INTRODUCTION TO SARKISOV PROGRAM

YIFEI CHEN, YANZE WANG

Dedicated to Professor Vyacheslav V. Shokurov on the occasion of his seventieth birthday

Abstract. In this paper, we describe three different Sarkisov program, and give examples.

Contents

1. Introduction

The purpose of this article is to show that two different log Mori fibre spaces as outputs of a klt pair can be linked by composition of Sarkisov links.

1.1. **Motivation and Main theorem.** The **Minimal model program (MMP)** aims to classify varieties up to birational equivalent classed, by finding a minimal model all or Mori fibre space. Let (X, B) be a (klt or lc) pair, and assume we can run $(K_X + B)$-MMP on it. Note that the varieties appear in the program are called **results** of the MMP, and the varieties where the MMP ends are called the **output** of the MMP.

(1) If $\kappa(X, B) \geqslant 0$, then we expected that MMP ends with a **minimal model**, i.e. a birational map $X \dashrightarrow Y$ such that $(K_Y + B_Y)$ is nef;
(2) If $\kappa(X, B) = -\infty$, then we expected that MMP ends with a log Mori fibre space, i.e. a birational map $X \dashrightarrow Y$ and a contraction $Y \to S$ such that $\dim Y < \dim X$ and $-(K_Y + B_Y)$ is relative ample.

However, for each case the output may not be unique.

For the first case, it is shown that two different minimal model can be linked by flops:

Theorem 1.1.1. [**?**, Theorem 1] *Let (W, B_W) be a $\mathbb{Q}$-factorial terminal pair, and $(X, B), (Y, D)$ are two minimal models of (W, B_W). Then the birational map $X \dashrightarrow Y$ may be factored as sequence of $(K_X + B)$ flops.*

For the second case, it is shown that:

Date: December 22, 2022.
1991 *Mathematics Subject Classification.* Primary 14E30, Secondary 14B05.

Theorem 1.1.2. *Let $f:(X,B)\to S$ and , $f':(X',B')\to S'$ be two MMP related $\mathbb{Q}$-factorial klt log Mori fibre spaces with induced induced birational map Φ:*

$$\begin{tikzcd}
(X,B) \arrow[r, dashed, "\Phi"] \arrow[d, "f"'] & (X',B') \arrow[d, "f'"] \\
S & S'
\end{tikzcd}$$

Then Φ can be decomposed into sequence of Sarkisov links.

Definition 1.1.3. *The following four types of birational maps $X \dashrightarrow X_1$ are called Sarkisov links:*

I:
$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d, "p"'] & X_1 \arrow[d, "f_1"] \\
X \arrow[d, "f"'] & S_1 \arrow[dl, "t"] \\
S &
\end{tikzcd}$$

II:
$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d, "p"'] & Z' \arrow[d, "q"] \\
X \arrow[d, "f"'] & X_1 \arrow[d, "f_1"] \\
S \arrow[r, "\sim"] & S_1
\end{tikzcd}$$

III:
$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, "f"'] & Z \arrow[d, "q"] \\
S \arrow[dr, "s"'] & X_1 \arrow[d, "f_1"] \\
 & S_1
\end{tikzcd}$$

IV:
$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[d, "f"'] & & X_1 \arrow[d, "f_1"] \\
S \arrow[dr, "s"'] & & S_1 \arrow[dl, "t"] \\
 & T &
\end{tikzcd}$$

Here, all $f:(X,B)\to S$ and $f_1:(X_1,B_1)\to S_1$ are log Mori fibre space, and all p,q are divisorial contractions, and all dash arrows are composition of flips (or flops in section 5).

The Sarkisov program has its origin in the birational classification of ruled surfaces [**?**]. Reid explained the original idea of Sarkisov [**?**]. The complete proof of Sarkisov program for terminal threefolds is given by Corti [**?**]. Andrea Bruno and Kenji Matsuki [**?**] generalized it to klt pairs of dimension 3. In fact, they showed that the Sarkisov program works for klt pairs of all dimensions, but in higher dimension it may not terminate. Due to the finiteness of weak log canonical model [**?**], Hacon gives another Sarkisov program called double scaling [**?**] which terminates in all dimensions. Liu Jihao generalized Hacon's method to generalized pairs [**?**]. As another application of [**?**], Hacon and M^{c}kern gave another proof [**?**] and is quite different.

1.2. **Using MMP.** Assume $f:(X,B)\to S'$ and $f':(X',B')\to S'$ are two Mori fibre spaces as outputs of (K_W+B_W)-MMP on W. The Sarkisov program constructs each Sarkisov link $X_i \dashrightarrow X_{i+1}$ inductively. Each Sarkisov link is given by running a special MMP called 2-ray game.

Sarkisov's Idea and Bruno and Matsuk's generalization: Take an ample $\mathbb{Q}$-divisor A' on S' such that $H'\sim -(K_{X'}+B')+f'^*A'$ is ample, then $(X',B'+H')$ is a weak log canonical model. Therefore we expect that X' is output of certain MMP from X. Let H be the strict transform of H' on X, then we run (K_X+B+cH)-MMP for max c such that

(1) K_X+B+cH is non-positive over S, and
(2) K_X+B+cH is log canonical (θ-canonical).

Then we construct the Sarkisov link in following cases:

(1) If c is not threshold of second condition, find a contraction $g_i:X_i\to T_i$ such that $\rho(X_i/T_i)=2$ and factor though $f_i:X_i\to S_i$. We run MMP on X_i over T_i, and obtains a Sarkisov link of type III or IV;
(2) If c is threshold of second condition, then find a divisorial extraction $p_i:Z_i\to X_i$, and then $\rho(Z_i/S_i)=2$. We run MMP on Z_i over S_i, and obtains a Sarkisov link of type I or II.

We run the whole program in a special collection $\mathcal{C}_\theta$ (Proposition 3.1.1). In this collection we can define Sarkisov degree, which decreses after composing a Sarkisov link $\psi_i:X_i\dashrightarrow X_{i+1}$.

Double scaling: Let W be the common resolution of two Mori fibre spaces. Take an ample $\mathbb{Q}$-divisor A on S such that $G\sim -(K_X+B)+f^*A$ is ample. Then $(X,B+G)$ and $(X',B'+H')$ are two weak log canonical models of W (for $K_W+B_W+G_W$ and $K_W+B_W+H_W$).

We expect that there are finitely many weak log canonical models $(X_i, B_i + g_iG_i + h_iH_i)$ of $(W, B_W + g_iG_W + h_iH_W)$ and $\psi_i : X_i \dashrightarrow X_{i+1}$ is a Sarkisov link given by 2-ray game.

1.3. **Using polytope.** This is a proof of main theorem that is not a program. Let W be the common log resolution of two Mori fibre spaces $X \to S$ and $Y \to T$. Take a finite dimensional subspace V of $\mathrm{WDiv}_{\mathbb{R}}(W)$ and an ample $\mathbb{Q}$-divisor A. Then $\{\mathcal{A}_i = \mathcal{A}_{A,f_i}\}$ is a partition of $\mathcal{E}_A(V)$, and each $\mathcal{A}_i$ corresponds to an ample model of W. There are morphisms betwwen the ample models of (W, D) for $D \in \mathcal{L}_A(V)$ (Theorem 5.1.2 and Theorem 5.1.4). In particular, if D contains in the boundary of $\mathcal{E}_A(V)$ and many polytopes of ample models, then those morphisms form a Sarkisov link (5.1.6).

Take a special 2-dimensional rational affine subspace V (by Lemma 5.2.1) such that

(1) S, T are ample models of W for some $D_S, D_T \in \mathcal{L}_A(V)$;
(2) There are finitely many points D_i "connecting" D_S and D_T, and these D_i corresponds to a Sarkisov link.
(3) $X \dashrightarrow Y$ is composition of these Sarkisov links.

2. Preliminary

In this article, all varieties are over complex number $\mathbb{C}$.

2.1. Models.

Definition 2.1.1. [?, 2.Notation and Conventions] *A rational map $f : X \dashrightarrow Y$ is called a **rational contraction** if there is a resolution $p : W \to X$ and $q : W \to Y$ of f such that p and q are contraction morphisms and p is birational. f is called a **birational contraction** if q is in addition birational and every p-exceptional divisor is q-exceptional. If in addition f^{-1} is also a **birational contraction**, then f is called a **small birational map**.*

Definition 2.1.2. [?, Definition 3.6.1]*Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties, and $p : W \to X$ and $q : W \to Y$ be a resolution of indeterminacy of f. Let D be a $\mathbb{R}$-Cartier divisor on X such that $D_Y = f_*D$ is also $\mathbb{R}$-Cartier. Then f is called **D-non-positive (respectively D-negative)** if*

- *f does not extract any divisor;*
- *$E = p^*D - q^*D_Y$ is effective and exceptional over Y (respectively $\mathrm{Supp}\, p_*E$ contains all f-exceptional divisors).*

Definition 2.1.3. [?, 13.2.Notation and conventions] *Let $f : X \dashrightarrow Y$ be a rational map of normal quasi-projective varieties over S, and D be a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X with $f_*D = D_Y$. Then f is called **D-trivial** if D is pull back of a $\mathbb{R}$-Cartier divisor on S.*

Recall the definitions of models in [?]

Definition 2.1.4. [?, Definition 3.6.5] *Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X. Let $f : X \dashrightarrow Y$ be a birational map over U, then Z is an **semiample model** for D over U if f is $K_X + D$-non-positive and $K_Y + f_*D$ is semiample over U.*

*Let $g : X \dashrightarrow Z$ be a rational map over U, then Z is an **ample model** for D over U if there is an ample divisor H over U on Z such that if $p : W \to X$ and $q : W \to Z$ resolves g, then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{R},U} q^*H + E$, where $E \geqslant 0$ and for any $B \in |p^*D/U|_{\mathbb{R}}$, then $B \geqslant E$.*

Definition 2.1.5. [?, Definition 3.6.7] *Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties, if $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then define:*

*(1) Y is **weak log canonical model** for $K_X + D$ over U if f is $K_X + D$-non-positive and $K_Y + f_*D$ is nef over U;*

(2) Y is ***log canonical model*** *for $K_X + D$ over U if f is $K_X + D$-non-positive and $K_Y + f_*D$ is ample over U;*
(3) Y is ***log terminal model*** *for $K_X + D$ over U if f is $K_X + D$-negative and $K_Y + f_*D$ is dlt and nef over U and Y is $\mathbb{Q}$-factorial.*

Lemma 2.1.6. [?, Lemma 3.6.6] *Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X.*

(1) If $g_i : X \dashrightarrow X_i, i = 1, 2$ are two ample models of D over U, then there is an isomorphism $h : X_1 \to X_2$ and $g_2 = h \circ g_1$.
(2) If $f : X \dashrightarrow Y$ is a semiample model of D over U, then the ample model $g : X \dashrightarrow Z$ of D over U exits and $g = h \circ f$, where $h : Y \to Z$ is a contraction and $f_*D \sim_{\mathbb{R},U} h^*H$. Here H is the ample divisor corresponding to the ample model Z.
(3) If $f : X \dashrightarrow Y$ is a birational map over U, then f is the ample model of D over U if and only if f is semiample model of D over U and f_*D is ample over U.

