

Sarkisov program

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1 Introduction

The purpose of this article is to show that two different Mori fibre spaces as outputs of a klt pair can be linked by composition of Sarkisov links.

1.1 Motivation and Main theorem

The **Minimal model program (MMP)** aims to classify varieties up to birational equivalent classed, by finding a minimal model all or Mori fibre space. Let (X, B) be a (klt or lc) pair, and assume we can run $(K_X + B)$ -MMP on it. Note that the varieties appear in the program are called **results** of the MMP, and the varieties where the MMP ends are called the **output** of the MMP.

1. If $\kappa(X, B) \geq 0$, then we expected that MMP ends with a **minimal model**, i.e. a birational map $X \dashrightarrow Y$ such that $(K_Y + B_Y)$ is nef;
2. If $\kappa(X, B) = -\infty$, then we expected that MMP ends with a Mori fibre space, i.e. a birational map $X \dashrightarrow Y$ and a contraction $Y \rightarrow S$ such that $\dim Y < \dim X$ and $-(K_Y + B_Y)$ is relative ample.

However, for each case the output may not be unique.

For the first case, it is shown that two different minimal model can be linked by flops:

Theorem 1.1.1. [9, Theorem 1] *Let (W, B_W) be a \mathbb{Q} -factorial terminal pair, and $(X, B), (Y, D)$ are two minimal models of (W, B_W) . Then the birational map $X \dashrightarrow Y$ may be factored as sequence of $(K_X + B)$ flops.*

For the second case, it is shown that:

Theorem 1.1.2. *Let $f : (X, B) \rightarrow S$ and $f' : (X', B') \rightarrow S'$ be two MMP related \mathbb{Q} -factorial klt log Mori fibre spaces with induced induced birational map Φ :*

$$\begin{array}{ccc} (X, B) & \xrightarrow{\Phi} & (X', B') \\ f \downarrow & & \downarrow f' \\ S & & S' \end{array}$$

Then Φ can be decomposed into sequence of Sarkisov links.

Definition 1.1.3. *The following four types of birational maps $X \dashrightarrow X_1$ are called Sarkisov links:*

$$\begin{array}{llll} \text{I:} & \begin{array}{ccc} Z & \dashrightarrow & X_1 \\ p \downarrow & & \downarrow f_1 \\ X & & S_1 \\ f \downarrow & \nearrow t & \\ S & & \end{array} & \text{II:} & \begin{array}{ccc} Z & \dashrightarrow & Z' \\ p \downarrow & & \downarrow q \\ X & & X_1 \\ f \downarrow & & \downarrow f_1 \\ S & \xrightarrow{\sim} & S_1 \end{array} & \text{III:} & \begin{array}{ccc} X & \dashrightarrow & Z \\ f \downarrow & & \downarrow q \\ S & & X_1 \\ & \searrow s & \downarrow f_1 \\ & & S_1 \end{array} & \text{IV:} & \begin{array}{ccc} X & \dashrightarrow & X_1 \\ f \downarrow & & \downarrow f_1 \\ S & & S_1 \\ & \searrow s & \nearrow t \\ & & T \end{array} \end{array}$$

Here, all $f : (X, B) \rightarrow S$ and $f_1 : (X_1, B_1) \rightarrow S_1$ are log Mori fibre space, and all p, q are divisorial contractions, and all dash arrows are composition of flips, flops and inverse flips.

1.2 Using MMP

Assume $f : (X, B) \rightarrow S'$ and $f' : (X', B') \rightarrow S'$ are two Mori fibre spaces as outputs of $(K_W + B_W)$ -MMP on W . The Sarkisov program constructs each Sarkisov link $X_i \dashrightarrow X_{i+1}$ inductively. For each X_i we shall find some W_i such that X_i and X_{i+1} are two Mori fibre spaces as outputs of certain MMP on W_i . Moreover, $W_i \dashrightarrow X_{i+1}$ is a 2-tay game. More precisely, there are two cases:

1. Find a contraction $g : X_i \rightarrow T_i$ such that $\rho(X_i/T_i) = 2$ and factor through $f_i : X_i \rightarrow S_i$, then we run MMP on X_i over T_i , and obtains a Sarkisov link of type III or IV. Here $W_i = X_i$;
2. Find a divisorial contraction $p : Z_i \rightarrow X_i$, and therefore $\rho(Z_i/S_i) = 2$. Then we run MMP on Z_i over S_i , and obtains a Sarkisov link of type I or II. Here $W_i = Z_i$.

In [6], original proof;
 In [7], double scaling;

1.3 Using polytope

1.4 Structure of the article

2 Preliminary

In this article, all varieties are over complex number \mathbb{C} .

2.1 Models

Definition 2.1.1. [8, 2. Notation and Conventions] A rational map $f : X \dashrightarrow S$ is called a **rational contraction** if there is a resolution $p : W \dashrightarrow X$ and $q : W \dashrightarrow Y$ of f such that p and q are contraction morphisms and p is birational. f is called a **birational contraction** if q is in addition birational and every p -exceptional divisor is q -exceptional. If in addition f^{-1} is also a **birational contraction**, then f is called a **small birational map**.

Definition 2.1.2. [4, Definition 3.6.1] Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties, and $p : W \rightarrow X$ and $q : W \rightarrow Y$ be a resolution of indeterminacy of f . Let D be a \mathbb{R} -Cartier divisor on X such $D_Y = f_*D$ is also \mathbb{R} -Cartier. Then f is called **D -non-positive** (**D -negative**) if

- f does not extract any divisor;
- $E = p^*D - q^*D_Y$ is effective and exceptional over Y (and $\text{Supp } p_*E$ contains all f -exceptional divisors).

Definition 2.1.3. [7, 13.2. Notation and conventions] Let $f : X \dashrightarrow Y$ be a rational map of normal quasi-projective varieties over S , and D be a \mathbb{R} -Cartier \mathbb{R} -divisor on X with $f_*D = D_Y$. Then f is called **D -trivial** if D is pull back of a \mathbb{R} -Cartier divisor on S .

Recall the definitions of models in [4]

Definition 2.1.4. [4, Definition 3.6.5] Let $\pi : (X, D) \rightarrow U$ be a projective morphism of normal quasi-projective varieties and let D be an \mathbb{R} -Cartier divisor on X . Let $f : X \dashrightarrow Y$ be a birational map over U , then Z is an **semiample model** for D over U if f is $K_X + D$ -non-positive and $K_Y + f_*D$ is semiample over U .

Let $g : X \dashrightarrow Z$ be a rational map over U , then Z is an **ample model** for D over U if there is an ample divisor over U on Z such that if $p : W \rightarrow X$ and $q : W \rightarrow Z$ resolves g , then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{R}, U} q^*H + E$, where $E \geq 0$ and for any $B \in |p^*D/U|_{\mathbb{R}}$, then $B \geq E$.

Definition 2.1.5. [4, Definition 3.6.7] Let $\pi : (X, D) \rightarrow U$ be a projective morphism of normal quasi-projective varieties, if $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then define:

1. Y is **weak log canonical model** for $K_X + D$ over U if f is $K_X + D$ -non-positive and $K_Y + f_*D$ is nef over U ;
2. Y is **log canonical model** for $K_X + D$ over U if f is $K_X + D$ -non-positive and $K_Y + f_*D$ is ample over U ;
3. Y is **log terminal model** for $K_X + D$ over U if f is $K_X + D$ -negative and $K_Y + f_*D$ is dlt and nef over U and Y is \mathbb{Q} -factorial.

Lemma 2.1.6. [4, Lemma 3.6.6] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties and let D be an \mathbb{R} -Cartier divisor on X .

1. If $g_i : X \dashrightarrow X_i, i = 1, 2$ are two ample models of D over U , then there is an isomorphism $h : X_1 \rightarrow X_2$ and $g_2 = h \circ g_1$.
2. If $f : X \dashrightarrow Y$ is a semiample model of D over U , then the ample model $g : X \dashrightarrow Z$ of D over U exists and $g = h \circ f$, where $h : Y \rightarrow Z$ is a contraction and $f_*D \sim_{\mathbb{R}, U} h^*H$.
3. If $f : X \dashrightarrow Y$ is a birational map over U , then f is the ample model of D over U if and only if f is semiample model of D over U and f_*D is ample over U .

