\delta_t, x_{2t}, f_0(.|\lambda_0)) &= \mathbb{P}(j \text{ chosen in market } t \text{ by } i|x_{1t}, x_{2t}, \xi_t) \\
&= \int_{\mathbb{R}} \frac{\exp\{x_{1jt}'\beta_0 + x_{2jt}'v + \xi_{jt}\}}{1 + \sum_{j'=1}^{J} \exp\{x_{1j't}'\beta_0 + x_{2j't}'v + \xi_{j't}\}} f_0(v|\lambda_0) dv \\
&= \sum_{l=1}^{L} p_{l0} \int_{\mathbb{R}} \frac{\exp\{\delta_{jt} + x_{2jt}'v\}}{1 + \sum_{j'=1}^{J} \exp\{\delta_{j't} + x_{2j't}'v\}} f_{l0}(v|\lambda_{l0}) dv
\end{aligned}$$

[39] If for some $l \neq l'$ we have $\mu_{l0} = \mu_{l'0}$ then the Gaussian mixture becomes observationally equivalent to an infinite number of other Gaussian mixtures

Reparametrization. The parameter λ associated with the mixture consists of the means, the standard deviation and the probability of each component. As highlighted by Ketz (2019) in the simple Gaussian case, the way we parametrize the model can greatly affect the asymptotic properties of the estimator as well as the quality of the estimation. In particular, he shows that the standard deviations σ should be reparametrized in order to avoid boundaries issues when σ close to 0. We follow this parametrization and perform the minimization with respect to $\{(+/-)\sqrt{\sigma_l}\}_{l=1}^{L}$ instead and $(\sigma_l)_{l=1}^{L}$ directly. An additional difficulty in the case of mixtures concerns the estimation of the probabilities associated to each component. These probabilities must all be between 0 and 1 and their sum must be equal to 1. To smoothly integrate these constraints, we perform the optimization with respect to $\gamma = (\gamma_2, \ldots, \gamma_L)$ with $p = (p_1, p_2, \ldots, p_L) = (\frac{1}{1+\sum_{l=2}^{L} exp(\gamma_l)}, \frac{exp(\gamma_2)}{1+\sum_{l=2}^{L} exp(\gamma_l)}, \ldots, \frac{exp(\gamma_L)}{1+\sum_{l=2}^{L} exp(\gamma_l)})$.

Estimation details. Apart from the modification in the computation of the market shares and the new parametrization of the model, the estimation procedure with a mixture follows closely the traditional one and the parameters of interest are estimated by minimizing a GMM criterion. Let $\mathcal{Q}(\theta)$ the GMM objective function:

$$\mathcal{Q}(\theta) = \hat{\xi}(\theta)' h_E(Z) W h_E(Z)' \hat{\xi}(\theta)$$

We now describe the derivation of the Gradient that we provide to the minimization program.

$$\frac{\partial \mathcal{Q}}{\partial \theta} = 2 \left[\frac{\partial \hat{\xi}(\theta)}{\partial \theta} \right]' h_E(Z) W h_E(Z)' \hat{\xi}(\theta)$$

Where $\frac{\partial \hat{\xi}(\theta)}{\partial \beta} = -x_1$ and where by the implicit function theorem we have $\hat{\rho}_j(\delta_t, x_{2t}, \lambda) - s_{jt} = 0 \ \ \forall j, t$ which implies:

$$\frac{\partial \hat{\xi}(\theta)}{\partial \lambda} = \frac{\partial \hat{\delta}(\theta)}{\partial \lambda} = - \left[\frac{\partial \hat{\rho}(\delta, x_2, \lambda)}{\partial \delta} \right]^{-1} \frac{\partial \hat{\rho}(\delta, x_2, \lambda)}{\partial \lambda}$$

- $\frac{\partial \rho}{\partial \delta}$ is a $JT \times JT$ diagonal by block matrix such that:

$$\frac{\partial \rho_j(\delta_t, x_{2t}, \lambda)}{\partial \delta_{kt}} = \begin{cases} \sum_l p_l \int \mathcal{T}_{jlt}(v) \left(1 - \mathcal{T}_{klt}(v)\right) \phi_l(v) dv & if \quad j = k \\ -\sum_l p_l \int \mathcal{T}_{jlt}(v) \mathcal{T}_{klt}(v) \phi_l(v) dv & if \quad j \neq k \end{cases}$$

with $\mathcal{T}_{jlt}(v) \equiv \frac{exp\{\delta_{jt}+x'_{2jt}v_l\}}{1+\sum_{j'=1}^{J} exp\{\delta_{j't}+x'_{2j't}v_l\}}$

- $\frac{\partial \rho}{\partial \lambda}$ is a $JT \times (3L-1)$ matrix such that:

$$\frac{\partial \rho_j\left(\delta_t, x_{2t}, \lambda\right)}{\partial \mu_l} = p_l \int \mathcal{T}_{jlt}\left(x_{2jt} - \sum_{j'} \mathcal{T}_{j'lt} x_{2j't}\right)\phi(v)dv$$

$$\frac{\partial \rho_j\left(\delta_t, x_{2t}, \lambda\right)}{\partial \sigma_l} = p_l \int \mathcal{T}_{jlt}\left(x_{2jt} - \sum_{j'} \mathcal{T}_{j'lt} x_{2j't}\right)v\phi(v)dv$$

$$\frac{\partial \rho_j\left(\delta_t, x_{2t}, \lambda\right)}{\partial \gamma_l} = \sum_{l'=1}^{L} \zeta(l,l') \int \mathcal{T}_{jlt}$$

With $\zeta(l,l') = \frac{-exp(\gamma_l)}{1+\sum_{k\neq 1} exp(\gamma_k)} \times \frac{exp(\gamma_{l'})}{1+\sum_{k\neq 1} exp(\gamma_k)} + \mathbf{1}\{l=l'\}\frac{exp(\gamma_l)}{1+\sum_{k\neq 1} exp(\gamma_k)} = -p_l \times p_{l'} + \mathbf{1}\{l=l'\}p_l$

C.6 Properties of the feasible approximations of the MPI

So far we have studied the properties of the MPI, which is an ideal instrument that cannot be derived in practice. Nevertheless, in light of the previous results, the MPI provides a useful upper bound on the power that can be reached using our specification test. More precisely, the asymptotic slope reached by the MPI can be interpreted as a power envelope on our specification test. Ideally, we want our specification test, with the approximated MPIs as instruments, to achieve slopes close to the ones reached by the MPI. For the sake of exposition, let us assume homoskedasticity. We now distinguish 2 situations.

First, we consider the case where the econometrician tests H_0 against the true alternative $\bar{H}_a$: $(f, \beta) = (f_a, \beta_a)$. This situation is not interesting in practice as the econometrician usually doesn't know the true alternative and doesn't want to specify an alternative. Nevertheless, it illustrates that in this specific case, we can (in theory) derive a consistent estimator of the MPI. Indeed, in this particular case, we can directly derive an analytical expression for the correction term $\Delta_{0,a}^{\xi_{jt}}$ either using its definition or the expression in 4.2. Next, we must to compute the conditional expectation of our the correction term with respect to z_{jt}. This step is quite challenging because the dimension of z_{jt} is large and because the correction term is heavily non-linear and non-separable with respect to the endogenous variables. In theory, a solution is to perform a Sieve non-parametric estimation of the conditional mean and under standard regularity conditions recover a consistent estimator of $\mathbb{E}[\Delta_{0,a}^{\xi_{jt}}|z_{jt}]$. Unfortunately, the rate of converge will be extremely slow given the dimension of z_{jt} and we don't recommend to do this in practice. Instead, we suggest to use the global approximation and to "exogenize" the endogenous variables by projecting them on the space spanned by a relevant subset of z_{jt}. As we show in the appendix, this strategy yields an estimator which converges faster to a first order approximation of the MPI.

Second, we consider the more realistic situation where the econometrician tests H_0 against an unspecified alternative. This is the situation of interest in this paper. In this case, we use the interval instruments that we developed in section 4 as an approximation of the MPI. Due to the different layers of approximations which intervene in the construction of these instruments and the absence of knowledge of f_a, it is quite difficult to establish conditions under which these instruments can reach the optimal slope of the MPI. A thorough analysis of the properties of these instruments is beyond the scope of this paper and may constitute an interesting starting point for future research. In the appendix, we present a preliminary investigation on the theoretical properties of the local approximation. In spite of the lack of theoretical analysis, our Monte Carlo exercises show that the interval instruments perform really well in finite sample.