By above lemma there is another definition of log canonical models:

Definition 2.1.7. *Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties and $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then Y is* ***log canonical model*** *if it is the ample model.*

This Lemma seems useless.

Furthermore, for big boundary, we have

Lemma 2.1.8. [?, Lemma 3.9.3]*Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties. Suppose (X, B) is a klt pair and B is big over U. If $f : X \dashrightarrow Y$ is a weak log canonical model over U then*

- *f is a semiample model over U;*
- *the ample model $g : X \dashrightarrow Z$ over U exits;*
- *there is a contraction $h : Y \to Z$ such that $K_Y + f_*B \sim_{\mathbb{R},U} h^*H$ for some ample $\mathbb{R}$-divisor H on Z over U.*

Definition 2.1.9. [?, Definition 1.1.4] *Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ defined over rational numbers. Fix an $\mathbb{R}$-divisor $A \geqslant 0$, and then define*

$$\mathcal{L}_A(V) = \{D = A + B : B \in V, K_X + D \text{ is log canonical and } B \geqslant 0\}$$
$$\mathcal{E}_{A,\pi}(V) = \{D \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\}$$

Given a birational contraction $f : X \dashrightarrow Y$, define

$$\mathcal{W}_{A,\pi,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is a weak log canonical model of } (X, D) \text{ over } U\}$$

Given a rational contraciton $g : X \dashrightarrow Z$ over U, define

$$\mathcal{A}_{A,\pi,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is the ample model of } (X, D) \text{ over } U\}$$

In addition, let $\mathcal{C}_{A,\pi,g}(V)$ denote the closure of $\mathcal{A}_{A,\pi,g}(V)$ in $\mathcal{L}_A(V)$.

If the base U is clear or it is a point, then we may ommite π and simply write $\mathcal{E}_A(V)$ and $\mathcal{A}_{A,f}$.

Theorem 2.1.10 (Finiteness of weak log canonical models)**.** [?, Theorem E] *Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and A be an general divisor relatively ample over U, and $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a klt pair (X, Δ_0). Then there are finitely many birational maps $f_i : X \dashrightarrow X_i$ such that if $f : X \dashrightarrow Y$ is a weak log canonical model of $K_X + D$ over U for some $D \in \mathcal{L}_A(V)$, then there is an isomorphism $h_i : X_i \to Y$ and $f = h_i \circ f_i$.*

2.2. MMP.

Definition 2.2.1. *Let (X,B) be a pair and let $f: Y \to X$ be a log resolution of (X,B). Suppose*
$$K_Y + C = f^*(K_X + B),$$
then the discrepancy of exceptional divisor E_i over X is
$$a(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$
Moreover, let
$$\operatorname{discrep}(X,B) := \inf\{a(E;X,B) : E \text{ is an exceptional divisor over } X\}$$
and
$$\operatorname{totdiscrep}(X,B) := \inf\{a(E;X,B) : E \text{ is a divisor over } X\}.$$

Theorem 2.2.2. [?, Corollary 1.4.2]*Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let (X,B) be a $\mathbb{Q}$-factorial klt pair where $K_X + B$ is $\mathbb{R}$-Cartier and B is π-big. Let $C \geqslant 0$ be an $\mathbb{R}$-divisor. If $K_X + B + C$ is klt and π-nef, then we may run $(K_X + B)$-MMP over U with scaling of C and terminates.*

Theorem 2.2.3. [?, Corollary 1.3.3]*Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let (X,B) be a $\mathbb{Q}$-factorial klt pair where $K_X + B$ is $\mathbb{R}$-Cartier. If $K_X + B + C$ is not π-peseudo-effective, then we may run $f : X \dashrightarrow Y$ a $(K_X + B)$-MMP over U and end with a Mori fibre space $g : Y \to Z$.*

Corollary 2.2.4. [?, Corollary 13.7] *and* [?, Corollary 1.4.3]*: Let (X,B) be a a klt pair and $\mathfrak{C}$ be any set of exceptional divisors such that contains only exceptional divisors E of discrepancy $a(E;X,B) \leqslant 0$. Then there is a birational morphism $f : Z \to X$ and a $\mathbb{Q}$-divisor B_Z such that:*

(1) (Z, B_Z) is klt;
(2) E is a f-exceptional divisor if and only if $E \in \mathfrak{C}$;
(3) $B_Z = \sum -a(E;X,B)$ and $f_* B_Z = B$ and $K_Z + B_Z = f^*(K_X + B)$.

*In particular, if we take $\mathfrak{C}$ containing all such divisors, then Z is called **terminalization** of X; if take $\mathfrak{C}$ containing only one such divisor, then $f : Z \to X$ is called a **divisorial extraction**.*

Definition 2.2.5. [?, Definition 3.3] *Two or more pairs $\{(X_i, B_i)\}$ are called **MMP-related** if they are results of $(K+B)$-MMP from a log smooth pair (W, B_W).*

Lemma 2.2.6. [?, Proposition 3.4] *Let $\{(X_l, B_l)\}$ be a finite set of $\mathbb{Q}$-factorial klt pairs such that birational to other, then TFAE:*

(1) They are MMP-related;
(2) There is a nonsingular pair (W, B_W) with snc boundary, and projective birational morphisms $f_l : W \to X_l$ dominating each X_l, such that $f_{l*} B_W = B_l$ and
$$K_W + B_W = f_l^*(K_{X_l} + B_l) + \sum_{\text{exceptional}} a_{li} E_{li}$$
with $a_{li} > 0$ for all f_i-exceptional divisors;
(3) For any two pairs $(X, B = \sum_i b_i B_i), (X', B' = \sum_j b'_j B'_j)$ in the set, $a(B_i; X', B') \geqslant -b_i$ and strict inequality holds if and only if B_i exceptional over X', and $a(B'_j; X, B) \geqslant -b'_j$ and strict inequality holds if and only if B'_j exceptional over X.

Let $K = K(X)$ be the function field, and let $\Sigma = \{\nu\}$ be the set of discrete valutions of the field .

Definition 2.2.7. [?, Definition 3.5] *Let $\theta : \Sigma \to [0,1)_{\mathbb{Q}}$ be a function. Then we can construct a collection $\mathcal{C}_\theta$ of pairs associated to θ, consists of klt pairs $(X, B = \sum a_i B_i)$ satisfying*

(1) $a_i = \theta(B_i)$;
(2) $a(E;X,B) > -\theta(E)$ for all E exceptional over X.

For example, if we take $\theta \equiv 0$ constant, the $\mathcal{C}_\theta$ is the collection of all terminal varieties Y without boundary birational to X. Furthermore, we can define the corresponding discrepancy:

Definition 2.2.8 (θ-discrepancy)**.** *Let (X,B) be a pair in the category $\mathcal{C}_\theta$ for some function θ and let $f: Y \to X$ be a log resolution of (X,B). Suppose*

$$K_Y + B_Y + C = f^*(K_X + B)$$

where $B_Y = (f^{-1})_*B + \sum_{E_i \text{ exceptional}} \theta(E_i)E_i$, then the θ-discrepancy of exceptional divisor E_i over X is

$$a_\theta(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$

Or equivalently, we have

$$a_\theta(E_i; X, B) = a(E_i; X, B) + \theta(E_i).$$

A pair (X,B) is called θ-canonical(θ-terminal) if $a_\theta(E;X,B) \geqslant 0$($a_\theta(E;X,B) > 0$) for all exceptional divisors E over X. Note that θ-canonical pair is not always in $\mathcal{C}_\theta$.

3. Original proof

3.1. Prepare.

First we fix a collection:

Proposition 3.1.1. [?, Lemma 3.6] *Let $f:(X,B)\to S, f':(X',B')\to S'$ be two $\mathbb{Q}$-factorial log Mori fibre spaces with only klt singularities and MMP-related, inducing a birational map Φ:*

$$\begin{array}{ccc} (X,B) & \overset{\Phi}{\dashrightarrow} & (X',B') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & & S' \end{array}$$

Suppose $B = \sum_i b_iB_i + \sum_j d_jD_j$ and $B' = \sum_j d'_jD_j + \sum_k b'_kB'_k$, where B_i are divisors on X but not on X', B'_k are divisors on X' but not on X, and D_j are divisors on both X and X'. By Lemma 2.2.6, $d_j = d'_j$. Take a rational number $\epsilon < 1$ such that $\epsilon > -\operatorname{totdiscrep}(X,B), -\operatorname{totdiscrep}(X',B')$, and take the function $\theta: \{\nu\} \to [0,1)_{\mathbb{Q}}$ as following:

- *$\theta(B_i) = b_i, \theta(D_j) = d_j, \theta(B'_k) = b'_k$;*
- *$\theta(E) = \epsilon$ if E is exceptional over both X and X';*
- *$\theta(D) = 0$ if D is a divisor on both X and X', but not a component of B or B'.*

Then the collection $\mathcal{C}_\theta$ satisfies

1. *(X,B) and (X',B') belongs to $\mathcal{C}_\theta$;*
2. *For any finitely many klt pairs $\{(X_l,B_l)\}$ in $\mathcal{C}_\theta$, there is an object $(Z,B_Z)\in\mathcal{C}_\theta$ and projective birational morphisms $Z\to X_l$ dominating each X_l as a process of (K_Z+B_Z)-MMP over X_l (thus over $\operatorname{Spec}\mathbb{C}$);*
3. *Any $(K+B)$-MMP starting from an object in $\mathcal{C}_\theta$ stays inside of $\mathcal{C}_\theta$, and so does any $(K+B+cH)$-MMP where H is base point free and $c\in\mathbb{Q}_{>0}$.*

Remark 3.1.2. *Let $\delta = 1-\epsilon$, then all pairs in $\mathcal{C}_\theta$ is δ-lc.*

With notations and assumptions in Proposition 3.1.1, we shall define the Sarkisov degree. We take a very ample divisor A' on S' and a sufficiently large and divisible integer $\mu' > 1$ such that

$$\mathcal{H}' = |-\mu'(K_{X'}+B') + f'^*A'|$$

is a very ample complete linear system on X' over $\operatorname{Spec}\mathbb{C}$. Let (W,B_W) be a common log resolution of X and X' in $\mathcal{C}_\theta$ with projective birational morphism $\sigma: W\to X$, $\sigma': W\to X'$ and $\sigma_*B_W = B, \sigma'_*B_W = B'$. Let $\mathcal{H}_W := \sigma'^*\mathcal{H}'$ and then $\mathcal{H} := (\Phi^{-1})_*\mathcal{H}' = \sigma_*\mathcal{H}_W$. Furthermore, if $\mathcal{H}$ is not base point free, then

$$\sigma^*\mathcal{H} = \mathcal{H}_W + F$$

where $F = \sum f_lF_l \geqslant 0$ is the fixed part. Take a general member H' of the linear system $\mathcal{H}'$ such that $H_W := \sigma'^*H' = (\sigma'^{-1})_*H' \in \mathcal{H}_W$, and let $H := (\Phi^{-1})_*H' = \sigma_*H_W$, then H if f-ample and $\sigma^*H = H_W + F$. By taking further resolution, we may assume H_W is smooth and crosses normally with exceptional locus of σ and σ'.