By above lemma there is another definition of log canonical models:

Definition 2.1.7. Let $\pi : (X, D) \rightarrow U$ be a projective morphism of normal quasi-projective varieties and $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then Y is **log canonical model** if it is the ample model.

Futhermore, for big boundary, we have

Lemma 2.1.8. [4, Lemma 3.9.3] Let $\pi : (X, D) \rightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose (X, B) is a klt pair and B is big over U . If $f : X \dashrightarrow Y$ is a weak log canonical model over U then

- f is a semiample model over U ;
- the ample model $g : X \dashrightarrow Z$ over U exists;
- there is a contraction $h : Y \rightarrow Z$ such that $K_Y + f_*B \sim_{\mathbb{R}, U} h^*H$ for some ample \mathbb{R} -divisor H on Z over U .

Definition 2.1.9. [4, Definition 1.1.4] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of $\text{WDiv}_{\mathbb{R}}(X)$ defined over rational numbers. Define

$$\begin{aligned}\mathcal{L}(V) &= \{D \in V : K_X + D \text{ is log canonical} \} \\ \mathcal{N}_{\pi}(V) &= \{D \in \mathcal{L} : K_X + D \text{ is nef over } U\}\end{aligned}$$

Moreover, if fix an \mathbb{R} -divisor $A \geq 0$, and then define

$$\begin{aligned} V_A &= \{D = A + B : B \in V\} \\ \mathcal{L}_A(V) &= \{D = A + B \in V_A : K_X + D \text{ is log canonical and } B \geq 0\} \\ \mathcal{E}_{A,\pi}(V) &= \{D = A + B \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\} \\ \mathcal{N}_{A,\pi}(V) &= \{D \in \mathcal{L}_A(V) : K_X + D \text{ is nef over } U\} \end{aligned}$$

Given a birational contraction $f : X \dashrightarrow Y$, define

$$\mathcal{W}_{A,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is weak log model of } (X, D) \text{ over } U\}$$

Given a rational contraction $g : X \dashrightarrow Z$ over U , define

$$\mathcal{A}_{A,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is ample model of } (X, D) \text{ over } U\}$$

In addition, let $\mathcal{C}_{A,g}(V)$ denote the closure of $\mathcal{A}_{A,g}(V)$

By [4, Lemma 3.7.2], if V is a rational subspace, then $\mathcal{L}_A(V)$ is a rational polytope.

Lemma 2.1.10. [4, Theorem E] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and A be an general divisor relatively ample over U , and $V \subset \text{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a klt pair (X, Δ_0) . Then there are finitely many birational maps $f_i : X \dashrightarrow X_i$ such that if $f : X \dashrightarrow Y$ is a weak log canonical model of $K_X + D$ over U for some $D \in \mathcal{L}_A(V)$, then there is an isomorphism $h_i : X_i \rightarrow Y$ and $f = h_i \circ f_i$.

Theorem 2.1.11. [4, Corollary 1.1.5] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and A be an general divisor relatively ample over U , and $V \subset \text{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a divisor $\Delta_0 \in V$ such that (X, Δ_0) is klt. Let A be a general ample \mathbb{Q} -divisor over U which has no components common with any element of V .

1. There are finitely many birational maps $f_i : X \dashrightarrow X_i$ over U such that

$$\mathcal{E}_{A,\pi}(V) = \bigcup_i \mathcal{W}_i$$

where $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$ is a rational polytope. Moreover, if $f : X \dashrightarrow Y$ is a log terminal model of $K_X + D$ over U for some $D \in \mathcal{E}_A(V)$, then $f = f_i$ for some i .

2. There are finitely many birational maps $g_j : X \dashrightarrow Z_j$ over U such that

$$\mathcal{E}_{A,\pi}(V) = \coprod_j \mathcal{A}_j$$

$\{\mathcal{A}_j = \mathcal{A}_{A,g_j}\}$ is a partition of $\mathcal{E}_A(V)$. \mathcal{A}_i is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;

3. For every f_i there is a g_j and a morphism $h_{ij} : Y_i \rightarrow Z_j$ such that $\mathcal{W}_i \subset \overline{\mathcal{A}_j}$.

In particular $\mathcal{E}_{A,\pi}$ is a rational polytope and $\overline{\mathcal{A}_j}$ is a finite union of rational polytopes.

2.2 MMP

Definition 2.2.1. Let (X, B) be a pair and let $f : Y \rightarrow X$ be a log resolution of (X, B) . Suppose

$$K_Y + C = f^*(K_X + B),$$

then the discrepancy of exceptional divisor E_i over X is

$$a(E_i; X, B) = -\text{mult}_{E_i} C.$$

Moreover, let

$$\text{discrep}(X, B) := \inf\{a(E; X, B) : E \text{ is an exceptional divisor over } X\}$$

and

$$\text{totdiscrep}(X, B) := \inf\{a(E; X, B) : E \text{ is a divisor over } X\}.$$

Theorem 2.2.2. [4, Corollary 1.4.2] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and let (X, B) be a \mathbb{Q} -factorial klt pair where $K_X + B$ is \mathbb{R} -Cartier and B is π -big. Let $C \geq 0$ be an \mathbb{R} -divisor. If $K_X + B + C$ is klt and π -nef, then we may run $(K_X + B)$ -MMP over U with scaling of C and terminates.

Theorem 2.2.3. [4, Corollary 1.3.3] Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties, and let (X, B) be a \mathbb{Q} -factorial klt pair where $K_X + B$ is \mathbb{R} -Cartier. If $K_X + B + C$ is not π -pseudo-effective, then we may run $f : X \dashrightarrow Y$ a $(K_X + B)$ -MMP over U and end with a Mori fibre space $g : Y \rightarrow Z$.

Corollary 2.2.4. [7, Corollary 13.7] and [4, Corollary 1.4.3]: Let (X, B) be a klt pair and \mathfrak{C} be any set of exceptional divisors such that contains only exceptional divisors E of discrepancy $a(E; X, B) \leq 0$. Then there is a birational morphism $f : Z \rightarrow X$ and a \mathbb{Q} -divisor B_Z such that:

1. (Z, B_Z) is klt;
2. E is a f -exceptional divisor if and only if $E \in \mathfrak{C}$;
3. $B_Z = \sum -a(E; X, B)$ and $f_* B_Z = B$ and $K_Z + B_Z = f^*(K_X + B)$.

In particular, if we take \mathfrak{C} containing all such divisors, then Z is called **terminalization** of X ; if take \mathfrak{C} containing only one such divisor, then $f : Z \rightarrow X$ is called a **divisorial extraction**.

Definition 2.2.5. [5, Definition 3.3] Two or more pairs $\{(X_i, B_i)\}$ are called **MMP-related** if they are results of $(K + B)$ -MMP from a log smooth pair (W, B_W) .

Lemma 2.2.6. [5, Proposition 3.4] Let $\{(X_l, B_l)\}$ be a finite set of \mathbb{Q} -factorial klt pairs such that birational to other, then TFAE:

1. They are MMP-related;
2. There is a nonsingular pair (W, B_W) with snc boundary, and projective birational morphisms $f_l : W \rightarrow X_l$ dominating each X_l , such that $f_{l*} B_W = B_l$ and

$$K_W + B_W = f_l^*(K_{X_l} + B_l) + \sum_{\text{exceptional}} a_{li} E_{li}$$

with $a_{li} > 0$ for all f_i -exceptional divisors;

3. For any two pairs $(X, B = \sum_i b_i B_i), (X', B' = \sum_j b'_j B'_j)$ in the set, $a(B_i; X', B') \geq -b_i$ and strict inequality holds if and only if B_i exceptional over X' , and $a(B'_j; X, B) \geq -b'_j$ and strict inequality holds if and only if B'_j exceptional over X .

