

# A NOTE ON THE SARKISOV PROGRAM

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*Dedicated to Professor Vyacheslav V. Shokurov on the occasion of his seventieth birthday*

ABSTRACT. The purpose of the note is to introduce three methods of Sarkisov program, which aims to factorize birational maps of log Mori fibre spaces.

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## 1. INTRODUCTION

The Minimal model program (MMP) aims to classify varieties up to birational equivalence. Conjectually, any variety is either birational to a minimal model or a Mori fibre space. The representative in each birational class is possibly not unique. It is natural to ask what is the relation of representatives of a birational class. Kawamata shows that two different minimal models can be linked by flops as follows:

**Theorem 1.1.** [Kaw08, Theorem 1] *Let  $(W, B_W)$  be a  $\mathbb{Q}$ -factorial terminal pair, and  $(X, B), (Y, D)$  are two minimal models of  $(W, B_W)$ . Then the birational map  $X \dashrightarrow Y$  may be factored as sequence of  $(K_X + B)$  flops.*

For Mori fibre spaces, Sarkisov program shows:

**Theorem 1.2.** *Let  $f : (X, B) \rightarrow S$  and  $f' : (X', B') \rightarrow S'$  be two MMP related  $\mathbb{Q}$ -factorial klt log Mori fibre spaces with the induced birational map  $\Phi$ :*

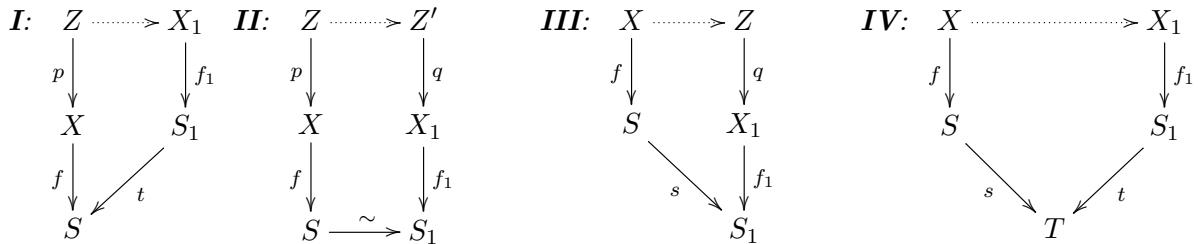
$$\begin{array}{ccc} (X, B) & \xrightarrow{\Phi} & (X', B') \\ f \downarrow & & \downarrow f' \\ S & & S' \end{array}$$

*Then modulo isomorphisms,  $\Phi$  can be decomposed into a sequence of the following four types of Sarkisov links:*

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*Date:* December 26, 2022.

*1991 Mathematics Subject Classification.* Primary 14E30, Secondary 14B05.



where all  $f : (X, B) \rightarrow S$  and  $f_1 : (X_1, B_1) \rightarrow S_1$  are log Mori fibre spaces, all  $p, q$  are divisorial contractions, and all dash arrows are composition of flips (or flops in Section 5).

The Sarkisov program has its origin in the birational classification of ruled surfaces [Sar82]. Reid [Reid91] explains the original idea of Sarkisov [Sar89]. The complete proof of Sarkisov program for terminal threefolds is given by Corti [Cor95]. Bruno and Matsuki [BM97] generalize it to klt pairs of dimension 3. Moreover, they outline the Sarkisov program for any dimensions. The original idea of Sarkisov has two parts: construction of Sarkisov links and termination of the program. The Sarkisov links are outcomes of 2-ray game of minimal model program. Sarkisov degree  $(\mu, \lambda, e)$  is designed for termination. The invariant  $\mu$  is related to boundedness of Fano varieties. The invariants  $\lambda$  and  $e$  are related to the singularities of linear systems. As important advances in minimal model program, such as MMP with scaling terminates by [BCHM], ACC of lc [HMX14], boundedness of Fano varieties [Bir19], [Bir21], only the termination of Sarkisov program of dimension greater than 3 is left open, which is related to the termination of flips. We call the method as original method, and introduce it in Section 3.

Due to the finiteness of weak log canonical models in [BCHM], Hacon [Hac12] gives another method for Sarkisov program, which is called double scaling and it terminates in all dimensions. Liu [Liu21] generalizes Hacon's method to generalized pairs. The method and the original method are running minimal model program. The difference is that the method of double scaling has a common roof, and each step of Sarkisov link is under the common roof, which makes MMP running in the case is in a smaller collection compared with the original method. Hence the method of double scaling can terminate. We call the method as double scaling and introduce it in Section 4.

By using the idea of Shokurov's polytope [Sho96], [CS11], Hacon and McKernan [HM13] give another method of Sarkisov program. Miyamoto [Miy19] uses the method to show Sarkisov program works for lc log surfaces or  $\mathbb{Q}$ -factorial log surfaces over an algebraically closed of any characteristic. We call the method as polytope method and introduce it in Section 5.

In Section 6, we give examples to illustrate each methods of Sarkisov program.

Sarkisov program has many applications. Such as the classical result for Cremona group of rank 2. That is, any birational automorphism of projective plane is a composition of automorphism of projective plane and a quadratic transform (see [KSC04] Chapter 2). Takahashi [Tak95] applies log Sarkisov program to dlt surfaces and obtains another proof for the classical algebraic result: any automorphism of affine plane is the composition of linear transformations and upper triangular transformations (see [Mat02], Chapter 13). For more applications, we refer to the nice book of Lamy ([Lam22]).

## 2. PRELIMINARIES

Throughout the note, all varieties are over  $\mathbb{C}$ , the field of complex numbers.

**2.1. MMP.** We call the varieties appeared in minimal model program the **results** of the MMP, and the varieties where the MMP ends the **outputs** of the MMP.

**Definition 2.1.** Let  $(X, B)$  be a pair and let  $f : Y \rightarrow X$  be a log resolution of  $(X, B)$ . Suppose

$$K_Y + C = f^*(K_X + B),$$

then the discrepancy of exceptional divisor  $E_i$  over  $X$  is

$$a(E_i; X, B) = -\text{mult}_{E_i} C.$$

Moreover, let

$$\text{discrep}(X, B) := \inf\{a(E; X, B) : E \text{ is an exceptional divisor over } X\}$$

and

$$\text{totdiscrep}(X, B) := \inf\{a(E; X, B) : E \text{ is a divisor over } X\}.$$

**Theorem 2.2.** [BCHM, Corollary 1.4.2] *Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties, and let  $(X, B)$  be a  $\mathbb{Q}$ -factorial klt pair where  $K_X + B$  is  $\mathbb{R}$ -Cartier and  $B$  is  $\pi$ -big. Let  $C \geq 0$  be an  $\mathbb{R}$ -divisor. If  $K_X + B + C$  is klt and  $\pi$ -nef, then we may run  $(K_X + B)$ -MMP over  $U$  with scaling of  $C$  and terminates.*

**Theorem 2.3.** [BCHM, Corollary 1.3.3] *Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties, and let  $(X, B)$  be a  $\mathbb{Q}$ -factorial klt pair where  $K_X + B$  is  $\mathbb{R}$ -Cartier. If  $K_X + B + C$  is not  $\pi$ -pseudo-effective, then we may run  $f : X \dashrightarrow Y$  a  $(K_X + B)$ -MMP over  $U$  and end with a Mori fibre space  $g : Y \rightarrow Z$ .*

**Corollary 2.4.** [Hac12, Corollary 13.7] and [BCHM, Corollary 1.4.3]: *Let  $(X, B)$  be a klt pair and  $\mathfrak{C}$  be any set of exceptional divisors such that contains only exceptional divisors  $E$  of discrepancy  $a(E; X, B) \leq 0$ . Then there is a birational morphism  $f : Z \rightarrow X$  and a  $\mathbb{Q}$ -divisor  $B_Z$  such that:*

- (1)  $(Z, B_Z)$  is klt;
- (2)  $E$  is an  $f$ -exceptional divisor if and only if  $E \in \mathfrak{C}$ ;
- (3)  $\text{mult}_E B_Z = -a(E; X, B)$  if  $E \in \mathfrak{C}$ , and  $f_* B_Z = B$  and  $K_Z + B_Z = f^*(K_X + B)$ .

*In particular, if we take  $\mathfrak{C}$  containing all such divisors, then  $Z$  is called **terminalization** of  $X$ ; if take  $\mathfrak{C}$  containing only one such divisor, then  $f : Z \rightarrow X$  is called a **divisorial extraction**.*

**Definition 2.5.** [BM97, Definition 3.3] *Two or more pairs  $\{(X_i, B_i)\}$  are called **MMP-related** if they are results of  $(K + B)$ -MMP from a log smooth pair  $(W, B_W)$ .*

**Lemma 2.6.** [BM97, Proposition 3.4] *Let  $\{(X_l, B_l)\}$  be a finite set of  $\mathbb{Q}$ -factorial klt pairs such that birational to other, then the following are equivalent:*

- (1) *They are MMP-related;*
- (2) *There is a log smooth pair  $(W, B_W)$ , and projective birational morphisms  $f_l : W \rightarrow X_l$  dominating each  $X_l$ , such that  $f_{l*} B_W = B_l$  and*

$$K_W + B_W = f_l^*(K_{X_l} + B_l) + \sum_{\text{exceptional}} a_{li} E_{li}$$

*with  $a_{li} > 0$  for all  $f_l$ -exceptional divisors  $E_{li}$ ;*

- (3) *For any two pairs  $(X, B = \sum_i b_i B_i), (X', B' = \sum_j b'_j B'_j)$  in the set,  $a(B_i; X', B') \geq -b_i$  and strict inequality holds if and only if  $B_i$  exceptional over  $X'$ , and  $a(B'_j; X, B) \geq -b'_j$  and strict inequality holds if and only if  $B'_j$  exceptional over  $X$ .*

## 2.2. Models.

**Definition 2.7.** [HM13, §2] *A rational map  $f : X \dashrightarrow Y$  is called a **rational contraction** if there is a resolution  $p : W \rightarrow X$  and  $q : W \rightarrow Y$  of  $f$  such that  $p$  and  $q$  are contraction morphisms and  $p$  is birational.  $f$  is called a **birational contraction** if  $q$  is in addition birational and every  $p$ -exceptional divisor is  $q$ -exceptional. If in addition  $f^{-1}$  is also a **birational contraction**, then  $f$  is called a **small birational map**.*

**Definition 2.8.** [BCHM, Definition 3.6.1] *Let  $f : X \dashrightarrow Y$  be a birational map of normal quasi-projective varieties, and  $p : W \rightarrow X$  and  $q : W \rightarrow Y$  be a resolution of indeterminacy of  $f$ . Let  $D$  be a  $\mathbb{R}$ -Cartier divisor on  $X$  such that  $D_Y = f_* D$  is also  $\mathbb{R}$ -Cartier. Then  $f$  is called  **$D$ -non-positive** ( **respectively  $D$ -negative**) if*

- $f$  does not extract any divisor;
- $E = p^*D - q^*D_Y$  is effective and exceptional over  $Y$  ( respectively  $\text{Supp } p_*E$  contains all  $f$ -exceptional divisors).

Recall the definitions of models in [BCHM]

**Definition 2.9.** [BCHM, Definition 3.6.5] Let  $\pi : (X, D) \rightarrow U$  be a projective morphism of normal quasi-projective varieties and let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ . Let  $f : X \dashrightarrow Y$  be a birational map over  $U$ , then  $Z$  is an **semiample model** for  $D$  over  $U$  if  $f$  is  $K_X + D$ -non-positive and  $K_Y + f_*D$  is semiample over  $U$ .

Let  $g : X \dashrightarrow Z$  be a rational map over  $U$ , then  $Z$  is an **ample model** for  $D$  over  $U$  if there is an ample divisor  $H$  over  $U$  on  $Z$  such that if  $p : W \rightarrow X$  and  $q : W \rightarrow Z$  resolves  $g$ , then  $q$  is a contraction morphism and we may write  $p^*D \sim_{\mathbb{R}, U} q^*H + E$ , where  $E \geq 0$  and for any  $B \in |p^*D/U|_{\mathbb{R}}$ , then  $B \geq E$ .