Now we can define the Sarkisov degree in $\mathcal{C}_\theta$ with respect to H' (or $\mathcal{H}'$):

Definition 3.1.3. [?, Definition 3.8] *Sarkisov degree of (X, B) with respect to H (or $\mathcal{H}$) in $\mathcal{C}_\theta$ is a triple (μ, λ, e) ordered lexicographically:*

- ***Nef threshold*** *μ: Let $C \subset X$ be a curve contracted by f, then*
$$\mu := -\frac{H.C}{(K_X + B).C}$$
i.e. $K_X + B + \frac{1}{\mu}H \equiv_S 0$;
- ***θ-canonical threshold*** *c* ***and*** *λ: $\lambda = 0$ if $\mathcal{H}$ is base point free; otherwise,*
$$c := \frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geqslant 0, E \text{ exceptionl over } X\}$$
- ***Number of*** *$(K_X + B_X + \frac{1}{\mu}H)$**-crepant divisors**: Let $e = 0$ if $\mathcal{H}$ is base point free (and hence $\lambda = 0$), otherwise*
$$e = \#\{E; E \text{ is } \sigma\text{-exceptional and } a_\theta(E; X, B + \frac{1}{\lambda}H) = 0\}$$

Remark 3.1.4. *(1) The Sarkisov degree is dependent on the choice of A', H' and θ.*

(2) Take a common log resolution $(W, B_W) \in \mathcal{C}_\theta$ with $B_W = \sum \theta(E)E$ and projective birational morphisms $\sigma : W \to X$, $\sigma' : W \to X'$. Since $\sigma^*\mathcal{H} = \mathcal{H}_W + \sum f_l F_l$, we have ramification formula:
$$K_W + B_W + tH_W = \sigma^*(K_X + B + tH) + \sum (a_l - tf_l)E_l$$
where $\sum a_l E_l$ is effective and supported on $\operatorname{Exc}\sigma$. Then $\lambda := \max\{\frac{f_l}{a_l}\}$. If $\mathcal{H}$ is base point free, then $\sum f_l F_l = 0$ and $\lambda = 0$.

(3) e is the number of components in $\sum(a_l - cf_l)E_l$ with coefficient 0 in the formula
$$K_W + B_W + \frac{1}{\lambda}H_W = \sigma^*(K_X + B + \frac{1}{\lambda}H) + \sum (a_l - \lambda f_l)E_l.$$
Such prime divisors $E_1 \dots E_e$ are called $(K_X + B_X + \frac{1}{\lambda}H)$-$\theta$-crepant.

We also need some extraction map in this category:

Lemma 3.1.5. *Using the notation in the definition of Sarkisov degree, then there is a contraction $f : Z \to X$ such that*

- *$(Z, B_Z) \in \mathcal{C}_\theta$ and $(Z, B_Z + \frac{1}{\lambda}H_Z)$ is θ-terminal and $\mathbb{Q}$-factorial;*
- *$\rho(Z) = \rho(X) + 1$;*
- *f is $(K_X + B_X + \frac{1}{\lambda}H_X)$-crepant, that is*
$$K_Z + B_Z + \frac{1}{\lambda}H_Z = f^*(K_X + B + \frac{1}{\lambda}H).$$

Proof. We follow the proof in [?, Proposition 1.6] but for klt pair case. Let $(W, B_W) \in \mathcal{C}_\theta$ and $\sigma : W \to X, \sigma' : W \to X'$ be the common resolution as in Definition 3.1.3 ,and suppose $E_1, \dots, E_e$ are $(K_X + B_X + \frac{1}{\mu}H)$-$\theta$-crepant divisors after renumbering. Then we have
$$K_W + B_W + \frac{1}{\lambda}H_W = \sigma^*(K_X + B + \frac{1}{\lambda}H) + \sum_{l=1}^{e} 0 \cdot E_l + \sum_{l>e}(a_l - \frac{1}{\lambda}f_l)E_l.$$
We run $(K_W + B_W + \frac{1}{\lambda}H_W)$-MMP on W over X with scaling of some ample divisor, then the MMP ends with a minimal model $p : (Y, B_Y + \frac{1}{\lambda}H_Y) \to X$ of $(W, B_W + \frac{1}{\lambda}H_W)$ over X and the exceptional locus is exactly $\cup_{i=1}^{e} E_i$ and p is crepant:
$$K_Y + B_Y + \frac{1}{\lambda}H_Y = p^*(K_X + B_X + \frac{1}{\lambda}H_X).$$

Then we run $(K_Y + B_Y)$-MMP on Y over X with scaling of some ample divisor. This ends with the minimal model (X, B) of (Y, B_Y) over X, and the last contraction in the MMP is $f : Z \to X$ as required. □

3.2. **Flowchart for the Log Sarkisov program.** We follow [**?**, Flowchart for the Sarkisov program] in this subsection.

If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is nef, the two Mori fibre spaces are isomorphic by following proposition and we stop here.

Theorem 3.2.1. *(Noether-Fano-Iskovskikh Criterion): Notations as in the definition of Sarkisov degree, then*

(1) $\mu \geqslant \mu'$*;*

(2) If $\mu \geqslant \lambda$ *and* $(K_X + B + \frac{1}{\mu}H)$ *is nef, then* Φ *is an isomorphisms of Mori fibre spaces, i.e., we have commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\sim", "\Phi"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "\sim"] & S' \end{tikzcd}$$

Proof. We follow the proof in [**?**],[**?**] and [**?**]:

(1) Only need to show $(K_X + B + \frac{1}{\mu'}H)$ is f-nef. Let $\sigma : W \to X$ and $\sigma' : W \to X'$ be the common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W =& \sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ =& \sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here $\{G_i\}, \{E_j\}$ are σ-exceptional divisors, and $\{E_j\}\{G'_k\}$ are σ'-exceptional divisors. Since $H_W = \sigma'^* H'$, $g'_k > 0$ (or there are no such G'_k). Then take a general curve $C \subset X$ contracted by f, such that its strict transform $\tilde{C}$ on W is disjoint from G_i, E_j, and is not contained in G'_k. Then we have:

$$\begin{aligned} C.\left(K_X + B + \frac{1}{\mu'}H\right) ==& \tilde{C}.\left(\sigma^*\left(K_W + B_W + \frac{1}{\mu'}H\right) + \sum g_i G_i + \sum e_j E_j\right) \\ =& \tilde{C}.\left(\sigma'^*\left(K_{X'} + B' + \frac{1}{\mu'}H'\right) + \sum e'_j E_j + \sum g'_k G'_k\right). \\ =& \tilde{C}.\sigma'^* f'^* A' + C.\left(\sum g'_k G'_k\right) \geqslant 0 \end{aligned}$$

This implies $(K_X + B + \frac{1}{\mu'}H)$ is f-nef and $\mu \geqslant \mu'$;

(2) First we show that $\mu = \mu'$. By 1, we only need to show $(K_{X'} + B' + \frac{1}{\mu}H')$ is f'-nef. Indeed, same as 1, we can take a curve C' on X' contracted by f', such that its strict transform $\tilde{C}'$ on W is disjoint from G'_i, E_j, and is not contained in G'_k and $C'.\left(K_{X'} + B' + \frac{1}{\mu}H'\right) \geqslant 0$.

Then we show there are isomorphic. Take a very ample divisor D on X and let D' be its strict transform on X'. D' is f'-ample, thus there exists $0 < d \ll 1$ such that the following holds:

- $K_X + B + \frac{1}{\mu}H + dD$ is ample;
- $K_{X'} + B' + \frac{1}{\mu}H' + dD'$ is ample.

Therefore X and X' are both log canonical models of $(W, B_W + \frac{1}{\mu}H_W + dD_W)$, hence $X \cong X'$. Furthermore, f and f' are contractions of same numerical class of curves, thus two log Mori fibre spaces are isomorphic.

□

Otherwise, if the condition does not hold:

Claim 3.2.2. *(1) If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef, then there is a contraction $f : X \to T$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type III or IV;*

(2) If $\lambda > \mu$, then there is a divisorial extraction $p : Z \to X$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type I or II.

Proof. (1) By assumption, $\lambda \leqslant \mu$ and $K_X+B+\frac{1}{\mu}H$ is not nef. Suppose f is the contraction with respect to a $(K_X + B)$-negative extremal ray $R = \overline{\mathrm{NE}}(X/S)$, then $(K_X + B + \frac{1}{\mu}H).R = 0$ by definition of μ. There is an extremal ray $P \subset \overline{\mathrm{NE}}(X)$ such that $(K_X + B + \frac{1}{\mu}H).P < 0$ and $F := P + R$ is an extremal face (Check [?, 5.4.2] for details). Take $0 < \delta \ll 1$ such that $(K_X+B+(\frac{1}{\mu}-\delta)H).P < 0$, then $(K_X+B+(\frac{1}{\mu}-\delta)H).R < 0$ since H is f-ample, and F is a $(K_X+B+(\frac{1}{\mu}-\delta)H)$-negative extremal face. Since $(X, B+(\frac{1}{\mu}-\delta)H)$ is klt, there is a contraction $g : X \to T$ with respect to F factorizing through $f : X \to S$. Since $(X, B + \frac{1}{\mu}H)$ is klt, and $\rho(X/T) = 2$, we can run $(K_X + B + \frac{1}{\mu}H)$-MMP on X with scaling of some ample divisor. Since $B+\frac{1}{\mu}H$ is relatively big, the MMP terminates. There are following cases:

(a) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$, and then followed by a Mori fibre space $(X_1, B_1 + \frac{1}{\mu}H_1) \to S_1$. Then $S_1 \cong T$ and this is a link of type III.