Let $K = K(X)$ be the function field, and let $\Sigma = \{\nu\}$ be the set of discrete valuations of the field

Definition 2.2.7. [5, Definition 3.5] Let $\theta : \Sigma \rightarrow [0, 1)_{\mathbb{Q}}$ be a function. Then we can construct a collection \mathcal{C}_θ of pairs associated to θ , consists of klt pairs $(X, B = \sum a_i B_i)$ satisfying

1. $a_i = \theta(B_i)$;
2. $a(E; X, B) > -\theta(E)$ for all E exceptional over X .

For example, if we take $\theta \equiv 0$ constant, the \mathcal{C}_θ is the collection of all terminal varieties Y without boundary birational to X . Furthermore, we can define the corresponding discrepancy:

Definition 2.2.8 (θ -discrepancy). Let (X, B) be a pair in the category \mathcal{C}_θ for some function θ and let $f : Y \rightarrow X$ be a log resolution of (X, B) . Suppose

$$K_Y + B_Y + C = f^*(K_X + B)$$

where $B_Y = (f^{-1})_* B + \sum_{E_i \text{ exceptional}} \theta(E_i) E_i$, then the θ -discrepancy of exceptional divisor E_i over X is

$$a_\theta(E_i; X, B) = -\text{mult}_{E_i} C.$$

Or equivalently, we have

$$a_\theta(E_i; X, B) = a(E_i; X, B) + \theta(E_i).$$

A pair (X, B) is called θ -canonical(θ -terminal) if $a_\theta(E; X, B) \geq 0$ ($a_\theta(E; X, B) > 0$) for all exceptional divisors E over X . Note that θ -canonical pair is not always in \mathcal{C}_θ .

3 Original proof

3.1 Prepare

First we fix a category:

Proposition 3.1.1. [5, Lemma 3.6] Let $f : (X, B) \rightarrow S, f' : (X', B') \rightarrow S'$ be two \mathbb{Q} -factorial log Mori fibre spaces with only klt singularities and MMP-related, inducing a birational map Φ :

$$\begin{array}{ccc} (X, B) & \xrightarrow{\Phi} & (X', B') \\ f \downarrow & & \downarrow f' \\ S & & S' \end{array}$$

Suppose $B = \sum_i b_i B_i + \sum_j d_j D_j$ and $B' = \sum_j d'_j D_j + \sum_k b'_k B'_k$, where B_i are divisors on X but not on X' , B'_k are divisors on X' but not on X , and D_j are divisors on both X and X' . By Lemma 2.2.6, $d_j = d'_j$. Take a rational number $\epsilon < 1$ such that $\epsilon > -\text{totdiscrep}(X, B), -\text{totdiscrep}(X', B')$, and take the function $\theta : \{\nu\} \rightarrow [0, 1)_{\mathbb{Q}}$ as following:

- $\theta(B_i) = b_i, \theta(D_j) = d_j, \theta(B'_k) = b'_k$;
- $\theta(E) = \epsilon$ if E is exceptional over both X and X' ;
- $\theta(D) = 0$ if D is a divisor on both X and X' , but not a component of B or B' .

Then the collection \mathcal{C}_θ satisfies

1. (X, B) and (X', B') belongs to \mathcal{C}_θ ;
2. For any finitely many klt pairs $\{(X_l, B_l)\}$ in \mathcal{C}_θ , there is an object $(Z, B_Z) \in \mathcal{C}_\theta$ and projective birational morphisms $Z \rightarrow X_l$ dominating each X_l as a process of $(K_Z + B_Z)$ -MMP over X_l (thus over $\text{Spec } \mathbb{C}$);
3. Any $(K + B)$ -MMP starting from an object in \mathcal{C}_θ stays inside of \mathcal{C}_θ , and so does any $(K + B + cH)$ -MMP where H is base point free and $c \in \mathbb{Q}_{>0}$.

Remark 3.1.2. Let $\delta = 1 - \epsilon$, then all pairs in \mathcal{C}_θ is δ -lc.

With notations and assumptions in Proposition 3.1.1, we shall define the Sarkisov degree. We take a very ample divisor A' on S' and a sufficiently large and divisible integer $\mu' > 1$ such that

$$\mathcal{H}' = |-\mu'(K_{X'} + B') + f'^*A'|$$

is a very ample complete linear system on X' over $\text{Spec } \mathbb{C}$. Let (W, B_W) be a common log resolution of X and X' in \mathcal{C}_θ with projective birational morphism $\sigma : W \rightarrow X$, $\sigma' : W \rightarrow X'$ and $\sigma_*B_W = B, \sigma'_*B_W = B'$. Let $\mathcal{H}_W := \sigma'^*\mathcal{H}'$ and then $\mathcal{H} := (\Phi^{-1})_*\mathcal{H}' = \sigma_*\mathcal{H}_W$. Furthermore, if \mathcal{H} is not base point free, then

$$\sigma^*\mathcal{H} = \mathcal{H}_W + F$$

where $F = \sum f_i F_i \geq 0$ is the fixed part. Take a general member H' of the linear system \mathcal{H}' such that $H_W := \sigma'^*H' = (\sigma'^{-1})_*H' \in \mathcal{H}_W$, and let $H := (\Phi^{-1})_*H' = \sigma_*H_W$, then H is f -ample and $\sigma^*H = H_W + F$. By taking further resolution, we may assume H_W is smooth and crosses normally with exceptional locus of σ and σ' .

Now we can define the Sarkisov degree in \mathcal{C}_θ with respect to H' (or \mathcal{H}'):

Definition 3.1.3. [5, Definition 3.8] *Sarkisov degree of (X, B) with respect to H (or \mathcal{H}) in \mathcal{C}_θ is a triple (μ, λ, e) ordered lexicographically:*

- **Nef threshold μ :** Let $C \subset X$ be a curve contracted by f , then

$$\mu := -\frac{H.C}{(K_X + B).C}$$

$$\text{i.e. } K_X + B + \frac{1}{\mu}H \equiv_S 0;$$

- **θ -canonical threshold c and λ :** $\lambda = 0$ if \mathcal{H} is base point free; otherwise,

$$c := \frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geq 0, E \text{ exceptional over } X\}$$

- **Number of $(K_X + B_X + \frac{1}{\mu}H)$ -crepant divisors:** Let $e = 0$ if \mathcal{H} is base point free (and hence $\lambda = 0$), otherwise

$$e = \#\{E; E \text{ is } \sigma\text{-exceptional and } a_\theta(E; X, B + \frac{1}{\lambda}H) = 0\}$$

Remark 3.1.4. 1. The Sarkisov degree is dependent on the choice of A' , H' and θ .

2. Take a common log resolution $(W, B_W) \in \mathcal{C}_\theta$ with $B_W = \sum \theta(E)E$ and projective birational morphisms $\sigma : W \rightarrow X$, $\sigma' : W \rightarrow X'$. Since $\sigma^*\mathcal{H} = \mathcal{H}_W + \sum f_l F_l$, we have ramification formula:

$$K_W + B_W + tH_W = \sigma^*(K_X + B + tH) + \sum (a_l - t f_l)E_l$$

where $\sum a_l E_l$ is effective and supported on $\text{Exc } \sigma$. Then $\lambda := \max\{\frac{f_l}{a_l}\}$. If \mathcal{H} is base point free, then $\sum f_l F_l = 0$ and $\lambda = 0$.

3. e is the number of components in $\sum (a_l - c f_l)E_l$ with coefficient 0 in the formula

$$K_W + B_W + \frac{1}{\lambda}H_W = \sigma^*(K_X + B + \frac{1}{\lambda}H) + \sum (a_l - \lambda f_l)E_l.$$

Such prime divisors $E_1 \dots E_e$ are called $(K_X + B_X + \frac{1}{\lambda}H)$ - θ -crepant.

