Sarkisov program

wyz

December 13, 2022

Contents

1 Introduction **1**
- 1.1 Motivation and Main theorem . . . 1
- 1.2 Using MMP . . . 2
- 1.3 Using polytope . . . 2
- 1.4 Structure of the article . . . 2

2 Preliminary **2**
- 2.1 Models . . . 2
- 2.2 MMP . . . 4

3 Original proof **5**
- 3.1 Prepare . . . 5
- 3.2 Flowchart for the Log Sarkisov program . . . 7
- 3.3 Termination . . . 8

4 Double scaling **10**
- 4.1 Prepare . . . 10
- 4.2 Construct Sarkisov links . . . 11
- 4.3 Termination . . . 13

5 Using the Polytope **13**
- 5.1 Morphisms between models . . . 14
- 5.2 Construction of Sarkisov links . . . 16

6 Application **17**

1 Introduction

The purpose of this article is to show that two different Mori fibre spaces as outputs of a klt pair can be linked by composition of Sarkisov links.

1.1 Motivation and Main theorem

The **Minimal model program (MMP)** aims to classify varieties up to birational equivalent classed, by finding a minimal model all or Mori fibre space. Let (X, B) be a (klt or lc) pair, and assume we can run $(K_X + B)$-MMP on it. Note that the varieties appear in the program are called **results** of the MMP, and the varieties where the MMP ends are called the **output** of the MMP.

1. If $\kappa(X, B) \geqslant 0$, then we expected that MMP ends with a **minimal model**, i.e. a birational map $X \dashrightarrow Y$ such that $(K_Y + B_Y)$ is nef;
2. If $\kappa(X, B) = -\infty$, then we expected that MMP ends with a Mori fibre space, i.e. a birational map $X \dashrightarrow Y$ and a contraction $Y \to S$ such that $\dim Y < \dim X$ and $-(K_Y + B_Y)$ is relative ample.

However, for each case the output may not be unique.

For the first case, it is shown that two different minimal model can be linked by flops:

Theorem 1.1.1. *[9, Theorem 1] Let (W, B_W) be a $\mathbb{Q}$-factorial terminal pair, and $(X, B), (Y, D)$ are two minimal models of (W, B_W). Then the birational map $X \dashrightarrow Y$ may be factored as sequence of $(K_X + B)$ flops.*

For the second case, it is shown that:

Theorem 1.1.2. *Let $f:(X,B)\to S$ and , $f':(X',B')\to S'$ be two MMP related $\mathbb{Q}$-factorial klt log Mori fibre spaces with induced induced birational map Φ:*

$$\begin{tikzcd}
(X,B) \arrow[r, dashrightarrow, "\Phi"] \arrow[d, "f"'] & (X',B') \arrow[d, "f'"] \\
S & S'
\end{tikzcd}$$

Then Φ can be decomposed into sequence of Sarkisov links.

Definition 1.1.3. *The following four types of birational maps $X \dashrightarrow X_1$ are called Sarkisov links:*

I:
$$\begin{tikzcd}
Z \arrow[r, dashrightarrow] \arrow[d, "p"'] & X_1 \arrow[d, "f_1"] \\
X \arrow[d, "f"'] & S_1 \arrow[dl, "t"] \\
S &
\end{tikzcd}$$

II:
$$\begin{tikzcd}
Z \arrow[r, dashrightarrow] \arrow[d, "p"'] & Z' \arrow[d, "q"] \\
X \arrow[d, "f"'] & X_1 \arrow[d, "f_1"] \\
S \arrow[r, "\sim"] & S_1
\end{tikzcd}$$

III:
$$\begin{tikzcd}
X \arrow[r, dashrightarrow] \arrow[d, "f"'] & Z \arrow[d, "q"] \\
S \arrow[dr, "s"'] & X_1 \arrow[d, "f_1"] \\
& S_1
\end{tikzcd}$$

IV:
$$\begin{tikzcd}
X \arrow[rr, dashrightarrow] \arrow[d, "f"'] & & X_1 \arrow[d, "f_1"] \\
S \arrow[dr, "s"'] & & S_1 \arrow[dl, "t"] \\
& T &
\end{tikzcd}$$

Here, all $f:(X,B)\to S$ and $f_1:(X_1,B_1)\to S_1$ are log Mori fibre space, and all p,q are divisorial contractions, and all dash arrows are composition of flips, flops and inverse flips.

1.2 Using MMP

Assume $f:(X,B)\to S'$ and $f':(X',B')\to S'$ are two Mori fibre spaces as outputs of (K_W+B_W)-MMP on W. The Sarkisov program constructs each Sarkisov link $X_i \dashrightarrow X_{i+1}$ inductively. For each X_i we shall find some W_i such that X_i and X_{i+1} are two Mori fibre spaces as outputs of certain MMP on W_i. Moreover, $W_i \dashrightarrow X_{i+1}$ is a 2-tay game. More precisely, there are two cases:

1. Find a contraction $g: X_i \to T_i$ such that $\rho(X_i/T_i)=2$ and factor though $f_i: X_i \to S_i$, then we run MMP on X_i over T_i, and obtains a Sarkisov link of type III or IV. Here $W_i = X_i$;
2. Find a divisorial contraction $p: Z_i \to X_i$, and therefore $\rho(Z_i/S_i)=2$. Then we run MMP on Z_i over S_i, and obtains a Sarkisov link of type I or II. Here $W_i = Z_i$.

In [6], original proof;

In [7], double scaling;

1.3 Using polytope

1.4 Structure of the article

2 Preliminary

In this article, all varieties are over complex number $\mathbb{C}$.

2.1 Models

Definition 2.1.1. *[8, 2.Notation and Conventions] A rational map $f: X\to S$ is called a **rational contraction** if there is a resolution $p: W \dashrightarrow X$ and $q: W\dashrightarrow Y$ of f such that p and q are contraction morphisms and p is birational. f is called a **birational contraction** if q is in addition birational and every p -exceptional divisor is q -exceptional. If in addition f^{-1} is also a **birational contraction**, then f is called a **small birational map**.*

Definition 2.1.2. *[4, Definition 3.6.1]Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties, and $p: W\to X$ and $q: W\to Y$ be a resolution of indeterminacy of fl. Let D be a $\mathbb{R}$-Cartier divisor on X such $D_Y = f_*D$ is also $\mathbb{R}$-Cartier. Then f is called D**-non-positive (D-negative)** if*

- *f does not extract any divisor;*
- *$E = p^*D - q^*D_Y$ is effecive and exceptional over Y (and $\operatorname{Supp} p_*E$ contains all f-exceptional divisors).*

Definition 2.1.3. *[7, 13.2.Notation and conventions] Let $f: X\dashrightarrow Y$ be a rational map of normal quasi-projective varieties over S, and D be a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor divisor on X with $f_*D = D_Y$. Then f is called D**-trivial** if D is pull back of a $\mathbb{R}$-Cartier divisor on S.*

Recall the definitions of models in [4]

Definition 2.1.4. *[4, Definition 3.6.5] Let $\pi:(X,D)\to U$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X. Let $f: X\dashrightarrow Y$ be a birational map over U, then Z is an **semiample model** for D over U if f is K_X+D-non-positive and K_Y+f_*D is semiample over U.*

*Let $g: X\dashrightarrow Z$ be a rational map over U, then Z is an **ample model** for D over U if there is a an ample divisor over U on Z such that if $p: W\to X$ and $q: W\to Z$ resolves g, then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{R},U} q^*H + E$, where $E\geqslant 0$ and for any $B\in |p^*D/U|_{\mathbb{R}}$, then $B\geqslant E$.*

Definition 2.1.5. *[4, Definition 3.6.7] Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties, if $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then define:*

1. *Y is **weak log canonical model** for $K_X + D$ over U if f is $K_X + D$-non-positive and $K_Y + f_*D$ is nef over U;*

2. *Y is **log canonical model** for $K_X + D$ over U if f is $K_X + D$-non-positive and $K_Y + f_*D$ is ample over U;*

3. *Y is **log terminal model** for $K_X + D$ over U if f is $K_X + D$-negative and $K_Y + f_*D$ is dlt and nef over U and Y is $\mathbb{Q}$-factorial.*

Lemma 2.1.6. *[4, Lemma 3.6.6] Let $\pi : X \to U$ ve a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X.*

1. *If $g_i : X \dashrightarrow X_i, i = 1, 2$ are two ample models of D over U, then there is an isomorphism $h : X_1 \to X_2$ and $g_2 = h \circ g_1$.*

2. *If $f : X \dashrightarrow Y$ is a semiample model of D over U, then the ample model $g : X \dashrightarrow Z$ of D over U exits and $g = h \circ f$, where $h : Y \to Z$ is a contraction and $f_*D \sim_{\mathbb{R},U} h^*H$.*

3. *If $f : X \dashrightarrow Y$ is a birational map over U, then f is the ample model of D over U if and only if f is semiample model of D over U and f_*D is ample over U.*

By above lemma there is another definition of log canonical models:

Definition 2.1.7. *Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties and $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then Y is **log canonical model** if it is the ample model.*

Futhermore, for big boundary, we have

Lemma 2.1.8. *[4, Lemma 3.9.3]Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties. Suppose (X, B) is a klt pair and B is big over U. If $f : X \dashrightarrow Y$ is a weak log canonical model over U then*

- *f is a semiample model over U;*
- *the ample model $g : X \dashrightarrow Z$ over U exits;*
- *there is a contraction $h : Y \to Z$ such that $K_Y + f_*B \sim_{\mathbb{R},U} h^*H$ for some ample $\mathbb{R}$-divisor H on Z over U.*

Definition 2.1.9. *[4, Definition 1.1.4] Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *defined over rational numbers. Define*

$$\mathcal{L}(V) = \{D \in V : K_X + D \text{ is log canonical }\}$$
$$\mathcal{N}_\pi(V) = \{D \in \mathcal{L} : K_X + D \text{ is nef over } U\}$$

Moreover, if fix an $\mathbb{R}$-divisor $A \geqslant 0$, and then define

$$V_A = \{D = A + B : B \in V\}$$
$$\mathcal{L}_A(V) = \{D = A + B \in V_A : K_X + D \text{ is log canonical and } B \geqslant 0\}$$
$$\mathcal{E}_{A,\pi}(V) = \{D = A + B \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\}$$
$$\mathcal{N}_{A,\pi}(V) = \{D \in \mathcal{L}_A(V) : K_X + D \text{ is nef over } U\}$$

Given a birational contraction $f : X \dashrightarrow Y$, define

$$\mathcal{W}_{A,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is weak log model of } (X, D) \text{ over } U\}$$

Given a rational contraciton $g : X \dashrightarrow Z$ over U, define

$$\mathcal{A}_{A,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is ample model of } (X, D) \text{ over } U\}$$

In addition, let $\mathcal{C}_{A,g}(V)$ denote the closure of $\mathcal{A}_{A,g}(V)$

By [4, Lemma 3.7.2], if V is a rational subspace, then $\mathcal{L}_A(V)$ is a rational polytope.