**Definition 2.10.** [BCHM, Definition 3.6.7] Let  $\pi : (X, D) \rightarrow U$  be a projective morphism of normal quasi-projective varieties, if  $K_X + D$  is log canonical and  $f : X \dashrightarrow Y$  is a birational map extracts no divisors, then define:

- (1)  $Y$  is **weak log canonical model** for  $K_X + D$  over  $U$  if  $f$  is  $K_X + D$ -non-positive and  $K_Y + f_*D$  is nef over  $U$ ;
- (2)  $Y$  is **log canonical model** for  $K_X + D$  over  $U$  if  $f$  is  $K_X + D$ -non-positive and  $K_Y + f_*D$  is ample over  $U$ ;
- (3)  $Y$  is **log terminal model** for  $K_X + D$  over  $U$  if  $f$  is  $K_X + D$ -negative and  $K_Y + f_*D$  is dlt and nef over  $U$  and  $Y$  is  $\mathbb{Q}$ -factorial.

**Lemma 2.11.** [BCHM, Lemma 3.6.6] Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties and let  $D$  be an  $\mathbb{R}$ -Cartier divisor on  $X$ .

- (1) If  $g_i : X \dashrightarrow X_i, i = 1, 2$  are two ample models of  $D$  over  $U$ , then there is an isomorphism  $h : X_1 \rightarrow X_2$  and  $g_2 = h \circ g_1$ .
- (2) If  $f : X \dashrightarrow Y$  is a semiample model of  $D$  over  $U$ , then the ample model  $g : X \dashrightarrow Z$  of  $D$  over  $U$  exists and  $g = h \circ f$ , where  $h : Y \rightarrow Z$  is a contraction and  $f_*D \sim_{\mathbb{R}, U} h^*H$ . Here  $H$  is the ample divisor corresponding to the ample model  $Z$ .
- (3) If  $f : X \dashrightarrow Y$  is a birational map over  $U$ , then  $f$  is the ample model of  $D$  over  $U$  if and only if  $f$  is a semiample model of  $D$  over  $U$  and  $f_*D$  is ample over  $U$ .

By above lemma there is another definition of log canonical models:

**Definition 2.12.** Let  $\pi : (X, D) \rightarrow U$  be a projective morphism of normal quasi-projective varieties,  $K_X + D$  log canonical and  $f : X \dashrightarrow Y$  a birational map extracts no divisors, then  $Y$  is **log canonical model** if it is the ample model.

Furthermore, for big boundaries, we have

**Lemma 2.13.** [BCHM, Lemma 3.9.3] Let  $\pi : (X, B) \rightarrow U$  be a projective morphism of normal quasi-projective varieties. Suppose  $(X, B)$  is a klt pair and  $B$  is big over  $U$ . If  $f : X \dashrightarrow Y$  is a weak log canonical model over  $U$  then

- $f$  is a semiample model over  $U$ ;
- the ample model  $g : X \dashrightarrow Z$  over  $U$  exists;
- there is a contraction  $h : Y \rightarrow Z$  such that  $K_Y + f_*B \sim_{\mathbb{R}, U} h^*H$  for some ample  $\mathbb{R}$ -divisor  $H$  on  $Z$  over  $U$ .

**Definition 2.14.** [BCHM, Definition 1.1.4] Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties, and let  $V$  be a finite dimensional affine subspace of  $\text{WDiv}_{\mathbb{R}}(X)$  defined over rational numbers. Fix an  $\mathbb{R}$ -divisor  $A \geq 0$ , and then define

$$\begin{aligned} \mathcal{L}_A(V) &= \{D = A + B : B \in V, K_X + D \text{ is log canonical and } B \geq 0\} \\ \mathcal{E}_{A, \pi}(V) &= \{D \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\} \end{aligned}$$

Given a birational contraction  $f : X \dashrightarrow Y$ , define

$$\mathcal{W}_{A,\pi,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is a weak log canonical model of } (X, D) \text{ over } U\}$$

Given a rational contraction  $g : X \dashrightarrow Z$  over  $U$ , define

$$\mathcal{A}_{A,\pi,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is the ample model of } (X, D) \text{ over } U\}$$

In addition, let  $\mathcal{C}_{A,\pi,g}(V)$  denote the closure of  $\mathcal{A}_{A,\pi,g}(V)$  in  $\mathcal{L}_A(V)$ .

If the base  $U$  is clear or it is a point, then we may omit  $\pi$  and simply write  $\mathcal{E}_A(V)$  and  $\mathcal{A}_{A,f}$ .

**Theorem 2.15** (Finiteness of weak log canonical models, [BCHM] Theorem E). *Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties, and  $A$  be a general divisor relatively ample over  $U$ , and  $V \subset \text{WDiv}_{\mathbb{R}}(X)$  be a finite dimensional rational subspace. Suppose that there is a klt pair  $(X, \Delta_0)$ . Then there are finitely many birational maps  $f_i : X \dashrightarrow X_i$  such that if  $f : X \dashrightarrow Y$  is a weak log canonical model of  $K_X + D$  over  $U$  for some  $D \in \mathcal{L}_A(V)$ , then there is an isomorphism  $h_i : X_i \rightarrow Y$  and  $f = h_i \circ f_i$ .*

### 3. ORIGINAL METHOD

Suppose  $f : (X, B) \rightarrow S'$  and  $f' : (X', B') \rightarrow S'$  are two Mori fibre spaces as outputs of  $(K_W + B_W)$ -MMP on  $W$ . The Sarkisov program constructs each Sarkisov link  $X_i \dashrightarrow X_{i+1}$  inductively. Each Sarkisov link is given by running a special MMP called 2-ray game. Take an ample divisor  $A'$  on  $S'$  such that  $H' \sim -\frac{1}{\mu'}(K_{X'} + B') + f'^*A'$  is ample, then  $(X', B' + \frac{1}{\mu}H')$  is a weak log canonical model. Therefore we expect that  $X'$  is output of certain MMP from  $X$ . Let  $H$  be the strict transform of  $H'$  on  $X$ , then we run  $(K_X + B + cH)$ -MMP for maximum  $c$  such that

- (1)  $K_X + B + cH$  is non-positive over  $S$ , and
- (2)  $K_X + B + cH$  is log canonical ( $\theta$ -canonical).

Then we construct the Sarkisov link in following cases:

- (1) If  $c$  is not threshold of second condition, find a contraction  $g_i : X_i \rightarrow T_i$  such that  $\rho(X_i/T_i) = 2$  and factor through  $f_i : X_i \rightarrow S_i$ . We run MMP on  $X_i$  over  $T_i$ , and obtains a Sarkisov link of type III or IV;
- (2) If  $c$  is threshold of second condition, then find a divisorial extraction  $p_i : Z_i \rightarrow X_i$ , and then  $\rho(Z_i/S_i) = 2$ . We run MMP on  $Z_i$  over  $S_i$ , and obtains a Sarkisov link of type I or II.

We run the whole program in a special collection  $\mathcal{C}_\theta$  (Proposition 3.3). In this collection we can define Sarkisov degree, which decreases after composing a Sarkisov link  $\psi_i : X_i \dashrightarrow X_{i+1}$ .

To show termination of this program, we need boundedness of fano varieties, termination of flips and chain condition for singularities in this collection. Unfortunately these only hold for klt 3-folds.

**3.1. Prepare.** First we fix a collection:

Let  $K = K(X)$  be the function field, and let  $\Sigma = \{\nu\}$  be the set of discrete valuations of the field.

**Definition 3.1.** [BM97, Definition 3.5] *Let  $\theta : \Sigma \rightarrow [0, 1]_{\mathbb{Q}}$  be a function. Then we can define a collection  $\mathcal{C}_\theta$  of pairs associated to  $\theta$ , consists of klt pairs  $(X, B = \sum a_i B_i)$  satisfying*

- (1)  $a_i = \theta(B_i)$ ;
- (2)  $a(E; X, B) > -\theta(E)$  for all  $E$  exceptional over  $X$ .

For example, if we take  $\theta \equiv 0$  constant, the  $\mathcal{C}_\theta$  is the collection of all terminal varieties  $Y$  without boundary birational to  $X$ . Furthermore, we can define the corresponding discrepancy:

**Definition 3.2** ( $\theta$ -discrepancy). Let  $\mathcal{C}_\theta$  be a collection of varieties as above. Let  $(X, B)$  be a pair with function field  $K(X) = K$ . Let  $f : Y \rightarrow X$  be a log resolution of  $(X, B)$ . Suppose

$$K_Y + B_Y + C = f^*(K_X + B)$$

where  $B_Y = (f^{-1})_*B + \sum_{E_i \text{ exc}} \theta(E_i)E_i$ , then the  $\theta$ -discrepancy of exceptional divisor  $E_i$  over  $X$  is

$$a_\theta(E_i; X, B) = -\text{mult}_{E_i} C.$$

Or equivalently, we have

$$a_\theta(E_i; X, B) = a(E_i; X, B) + \theta(E_i).$$

A pair  $(X, B)$  is called  $\theta$ -canonical ( $\theta$ -terminal) if  $a_\theta(E; X, B) \geq 0$  ( $a_\theta(E; X, B) > 0$ ) for all exceptional divisors  $E$  over  $X$ . Note that  $\theta$ -canonical pair is not always in  $\mathcal{C}_\theta$ .

In our problem, there is a collection corresponds to the two log Mori fibre spaces:

**Proposition 3.3.** [BM97, Lemma 3.6] Let  $f : (X, B) \rightarrow S, f' : (X', B') \rightarrow S'$  be two  $\mathbb{Q}$ -factorial log Mori fibre spaces with only klt singularities and MMP-related, inducing a birational map  $\Phi$ :

$$\begin{array}{ccc} (X, B) & \xrightarrow{\Phi} & (X', B') \\ f \downarrow & & \downarrow f' \\ S & & S' \end{array}$$

Suppose  $B = \sum_i b_i B_i + \sum_j d_j D_j$  and  $B' = \sum_j d'_j D_j + \sum_k b'_k B'_k$ , where  $B_i$  are divisors on  $X$  but not on  $X'$ ,  $B'_k$  are divisors on  $X'$  but not on  $X$ , and  $D_j$  are divisors on both  $X$  and  $X'$ . By Lemma 2.6,  $d_j = d'_j$ . Take a rational number  $\epsilon < 1$  such that  $\epsilon > -\text{totdiscrep}(X, B), -\text{totdiscrep}(X', B')$ , and take the function  $\theta : \{\nu\} \rightarrow [0, 1]_{\mathbb{Q}}$  as following:

- $\theta(B_i) = b_i, \theta(D_j) = d_j, \theta(B'_k) = b'_k$ ;
- $\theta(E) = \epsilon$  if  $E$  is exceptional over both  $X$  and  $X'$ ;
- $\theta(D) = 0$  if  $D$  is a divisor on both  $X$  and  $X'$ , but not a component of  $B$  or  $B'$ .

Then the collection  $\mathcal{C}_\theta$  satisfies

- (1)  $(X, B)$  and  $(X', B')$  belongs to  $\mathcal{C}_\theta$ ;
- (2) For any finitely many klt pairs  $\{(X_l, B_l)\}$  in  $\mathcal{C}_\theta$ , there is an object  $(Z, B_Z) \in \mathcal{C}_\theta$  and projective birational morphisms  $Z \rightarrow X_l$  dominating each  $X_l$  as a process of  $(K_Z + B_Z)$ -MMP over  $X_l$  (thus over  $\text{Spec } \mathbb{C}$ );
- (3) Any  $(K + B)$ -MMP starting from an object in  $\mathcal{C}_\theta$  stays inside of  $\mathcal{C}_\theta$ , and so does any  $(K + B + cH)$ -MMP where  $H$  is base point free and  $c \in \mathbb{Q}_{>0}$ .

**Remark 3.4.** Let  $\delta = 1 - \epsilon$ , then all pairs in  $\mathcal{C}_\theta$  are  $\delta$ -lc.

With notations and assumptions in Proposition 3.3, we shall define the Sarkisov degree. Take a very ample divisor  $A'$  on  $S'$  and a sufficiently large and divisible integer  $\mu' > 1$  such that

$$\mathcal{H}' = |-\mu'(K_{X'} + B') + f'^*A'|$$

is a very ample complete linear system on  $X'$  over  $\text{Spec } \mathbb{C}$ . Let  $(W, B_W)$  be a common log resolution of  $X$  and  $X'$  in  $\mathcal{C}_\theta$  with projective birational morphism  $\sigma : W \rightarrow X, \sigma' : W \rightarrow X'$  and  $\sigma_*B_W = B, \sigma'_*B_W = B'$ . Let  $\mathcal{H}_W := \sigma'^*\mathcal{H}'$  and then  $\mathcal{H} := (\Phi^{-1})_*\mathcal{H}' = \sigma_*\mathcal{H}_W$ . Furthermore, if  $\mathcal{H}$  is not base point free, then

$$\sigma^*\mathcal{H} = \mathcal{H}_W + F$$

where  $F = \sum f_l F_l \geq 0$  is the fixed part. Take a general member  $H'$  of the linear system  $\mathcal{H}'$  such that  $H_W := \sigma'^*H' = (\sigma'^{-1})_*H' \in \mathcal{H}_W$ , and let  $H := (\Phi^{-1})_*H' = \sigma_*H_W$ , then  $H$  is  $f$ -ample and  $\sigma^*H = H_W + F$ . By taking further resolution, we may assume  $H_W$  is smooth and crosses normally with exceptional locus of  $\sigma$  and  $\sigma'$ .