Approximation properties of the interval instruments. The interval instruments, ie the approximation of the MPI denoted as $\hat{h}_T^*$, work well in practice in the sense that they yield a valid test which is powerful. However it is difficult to prove that the speed of divergence of our test when using them is as large as the speed of divergence when using the true MPI without further assumptions. As described in the previous subsection there are 3 levels of approximation to the MPI: First only the 1st order of the expansion of the difference between the true error and generated error is considered, another term $\mathcal{R}_0$ remains[40]; Second the conditional expectation with respect to the full set of instruments is approximated via projections; Third the integral which appears within this 1st order approximation is estimated via a Riemann sum of N points. Consequently if $\mathcal{R}_0$ is

[40] we have obtained the formula of the approximation of the difference between the true error and the generated error up to the second order

negligible, if N is very large, and if projecting the difference between the generated error and the true error is equivalent to taking its conditional expectation with respect to the instruments, then the slope $C_{\hat{h}_T^*}$ is equal to $C_{h_T^*}$. This result is summarized in the following proposition:

Proposition 8.

Under Assumption B and C, and assuming strict homoskedasticity $\mathbb{E}(\xi_{t0}\xi_{t0}'|z_t) = I_J$ then under $H_1 : f \notin \mathcal{F}_0$ there exists some sequence $(\alpha)_{l=1}^N$ such that

$$\hat{h}_T^*(z_{jt})'\alpha + err_{jt} + e\tilde{r}r_{jt} \underset{N\to+\infty}{\longrightarrow} h_T^*(z_{jt})$$

almost surely, for some errors $err_{jt} \in \mathbb{R}^N$ and $e\tilde{r}r_{jt} \in \mathbb{R}$ such that $e\tilde{r}r_{jt} \underset{N\to+\infty}{\overset{as}{\longrightarrow}} 0$. As a consequence

$$C_{\hat{h}_T^*} = \mathbb{E}\left(\sum_j \alpha'\hat{h}_T^*(z_{jt})\hat{h}_T^*(z_{jt})'\alpha\right) + error + err\tilde{o}r$$

for some $error \in \mathbb{R}$ and some $err\tilde{o}r \in \mathbb{R}$ such that $err\tilde{o}r \underset{N\to+\infty}{\overset{as}{\longrightarrow}} 0$. If $error = 0$ then

$$C_{\hat{h}_T^*} \underset{N\to+\infty}{\longrightarrow} C_{h_T^*} = \mathbb{E}\left(\sum_j \mathbb{E}(\tilde{\Delta}_{jt}(s_t, x_{2t}, \mathcal{F}_0, f)|z_{jt})^2\right)$$

To further comment on this result err_{jt} ($error$) corresponds to the first and second errors of approximations of the MPI described above and $e\tilde{r}r_{jt}$ ($err\tilde{o}r$) corresponds to the third; On the other hand $(\alpha)_{l=1}^N$ is a sequence of integration weights whose empirical mean converge to 0. In addition there are 2 conditions necessary for $error = 0$. The first and most important one is that $\mathcal{R}_0$ shoud be close to 0, in other words the 1st order approximation should explain most of the difference between the generated error and the true error. The second condition for $error$ to be close to 0 is very likely to be satisfied in practice: We need to be able to approximate well the conditional expectation with respect to z_{jt} of the 1st order approximation of $\tilde{\Delta}$. As noted by Reynaert and Verboven (2014), because most product characteristics are uncorrelated with the unobserved product characteristics, using a Sieve estimator of the conditional expectation or a more practical method as is described in our paper or theirs does not seem to make a lot of difference. If these two conditions are satisfied then err_{jt} is small and therefore $error$ is small.

C.6.1 Proof of Proposition 8

Using the strict homoskedasticity assumption then from ?? we know that

$$C_{h_T^*} = \mathbb{E}(\mathbb{E}(\tilde{\Delta}_t(\mathcal{F}_0, f)|z_t)'\mathbb{E}(\tilde{\Delta}_t(\mathcal{F}_0, f)|z_t)) = \mathbb{E}(h_T^*(z_t)'h_T^*(z_t))$$

Our goal is therefore to prove that under some conditions

$$\underset{N\to+\infty}{lim} C_{\hat{h}_T^*} = C_{h_T^*}$$

and we do so in four steps: First we prove that there exists some (err_1, err_2, err_3) such that

$$\hat{h}_T^*(z_t)'\alpha + err_{1t} + err_{2t} + err_{3t} \underset{N\to+\infty}{\to} h_T^*(z_t)$$

Second we show that $err_{3t} \underset{N\to+\infty}{\to} 0$ almost surely; Third we show that there exists some $\tilde{h}_T^*(z_t)$ and some $(error, er\tilde{r}or)$ such that

$$C_{\hat{h}_T^*} = C_{\tilde{h}_T^*}, \qquad C_{h_T^*} = \alpha'\mathbb{E}(\tilde{h}_T^*(z_t)'\tilde{h}_T^*(z_t))\alpha + error + er\tilde{r}or$$

and $er\tilde{r}or \underset{N\to+\infty}{\overset{as}{\to}} 0$; Fourth we conclude. We prove each point in order:

- Denote and recall

$$\begin{aligned}
\eta_{jt} &= \int \frac{exp(\hat{\delta}_{jt}^0 + x_{2jt}'v)}{1 + \sum_{k=1}^J exp(\hat{\delta}_0 + x_{2kt}'v)}(f(v) - f_0(v|\lambda_0))dv \\
\hat{\eta}_{jt,l} &= \frac{exp(\hat{\delta}_{jt}^0 + x_{2jt}'v_l)}{1 + \sum_{k=1}^J exp(\hat{\delta}_0 + x_{2kt}'v_l)} \\
\Rightarrow \hat{\eta}_{jt}'\alpha &= \sum_l \alpha_l \frac{exp(\hat{\delta}_{jt}^0 + x_{2jt}'v_l)}{1 + \sum_{k=1}^J exp(\hat{\delta}_0 + x_{2kt}'v_l)} \\
M_t(\cdot) &= x_{1t}\left(\mathbb{E}\left[\sum_j x_{1jt}h_E(z_{jt})'\right]W\mathbb{E}\left[\sum_j h_E(z_{jt})x_{1jt}'\right]\right)^{-1}\mathbb{E}\left[\sum_j x_{1jt}h_E(z_{jt})'\right]W\mathbb{E}\left[\sum_j h_E(z_{jt})\cdot\right] \\
\hat{M} &= \hat{x}_1\left[x_1'h_E(z)(h_E(z)'h_E(z))^{-1}h_E(z)'x_1\right]^{-1}x_1'h_E(z)\hat{W}h_E(z)' \\
M_{t,\partial\rho}^{-1} &= \left(\frac{\partial\rho(\delta_t^0, x_{2t}, f_0(\cdot|\lambda_0)}{\partial\delta}\right)^{-1} \\
\hat{M}_{t,\partial\rho}^{-1} &= \left(\frac{\partial\rho(\hat{\delta}_t^0, \hat{x}_{2t}, f_0(\cdot|\hat{\lambda})}{\partial\delta}\right)^{-1}
\end{aligned}$$

where $(\hat{x}_1, \hat{x}_2, \hat{\delta}^0)$ are transformations of (x_1, x_2, δ^0) (for instance their projection on the instruments) as described in **??**. Then define $\hat{M}_{\partial\rho}$ of dimension $(J \times T) \times (J \times T)$ which is block diagonal with T blocks of dimension $J \times J$ equal to $\hat{M}^{-1}_{t,\partial\rho}$, and define $\hat{\eta}$ which is the stacked versions of $\hat{\eta}_{jt}$. Consequently