Lemma 2.1.10. *[4, Theorem E] Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and A be an general divisor relatively ample over U, and $V \subset \mathrm{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a klt pair (X, Δ_0). Then there are finitely many birational maps $f_i : X \dashrightarrow X_i$ such that if $f : X \dashrightarrow Y$ is a weak log canonical model of $K_X + D$ over U for some $D \in \mathcal{L}_A(V)$, then there is an isomorphism $h_i : X_i \to Y$ and $f = h_i \circ f_i$.*

Theorem 2.1.11. *[4, Corollary 1.1.5]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and A be an general divisor relatively ample over U, and $V \subset \mathrm{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a divisor $\Delta_0 \in V$ such that (X, Δ_0) is klt. Let A be a general ample $\mathbb{Q}$-divisor over U which has no components common with any element of V.*

1. *There are finitely many birational maps $f_i : X \dashrightarrow X_i$ over U such that*
$$\mathcal{E}_{A,\pi}(V) = \bigcup_i \mathcal{W}_i$$
where $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$ is a rational polytope. Moreover, if $f : X \dashrightarrow Y$ is a log terminal model of $K_X + D$ over U for some $D \in \mathcal{E}_A(V)$, then $f = f_i$ for some i.

2. *There are finitely many birational maps $g_j : X \dashrightarrow Z_j$ over U such that*
$$\mathcal{E}_{A,\pi}(V) = \coprod_j \mathcal{A}_j$$
$\{\mathcal{A}_j = \mathcal{A}_{A,g_j}\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;

3. *For every f_i there is a g_j and a morphism $h_{ij} : Y_i \to Z_j$ such that $\mathcal{W}_i \subset \overline{\mathcal{A}_j}$.*

In particular $\mathcal{E}_{A,\pi}$ is a rational polytope and $\overline{\mathcal{A}_j}$ is a finite union of raional polytopes.

2.2 MMP

Definition 2.2.1. *Let (X, B) be a pair and let $f : Y \to X$ be a log resolution of (X, B). Suppose*
$$K_Y + C = f^*(K_X + B),$$
then the discrepancy of exceptional divisor E_i over X is
$$a(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$
Moreover, let
$$\operatorname{discrep}(X, B) := \inf\{a(E; X, B) : E \text{ is an expectional divisor over } X\}$$
and
$$\operatorname{totdiscrep}(X, B) := \inf\{a(E; X, B) : E \text{ is a divisor over } X\}.$$

Theorem 2.2.2. *[4, Corollary 1.4.2]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let (X, B) be a $\mathbb{Q}$-factorial klt pair where $K_X + B$ is $\mathbb{R}$-Cartier and B is π-big. Let $C \geqslant 0$ be an $\mathbb{R}$-divisor. If $K_X + B + C$ is klt and π-nef, then we may run $(K_X + B)$-MMP over U with scaling of C and terminates.*

Theorem 2.2.3. *[4, Corollary 1.3.3]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let (X, B) be a $\mathbb{Q}$-factorial klt pair where $K_X + B$ is $\mathbb{R}$-Cartier. If $K_X + B + C$ is not π-peseudo-effective, then we may run $f : X \dashrightarrow Y$ a $(K_X + B)$-MMP over U and end with a Mori fibre space $g : Y \to Z$.*

Corollary 2.2.4. *[7, Corollary 13.7] and [4, Corollary 1.4.3]: Let (X, B) be a a klt pair and $\mathfrak{C}$ be any set of exceptional divisors such that contains only exceptional divisors E of discrepancy $a(E; X, B) \leqslant 0$. Then there is a birational morphism $f : Z \to X$ and a $\mathbb{Q}$-divisor B_Z such that:*

1. *(Z, B_Z) is klt;*

2. *E is a f-exceptional divisor if and only if $E \in \mathfrak{C}$;*

3. *$B_Z = \sum -a(E; X, B)$ and $f_* B_Z = B$ and $K_Z + B_Z = f^*(K_X + B)$.*

*In particular, if we take $\mathfrak{C}$ containing all such divisors, then Z is called **terminalization** of X; if take $\mathfrak{C}$ containing only one such divisor, then $f : Z \to X$ is called a **divisorial extraction**.*

Definition 2.2.5. *[5, Definition 3.3] Two or more pairs $\{(X_i, B_i)\}$ are called **MMP-related** if they are results of $(K+B)$-MMP from a log smooth pair (W, B_W).*

Lemma 2.2.6. *[5, Proposition 3.4] Let $\{(X_l, B_l)\}$ be a finite set of $\mathbb{Q}$-factorial klt pairs such that birational to other, then TFAE:*

1. *They are MMP-related;*

2. *There is a nonsingular pair (W, B_W) with snc boundary, and projective birational morphisms $f_l : W \to X_l$ dominating each X_l, such that $f_{l*} B_W = B_l$ and*
$$K_W + B_W = f_l^*(K_{X_l} + B_l) + \sum_{exceptional} a_{li} E_{li}$$
with $a_{li} > 0$ for all f_i-exceptional divisors;

3. *For any two pairs* $(X, B=\sum_i b_iB_i), (X', B'=\sum_j b'_jB'_j)$ *in the set,* $a(B_i; X', B') \geqslant -b_i$ *and strict inequality holds if and only if* B_i *exceptional over* X', *and* $a(B'_j; X, B) \geqslant -b'_j$ *and strict inequality holds if and only if* B'_j *exceptional over* X.

Let $K = K(X)$ be the function field, and let $\Sigma = \{\nu\}$ be the set of discrete valutions of the field .

Definition 2.2.7. *[5, Definition 3.5] Let* $\theta : \Sigma \to [0,1)_{\mathbb{Q}}$ *be a function. Then we can construct a collection* $\mathcal{C}_\theta$ *of pairs associated to* θ, *consists of klt pairs* $(X, B = \sum a_iB_i)$ *satisfying*

1. $a_i = \theta(B_i)$;
2. $a(E; X, B) > -\theta(E)$ *for all* E *exceptional over* X.

For example, if we take $\theta \equiv 0$ constant, the $\mathcal{C}_\theta$ is the collection of all terminal varieties Y without boundary birational to X. Furthermore, we can define the corresponding discrepancy:

Definition 2.2.8 (θ-discrepancy)**.** *Let* (X, B) *be a pair in the category* $\mathcal{C}_\theta$ *for some function* θ *and let* $f : Y \to X$ *be a log resolution of* (X, B). *Suppose*

$$K_Y + B_Y + C = f^*(K_X + B)$$

where $B_Y = (f^{-1})_*B + \sum_{E_i\ exceptional}\theta(E_i)E_i$, *then the* θ*-discrepancy of exceptional divisor* E_i *over* X *is*

$$a_\theta(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$

Or equivalently, we have

$$a_\theta(E_i; X, B) = a(E_i; X, B) + \theta(E_i).$$

A pair (X, B) *is called* θ*-canonical(*θ*-terminal) if* $a_\theta(E; X, B) \geqslant 0$ *(*$a_\theta(E; X, B) > 0$ *) for all exceptional divisors* E *over* X. *Note that* θ*-canonical pair is not always in* $\mathcal{C}_\theta$.

3 Original proof

3.1 Prepare

First we fix a category:

Proposition 3.1.1. *[5, Lemma 3.6] Let* $f : (X, B) \to S, f' : (X', B') \to S'$ *be two* $\mathbb{Q}$*-factorial log Mori fibre spaces with only klt singularities and MMP-related, inducing a birational map* Φ*:*

$$\begin{tikzcd} (X,B) \arrow[r, dashed, "\Phi"] \arrow[d, "f"'] & (X',B') \arrow[d, "f'"] \\ S & S' \end{tikzcd}$$

Suppose $B = \sum_i b_iB_i + \sum_j d_jD_j$ *and* $B' = \sum_j d'_jD_j + \sum_k b'_kB'_k$, *where* B_i *are divisors on* X *but not on* X', B'_k *are divisors on* X' *but not on* X, *and* D_j *are divisors on both* X *and* X'. *By Lemma 2.2.6,* $d_j = d'_j$. *Take a rational number* $\epsilon < 1$ *such that* $\epsilon > -\operatorname{totdiscrep}(X, B), -\operatorname{totdiscrep}(X', B')$, *and take the function* $\theta : \{\nu\} \to [0,1)_{\mathbb{Q}}$ *as following:*

- $\theta(B_i) = b_i, \theta(D_j) = d_j, \theta(B'_k) = b'_k$;
- $\theta(E) = \epsilon$ *if* E *is exceptional over both* X *and* X';
- $\theta(D) = 0$ *if* D *is a divisor on both* X *and* X', *but not a component of* B *or* B'.

Then the collection $\mathcal{C}_\theta$ *satisfies*

1. (X, B) *and* (X', B') *belongs to* $\mathcal{C}_\theta$;
2. *For any finitely many klt pairs* $\{(X_l, B_l)\}$ *in* $\mathcal{C}_\theta$, *there is an object* $(Z, B_Z) \in \mathcal{C}_\theta$ *and projective birational morphisms* $Z \to X_l$ *dominating each* X_l *as a process of* $(K_Z + B_Z)$*-MMP over* X_l *(thus over* $\operatorname{Spec}\mathbb{C}$*);*
3. *Any* $(K + B)$*-MMP starting from an object in* $\mathcal{C}_\theta$ *stays inside of* $\mathcal{C}_\theta$, *and so does any* $(K + B + cH)$*-MMP where* H *is base point free and* $c \in \mathbb{Q}_{>0}$.

Remark 3.1.2. *Let* $\delta = 1 - \epsilon$, *then all pairs in* $\mathcal{C}_\theta$ *is* δ*-lc.*

With notations and assumptions in Proposition 3.1.1, we shall define the Sarkisov degree. We take a very ample divisor A' on S' and a sufficiently large and divisible integer $\mu' > 1$ such that

$$\mathcal{H}' = |-\mu'(K_{X'} + B') + f'^*A'|$$

is a very ample complete linear system on X' over $\operatorname{Spec}\mathbb{C}$. Let (W, B_W) be a common log resolution of X and X' in $\mathcal{C}_\theta$ with projective birational morphism $\sigma : W \to X$, $\sigma' : W \to X'$ and $\sigma_* B_W = B, \sigma'_* B_W = B'$. Let $\mathcal{H}_W := \sigma'^* \mathcal{H}'$ and then $\mathcal{H} := (\Phi^{-1})_* \mathcal{H}' = \sigma_* \mathcal{H}_W$. Furthermore, if $\mathcal{H}$ is not base point free, then

$$\sigma^* \mathcal{H} = \mathcal{H}_W + F$$

where $F = \sum f_l F_l \geqslant 0$ is the fixed part. Take a general member H' of the linear system $\mathcal{H}'$ such that $H_W := \sigma'^* H' = (\sigma'^{-1})_* H' \in \mathcal{H}_W$, and let $H := (\Phi^{-1})_* H' = \sigma_* H_W$, then H if f-ample and $\sigma^* H = H_W + F$. By taking further resolution, we may assume H_W is smooth and crosses normally with exceptional locus of σ and σ'.