(b) After finitely many flips $X \dashrightarrow X_1$, first non-flip contraction is a Mori fibre space $f_1 : X_1 \to S_1$. This is a link of type IV.

(c) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$ with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where $e > 0$ and $E = \mathrm{Exc}\ p$ and $g_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to T$ is a log minimal model of $(X, B+\frac{1}{\mu}H)$ over T . In fact the only ray of $\overline{\mathrm{NE}}(X_1/T)$ is $(K_{X_1}+B_1+\frac{1}{\mu}H_1)$-trivial and hence is $(K_{X_1} + B_1)$-negative, therefore $(X_1, B_1)/T$ is a log Mori fibre space. Take $S_1 = T$, then this is a link of type III:

(d) After finitely many flips $X \dashrightarrow X_1$, $(K_X + B + \frac{1}{\mu}H)$-MMP ends with a log minimal model $(X_1, B_1+\frac{1}{\mu}H_1)$ over T. Then there is an extremal ray R of $\overline{\mathrm{NE}}(X_1/T)$, which is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and $(K_{X_1} + B_1)$-negative. Let $f_1 : X_1 \to S_1$ be the contraction with respect to R. This is a link of type IV. In fact, $X \dashrightarrow S_1$ is the ample model of $K_X + B + \frac{1}{\mu}H$.

(2) By assumption, $\lambda > \mu$. Take an extraction $p : (Z, B_Z, H_Z) \to (X, B, H)$ as in Lemma 3.1.5. That is, (Z, B_Z) is θ-terminal and $p^*(K_X+B+\frac{1}{\lambda}H) = K_Z+B_Z+\frac{1}{\lambda}H_Z$ where $B_Z = \sum \theta(E_\nu)E_\nu$ and $E = \mathrm{Exc}\, p$ is a prime divisor on Z. Then we run $(K_Z+B_Z+\frac{1}{\lambda}H_Z)$-MMP on Z over S with scaling of some ample divisor. Since Z is covered by $(K_Z+B_Z+\frac{1}{\lambda}H_Z)$-negative curves, $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ is not relatively pseudo-effective. Hence this ends with a Mori fibre space by Theorem 2.2.3. There are two cases:

(a) After finitely many flips $Z \dashrightarrow Z'$, the first non-flip contraction is a divisorial contraction $q : Z' \to X_1$. Then $X_1 \to S$ is a log Mori fibre space of (X, B) and $(X, B + \frac{1}{\lambda}H)$. Let $S_1 = S$ and this is a link of type II.

(b) After finitely many flips $Z \dashrightarrow X_1$, first non-flip contraction is a fibering contraction $f_1 : X_1 \to S_1$. Since $(K_{X_1}+B_1+\frac{1}{\lambda}H_1)$ is f_1-negative and H_1 is f_1- ample, $(K_{X_1}+B_1)$ is f_1-negative, and $(X_1, B_1)/Y$ is a log Mori fibre space. Take $S_1 = Y$ and this is a link of type I.

□

Remark 3.2.3. *(1) (a) For case 1a and 1b, since $K_{X_1}+B_1+\frac{1}{\mu}H_1$ is f_1-negative, we have $\mu_1<\mu$.*

(b) For case 1c and 1d, Since $(K_{X_1}+B_1+\frac{1}{\mu}H_1)$ is trivial on the ray $R=\overline{\mathrm{NE}}(X_1/S_1)$ for both cases, we have $\mu_1=\mu$. Notice that $(X_1,B_1+\frac{1}{\mu}H_1)$ stays θ-canonical, we have $\lambda_1\leqslant\mu=\mu_1$, thus next link goes back to case 1. Furthermore, for case 1c we have $\rho(X_1)=\rho(X)-1$.

(2) For case 2:

(a) For both case 2a and 2b, we have $\mu_1\leqslant\mu$ with equality holds if and only if
- *either* $\dim S_i<\dim S_{i+1}$
- *or* $\dim S_i=\dim S_{i+1}$ *and the link is square.*

(b) We have $\lambda_1\leqslant\lambda$ and if $\lambda_1=\lambda$, then $e_1<e$.

3.3. **Termination.** To prove the termination by contraction, we need following in $\mathcal{C}_\theta$:
(1) discreteness of nef threshold μ;
(2) termination of flips;
(3) ascending chain condition of log canonical threshold;
(4) finiteness of local log canonical threshold.

Suppose there is an infinite sequence, i.e. there are infinitely many X_i and birational maps obtained from the program:

$$X=X_0\dashrightarrow X_1\dashrightarrow\cdots\dashrightarrow X_i\dashrightarrow\cdots\dashrightarrow X'$$

(1) Discreteness of nef threshold holds for all dimensions, by boundedness of δ-lc fano varieties ([**?**, Theorem 1.1]). Therefore we may assume μ_i is constant, that is, $\mu=\mu_0=\mu_i$ for all i.
(2) Termination of flips holds only for threefolds and peseudo-effective fourfolds. Therefore, for higher dimensions, we cannot exclude the possibility that the program gives an infinite sequence of Sarkisov links of type IV.
(3) Ascending chain condition of log canonical threshold holds for all dimensions [HMX14].
(4) Finiteness of local log canonical threshold doesnot hold for higher dimensions. Therefore we cannot exclude the possibility that the program gives an infinite sequence of Sarkisov links of type II.

4. Double scaling

This section we follows [**?**, 13.The Sarkisov program] and [**?**].

4.1. **Prepare.** Let (W,B_W) be a $\mathbb{Q}$-factorial klt pair and $f:(X,B)\to S$ and $f':(X',B')\to S'$ be two different log Mori fibre spaces as outputs of (K_W+B_W)-MMP. To modify the beginning setting, we need more conventions and lemmas:

Definition 4.1.1. *Let $f:X\dashrightarrow Y$ be a birational map of normal quasi-projective varieties. If*
- *f does not extract divisors;*
- *$a(E;X,B_X)\leqslant a(E;Y,B_Y)$ for all divisors E over X.*

then we denote $(X,B)\geqslant(Y,B_Y)$.

In particular, for terminal pairs, we have following lemma:

Lemma 4.1.2. [**?**, Lemma 13.8] *Let $f:W\dashrightarrow X$ be a birational map where (W,B_W) is terminal. If*
- *f does not extract divisors;*
- *K_X+B is nef, where $B=f_*B_W$;*
- *$a(E;X,B)\geqslant a(E;W,B_W)$ for all divisors $E\subset W$,*

then
- *$(W,B_W)\geqslant(X,B)$.*

- *(X,B) is klt*
- *If $Z \to X$ is a divisorial extraction of a divisor E with $a(E;X,B) \leqslant 0$, then E is a divisor on W ;*
- *If $Z \to X$ is terminalization of (X,B), then $W \dashrightarrow Z$ extracts no divisors.*

Conversely, start from a klt pair and non-positive map, we have

Lemma 4.1.3. [?, Lemma 3.5] *Let $\sigma : (W, B_W) \dashrightarrow (X,B)$ be a $K_W + B_W$-non-positive birational map such that $\sigma_*(K_W + B_W) = K_X + B$ and (W, B_W) is a $\mathbb{Q}$-factorial klt pair. Then there is a resolution of indeterminacy $\pi : \tilde{W} \to W$ and $\tilde{\sigma} : \tilde{W} \to X$ such that*

- *$(\tilde{W}, B_{\tilde{W}})$ is $\mathbb{Q}$-factorial terminal and $\tilde{\sigma}_* B_{\tilde{W}} = B$,*
- *$\tilde{\sigma}$ is $(K_{\tilde{W}} + B_{\tilde{W}})$-non-positive and $(\tilde{W}, B_{\tilde{W}}) \geqslant (X,B)$.*

By Lemma 4.1.3, we replace (W, B_W) by its log resolution such that (W, B_W) is terminal and $\sigma : W \to X$ and $\sigma' : W \to X'$ are $(K_W + B_W)$-non-positive morphisms, and $(W, B_W) \geqslant (X,B), (X', B')$.

Take very general ample $\mathbb{Q}$-divisors A and A' on S and S' such that $G \sim_{\mathbb{Q}} -(K_X + B) + f^*A$ and $H \sim_{\mathbb{Q}} -(K_{X'} + B') + f'^*A'$ are two ample $\mathbb{Q}$-divisors. Moreover, we may assume G and H satisfying $G_W := \sigma^* G = \sigma_*^{-1} G$ and $H_W := \sigma'^* H = \sigma_*'^{-1} H$. Therefore $\sigma_*(K_W + B_W + G_W) = K_X + B + G$ is nef, and Lemma 4.1.2 holds. Furthermore, we may assume $(W, B_W + gG_W + hH_W)$ is log smooth and terminal for all $0 \leqslant g, h \leqslant 2$ by taking furthermore blowing up if necessary. Then we have:

Theorem 4.1.4 (Sarkisov program with double scaling)**.** *Notations as above, there is a finite sequence of Sarkisov links*

$$\begin{tikzcd}
X = X_0 \arrow[r, dotted] \arrow[d, "f=f_0"'] & X_1 \arrow[r, dotted] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"'] & \cdots \arrow[r, dotted] & X_N = X' \arrow[d, "f_N"'] \\
S = S_0 & S_1 & S_2 & & S_N = S'
\end{tikzcd}$$

and rational numbers

$$\begin{aligned} 1 = g_0 \geqslant g_1 \geqslant \cdots \geqslant g_N &= 0 \\ 0 = h_0 \leqslant h_1 \leqslant \cdots \leqslant h_N &= 1 \end{aligned}$$

such that

1. *For each i, $\sigma_i : W \dashrightarrow X_i$ is $(K_W + B_W + g_i G_W + h_i H_W)$-non-positive, and $(K_{X_i} + B_i + g_i G_i + h_i H_i) = \sigma_{i*}(K_W + B_W + g_i G_W + h_i H_W)$ is nef and is relatively trivial over S_i;*
2. *$(W, B_W + g_i G_W + h_i H_W) \geqslant (X_i, B_i + g_i G_i + h_i H_i)$;*
3. *each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is given by a sequence of $(K_{X_i} + B_i + g_i G_i + h_i H_i)$-trivial maps.*
4. *The last link $X_N \to S_N$ is isomorphic to $X' \to S'$*

4.2. **Construct Sarkisov links.** This subsection we construct the links inductively. Suppose we have $\sigma_i : W \dashrightarrow X_i$ as in Theorem 4.1.4, that is

- $f_i : (X_i, B_i) \to S_i$ is a log Mori fibre space and $\sigma_{i*} B_W = B_i$;
- $\sigma_i : W \dashrightarrow X_i$ is $(K_W + B_W + g_i G_i + h_i H_W)$-non-positive birational map, and $(K_{X_i} + B_i + g_i G + h_i H) = \sigma_{i*}(K_W + B_W + g_i G_W + h_i H_W)$ is nef and relatively trivial over S_i;
- $(W, B_W + g_i G_W + h_i H_W) \geqslant (X_i, B_i + g_i G_i + h_i H_i)$;
- $0 \leqslant g_i, h_i \leqslant 1$ are rational numbers.