Now we can define the Sarkisov degree in  $\mathcal{C}_\theta$  with respect to  $H'$  (or  $\mathcal{H}'$ ):

**Definition 3.5.** [BM97, Definition 3.8] *Sarkisov degree of  $(X, B)$  with respect to  $H$  (or  $\mathcal{H}$ ) in  $\mathcal{C}_\theta$  is a triple  $(\mu, \lambda, e)$  ordered lexicographically:*

- **Nef threshold  $\mu$ :** Let  $C \subset X$  be a curve contracted by  $f$ , then

$$\mu := -\frac{H.C}{(K_X + B).C}$$

i.e.  $K_X + B + \frac{1}{\mu}H \equiv_S 0$ ;

- **$\theta$ -canonical threshold  $c$  and  $\lambda$ :**  $\lambda = 0$  if  $\mathcal{H}$  is base point free; otherwise,

$$c := \frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geq 0, E \text{ exceptional over } X\}$$

- **Number of  $(K_X + B_X + \frac{1}{\mu}H)$ -crepant divisors:** Let  $e = 0$  if  $\mathcal{H}$  is base point free (and hence  $\lambda = 0$ ), otherwise

$$e = \#\{E; E \text{ is } \sigma\text{-exceptional and } a_\theta(E; X, B + \frac{1}{\lambda}H) = 0\}$$

**Remark 3.6.** (1) The Sarkisov degree is dependent on the choice of  $A', H'$  and  $\theta$ .

- (2) Take a common log resolution  $(W, B_W) \in \mathcal{C}_\theta$  with  $B_W = \sum \theta(E)E$  and projective birational morphisms  $\sigma : W \rightarrow X$ ,  $\sigma' : W \rightarrow X'$ . Since  $\sigma^*\mathcal{H} = \mathcal{H}_W + \sum f_l F_l$ , we have ramification formula:

$$K_W + B_W + tH_W = \sigma^*(K_X + B + tH) + \sum (a_l - t f_l) E_l$$

where  $\sum a_l E_l$  is effective and supported on  $\text{Exc } \sigma$ . Then  $\lambda := \max\{\frac{f_l}{a_l}\}$ . If  $\mathcal{H}$  is base point free, then  $\sum f_l F_l = 0$  and  $\lambda = 0$ .

- (3)  $e$  is the number of components in  $\sum (a_l - c f_l) E_l$  with coefficient 0 in the formula

$$K_W + B_W + \frac{1}{\lambda} H_W = \sigma^*(K_X + B + \frac{1}{\lambda} H) + \sum (a_l - \lambda f_l) E_l.$$

Such prime divisors  $E_1, \dots, E_e$  are called  $(K_X + B_X + \frac{1}{\lambda}H)$ - $\theta$ -crepant.

We also need some extraction maps in this collection

**Lemma 3.7.** Using the notation in the definition of Sarkisov degree, then there is a contraction  $f : Z \rightarrow X$  such that

- $(Z, B_Z) \in \mathcal{C}_\theta$  and  $(Z, B_Z + \frac{1}{\lambda}H_Z)$  is  $\theta$ -terminal and  $\mathbb{Q}$ -factorial;
- $\rho(Z) = \rho(X) + 1$ ;
- $f$  is  $(K_X + B_X + \frac{1}{\lambda}H_X)$ -crepant, that is

$$K_Z + B_Z + \frac{1}{\lambda}H_Z = f^*(K_X + B + \frac{1}{\lambda}H).$$

*Proof.* We follow the proof in [BM97, Proposition 1.6] but for klt pair case. Let  $(W, B_W) \in \mathcal{C}_\theta$  and  $\sigma : W \rightarrow X, \sigma' : W \rightarrow X'$  be the common resolution as in Definition 3.5, and suppose  $E_1, \dots, E_e$  are  $(K_X + B_X + \frac{1}{\mu}H)$ - $\theta$ -crepant divisors after renumbering. Then we have

$$K_W + B_W + \frac{1}{\lambda}H_W = \sigma^*(K_X + B + \frac{1}{\lambda}H) + \sum_{l=1}^e 0 \cdot E_l + \sum_{l>e} (a_l - \frac{1}{\lambda}f_l) E_l.$$

We run  $(K_W + B_W + \frac{1}{\lambda}H_W)$ -MMP on  $W$  over  $X$  with scaling of some ample divisor, then the MMP ends with a minimal model  $p : (Y, B_Y + \frac{1}{\lambda}H_Y) \rightarrow X$  of  $(W, B_W + \frac{1}{\lambda}H_W)$  over  $X$  and the exceptional locus is exactly  $\cup_{i=1}^e E_i$  and  $p$  is crepant:

$$K_Y + B_Y + \frac{1}{\lambda}H_Y = p^*(K_X + B_X + \frac{1}{\lambda}H_X).$$

Then we run  $(K_Y + B_Y)$ -MMP on  $Y$  over  $X$  with scaling of some ample divisor. This ends with the minimal model  $(X, B)$  of  $(Y, B_Y)$  over  $X$ , and the last contraction in the MMP is  $f : Z \rightarrow X$  as required.  $\square$

**3.2. Flowchart for the Log Sarkisov program.** We follow [BM97, Flowchart for the Sarkisov program] in this subsection.

If  $\lambda \leq \mu$  and  $K_X + B + \frac{1}{\mu}H$  is nef, the two Mori fibre spaces are isomorphic by following proposition and we stop here:

**Theorem 3.8.** (*Noether-Fano-Iskovskikh Criterion*): *Notations as in the definition of Sarkisov degree, then*

- (1)  $\mu \geq \mu'$ ;
- (2) If  $\mu \geq \lambda$  and  $(K_X + B + \frac{1}{\mu}H)$  is nef, then  $\Phi$  is an isomorphisms of Mori fibre spaces, i.e., we have commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow[\Phi]{\sim} & X' \\ f \downarrow & & \downarrow f' \\ S & \xrightarrow{\sim} & S' \end{array}$$

*Proof.* We follow the proof in [Hac12],[Liu21] and [Cor95]:

- (1) Only need to show  $(K_X + B + \frac{1}{\mu'}H)$  is  $f$ -nef. Let  $\sigma : W \rightarrow X$  and  $\sigma' : W \rightarrow X'$  be the common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W &= \sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ &= \sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here  $\{G_i\}, \{E_j\}$  are  $\sigma$ -exceptional divisors, and  $\{E_j\}, \{G'_k\}$  are  $\sigma'$ -exceptional divisors. Since  $H_W = \sigma'^*H'$ ,  $g'_k > 0$  (or there are no such  $G'_k$ ). Then take a general curve  $C \subset X$  contracted by  $f$ , such that its strict transform  $\tilde{C}$  on  $W$  is disjoint from  $G_i, E_j$ , and is not contained in  $G'_k$ . Then we have:

$$\begin{aligned} C \cdot \left( K_X + B + \frac{1}{\mu'}H \right) &= \tilde{C} \cdot \left( \sigma^* \left( K_W + B_W + \frac{1}{\mu'}H \right) + \sum g_i G_i + \sum e_j E_j \right) \\ &= \tilde{C} \cdot \left( \sigma'^* \left( K_{X'} + B' + \frac{1}{\mu'}H' \right) + \sum e'_j E_j + \sum g'_k G'_k \right) \\ &= \tilde{C} \cdot \sigma'^* f'^* A' + C \cdot \left( \sum g'_k G'_k \right) \geq 0 \end{aligned}$$

This implies  $(K_X + B + \frac{1}{\mu'}H)$  is  $f$ -nef and  $\mu \geq \mu'$ ;

- (2) First we show that  $\mu = \mu'$ . By 1, we only need to show  $(K_{X'} + B' + \frac{1}{\mu}H')$  is  $f'$ -nef. Indeed, same as 1, we can take a curve  $C'$  on  $X'$  contracted by  $f'$ , such that its strict transform  $\tilde{C}'$  on  $W$  is disjoint from  $G'_i, E_j$ , and is not contained in  $G'_k$  and  $C' \cdot (K_{X'} + B' + \frac{1}{\mu}H') \geq 0$ .

Then we show there are isomorphic. Take a very ample divisor  $D$  on  $X$  and let  $D'$  be its strict transform on  $X'$ .  $D'$  is  $f'$ -ample, thus there exists  $0 < d \ll 1$  such that the following holds:

- $K_X + B + \frac{1}{\mu}H + dD$  is ample;
- $K_{X'} + B' + \frac{1}{\mu}H' + dD'$  is ample.

Therefore  $X$  and  $X'$  are both log canonical models of  $(W, B_W + \frac{1}{\mu}H_W + dD_W)$ , hence  $X \cong X'$ . Furthermore,  $f$  and  $f'$  are contractions of same numerical class of curves, thus two log Mori fibre spaces are isomorphic.

□

Otherwise, if the condition does not hold:



- Claim 3.9.** (1) If  $\lambda \leq \mu$  and  $K_X + B + \frac{1}{\mu}H$  is not nef, then there is a contraction  $f : X \rightarrow T$  and a Sarkisov link  $\psi_1 : X \dashrightarrow X_1$  of type III or IV;  
 (2) If  $\lambda > \mu$ , then there is a divisorial extraction  $p : Z \rightarrow X$  and a Sarkisov link  $\psi_1 : X \dashrightarrow X_1$  of type I or II.

*Proof.* (1) By assumption,  $\lambda \leq \mu$  and  $K_X + B + \frac{1}{\mu}H$  is not nef. Suppose  $f$  is the contraction with respect to a  $(K_X + B)$ -negative extremal ray  $R = \overline{\text{NE}}(X/S)$ , then  $(K_X + B + \frac{1}{\mu}H).R = 0$  by definition of  $\mu$ . There is an extremal ray  $P \subset \overline{\text{NE}}(X)$  such that  $(K_X + B + \frac{1}{\mu}H).P < 0$  and  $F := P + R$  is an extremal face (Check [Cor95, 5.4.2] for details). Take  $0 < \delta \ll 1$  such that  $(K_X + B + (\frac{1}{\mu} - \delta)H).P < 0$ , then  $(K_X + B + (\frac{1}{\mu} - \delta)H).R < 0$  since  $H$  is  $f$ -ample, and  $F$  is a  $(K_X + B + (\frac{1}{\mu} - \delta)H)$ -negative extremal face. Since  $(X, B + (\frac{1}{\mu} - \delta)H)$  is klt, there is a contraction  $g : X \rightarrow T$  with respect to  $F$  factorizing through  $f : X \rightarrow S$ . Since  $(X, B + \frac{1}{\mu}H)$  is klt, and  $\rho(X/T) = 2$ , we can run  $(K_X + B + \frac{1}{\mu}H)$ -MMP on  $X$  with scaling of some ample divisor. Since  $B + \frac{1}{\mu}H$  is relatively big, the MMP terminates. There are following cases:

- (a) After finitely many flips  $X \dashrightarrow Z$ , first non-flip contraction is a divisorial contraction  $p : Z \rightarrow X_1$ , and then followed by a log Mori fibre space  $f_1 : X_1 \rightarrow S_1$  for  $(X_1, B_1 + \frac{1}{\mu}H_1)$  and  $(X_1, B_1)$ . Then  $S_1 \cong T$  and this is a link of type III.
- (b) After finitely many flips  $X \dashrightarrow X_1$ , first non-flip contraction is a log Mori fibre space  $f_1 : X_1 \rightarrow S_1$  for  $(X_1, B_1 + \frac{1}{\mu}H_1)$  and  $(X_1, B_1)$ . This is a link of type IV.
- (c) After finitely many flips  $X \dashrightarrow Z$ , first non-flip contraction is a divisorial contraction  $p : Z \rightarrow X_1$  with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where  $e > 0$  and  $E = \text{Exc } p$  and  $g_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \rightarrow T$  is a log minimal model of  $(X, B + \frac{1}{\mu}H)$  over  $T$ . In fact the only ray of  $\overline{\text{NE}}(X_1/T)$  is  $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ -trivial and hence is  $(K_{X_1} + B_1)$ -negative, therefore  $(X_1, B_1)/T$  is a log Mori fibre space. Take  $S_1 = T$ , then this is a link of type III:

- (d) After finitely many flips  $X \dashrightarrow X_1$ ,  $(K_X + B + \frac{1}{\mu}H)$ -MMP ends with a log minimal model  $(X_1, B_1 + \frac{1}{\mu}H_1)$  over  $T$ . Then there is an extremal ray  $R$  of  $\overline{\text{NE}}(X_1/T)$ , which is  $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ -trivial and  $(K_{X_1} + B_1)$ -negative. Let  $f_1 : X_1 \rightarrow S_1$  be the contraction with respect to  $R$ . This is a link of type IV. In fact,  $X \dashrightarrow S_1$  is the ample model of  $K_X + B + \frac{1}{\mu}H$ .
- (2) By assumption,  $\lambda > \mu$ . Take an extraction  $p : (Z, B_Z, H_Z) \rightarrow (X, B, H)$  as in Lemma 3.7. That is,  $(Z, B_Z)$  is  $\theta$ -terminal and  $p^*(K_X + B + \frac{1}{\lambda}H) = K_Z + B_Z + \frac{1}{\lambda}H_Z$  where  $B_Z = \sum \theta(E_\nu)E_\nu$  and  $E = \text{Exc } p$  is a prime divisor on  $Z$ . Then we run  $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ -MMP on  $Z$  over  $S$  with scaling of some ample divisor. Since  $Z$  is covered by  $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ -negative curves,  $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$  is not relatively pseudo-effective. Hence this MMP ends with a log Mori fibre space by Theorem 2.3. There are two cases:
- (a) After finitely many flips  $Z \dashrightarrow Z'$ , the first non-flip contraction is a divisorial contraction  $q : Z' \rightarrow X_1$ . Then  $X_1 \rightarrow S$  is a log Mori fibre space of  $(X, B)$  and  $(X, B + \frac{1}{\lambda}H)$ . Let  $S_1 = S$  and this is a link of type II.
  - (b) After finitely many flips  $Z \dashrightarrow X_1$ , first non-flip contraction is a fibering contraction  $f_1 : X_1 \rightarrow S_1$ . Since  $(K_{X_1} + B_1 + \frac{1}{\lambda}H_1)$  is  $f_1$ -negative and  $H_1$  is  $f_1$ -ample,  $(K_{X_1} + B_1)$  is  $f_1$ -negative, and  $(X_1, B_1)/Y$  is a log Mori fibre space. Take  $S_1 = Y$  and this is a link of type I.

□

- Remark 3.10.** (1) (a) For case 1a and 1b, since  $K_{X_1} + B_1 + \frac{1}{\mu}H_1$  is  $f_1$ -negative, we have  $\mu_1 < \mu$ .  
 (b) For case 1c and 1d, Since  $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$  is trivial on the ray  $R = \overline{\text{NE}}(X_1/S_1)$  for both cases, we have  $\mu_1 = \mu$ . Notice that  $(X_1, B_1 + \frac{1}{\mu}H_1)$  stays  $\theta$ -canonical, we have  $\lambda_1 \leq \mu = \mu_1$ , thus next link goes back to case 1. Furthermore, for case 1c we have  $\rho(X_1) = \rho(X) - 1$ .  
 (2) For case 2:  
 (a) For both case 2a and 2b, we have  $\mu_1 \leq \mu$  with equality holds if and only if  
 • either  $\dim S_i < \dim S_{i+1}$   
 • or  $\dim S_i = \dim S_{i+1}$  and the link is square.  
 (b) We have  $\lambda_1 \leq \lambda$  and if  $\lambda_1 = \lambda$ , then  $e_1 < e$ .

**3.3. Termination.** To prove the termination by contradiction, we need following in  $\mathcal{C}_\theta$ :

- (1) discreteness of nef threshold  $\mu$ ;
- (2) termination of flips;
- (3) ascending chain condition of log canonical threshold;
- (4) finiteness of local log canonical threshold.

Suppose there is an infinite sequence, i.e. there are infinitely many  $X_i$  and birational maps obtained from the program:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X'$$

- (1) Discreteness of nef threshold holds for all dimensions, by boundedness of  $\delta$ -lc fano varieties ([Bir21, Theorem 1.1]). Therefore we may assume  $\mu_i$  is constant, that is,  $\mu = \mu_0 = \mu_i$  for all  $i$ .
- (2) Termination of flips holds only for threefolds and pseudo-effective fourfolds. Therefore, for higher dimensions, we cannot exclude the possibility that the program gives an infinite sequence of Sarkisov links of type IV.
- (3) Ascending chain condition of log canonical threshold holds for all dimensions [HMX14].
- (4) Finiteness of local log canonical threshold doesnot hold for higher dimensions. Therefore we cannot exclude the possibility that the program gives an infinite sequence of Sarkisov links of type II.

#### 4. DOUBLE SCALING

This section we follows [Hac12, §13] and [Liu21]. Let  $W$  be the common resolution of two log Mori fibre spaces. Take an ample  $\mathbb{Q}$ -divisor  $A$  on  $S$  such that  $G \sim -(K_X + B) + f^*A$  is ample. Similarly, take an ample  $\mathbb{Q}$ -divisor  $A'$  on  $S'$  such that  $H' \sim -(K_{X'} + B') + f'^*A'$  is ample. Then  $(X, B + G)$  and  $(X', B' + H')$  are two weak log canonical models of  $W$  (for  $K_W + B_W + G_W$  and  $K_W + B_W + H_W$ ). We expect that there are finitely many weak log canonical models  $(X_i, B_i + g_i G_i + h_i H_i)$  of  $(W, B_W + g_i G_W + h_i H_W)$  and  $\psi_i : X_i \dashrightarrow X_{i+1}$  is a Sarkisov link given by 2-ray game.

In this method, we run the program in a much smaller collection of varieties. That is, all pairs with a  $(K_W + D)$ -non-positive birational contraction  $W \dashrightarrow X_i$ , where  $D$  varies in a compact subset  $\mathcal{E}_A(V)$  of  $\text{WDiv}_{\mathbb{R}}(W)$ . In this collection, the termination of the program follows after the finiteness of weak log canonical models.

**4.1. Prepare.** Let  $(W, B_W)$  be a  $\mathbb{Q}$ -factorial klt pair and  $f : (X, B) \rightarrow S$  and  $f' : (X', B') \rightarrow S'$  be two different log Mori fibre spaces as outputs of  $(K_W + B_W)$ -MMP. To modify the beginning setting, we need more conventions and lemmas:

**Definition 4.1.** Let  $f : X \dashrightarrow Y$  be a birational map of normal quasi-projective varieties. If

- $f$  does not extract divisors;
- $a(E; X, B_X) \leq a(E; Y, B_Y)$  for all divisors  $E$  over  $X$ .

then we denote  $(X, B) \geq (Y, B_Y)$ .

In particular, for terminal pairs, we have following lemma:

**Lemma 4.2.** [Hac12, Lemma 13.8] *Let  $f : W \dashrightarrow X$  be a birational map where  $(W, B_W)$  is terminal. If*

- *$f$  does not extract divisors;*
- *$K_X + B$  is nef, where  $B = f_* B_W$ ;*
- *$a(E; X, B) \geq a(E; W, B_W)$  for all divisors  $E \subset W$ ,*

*then*

- *$(W, B_W) \geq (X, B)$ .*
- *$(X, B)$  is klt*
- *If  $Z \rightarrow X$  is a divisorial extraction of a divisor  $E$  with  $a(E; X, B) \leq 0$ , then  $E$  is a divisor on  $W$  ;*
- *If  $Z \rightarrow X$  is terminalization of  $(X, B)$ , then  $W \dashrightarrow Z$  extracts no divisors.*

Conversely, start from a klt pair and a non-positive map, we have

**Lemma 4.3.** [Liu21, Lemma 3.5] *Let  $\sigma : (W, B_W) \dashrightarrow (X, B)$  be a  $K_W + B_W$ -non-positive birational map such that  $\sigma_*(K_W + B_W) = K_X + B$  and  $(W, B_W)$  is a  $\mathbb{Q}$ -factorial klt pair. Then there is a resolution of indeterminacy  $\pi : \tilde{W} \rightarrow W$  and  $\tilde{\sigma} : \tilde{W} \rightarrow X$  such that*

- *$(\tilde{W}, B_{\tilde{W}})$  is  $\mathbb{Q}$ -factorial terminal and  $\tilde{\sigma}_* B_{\tilde{W}} = B$ ,*
- *$\tilde{\sigma}$  is  $(K_{\tilde{W}} + B_{\tilde{W}})$ -non-positive and  $(\tilde{W}, B_{\tilde{W}}) \geq (X, B)$ .*

By Lemma 4.3, we replace  $(W, B_W)$  by its log resolution such that  $(W, B_W)$  is terminal and  $\sigma : W \rightarrow X$  and  $\sigma' : W \rightarrow X'$  are  $(K_W + B_W)$ -non-positive morphisms, and  $(W, B_W) \geq (X, B), (X', B')$ .

Take very general ample  $\mathbb{Q}$ -divisors  $A$  and  $A'$  on  $S$  and  $S'$  such that  $G \sim_{\mathbb{Q}} -(K_X + B) + f^* A$  and  $H \sim_{\mathbb{Q}} -(K_{X'} + B') + f'^* A'$  are two ample  $\mathbb{Q}$ -divisors. Moreover, we may assume  $G$  and  $H$  satisfying  $G_W := \sigma^* G = \sigma_*^{-1} G$  and  $H_W := \sigma'^* H = \sigma'^{-1} H$ . Therefore  $\sigma_*(K_W + B_W + G_W) = K_X + B + G$  is nef, and Lemma 4.2 holds. Furthermore, we may assume  $(W, B_W + gG_W + hH_W)$  is log smooth and terminal for all  $0 \leq g, h \leq 2$  by taking furthermore blowing up if necessary. Then we have:

**Theorem 4.4** (Sarkisov program with double scaling). [Hac12, Claim 13.12] *Notations as above, there is a finite sequence of Sarkisov links*

$$\begin{array}{ccccccc}
 X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \cdots & \dashrightarrow & X_N = X' \\
 f=f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & & f_N \downarrow \\
 S = S_0 & & S_1 & & S_2 & & & S_N = S'
 \end{array}$$

and rational numbers

$$\begin{aligned}
 1 = g_0 &\geq g_1 \geq \cdots \geq g_N = 0 \\
 0 = h_0 &\leq h_1 \leq \cdots \leq h_N = 1
 \end{aligned}$$

such that

- (1) *For each  $i$ ,  $\sigma_i : W \dashrightarrow X_i$  is  $(K_W + B_W + g_i G_W + h_i H_W)$ -non-positive, and  $(K_{X_i} + B_i + g_i G_i + h_i H_i) = \sigma_{i*}(K_W + B_W + g_i G_W + h_i H_W)$  is nef and is relatively trivial over  $S_i$ ;*
- (2)  *$(W, B_W + g_i G_W + h_i H_W) \geq (X_i, B_i + g_i G_i + h_i H_i)$ ;*
- (3) *each Sarkisov link  $X_i \dashrightarrow X_{i+1}$  is given by a sequence of  $(K_{X_i} + B_i + g_i G_i + h_i H_i)$ -trivial maps.*
- (4) *The last link  $X_N \rightarrow S_N$  is isomorphic to  $X' \rightarrow S'$*

Here trivial map means:

**Definition 4.5.** [Hac12, §13.2] Let  $f : X \dashrightarrow Y$  be a rational map of normal quasi-projective varieties over  $S$ , and  $D$  be a  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor on  $X$  with  $f_*D = D_Y$ . Then  $f$  is called  *$D$ -trivial* if  $D$  is pull back of a  $\mathbb{R}$ -Cartier divisor on  $S$ .