We also need some extraction map in this category:

Lemma 3.1.5. Using the notation in the definition of Sarkisov degree, then there is a contraction $f : Z \rightarrow X$ such that

- $(Z, B_Z) \in \mathcal{C}_\theta$ and $(Z, B_Z + \frac{1}{\lambda}H_Z)$ is θ -terminal and \mathbb{Q} -factorial;
- $\rho(Z) = \rho(X) + 1$;
- f is $(K_X + B_X + \frac{1}{\lambda}H_X)$ -crepant, that is

$$K_Z + B_Z + \frac{1}{\lambda}H_Z = f^*(K_X + B + \frac{1}{\lambda}H).$$

Proof. We follow the proof in [5, Proposition 1.6] but for klt pair case. Let $(W, B_W) \in \mathcal{C}_\theta$ and $\sigma : W \rightarrow X, \sigma' : W \rightarrow X'$ be the common resolution as in Definition 3.1.3, and suppose E_1, \dots, E_e are $(K_X + B_X + \frac{1}{\mu}H)$ - θ -crepant divisors after renumbering. Then we have

$$K_W + B_W + \frac{1}{\lambda}H_W = \sigma^*(K_X + B + \frac{1}{\lambda}H) + \sum_{l=1}^e 0 \cdot E_l + \sum_{l>e} (a_l - \frac{1}{\lambda}f_l)E_l.$$

We run $(K_W + B_W + \frac{1}{\lambda}H_W)$ -MMP on W over X with scaling of some ample divisor, then the MMP ends with a minimal model $p : (Y, B_Y + \frac{1}{\lambda}H_Y) \rightarrow X$ of $(W, B_W + \frac{1}{\lambda}H_W)$ over X and the exceptional locus is exactly $\cup_{i=1}^e E_i$ and p is crepant:

$$K_Y + B_Y + \frac{1}{\lambda}H_Y = p^*(K_X + B_X + \frac{1}{\lambda}H_X).$$

Then we run $(K_Y + B_Y)$ -MMP on Y over X with scaling of some ample divisor. This ends with the minimal model (X, B) of (Y, B_Y) over X , and the last contraction in the MMP is $f : Z \rightarrow X$ as required. \square

3.2 Flowchart for the Log Sarkisov program

We follow [5, Flowchart for the Sarkisov program] in this subsection.

If $\lambda \leq \mu$ and $K_X + B + \frac{1}{\mu}H$ is nef, the two Mori fibre spaces are isomorphic (shown in next subsection by proposition 3.3.5) and we stop here. Otherwise:

- Claim 3.2.1.** 1. If $\lambda \leq \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef, then there is a contraction $f : X \rightarrow T$ and a Sarkisov link $\phi_1 : X \dashrightarrow X_1$ of type III or IV; ...
2. If $\lambda > \mu$, then there is a divisorial extraction $p : Z \rightarrow X$ and a Sarkisov link $\phi_1 : X \dashrightarrow X_1$ of type I or II.

Proof. 1. Suppose f is the contraction with respect to a $(K_X + B)$ -negative extremal ray $R = \overline{\text{NE}}(X/S)$, then $(K_X + B + \frac{1}{\mu}H) \cdot R = 0$ by definition of μ . There is an extremal ray $P \subset \overline{\text{NE}}(X)$ such that $(K_X + B + \frac{1}{\mu}H) \cdot P < 0$ and $F := P + R$ is an extremal face (Check [6, 5.4.2] for details). Take $0 < t \ll 1$ such that $(K_X + B + (\frac{1}{\mu} - t)H) \cdot P < 0$, then $(K_X + B + (\frac{1}{\mu} - t)H) \cdot R < 0$ since H is f -ample, and F is a $(K_X + B + (\frac{1}{\mu} - t)H)$ -negative extremal face. Since $(X, B + (\frac{1}{\mu} - t)H)$ is klt, there is a contraction $g : X \rightarrow T$ with respect to F factorizing through $f : X \rightarrow S$. Since $(X, B + \frac{1}{\mu}H)$ is klt, and $\rho(X/T) = 2$, we can run $(K_X + B + \frac{1}{\mu}H)$ -MMP on X with scaling of some ample divisor C . Since $B + \frac{1}{\mu}H$ is relatively big, the MMP terminates. There are following cases:

- (a) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \rightarrow X_1$, and then followed by a Mori fibre space $(X_1, B_1 + \frac{1}{\mu}H_1) \rightarrow S_1$. Then $S_1 \cong T$ and this is a link of type III.
- (b) After finitely many flips $X \dashrightarrow X_1$, first non-flip contraction is a Mori fibre space $f_1 : X_1 \rightarrow S_1$. This is a link of type IV.
- (c) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \rightarrow X_1$ with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where $e > 0$ and $E = \text{Exc } p$ and $g_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \rightarrow T$ is a log minimal model of $(X, B + \frac{1}{\mu}H)$ over T . In fact the only ray of $\overline{\text{NE}}(X_1/T)$ is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ -trivial and hence is $(K_{X_1} + B_1)$ -negative, therefore $(X_1, B_1)/T$ is a log Mori fibre space. Take $S_1 = T$, then this is a link of type III:

- (d) After finitely many flips $X \dashrightarrow X_1$, $(K_X + B + \frac{1}{\mu}H)$ -MMP ends with a log minimal model $(X_1, B_1 + \frac{1}{\mu}H_1)$ over T . Then there is an extremal ray R of $\overline{\text{NE}}(X_1/T)$, which is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ -trivial and $(K_{X_1} + B_1)$ -negative. Let $f_1 : X_1 \rightarrow S_1$ be the contraction with respect to R . This is a link of type IV. In fact, $X \dashrightarrow S_1$ is the ample model of $K_X + B + \frac{1}{\mu}H$.
2. Take an extraction $p : (Z, B_Z, H_Z) \rightarrow (X, B, H)$ as in Lemma 3.1.5. That is, (Z, B_Z) is θ -terminal and $p^*(K_X + B + \frac{1}{\lambda}H) = K_Z + B_Z + \frac{1}{\lambda}H_Z$ where $B_Z = \sum \theta(E_\nu)E_\nu$ and $E = \text{Exc } p$ is a prime divisor on Z . Then we run $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ -MMP on Z over S with scaling of some

ample divisor C . Since Z is covered by $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ -negative curves, $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ is not relatively pseudo-effective. Hence this ends with a Mori fibre space by Theorem 2.2.3. There are two cases:

- (a) After finitely many flips $Z \dashrightarrow Z'$, the first non-flip contraction is a divisorial contraction $q : Z' \rightarrow X_1$. Then $X_1 \rightarrow S$ is a log Mori fibre space of (X, B) and $(X, B + \frac{1}{\lambda}H)$. Let $S_1 = S$ and this is a link of type II.
- (b) After finitely many flips $Z \dashrightarrow X_1$, first non-flip contraction is a fibering contraction $f_1 : X_1 \rightarrow S_1$. Since $(K_{X_1} + B_1 + \frac{1}{\lambda}H_1)$ is f_1 -negative and H_1 is f_1 -ample, $(K_{X_1} + B_1)$ is f_1 -negative, and $(X_1, B_1)/Y$ is a log Mori fibre space. Take $S_1 = Y$ and this is a link of type I.

□

Remark 3.2.2. 1. (a) For case 1a and 1b, since $K_{X_1} + B_1 + \frac{1}{\mu}H_1$ is f_1 -negative, we have $\mu_1 < \mu$.

- (b) For case 1c and 1d, Since $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ is trivial on the ray $R = \overline{\text{NE}}(X_1/S_1)$ for both cases, we have $\mu_1 = \mu$. Notice that $(X_1, B_1 + \frac{1}{\mu}H_1)$ stays θ -canonical, we have $\lambda_1 \leq \mu = \mu_1$, thus this goes back to case 1. Furthermore, for case 1c we have $\rho(X_1) = \rho(X) - 1$.

2. For case 2:

- (a) For both case 2a and 2b, we have $\mu_1 \leq \mu$ with equality holds if and only if
 - either $\dim S_i < \dim S_{i+1}$
 - or $\dim S_i = \dim S_{i+1}$ and the link is square
- (b) We have $\lambda_1 \leq \lambda$ and if $\lambda_1 = \lambda$, then $e_1 < e$.

3.3 Termination

We need following theorems:

Theorem 3.3.1. [3, Theorem 1.1] Let d be a natural number and δ be a positivity real number, then the projective varieties X such that

- (X, B) is a δ -lc pair of dimension d for some boundary B , and
- $-(K_X + B)$ nef and big,

form a bounded family.