Then we need to show that there is a Sarkisov link $X_i \dashrightarrow X_{i+1}$ satisfying Theorem 4.1.4. Similarly with Sarkisov degree, we have following notations:

Definition 4.2.1. *Let C_i be a general f_i-vertical curve on X_i, then*

- *$r_i := \frac{H_i.C_i}{G_i.C_i}$;*

- *Let* Γ *be the set of* $t \in [0, \frac{g_i}{r_i}]$ *such that*
  (1) $(W, B_W + g_i G_W + h_i H_W + t(H_W - r_i G_W)) \geqslant (X_i, B_i + g_i G_i + h_i H_i + t\,(H_i - r_i G_i))$
  (2) $K_{X_i} + B_i + g_i G + h_i H + t(H_i - r_i G_i)$ *is nef;*
  Let $s_i = \max \Gamma$*;*
- *Let* $D_{W,i} = B_W + g_i G_W + h_i H_W$ *and* $D_i = B_i + g_i G_i + h_i H_i$*. Let* $D_{W,i}(t) = B_W + g_i G_W + h_i H_W + t(G_W - r_i H_W)$ *and* $D_i(t) = B_i + g_i G_i + h_i H_i + t\,(G_i - r_i H_i))$*. Let* $g_{i+1} = g_i - r_i s_i$ *and* $h_{i+1} = h_i + s_i$*. Note that* $D_{W,i+1} = D_{W,i}(s_i)$*.*

Then we have (check [**?**, Lemma 4.4] for details)

(1) $r_i > 0$;
(2) either $\Gamma = \{0\}$ or is a closed interval;
(3) $g_{i+1} = g_i \Leftrightarrow h_{i+1} = h_i \Leftrightarrow s_i = 0$;

Construct links.: If $s_i = \frac{g_i}{r_i}$, then $g_{i+1} = 0$. Let $N = i+1$ and let $f_N : X_N = X_i \to S_N = S_i$, then $X_N \to S_N$ is isomorphic to $f' : X' \to S'$ (see Proposition 4.3.2) and we stop. Otherwise, if $s_i < \frac{g_i}{r_i}$, then we construct the Sarkisov link $X_i \dashrightarrow X_{i+1}$ in following cases:

(1) Suppose s_i is not the threshold of condition 1 of Γ. That is, there exists $0 < \epsilon \ll 1$, such that for any divisor E on W, we have
$$a(E; X_i, D_i(s_i + \epsilon)) \geqslant a(E; W, D_{W,i}(s_i + \epsilon))$$
and $K_{X_i} + D_i(s_i + \epsilon)$ is not nef. Then there is a 2-dimensional $(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$-negative extremal face F for some $0 < \delta \ll \epsilon$, spaned by $R = \mathbb{R}_{\geqslant 0}[C_i]$ and another extremal ray P. Hence there is a contraction $X_i \to T_i$ corresponding to F factoring through f_i. Then we run $(K_{X_i} + D_i(s_i + \epsilon))$-MMP on X_i with scaling over T_i. After finitely many flips, we either have a $(K_{X_i} + D_i(s_i + \epsilon))$ minimal model, a divisorial contraction, or a Mori fibre space over T_i:
   (a) After finitely many flips $X_i \dashrightarrow X_{i+1}$ there is a log Mori fibre space $X_{i+1} \to S_{i+1}$, and this is a link of type III.
   (b) After finitely many flips $X_i \dashrightarrow Z_i$ there is a divisorial contraction $Z_i \to X_{i+1}$, then let $S_{i+1} = T_i$ and $X_{i+1} \to S_{i+1}$ is a log Mori fibre space and this is a link of type IV.
   (c) After finitely many flips $X_i \dashrightarrow X_{i+1}$, the contraction $X_{i+1} \to T_i$ is a log minimal model of $(X_i, D_i\,(s_i + \epsilon))$ over T_i. Let C' be the strict transform of C_i on X_{i+1}, then $(K_{X_{i+1}} + D_{i+1}(\epsilon)).C' = 0$ and $(K_{X_{i+1}} + B_{i+1}).C' < 0$, therefore there is a contraction $X_{i+1} \to S_{i+1}$ which is a log Mori fibre space. And this is a link of type IV.

(2) Suppose s_i is the threshold of condition 1 of Γ, that is, there exists $0 < \epsilon \ll 1$ and a σ_i-exceptional divisor E_i on W such that
$$a(E_i; X_i, D_i(s_i + \epsilon)) < a(E_i; W, D_{W,i}(s_i + \epsilon)).$$
In this case, we have
$$a(E_i; X_i, D_i(s_i)) = a(E_i; W, D_{W,i}(s_i)) = -\operatorname{mult}_{E_i}(D_{W,i}(s_i)) \leqslant 0.$$
   Let $p_i : Z_i \to X_i$ be the divisorial extraction of the divisor E_i as in Corollary 2.2.4, and suppose $K_{Z_i} + D_{Z_i}(s_i) = K_{Z_i} + B_{Z_i} + g_{i+1} G_{Z_i} + h_{i+1} H_{Z_i} = p_i^*\,(K_{X_i} + D_i\,(s_i + \epsilon))$. Take a sufficiently small δ such that $0 < \delta \ll \epsilon \ll 1$ and
$$K_{Z_i} + \Delta_i = p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$$
is klt. Then we run $(K_{Z_i} + \Delta_i)$-MMP on Z_i over S_i. Since Z_i is covered by $(K_{Z_i} + \Delta_i)$-negative curves, it follows that $(K_{Z_i} + \Delta_i)$ is not pseudo-effective over S_i, and this MMP ends with a log Mori fibre space. Moreover, this is a MMP for $p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta' G_i)$ for all $0 < \delta' \leqslant \delta$. After finitely many flips, we either have a $(K_{Z_i} + \Delta_i)$ log Mori fibre space or a $(K_{Z_i} + \Delta_i)$ divisorial contraction.

(a) After finitely many flips $Z_i \dashrightarrow X_{i+1}$ there is a Mori fibre space $X_{i+1} \to S_{i+1}$, and this is a link of type I. In this case we have $\rho(X_{i+1}) = \rho(X_i) + 1$.
(b) After finitely many flips $Z_i \dashrightarrow Z'_{i+1}$ there is a divisorial contraction $q_i : Z'_{i+1} \to X_{i+1}$, and then a logMori fibre space $X_{i+1} \to S_i =: S_{i+1}$. This is a link of type II.

Claim 4.2.2. *By* [**?**, Lemma 13.14-17] *and* [**?**, Lemma 4.2] *, we have:*

(1) $r_i \leqslant r_{i+1}$. Moreover, in case 1a, we have $r_i < r_{i+1}$.
(2) Since the birational map $X_i \dashrightarrow X_{i+1}$ is over T_i (over S_i) and $(K_{X_i} + D_i(s_i))$ is numerically trivial over T_i (over S_i) in case 1 (case 2), it follows that $a(E; X_i, D_i(s_i)) = a(E; X_{i+1}, D_{i+1})$ for any divisors E over W and so the inequality

$$a(E; X_{i+1}, D_{i+1}) \geqslant a(E; W, D_{W,i+1})$$

(3) In case 1, for any divisor $E \subset W$, we have $a(E; X_i, D_i(s_i + \epsilon)) \leqslant a(E; X_{i+1}, D_{i+1}(\epsilon))$ for all $0 < \epsilon \ll 1$. Moreover, since $X_i \ncong X_{i+1}$, there is a divisor F over W such that $a(F; X_i, D_i(s_i + \epsilon)) < a(F; X_{i+1}, D_{i+1}(\epsilon))$.
(4) In case 2, for any divisor $E \subset W$, we have $a(E; X_i, D_i(s_i+\epsilon)-\delta G_i) \leqslant a(E; X_{i+1}, D_{i+1}(\epsilon)-\delta G_{i+1})$ for all $0 < \epsilon \ll 1$. Moreover, since $X_i \ncong X_{i+1}$, there is a divisor F over W such that $a(F; X_i, D_i(s_i + \epsilon) - \delta G_i) < a(F; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$.
(5) $h_i \leqslant 1$, and $h_i = 1$ if and only if $g_i = 0$;

4.3. **Termination.**

Lemma 4.3.1. [**?**, Lemma 13.18 and Lemma 13.19] *(or* [**?**, Lemma 4.9]*) Suppose we construct a sequence of Sarkisov links:*

$$\begin{tikzcd} X = X_0 \arrow[r, dashed] \arrow[d] & X_1 \arrow[r, dashed] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r, dashed] & X_i \arrow[r] \arrow[d] & \cdots, \\ S = S_0 & S_1 & S_2 & & S_i & \end{tikzcd}$$

then

(1) there are only finitely many possibilities of $f_i : X_i \to S_i$ up to isomorphism;
(2) the Sarkisov program with double scalling of (G_W, H_W) terminates. That is, there exits an integer $N > 0$ such that $g_N = 0$.