$$\begin{aligned}
h^*_T(z_t) = \mathbb{E}(\tilde{\Delta}(\mathcal{F}_0, f)|z_t) &= \mathbb{E}\left((id - M_t)\Delta(s_t, x_{2t}, \mathcal{F}_0, f)|z_t\right) \\
&= \mathbb{E}\left((id - M_t)(M^{-1}_{t,\partial\rho}\eta_t + \mathcal{R}_0)|z_t\right) \\
\hat{h}^*_T(z_t)\alpha &= A_t \hat{h}_T(z)\alpha \\
&= A_t(I_{J\times T} - \hat{M})\hat{\Delta}'_N\alpha \\
&= A_t(I_{J\times T} - \hat{M})\hat{M}^{-1}_{\partial\rho}\hat{\eta}\alpha
\end{aligned}$$

where A_t is the matrix which picks the J observations in t, ie A_t is a $J \times (J \times T)$ matrix of zeros except the block from column $(J-1)t + 1$ to Jt which is equal to I_J. In other words

$$\begin{aligned}
h^*_T(z_t) &= \hat{h}^*_T(z_t)\alpha + \mathbb{E}\left((id - M_t)\mathcal{R}_0|z_t\right) \\
&\quad + \left[\mathbb{E}((id - M_t)M^{-1}_{t,\partial\rho}\eta_t|z_t) - A_t(I_{J\times T} - \hat{M})\hat{M}^{-1}_{\partial\rho} \lim_{N\to+\infty} \hat{\eta}\alpha\right] \\
&\quad + \left[\lim_{N\to+\infty} (\hat{h}^*_T(z_t)\alpha) - \hat{h}^*_T(z_t)\alpha\right] \\
&\equiv \hat{h}^*_T(z_t)\alpha + err_{1t} + err_{2t} + err_{3t}
\end{aligned}$$

- Next clearly if $(v_l, c_{l,N})_{l=1}^N$ are chosen so that $\forall l \quad v_l$ is in the support of $f(\cdot) - f_0(\cdot|\hat{\lambda})$ and $v_{l+1} - v_{l+1} = c_{l,N} \underset{N\to+\infty}{\to} 0$ then the Riemann sum

$$\hat{\eta}'_{jt}\alpha = \sum_l \alpha_l \frac{exp(\hat{\delta}^0_{jt} + x'_{2jt}v_l)}{1 + \sum_{k=1}^J exp(\hat{\delta}_0 + x'_{2kt}v_l)} = \sum_l \frac{c_{l,N}}{N}(f(v_l) - f_0(v_l|\hat{\lambda}))\frac{exp(\hat{\delta}^0_{jt} + x'_{2jt}v_l)}{1 + \sum_{k=1}^J exp(\hat{\delta}_0 + x'_{2kt}v_l)}$$

converges to $\int \frac{exp(\hat{\delta}^0_{jt}+x'_{2jt}v)}{1+\sum_{k=1}^J exp(\hat{\delta}_0+x'_{2kt}v)}(f(v) - f_0(v|\hat{\lambda}))dv$ almost surely when $N \to +\infty$, see arguments for the convergence of Riemann sums. This integral exists by Assumption C(ii) and implicitly by Assumption **??**(i)-(ii). Therefore for any $t \quad err_{3t} \underset{N\to+\infty}{\to} 0$ almost surely, which corresponds to $\tilde{err}_t$ in the Proposition.

- Next for a fixed N, by Assumption **??** using the LLN and the CMT, there exists some $\tilde{h}^*_T(z_t)$ which is the "probability limit" of $\hat{h}^*_T(z_t)$ in the sense that

$$\frac{1}{T}S(\hat{h}^*_T, \mathcal{F}_0, \theta_0) = \frac{1}{T}S(\tilde{h}^*_T, \mathcal{F}_0, \theta_0) + o_P(1), \qquad \tilde{h}^*_T(z_t) = (id - \tilde{M})\tilde{M}^{-1}_{t,\partial\rho}\tilde{\eta}_t, \qquad C_{\hat{h}^*_T} = C_{\tilde{h}^*_T}$$

where $BLP(\cdot|z_t)$ is the best linear projection operator and

$$\tilde{M}_t(\cdot) = \tilde{x}_{1t}\left(\mathbb{E}\left[\sum_j x_{1jt}h_E(z_{jt})'\right]W\mathbb{E}\left[\sum_j h_E(z_{jt})x'_{1jt}\right]\right)^{-1}\mathbb{E}\left[\sum_j x_{1jt}h_E(z_{jt})'\right]W\mathbb{E}\left[\sum_j h_E(z_{jt})\,\cdot\,\right]$$

$$\tilde{M}_{t,\partial\rho} = \left(\frac{\partial\rho(\tilde{\delta}^0_t,\tilde{x}_{2t},\mathcal{F}_0,\lambda_0)}{\partial\delta}\right)^{-1}$$

$$\tilde{\eta}_{jt,l} = \frac{exp(\tilde{\delta}^0_{jt}+\tilde{x}'_{2jt}v_l)}{1+\sum_k exp(\tilde{\delta}^0_{kt}+\tilde{x}'_{2kt}v_l)}$$

$$\tilde{\delta}^0_t = \delta^0_t - BLP(\delta^0_t|z_t)$$

$$\tilde{x}_{1t} = x_{1t} - BLP(x_{1t}|z_t)$$

$$\tilde{x}_{2t} = x_{2t} - BLP(x_{2t}|z_t)$$

As a consequence $h^*_T(z_t)$ rewrites

$$\begin{aligned}
h^*_T(z_t) &= \tilde{h}^*_T(z_t)\alpha + \mathbb{E}\left((id-M_t)\mathcal{R}_0|z_t\right)\\
&\quad+\left[\mathbb{E}((id-M_t)M^{-1}_{t,\partial\rho}\eta_t|z_t) - (id-\tilde{M}_t)(\tilde{M}^{-1}_{t,\partial\rho}\lim_{N\to+\infty}\tilde{\eta}_t\alpha)\right]\\
&\quad+\left[\lim_{N\to+\infty}(\tilde{h}^*_T(z_t)\alpha)-\tilde{h}^*_T(z_t)\alpha\right]\\
&\equiv \tilde{h}^*_T(z_t)\alpha + e\tilde{r}r_{1t} + e\tilde{r}r_{2t} + e\tilde{r}r_{3t}\\
\Rightarrow C_{h^*_T(z_t)} &\equiv \alpha'\mathbb{E}(\tilde{h}^*_T(z_t)'\tilde{h}^*_T(z_t))\alpha + er\tilde{r}or + error
\end{aligned}$$

where $er\tilde{r}or$ is a function of $e\tilde{r}r_{3t}$ and therefore converges to 0 almost surely as $N\to+\infty$ and $error$ is a function of $\tilde{h}^*_T(z_t)$, $e\tilde{r}r_{1t}$ and $e\tilde{r}r_{2t}$.

- From the previous point if $e\tilde{r}r_{1t} = e\tilde{r}r_{2t} = 0$, ie

$$\mathcal{R}_0 = 0, \qquad \left[\mathbb{E}((id-M_t)M^{-1}_{t,\partial\rho}\eta_t|z_t) - (id-\tilde{M}_t)(\tilde{M}^{-1}_{t,\partial\rho}\lim_{N\to+\infty}\tilde{\eta}_t\alpha)\right] = 0$$