**4.2. Construct Sarkisov links.** This subsection we construct the links inductively. Suppose we have  $\sigma_i : W \dashrightarrow X_i$  as in Theorem 4.4, that is

- $f_i : (X_i, B_i) \rightarrow S_i$  is a log Mori fibre space and  $\sigma_{i*}B_W = B_i$ ;
- $\sigma_i : W \dashrightarrow X_i$  is  $(K_W + B_W + g_iG_i + h_iH_W)$ -non-positive birational map, and  $(K_{X_i} + B_i + g_iG + h_iH) = \sigma_{i*}(K_W + B_W + g_iG_W + h_iH_W)$  is nef and relatively trivial over  $S_i$ ;
- $(W, B_W + g_iG_W + h_iH_W) \geq (X_i, B_i + g_iG_i + h_iH_i)$ ;
- $0 \leq g_i, h_i \leq 1$  are rational numbers.

Then we need to show that there is a Sarkisov link  $X_i \dashrightarrow X_{i+1}$  satisfying Theorem 4.4. Similarly with Sarkisov degree, we have following notations:

**Definition 4.6.** Let  $C_i$  be a general  $f_i$ -vertical curve on  $X_i$ , then

- $r_i := \frac{H_i \cdot C_i}{G_i \cdot C_i}$ ;
- Let  $\Gamma$  be the set of  $t \in [0, \frac{g_i}{r_i}]$  such that
  - (1)  $(W, B_W + g_iG_W + h_iH_W + t(H_W - r_iG_W)) \geq (X_i, B_i + g_iG_i + h_iH_i + t(H_i - r_iG_i))$
  - (2)  $K_{X_i} + B_i + g_iG + h_iH + t(H_i - r_iG_i)$  is nef;
 Let  $s_i = \max \Gamma$ ;
- Let  $D_{W,i} = B_W + g_iG_W + h_iH_W$  and  $D_i = B_i + g_iG_i + h_iH_i$ . Let  $D_{W,i}(t) = B_W + g_iG_W + h_iH_W + t(G_W - r_iH_W)$  and  $D_i(t) = B_i + g_iG_i + h_iH_i + t(G_i - r_iH_i)$ . Let  $g_{i+1} = g_i - r_i s_i$  and  $h_{i+1} = h_i + s_i$ . Note that  $D_{W,i+1} = D_{W,i}(s_i)$ .

Then we have (check [Liu21, Lemma 4.4] for details)

- (1)  $r_i > 0$ ;
- (2) either  $\Gamma = \{0\}$  or is a closed interval;
- (3)  $g_{i+1} = g_i \Leftrightarrow h_{i+1} = h_i \Leftrightarrow s_i = 0$ ;

**Construct links.:** If  $s_i = \frac{g_i}{r_i}$ , then  $g_{i+1} = 0$ . Let  $N = i + 1$  and let  $f_N : X_N = X_i \rightarrow S_N = S_i$ , then  $X_N \rightarrow S_N$  is isomorphic to  $f' : X' \rightarrow S'$  (see Proposition 4.9) and we stop. Otherwise, if  $s_i < \frac{g_i}{r_i}$ , then we construct the Sarkisov link  $X_i \dashrightarrow X_{i+1}$  in following cases:

- (1) Suppose  $s_i$  is not the threshold of condition 1 of  $\Gamma$ . That is, there exists  $0 < \epsilon \ll 1$ , such that for any divisor  $E$  on  $W$ , we have

$$a(E; X_i, D_i(s_i + \epsilon)) \geq a(E; W, D_{W,i}(s_i + \epsilon))$$

and  $K_{X_i} + D_i(s_i + \epsilon)$  is not nef. Then there is a 2-dimensional  $(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$ -negative extremal face  $F$  for some  $0 < \delta \ll \epsilon$ , spanned by  $R = \mathbb{R}_{\geq 0}[C_i]$  and another extremal ray  $P$ . Hence there is a contraction  $X_i \rightarrow T_i$  corresponding to  $F$  factoring through  $f_i$ . Then we run  $(K_{X_i} + D_i(s_i + \epsilon))$ -MMP on  $X_i$  with scaling over  $T_i$ . After finitely many flips, we either have a  $(K_{X_i} + D_i(s_i + \epsilon))$  minimal model, a divisorial contraction, or a log Mori fibre space over  $T_i$ :

- (a) After finitely many flips  $X_i \dashrightarrow X_{i+1}$  there is a log Mori fibre space  $X_{i+1} \rightarrow S_{i+1}$ , and this is a link of type III.
- (b) After finitely many flips  $X_i \dashrightarrow Z_i$  there is a divisorial contraction  $Z_i \rightarrow X_{i+1}$ , then let  $S_{i+1} = T_i$  and  $X_{i+1} \rightarrow S_{i+1}$  is a log Mori fibre space and this is a link of type IV.
- (c) After finitely many flips  $X_i \dashrightarrow X_{i+1}$ , the contraction  $X_{i+1} \rightarrow T_i$  is a log minimal model of  $(X_i, D_i(s_i + \epsilon))$  over  $T_i$ . Let  $C'$  be the strict transform of  $C_i$  on  $X_{i+1}$ , then  $(K_{X_{i+1}} + D_{i+1}(\epsilon)).C' = 0$  and  $(K_{X_{i+1}} + B_{i+1}).C' < 0$ , therefore there is a contraction  $X_{i+1} \rightarrow S_{i+1}$  which is a log Mori fibre space. And this is a link of type IV.

- (2) Suppose  $s_i$  is the threshold of condition 1 of  $\Gamma$ , that is, there exists  $0 < \epsilon \ll 1$  and a  $\sigma_i$ -exceptional divisor  $E_i$  on  $W$  such that

$$a(E_i; X_i, D_i(s_i + \epsilon)) < a(E_i; W, D_{W,i}(s_i + \epsilon)).$$

In this case, we have

$$a(E_i; X_i, D_i(s_i)) = a(E_i; W, D_{W,i}(s_i)) = -\text{mult}_{E_i}(D_{W,i}(s_i)) \leq 0.$$

Let  $p_i : Z_i \rightarrow X_i$  be the divisorial extraction of the divisor  $E_i$  as in Corollary 2.4, and suppose  $K_{Z_i} + D_{Z_i}(s_i) = K_{Z_i} + B_{Z_i} + g_{i+1}G_{Z_i} + h_{i+1}H_{Z_i} = p_i^*(K_{X_i} + D_i(s_i + \epsilon))$ . Take a sufficiently small  $\delta$  such that  $0 < \delta \ll \epsilon \ll 1$  and

$$K_{Z_i} + \Delta_i = p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$$

is klt. Then we run  $(K_{Z_i} + \Delta_i)$ -MMP on  $Z_i$  over  $S_i$ . Since  $Z_i$  is covered by  $(K_{Z_i} + \Delta_i)$ -negative curves, it follows that  $(K_{Z_i} + \Delta_i)$  is not pseudo-effective over  $S_i$ , and this MMP ends with a log Mori fibre space. Moreover, this is a MMP for  $p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta' G_i)$  for all  $0 < \delta' \leq \delta$ . After finitely many flips, we either have a  $(K_{Z_i} + \Delta_i)$  log Mori fibre space or a  $(K_{Z_i} + \Delta_i)$  divisorial contraction.

- (a) After finitely many flips  $Z_i \dashrightarrow X_{i+1}$  there is a log Mori fibre space  $X_{i+1} \rightarrow S_{i+1}$ , and this is a link of type I. In this case we have  $\rho(X_{i+1}) = \rho(X_i) + 1$ .
- (b) After finitely many flips  $Z_i \dashrightarrow Z'_{i+1}$  there is a divisorial contraction  $q_i : Z'_{i+1} \rightarrow X_{i+1}$ , and then a logMori fibre space  $X_{i+1} \rightarrow S_i =: S_{i+1}$ . This is a link of type II.

**Claim 4.7.** *By [Hac12, Lemma 13.14-17] and [Liu21, Lemma 4.2], we have:*

- (1)  $r_i \leq r_{i+1}$ . Moreover, in case 1a, we have  $r_i < r_{i+1}$ .
- (2) Since the birational map  $X_i \dashrightarrow X_{i+1}$  is over  $T_i$  (over  $S_i$ ) and  $(K_{X_i} + D_i(s_i))$  is numerically trivial over  $T_i$  (over  $S_i$ ) in case 1 (case 2), it follows that  $a(E; X_i, D_i(s_i)) = a(E; X_{i+1}, D_{i+1})$  for any divisors  $E$  over  $W$  and so the inequality

$$a(E; X_{i+1}, D_{i+1}) \geq a(E; W, D_{W,i+1})$$

- (3) In case 1, for any divisor  $E \subset W$ , we have  $a(E; X_i, D_i(s_i + \epsilon)) \leq a(E; X_{i+1}, D_{i+1}(\epsilon))$  for all  $0 < \epsilon \ll 1$ . Moreover, since  $X_i \not\cong X_{i+1}$ , there is a divisor  $F$  over  $W$  such that  $a(F; X_i, D_i(s_i + \epsilon)) < a(F; X_{i+1}, D_{i+1}(\epsilon))$ .
- (4) In case 2, for any divisor  $E \subset W$ , we have  $a(E; X_i, D_i(s_i + \epsilon) - \delta G_i) \leq a(E; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$  for all  $0 < \epsilon \ll 1$ . Moreover, since  $X_i \not\cong X_{i+1}$ , there is a divisor  $F$  over  $W$  such that  $a(F; X_i, D_i(s_i + \epsilon) - \delta G_i) < a(F; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$ .
- (5)  $h_i \leq 1$ , and  $h_i = 1$  if and only if  $g_i = 0$ ;

#### 4.3. Termination.

**Lemma 4.8.** [Hac12, Lemma 13.18-19] (or [Liu21, Lemma 4.9]) *Suppose we construct a sequence of Sarkisov links:*

$$\begin{array}{ccccccc} X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \longrightarrow & \cdots \dashrightarrow X_i \longrightarrow \cdots, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S = S_0 & & S_1 & & S_2 & & S_i \end{array}$$

then

- (1) there are only finitely many possibilities of  $f_i : X_i \rightarrow S_i$  up to isomorphism;
- (2) the Sarkisov program with double scaling of  $(G_W, H_W)$  terminates. That is, there exists an integer  $N > 0$  such that  $g_N = 0$ .

*Proof.* (1) This essentially follows from the finiteness of weak log canonical model (Theorem 2.15). We construct the subspace  $V$  of  $\text{WDiv}_{\mathbb{R}}(W)$  as following:

- (a) If  $h_k > 0$  for some  $k$ : Since  $H_W$  is nef and big, take an ample  $\mathbb{Q}$ -divisor  $A_W$  and an effective  $\mathbb{Q}$ -divisor  $C_W$  such that  $H_W \sim_{\mathbb{Q}} A_W + C_W$ . Let  $V$  be the affine space spanned by components of  $B_W, G_W, H_W, C_W$ , then for  $i > k$ :

$$B_W + g_i G_W + h_i H_W \sim_{\mathbb{Q}} h_k A_W + B_W + g_i G_W + (h_i - h_k) H_W + h_k C_W =: \Delta_i \in \mathcal{L}_{h_k A_W}(V)$$

- (b) If  $h_k = 0$  for all  $k$ , then  $h_i \equiv 0$  and  $g_i \equiv 1$ . Since  $G_W$  is nef and big, take an ample  $\mathbb{Q}$ -divisor  $A_W$  and an effective  $\mathbb{Q}$ -divisor  $C_W$  such that  $G_W \sim_{\mathbb{Q}} A_W + C_W$ . Let  $V$  be the affine space spanned by components of  $B_W, C_W$ , then

$$B_W + G_W \sim_{\mathbb{Q}} A_W + B_W + C_W =: \Delta_i \in \mathcal{L}_{A_W}(V)$$

Then all  $X_i$  are weak log canonical models of  $(W, \Delta_i)$ . By finiteness of weak log canonical models, there are finitely many  $\sigma_i : W \dashrightarrow X_i$  up to isomorphism. Then we shall show that for  $\sigma_i : W \dashrightarrow X_i$  there are finitely many log Mori fibre spaces in the sequence up to isomorphism. Indeed, we may assume that there is a  $k$  such that  $X_i \cong X_k$  for all  $i > k$ , and  $f_i$  is the contraction corresponding to an extremal ray  $R_i \subset \overline{\text{NE}}(X_k)$ . Then we have  $(K_{X_k} + B_k) \cdot R_i < 0$  and  $(K_{X_k} + B_k + g_i G_k + h_i H_k) \cdot R_i = 0$ . Furthermore,  $H_k$  and  $G_k$  are relatively ample over  $S_i$  for all  $i > k$ .