Lemma 3.3.2. [2, Lemma 2.24] Let \mathcal{P} be a bounded set of couples. Then there is a natural number I depending only on \mathcal{P} satisfying the following: Assume X is projective with klt singularities and $M \geq 0$ an integral divisor on X so that $(X, \text{Supp } M) \in \mathcal{P}$, then IK_X and IM are cartier.

Corollary 3.3.3. The nef threshold μ with respect to θ is discrete.

Proof. Notice that all pairs in \mathcal{C}_θ are δ -lc, then the general fibre of (F_i, B_{F_i}) of $(X_i, B_i) \rightarrow S_i$ is also δ -lc with $\dim F_i \leq \dim X_i$. Thus they form a bounded family by Theorem 3.3.1. Take the integral I in Lemma 3.3.2, then $I(K_{F_i} + B_{F_i})$ is Cartier. Take a rational curve C_{F_i} in $\overline{\text{NE}}(F_i)$, then

$$0 < -I(K_{F_i} + B_{F_i}) \cdot C_{F_i} \leq 2I \dim F_i$$

Notice that $\mu = \frac{IH_{F_i} \cdot C_{F_i}}{-I(K_{F_i} + B_{F_i}) \cdot C_{F_i}}$, where $H_{F_i} \cdot C_{F_i}$ and $-I(K_{F_i} + B_{F_i}) \cdot C_{F_i}$ are integers, thus

$$\mu \in \frac{1}{(2I \dim F_i)!} \mathbb{N}.$$

□

We prove the termination by contraction. Otherwise, if there is an infinite sequence, i.e. there are infinitely many X_i and birational maps obtained from the program:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X'$$

Then we have:

- Since $\mu' \leq \mu_{i+1} \leq \mu_i$, and as is shown in 3.3.3 that $\{\mu_i\}$ is discreteness, there is an integer N such that $\mu_i = \mu_N$ for all $i > N$. In fact, we may assume $N = 0$ and $\mu_i = \mu_0 = \mu$ for all i ;
- Notice that for case 1a and 1b, we have $\mu_{i+1} < \mu_i$, thus there is no such links in the infinite sequence. If there is a link as case 1c or 1d, then $\mu_{i+1} = \mu_i = \mu$ and $\lambda_{i+1} \leq \mu$, thus next link must be case 1c or 1d again, and all links following must be case 1c or 1d. For case 1c we have $\rho(X_{i+1}) = \rho(X_i) - 1$, therefore there are only finitely many such links, and all links after are case 1d;
- Each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is obtained by $(K + B + \frac{1}{\mu}H)$ -MMP with scaling of a \mathbb{Q} -divisor C_i . But we can choose C_{i+1} to be the strict transform of C_i in X_{i+1} , then the whole sequence is $(K + B + \frac{1}{\mu}H)$ -MMP with scaling of a \mathbb{Q} -divisor C_0 , and this ends. Therefore there are no links of case 1c or 1d, and i.e. all links are of case 2.
- For case 2, recall that $\mu_{i+1} = \mu_i$ implies that

$$\begin{aligned} & \text{either } \dim S_i < \dim S_{i+1} \\ & \text{or } \dim S_i = \dim S_{i+1} \text{ and the link is square} \end{aligned}$$

and notice that $\dim S_i < \dim X$, hence we may assume $\dim S_i = \dim S_0$ (Note that $\dim S_0 \neq 0$, otherwise all X_i are isomorphic, which is absurd).

We are left to show that there is no infinite sequence with stationary μ_i and $\dim S_i$. Since for case 2, $\lambda_{i+1} \leq \lambda_i$ and $\lambda_{i+1} = \lambda_i$ implies $e_{i+1} < e_i$, furthermore $\frac{1}{\lambda_i} \leq \frac{1}{\mu_0}$, we have

$$c := \lim_i \frac{1}{\lambda_i} > \frac{1}{\lambda_i} = c_i$$

We prove it in several steps:

Step 1: Claim that $(X_i, B_i + cH_i)$ and $(Z_i, B_i + cH_i)$ are log canonical for all $i \gg 0$. Otherwise, let

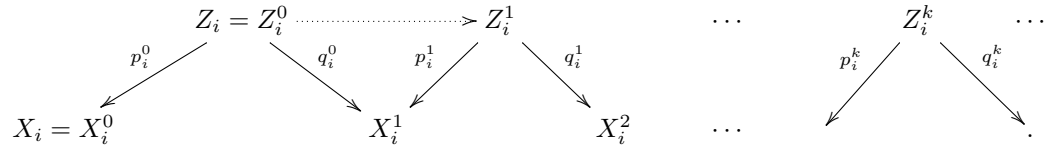
$$\alpha_i = \text{lct}(X_i, B_i; H_i)$$

then there are infinitely many i such that $c > \alpha_i$. By definition of λ_i , we have $\alpha_i > c_i$. Notice that c_i accumulates from below to c and never equals, there are infinitely many α_i , which contradicts to acc conditioation of lct :

Theorem 3.3.4. [1, Theorem 1.1] Fix a positive integral n , $I \subset [0, 1]$ and a subset J of positive real numbers. If I, J satisfy the DCC, then $\text{LCT}_n(I, J)$ satisfies ACC.

The same argument applies to $(Z_i, B_i + cH_i)$. Therefore, we may assume all pairs are log canonical.

Step 2: For each link there are flips



This step we will show that such 2-ray game of $(K + B + c_i H)$ -MMP on Z_i is also a 2-ray game of $(K + B + cH)$ -MMP.

Let $P^k = \overline{\text{NE}}(Z^k/X^k)$ and $Q^k = \overline{\text{NE}}(Z^k/X^{k+1})$, then P^k is $(K_{Z^k} + B_{Z^k} + c_0 H_{Z^k})$ -positive and $(K_{Z^k} + B_{Z^k} + c_0 H_{Z^k})$ -negative. Need to show this also holds for each $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$. Prove this by induction on k .

Since $c > c_i$, we have

$$K_Z + B_Z + cH_Z = p^*(K_X + B + cH) - aE \quad (a > 0)$$

By negativity lemma, there is a curve C_Z on Z mapping to a point on X , and $E.C_Z < 0$, thus we have $(K_Z + B_Z + cH_Z).P^0 > 0$, where $P^0 = \mathbb{R}_{\geq 0}[C_Z] = \overline{\text{NE}}(Z/X)$. Suppose $(K_{Z^k} + B_{Z^k} + cH_{Z^k}).P^k > 0$, then $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$ is not nef over S . In particular, P^k is positive, and the other extremal ray Q^k is negative. This implies step 2. Furthermore, by decreasing of canonical divisor, we have

$$a(\nu; X_i, B_i + cH_i) \leq a(\nu; X, B + cH)$$

and strictly inequality holds if and only if $X_l \dashrightarrow X_{l+1}$ is not an isomorphism at center of ν on X_l for some $l < i$

Step 3: Claim that $(X_i, B_i + cH_i)$ is klt for all $i \gg 0$. Otherwise, if there are infinitely many i such that $(X_i, B_i + cH_i)$ is not klt, since they are all log canonical, this is equivalent to say there infinitely many i and ν_i such that

$$-1 = a(\nu_i; X_i, B_i + cH_i) \geq a(\nu_i; X_0, B_0 + cH_0) \geq -1$$

Therefore $a(\nu; X_i, B_i + cH_i) = -1$ and $X_0 \dashrightarrow X_i$ isomorphism at the center $z(\nu_i, X)$. Thus the local θ -canonical thresholds are same

$$ct_\theta(\nu_i; X, B; H) = ct_\theta(\nu_i; X_i, B_i; H_i)$$

On the other hand, by definition

$$c_i \leq ct_\theta(\nu_i; X_i, B_i; H_i)$$

and since $(X, B + cH)$ is not klt along $z(\nu_i, X)$, it is not θ -canonical, thus

$$ct_\theta(\nu_i; X_i, B_i; H_i) < c$$

Therefore

$$c_i \leq ct_\theta(\nu_i; X, B; H) < c$$

But the set $\{ct_\theta(x; X, B; H); x \in X\}$ is finite, a contradiction! We may assume $(X_i, B_i + cH_i)$ are all klt.