Then $h^*_T(z_t) \underset{N\to+\infty}{=} \tilde{h}^*_T(z_t)\alpha$ thus $C_{h^*_T} \underset{N\to+\infty}{=} C_{\tilde{h}^*_T\alpha} = C_{\hat{h}^*_T\alpha}$. Finally using the properties of best linear projections it can be shown that $C_{\hat{h}^*_T} = C_{\tilde{h}^*_T} \geqslant C_{\tilde{h}^*_T\alpha} = C_{\hat{h}^*_T\alpha}$ so that $\lim_{N\to+\infty} C_{\hat{h}^*_T} = C_{h^*_T\alpha}$ because $C_{h^*_T\alpha}$ also constitutes an upper bound on $C_{\hat{h}^*_T}$. Indeed

$$\begin{aligned}
C_{\tilde{h}^*_T} &= \mathbb{E}\left(\tilde{\Delta}_t(\mathcal{F}_0,f)'\tilde{h}^*_T(z_t)\right)\mathbb{E}\left(\tilde{h}^*_T(z_t)'\xi_{0t}\xi'_{0t}\tilde{h}^*_T(z_t)\right)^{-1}\mathbb{E}\left(\tilde{h}^*_T(z_t)'\tilde{\Delta}_t(\mathcal{F}_0,f)\right)\\
&= \mathbb{E}\left(\tilde{\Delta}_t(\mathcal{F}_0,f)'\tilde{h}^*_T(z_t)\right)\mathbb{E}\left(\tilde{h}^*_T(z_t)'\tilde{h}^*_T(z_t)\right)^{-1}\mathbb{E}\left(\tilde{h}^*_T(z_t)'\tilde{\Delta}_t(\mathcal{F}_0,f)\right)\\
&\geqslant C_{\tilde{h}^*_T\alpha} = \mathbb{E}\left(\tilde{\Delta}_t(\mathcal{F}_0,f)'\tilde{h}^*_T(z_t)\right)\alpha\mathbb{E}\left(\alpha'\tilde{h}^*_T(z_t)'\tilde{h}^*_T(z_t)\alpha\right)^{-1}\alpha\mathbb{E}\left(\tilde{h}^*_T(z_t)'\tilde{\Delta}_t(\mathcal{F}_0,f)\right)
\end{aligned}$$

where the first second equality is due to the fact that we assume strict exogeneity $\mathbb{E}(\xi_{0t}\xi_{0t}|z_t) = I_J$, and the inequality is due to the fact that the best linear projection of $\tilde{\Delta}_t(\mathcal{F}_0, f)$ on the subspace $\tilde{h}^*_T(z_t)\alpha$ always has lower second moment compared to the best linear projection of $\tilde{\Delta}_t(\mathcal{F}_0, f)$ on the space $\tilde{h}^*_T(z_t)$.

D Monte Carlo experiments

D.1 Counterfactuals under an alternative distribution

Expressions for price and cross-price elasticities as a function of p_1 in the simulation exercise presented in section 6.2

- Price elasticity:

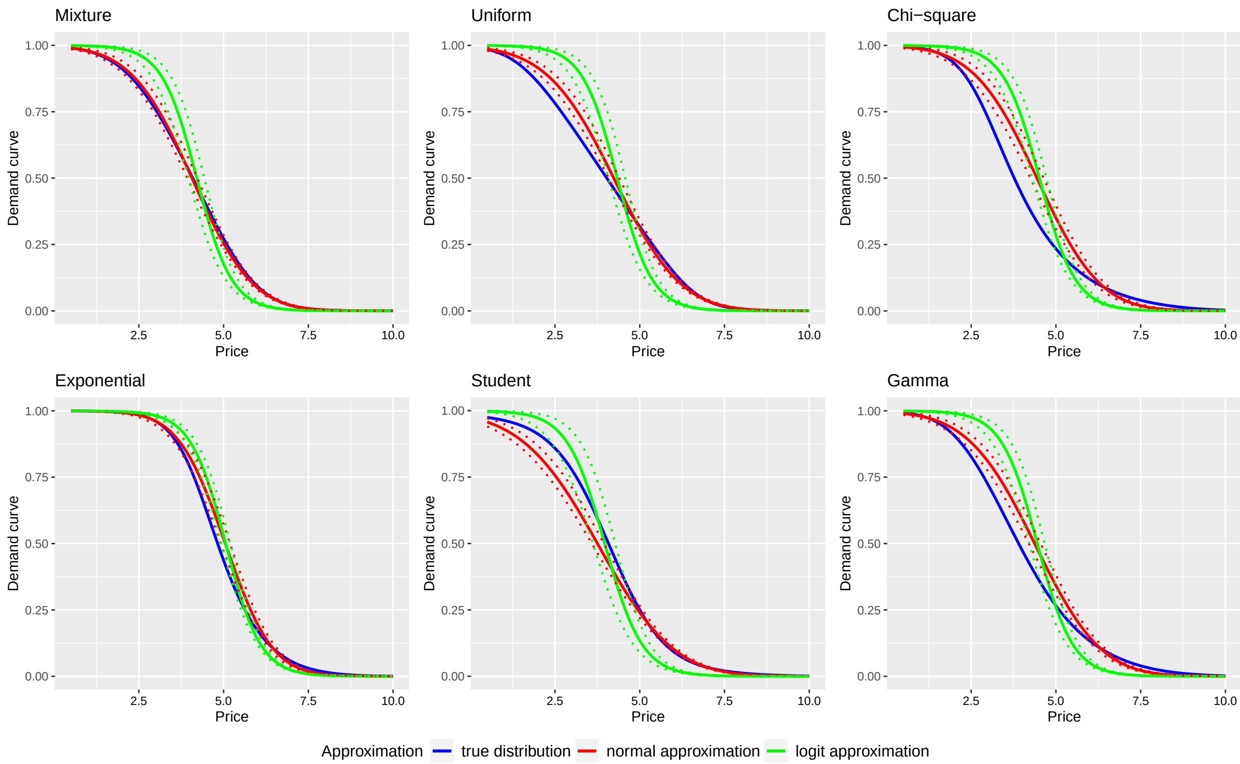
$$\xi_1 = \frac{p_1}{s_1}\frac{\partial s_1}{\partial p_1} = \int -\alpha \left(1 - \frac{\exp\{u_{i1}\}}{1 + \sum_{j' \in \{1,2\}} \exp\{u_{ij'}\}}\right) \frac{exp\{u_{i1}\}}{1 + \sum_{j' \in \{1,2\}} \exp\{u_{ij'}\}} f_\theta(v) dv \phi(\alpha) d\alpha$$

- Cross price elasticity:

$$\xi_{2/1} = \frac{p_1}{s_2}\frac{\partial s_2}{\partial p_1} = \int \alpha \left(\frac{\exp\{u_{i1}\}}{1 + \sum_{j' \in \{1,2\}}^J \exp\{u_{ij'}\}}\right) \frac{\exp\{u_{i2}\}}{1 + \sum_{j' \in \{1,2\}}^J \exp\{u_{ij'}\}} f_\theta(v) dv \phi(\alpha) d\alpha$$

Here, we plot the demand functions generates by the different approximations of the true densities

Figure 11: Demand function



D.2 Finite sample performance of the test

Practical implementation of the test. For each setting, we estimate the model for 1000 replications. Minimization is performed with nloptr (algorithm: NLOPT-LD-LBFGS). We provide an analytical gradient. The Threshold for the outer loop is 1e-9 while the threshold for the inner loop is 1e-13. We use squarem and a C++ implementation for the computation of the market shares to speed up the contraction. We also parallelize the contraction over markets using 7 independent cores. Now we formally describe the instruments included in each test.

Instruments

- J(1): differentiation instruments + exogenous characteristics (polynomial terms) + cost shifters (15 instruments/ degrees of overidentification:8)

- I(1): first stage instruments: instruments J(1). testing instruments: Interval Instruments: 7 instruments. Points chosen as follows: $\{\hat{\mu}, (\hat{\mu} + k(max(0.25, \hat{\sigma})), k(max(0.25, \hat{\sigma}))\}$ $(for\ k = 1, 2, 3)$

- J(2): first stage: instruments: instruments J(1). Second stage instruments: optimal instruments (approximation of $\mathbb{E}\left[\frac{\partial \rho_j^{-1}(s_t, x_{2t}, \lambda)}{\partial \lambda}\middle| z_t\right]$)+ exogenous characteristics (polynomial terms) + cost shifters (12 instruments)

- I(2): first stage instruments: instruments J(2). Testing instruments: Interval Instruments: 7 instruments. Points chosen as follows: $\{\hat{\mu}, (\hat{\mu} + k(max(0.25, \hat{\sigma})), k(max(0.25, \hat{\sigma}))\}$ $(for\ k = 1, 2, 3)$

Power against local alternatives. We now assess the local power properties of our test by assuming that the random coefficient v_i is distributed according to a local alternative. Namely, we assume $v_i \sim \left(1 - \frac{1}{\sqrt{T}}\right) \mathcal{N}(2, 1) + \frac{1}{\sqrt{T}} Y$ where Y is an alternative distribution including exponential, Chi-square, Student, Uniform. We ensure that Y has mean 2 and variance 1. The results are reported in 25. First, we can observe that except for the uniform local alternative, our test appears to have non-trivial power against all the other local alternatives. For the exponential and chi-square distributions, it is clear that our test with interval instruments outperforms the Sargan-J test with traditional instruments. For the student local alternative, the results seem quite unstable for small sample sizes but as T increases, interval instruments also seem to perform better. For the uniform alternative, it appears that we don't have power against this local alternative.