- (a) If  $h_k > 0$ : Since  $H_k$  is big, we have  $h_k H_k = A_k + E_k$  for some ample  $\mathbb{Q}$ -divisor  $A_k$  and effective  $\mathbb{Q}$ -divisor  $E_k$ . Let  $B'_k = B_k + (1 - \epsilon)h_k H_k + \epsilon E_k$  for sufficiently small  $\epsilon$  such that  $(X_k, B'_k)$  is klt, then  $(K_{X_k} + B'_k) \cdot R_i < 0$  and  $(K_{X_k} + B'_k + \epsilon A_k) \cdot R_i < 0$  for all  $i > k$ . By Cone theorem, we have

$$\overline{\text{NE}}(X_k) = \overline{\text{NE}}(X_k)_{K_{X_k} + B'_k + \epsilon A_k \geq 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_{\alpha}$$

All extremal rays  $R_i$  corresponding to  $f_i$  for  $i > k$  are in the finite set  $\{R_{\alpha}\}_{\alpha \in \Lambda}$ , thus there are finitely many log Mori fibre spaces  $f_i : X_i \rightarrow S_i$  of  $X_k$ .

- (b) If  $h_i = 0$  for all  $i$ , and hence  $g_i = 1$  for all  $i$ . Since  $G_i$  is big, we have  $G_k = A_k + E_k$  for some ample  $\mathbb{Q}$ -divisor  $A_k$  and effective  $\mathbb{Q}$ -divisor  $E_k$ . Let  $B'_k = B_k + (1 - \epsilon)G_k + \frac{\epsilon}{2}E_k$  for sufficiently small  $\epsilon$  such that  $(X_k, B'_k)$  is klt, then  $(K_{X_k} + B'_k) \cdot R_i < 0$  and  $(K_{X_k} + B'_k + \frac{\epsilon}{2}A_k) \cdot R_i < 0$  for all  $i > k$ . By Cone theorem, we have

$$\overline{\text{NE}}(X_k) = \overline{\text{NE}}(X_k)_{K_{X_k} + B'_k + \frac{\epsilon}{2}A_k \geq 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_{\alpha}$$

Again, there are finitely many log Mori fibre spaces  $f_i : X_i \rightarrow S_i$  of  $X_k$ .

- (2) Assume this sequence of links is infinite, then there exists an  $i$  such that there are infinitely many  $j > i$  such that  $f_i : X_i \rightarrow S_i$  and  $f_j : X_j \rightarrow S_j$  are isomorphic. Then we have  $g_{i+1} = g_{j+1}$  and  $h_{i+1} = h_{j+1}$ . Since sequences of  $h_k$  and  $g_k$  are monotone, we have  $h_{i+1} = h_k$  and  $g_{i+1} = g_k$  for all  $k > i$ . Suppose  $X_i \dashrightarrow X_{i+1}$  is a link in case 1 of the Construction in 4.2, then the next link is also in case 1, and all the links after are in case 1. Note that  $X_i \cong X_j$  and therefore  $\rho(X_i) = \rho(X_j)$ , the links are all of type IV. But this contradicts 3 of Claim 4.7. Therefore there are no link of type III or IV after  $X_i$ . In other words, the links after  $X_i$  are all type I or II in case 2.

Since  $\rho(X_i) = \rho(X_j)$ ,  $X_i$  and  $X_j$  are linked by the Sarkisov links of type II. But this contradicts 4 of Claim 4.7.

□

**Proposition 4.9.**  $X_N \rightarrow S_N$  is isomorphic to  $X' \rightarrow S'$ .

*Proof.* Similarly with 2 of Theorem 3.8, we have  $h_N = 1$  and they are isomorphic. □

## 5. USING THE POLYTOPE

In this section we follow [HM13]. This proof of Theorem 1.2 is not a program. Let  $W$  be the common log resolution of two Mori fibre spaces  $X \rightarrow S$  and  $Y \rightarrow T$ . Take a finite dimensional affine subspace  $V$  of  $\text{WDiv}_{\mathbb{R}}(W)$  and an ample  $\mathbb{Q}$ -divisor  $A$ . Then  $\{\mathcal{A}_i = \mathcal{A}_{A, f_i}\}$  is a partition of

$\mathcal{E}_A(V)$ , and each  $\mathcal{A}_i$  corresponds to an ample model of  $W$ . There are morphisms between these ample models (Theorem 5.2 and Theorem 5.4). Moreover, let  $\mathcal{C}_i$  be the closure of  $\mathcal{A}_i$ . If  $D$  is contained in  $\partial\mathcal{E}_A(V)$  and two or more  $\mathcal{C}_i$ , then those morphisms form a Sarkisov link (Theorem 5.6).

We can find a special 2-dimensional rational affine subspace  $V$  (by Lemma 5.7) such that

- (1)  $S, T$  are ample models of  $W$  for some  $D_S, D_T \in \mathcal{L}_A(V)$ ;
- (2) There are finitely many points  $D_i$  between  $D_S$  and  $D_T$ , and each  $D_i$  corresponds to a Sarkisov link.

Then  $X \dashrightarrow Y$  is composition of these Sarkisov links, and the Theorem 1.2 follows.

**5.1. Construction of Sarkisov links.** In this subsection we construct one Sarkisov link. First we show the partition of  $\mathcal{E}_A(V)$  corresponding to ample models and morphisms between these ample models.

**Theorem 5.1.** [BCHM, Corollary 1.1.5] *Let  $\pi : X \rightarrow U$  be a projective morphism of normal quasi-projective varieties, and  $V \subset \text{WDiv}_{\mathbb{R}}(X)$  be a finite dimensional rational subspace. Suppose that there is a divisor  $\Delta_0 \in V$  such that  $(X, \Delta_0)$  is klt. Let  $A$  be a general ample  $\mathbb{Q}$ -divisor over  $U$  which has no components common with any element of  $V$ .*

- (1) *There are finitely many birational maps  $f_i : X \dashrightarrow X_i$  over  $U$  such that*

$$\mathcal{E}_{A,\pi}(V) = \bigcup_i \mathcal{W}_i$$

*where  $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$  is a rational polytope. Moreover, if  $f : X \dashrightarrow Y$  is a log terminal model of  $K_X + D$  over  $U$  for some  $D \in \mathcal{E}_{A,\pi}(V)$ , then  $f = f_i$  for some  $i$ .*

- (2) *There are finitely many rational maps  $g_j : X \dashrightarrow Z_j$  over  $U$  such that*

$$\mathcal{E}_{A,\pi}(V) = \coprod_j \mathcal{A}_j$$

*$\{\mathcal{A}_j = \mathcal{A}_{A,\pi,g_j}\}$  is a partition of  $\mathcal{E}_A(V)$ ;*

- (3) *For every  $f_i$  there is a  $g_j$  and a morphism  $h_{ij} : Y_i \rightarrow Z_j$  such that  $\mathcal{W}_i \subset \overline{\mathcal{A}_j}$ .*

**Theorem 5.2.** [HM13, Theorem 3.3] *Let  $W$  be a smooth projective variety, and  $V$  be a finite dimensional affine subspace of  $\text{WDiv}_{\mathbb{R}}(W)$  defined over rational numbers and fix an ample effective  $\mathbb{Q}$ -divisor  $A$ . Suppose that there is an element  $D_0$  of  $\mathcal{L}_A(V)$  such that  $K_W + D_0$  is big and klt. Then there are finitely many rational contractions  $f_i : W \dashrightarrow X_i$  such that*

- (1)  *$\{\mathcal{A}_i = \mathcal{A}_{A,f_i}\}$  is a partition of  $\mathcal{E}_A(V)$ .  $\mathcal{A}_i$  is a finite union of interiors of rational polytopes. If  $f_i$  is birational then  $\mathcal{C}_i = \mathcal{C}_{A,f_i}$  is a rational polytope;*
- (2) *If  $i, j$  are two indices such that  $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$  then there is a contraction  $f_{ij} : X_i \rightarrow X_j$  such that  $f_j = f_{ij} \circ f_i$ ;*
- (3) *Suppose in addition  $V$  spans Neron-Severi group of  $W$ . Pick  $i$  such that a connected component  $\mathcal{C}$  of  $\mathcal{C}_i$  intersects the interior of  $\mathcal{L}_A(V)$ , the following are equivalent:*
  - (a)  *$\mathcal{C}$  spans  $V$ ;*
  - (b) *If  $D \in \mathcal{A}_i \cap \mathcal{C}$  then  $f_i$  is a log terminal model of  $K_W + D$ ;*
  - (c)  *$f_i$  is birational and  $X_i$  is  $\mathbb{Q}$ -factorial.*
- (4) *Suppose in addition  $V$  spans Neron-Severi group of  $W$ . If  $i, j$  are two indices such that  $\mathcal{C}_i$  spans  $V$  and  $D$  is a general point of  $\mathcal{A}_j \cap \mathcal{C}_i$  which is also a point of interior of  $\mathcal{L}_A(V)$ , then  $\mathcal{C}_i$  and  $\overline{\text{NE}}(X_i/X_j)^* \times \mathbb{R}^k$  are locally isomorphic in a neighbourhood of  $D$ , for some  $k \geq 0$ . Furthermore  $\rho(X_i/X_j) = \dim \mathcal{C}_i - \dim \mathcal{C}_j \cap \mathcal{C}_i$ .*

**Lemma 5.3.** [HM13, Corollary 3.4] *If  $V$  spans Neron-Severi group of  $W$ , then there is a Zariski dense open subset  $U$  of the Grassmannian  $G(r, V)$  of real affine subspace of dimension  $r$  such that any  $[V'] \in U$  defined on rational numbers satisfies (1-4) of Theorem 5.2*

*Proof.* Let  $U \subset G(r, V)$  be the set of real affine subspace  $V'$  of  $V$  of dimension  $r$ , which contain no face of any  $\mathcal{C}_i$  of  $\mathcal{L}(V)$ . In particular, the interior of  $\mathcal{L}_A(V')$  is contained in the interior of  $\mathcal{L}_A(V)$ . Clearly that any  $V' \in U$  defined over rationals satisfies (1-4) of 5.2.  $\square$

By above Lemma, from now on in this subsection, we always assume that  $V$  has dimension 2 and satisfies Theorem 5.2. The following lemma shows that the morphism in 2 of Theorem 5.2 can be divisorial contraction, small contraction or log Mori fibre space. And in some cases they forms a flop.

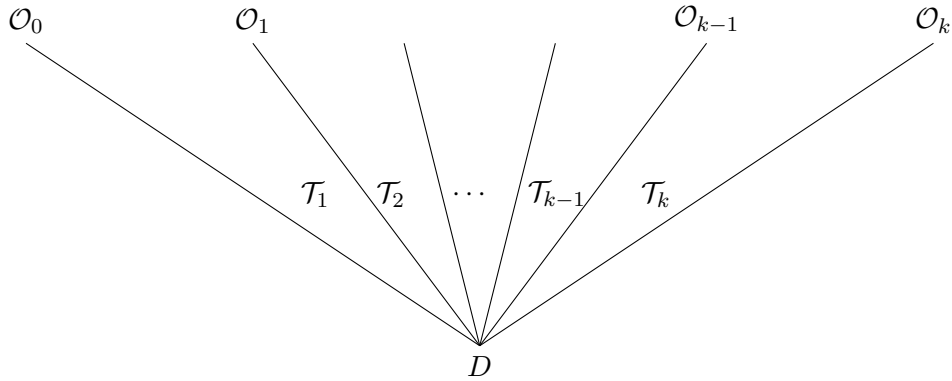
**Lemma 5.4.** [HM13, Lemma 3.5] *Let  $f : W \dashrightarrow X$  and  $g : W \dashrightarrow Y$  be two rational contractions such that  $\mathcal{C}_{A,f}$  is of dimension 2 and  $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$  is of dimension 1. Assume  $\rho(X) \geq \rho(Y)$  and  $\mathcal{O}$  is not contained in the boundary of  $\mathcal{L}_A(V)$ . Let  $D$  be an interior point of  $\mathcal{O}$  and  $B = f_*D$ . Then there is a rational contraction  $\pi : X \dashrightarrow Y$  and  $g = \pi \circ f$  such that either*

- (1)  $\rho(X) = \rho(Y) + 1$  and  $\pi$  is  $(K_X + B)$ -trivial, and either
  - (a)  $\pi$  is birational and  $\mathcal{O}$  is not contained in the boundary of  $\mathcal{E}_A(V)$ , and either
    - (i)  $\pi$  is a divisorial contraction and  $\mathcal{O} \neq \mathcal{C}_{A,g}$ , or
    - (ii)  $\pi$  is a small contraction and  $\mathcal{O} = \mathcal{C}_{A,g}$  or
  - (b)  $\pi$  is a log Mori fibre space, and  $\mathcal{O} = \mathcal{C}_{A,g}$  is contained in the boundary of  $\mathcal{E}_A(V)$ , or
- (2)  $\rho(X) = \rho(Y)$ , and  $\pi$  is a  $(K_X + B)$ -flop and  $\mathcal{O} \neq \mathcal{C}_{A,g}$  is not contained in the boundary of  $\mathcal{E}_A(V)$ .

