Parrondo's game

Final project for Stochastic Simulation Methods

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1 Parrondo's paradox

Juan Parrondo, in 1996, stated the paradox that says, *There exist pairs of games, each with a higher probability of losing than winning, for which it is possible to construct a winning strategy by playing the games alternately.* Consider we have two games, A and B. If we play game A or game B individually, we are guaranteed to lose in the long run, but if we alternate back and forth between the two games, we are guaranteed to win. This means grandma was wrong, and it is actually possible for two wrongs to make a right.

1.1 The original game

In the original version of the game, we toss a coin to choose which game to play. Suppose that the probability of playing game A is γ, called the mixing probability, and that of playing game B is $1-\gamma$. Further assume a quantity $C(t)$, which is the capital of the player. Here, $t = 0, 1, 2, 3\cdots$ represent the time at which a turn is played. Winning a game earns us \$1 and loosing a game will cost us \$1. So, if we win or loose a game at time t, we will have $C(t+1) = C(t) \pm 1$ capital for the next turn to play. These games are said to be fair, losing or winning when the average capital of the player stabilizes, decreases or increases respectively. The probabilities of winning game A and Game B are listed below.

- If we are playing Game A, the probability of winning is $1/2 - \epsilon$, with $\epsilon \geqslant 0$.
- If we are playing Game B the probability of winning depends on the capital of the player.If the capital of a player is multiple of 3 we use a bad-coin with winning probability $1/10 - \epsilon$, otherwise we use good-coin with probability of winning $3/4 - \epsilon$.

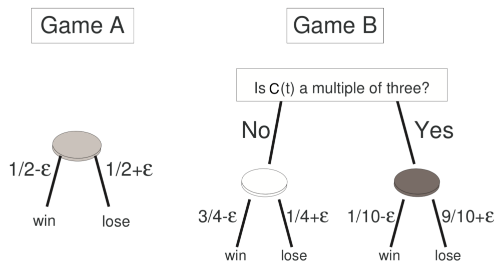

Figure 1: Rules of the games.

Both games are fair if $\epsilon = 0$, which means $\Delta C(t) = 0$. Whereas, both games have tendency to loose, $\triangle C(t) < 0$, when $\epsilon > 0$. Surprisingly, if we play these games alternatively, $0 < \gamma < 1$, in a combination of $A + B$, we end up with $\Delta C(t) > 0$.

2 Collective game

Different versions of the game has been studied for N players, [1, 2] with game A being the same as original but game B has been redefined each time and desired results has been obtained. The game that we are considering here [3] is a system of N players, a player is said to be winner or loser when he has won or lost, respectively, his last game. The winning probability can have three possible values, determined by the actual number of winners n within the total number of players N, in the following way,

$$p_n^B = \begin{cases} p_B^1, & n > \lceil \frac{2N}{3} \rceil \\ p_B^2, & \lceil \frac{N}{3} \rceil \leq n \leq \lceil \frac{2N}{3} \rceil \\ p_B^3, & n < \lceil \frac{2N}{3} \rceil, \end{cases}$$

where the notation $\lceil x \rceil$ means nearest integer to x. For our study of Parrondo's effect, the main quantity of interest is $J^{(A+B)}(t)$ which is the probability current at any time t for the collection of N players for the stochastic game $A + B$. Since the winning probability of game B only depends on the total number of winners, we can say that our system has $N + 1$ different states $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_N$. The state σ_n is the configuration where n players are labeled as winners and $N - n$ as losers. Transitions between these states are of three types, see Fig 2.

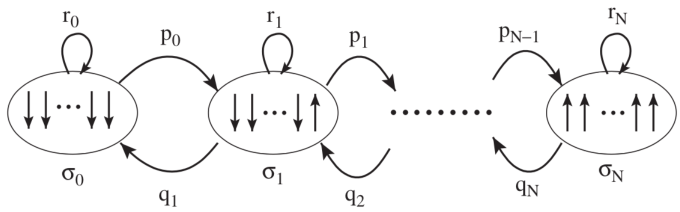

Figure 2: Three transition probabilities are associated with each state. The up and down arrows represents the winners and loosers respectively.

- The forward transition probability p_n, when a winner is added to the already existing number of winners.
- The backward transition probability q_n, when a looser is added to the already existing number of losers.
- r_n is the probability for remaining in the same state.