[insert new table]

Table 25: Empirical power, local alternatives (1000 replications)

Number of markets	T=50				T=100				T=200			
Test type	J test(1)	I test(1)	J test(2)	I test(2)	J test(1)	I test(1)	J test(2)	I test(2)	J test(1)	I test(1)	J test(2)	I test(2)
Exponential	0.266	0.704	0.227	0.677	0.222	0.869	0.272	0.868	0.236	0.982	0.394	0.975
Chi-square	0.217	0.219	0.134	0.174	0.13	0.167	0.096	0.151	0.099	0.171	0.086	0.15
Student	0.212	0.139	0.33	0.436	0.115	0.115	0.127	0.093	0.082	0.13	0.134	0.312
Uniform	0.198	0.1	0.126	0.074	0.107	0.062	0.095	0.051	0.073	0.049	0.084	0.044

D.3 Finite sample performance of Interval instruments for estimation

Practical implementation of the estimation procedure. To assess the performance of our instruments in estimating the non-linear parameters with a flexible distribution of random coefficients, we simulate data with a distribution of random coefficients following a mixture of gaussians and we estimate the parameters of this mixture. For each setting, we estimate the model for 1050 replications. We select the replications with an objective function below a certain threshold (in order to avoid local minima). Minimization is performed with nloptr (algorithm: NLOPT-LD-LBFGS). We provide an analytical gradient, which we describe subsequently. The Threshold for the outer loop is 1e-9 while the threshold for the inner loop is 1e-13. We use squarem and a C++ implementation for the computation of the market shares to speed up the contraction. We also parallelize the contraction over markets using 7 independent core. Before we formally define the different sets of instruments, let us present the estimation procedure when the distribution of random coefficients is assumed to be a mixture.

Instruments Now we formally describe the instruments present in each different sets used for estimation

- Differentiation instruments: differentiation instruments + exogenous characteristics (polynomial terms) + cost shifters (20 instruments)

- Optimal instruments are computed in two stages. The first stage instruments consist of differentiation instruments and exogenous characteristics (polynomial terms). Second stage instruments consist of polynomial terms of exogenous characteristics and the approximation of optimal instruments proposed in Reynaert and Verboven (2014) (approximation of $\mathbb{E}\left[\frac{\partial\rho_j^{-1}(s_t,x_{2t},\lambda)}{\partial\lambda}\middle|z_t\right]$). The set called optimal instruments includes 15 instruments.

- Interval Instruments are computed in two stages. The first stage instruments consist of differentiation instruments and exogenous characteristics (polynomial terms). Second stage instruments are the interval instruments couples with some exogenous characteristics. A total of 23 instruments. The points in the support to compute the interval instruments are chose as follows: we take equally spaced points in the interval $\{\beta_{3L} - 0.5(\beta_{3H} - \beta_{3L}), \beta_{3H} + 0.5(\beta_{3H} - \beta_{3L})\}$.

Comparison of the performance between the different sets of instruments. We now report the mean biases and the empirical $\sqrt{MSE}$ of the estimates for each set of instruments and for different sample sizes. We also plot the distributions of estimates for the non-linear parameters for the different sets of instruments. First, we plot the distribution of estimates obtained when the set of differentiation instruments from Gandhi and Houde (2019) is used with a sample of $T = 200$ markets and $J = 12$ products. We observe that despite a relatively large sample, the differentiation instruments perform rather poorly in estimating the non-linear parameters associated with the mixture of Gaussians. In particular, the estimates of the standard deviation parameters associated to each component are very dispersed and a large portion of the estimates are bunched at zero. Second, we plot the distribution of non-linear estimates obtained with the optimal instruments from Reynaert and Verboven (2014). They tend to perform better than the differentiation instruments as we can see that the estimates are more concentrated around the true value. Yet, it is important to emphasize that the optimal instruments display large failure rates caused by perfect colinearity of the instruments. We report the percentage of replications that subject to perfect colinearity issues for each sample size $(39\%, 34\%, 31\%, 26\%, 23\%)$. Finally, we plot the distribution of estimates for the non linear parameters when we use the interval instruments developed in section **??**. It appears clearly that the interval instruments yield a more concentrated distribution of estimates than the two other sets of instruments. For the sake of conciseness, we do not report the results with a mixture with 3 components but the observations we make with two components are even more exacerbated.

Table 26: Estimation mixture with "differentiation" instruments (1000 replications)

	Parameter	β_0	α	β_1	β_2	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L
Sample size	true	2	-2	1.5	1	-2	0.5	4	0.5	0.25
T=50, J=12	bias	-0.12	0.022	-0.016	-0.018	0.214	0.184	-0.022	-0.045	0.027
	$\sqrt{MSE}$	0.308	0.06	0.215	0.215	0.633	0.734	0.281	0.35	0.075
T=50, J=20	bias	-0.064	0.011	-0.01	-0.011	0.189	0.347	0.022	-0.081	0.025
	$\sqrt{MSE}$	0.231	0.044	0.165	0.166	0.566	0.887	0.184	0.291	0.059
T=100, J=12	bias	-0.058	0.01	-0.012	-0.012	0.233	0.226	0.02	-0.066	0.027
	$\sqrt{MSE}$	0.204	0.041	0.147	0.148	0.592	0.703	0.256	0.305	0.072
T=100, J=20	bias	-0.04	0.006	-0.007	-0.007	0.198	0.423	0.047	-0.101	0.025
	$\sqrt{MSE}$	0.165	0.032	0.117	0.116	0.552	0.89	0.164	0.27	0.055
T=200, J=12	bias	-0.038	0.007	-0.003	-0.003	0.184	0.167	0.011	-0.049	0.019
	$\sqrt{MSE}$	0.152	0.03	0.11	0.11	0.466	0.601	0.176	0.262	0.053

Table 27: Estimation mixture with "Optimal" instruments(1000 replications)

	Parameter	β_0	α	β_1	β_2	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L
Sample size	true	2	-2	1.5	1	-2	0.5	4	0.5	0.25
T=50, J=12	bias	-0.09	0.016	-0.012	-0.013	0.076	0.059	0.026	-0.111	0.01
	$\sqrt{MSE}$	0.296	0.057	0.234	0.232	0.361	0.483	0.212	0.281	0.036
T=50, J=20	bias	-0.046	0.007	0	0.001	0.074	0.11	0.028	-0.089	0.01
	$\sqrt{MSE}$	0.225	0.044	0.178	0.176	0.328	0.563	0.163	0.228	0.033
T=100, J=12	bias	-0.041	0.007	-0.004	-0.003	0.054	0.037	0.019	-0.066	0.007
	$\sqrt{MSE}$	0.202	0.039	0.157	0.158	0.279	0.4	0.154	0.211	0.028
T=100, J=20	bias	-0.029	0.004	-0.003	-0.003	0.074	0.107	0.033	-0.074	0.01
	$\sqrt{MSE}$	0.153	0.03	0.126	0.124	0.311	0.52	0.129	0.194	0.034
T=200, J=12	bias	-0.029	0.005	-0.001	-0.001	0.026	0.011	0.021	-0.061	0.004
	$\sqrt{MSE}$	0.136	0.026	0.111	0.111	0.184	0.313	0.113	0.172	0.018

Table 28: Estimation mixture with Global Interval instruments(1000 replications)