Now we can define the Sarkisov degree in $\mathcal{C}_\theta$ with respect to H' (or $\mathcal{H}'$):

Definition 3.1.3. *[5, Definition 3.8] Sarkisov degree of (X, B) with respect to H (or $\mathcal{H}$) in $\mathcal{C}_\theta$ is a triple (μ, λ, e) ordered lexicographically:*

- ***Nef threshold** μ: Let $C \subset X$ be a curve contracted by f, then*

$$\mu := -\frac{H.C}{(K_X + B).C}$$

i.e. $K_X + B + \frac{1}{\mu} H \equiv_S 0$;

- ***θ-canonical threshold** c **and** λ: $\lambda = 0$ if $\mathcal{H}$ is base point free; otherwise,*

$$c := \frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geqslant 0, E \text{ exceptionl over } X\}$$

- ***Number of** $(K_X + B_X + \frac{1}{\mu} H)$**-crepant divisors**: Let $e = 0$ if $\mathcal{H}$ is base point free (and hence $\lambda = 0$), otherwise*

$$e = \#\{E; E \text{ is } \sigma\text{-exceptional and } a_\theta(E; X, B + \frac{1}{\lambda} H) = 0\}$$

Remark 3.1.4.

1. *The Sarkisov degree is dependent on the choice of A', H' and θ.*
2. *Take a common log resolution $(W, B_W) \in \mathcal{C}_\theta$ with $B_W = \sum \theta(E) E$ and projective birational morphisms $\sigma : W \to X$, $\sigma' : W \to X'$. Since $\sigma^* \mathcal{H} = \mathcal{H}_W + \sum f_l F_l$, we have ramification formula:*

$$K_W + B_W + tH_W = \sigma^*(K_X + B + tH) + \sum (a_l - t f_l) E_l$$

where $\sum a_l E_l$ is effective and supported on $\operatorname{Exc}\sigma$. Then $\lambda := \max\{\frac{f_l}{a_l}\}$. If $\mathcal{H}$ is base point free, then $\sum f_l F_l = 0$ and $\lambda = 0$.

3. *e is the number of components in $\sum (a_l - c f_l) E_l$ with coefficient 0 in the formula*

$$K_W + B_W + \frac{1}{\lambda} H_W = \sigma^*(K_X + B + \frac{1}{\lambda} H) + \sum (a_l - \lambda f_l) E_l.$$

Such prime divisors $E_1 \dots E_e$ are called $(K_X + B_X + \frac{1}{\lambda} H)$-$\theta$-crepant.

We also need some extraction map in this category:

Lemma 3.1.5. *Using the notation in the definition of Sarkisov degree, then there is a contraction $f : Z \to X$ such that*

- *$(Z, B_Z) \in \mathcal{C}_\theta$ and $(Z, B_Z + \frac{1}{\lambda} H_Z)$ is θ-terminal and $\mathbb{Q}$-factorial;*
- *$\rho(Z) = \rho(X) + 1$;*
- *f is $(K_X + B_X + \frac{1}{\lambda} H_X)$-crepant, that is*

$$K_Z + B_Z + \frac{1}{\lambda} H_Z = f^*(K_X + B + \frac{1}{\lambda} H).$$

Proof. We follow the proof in [5, Proposition 1.6] but for klt pair case. Let $(W, B_W) \in \mathcal{C}_\theta$ and $\sigma : W \to X, \sigma' : W \to X'$ be the common resolution as in Definition 3.1.3 ,and suppose $E_1, \dots, E_e$ are $(K_X + B_X + \frac{1}{\mu} H)$-$\theta$-crepant divisors after renumbering. Then we have

$$K_W + B_W + \frac{1}{\lambda} H_W = \sigma^*(K_X + B + \frac{1}{\lambda} H) + \sum_{l=1}^{e} 0 \cdot E_l + \sum_{l > e} (a_l - \frac{1}{\lambda} f_l) E_l.$$

We run $(K_W + B_W + \frac{1}{\lambda} H_W)$-MMP on W over X with scaling of some ample divisor, then the MMP ends with a minimal model $p : (Y, B_Y + \frac{1}{\lambda} H_Y) \to X$ of $(W, B_W + \frac{1}{\lambda} H_W)$ over X and the exceptional locus is exactly $\cup_{i=1}^{e} E_i$ and p is crepant:

$$K_Y + B_Y + \frac{1}{\lambda} H_Y = p^*(K_X + B_X + \frac{1}{\lambda} H_X).$$

Then we run $(K_Y + B_Y)$-MMP on Y over X with scaling of some ample divisor. This ends with the minimal model (X, B) of (Y, B_Y) over X, and the last contraction in the MMP is $f : Z \to X$ as required. ∎

3.2 Flowchart for the Log Sarkisov program

We follow [5, Flowchart for the Sarkisov program] in this subsection.

If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is nef, the two Mori fibre spaces are isomorphic (shown in next subsection by propostion 3.3.5) and we stop here. Otherwise:

Claim 3.2.1. *1. If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef, then there is a contraction $f : X \to T$ and a Sarkisov link $\phi_1 : X \dashrightarrow X_1$ of type III or IV; ...*

2. If $\lambda > \mu$, then there is a divisorial extraction $p : Z \to X$ and a Sarkisov link $\phi_1 : X \dashrightarrow X_1$ of type I or II.

Proof. 1. Suppose f is the contraction with respect to a $(K_X + B)$-negative extremal ray $R = \overline{\mathrm{NE}}(X/S)$, then $(K_X + B + \frac{1}{\mu}H).R = 0$ by definition of μ. There is an extremal ray $P \subset \overline{\mathrm{NE}}(X)$ such that $(K_X + B + \frac{1}{\mu}H).P < 0$ and $F := P + R$ is an extremal face (Check [6, 5.4.2] for details). Take $0 < t \ll 1$ such that $(K_X + B + (\frac{1}{\mu} - t)H).P < 0$, then $(K_X + B + (\frac{1}{\mu} - t)H).R < 0$ since H is f-ample, and F is a $(K_X + B + (\frac{1}{\mu} - t)H)$-negative extremal face. Since $(X, B + (\frac{1}{\mu} - t)H)$ is klt, there is a contraction $g : X \to T$ with respect to F factorizing through $f : X \to S$. Since $(X, B + \frac{1}{\mu}H)$ is klt, and $\rho(X/T) = 2$, we can run $(K_X + B + \frac{1}{\mu}H)$-MMP on X with scaling of some ample divisor C. Since $B + \frac{1}{\mu}H$ is relatively big, the MMP terminates. There are following cases:

(a) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$, and then followed by a Mori fibre space $(X_1, B_1 + \frac{1}{\mu}H_1) \to S_1$. Then $S_1 \cong T$ and this is a link of type III.

(b) After finitely many flips $X \dashrightarrow X_1$, first non-flip contraction is a Mori fibre space $f_1 : X_1 \to S_1$. This is a link of type IV.

(c) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$ with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where $e > 0$ and $E = \mathrm{Exc}\, p$ and $g_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to T$ is a log minimal model of $(X, B + \frac{1}{\mu}H)$ over T . In fact the only ray of $\overline{\mathrm{NE}}(X_1/T)$ is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and hence is $(K_{X_1} + B_1)$-negative, therefore $(X_1, B_1)/T$ is a log Mori fibre space. Take $S_1 = T$, then this is a link of type III:

(d) After finitely many flips $X \dashrightarrow X_1$, $(K_X + B + \frac{1}{\mu}H)$-MMP ends with a log minimal model $(X_1, B_1 + \frac{1}{\mu}H_1)$ over T. Then there is an extremal ray R of $\overline{\mathrm{NE}}(X_1/T)$, which is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and $(K_{X_1} + B_1)$-negative. Let $f_1 : X_1 \to S_1$ be the contraction with respect to R. This is a link of type IV. In fact, $X \dashrightarrow S_1$ is the ample model of $K_X + B + \frac{1}{\mu}H$.

2. Take an extraction $p : (Z, B_Z, H_Z) \to (X, B, H)$ as in Lemma 3.1.5. That is, (Z, B_Z) is θ-terminal and $p^*(K_X + B + \frac{1}{\lambda}H) = K_Z + B_Z + \frac{1}{\lambda}H_Z$ where $B_Z = \sum \theta(E_\nu)E_\nu$ and $E = \mathrm{Exc}\, p$ is a prime divisor on Z. Then we run $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$-MMP on Z over S with scaling of some ample divisor C. Since Z is covered by $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$-negative curves, $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ is not relatively pseudo-effective. Hence this ends with a Mori fibre space by Theorem 2.2.3. There are two cases:

(a) After finitely many flips $Z \dashrightarrow Z'$, the first non-flip contraction is a divisorial contraction $q : Z' \to X_1$. Then $X_1 \to S$ is a log Mori fibre space of (X, B) and $(X, B + \frac{1}{\lambda}H)$. Let $S_1 = S$ and this is a link of type II.

(b) After finitely many flips $Z \dashrightarrow X_1$, first non-flip contraction is a fibering contraction $f_1 : X_1 \to S_1$. Since $(K_{X_1} + B_1 + \frac{1}{\lambda}H_1)$ is f_1-negative and H_1 is f_1- ample, $(K_{X_1} + B_1)$ is f_1-negative, and $(X_1, B_1)/Y$ is a log Mori fibre space. Take $S_1 = Y$ and this is a link of type I.

□

Remark 3.2.2. *1. (a) For case 1a and 1b, since $K_{X_1} + B_1 + \frac{1}{\mu}H_1$ is f_1-negative, we have $\mu_1 < \mu$.*

(b) For case 1c and 1d, Since $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ is trivial on the ray $R = \overline{\mathrm{NE}}(X_1/S_1)$ for both cases, we have $\mu_1 = \mu$. Notice that $(X_1, B_1 + \frac{1}{\mu}H_1)$ stays θ-canonical, we have $\lambda_1 \leqslant \mu = \mu_1$, thus this goes back to case 1. Furthermore, for case 1c we have $\rho(X_1) = \rho(X) - 1$.

2. For case 2:

(a) For both case 2a and 2b, we have $\mu_1 \leqslant \mu$ with equality holds if and only if

- *either* $\dim S_i < \dim S_{i+1}$
- *or* $\dim S_i = \dim S_{i+1}$ *and the link is square*

(b) We have $\lambda_1 \leqslant \lambda$ and if $\lambda_1 = \lambda$, then $e_1 < e$.

3.3 Termination

We need following theorems:

Theorem 3.3.1. *[3, Theorem 1.1] Let d be a natural number and δ be a positivity real number, then the projective varieties X such that*

- *(X, B) is a δ-lc pair of dimension d for some boundary B, and*
- *$-(K_X + B)$ nef and big,*

form a bounded family.