Proof. (1) This essentially follows from the finiteness of weak log canonical model (Theorem 2.1.10). We construct the subspace V of $\mathrm{WDiv}_{\mathbb{R}}(W)$ as following:

(a) If $h_k > 0$ for some k: Since H_W is nef and big, take an ample $\mathbb{Q}$-divisor A_W and an effective $\mathbb{Q}$-divisor C_W such that $H_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spaned by components of B_W, G_W, H_W, C_W, then for $i > k$:

$$B_W + g_i G_W + h_i H_W \sim_{\mathbb{Q}} h_k A_W + B_W + g_i G_W + (h_i - h_k) H_W + h_k C_W =: \Delta_i \in \mathcal{L}_{h_k A_W}(V)$$

(b) If $h_k = 0$ for all k, then $h_i \equiv 0$ and $g_i \equiv 1$. Since G_W is nef and big, take an ample $\mathbb{Q}$-divisor A_W and an effective $\mathbb{Q}$-divisor C_W such that $G_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spaned by components of B_W, C_W, then

$$B_W + G_W \sim_{\mathbb{Q}} A_W + B_W + C_W =: \Delta_i \in \mathcal{L}_{A_W}(V)$$

Then all X_i are weak log canonical models of (W, Δ_i). By finiteness of weak log canonical models, there are finitely many $\sigma_i : W \dashrightarrow X_i$ up to isomorpism. Then we shall show that for $\sigma_i : W \dashrightarrow X_i$ there are finitely many log Mori fibre spaces in the sequence up to isomorpism. Indeed, we may assume that there is a k such that $X_i \cong X_k$ for all $i > k$, and f_i is the contraction corresponding to an extremal ray $R_i \subset \overline{\mathrm{NE}}(X_k)$. Then we have $(K_{X_k} + B_k).R_i < 0$ and $(K_{X_k} + B_k + g_i G_k + h_i H_k).R_i = 0$. Furthermore, H_k and G_k are relatively ample over S_i for all $i > k$.

(a) If $h_k > 0$: Since H_k is big, we have $h_k H_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor H_k and effective $\mathbb{Q}$-divisor E_k. Let $B'_k = B_k + (1-\epsilon)h_k H_k + \epsilon E_k$ for sufficiently small ϵ such that (X_k, B'_k) is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \epsilon A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k}+B'_k+\epsilon A_k \geqslant 0} + \sum_{\alpha\in\Lambda \text{ finite set}} R_\alpha$$

All extremal rays R_i corresponding to f_i for $i > k$ are in the finite set $\{R_\alpha\}_{\alpha\in\Lambda}$, thus there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of X_k.

(b) If $h_i = 0$ for all i, and hence $g_i = 1$ for all i. Since G_i is big, we have $G_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor A_k and effective $\mathbb{Q}$-divisor E_k. Let $B'_k = B_k + (1-\epsilon)G_k + \frac{\epsilon}{2}E_k$ for sufficiently small ϵ such that (X_k, B'_k) is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \frac{\epsilon}{2}A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k}+B'_k+\frac{\epsilon}{2}A_k \geqslant 0} + \sum_{\alpha\in\Lambda \text{ finite set}} R_\alpha$$

Again, there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of X_k.

(2) Assume this sequence of links is infinite, then there exits an i such that there are infiniely many $j > i$ such that $f_i : X_i \to S_i$ and $f_j : X_j \to S_j$ are isomorphic. Then we have $g_{i+1} = g_{j+1}$ and $h_{i+1} = h_{j+1}$. Since sequences of h_k and g_k are monotone, we have $h_{i+1} = h_k$ and $g_{i+1} = g_k$ for all $k > i$. Suppose $X_i \dashrightarrow X_{i+1}$ is a link in case 1 of the Construction in 4.2, then the next link is also in case 1, and all the links after are in case 1. Note that $X_i \cong X_j$ and therefore $\rho(X_i) = \rho(X_j)$, the links are all of type IV. But this contracts 3 of Claim 4.2.2. Therefore there are no link of type III or IV after X_i. In other words, the links after X_i are all type I or II in case 2.

Since $\rho(X_i) = \rho(X_j)$, X_i and X_j are linked by the Sarkisov links of type II. But this contracts 4 of Claim 4.2.2.

□

Proposition 4.3.2. *$X_N \to S_N$ is isomorphic to $X' \to S'$.*

Proof. By 2 of Theorem 3.2.1, we have $h_N = 1$ and they are isomorphic. □

5. Using the Polytope

In this section we follows [?].

5.1. **Construction of Sarkisov links.** In this subsection we construct one Sarkisov link. First we show the partition of $\mathcal{E}_A(V)$ corresponding to ample models and morphisms between these ample models.

Theorem 5.1.1. [?, Corollary 1.1.5]*Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a divisor $\Delta_0 \in V$ such that (X, Δ_0) is klt. Let A be a general ample $\mathbb{Q}$-divisor over U which has no components common with any element of V.*

(1) There are finitely many birational maps $f_i : X \dashrightarrow X_i$ over U such that

$$\mathcal{E}_{A,\pi}(V) = \bigcup_i \mathcal{W}_i$$

where $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$ is a rational polytope. Moreover, if $f : X \dashrightarrow Y$ is a log terminal model of $K_X + D$ over U for some $D \in \mathcal{E}_{A,\pi}(V)$, then $f = f_i$ for some i.

(2) There are finitely many rational maps $g_j : X \dashrightarrow Z_j$ over U such that

$$\mathcal{E}_{A,\pi}(V) = \coprod_j \mathcal{A}_j$$

$\{\mathcal{A}_j = \mathcal{A}_{A,\pi,g_j}\}$ is a partition of $\mathcal{E}_A(V)$;

(3) For every f_i there is a g_j and a morphism $h_{ij} : Y_i \to Z_j$ such that $\mathcal{W}_i \subset \overline{\mathcal{A}_j}$.

Theorem 5.1.2. [?, Theorem 3.3] *Let W be a smooth projective variety, and V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective $\mathbb{Q}$-divisor A. Suppose that there is an element D_0 of $\mathcal{L}_A(V)$ such that $K_W + D_0$ is big and klt. Then there are finitely many rational contractions $f_i : W \dashrightarrow X_i$ such that*

(1) $\{\mathcal{A}_i = \mathcal{A}_{A,f_i}\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;

(2) If i, j are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ then there is a contraction $f_{ij} : X_i \to X_j$ such that $f_j = f_{ij} \circ f_i$;

(3) Suppose in addition V spans Neron-Severi group of W. Pick i such that a connected component $\mathcal{C}$ of $\mathcal{C}_i$ intersects the interior of $\mathcal{L}_A(V)$, the following are equivalent:
   (a) $\mathcal{C}$ spans V;
   (b) If $D \in \mathcal{A}_i \cap \mathcal{C}$ then f_i is a log terminal model of $K_W + D$;
   (c) f_i is birational and X_i is $\mathbb{Q}$-factorial.

(4) Suppose in addition V spans Neron-Severi group of W. If i, j are two indices such that $\mathcal{C}_i$ spans V and D is a general point of $\mathcal{A}_j \cap \mathcal{C}_i$ which is also a point of interior of $\mathcal{L}_A(V)$, then $\mathcal{C}_i$ and $\overline{\mathrm{NE}}(X_i/X_j)^* \times \mathbb{R}^k$ are locally isomorphic in a neighbourhood of D, for some $k \geqslant 0$. Furthermore $\rho(X_i/X_j) = \dim \mathcal{C}_i - \dim \mathcal{C}_j \cap \mathcal{C}_i$.

Lemma 5.1.3. [?, Corollary 3.4]*If V spans Neron-Severi group of W, then there is a Zariski dense open subset U of the Grassmannian $G(r, V)$ of real affine subspace of dimension r such that any $[V'] \in U$ defined on rational numbers satisfies (1-4) of Theorem 5.1.2*

Proof. Let $U \subset G(r, V)$ be the set of real affine subspace V' of V of dimension r, which contain no face of any $\mathcal{C}_i$ of $\mathcal{L}(V)$. In particular, the interior of $\mathcal{L}_A(V')$ is contained in the interior of $\mathcal{L}_A(V)$. Clearly that any $V' \in U$ defined over rationals satisfies (1-4) of 5.1.2. □

By above Lemma, from now on in this subsection, we always assume that V has dimension 2 and satisfies Theorem 5.1.2. The following lemma shows that the morphism in 2 of Theorem 5.1.2 can be divisorial contraction, small contraction or Mori fibre space. And in some cases they forms a flop.

Lemma 5.1.4. [?, Lemma 3.5] *Let $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ be two rational contracions such that $\mathcal{C}_{A,f}$ is of dimension 2 and $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$ is of dimension 1. Assume $\rho(X) \geqslant \rho(Y)$ and $\mathcal{O}$ is not contained in the boundary of $\mathcal{L}_A(V)$. Let D be an interior point of $\mathcal{O}$ and $B = f_*D$. Then there is a rational contraction $\pi : X \dashrightarrow Y$ and $g = \pi \circ f$ such that either*

(1) $\rho(X) = \rho(Y) + 1$ and π is $(K_X + B)$-trivial, and either
   (a) π is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$, and either
      (i) π is a divisorial contraction and $\mathcal{O} \neq \mathcal{C}_{A,g}$, or
      (ii) π is a small contraction and $\mathcal{O} = \mathcal{C}_{A,g}$ or
   (b) π is a log Mori fibre space, and $\mathcal{O} = \mathcal{C}_{A,g}$ is contained in the boundary of $\mathcal{E}_A(V)$, or

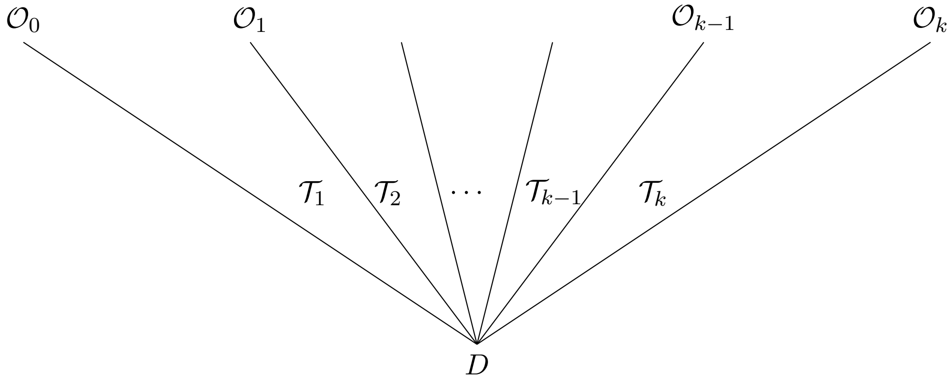
(2) $\rho(X) = \rho(Y)$, and π is a $(K_X + B)$-flop and $\mathcal{O} \neq \mathcal{C}_{A,g}$ is not contained in the boundary of $\mathcal{E}_A(V)$.