Step 4: Note that $E_i = \text{Exc}(p_i)$ are all distinct. Otherwise, assume $E_i = E_j$ for some $i < j$, then Z_i and Z_j are isomorphic in a neighborhood of E_i and E_j , thus

$$a(E_i; X_i, B_i + cH_i) = a(E_j; X_j, B_j + cH_j)$$

However, since $E_i = E_j$ is not a divisor on X_j , there is $k < j$ such that E_j is contracted by $Z'_k \rightarrow X_{k+1}$, therefore $X_k \dashrightarrow X_{k+1}$ is not isomorphic at E_j , hence

$$a(E_i; X_i, B_i + cH_i) \leq a(E_j; X_k, B_k + cH_k) < a(E_j; X_{k+1}, B_{k+1} + cH_{k+1}) \leq a(E_j; X_j, B_j + cH_j)$$

which is a contradiction.

Since $(X, B + cH)$ is klt, then there are only finitely many E_i with $a(E_i, X, B + cH) < 0$. But there are in fact infinitely many

$$a(E_i; X, B + cH) \leq a(E_i; X_i, B_i + cH_i) < -\theta(E) \leq 0,$$

a contradiction!

At last we have the Noether-Fano-Iskovskikh criterion to show when they are isomorphic:

Theorem 3.3.5. (*Noether-Fano-Iskovskikh Criterion*): *Notations as in the definition of Sarkisov degree, then*

1. $\mu \geq \mu'$;
2. If $\mu \geq \lambda$ and $(K_X + B + \frac{1}{\mu}H)$ is nef, then Φ is an isomorphism of Mori fibre space, i.e., we have commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow[\Phi]{\sim} & X' \\ f \downarrow & & \downarrow f' \\ S & \xrightarrow{\sim} & S' \end{array}$$

Proof. We follow the proof in [7],[10] and [6]:

1. Only need to show $(K_X + B + \frac{1}{\mu'}H)$ is f -nef. Let $\sigma : W \rightarrow X$ and $\sigma' : W \rightarrow X'$ be the common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W &= \sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ &= \sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here $\{G_i\}, \{E_j\}$ are σ -exceptional divisors, and $\{E_j\}, \{G'_k\}$ are σ' -exceptional divisors. Since $H_W = \sigma'^*H'$, $g'_k > 0$ (or there are no such G'_k). Then take a general curve $C \subset X$ contracted by f , such that its strict transform \tilde{C} on W is disjoint from G_i, E_j , and is not contained in G'_k . Then we have:

$$\begin{aligned} C \cdot \left(K_X + B + \frac{1}{\mu'}H \right) &= \tilde{C} \cdot \sigma^* \left(K_X + B + \frac{1}{\mu'}H \right) \\ &= \tilde{C} \cdot \left(\sigma^* \left(K_W + B_W + \frac{1}{\mu'}H \right) + \sum g_i G_i + \sum e_j E_j \right) \\ &= \tilde{C} \cdot \left(K_W + B_W + \frac{1}{\mu'}H_W \right) \\ &= \tilde{C} \cdot \left(\sigma'^* \left(K_{X'} + B' + \frac{1}{\mu'}H' \right) + \sum e'_j E_j + \sum g'_k G'_k \right) \\ &= \tilde{C} \cdot \sigma'^* f'^* A' + C \cdot \left(\sum g'_k G'_k \right) \\ &\geq 0 \end{aligned}$$

This implies $(K_X + B + \frac{1}{\mu'}H)$ is f -nef and $\mu \geq \mu'$;

2. First we show that $\mu = \mu'$. By 1, we only need to show $(K_{X'} + B' + \frac{1}{\mu}H')$ is f' -nef. Indeed, same as 1, we can take a curve C' on X' contracted by f' , such that its strict transform \tilde{C}' on W is disjoint from G'_i, E_j , and is not contained in G'_k and $C' \cdot \left(K_{X'} + B' + \frac{1}{\mu}H' \right) \geq 0$.

Then we show there are isomorphic. Take a very ample divisor D on X and let D' be its strict transform on X' . D' is f' -ample, thus there exists $0 < d \ll 1$ such that the following holds:

- $K_X + B + \frac{1}{\mu}H_X + dD$ is ample;
- $K_{X'} + B' + \frac{1}{\mu}H' + dD'$ is ample.

Therefore X and X' are both log canonical models of $(W, B_W + \frac{1}{\mu}H_W + dD_W)$, hence $X \cong X'$. and $S \cong S'$.

□

4 Double scaling

This section we follows [7, 13. The Sarkisov program] and [10].

4.1 Prepare

Let (W, B_W) be a \mathbb{Q} -factorial klt pair and $f : (X, B) \rightarrow S$ and $f' : (X', B') \rightarrow S'$ be two different log Mori fibre spaces as outputs of $(K_W + B_W)$ -MMP. To modify the beginning setting, we need more conventions and lemmas:

Definition 4.1.1. *Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties. If*

- *f does not extract divisors;*
- *$a(E; X, B_X) \leq a(E; Y, B_Y)$ for all divisors E over X .*

then we denote $(X, B) \geq (Y, B_Y)$.

In particular, for terminal pairs, we have following lemma:

Lemma 4.1.2. *[7, Lemma 13.8] Let $f : W \dashrightarrow X$ be a birational map where (W, B_W) is terminal. If*

- *f does not extract divisors;*
- *$K_X + B$ is nef, where $B = f_* B_W$;*
- *$a(E; X, B) \geq a(E; W, B_W)$ for all divisors $E \subset W$,*

then

- *$(W, B_W) \geq (X, B)$.*
- *(X, B) is klt*
- *If $Z \rightarrow X$ is a divisorial extraction of a divisor E with $a(E; X, B) \leq 0$, then E is a divisor on W ;*
- *If $Z \rightarrow X$ is terminalization of (X, B) , then $W \dashrightarrow Z$ extracts no divisors.*

Conversely, start from a klt pair and non-positive map, we have

Lemma 4.1.3. *[10, Lemma 3.5] Let $\sigma : (W, B_W) \dashrightarrow (X, B)$ be a $K_W + B_W$ -non-positive birational map such that $\sigma_*(K_W + B_W) = K_X + B$ and (W, B_W) is a \mathbb{Q} -factorial klt pair. Then there is a resolution of indeterminacy $\pi : \tilde{W} \rightarrow W$ and $\tilde{\sigma} : \tilde{W} \rightarrow X$ such that*

- *$(\tilde{W}, B_{\tilde{W}})$ is \mathbb{Q} -factorial terminal and $\tilde{\sigma}_* B_{\tilde{W}} = B$,*
- *$\tilde{\sigma}$ is $(K_{\tilde{W}} + B_{\tilde{W}})$ -non-positive and $(\tilde{W}, B_{\tilde{W}}) \geq (X, B)$.*

By Lemma 4.1.3, we replace (W, B_W) by its log resolution such that (W, B_W) is terminal and $\sigma : W \rightarrow X$ and $\sigma' : W \rightarrow X'$ are $(K_W + B_W)$ -non-positive morphisms, and $(W, B_W) \geq (X, B), (X', B')$.