Lemma 3.3.2. *[2, Lemma 2.24] Let $\mathcal{P}$ be a bounded set of couples. Then there is a natural number I depending only on $\mathcal{P}$ satisfying the following: Assume X is projective with klt singularities and $M \geqslant 0$ an integral divisor on X so that $(X, \operatorname{Supp} M) \in \mathcal{P}$, then IK_X and IM are cartier.*

Corollary 3.3.3. *The nef threshold μ with respect to θ is discrete.*

Proof. Notice that all pairs in $\mathcal{C}_\theta$ are δ-lc, then the general fibre of (F_i, B_{F_i}) of $(X_i, B_i) \to S_i$ is also δ-lc with $\dim F_i \leqslant \dim X_i$. Thus they form a bounded family by Theorem 3.3.1. Take the integral I in Lemma 3.3.2, then $I(K_{F_i} + B_{F_i})$ is Cartier. Take a rational curve C_{F_i} in $\overline{\mathrm{NE}}(F_i)$, then

$$0 < -I(K_{F_i} + B_F).C_{F_i} \leqslant 2I \dim F_i$$

Notice that $\mu = \frac{IH_{F_i}.C_{F_i}}{-I(K_{F_i}+B_{F_i}).C_{F_i}}$, where $H_{F_i}.C_{F_i}$ and $-I(K_{F_i} + B_{F_i}).C_{F_i}$ are integers, thus

$$\mu \in \frac{1}{(2I \dim F_i)!}\mathbb{N}.$$

□

We prove the termination by contraction. Otherwise, if there is an infinite sequence, i.e. there are infinitely many X_i and birational maps obtained from the program:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X'$$

Then we have:

- Since $\mu' \leqslant \mu_{i+1} \leqslant \mu_i$, and as is shown in 3.3.3 that $\{\mu_i\}$ is discreteness, there is an integer N such that $\mu_i = \mu_N$ for all $i > N$. In fact, we may assume $N = 0$ and $\mu_i = \mu_0 = \mu$ for all i;
- Notice that for case 1a and 1b, we have $\mu_{i+1} < \mu_i$, thus there is no such links in the infinite sequence. If there is a link as case 1c or 1d, then $\mu_{i+1} = \mu_i = \mu$ and $\lambda_{i+1} \leqslant \mu$, thus next link must be case 1c or 1d again, and all links following must be case 1c or 1d. For case 1c we have $\rho(X_{i+1}) = \rho(X_i) - 1$, therefore there are only finitely many such links, and all links after are case 1d;
- Each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is obtained by $(K + B + \frac{1}{\mu}H)$-MMP with scaling of a $\mathbb{Q}$-divisor C_i. But we can choose C_{i+1} to be the strict transform of C_i in X_{i+1}, then the whole sequence is $(K + B + \frac{1}{\mu}H)$-MMP with scaling of a $\mathbb{Q}$-divisor C_0, and this ends. Therefore there are no links of case 1c or 1d, and i.e. all links are of case 2.
- For case 2, recall that $\mu_{i+1} = \mu_i$ implies that

  $$\begin{aligned} &\text{either } \dim S_i < \dim S_{i+1} \\ &\text{or } \dim S_i = \dim S_{i+1} \text{ and the link is square} \end{aligned}$$

  and notice that $\dim S_i < \dim X$, hence we may assume $\dim S_i = \dim S_0$ (Note that $\dim S_0 \neq 0$, otherwise all X_i are isomorphic, which is absurd).

We are left to show that there is no infinite sequence with stationary μ_i and $\dim S_i$. Since for case 2, $\lambda_{i+1} \leqslant \lambda_i$ and $\lambda_{i+1} = \lambda_i$ implies $e_{i+1} < e_i$, furthermore $\frac{1}{\lambda_i} \leqslant \frac{1}{\mu_0}$, we have

$$c := \lim_i \frac{1}{\lambda_i} > \frac{1}{\lambda_i} = c_i$$

We prove it in servel steps:

Step 1: Claim that $(X_i, B_i + cH_i)$ and $(Z_i, B_i + cH_i)$ are log canonical for all $i \gg 0$. Otherwise, let

$$\alpha_i = \operatorname{lct}(X_i, B_i; H_i)$$

then there are infinitely many i such that $c > \alpha_i$. By definition of λ_i, we have $\alpha_i > c_i$. Notice that c_i accumulates from below to c and never equals, there are infinitely many α_i, which contradicts to acc conditation of lct:

Theorem 3.3.4. *[1, Theorem 1.1] Fix a positive integral n, $I \subset [0,1]$ and a subset J of positive real numbers. If I, J satisfy the DCC, then $\mathrm{LCT}_n(I,J)$ satisfies ACC.*

The same argument applies to $(Z_i, B_i + cH_i)$. Therefore, we may assume all pairs are log canonical.

Step 2: For each link there are flips

$$\begin{tikzcd}
& Z_i = Z_i^0 \arrow[dl, "p_i^0"'] \arrow[dr, "q_i^0"] \arrow[rr, dotted] & & Z_i^1 \arrow[dl, "p_i^1"'] \arrow[dr, "q_i^1"] & & \cdots & & Z_i^k \arrow[dl, "p_i^k"'] \arrow[dr, "q_i^k"] & \cdots \\
X_i = X_i^0 & & X_i^1 & & X_i^2 & \cdots & {} & & .
\end{tikzcd}$$

This step we will show that such 2-ray game of $(K+B+c_iH)$-MMP on Z_i is also a 2-ray game of $(K+B+cH)$-MMP. Let $P^k = \overline{\mathrm{NE}}(Z^k/X^k)$ and $Q^k = \overline{\mathrm{NE}}(Z^k/X^{k+1})$, then P^k is $(K_{Z^k}+B_{Z^k}+c_0H_{Z^k})$-positive and $(K_{Z^k}+B_{Z^k}+c_0H_{Z^k})$-negative. Need to show this also holds for each $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$. Prove this by induction on k.

Since $c > c_i$, we have

$$K_Z + B_Z + cH_Z = p^*(K_X + B + cH) - aE\ (a > 0)$$

By negativity lemma, there is a curve C_Z on Z mapping to a point on X, and $E.C_Z < 0$, thus we have $(K_Z + B_Z + cH_Z).P^0 > 0$, where $P^0 = \mathbb{R}_{\geqslant 0}[C_Z] = \overline{\mathrm{NE}}(Z/X)$. Suppose $(K_{Z^k} + B_{Z^k} + cH_{Z^k}).P^k > 0$, then $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$ is not nef over S. In particular, P^k is positive, and the other extremal ray Q^k is negative. This implies step 2. Furthermore, by decreasing of canonical divisor, we have

$$a(\nu; X_i, B_i + cH_i) \leqslant a(\nu; X, B + cH)$$

and strictly inequality holds if and only if $X_l \dashrightarrow X_{l+1}$ is not an isomorphism at center of ν on X_l for some $l < i$

Step 3: Claim that $(X_i, B_i + cH_i)$ is klt for all $i \gg 0$. Otherwise, if there are infinitely many i such that $(X_i, B_i + cH_i)$ is not klt, since they are all log canonical, this is equivalent to say there infinitely many i and ν_i such that

$$-1 = a(\nu_i; X_i, B_i + cH_i) \geqslant a(\nu_i; X_0, B_0 + cH_0) \geqslant -1$$

Therefore $a(\nu; X_i, B_i + cH_i) = -1$ and $X_0 \dashrightarrow X_i$ isomorphism at the center $z(\nu_i, X)$. Thus the local θ-canonical threholds are same

$$ct_\theta(\nu_i; X, B; H) = ct_\theta(\nu_i; X_i, B_i; H_i)$$

On the other hand, by definition

$$c_i \leqslant ct_\theta(\nu_i; X_i, B_i; H_i)$$

and since $(X, B + cH)$ is not klt along $z(\nu_i, X)$, it is not θ-canonical, thus

$$ct_\theta(\nu_i; X_i, B_i; H_i) < c$$

Therefore

$$c_i \leqslant ct_\theta(\nu_i; X, B; H) < c$$

But the set $\{ct_\theta(x; X, B; H); x \in X\}$ is finite, a contradiction! We may assume $(X_i, B_i + cH_i)$ are all klt.

Step 4: Note that $E_i = \mathrm{Exc}(p_i)$ are all distinct. Otherwise, assume $E_i = E_j$ for some $i < j$, then Z_i and Z_j are isomorphic in a neighborhood of E_i and E_j, thus

$$a(E_i; X_i, B_i + cH_i) = a(E_j; X_j, B_j + cH_j)$$

However, since $E_i = E_j$ is not a divisor on X_j, there is $k < j$ such that E_j is contracted by $Z'_k \to X_{k+1}$, therefore $X_k \dashrightarrow X_{k+1}$ is not isomorphic at E_j, hence

$$a(E_i; X_i, B_i + cH_i) \leqslant a(E_j; X_k, B_k + cH_k) < a(E_j; X_{k+1}, B_{k+1} + cH_{k+1}) \leqslant a(E_j; X_j, B_j + cH_j)$$

which is a contradiction.

Since $(X, B + cH)$ is klt, then there are only finitely many E_i with $a(E_i, X, B + cH) < 0$. But there are in fact infinitely many

$$a(E_i; X, B + cH) \leqslant a(E_i; X_i, B_i + cH_i) < -\theta(E) \leqslant 0,$$

a contradiction!

At last we have the Noether-Fano-Iskovskikh criterion to show when they are isomorphic:

Theorem 3.3.5. *(Noether-Fano-Iskovskikh Criterion): Notations as in the definition of Sarkisov degree, then*

1. $\mu \geqslant \mu'$;

2. *If $\mu \geqslant \lambda$ and $(K_X+B+\frac{1}{\mu}H)$ is nef, then Φ is an isomorphism of Mori fibre space, i.e., we have commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\sim", "\Phi"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "\sim"] & S' \end{tikzcd}$$

Proof. We follow the proof in [7],[10] and [6]:

1. Only need to show $(K_X + B + \frac{1}{\mu'}H)$ is f-nef. Let $\sigma : W \to X$ and $\sigma' : W \to X'$ be the common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W =&\sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ =&\sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here $\{G_i\}, \{E_j\}$ are σ-exceptional divisors, and $\{E_j\}\{G'_k\}$ are σ'-exceptional divisors. Since $H_W = \sigma'^* H'$, $g'_k > 0$ (or there are no such G'_k). Then take a general curve $C \subset X$ contracted by f, such that its strict transform $\tilde{C}$ on W is disjoint from G_i, E_j, and is not contained in G'_k. Then we have:

$$\begin{aligned} C.\left(K_X + B + \frac{1}{\mu'}H\right) =&\tilde{C}.\sigma^*\left(K_X + B + \frac{1}{\mu'}H\right) \\ =&\tilde{C}.\left(\sigma^*\left(K_W + B_W + \frac{1}{\mu'}H\right) + \sum g_i G_i + \sum e_j E_j\right) \\ =&\tilde{C}.\left(K_W + B_W + \frac{1}{\mu'}H_W\right) \\ =&\tilde{C}.\left(\sigma'^*\left(K_{X'} + B' + \frac{1}{\mu'}H'\right) + \sum e'_j E_j + \sum g'_k G'_k\right) \\ =&\tilde{C}.\sigma'^* f'^* A' + C.\left(\sum g'_k G'_k\right) \\ \geqslant& 0 \end{aligned}.$$

This implies $(K_X + B + \frac{1}{\mu'}H)$ is f-nef and $\mu \geqslant \mu'$;

2. First we show that $\mu = \mu'$. By 1, we only need to show $(K_{X'} + B' + \frac{1}{\mu}H')$ is f'-nef. Indeed, same as 1, we can take a curve C' on X' contracted by f', such that its strict transform $\tilde{C}'$ on W is disjoint from G'_i, E_j, and is not contained in G'_k and $C'.\left(K_{X'} + B' + \frac{1}{\mu}H'\right) \geqslant 0$.