	Parameter	β_0	α	β_1	β_2	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L
Sample size	true	2	-2	1.5	1	-2	0.5	4	0.5	0.25
T=50, J=12	bias	-0.154	0.029	-0.043	-0.045	0.017	0	-0.045	0.004	0.005
	$\sqrt{MSE}$	0.341	0.067	0.257	0.258	0.277	0.391	0.227	0.259	0.024
T=50, J=20	bias	-0.092	0.017	-0.02	-0.021	0.013	0.042	-0.018	-0.003	0.004
	$\sqrt{MSE}$	0.245	0.048	0.19	0.19	0.248	0.415	0.166	0.22	0.021
T=100, J=12	bias	-0.07	0.013	-0.017	-0.019	0.004	-0.012	-0.027	0.005	0.002
	$\sqrt{MSE}$	0.2	0.039	0.161	0.161	0.167	0.282	0.157	0.201	0.013
T=100, J=20	bias	-0.047	0.008	-0.006	-0.007	-0.009	-0.005	-0.008	-0.009	0.001
	$\sqrt{MSE}$	0.158	0.031	0.13	0.129	0.115	0.264	0.115	0.169	0.005
T=200, J=12	bias	-0.039	0.007	-0.004	-0.003	-0.006	-0.027	-0.015	-0.001	0.001
	$\sqrt{MSE}$	0.141	0.027	0.109	0.109	0.088	0.219	0.108	0.164	0.003

Table 29: Estimation mixture with Local Interval instruments(1000 replications)

	Parameter	β_0	α	β_1	β_2	β_{3L}	σ_{3L}	β_{3H}	σ_{3H}	p_L
Sample size	true	2	-2	1.5	1	-2	0.5	4	0.5	0.25
T=50, J=12	bias	-0.134	0.025	-0.023	-0.024	-0.006	-0.005	-0.039	-0.001	0.003
	$\sqrt{MSE}$	0.307	0.059	0.26	0.259	0.251	0.34	0.214	0.244	0.019
T=50, J=12	bias	-0.084	0.016	-0.024	-0.025	0.019	0.033	-0.023	0.01	0.003
	$\sqrt{MSE}$	0.245	0.047	0.188	0.186	0.228	0.38	0.15	0.184	0.018
T=50, J=12	bias	-0.075	0.015	-0.018	-0.016	0	0	-0.028	0.007	0.001
	$\sqrt{MSE}$	0.199	0.039	0.159	0.16	0.127	0.225	0.143	0.164	0.005
T=50, J=12	bias	-0.039	0.007	-0.011	-0.011	-0.003	0.004	-0.01	0.004	0.001
	$\sqrt{MSE}$	0.162	0.032	0.129	0.129	0.104	0.226	0.103	0.125	0.004
T=50, J=12	bias	-0.037	0.007	-0.008	-0.007	0.002	-0.007	-0.016	0.006	0.001
	$\sqrt{MSE}$	0.136	0.026	0.11	0.109	0.091	0.174	0.099	0.123	0.003

Figure 12: Distribution of estimates for non-linear parameters with "Differentiation" instruments ($T = 200, J = 12$)

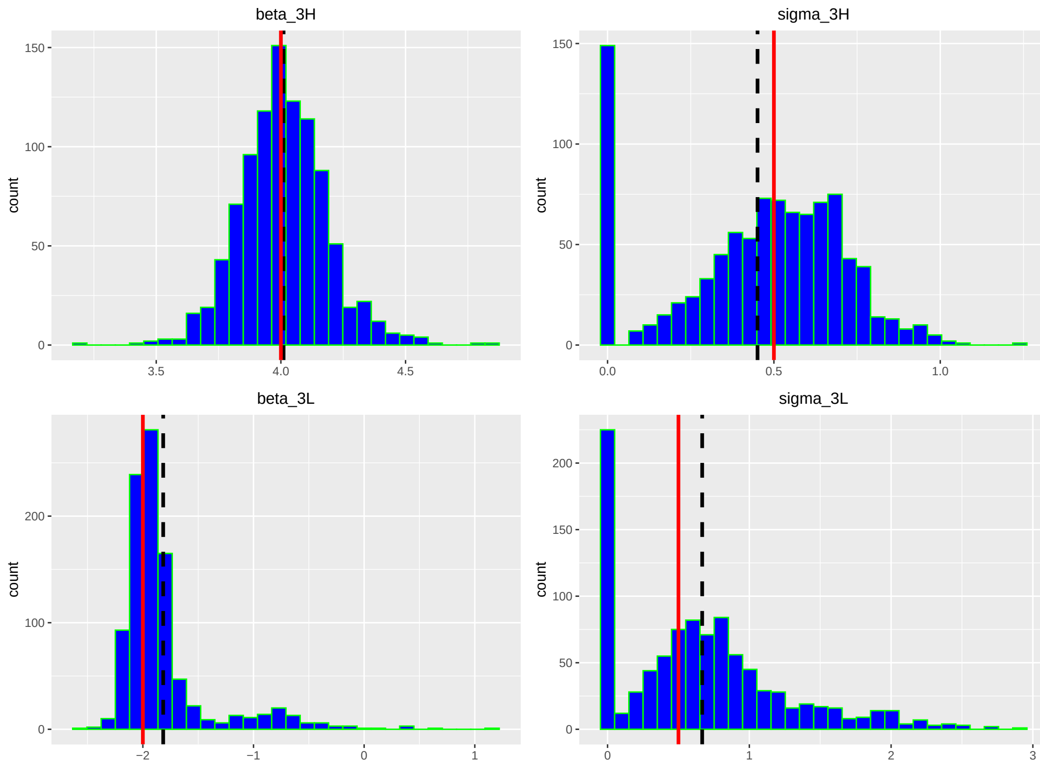


Figure 13: Distribution of estimates for non-linear parameters with "Optimal" instruments ($T = 200, J = 12$)

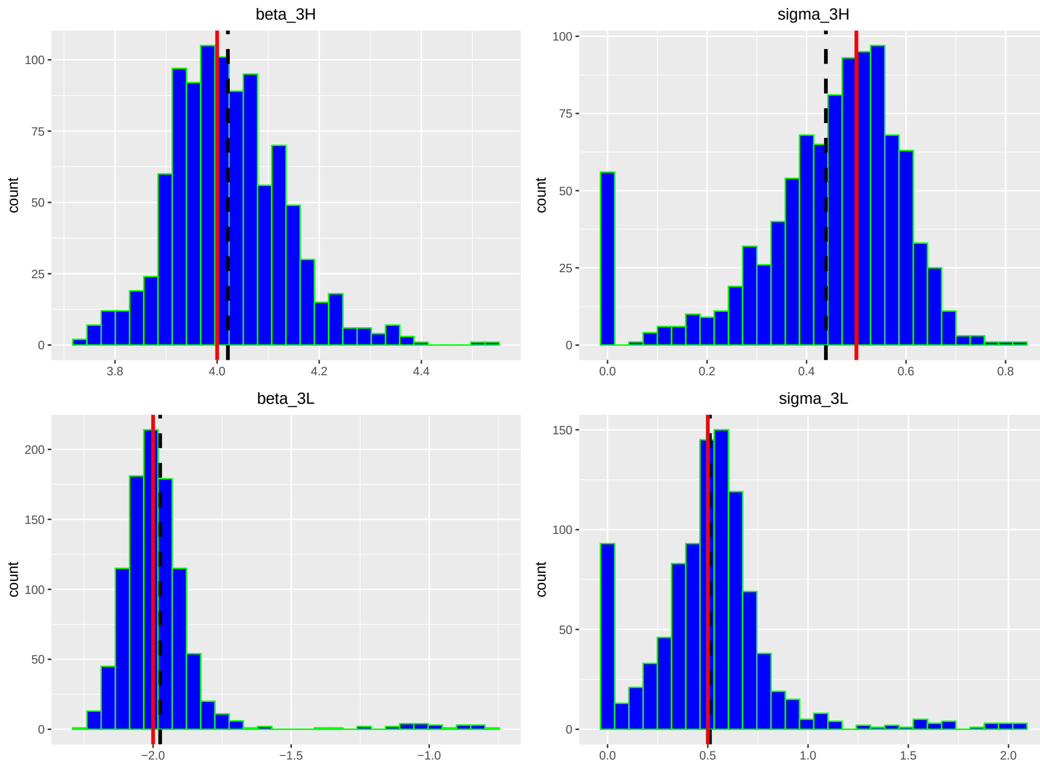


Figure 14: Distribution of estimates for non-linear parameters with “Global Interval” instruments $(T = 200, J = 12)$