Lemma 5.1.5. [?, Lemma 3.6] *Let $f : W \dashrightarrow X$ be a birational contraction between $\mathbb{Q}$-factorial varieties. Suppose (W, D) and $(W, D + A)$ are both klt. If f is ample model of $(W, D + A)$ and A is ample, then f is a result of $(K_W + D)$-MMP.*

This lemma gunrantee that every variety in the Sarkisov links constructed later is a result of (W, B_W)-MMP.

Finally we show there is a Sarkisov link corresponding to certain $D \in \mathcal{E}_A(V)$. Let $D = A + B$ be a point of the boundary of $\mathcal{E}_A(V)$ in the interior of $\mathcal{L}_A(V)$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be the polytopes $\mathcal{C}_i$ of dimension 2 containing D. Possibly reordering, we may assume that the intersection $\mathcal{O}_0$ and $\mathcal{O}_k$ of $\mathcal{T}_1$ and $\mathcal{T}_k$ with boundary of $\mathcal{E}_A(V)$ and $\mathcal{O}_i = \mathcal{T}_i \cap \mathcal{T}_{i+1}$ are one dimensional. Let

$f_i : W \dashrightarrow X_i$ be the birational contraction associated to $\mathcal{T}_i$ and $g_i : W \dashrightarrow S_i$ be the rational contraction associated to $\mathcal{O}_i$.



Set $f = f_1 : W \dashrightarrow X, g = f_k : W \dashrightarrow Y$ and $\phi : X \to S = S_0, \psi : Y \to T = S_k$ and $X' = X_2, Y' = X_{k-1}$ and let $W \dashrightarrow R$ be the ample model of D. Then

Theorem 5.1.6. [**?**, Theorem 3.7] *Suppose B_W is a divisor such that $K_W + B_W$ is klt and $D - B_W$ is ample. Then ϕ and ψ are log Mori fibre spaces as outputs of $(K_W + B_W)$-MMP and connected by a Sarkisov link if D is contained in more than two polytopes.*

Proof. WMA $k \geqslant 3$ and we have

$$\begin{tikzcd}
X' \arrow[r, dotted] \arrow[d, dotted, "p"'] & Y' \arrow[d, dotted, "q"] \\
X \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\
S \arrow[dr, "s"] & T \arrow[d, "t"'] \\
 & R
\end{tikzcd}$$

Note that $\rho(X_i/R) \leqslant 2$ and $\rho(X/S) = \rho(Y/T) = 1$. Thus

(1) s is identity and p is a divisorial contraction (extraction), or
(2) s is a contraction and p is a flop.

The same holds for q and t. And the map $X' \to Y'$ is clear the composition of flops. This gives 4 types of links. □

5.2. **Decomposition into Sarkisov links.** We need a special resolution W and a special affine subspace $V \subset \mathrm{WDiv}(W)$.

Lemma 5.2.1. [**?**, Lemma 4.1] *Let $\phi : X \to S$ and $\psi : Y \to T$ be two MMP related Mori fibre spaces corresponding to two klt projective varieties (X, B_X) and (Y, B_Y). Then we may find a smooth projective variety W, two birational morphisms $f : W \to X$ and $g : W \to Y$, a klt pair (W, B_W), an ample $\mathbb{Q}$-divisor A on W and a two dimensional rational affine subspace V of $\mathrm{WDiv}_{\mathbb{R}}(W)$ such that*

(1) If $D \in \mathcal{L}_A(V)$ then $D - B_W$ is ample;
(2) $\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$;
(3) V satisfies (1-4) of Theorem 5.1.2;
(4) $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are two dimensional;
(5) $\mathcal{C}_{A,\phi\circ f}$ and $\mathcal{C}_{A,\psi\circ g}$ are one dimensional.

Proof. By assumption there is a $\mathbb{Q}$-factorial klt pair (W, B_W) such that $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$are outputs of $(K_W + B_W)$-MMP. Let $p' : W' \to W$ be any log resolution such that resolves the indeterminacy of f and g, then we may write

$$K_{W'} + B_{W'} = p'^*(K_W + B_W) + E'$$

where $E' \geqslant 0$ and $B_{W'} \geqslant 0$ have no common components, and E' is exceptional and $p'_* B_{W'} = B_W$. Pick a divisor $-F$ which is ample over W with $\operatorname{Supp} F = \operatorname{Exc} p'$ such that $K_{W'} + B_{W'} + F$ is klt. As p' is $(K_{W'} + B_{W'} + F)$-negative and $(K_W + B_W)$ is klt and W is $\mathbb{Q}$-factorial, the $(K_{W'} + B_{W'} + F)$-MMP over W terminates with the pair (W, B_W). Replacing (W, B_W) by $(W', B_{W'} + F)$ we may assume that (W, B_W) is log smooth and f, g are morphisms.

Pick general ample $\mathbb{Q}$-divisors $A, H_1, H_2, \ldots, H_k$ on W such that $H_1, \ldots, H_k$ generate the Neron-Severi group of W. Let $H = A + H_1 + \ldots + H_k$. Pick sufficiently ample divisors A_S on S and A_T on T such that

$$-(K_X + B_X) + \phi^* A_S \text{ and } -(K_Y + B_Y) + \psi^* A_T$$

are both ample. Pick a rational number $0 < \delta < 1$ such that

$$-(K_X + B_X + \delta f_* H) + \phi^* A_S \text{ and } -(K_Y + B_Y + \delta g_* H) + \psi^* A_T$$

are both ample and f and g are both $(K_W + B_W + \delta H)$-negative. Replacing H by δH we may assume that $\delta = 1$. Now pick a $\mathbb{Q}$-divisor $B_0 \leqslant B_W$ such that $A + (B_0 - B_W), -(K_X + f_* B_0 + f_* H) + \phi^* A_S$ and $-(K_Y + g_* B_0 + f_* H) + \psi^* A_T$ are all ample and f and g are both $(K_W + B_0 + \delta H)$-negative.

Pick general ample $\mathbb{Q}$-divisors $F_1 \geqslant 0$ and $G_1 \geqslant 0$ such that

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_* B_0 + f_* H) + \phi^* A_S \text{ and } G_1 \sim_{\mathbb{Q}} -(K_Y + g_* B_0 + g_* H) + \psi^* A_T$$

and

$$K_W + B_0 + H + F + G$$

is klt, where $F = f^* F_1$ and $G = g^* G_1$.

Let V_0 be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ which is the translate by B_0 of the vector subspace spaned by $H_1, \ldots, H_k, F, G$. Supppose that $D = A + B \in \mathcal{L}_A(V_0)$. Then

$$D - B_W = (A + B_0 - B_W) + (B - B_0)$$

is ample, as $B - B_0$ is nef by definition of V_0. Note that

$$B_0 + F + H \in \mathcal{A}_{A, \phi \circ f}(V_0), B_0 + G + H \in \mathcal{A}_{\psi \circ g}(V_0)$$

and f, respectively g, is a weak log canonical model of $K_W + B_0 + F + H$, respectively $K_W + B_0 + G + H$. Thus Theorem 5.1.2 implies that V_0 satisfies (1-4) of Theorem 5.1.2.

Since $H_1, \ldots, H_k$ generated the Neron-Severi group of W we may find constants $h_1, \ldots, h_k$ such that $G \equiv \sum_{i=1}^k h_i H_i$. Then there is $0 < \delta \ll 1$ such that $B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \in \mathcal{L}_A(V_0)$ and

$$B_0 + F + \delta G + H - \delta(\sum_i^k h_i H_i) \equiv B_0 + F + H.$$

Thus $\mathcal{A}_{A, \phi \circ f}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. Similarly $\mathcal{A}_{A, \psi \circ g}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. In particular $\mathcal{A}_{A, \phi \circ f}$ and $\mathcal{A}_{A, \psi \circ g}$ span affine hyperplanes of V_0, since $\rho(X/S) = \rho(Y/T) = 1$.

Let V_1 be the translate by B_0 of the two dimensional vector space spaned by $F + H - A$ and $G + H - A$. Let V be a small general perturbation of V_1 as in Lemma 5.1.3, which is defined over rationals. This is the affine subspace we need. $\square$

Then we can prove the main theorem

Proof of the main theorem. Let $(W, B_W), A$ and V as in the lemma 5.2.1. Pick $D_0 \in \mathcal{A}_{A,\phi\circ f}$ and $D_1 \in \mathcal{C}_{A,g}$ belonging to the interior of $\mathcal{L}_A(V)$. As V is two dimensional, removing D_0 and D_1 divides the boundary of $\mathcal{E}_A(V)$ into two parts. The part which consists entirely of divisors which are not big is contained in the interior of $\mathcal{L}_A(V)$. Consider tracing this boundary from D_0 to D_1. Then there are finitely many $2 \leqslant i \leqslant N$ points D_i which are contained in more than two polytopes $\mathcal{C}_{A,f_i}(V)$. By lemma 5.1.6, each point D_i gives a Sarkisov link. And the birational map $X \dashrightarrow Y$ is composition of such links. $\square$

6. Examples

This section we give an example for each method.

6.1. **Original method.** Let $X = \mathbb{P}^2$ with coordinates $(x_0 : x_1 : x_2)$ and $X' = \mathbb{P}^2$ with coordinates $(y_0 : y_1 : y_2)$. Denote $B = \{x_0 = 0\}$ and $B' = \{y_0 = 0\}$. Take a rational map $\Phi : X \dashrightarrow X'$ defined by

$$\Phi : (x_0 : x_1 : x_2) \dashrightarrow (x_0^2 : x_0x_1 : x_1^2 + x_0x_2)$$

The common resolution $\sigma : W \to X$ and $\sigma' : W \to X'$ are both composition of three blowing-ups at indetermined points. More precisely, $\pi_1 : W_1 \to X$ is blowing-up at $P_0 \in B$. Identifies B with its strict transform on W_1 and let E_1 be the exceptional divisor. $\pi_2 : W_2 \to W_1$ is blowing-up at $P_1 = E_1 \cap B$. Identifies B and E_1 with their strict transforms on W_2 and let E_2 be the exceptional divisor. $\pi_3 : W = W_3 \to W_2$ is blowing-up at $P_2 \in E_2 \setminus (B \cup E_1)$. Identifies B, E_1 and E_2 with their strict transforms on W_3 and let E_3 be the exceptional divisor. Then $\sigma = \pi_3 \circ \pi_2 \circ \pi_1$ and $W = W_3$ is the common resolution. Moreover $\sigma' : W \to X'$ is composition of blowing-down curves in the order of B, E_2, E_1, and $\sigma'_*(E_3) = B'$. We denote as $W = W'_3 \xrightarrow{\pi'_3} W'_2 \xrightarrow{\pi'_2} W'_1 \xrightarrow{\pi'_1} X'$

We give some notations of varieties:

- Let $W_2 \to Z_0$ be the contraction of E_1 on W_2, then $Z_0 \to X_1$ is the contracion of B and $Z_0 \to X$ is the extraction of E_2 on X;
- Let $W'_2 \to Z_1$ be the contraction of E_1 on W'_2, then $Z_1 \to X_1$ is the extraction of E_3 on X_1, and $Z_1 \to X'$ is the contracion of E_2;
- $W \to Z$ be the contraction of E_1 and E_2 on W, then $Z \to X$ is the extraction of E_3 and $Z \to X'$ is the contraction of B.