**Lemma 5.5.** [HM13, Lemma 3.6] *Let  $f : W \dashrightarrow X$  be a birational contraction between  $\mathbb{Q}$ -factorial varieties. Suppose  $(W, D)$  and  $(W, D+A)$  are both klt. If  $f$  is ample model of  $(W, D+A)$  and  $A$  is ample, then  $f$  is a result of  $(K_W + D)$ -MMP.*

This lemma guarantee that every variety in the Sarkisov links constructed later is a result of  $(W, B_W)$ -MMP.

Finally we show there is a Sarkisov link corresponding to certain  $D \in \mathcal{E}_A(V)$ . Let  $D = A + B$  be a point of the boundary of  $\mathcal{E}_A(V)$  in the interior of  $\mathcal{L}_A(V)$ . Let  $\mathcal{T}_1, \dots, \mathcal{T}_k$  be the polytopes  $\mathcal{C}_i$  of dimension 2 containing  $D$ . Possibly reordering, we may assume that the intersection  $\mathcal{O}_0$  and  $\mathcal{O}_k$  of  $\mathcal{T}_1$  and  $\mathcal{T}_k$  with boundary of  $\mathcal{E}_A(V)$  and  $\mathcal{O}_i = \mathcal{T}_i \cap \mathcal{T}_{i+1}$  are one dimensional. Let  $f_i : W \dashrightarrow X_i$  be the birational contraction associated to  $\mathcal{T}_i$  and  $g_i : W \dashrightarrow S_i$  be the rational contraction associated to  $\mathcal{O}_i$ .



Set  $f = f_1 : W \dashrightarrow X, g = f_k : W \dashrightarrow Y$  and  $\phi : X \rightarrow S = S_0, \psi : Y \rightarrow T = S_k$  and  $X' = X_2, Y' = X_{k-1}$  and let  $W \dashrightarrow R$  be the ample model of  $D$ . Then

**Theorem 5.6.** [HM13, Theorem 3.7] *Suppose  $B_W$  is a divisor such that  $K_W + B_W$  is klt and  $D - B_W$  is ample. Then  $\phi$  and  $\psi$  are log Mori fibre spaces as outputs of  $(K_W + B_W)$ -MMP and connected by a Sarkisov link if  $D$  is contained in more than two polytopes.*



*Proof.* WMA  $k \geq 3$  and we have

$$\begin{array}{ccc}
 X' & \cdots \rightarrow & Y' \\
 \downarrow p & & \downarrow q \\
 X & & Y \\
 \downarrow \phi & & \downarrow \psi \\
 S & & T \\
 & \searrow s \quad \swarrow t & \\
 & R &
 \end{array}$$

Note that  $\rho(X_i/R) \leq 2$  and  $\rho(X/S) = \rho(Y/T) = 1$ . Thus

- (1)  $s$  is identity and  $p$  is a divisorial contraction (extraction), or
- (2)  $s$  is a contraction and  $p$  is a flop.

The same holds for  $q$  and  $t$ . And the map  $X' \rightarrow Y'$  is clear the composition of flops. This gives 4 types of links.  $\square$

**5.2. Decomposition into Sarkisov links.** We need a special resolution  $W$  and a special affine subspace  $V \subset \text{WDiv}(W)$ .

**Lemma 5.7.** [HM13, Lemma 4.1] *Let  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$  be two MMP related log Mori fibre spaces corresponding to two klt projective varieties  $(X, B_X)$  and  $(Y, B_Y)$ . Then we may find a smooth projective variety  $W$ , two birational morphisms  $f : W \rightarrow X$  and  $g : W \rightarrow Y$ , a klt pair  $(W, B_W)$ , an ample  $\mathbb{Q}$ -divisor  $A$  on  $W$  and a two dimensional rational affine subspace  $V$  of  $\text{WDiv}_{\mathbb{R}}(W)$  such that*

- (1) *If  $D \in \mathcal{L}_A(V)$  then  $D - B_W$  is ample;*
- (2)  *$\mathcal{A}_{A, \phi \circ f}$  and  $\mathcal{A}_{A, \psi \circ g}$  are not contained in the boundary of  $\mathcal{L}_A(V)$ ;*
- (3)  *$V$  satisfies (1-4) of Theorem 5.2;*
- (4)  *$\mathcal{C}_{A, f}$  and  $\mathcal{C}_{A, g}$  are two dimensional;*
- (5)  *$\mathcal{C}_{A, \phi \circ f}$  and  $\mathcal{C}_{A, \psi \circ g}$  are one dimensional.*

*Proof.* By assumption there is a  $\mathbb{Q}$ -factorial klt pair  $(W, B_W)$  such that  $f : W \dashrightarrow X$  and  $g : W \dashrightarrow Y$  are outputs of  $(K_W + B_W)$ -MMP. Let  $p' : W' \rightarrow W$  be any log resolution such that resolves the indeterminacy of  $f$  and  $g$ , then we may write

$$K_{W'} + B_{W'} = p'^*(K_W + B_W) + E'$$

where  $E' \geq 0$  and  $B_{W'} \geq 0$  have no common components, and  $E'$  is exceptional and  $p'_* B_{W'} = B_W$ . Pick a divisor  $-F$  which is ample over  $W$  with  $\text{Supp } F = \text{Exc } p'$  such that  $K_{W'} + B_{W'} + F$  is klt. As  $p'$  is  $(K_{W'} + B_{W'} + F)$ -negative and  $(K_W + B_W)$  is klt and  $W$  is  $\mathbb{Q}$ -factorial, the  $(K_{W'} + B_{W'} + F)$ -MMP over  $W$  terminates with the pair  $(W, B_W)$ . Replacing  $(W, B_W)$  by  $(W', B_{W'} + F)$  we may assume that  $(W, B_W)$  is log smooth and  $f, g$  are morphisms.

Pick general ample  $\mathbb{Q}$ -divisors  $A, H_1, H_2, \dots, H_k$  on  $W$  such that  $H_1, \dots, H_k$  generate the Neron-Severi group of  $W$ . Let  $H = A + H_1 + \dots + H_k$ . Pick sufficiently ample divisors  $A_S$  on  $S$  and  $A_T$  on  $T$  such that

$$-(K_X + B_X) + \phi^* A_S \text{ and } -(K_Y + B_Y) + \psi^* A_T$$

are both ample. Pick a rational number  $0 < \delta < 1$  such that

$$-(K_X + B_X + \delta f_* H) + \phi^* A_S \text{ and } -(K_Y + B_Y + \delta g_* H) + \psi^* A_T$$

are both ample and  $f$  and  $g$  are both  $(K_W + B_W + \delta H)$ -negative. Replacing  $H$  by  $\delta H$  we may assume that  $\delta = 1$ . Now pick a  $\mathbb{Q}$ -divisor  $B_0 \leq B_W$  such that  $A + (B_0 - B_W)$ ,  $-(K_X + f_* B_0 + f_* H) + \phi^* A_S$  and  $-(K_Y + g_* B_0 + g_* H) + \psi^* A_T$  are all ample and  $f$  and  $g$  are both  $(K_W + B_0 + \delta H)$ -negative.

Pick general ample  $\mathbb{Q}$ -divisors  $F_1 \geq 0$  and  $G_1 \geq 0$  such that

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_*B_0 + f_*H) + \phi^*A_S \text{ and } G_1 \sim_{\mathbb{Q}} -(K_Y + g_*B_0 + g_*H) + \psi^*A_T$$

and

$$K_W + B_0 + H + F + G$$

is klt, where  $F = f^*F_1$  and  $G = g^*G_1$ .

Let  $V_0$  be the affine subspace of  $\text{WDiv}_{\mathbb{R}}(W)$  which is the translate by  $B_0$  of the vector subspace spanned by  $H_1, \dots, H_k, F, G$ . Suppose that  $D = A + B \in \mathcal{L}_A(V_0)$ . Then

$$D - B_W = (A + B_0 - B_W) + (B - B_0)$$

is ample, as  $B - B_0$  is nef by definition of  $V_0$ . Note that

$$B_0 + F + H \in \mathcal{A}_{A, \phi \circ f}(V_0), B_0 + G + H \in \mathcal{A}_{A, \psi \circ g}(V_0)$$

and  $f$ , respectively  $g$ , is a weak log canonical model of  $K_W + B_0 + F + H$ , respectively  $K_W + B_0 + G + H$ . Thus Theorem 5.2 implies that  $V_0$  satisfies (1-4) of Theorem 5.2.

Since  $H_1, \dots, H_k$  generated the Neron-Severi group of  $W$  we may find constants  $h_1, \dots, h_k$  such that  $G \equiv \sum_{i=1}^k h_i H_i$ . Then there is  $0 < \delta \ll 1$  such that  $B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \in \mathcal{L}_A(V_0)$  and

$$B_0 + F + \delta G + H - \delta\left(\sum_i^k h_i H_i\right) \equiv B_0 + F + H.$$

Thus  $\mathcal{A}_{A, \phi \circ f}$  is not contained in the boundary of  $\mathcal{L}_A(V_0)$ . Similarly  $\mathcal{A}_{A, \psi \circ g}$  is not contained in the boundary of  $\mathcal{L}_A(V_0)$ . In particular  $\mathcal{A}_{A, \phi \circ f}$  and  $\mathcal{A}_{A, \psi \circ g}$  span affine hyperplanes of  $V_0$ , since  $\rho(X/S) = \rho(Y/T) = 1$ .

Let  $V_1$  be the translate by  $B_0$  of the two dimensional vector space spanned by  $F + H - A$  and  $G + H - A$ . Let  $V$  be a small general perturbation of  $V_1$  as in Lemma 5.3, which is defined over rationals. This is the affine subspace we need.  $\square$

Then we can prove the main theorem

*Proof of Theorem 1.2.* Let  $(W, B_W), A$  and  $V$  as in the lemma 5.7. Pick  $D_0 \in \mathcal{A}_{A, \phi \circ f}$  and  $D_1 \in \mathcal{C}_{A, g}$  belonging to the interior of  $\mathcal{L}_A(V)$ . As  $V$  is two dimensional, removing  $D_0$  and  $D_1$  divides the boundary of  $\mathcal{E}_A(V)$  into two parts. The part which consists entirely of divisors which are not big is contained in the interior of  $\mathcal{L}_A(V)$ . Consider tracing this boundary from  $D_0$  to  $D_1$ . Then there are finitely many  $2 \leq i \leq N$  points  $D_i$  which are contained in more than two polytopes  $\mathcal{C}_{A, f_i}(V)$ . By lemma 5.6, each point  $D_i$  gives a Sarkisov link. And the birational map  $X \dashrightarrow Y$  is composition of such links.  $\square$

## 6. EXAMPLES

In this section we give an example for each method.