   Then we show there are isomorphic. Take a very ample divisor D on X and let D' be its strict transform on X'. D' is f'-ample, thus there exists $0 < d \ll 1$ such that the following holds:

   - $K_X + B + \frac{1}{\mu}H_X + dD$ is ample;
   - $K_{X'} + B' + \frac{1}{\mu}H' + dD'$ is ample.

   Therefore X and X' are both log canonical models of $(W, B_W + \frac{1}{\mu}H_W + dD_W)$, hence $X \cong X'$. and $S \cong S'$.

□

4 Double scaling

This section we follows [7, 13.The Sarkisov program] and [10].

4.1 Prepare

Let (W, B_W) be a $\mathbb{Q}$-factorial klt pair and $f : (X, B) \to S$ and $f' : (X', B') \to S'$ be two different log Mori fibre spaces as outputs of $(K_W + B_W)$-MMP. To modify the beginning setting, we need more conventions and lemmas:

Definition 4.1.1. *Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties. If*

- *f does not extract divisors;*
- *$a(E; X, B_X) \leqslant a(E; Y, B_Y)$ for all divisors E over X.*

then we denote $(X, B) \geqslant (Y, B_Y)$.

In particular, for terminal pairs, we have following lemma:

Lemma 4.1.2. *[7, Lemma 13.8] Let $f : W \dashrightarrow X$ be a birational map where (W, B_W) is terminal. If*

- *f does not extract divisors;*
- *$K_X + B$ is nef, where $B = f_*B_W$;*
- *$a(E; X, B) \geqslant a(E; W, B_W)$ for all divisors $E \subset W$,*

then

- *$(W, B_W) \geqslant (X, B)$.*
- *(X, B) is klt*
- *If $Z \to X$ is a divisorial extraction of a divisor E with $a(E; X, B) \leqslant 0$, then E is a divisor on W ;*
- *If $Z \to X$ is terminalization of (X, B), then $W \dashrightarrow Z$ extracts no divisors.*

Conversely, start from a klt pair and non-positive map, we have

Lemma 4.1.3. *[10, Lemma 3.5] Let $\sigma : (W, B_W) \dashrightarrow (X, B)$ be a $K_W + B_W$-non-positive birational map such that $\sigma_*(K_W + B_W) = K_X + B$ and (W, B_W) is a $\mathbb{Q}$-factorial klt pair. Then there is a resolution of indeterminacy $\pi : \tilde{W} \to W$ and $\tilde{\sigma} : \tilde{W} \to X$ such that*

- *$(\tilde{W}, B_{\tilde{W}})$ is $\mathbb{Q}$-factorial terminal and $\tilde{\sigma}_* B_{\tilde{W}} = B$,*
- *$\tilde{\sigma}$ is $(K_{\tilde{W}} + B_{\tilde{W}})$-non-positive and $(\tilde{W}, B_{\tilde{W}}) \geqslant (X, B)$.*

By Lemma 4.1.3, we replace (W, B_W) by its log resolution such that (W, B_W) is terminal and $\sigma : W \to X$ and $\sigma' : W \to X'$ are $(K_W + B_W)$-non-positive morphisms, and $(W, B_W) \geqslant (X, B), (X', B')$.

Take very general ample $\mathbb{Q}$-divisors A and A' on S and S' such that $G \sim_{\mathbb{Q}} -(K_X+B)+f^*A$ and $H \sim_{\mathbb{Q}} -(K_{X'}+B')+f'^*A'$ are two ample $\mathbb{Q}$-divisors. Moreover, we may assume G and H satisfying $G_W := \sigma^*G = \sigma_*^{-1}G$ and $H_W := \sigma'^*H = \sigma_*'^{-1}H$. Therefore $\sigma_*(K_W + B_W + G_W) = K_X + B + G$ is nef, and Lemma 4.1.2 holds. Furthermore, we may assume $(W, B_W + gG_W + hH_W)$ is log smooth and terminal for all $0 \leqslant g, h \leqslant 2$ by taking furthermore blowing up if necessary. Then we have:

Theorem 4.1.4 (Sarkisov program with double scaling)**.** *Notations as above, there is a finite sequence of Sarkisov links*

$$\begin{tikzcd}
X = X_0 \arrow[r, dotted] \arrow[d, "f=f_0"'] & X_1 \arrow[r, dotted] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"'] & \cdots \arrow[r, dotted] & X_N = X' \arrow[d, "f_N"'] \\
S = S_0 & S_1 & S_2 & & S_N = S'
\end{tikzcd}$$

and rational numbers

$$\begin{aligned}
1 = g_0 \geqslant g_1 \geqslant \cdots \geqslant g_N &= 0 \\
0 = h_0 \leqslant h_1 \leqslant \cdots \leqslant h_N &= 1
\end{aligned}$$

such that

1. *$p_i : W \dashrightarrow X_i$ is $(K_W + B_W + g_iG_W + h_iH_W)$-non-positive, and $(K_{X_i} + B_i + g_iG_i + h_iH_i) = p_{i*}(K_W + B_W + g_iG_W + h_iH_W)$ is nef and is relatively trivial over S_i;*
2. *$(W, B_W + g_iG_W + h_iH_W) \geqslant (X_i, B_i + g_iG_i + h_iH_i)$;*
3. *each Sarkisov link is given by a sequence of $(K_{X_i} + B_i + g_iG_i + h_iH_i)$-trivial maps.*
4. *The last link $X_N \to S_N$ is isomorphic to $X' \to S'$*

4.2 Construct Sarkisov links

This subsection we construct the links inductively. Suppose we have $\sigma_i : W \dashrightarrow X_i$ as in Theorem 4.1.4, that is

- $f_i : (X_i, B_i) \to S_i$ is a log Mori fibre space and $\sigma_{i*}B_W = B_i$;
- $\sigma_i : W \dashrightarrow X_i$ is $(K_W + g_iG_i + h_iH_W)$-non-positive biratinal map, and $(K_{X_i} + B_i + g_iG + h_iH) = \sigma_{i*}(K_W + g_iG_W + h_iH_W)$ is nef and is relatively trivial over S_i;
- $(W, B_W + g_iG_W + h_iH_W) \geqslant (X_i, B_i + g_iG_i + h_iH_i)$;
- $0 \leqslant g_i, h_i \leqslant 1$ are rational numbers.

Then we need to show that there is a Sarkisov link $X_i \dashrightarrow X_{i+1}$ satisfying the theorem 4.1.4. Similarly with Sarkisov degree, we have following notations:

Definition 4.2.1. *Let C_i be a general f_i-vertical curve on X_i, then*

- $r_i := \frac{H_i.C_i}{G_i.C_i}$;
- *Let* Γ *be the set of* $t \in [0, \frac{g_i}{r_i}]$ *such that*
  1. $(W, B_W + g_iG_W + h_iH_W + t(H_W - r_iG_W)) \geqslant (X_i, B_i + g_iG_i + h_iH_i + t\,(H_i - r_iG_i))$
  2. $K_{X_i} + B_i + g_iG + h_iH + t(H_i - r_iG_i)$ *is nef;*

  And Let $s_i = \max\Gamma$*;*
- *Let* $D_{W,i} = B_W + g_iG_W + h_iH_W$ *and* $D_i = B_i + g_iG_i + h_iH_i +$*. Let* $D_{W,i}(t) = B_W + g_iG_W + h_iH_W + t(G_W - r_iH_W)$ *and* $D_i(t) = B_i + g_iG_i + h_iH_i + t\,(G_i - r_iH_i))$*. Let* $g_{i+1} = g_i - r_is_i$ *and* $h_{i+1} = h_i + s_i$*. Note that* $D_{W,i+1} = D_{W,i}(s_i)$*.*

Then we have (check [10, Lemma 4.4] for details)

1. $r_i > 0$;
2. either $\Gamma = \{0\}$ or is a closed interval;
3. $g_{i+1} = g_i \Leftrightarrow h_{i+1} = h_i \Leftrightarrow s_i = 0$;

Construct links: If $s_i = \frac{g_i}{r_i}$, then $g_{i+1} = 0$. Let $N = i+1$ and let $f_N : X_N = X_i \to S_N = S_i$, then $X_N \to S_N$ is isomorphic to $f' : X' \to S'$ (see Proposition 4.3.2) and we stop. Otherwise, if $s_i < \frac{g_i}{r_i}$, then we construct the Sarkisov link $X_i \dashrightarrow X_{i+1}$ in following cases:

1. Suppose s_i is not the threshold of condition 1 of Γ. That is, there exists $0 < \epsilon \ll 1$, such that for any divisor E on W, we have
$$a(E; X_i, D_i(s_i + \epsilon)) \geqslant a(E; W, D_{W,i}(s_i + \epsilon))$$
and $K_{X_i} + D_i(s_i + \epsilon)$ is not nef. Then there is a 2-dimensional $(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$-negative extremal face F for some $0 < \delta \ll \epsilon$, spaned by $R = \mathbb{R}_{\geqslant 0}[C_i]$ and another extremal ray P. Hence there is a contraction $X_i \to T_i$ corresponding to F factoring through f_i. Then we run $(K_{X_i} + D_i(s_i + \epsilon))$-MMP on X_i with scaling over T_i. After finitely many flips, we either have a $(K_{X_i} + D_i(s_i + \epsilon))$ minimal model, a divisorial contraction, or a Mori fibre space over T_i:
   (a) After finitely many flips $X_i \dashrightarrow X_{i+1}$ there is a log Mori fibre space $X_{i+1} \to S_{i+1}$, and this is a link of type III.
   (b) After finitely many flips $X_i \dashrightarrow Z_i$ there is a divisorial contraction $Z_i \to X_{i+1}$, then let $S_{i+1} = T_i$ and $X_{i+1} \to S_{i+1}$ is a log Mori fibre space and this is a link of type IV.
   (c) After finitely many flips $X_i \dashrightarrow X_{i+1}$, the contraction $X_{i+1} \to T_i$ is a log minimal model of $(X_i, D_i\,(s_i + \epsilon))$ over T_i. Let C' be the strict transform of C_i on X_{i+1}, then $(K_{X_{i+1}} + D_{i+1}(\epsilon)).C' = 0$ and $(K_{X_{i+1}} + B_{i+1}).C' < 0$, therefore there is a contraction $X_{i+1} \to S_{i+1}$ which is a log Mori fibre space. And this is a link of type IV.
2. Suppose s_i is the threshold of conditation 2 of Γ, that is, there exists $0 < \epsilon \ll 1$ and a σ_i-exceptional divisor E_i on W such that
$$a(E_i; X_i, D_i(s_i + \epsilon)) < a(E_i; W, D_{W,i}(s_i + \epsilon)).$$
In this case, we have
$$a(E_i; X_i, D_i(s_i)) = a(E_i; W, D_{W,i}(s_i)) = -\operatorname{mult}_{E_i}(D_{W,i}(s_i)) \leqslant 0.$$
Let $p_i : Z_i \to X_i$ be the diviosrial extraction of divisor E_i as in Corollary 2.2.4, and suppose $K_{Z_i} + D_{Z_i}(s_i) = K_{Z_i} + B_{Z_i} + g_{i+1}G_{Z_i} + h_{i+1}H_{Z_i} = p_i^*\,(K_{X_i} + D_i\,(s_i + \epsilon))$. Take a sufficiently small δ_i such that $0 < \delta \ll \epsilon \ll 1$ and
$$K_{Z_i} + \Delta_i = p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$$
is klt. Then we run $(K_{Z_i} + \Delta_i)$-MMP on Z_i over S_i. Since Z_i is covered by $(K_{Z_i} + \Delta_i)$-negative curves, it follows that $(K_{Z_i} + \Delta_i)$ is not pseudo-effective over S_i, and this MMP ends with a Mori fibre space. Moreover, this is a MMP for $p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta' G_i)$ for all $0 < \delta' \leqslant \delta$. After finitely many flips, we either have a $(K_{Z_i} + \Delta_i)$ Mori fibre space or a $(K_{Z_i} + \Delta_i)$ divisorial contraction.
   (a) After finitely many flips $Z_i \dashrightarrow X_{i+1}$ there is a Mori fibre space $X_{i+1} \to S_{i+1}$, and this is a link of type I. In this case we have $\rho(X_{i+1}) = \rho(X_i) + 1$.
   (b) After finitely many flips $Z_i \dashrightarrow Z'_{i+1}$ there is a divisorial contraction $q_i : Z'_{i+1} \to X_{i+1}$, and then a Mori fibre space $X_{i+1} \to S_i =: S_{i+1}$. This is a link of type II.

Claim 4.2.2. *1.* $r_i \leqslant r_{i+1}$*. Morever, in case 1a, we have* $r_i < r_{i+1}$*.*

2. *Since the birational map* $X_i \dashrightarrow X_{i+1}$ *is over* T_i *(over* S_i*) and* $(K_{X_i} + D_i(s_i))$ *is numerically trival over* T_i *(over* S_i*) in case 1 (case 2), it follows that* $a(E; X_i, D_i(s_i)) = a(E; X_{i+1}, D_{i+1})$ *for any divisors* E *over* W *and so the inequality*
$$a(E; X_{i+1}, D_{i+1}) \geqslant a(E; W, D_{W,i+1})$$
3. *In case 1, for any divisor* $E \subset W$*, we have* $a(E; X_i, D_i(s_i + \epsilon)) \leqslant a(E; X_{i+1}, D_{i+1}(\epsilon))$ *for all* $0 < \epsilon \ll 1$*. Moreover, since* $X_i \ncong X_{i+1}$*, there is a divisor* F *over* W *such that* $a(F, X_i, D_i(s_i + \epsilon)) < a(F, X_{i+1}, D_{i+1}(\epsilon))$*.*
4. *In case 2, for any divisor* $E \subset W$*, we have* $a(E; X_i, D_i(s_i + \epsilon) - \delta G_i) \leqslant a(E; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$ *for all* $0 < \epsilon \ll 1$*. Moreover, since* $X_i \ncong X_{i+1}$*, there is a divisor* F *over* W *such that* $a(F; X_i, D_i(s_i + \epsilon) - \delta G_i) < a(F; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$*.*
5. $h_i \leqslant 1$*, and* $h_i = 1$ *if and only if* $g_i = 0$*;*

4.3 Termination

Lemma 4.3.1. *[8, Lemma 13.18 and Lemma 13.19] (or [10, Lemma 4.9]) Suppose we construct a sequence of Sarkisov links:*

$$\begin{tikzcd} X = X_0 \arrow[r, dashed] \arrow[d] & X_1 \arrow[r, dashed] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r, dashed] & X_i \arrow[r] \arrow[d] & \cdots, \\ S = S_0 & S_1 & S_2 & & S_i & \end{tikzcd}$$

then

1. *There are only finitely many possibilities of $f_i : X_i \to S_i$ up to isomorphism;*
2. *The Sarkisov program with double scalling of (G_W, H_W) terminates. That is, there exits an integer $N > 0$ such that $g_N = 0$.*

Proof. 1. This follows from the finiteness of weak log canonical model (Theorem 2.1.10). We construct the space V of $\mathrm{WDiv}_{\mathbb{R}}(W)$ as following:

(a) If $h_k > 0$ for some k: Since H_W is nef and big, take an ample $\mathbb{Q}$-divisor A_W and effecive $\mathbb{Q}$-divisor C_W such that $H_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spaned by components of B_W, G_W, H_W, C_W, then

$$B_W + g_i G_W + h_i H_W \sim_{\mathbb{Q}} h_k A_W + B_W + g_i G_W + (h_i - h_k) H_W + h_k C_W =: \Delta_i \in \mathcal{L}_{h_k A_W}(V)$$

(b) If $h_k = 0$ for all k, then $h_i \equiv 0$ and $g_i \equiv 1$. Since G_W is nef and big, take an ample $\mathbb{Q}$-divisor A_W and effecive $\mathbb{Q}$-divisor C_W such that $G_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spaned by components of B_W, C_W, then

$$B_W + G_W \sim_{\mathbb{Q}} A_W + B_W + C_W =: \Delta_i \in \mathcal{L}_{A_W}(V)$$

Then all X_i are weak log canonical models of (W, Δ_i). By finiteness of weak log canonical models, there are finitely many $\sigma_i : W \dashrightarrow X_i$ up to isomorpism. However even $X_i \cong X_j$, they may have different log Mori fibre spaces, i.e. f_i and f_j are not same. But such log Mori fibre spaces of same X_i are finite. Indeed, we may assume that there is an k such that $X_i \cong X_k$ for all $k > i$, and f_i is the contraction corresponding to an extremal ray $R_i \subset \overline{\mathrm{NE}}(X_i)$. Then we have $(K_{X_k} + B_k).R_i < 0$ and $(K_{X_k} + B_k + g_i G_i + h_i H_i).R_i = 0$. Furthermore, H_k and G_k are relatively ample over S_i for all $i > k$.

(a) If $h_k > 0$: Since H_k is big, we have $h_k H_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor H_k and effective $\mathbb{Q}$-divisor E_k. Let $B_k' = B_k + (1-\epsilon) h_k H_k + \epsilon E_k$ for sufficiently small ϵ such that (X, B') is klt, then $(K_{X_k} + B_k').R_i < 0$ and $(K_{X_k} + B_k' + \epsilon A_k).R_i < 0$ for all $i > k$. By cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k} + B_k' + \epsilon A_k \geqslant 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_\alpha$$

All extremal ray R_i corresponding to f_i for $i > k$ are in the finite set $\{R_\alpha\}_{\alpha \in \Lambda}$, thus there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of X_k.

(b) If $h_k = 0$ for all k, and hence $g_k = 1$ for all k. Since G_k is big, we have $G_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor H_k and effective $\mathbb{Q}$-divisor E_k. Let $B_k' = B_k + (1-\epsilon) G_k + \frac{\epsilon}{2} E_k$ for sufficiently small ϵ such that (X, B') is klt, then $(K_{X_k} + B_k').R_i < 0$ and $(K_{X_k} + B_k' + \frac{\epsilon}{2} A_k).R_i < 0$ for all $i > k$. By cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k} + B_k' + \frac{\epsilon}{2} A_k \geqslant 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_\alpha$$

Again,there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of X_k.

2. Assmue this sequence of links is infinite, then there are $i < j$ such that $f_i : X_i \to S_i$ and $f_j : X_j \to S_j$ are isomorphic. Then we have $g_{i+1} = g_{j+1}$ and $h_{i+1} = h_{j+1}$. Since sequences of h_k and g_k are monotone, we have $h_{i+1} = h_k$ and $g_{i+1} = g_k$ for all $k > i$. Suppose $X_i \dashrightarrow X_{i+1}$ is a link in case 1, then the next link is also in case 1, and all the links after are in case 1. Note that $X_i \cong X_j$ and therefore $\rho(X_i) = \rho(X_j)$, the links are all of type IV. But this contracts 2 of Claim4.2.2. Therefore there are no link of type III or IV after X_i. In other word, the links after X_i are all type I or II in case 2.

Since $\rho(X_i) = \rho(X_j)$, X_i and X_j is linked by the Sarkisov links of type II. But this contracts 4 of Claim 4.2.2. □

Lemma 4.3.2. *$X_N \to S_N$ is isomorphic to $X' \to S'$.*

Proof. By 2 of Theorem 3.3.5, we have $h_N = 1$ and they are isomorphic. □

5 Using the Polytope

In this section we follows [8].

5.1 Morphisms between models

Let W be a smooth projective variety, and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective $\mathbb{Q}$-divisor A. In this subsection we describe the models X_i of $(W, A+B)$ for some $B \in V$ and rational maps between X_i.

Theorem 5.1.1. *[8, Theorem 3.3] Let W be a smooth projective variety, and V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective $\mathbb{Q}$-divisor A. Suppose that there is an element D_0 of $\mathcal{L}_A(V)$ such that $K_W + D_0$ is big and klt. Then there are finitely many rational contractions $f_i : W \dashrightarrow X_i$ such that*

1. *$\{\mathcal{A}_i = \mathcal{A}_{A,f_i}\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;*
2. *If i, j are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ then there is a contraction $f_{ij} : X_i \to X_j$ and $f_j = f_{ij} \circ f_i$;*
3. *Suppose in addition V spans $\mathrm{NS}(W)$. Pick i such that a connected components $\mathcal{C}$ of $\mathcal{C}_i$ intersects the interior of $\mathcal{L}_A(V)$, TFAE:*
   - *(a) $\mathcal{C}$ spans V;*
   - *(b) If $D \in \mathcal{A}_i \cap \mathcal{C}$ then f_i is a log terminal model of $K_W + D$;*
   - *(c) f_i is birational and X_i is $\mathbb{Q}$-factorial.*
4. *Suppose in addition V spans $\mathrm{NS}(W)$. If i, j are two indices such that $\mathcal{C}_i$ spans V and D is a general point of $\mathcal{A}_j \cap \mathcal{C}_i$ which is also a point of interior of $\mathcal{L}_A(V)$, then $\mathcal{C}_i$ and $\overline{\mathrm{NE}}(X_i/X_j)^* \times \mathbb{R}^k$ for some $k \leqslant 0$. Furthermore $\rho(X_i/X_j)$ equals the differece in the dimensions of $\mathcal{C}_i$ and $\mathcal{C}_j \cap \mathcal{C}_i$.*

Proof. 1. is proved in [4].