Moreover, say $P_n(t)$ represent the probability of finding the system in configuration σ_n at the $t^{(th)}$ round played, then the governing time evolution equation can then be written as,

$$P_n(t+1) = p_{n-1}P_{n-1}(t) + r_nP_n(t) + q_{n+1}P_{n+1}(t). \tag{1}$$

Here, $P_n(t+1)$ is the probability for the system to be in state σ_n at time $t+1$, with transition probabilities from neighboring states is p_{n-1}, q_{n+1} and the one of remaining in the same state is r_n. These transition probabilities can then be explained as followings. We start with p_n,

$$p_n = \frac{N-n}{N}[\gamma p^A + (1-\gamma)p_n^B] \tag{2}$$

In state σ_n there are $N-n$ losers and n winners.The way to move to state σ_{n+1} is by choosing a player labeled as a loser with probability $\frac{N-n}{N}$, and that player is winning the game. So if the probability of playing game A is γ and that of game B is $1-\gamma$, the combined winning probability will be given by $\gamma p^A + (1-\gamma)p_n^B$.

$$q_n = \frac{n}{N}[\gamma(1-p^A) + (1-\gamma)(1-p_n^B)] \tag{3}$$

Now, let In the state σ_n there are n winners, the way to move to state σ_{n-1} is by choosing a player labeled as a winner with probability $\frac{n}{N}$, and that player is loosing the game. So if the probability of playing game A is γ and that of game B is $1-\gamma$, the combined loosing probability will be given by $\gamma(1-p^A) + (1-\gamma)(1-p_n^B)$.

$$r_n = \frac{n}{N}[\gamma p^A + (1-\gamma)p_n^B)] + \frac{N-n}{N} \tag{4}$$

Similarly ,the expression for r_n is also justified, as for staying in the same state, no winner or loser is added.

The set of these transition probabilities (p_n, q_n, r_n) must satisfy the normalization condition $p_n + q_n + r_n = 1$, which means that at any time t the probabilities $P_n(t) = \sum_{n=0}^{N} P_n(t) = 1$.

Finally, the winning probability of the stochastic game $A+B$ with mixing probability γ is given by,

$$p_{win}^{A+B}(t) = \sum_{n=0}^{N}[\gamma p^A + (1-\gamma)p_n^B]P_n(t). \tag{5}$$

This gives us the time dependent current $J^{A+B}(t) = 2p_{win}^{A+B}(t) - 1$, that is nothing but the difference of win and lose probability for the game $A+B$. A stabilizing capital is then the case when $J^{A+B}(t) = 0$, while $J^{A+B}(t) > 0$ means more winners then losers and hence we are gaining the capital. The mixing probability γ equal to 1 or 0 will retrieve properties of pure game A and B respectively.

3 Analysis and results

As the game is stochastic, each play will result in a different trajectory, For instance, the plot in Fig. 3 shows the evolution for $N = 10$, starting from a random initial state. First reaction method is used for solving the master equation. The simulation is then executed for $N = 100$ players for time steps $10,000$, with $\gamma = 0.3, p_A = 0.5, p_B^1 = 0.79, p_B^2 = 0.65$ and $p_B^3 = 0.15$. The first thing to check is the conservation of total probability, $\sum_{n=0}^{N} P_n(t) = 1$, see Fig.4. This check is essential to compute the probability current that will be computed later. From this figure, it can be clearly seen that the probability sum is 1 for the entire time duration. It is also of importance to look at the average trajectory of the system to find the stationary region. Averaging over 1000 trajectories gives us the average number of winners at any time. We can see that the winners are increasing in number and achieve a stationary state for $t > 4000$ (Fig. 5).

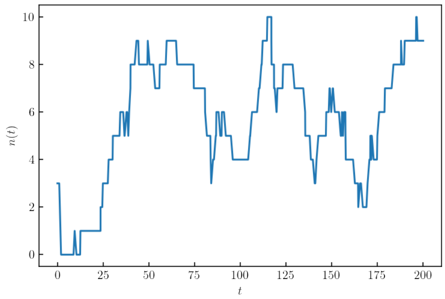

Figure 3: A single trajectory for $N = 10$ and $t = 200$.