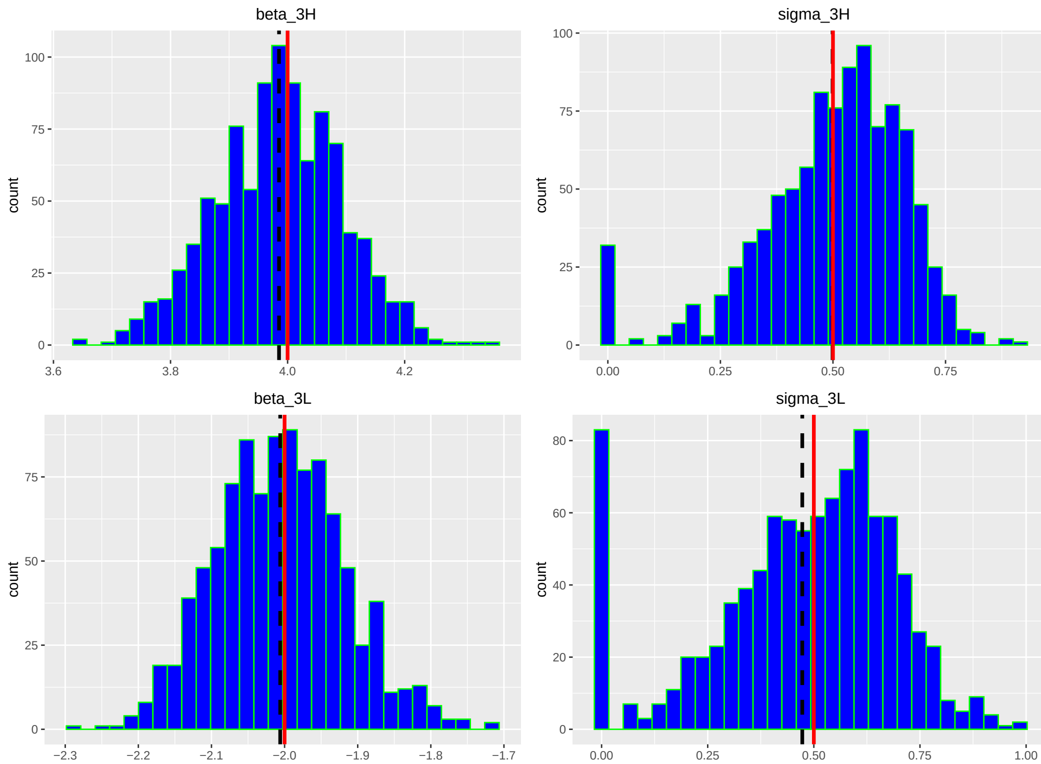
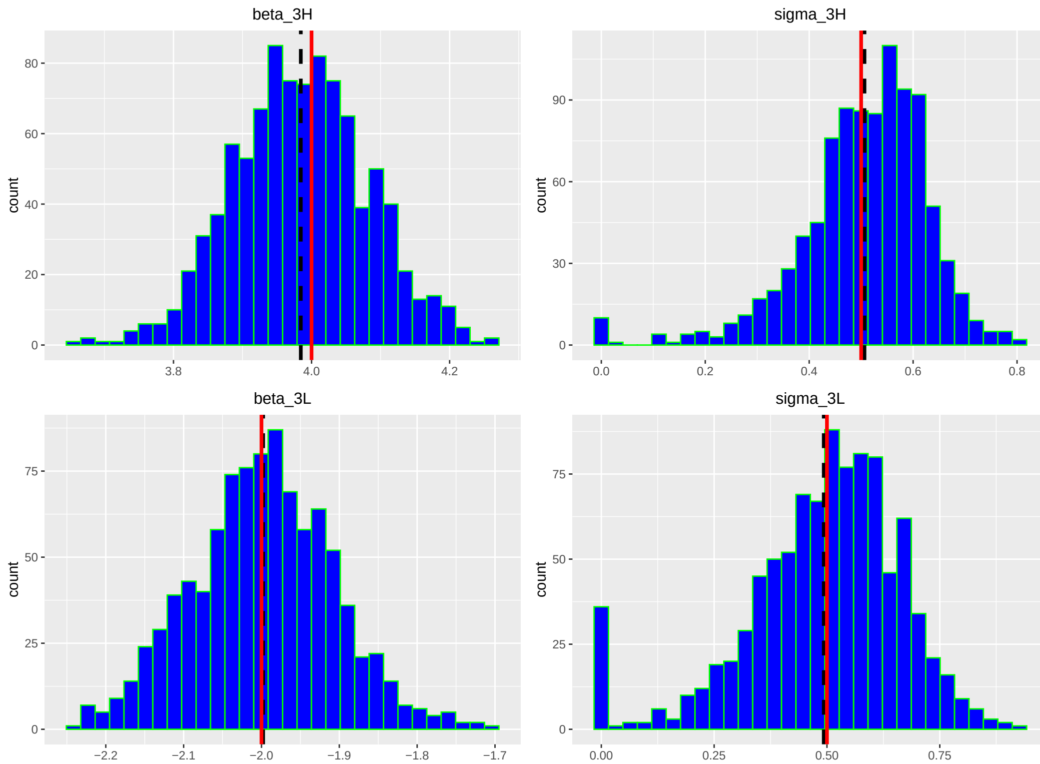


Figure 15: Distribution of estimates for non-linear parameters with "Local interval" instruments ($T = 200, J = 12$)



D.3.1 Estimation with a single Gaussian

Table 30: Estimation with a single Gaussian (1000 replications)

	Instruments	Differentiation						"Optimal"						Interval Global						Interval Local					
	Parameter	β_0	α	β_1	β_2	β_3	σ_3	β_0	α	β_1	β_2	β_3	σ_3	β_0	α	β_1	β_2	β_3	σ_3	β_0	α	β_1	β_2	β_3	σ_3
Sample size	true	2	-2	1.5	1	1.5	0.5	2	-2	1.5	1	1.5	0.5	2	-2	1.5	1	1.5	0.5	2	-2	1.5	1	1.5	0.5
T=50, J=12	bias	-0.16	0.032	-0.031	-0.028	-0.032	-0.004	-0.09	0.018	-0.016	-0.014	-0.018	-0.003	-0.15	0.03	-0.028	-0.026	-0.03	-0.004	-0.15	0.03	-0.028	-0.026	-0.03	-0.001
	$\sqrt{MSE}$	0.292	0.057	0.212	0.209	0.138	0.069	0.27	0.053	0.214	0.211	0.138	0.067	0.288	0.056	0.212	0.209	0.138	0.066	0.286	0.056	0.212	0.209	0.138	0.064
T=50, J=20	bias	-0.091	0.018	-0.022	-0.022	-0.015	0.001	-0.047	0.009	-0.013	-0.013	-0.006	0.001	-0.084	0.017	-0.021	-0.021	-0.013	0	-0.086	0.017	-0.021	-0.021	-0.014	0.002
	$\sqrt{MSE}$	0.209	0.041	0.159	0.16	0.106	0.05	0.199	0.039	0.16	0.161	0.106	0.05	0.206	0.041	0.16	0.16	0.106	0.052	0.208	0.041	0.159	0.16	0.106	0.052
T=100, J=12	bias	-0.088	0.017	-0.001	0	-0.027	0.001	-0.052	0.01	0.007	0.007	-0.02	0.001	-0.082	0.016	0	0.001	-0.026	0.001	-0.074	0.014	-0.016	-0.016	-0.013	0.001
	$\sqrt{MSE}$	0.199	0.039	0.146	0.145	0.1	0.045	0.189	0.037	0.148	0.147	0.099	0.047	0.197	0.039	0.146	0.146	0.1	0.044	0.185	0.036	0.151	0.152	0.099	0.044
T=100, J=20	bias	-0.043	0.009	-0.011	-0.012	-0.006	-0.001	-0.021	0.004	-0.007	-0.008	-0.002	-0.001	-0.04	0.008	-0.011	-0.012	-0.006	-0.001	-0.035	0.007	-0.01	-0.009	-0.004	0
	$\sqrt{MSE}$	0.145	0.028	0.115	0.114	0.075	0.035	0.141	0.028	0.115	0.114	0.075	0.035	0.145	0.028	0.115	0.114	0.076	0.035	0.14	0.027	0.116	0.115	0.076	0.035
T=100, J=20	bias	-0.038	0.007	-0.012	-0.012	-0.004	0.001	-0.017	0.003	-0.006	-0.007	-0.001	0	-0.032	0.006	-0.009	-0.01	-0.004	0	-0.033	0.006	-0.009	-0.01	-0.004	0.001
	$\sqrt{MSE}$	0.132	0.026	0.11	0.11	0.073	0.032	0.127	0.025	0.109	0.109	0.069	0.032	0.129	0.026	0.109	0.109	0.069	0.032	0.129	0.026	0.109	0.109	0.069	0.031

E Empirical application

First stage regression: instruments on price.