2. Pick a divisor $D \in \mathcal{A}_j \cap \mathcal{C}_i$ and $D' \in \mathcal{A}_i$ such that
$$D_t = D + t(D' - D) \in \mathcal{A}_i$$
for $t \in (0, 1]$. By finiteness of log terminal models, we may find a positive contast $\delta > 0$ and a birational contraction $f : W \dashrightarrow X$ which is a log terminal model of $K_W + D_t$ for $t \in (0, \delta]$. Replacing $D' = D_1$ by D_δ we may assume $\delta' = 1$. If we set
$$B_t = f_* D_t,$$
then $K_X + \Delta_t$ is klt and nef, and f is $K_W + D_t$ non-positive for $t \in [0, 1]$. As D_t is big the base point free theorem inplies that $K_X + B_t$ is semiample and so there is an induced contraction morphism $g_i : X \to X_i$ together with ample divisors $H_{1/2}$ and H_1 such that
$$K_X + B_{1/2} = g_i^* H_{1/2}, K_X + B_1 = g_i^* H_1$$
If we set
$$H_t = (2t - 1)H_1 + 2(1 - t)H_{1/2}$$
then
$$\begin{aligned} K_X + B_t &= (2t - 1)(K_X + B_1) + 2(1 - t)H_{(1/2)} \\ &= (2t - 1)g_i^* H_1 + 2(1 - t)g_i^* H_{1/2} \\ &= g_i^* H_t \end{aligned}$$
for all $t \in [0, 1]$. As $K_X + B_0$ is semiample, it follows that H_0 is semiample and the associated contraction $f_{i,j} : X_i \to X_j$ is the required morphism.

3. Suppose that $\mathcal{C}$ spans V. Pick D in the interior of $\mathcal{C} \cap \mathcal{A}_i$. Let $f : W \dashrightarrow X$ be a log termianl model of $(K_W + D)$, then $f = f_j$ for some index $1 \leqslant j \leqslant k$ and that $D \in \mathcal{C}_j$. But then $A_i \cap \mathcal{A}_j \neq \emptyset$ so that $i = j$. If f_i is a log terminal model of $K_W + D$ then f_i is birational and X is $\mathbb{Q}$-factorial. Finally suppose that f_i is birational and X_i is $\mathbb{Q}$-factorial. Fix $D \in \mathcal{A}_i$. Pick any divisor $G \in V$ such that $-G$ is ample and $K_{X_i} + f_{i*}(D + G)$ is ample and $D + G \in \mathcal{L}_A(V)$. Then f_i is $(K_W + D + G)$-negative and so $D + G \in \mathcal{A}_i$. But $\mathcal{C}_i$ spans V, this implies (3).

4. Let $f = f_i$ and $X = X_i$. As $\mathcal{C}_i$ spans V, (3) implies that f is birational and X is $\mathbb{Q}$-factorial so that f is a $\mathbb{Q}$-factorial weak log canonical model of $K_W + D$. Suppose that $E_1, E_2, \ldots, E_k$ are the divisors contracted by f. Pick $F_i \in V$ numerically equivalent to E_i. If we let $E_0 = \sum_{i=1}^k E_i$ and $F_0 = \sum_{i=1}^k F_i$, then E_0 and F_0 are numerically equivalent. As D belongs to ineterior of $\mathcal{L}_A(V)$ we may find $\delta > 0$ such that $K_W + D + \delta F_0$ and $K_W + D + \delta B_0$ are both klt. Then f is $(K_W + D + \delta E_0)$-negative and so f is a log terminal model of $(K_W + D + \delta E_0)$ and f_j is the ample model of $K_W + D + \delta B_0$. In paricular $D + \delta F_0 \in \mathcal{A}_j \cap \mathcal{C}_i$. As we are supposing that D is general in $\mathcal{A}_j \cap \mathcal{C}_i$, in fact f must be a log terminal model of $K_W + D$, and f is $(K_W + D)$-negative.

   Pick $\epsilon > 0$ such that if $G \in V$ and $||G - D|| < \epsilon$ then G belongs to the interior of $\mathcal{L}_A(V)$ and f is $(K_W + G)$-negative. Then $G \in \mathcal{C}_i$ simply means $K_X + H = f_*(K_W + G)$ is nef. Let V_X be the affine supspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ given by pushing forward the elements of V and let
$$\mathcal{N} = \{H \in V_X : K_X + H \text{ is nef }\}.$$

Given $(a_1, \dots a_k) \in \mathbb{R}^k$ and let $F = \sum a_i F_i$ and $E = \sum a_i E_i$. If $||F|| < \epsilon$ then $K_X + H \in \mathcal{N}$ if and only if $K_X + H + f_* F \in \mathcal{N}$. In particular $\mathcal{C}_i$ is locally isomorphic to $\mathcal{N} \times \mathbb{R}^k$.

But since f_j is the ample model of $K_W + D$, in fact we can choose ϵ sufficiently small such that $K_X + H$ is nef if and only if $K_X + H$ is nef over X_j. There is a surjective affine linear map from V_X to the space of Weil divisor on X modulo numerical equivalence over X_j and this induces an isomorphism

$$\mathcal{N} \cong \overline{\mathrm{NE}}(X/X_j)^* \times \mathbb{R}^l,$$

in a neighbourhood of $f_* D$.

Note that $K_X + f_* D$ is numerical trival over X_j. As $f_* D$ is big and $K_X + f_* D$ is klt we may find an ample $\mathbb{Q}$-divisor A' and a divisor $B' \geqslant 0$ such that

$$K_X + A' + B' \sim_{\mathbb{R}} K_X + f_* D$$

is klt. But then

$$-(K_X + B') \sim_{\mathbb{R}} -(K_X + H) + A'$$

is ample over X_j. Hence $f_{ij} : X \to X_j$ is a Fano fibration and so by cone theorem

$$\rho(X_i/X_j) = \dim \mathcal{N}$$

This is (4).

□

Lemma 5.1.2. *[8, Corollary 3.4]If V spans $\mathrm{NS}(W)$, then there is a Zariski dense open subset U of the Grassmannian $G(r, V)$ of real affine subspace of dimension r such that any $[V'] \in U$ defined on ratinal numbers statify (1-4) of 5.1.1*

Proof. Let $U \subset G(r, V)$ be the set of real affine subspace V' of V of dimension r, which contion any sub no face of any $\mathcal{C}_i$ or $\mathcal{L}(V)$. In particular, the interior of $\mathcal{L}_A(V')$ is contained in the interior of $\mathcal{L}_A(V)$. Clearly that any $V' \in U$ satisfies (1-4) of 5.1.1. □

From now on in this subsection, we always assume that V has dimension 2 and statisfies 5.1.1.

Lemma 5.1.3. *[8, Lemma 3.5] Let $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ be two rational contracions such that $\mathcal{C}_{A,f}$ is dimension 2 and $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$ is dimension 1. Assume $\rho(X) \geqslant \rho(Y)$ and $\mathcal{O}$ is not contained in the boundary of $\mathcal{L}_A(V)$. Let D be an interior point of $\mathcal{O}$ and $B = f_* D$. Then there is a rational contraction $\pi : X \dashrightarrow Y$ and $g = \pi \circ f$ such that either*

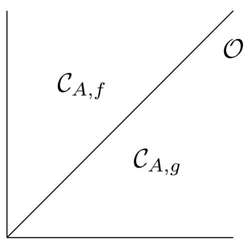
1. *$\rho(X) = \rho(Y) + 1$ and π is $(K_X + B)$-trival, and either*
   (a) *π if birational and $\mathcal{O}$ is not contained in the boudary of $\mathcal{E}_A(V)$, and either*
      i. *π is a divisorial contraction and $\mathcal{O} \neq \mathcal{C}_{A,g}$, or*
      ii. *π is a small contraction and $\mathcal{O} = \mathcal{C}_{A,g}$*

   or

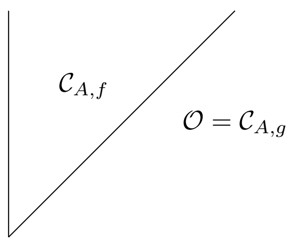
   (b) *π is a Mori fibre space, and $\mathcal{O} = \mathcal{C}_{A,g}$ is contained in the boundary of $\mathcal{E}_A(V)$*

   or

2. *$\rho(X) = \rho(Y)$, and π is a $(K_X + B)$-flop and $\mathcal{O} \neq \mathcal{C}_{A,g}$ is not contained in the boundary of $\mathcal{E}_A(V)$.*



or



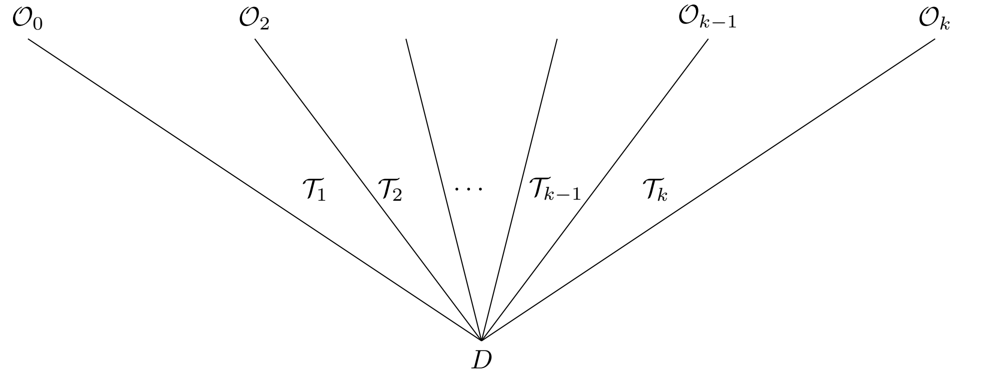
Proof. By assumption f is birational and X is $\mathbb{Q}$-factorial. Let $h : W \dashrightarrow S$ be the ample model corresponding to $K_W + D$. Since D is not a point of the boundary of $\mathcal{L}_A(V)$, if D belongs to the boundary of $\mathcal{E}_A$ then $K_W + D$ is not big and so h is not birational. As $\mathcal{O}$ is a subset of both $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ there are morphisms $p : X \to S$ and $q : Y \to S$ of relative Picard number at most one. There are therefore only two cases

1. $\rho(X) = \rho(Y) + 1$, or
2. $\rho(X) = \rho(Y)$

Suppose we are in the first case, then q is the identity and $\pi : X \to Y$ is a contraction morphism such that $g = p \circ f$. Suppose that π is birational, then h is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$. If π is divisorial then Y is $\mathbb{Q}$-factorial and so $\mathcal{O} \neq \mathcal{C}_{A,g}$. If π is a small contraction then π is not $\mathbb{Q}$-factorial and so $\mathcal{C}_{A,g} = \mathcal{O}$ is one dimensional. If π is a Mori fibre space then $\mathcal{O}$ is contgained in the boundary of $\mathcal{E}_A(V)$ and $\mathcal{O} = \mathcal{C}_{A,g}$.