That is

$$\begin{tikzcd}
 & & & W_3 \arrow[r, equal] \arrow[dl] & W \arrow[r, equal] \arrow[ddd] & W'_3 \arrow[dr] & & & \\
 & & W_2 \arrow[dl] \arrow[dr] & & & & W'_2 \arrow[dl] \arrow[dr] & & \\
 & W_1 \arrow[dl] & & Z_0 \arrow[dlll] \arrow[dr] & & Z_1 \arrow[dl] \arrow[drrr] & & W'_1 \arrow[dr] & \\
X_0 & & & & X_1 & & & & X'
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & W \arrow[ddl, "\sigma"'] \arrow[d] \arrow[ddr, "\sigma'"] & \\
 & Z \arrow[dl] \arrow[dr] & \\
X & & X'
\end{tikzcd}$$

Consider the pairs (X, bB) and $(X', b'B')$, and take the function θ:

- $\theta(B) = b$ and $\theta(B') = b'$;
- $\theta(E_1) = \theta(E_2) = e$ with $b, b' < e < 1$.

Then we have ramification fomulas:

$$\begin{array}{rclcccccc} K_W + B_W & = & \sigma^*(K_X + bB) & + & (3-2b+b')E_3 & + & (1-b+e)E_1 & + & (2-2b+e)E_2 \\ & = & \sigma'^*(K_{X'} + b'B') & + & (3-2b'+b)B & + & (1-b'+e)E_1 & + & (2-2b'+e)E_2 \end{array}$$

Set $\mathcal{H}' = |\mathcal{O}(1)|$. Different choices of θ and e gives different Sarkisov programs.

(1) If $2b + 2b' \geqslant 3e > 0$,then Φ is composition of 2 Sarkisov links ψ_0, ψ_1 of type II:

$$\begin{tikzcd} & Z_0 \arrow[dl] \arrow[dr] & & Z_1 \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "\psi_0"] \arrow[d] & & X_1 \arrow[rr, dashed, "\psi_1"] \arrow[d] & & X' \arrow[d] \\ pt & & pt & & pt \end{tikzcd}$$

(2) If $2b + 2b' < 3e$, then Φ is just one Sarkisov link Φ of type II:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr, dashed, "\Phi"] \arrow[d] & & X' \arrow[d] \\ pt & & pt \end{tikzcd}$$

(3) If $e = b = b' = 0$, then Φ is composition of 4 Sarkisov links ψ_i:

$$\begin{tikzcd} & & W_2 \arrow[dl] \arrow[dr] & & W_2' \arrow[dl] \arrow[dr] & & \\ & X_1 = \mathbb{F}_1 \arrow[rr, dashed, "\psi_1"] \arrow[d] & & X_2 = \mathbb{F}_2 \arrow[rr, dashed, "\psi_2"] \arrow[d] & & X_3 = \mathbb{F}_1 \arrow[d] \arrow[dr, "\psi_3"] & \\ X = \mathbb{P}^2 \arrow[ur, dashed, "\psi_0"] \arrow[d] & \mathbb{P}^1 \arrow[rr, equal] \arrow[dl] & & \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \arrow[dr] & X_4 = \mathbb{P}^2 \arrow[d] \\ pt & & & & & & pt \end{tikzcd}$$

6.2. **Double scaling.** Notations and assumptions as 6.1, let $B_W = \frac{1}{2}(B + E_1 + E_3)$ and consider pairs $(X, \frac{1}{2}B)$ and $(X', \frac{1}{2}B')$. Then we have $G = G_0 \sim_{\mathbb{Q}} \frac{5}{2}B$ and $H' \sim_{\mathbb{Q}} \frac{5}{2}B'$.

(1) $r_0 = 2$ and $s_0 = \frac{1}{5}$. X_1 is weak log canonical model of $(W, B_W + \frac{3}{5}G_W + \frac{1}{5}H_W)$;

(2) $r_1 = 1$ and $s_1 = \frac{2}{5}$. $X_2 = X'$ is weak log canonical model of $(W, B_W + \frac{1}{5}G_W + \frac{3}{5}H_W)$.

This gives the same decomposition as case 1 in 6.1.

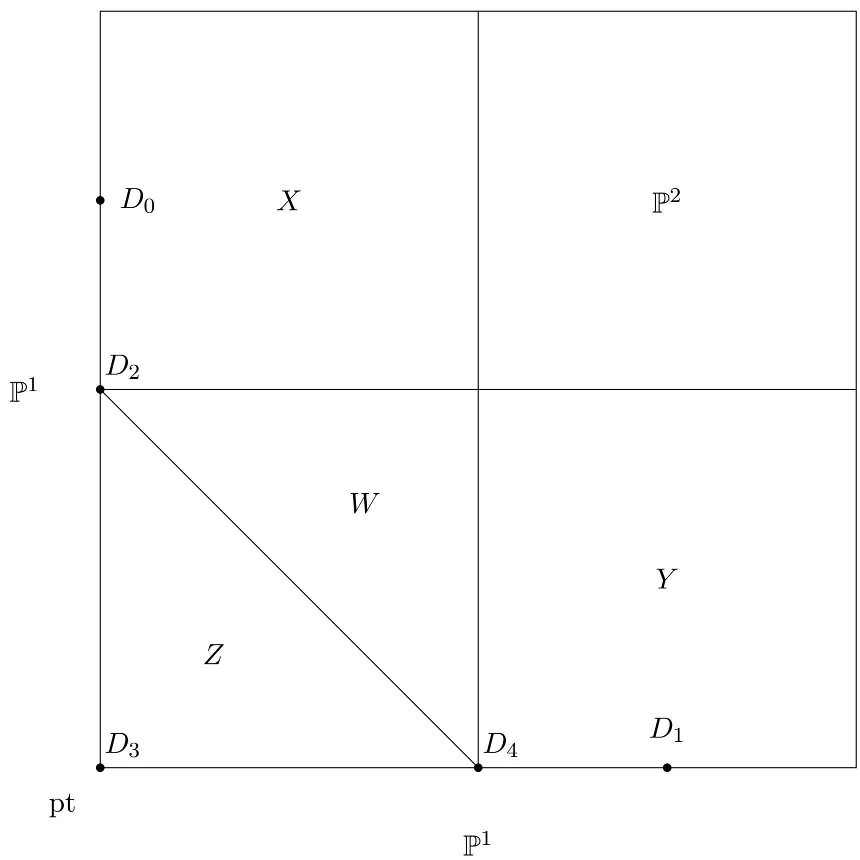
6.3. **Polytope.** Let P, Q be two different points on $\mathbb{P}^2$ and let L be the line passing through P and Q. Let $p : X \to \mathbb{P}^2$ be the blowing up at P and E_1 be the exceptional divisor. Let $q : Y \to \mathbb{P}^2$ be the blowing up at Q and E_2 be the exceptional divisor. Let $W \to \mathbb{P}^2$ be the blowing up of P and Q, then we have contracions $f : W \to X$ and $g : W \to Y$. Identifies

L, E_1 and E_2 with their strict transfoms on W. Let $h : W \to Z$ be the contracion of L, then $Z \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$.

$$\begin{tikzcd}
S \cong \mathbb{P}^1 & X \cong \mathbb{F}_1 \arrow[l, "\phi"'] \arrow[r, "p"] & \mathbb{P}^2 \\
 & W \arrow[u, "f"] \arrow[r, "g"'] \arrow[dl, "h"'] & Y \cong \mathbb{F}_1 \arrow[u, "q"'] \arrow[d, "\psi"] \\
Z \cong \mathbb{F}_0 \arrow[uu] \arrow[rr] & & T \cong \mathbb{P}^1
\end{tikzcd}$$

Note that $X \cong \mathbb{F}_1$, there is a Mori fibre space $\phi : X \to S \cong \mathbb{P}^1$. Similarly there is another Mori fibre space $\psi : Y \to T \cong \mathbb{P}^1$. There is a birational map $\Phi : X \dashrightarrow Y$ induced by p and q. In fact, if we take $B_W = \frac{1}{4}L$ on W, then f and g are two log Mori fibre spaces as outputs of $(K_W + B_W)$-MMP on W.

Take $A \sim_{\mathbb{Q}} -K_W + \frac{1}{4}L$, and let V be the translate by $\frac{1}{4}L$ of the 2-dimensional vector space spaned by E_1 and E_2. Then we have $\mathcal{L}_A(V) = \mathcal{E}_A(V)$. Furthermore, $K_W + D \sim_{\mathbb{Q}} \frac{1}{2}L + aE_1 + bE_2$ for $0 \leqslant a, b \leqslant 1$ if $D \in \mathcal{E}_A(V)$. The partition of $\mathcal{E}_A(V)$ is



Then D_0 and D_1 correspond to log Mori fibre spaces $\phi : X \to S$ and $\psi : Y \to T$. D_2, D_3 and D_4 correspond to three Sarkisov links. Therefore we have decomposition of $\Phi : X \dashrightarrow Y$ as

$$\begin{tikzcd}
& W \arrow[dl] \arrow[dr] & & & & W \arrow[dl] \arrow[dr] & \\
X \arrow[d] & & Z \arrow[d] \arrow[rr, equal] & & Z \arrow[d] & & Y \arrow[d] \\
\mathbb{P}^1 & & \mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] & & \mathbb{P}^1 \\
& & & pt & & &
\end{tikzcd}$$

References

[Bir19] C. Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, p. 345-463.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[HMX14] C. D. Hacon, J. McKernan, and C. Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523–571.

[Sho06] V. V. Shokurov *3-fold log models*, J. Math. Sci., **81**(3):2667–2699,1996.

UCAS
Email address: wangyanze@amss.ac.cn

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, 200438, China
Email address: hanjingjun@fudan.edu.cn