Table 31: Estimation results - Logit and Nested Logit

	OLS		*IV*		
	(1)	(2)	(3)	(4)	(5)
Price/income	-0.338^{***}	-3.180^{***}	-2.749^{***}	-2.453^{***}	-2.451^{***}
	(0.041)	(0.130)	(0.124)	(0.050)	(0.048)
log(within market shares)				0.433^{***}	0.438^{***}
				(0.006)	(0.006)
Fuel Cost	-0.337^{***}	-0.206^{***}	-0.259^{***}	-0.122^{***}	-0.147^{***}
	(0.022)	(0.017)	(0.023)	(0.011)	(0.014)
Size(m^2)	-0.018	0.017	0.154^{***}	-0.006	0.102^{***}
	(0.038)	(0.039)	(0.040)	(0.024)	(0.024)
Horsepower(KW/100)	0.090	3.652^{***}	3.067^{***}	2.344^{***}	2.269^{***}
	(0.092)	(0.176)	(0.173)	(0.075)	(0.073)
Foreign	0.370^{***}	0.162^{**}	0.176^{***}	-0.061	-0.080^{*}
	(0.064)	(0.070)	(0.068)	(0.043)	(0.041)
Height(m)	0.612^{***}	-0.712^{***}	1.057^{***}	-0.463^{***}	0.654^{***}
	(0.220)	(0.200)	(0.230)	(0.123)	(0.137)
Gasoline	1.366^{***}	0.604^{***}	0.964^{***}	0.290^{***}	0.454^{***}
	(0.059)	(0.063)	(0.064)	(0.039)	(0.039)
Fuel cost× income	-0.005	-0.022^{***}	-0.035^{***}	-0.021^{***}	-0.029^{***}
	(0.004)	(0.004)	(0.005)	(0.002)	(0.003)
Size × income	-0.005^{***}	-0.002^{***}	-0.006^{***}	0.0004	-0.002^{***}
	(0.001)	(0.001)	(0.001)	(0.0005)	(0.0005)
Horsepower× income	0.005^{***}	-0.036^{***}	-0.028^{***}	-0.027^{***}	-0.026^{***}
	(0.002)	(0.002)	(0.002)	(0.001)	(0.001)
Horsepower× time	-0.038^{***}	-0.041^{***}	-0.035^{***}	-0.020^{***}	-0.009^{**}
	(0.006)	(0.006)	(0.006)	(0.003)	(0.004)
Foreign × income	-0.020^{***}	-0.016^{***}	-0.016^{***}	-0.009^{***}	-0.008^{***}
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Height × income	-0.004	0.030^{***}	-0.003	0.014^{***}	-0.006^{**}
	(0.004)	(0.004)	(0.004)	(0.002)	(0.003)
Height × density	-0.039^{***}	-0.005^{***}	-0.039^{***}	-0.003^{***}	-0.021^{***}
	(0.004)	(0.0004)	(0.004)	(0.0002)	(0.002)
gazoline× income	-0.017^{***}	-0.007^{***}	-0.012^{***}	-0.003^{***}	-0.005^{***}
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Constant	-8.358^{***}	-10.781^{***}	-9.908^{***}	-7.213^{***}	-6.734^{***}
	(0.165)	(0.177)	(0.188)	(0.103)	(0.106)
State FE & Year FE	✓		✓		✓
Observations	39,888	39,888	39,888	39,888	39,888
R^2	0.372	0.285	0.326	0.712	0.746
Adjusted R^2	0.371	0.284	0.325	0.711	0.746

Note: Brand and car body FE's are included in each regression

*p<0.1; **p<0.05; ***p<0.01

Baseline specifications: logit and nested logit.

Construction of the interval instruments

- **Discretization of the support**
- **normalization of the instruments**

E.0.1 Results differentiation instruments

Table 32: counterfactual quantities under different specifications on RCs (20 most popular cars)

Counterfactual quantity		Price elasticity			Curvature			Marginal cost			Mark-up			Pass-through		
car	Manufacturer	Logit	Gaussian	Mixture	Logit	Gaussian	Mixture	Logit	Gaussian	Mixture	Logit	Gaussian	Mixture	Logit	Gaussian	Mixture
Golf	Volkswagen	-1.09	-0.72	-3.08	1.00	1.22	1.14	1260	-19996	15634	24098	45354	9724	1260	9383	15634
Polo	Volkswagen	-0.74	-0.54	-2.42	1.00	1.22	1.06	-6643	-23599	8833	23819	40775	8342	-6643	2430	8833
Passat	Volkswagen	-1.43	-0.92	-2.33	1.00	1.26	1.54	9488	-11624	18597	24631	45744	15523	9488	18741	18597
Corsa	PSA	-0.66	-0.48	-2.18	1.00	1.21	1.04	-8432	-19233	8096	24088	34889	7560	-8432	2921	8096
Fiesta	Ford	-0.62	-0.46	-2.09	1.00	1.21	1.04	-8983	-18338	7337	23487	32841	7166	-8983	2476	7337
Tiguan	Volkswagen	-1.32	-0.86	-2.35	1.00	1.26	1.51	6831	-13075	16952	24118	44024	13997	6831	16052	16952
Golf	Volkswagen	-1.17	-0.79	-3.21	1.00	1.26	1.19	3128	-18268	16930	23828	45224	10026	3128	11160	16930
up!	Volkswagen	-0.53	-0.40	-1.83	1.00	1.21	1.03	-11231	-26365	4203	23278	38412	7843	-11231	-2068	4203
Tiguan	Volkswagen	-1.34	-0.88	-3.27	1.00	1.28	1.28	7051	-14623	19870	23842	45515	11023	7051	15281	19870
1er-Reihe	BMW	-1.16	-0.79	-3.19	1.00	1.27	1.20	3845	-9889	19500	25138	38873	9484	3845	15557	19500
Octavia	Volkswagen	-1.23	-0.82	-2.37	1.00	1.25	1.45	4629	-14429	16006	24211	43269	12835	4629	14030	16006
A4	Volkswagen	-1.56	-0.98	-2.32	1.00	1.29	1.57	13209	-9341	21118	25865	48415	17957	13209	23033	21118
Clio	Renault	-0.73	-0.54	-2.41	1.00	1.23	1.07	-6240	-16328	9580	23120	33208	7299	-6240	4862	9580
T-Roc	Volkswagen	-0.87	-0.63	-2.75	1.00	1.24	1.10	-3645	-21799	11476	23798	41951	8676	-3645	5135	11476
Kuga	Ford	-1.16	-0.79	-3.18	1.00	1.26	1.20	3654	-9068	18464	23684	36406	8874	3654	14735	18464
Golf	Volkswagen	-1.10	-0.75	-2.37	1.00	1.24	1.38	1548	-16928	14092	23929	42405	11385	1548	10801	14092
A-Klasse	Daimler	-1.28	-0.84	-3.23	1.00	1.28	1.26	6608	-9017	21236	25066	40692	10439	6608	17820	21236
Golf	Volkswagen	-1.05	-0.72	-2.36	1.00	1.25	1.36	417	-17768	13561	24177	42362	11033	417	9931	13561
Golf	Volkswagen	-1.18	-0.80	-3.24	1.00	1.26	1.19	3202	-18681	17037	23921	45804	10087	3202	11097	17037
Octavia	Volkswagen	-1.05	-0.73	-3.04	1.00	1.25	1.15	380	-18631	14941	23862	42872	9300	380	9127	14941

Counterfactual quantities under different specifications