Now suppose we are in the second case. Since $\rho(X/S) = \rho(Y/S) = 1$, we know that p, q are not divisoriacontractions as $\mathcal{O}$ is one dimensinal and p, q are not Mori fibre spaces as $\mathcal{O}$ is cannot be contained in the boundary of $\mathcal{E}_A(V)$. Hence p, q are small and the the rest is clear. □

Let $D = A + B$ be a point of boundary of $\mathcal{E}_A(V)$ in the interior of $\mathcal{L}_A(V)$. Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be the polytopes $\mathcal{C}_i$ of dimension 2 containing D. Let $\mathcal{O}_0$ and $\mathcal{O}_k$ be the intersection of $\mathcal{T}_0$ and $\mathcal{T}_k$ with boundary of $\mathcal{E}_A(V)$, and let $\mathcal{O}_i = \mathcal{O}_i \cap \mathcal{O}_{i+1}$. Let $f_i : W \to X_i$ be the rational contraction associated to $\mathcal{T}_i$ and $g_i : W \to S_i$ be the rational contraction associated to $\mathcal{O}_i$.



Set $f = f_1 : W \dashrightarrow X, g = f_k : W \dashrightarrow Y$ and $\phi : X \to S = S_0, \psi : Y \to T = S_k$ and $X' = X_2, Y' = X_{k-1}$ and let $W \dashrightarrow R$ be the ample model of D. Then

Theorem 5.1.4. *[8, Theorem 3.7] Suppose B_W is a divisor such that $K_Z + B_W$ is klt and $D - B_W$ is ample. Then ϕ and ψ are Mori fibre spaces as outputs of $(K_Z + B_W)$-MMP and connected by a Sarkisov link if D is contained in more than two polytopes.*

Proof. WMA $k \geqslant 3$ and we have

$$\begin{tikzcd}
X' \arrow[rr, dotted] \arrow[d, dotted, "p"'] & & Y' \arrow[d, dotted, "q"] \\
X \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
S \arrow[dr, "s"] & & T \arrow[dl, "t"'] \\
& R &
\end{tikzcd}$$

Note that $\rho(X_i/R) \leqslant 2$ and $\rho(X/S) = \rho(Y/T) = 1$. Thus

1. s is identity and p is a divisorial contraction (extraction), or
2. s is a contraction and p is a flop.

The same holds for q and t. And the map $X' \to Y'$ is clear the composition of flops. This gives 4 types of links. $\square$

5.2 Construction of Sarkisov links

Lemma 5.2.1. *[8, Lemma 3.6] Let $f : W \dashrightarrow X$ be a birational contraction between $\mathbb{Q}$-factorial varieties. Suppose (W, D) and $(W, D + A)$ are both klt. If f is ample model of $(W, D + A)$ and A is ample, then f is result of running $(K_W + D)$-MMP.*

This lemma gunrantee that every variety in the Sarkisov links constructed later is a MMP result of (W, B_W). We need a special resolution W and an affine subspace $V \subset \mathrm{WDiv}(W)$ such that we can find two Mori fibre spaces X/S and Y/T and vertexs connecting them. The following lemma shows the desired affine subspace exits.

Lemma 5.2.2. *[8, Lemma 4.1] Let $\phi : X \to S$ and $\psi : Y \to T$ be two MMP related Mori fibre space corresponding to two klt projective varieties (X, B_X) and (Y, B_Y). Then we may find a smooth projective variety W, two biratinal morphism $f : W \to X$ and $g : W \to Y$, a klt pair (W, B_W), an ample $\mathbb{Q}$-divisor A on W and a two dimensional rational affine subspace V of $\mathrm{WDiv}_{\mathbb{R}}(W)$ such that*

1. *If $D \in \mathcal{L}_A(V)$ then $D - B_W$ is ample;*
2. *$\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$;*
3. *V satisfy 5.1.1;*
4. *$\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are two dimensional;*
5. *$\mathcal{C}_{A,\phi\circ f}$ and $\mathcal{C}_{A,\psi\circ g}$ are one dimensional.*

Proof. By assumption there is a $\mathbb{Q}$-factorial klt pair (W, B_W) such that $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$are both outcomes of $(K_W + B_W)$-MMP. Let $p' : W' \to W$ be any log resolution such that resolves the indeterminacy of f and g, then we may write

$$K_{W'} + B_{W'} = p'^*(K_W + B_W) + E'$$

where $E' \geqslant 0$ and $B_{W'} \geqslant 0$ have no common components, and E' is exceptional and $p'_*B_{W'} = B_W$. Pick a divisor $-F$ which is ample over W with support equal to the full exceptional locus such that $K_{W'} + B_{W'} + F$ is klt. As p' is $(K_{W'}B_{W'} + F)$-negative and $(K_W + B_W)$ is klt and W is $\mathbb{Q}$-factorial, the $(K_{W'} + B_{W'} + F)$-MMP over W terminates with the pair (W, B_W). Replacing (W, B_W) by $(W', B_{W'} + F)$ we may assume that (W, B_W) is log smooth and f, g are morphisms.

Pick general ample $\mathbb{Q}$-divisors $A, H_1, H_2, \ldots, H_k$ on W such that $H_1, \ldots, H_k$ generate te Neron-Seberi group of W. Let

$$H = A + H_1 + \ldots H_k$$

Pick sufficiently ample divisor A_S on S and A_T on T such that

$$-(K_X + B_X) + \phi^* A_S \text{ and } -(K_Y + B_Y)\psi^* A_T$$

are both ample. Pick a rational number $0 < \delta < 1$ such that

$$-(K_X + B_X + \delta f_* H) + \phi^* A_S \text{ and } -(K_Y + B_Y + \delta g_* H) + \psi^* A_T$$

are both ample and $(K_W + B_W + \delta H)$ is both f and g negative. Replacing H by δH we may assume that $\delta = 1$. Now pick a $\mathbb{Q}$-divisor $B_0 \leqslant B_W$ such that $A + (B_0 - B_W), -(K_X + f_* B_0 + f_* H) + \phi^* A_S$ and $-(K_Y + g_* B_0 + f_* H) + \psi^* A_T$ are all ample and $(K_W + B_0 + H)$ is both f and g negative.

Pick general ample $\mathbb{Q}$-divisors $F_1 \geqslant 0$ and $G_1 \geqslant 0$ such that

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_* B_0 + f_* H) + \phi^* A_S \text{ and } G_1 \sim_{\mathbb{Q}} -(K_Y + g_* B_0 + g_* H) + \psi^* A_T$$

and

$$K_W + B_0 + H + F + G$$

is klt, where $F = f^* F_1$ and $G = g^* G_1$.

Let V_0 be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ which is the tanslate by B_0 of the vector subspace spaned by $H_1, \ldots, H_k, F, G$. Supppose that $D = A + B \in \mathcal{L}_A(V_0)$. Then

$$D - B_W = (A + B_0 - B_W) + (B - B_0)$$

is ample, as $B - B_0$ is nef by definition of V_0. Note the

$$B_0 + F + H \in \mathcal{A}_{A,\phi\circ f}(V_0), B_0 + G + H \in \mathcal{A}_{\psi\circ g}(V_0)$$

and f, respectively g, is a weak log canonical model of $K_W + B_0 + F + H$, respectively $K_W + B_0 + G + H$. Thus theorem 5.1.1 implies that V_0 satisfies (1-4) of 5.1.1.

Since $H_1, \ldots, H_k$ generated the Neron-Severi group of W we may find constants $h_1, \ldots, h_k$ such that $G \equiv \sum_{i=1}^k h_i H_i$. Then there is $0 < \delta \ll 1$ such that $B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \in \mathcal{L}_A(V_0)$ and

$$B_0 + F + \delta G + H - \delta(\sum_i^k h_i H_i) \equiv B_0 + F + H.$$

Thus $\mathcal{A}_{A,\phi\circ f}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. Similarly $\mathcal{A}_{A,\psi\circ g}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. In particular $\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ span affine hyperplanes of V_0, since $\rho(X) = \rho(Y) = 1$.

Let V_1 be the tranlate by B_0 of two dimensional bector space spaned by $F + H - A$ and $F + G - A$. Let V be a small general pertubation of V_1, which is defined over rationals. This is the affine subspace we need. □

Then we can prove the main theorem

Proof of 1.1.2. Let $(W, B_W), A$ and V as in the lemma 5.2.2. Pick $D_0 \in \mathcal{A}_{A,\phi\circ f}$ and $D_1 \in \mathcal{C}_{A,g}$ belonging to the interior of $\mathcal{L}_A(V)$. As V is two dimensional, removing D_0 and D_1 divides the boundary of $\mathcal{E}_A(V)$ into two parts. The part which consists entirely of divisors which are not big is contained in the interior of $\mathcal{L}_A(V)$. Consider tracing this boundary from D_0 to D_1. Then there are finitely many $2 \leqslant i \leqslant N$ points D_i which are contained in more than two polytopes $\mathcal{C}_{A,f_i}(V)$. By lemma 5.1.4, each point D_i gives a Sarkisov link. And the birational map $X \dashrightarrow Y$ is composition of such links. □

6 Application

References

[1] Caucher Birkar. Ascending chain condition for log canonical thresholds and termination of log flips. Duke Mathematical Journal, 136(1), January 2007.

[2] Caucher Birkar. Anti-pluricanonical systems on Fano varieties, April 2019.

[3] Caucher Birkar. Singularities of linear systems and boundedness of Fano varieties, December 2020.

[4] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James M^cKernan. Existence of minimal models for varieties of log general type. Journal of the American Mathematical Society, 23(2):405–468, November 2009.

[5] Andrea Bruno and Kenji Matsuki. Log Sarkisov Program, February 1995.

[6] Alessio Corti. Factoring birational maps of threefolds after Sarkisov. Journal of Algebraic Geometry, 1995(4):223–254, 1995.

[7] Christopher D. Hacon. The Minimal model program for Varieties of log general type. Wiadomości Matematyczne, 48(2):49, June 2012.

[8] Christopher D. Hacon and James M^{c}Kernan. The Sarkisov program, June 2011.

[9] Yujiro Kawamata. Flops Connect Minimal Models. Publications of the Research Institute for Mathematical Sciences, 44(2):419–423, 2008.

[10] Jihao Liu. Sarkisov program for generalized pairs, September 2019.