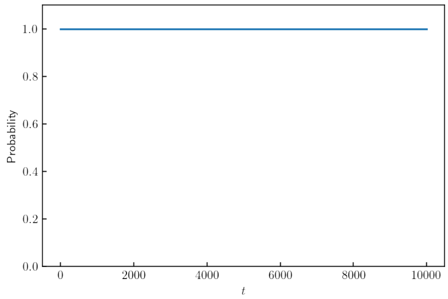

Figure 4: For $N = 100$ and $t = 10,000$, the sum of probabilities over time.

3.1 Probability current

The probability current can now be evaluated by taking the difference of winning and loosing probabilities. The evolution of this quantity $J^{A+b}(t)$ is plotted in the Fig. 6. Starting from some random state, the number of winners are increasing and so is the current moving towards some positive stationary value. The current finally cross the boundary $J^{A+B}(t) = 0$, and the system is then is gaining states till the end of time. This means that the system is gaining capital for all upcoming turns which is called the Parrondo's effect.

3.2 Average gain

Starting from a random state, we assign the gain $C_n(t) = 0$ for this state. Now at the next turn, if the system move to a new state of increasing winners, we add \$1 to the capital. If the state is not changed, the capital remains the same, and finally If we lose a winner, the gain is decreased by \$1. At the end of time cycle, we accumulate, $C(t) = \sum_n C_n(t)$. The average gain per player is then given by $\langle C(t)\rangle/N$. The average gain per player and its variance plots can be seen in Fig. 7.

References

[1] R. Toral, Cooperative Parrondo's games, Fluctuations Noise Lett. 1 (2001) L7–L12.

[2] R. Toral, Capital redistribution brings wealth by Parrondo's paradox, Fluctuations Noise Lett. 2 (2002) L305–L311.

[3] P. Amengual *et al.* Physica A 371 (2006) 641–648.

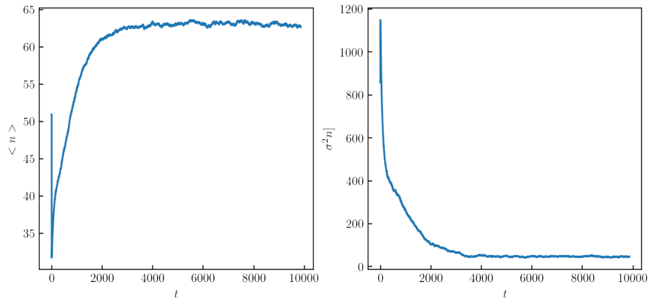

Figure 5: For $N = 100$ and $t = 10,000$, the average and variance over 1000 trajectories.

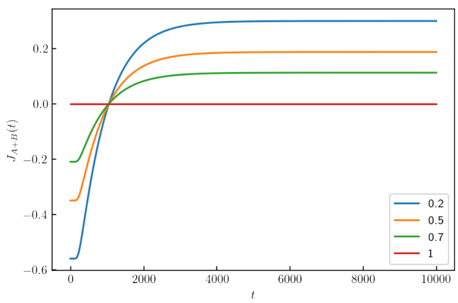

Figure 6: $J_{A+B}(t)$ for several values of mixing parameter γ.

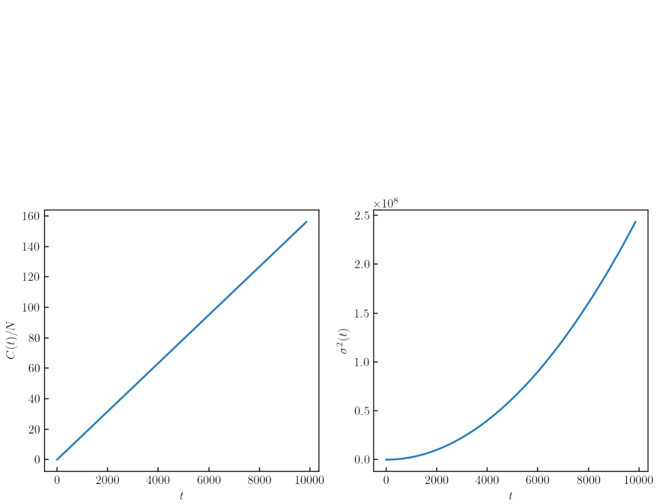

Figure 7: Average gain for $\gamma = 0.3$.