**6.1. Original method.** Let  $X = \mathbb{P}^2$  with coordinates  $(x_0 : x_1 : x_2)$  and  $X' = \mathbb{P}^2$  with coordinates  $(y_0 : y_1 : y_2)$ . Denote  $B = \{x_0 = 0\}$  and  $B' = \{y_0 = 0\}$ . Take a rational map  $\Phi : X \dashrightarrow X'$  defined by

$$\Phi : (x_0 : x_1 : x_2) \dashrightarrow (x_0^2 : x_0 x_1 : x_1^2 + x_0 x_2)$$

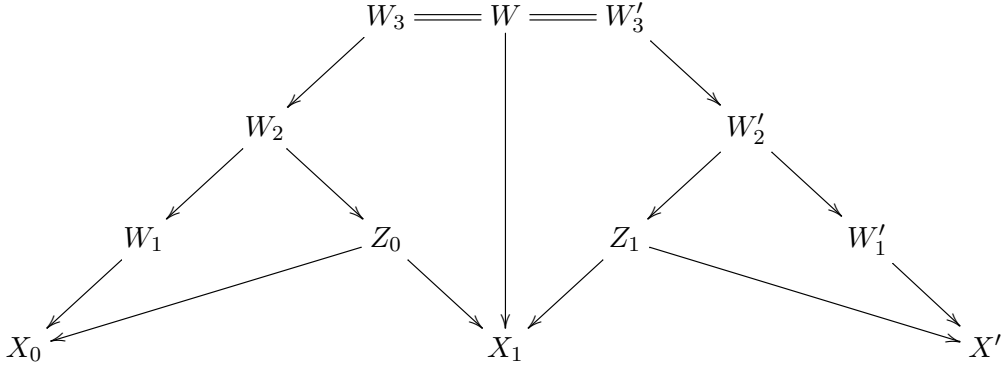
There is a common resolution  $\sigma : W \rightarrow X$  and  $\sigma' : W \rightarrow X'$ , which are both composition of three blowing-ups at indeterminacy points. More precisely,  $\pi_1 : W_1 \rightarrow X$  is blowing-up at indeterminacy point  $P_0 \in B$  of  $\Phi$ . Identifies  $B$  with its strict transform on  $W_1$  and let  $E_1$  be the exceptional divisor of  $\pi_1$ .  $\pi_2 : W_2 \rightarrow W_1$  is blowing-up at  $P_1 = E_1 \cap B$ . Identifies  $B$  and  $E_1$  with their strict transforms on  $W_2$  and let  $E_2$  be the exceptional divisor of  $\pi_2$ .  $\pi_3 : W = W_3 \rightarrow W_2$  is blowing-up at  $P_2 \in E_2 \setminus (B \cup E_1)$ . Identifies  $B, E_1$  and  $E_2$  with their strict transforms on  $W_3$  and let  $E_3$  be the exceptional divisor of  $\pi_3$ . Then  $\sigma = \pi_3 \circ \pi_2 \circ \pi_1$  and  $W = W_3$  is a common

resolution of  $\Phi$ . Moreover  $\sigma' : W \rightarrow X'$  is composition of blowing-down curves in the order of  $B, E_2, E_1$ , and  $\sigma'_*(E_3) = B'$ . We denote as  $W = W'_3 \xrightarrow{\pi'_3} W'_2 \xrightarrow{\pi'_2} W'_1 \xrightarrow{\pi'_1} X'$

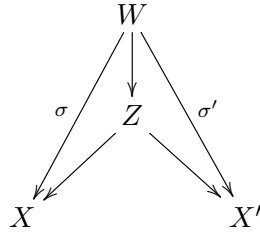
We give some notations of varieties:

- Let  $W_2 \rightarrow Z_0$  be the contraction of  $E_1$  on  $W_2$ , then  $Z_0 \rightarrow X_1$  is the contraction of  $B$  and  $Z_0 \rightarrow X_0$  is the extraction of  $E_2$  on  $X$ ;
- Let  $W'_2 \rightarrow Z_1$  be the contraction of  $E_1$  on  $W'_2$ , then  $Z_1 \rightarrow X_1$  is the extraction of  $E_3$  on  $X_1$ , and  $Z_1 \rightarrow X'$  is the contraction of  $E_2$ ;
- $W \rightarrow Z$  be the contraction of  $E_1$  and  $E_2$  on  $W$ , then  $Z \rightarrow X$  is the extraction of  $E_3$  and  $Z \rightarrow X'$  is the contraction of  $B$ .

That is



and



Consider the pairs  $(X, bB)$  and  $(X', b'B')$ , and take the function  $\theta$  such that:

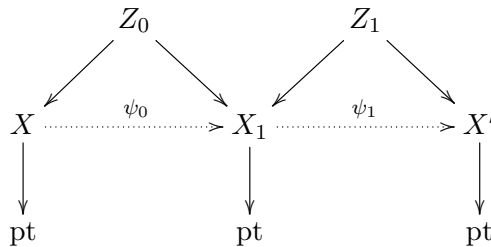
- $\theta(B) = b$  and  $\theta(B') = b'$ ;
- $\theta(E_1) = \theta(E_2) = \epsilon$  with  $b, b' < \epsilon < 1$ .

Then we have ramification fomulas:

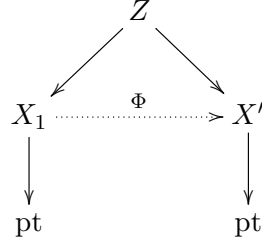
$$\begin{aligned} K_W + B_W &= \sigma^*(K_X + bB) + (3 - 2b + b')E_3 + (1 - b + \epsilon)E_1 + (2 - 2b + \epsilon)E_2 \\ &= \sigma'^*(K_{X'} + b'B') + (3 - 2b' + b)B + (1 - b' + \epsilon)E_1 + (2 - 2b' + \epsilon)E_2 \end{aligned}$$

Let  $\mathcal{H}' = |\mathcal{O}(1)|$  be the very ample complete linear system on  $X'$ , then  $H \in |\mathcal{O}_X(2)|$ . Different choices of  $\theta$  and  $e$  gives different decompositions:

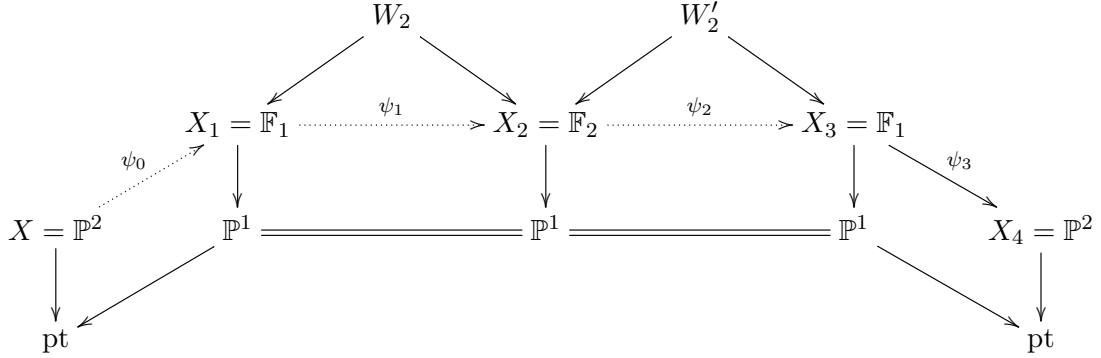
- (1) If  $2b + 2b' \geq 3\epsilon > 0$ , then  $\Phi$  is the composition of two Sarkisov links  $\psi_0, \psi_1$  of type II:



(2) If  $2b + 2b' < 3\epsilon$ , then  $\Phi$  is just one Sarkisov link of type II:



(3) If  $\epsilon = b = b' = 0$ , then  $\Phi$  is the composition of four Sarkisov links  $\psi_i$ :

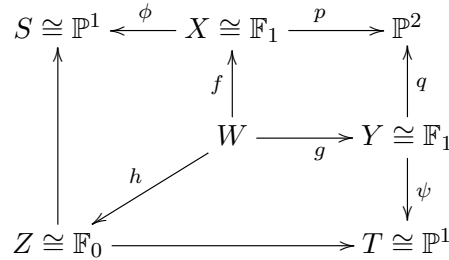


**6.2. Double scaling.** Notations and assumptions as 6.1, let  $B_W = \frac{1}{2}(B + E_1 + E_3)$  and consider pairs  $(X, \frac{1}{2}B)$  and  $(X', \frac{1}{2}B')$ . Then we have  $G = G_0 \sim_{\mathbb{Q}} \frac{5}{2}B$  and  $H' \sim_{\mathbb{Q}} \frac{5}{2}B'$ .

- (1)  $r_0 = 2$  and  $s_0 = \frac{1}{5}$ .  $X_1$  is a weak log canonical model of  $(W, B_W + \frac{3}{5}G_W + \frac{1}{5}H_W)$ ;
- (2)  $r_1 = 1$  and  $s_1 = \frac{2}{5}$ .  $X_2 = X'$  is weak log canonical model of  $(W, B_W + \frac{1}{5}G_W + \frac{3}{5}H_W)$ .

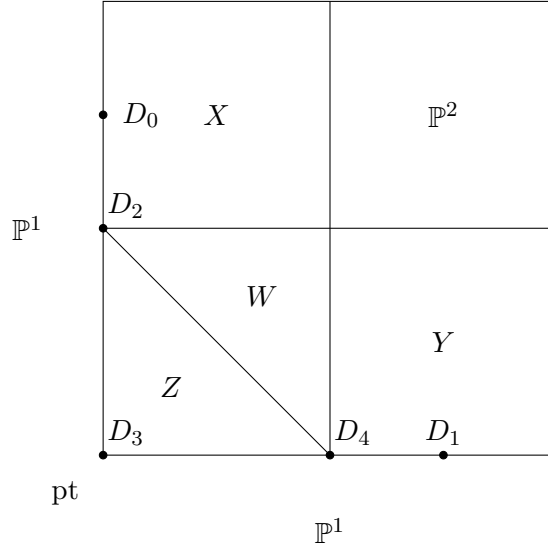
This gives the same decomposition as case (1) in Section 6.1.

**6.3. Polytope.** Let  $P, Q$  be two different points on  $\mathbb{P}^2$  and let  $L$  be the line passing through  $P$  and  $Q$ . Let  $p : X \rightarrow \mathbb{P}^2$  be the blowing up at  $P$  and  $E_1$  be the exceptional divisor. Let  $q : Y \rightarrow \mathbb{P}^2$  be the blowing up at  $Q$  and  $E_2$  be the exceptional divisor. Let  $W \rightarrow \mathbb{P}^2$  be the blowing up of  $P$  and  $Q$ , then we have contractions  $f : W \rightarrow X$  and  $g : W \rightarrow Y$ . Identifies  $L, E_1$  and  $E_2$  with their strict transforms on  $W$ . Let  $h : W \rightarrow Z$  be the contraction of  $L$ , then  $Z \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ .

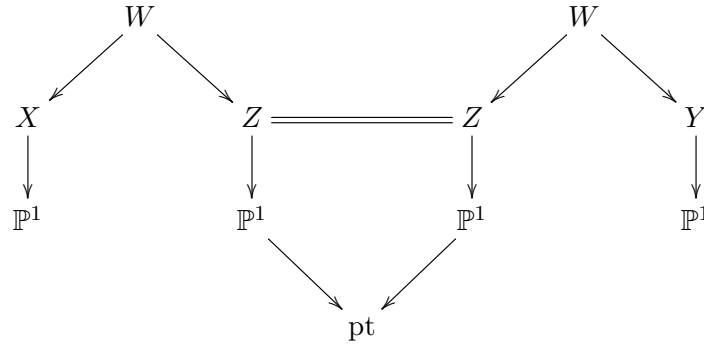


Note that  $X \cong \mathbb{F}_1$ , there is a Mori fibre space  $\phi : X \rightarrow S \cong \mathbb{P}^1$ . Similarly there is another Mori fibre space  $\psi : Y \rightarrow T \cong \mathbb{P}^1$ . There is a birational map  $\Phi : X \dashrightarrow Y$  induced by  $p$  and  $q$ . In fact, if we take  $B_W = \frac{1}{4}L$  on  $W$ , then  $f$  and  $g$  are two log Mori fibre spaces as outputs of  $(K_W + B_W)$ -MMP on  $W$ .

Take  $A \sim_{\mathbb{Q}} -K_W + \frac{1}{4}L$ , and let  $V$  be the translate by  $\frac{1}{4}L$  of the 2-dimensional vector space spanned by  $E_1$  and  $E_2$ . Then we have  $\mathcal{L}_A(V) = \mathcal{E}_A(V)$ . Furthermore,  $K_W + D \sim_{\mathbb{Q}} \frac{1}{2}L + aE_1 + bE_2$  for  $0 \leq a, b \leq 1$  if  $D \in \mathcal{E}_A(V)$ . The partition of  $\mathcal{E}_A(V)$  is



Then  $D_0$  and  $D_1$  correspond to log Mori fibre spaces  $\phi : X \rightarrow S$  and  $\psi : Y \rightarrow T$ .  $D_2, D_3$  and  $D_4$  correspond to three Sarkisov links. Therefore we have a decomposition of  $\Phi : X \dashrightarrow Y$  as



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