Take very general ample \mathbb{Q} -divisors A and A' on S and S' such that $G \sim_{\mathbb{Q}} -(K_X + B) + f^* A$ and $H \sim_{\mathbb{Q}} -(K_{X'} + B') + f'^* A'$ are two ample \mathbb{Q} -divisors. Moreover, we may assume G and H satisfying $G_W := \sigma^* G = \sigma_*^{-1} G$ and $H_W := \sigma'^* H = \sigma'^{-1} H$. Therefore $\sigma_*(K_W + B_W + G_W) = K_X + B + G$ is nef, and Lemma 4.1.2 holds. Furthermore, we may assume $(W, B_W + gG_W + hH_W)$ is log smooth and terminal for all $0 \leq g, h \leq 2$ by taking furthermore blowing up if necessary. Then we have:

Theorem 4.1.4 (Sarkisov program with double scaling). *Notations as above, there is a finite sequence of Sarkisov links*

$$\begin{array}{ccccccc}
X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \cdots & \dashrightarrow & X_N = X' \\
f=f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & & f_N \downarrow \\
S = S_0 & & S_1 & & S_2 & & & S_N = S'
\end{array}$$

and rational numbers

$$\begin{aligned}
1 = g_0 &\geq g_1 \geq \cdots \geq g_N = 0 \\
0 = h_0 &\leq h_1 \leq \cdots \leq h_N = 1
\end{aligned}$$

such that

1. For each i , $\sigma_i : W \dashrightarrow X_i$ is $(K_W + B_W + g_i G_W + h_i H_W)$ -non-positive, and $(K_{X_i} + B_i + g_i G_i + h_i H_i) = \sigma_{i*}(K_W + B_W + g_i G_W + h_i H_W)$ is nef and is relatively trivial over S_i ;
2. $(W, B_W + g_i G_W + h_i H_W) \geq (X_i, B_i + g_i G_i + h_i H_i)$;
3. each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is given by a sequence of $(K_{X_i} + B_i + g_i G_i + h_i H_i)$ -trivial maps.
4. The last link $X_N \rightarrow S_N$ is isomorphic to $X' \rightarrow S'$

4.2 Construct Sarkisov links

This subsection we construct the links inductively. Suppose we have $\sigma_i : W \dashrightarrow X_i$ as in Theorem 4.1.4, that is

- $f_i : (X_i, B_i) \rightarrow S_i$ is a log Mori fibre space and $\sigma_{i*} B_W = B_i$;
- $\sigma_i : W \dashrightarrow X_i$ is $(K_W + g_i G_i + h_i H_W)$ -non-positive birational map, and $(K_{X_i} + B_i + g_i G_i + h_i H_i) = \sigma_{i*}(K_W + B_W + g_i G_W + h_i H_W)$ is nef and is relatively trivial over S_i ;
- $(W, B_W + g_i G_W + h_i H_W) \geq (X_i, B_i + g_i G_i + h_i H_i)$;
- $0 \leq g_i, h_i \leq 1$ are rational numbers.

Then we need to show that there is a Sarkisov link $X_i \dashrightarrow X_{i+1}$ satisfying Theorem 4.1.4. Similarly with Sarkisov degree, we have following notations:

Definition 4.2.1. Let C_i be a general f_i -vertical curve on X_i , then

- $r_i := \frac{H_i \cdot C_i}{G_i \cdot C_i}$;
- Let Γ be the set of $t \in [0, \frac{g_i}{r_i}]$ such that
 1. $(W, B_W + g_i G_W + h_i H_W + t(H_W - r_i G_W)) \geq (X_i, B_i + g_i G_i + h_i H_i + t(H_i - r_i G_i))$
 2. $K_{X_i} + B_i + g_i G_i + h_i H_i + t(H_i - r_i G_i)$ is nef;

Let $s_i = \max \Gamma$;

- Let $D_{W,i} = B_W + g_i G_W + h_i H_W$ and $D_i = B_i + g_i G_i + h_i H_i$. Let $D_{W,i}(t) = B_W + g_i G_W + h_i H_W + t(G_W - r_i H_W)$ and $D_i(t) = B_i + g_i G_i + h_i H_i + t(G_i - r_i H_i)$. Let $g_{i+1} = g_i - r_i s_i$ and $h_{i+1} = h_i + s_i$. Note that $D_{W,i+1} = D_{W,i}(s_i)$.

Then we have (check [10, Lemma 4.4] for details)

1. $r_i > 0$;
2. either $\Gamma = \{0\}$ or is a closed interval;
3. $g_{i+1} = g_i \Leftrightarrow h_{i+1} = h_i \Leftrightarrow s_i = 0$;

Construct links.: If $s_i = \frac{g_i}{r_i}$, then $g_{i+1} = 0$. Let $N = i + 1$ and let $f_N : X_N = X_i \rightarrow S_N = S_i$, then $X_N \rightarrow S_N$ is isomorphic to $f' : X' \rightarrow S'$ (see Proposition 4.3.2) and we stop. Otherwise, if $s_i < \frac{g_i}{r_i}$, then we construct the Sarkisov link $X_i \dashrightarrow X_{i+1}$ in following cases:

1. Suppose s_i is not the threshold of condition 1 of Γ . That is, there exists $0 < \epsilon \ll 1$, such that for any divisor E on W , we have

$$a(E; X_i, D_i(s_i + \epsilon)) \geq a(E; W, D_{W,i}(s_i + \epsilon))$$

and $K_{X_i} + D_i(s_i + \epsilon)$ is not nef. Then there is a 2-dimensional $(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$ -negative extremal face F for some $0 < \delta \ll \epsilon$, spanned by $R = \mathbb{R}_{\geq 0}[C_i]$ and another extremal ray P . Hence there is a contraction $X_i \rightarrow T_i$ corresponding to F factoring through f_i . Then we run $(K_{X_i} + D_i(s_i + \epsilon))$ -MMP on X_i with scaling over T_i . After finitely many flips, we either have a $(K_{X_i} + D_i(s_i + \epsilon))$ minimal model, a divisorial contraction, or a Mori fibre space over T_i :

- (a) After finitely many flips $X_i \dashrightarrow X_{i+1}$ there is a log Mori fibre space $X_{i+1} \rightarrow S_{i+1}$, and this is a link of type III.
 - (b) After finitely many flips $X_i \dashrightarrow Z_i$ there is a divisorial contraction $Z_i \rightarrow X_{i+1}$, then let $S_{i+1} = T_i$ and $X_{i+1} \rightarrow S_{i+1}$ is a log Mori fibre space and this is a link of type IV.
 - (c) After finitely many flips $X_i \dashrightarrow X_{i+1}$, the contraction $X_{i+1} \rightarrow T_i$ is a log minimal model of $(X_i, D_i(s_i + \epsilon))$ over T_i . Let C' be the strict transform of C_i on X_{i+1} , then $(K_{X_{i+1}} + D_{i+1}(\epsilon)).C' = 0$ and $(K_{X_{i+1}} + B_{i+1}).C' < 0$, therefore there is a contraction $X_{i+1} \rightarrow S_{i+1}$ which is a log Mori fibre space. And this is a link of type IV.
2. Suppose s_i is the threshold of condition 1 of Γ , that is, there exists $0 < \epsilon \ll 1$ and a σ_i -exceptional divisor E_i on W such that

$$a(E_i; X_i, D_i(s_i + \epsilon)) < a(E_i; W, D_{W,i}(s_i + \epsilon)).$$

In this case, we have

$$a(E_i; X_i, D_i(s_i)) = a(E_i; W, D_{W,i}(s_i)) = -\text{mult}_{E_i}(D_{W,i}(s_i)) \leq 0.$$

Let $p_i : Z_i \rightarrow X_i$ be the divisorial extraction of the divisor E_i as in Corollary 2.2.4, and suppose $K_{Z_i} + D_{Z_i}(s_i) = K_{Z_i} + B_{Z_i} + g_{i+1}G_{Z_i} + h_{i+1}H_{Z_i} = p_i^*(K_{X_i} + D_i(s_i + \epsilon))$. Take a sufficiently small δ such that $0 < \delta \ll \epsilon \ll 1$ and

$$K_{Z_i} + \Delta_i = p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta G_i)$$

is klt. Then we run $(K_{Z_i} + \Delta_i)$ -MMP on Z_i over S_i . Since Z_i is covered by $(K_{Z_i} + \Delta_i)$ -negative curves, it follows that $(K_{Z_i} + \Delta_i)$ is not pseudo-effective over S_i , and this MMP ends with a log Mori fibre space. Moreover, this is a MMP for $p_i^*(K_{X_i} + D_i(s_i + \epsilon) - \delta' G_i)$ for all $0 < \delta' \leq \delta$. After finitely many flips, we either have a $(K_{Z_i} + \Delta_i)$ log Mori fibre space or a $(K_{Z_i} + \Delta_i)$ divisorial contraction.

- (a) After finitely many flips $Z_i \dashrightarrow X_{i+1}$ there is a Mori fibre space $X_{i+1} \rightarrow S_{i+1}$, and this is a link of type I. In this case we have $\rho(X_{i+1}) = \rho(X_i) + 1$.
- (b) After finitely many flips $Z_i \dashrightarrow Z'_{i+1}$ there is a divisorial contraction $q_i : Z'_{i+1} \rightarrow X_{i+1}$, and then a logMori fibre space $X_{i+1} \rightarrow S_i =: S_{i+1}$. This is a link of type II.

Claim 4.2.2. 1. $r_i \leq r_{i+1}$. Moreover, in case 1a, we have $r_i < r_{i+1}$.

2. Since the birational map $X_i \dashrightarrow X_{i+1}$ is over T_i (over S_i) and $(K_{X_i} + D_i(s_i))$ is numerically trivial over T_i (over S_i) in case 1 (case 2), it follows that $a(E; X_i, D_i(s_i)) = a(E; X_{i+1}, D_{i+1})$ for any divisors E over W and so the inequality

$$a(E; X_{i+1}, D_{i+1}) \geq a(E; W, D_{W,i+1})$$

3. In case 1, for any divisor $E \subset W$, we have $a(E; X_i, D_i(s_i + \epsilon)) \leq a(E; X_{i+1}, D_{i+1}(\epsilon))$ for all $0 < \epsilon \ll 1$. Moreover, since $X_i \not\cong X_{i+1}$, there is a divisor F over W such that $a(F; X_i, D_i(s_i + \epsilon)) < a(F; X_{i+1}, D_{i+1}(\epsilon))$.
4. In case 2, for any divisor $E \subset W$, we have $a(E; X_i, D_i(s_i + \epsilon) - \delta G_i) \leq a(E; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$ for all $0 < \epsilon \ll 1$. Moreover, since $X_i \not\cong X_{i+1}$, there is a divisor F over W such that $a(F; X_i, D_i(s_i + \epsilon) - \delta G_i) < a(F; X_{i+1}, D_{i+1}(\epsilon) - \delta G_{i+1})$.
5. $h_i \leq 1$, and $h_i = 1$ if and only if $g_i = 0$;

4.3 Termination

Lemma 4.3.1. [7, Lemma 13.18 and Lemma 13.19] (or [10, Lemma 4.9]) Suppose we construct a sequence of Sarkisov links:

$$\begin{array}{ccccccc} X = X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \longrightarrow & \dots \dashrightarrow X_i \longrightarrow \dots, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S = S_0 & & S_1 & & S_2 & & S_i \end{array}$$

then

- 1. there are only finitely many possibilities of $f_i : X_i \rightarrow S_i$ up to isomorphism;
- 2. the Sarkisov program with double scaling of (G_W, H_W) terminates. That is, there exists an integer $N > 0$ such that $g_N = 0$.

Proof. 1. This essentially follows from the finiteness of weak log canonical model (Theorem 2.1.10). We construct the subspace V of $\text{WDiv}_{\mathbb{R}}(W)$ as following:

- (a) If $h_k > 0$ for some k : Since H_W is nef and big, take an ample \mathbb{Q} -divisor A_W and an effective \mathbb{Q} -divisor C_W such that $H_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spanned by components of B_W, G_W, H_W, C_W , then for $i > k$:

$$B_W + g_i G_W + h_i H_W \sim_{\mathbb{Q}} h_k A_W + B_W + g_i G_W + (h_i - h_k) H_W + h_k C_W =: \Delta_i \in \mathcal{L}_{h_k A_W}(V)$$

- (b) If $h_k = 0$ for all k , then $h_i \equiv 0$ and $g_i \equiv 1$. Since G_W is nef and big, take an ample \mathbb{Q} -divisor A_W and an effective \mathbb{Q} -divisor C_W such that $G_W \sim_{\mathbb{Q}} A_W + C_W$. Let V be the affine space spanned by components of B_W, C_W , then

$$B_W + G_W \sim_{\mathbb{Q}} A_W + B_W + C_W =: \Delta_i \in \mathcal{L}_{A_W}(V)$$

Then all X_i are weak log canonical models of (W, Δ_i) . By finiteness of weak log canonical models, there are finitely many $\sigma_i : W \dashrightarrow X_i$ up to isomorphism. Then we shall show that for $\sigma_i : W \dashrightarrow X_i$ there are finitely many log Mori fibre spaces in the sequence up to isomorphism. Indeed, we may assume that there is a k such that $X_i \cong X_k$ for all $i > k$, and f_i is the contraction corresponding to an extremal ray $R_i \subset \overline{\text{NE}}(X_k)$. Then we have $(K_{X_k} + B_k).R_i < 0$ and $(K_{X_k} + B_k + g_i G_k + h_i H_k).R_i = 0$. Furthermore, H_k and G_k are relatively ample over S_i for all $i > k$.

- (a) If $h_k > 0$: Since H_k is big, we have $h_k H_k = A_k + E_k$ for some ample \mathbb{Q} -divisor A_k and effective \mathbb{Q} -divisor E_k . Let $B'_k = B_k + (1 - \epsilon)h_k H_k + \epsilon E_k$ for sufficiently small ϵ such that (X_k, B'_k) is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \epsilon A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\text{NE}}(X_k) = \overline{\text{NE}}(X_k)_{K_{X_k} + B'_k + \epsilon A_k \geq 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_{\alpha}$$

All extremal rays R_i corresponding to f_i for $i > k$ are in the finite set $\{R_{\alpha}\}_{\alpha \in \Lambda}$, thus there are finitely many log Mori fibre spaces $f_i : X_i \rightarrow S_i$ of X_k .

- (b) If $h_i = 0$ for all i , and hence $g_i = 1$ for all i . Since G_i is big, we have $G_k = A_k + E_k$ for some ample \mathbb{Q} -divisor A_k and effective \mathbb{Q} -divisor E_k . Let $B'_k = B_k + (1 - \epsilon)G_k + \frac{\epsilon}{2}E_k$ for sufficiently small ϵ such that (X_k, B'_k) is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \frac{\epsilon}{2}A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\text{NE}}(X_k) = \overline{\text{NE}}(X_k)_{K_{X_k} + B'_k + \frac{\epsilon}{2}A_k \geq 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_{\alpha}$$

Again, there are finitely many log Mori fibre spaces $f_i : X_i \rightarrow S_i$ of X_k .

2. Assume this sequence of links is infinite, then there exist i such that there are infinitely many $j > i$ such that $f_i : X_i \rightarrow S_i$ and $f_j : X_j \rightarrow S_j$ are isomorphic. Then we have $g_{i+1} = g_{j+1}$ and $h_{i+1} = h_{j+1}$. Since sequences of h_k and g_k are monotone, we have $h_{i+1} = h_k$ and $g_{i+1} = g_k$ for all $k > i$. Suppose $X_i \dashrightarrow X_{i+1}$ is a link in case 1 of the Construction in 4.2, then the next link is also in case 1, and all the links after are in case 1. Note that $X_i \cong X_j$ and therefore $\rho(X_i) = \rho(X_j)$, the links are all of type IV. But this contradicts 2 of Claim 4.2.2. Therefore there are no link of type III or IV after X_i . In other words, the links after X_i are all type I or II in case 2.

Since $\rho(X_i) = \rho(X_j)$, X_i and X_j are linked by the Sarkisov links of type II. But this contracts 4 of Claim 4.2.2. □

Proposition 4.3.2. $X_N \rightarrow S_N$ is isomorphic to $X' \rightarrow S'$.

Proof. By 2 of Theorem 3.3.5, we have $h_N = 1$ and they are isomorphic. □

5 Using the Polytope

In this section we follow [8].

5.1 Morphisms between models

Let W be a smooth projective variety, and let V be a finite dimensional affine subspace of $\text{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective \mathbb{Q} -divisor A . In this subsection we describe the models X_i of $(W, A + B)$ for some $B \in V$ and rational maps between X_i .

Theorem 5.1.1. [8, Theorem 3.3] *Let W be a smooth projective variety, and V be a finite dimensional affine subspace of $\text{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective \mathbb{Q} -divisor A . Suppose that there is an element D_0 of $\mathcal{L}_A(V)$ such that $K_W + D_0$ is big and klt. Then there are finitely many rational contractions $f_i : W \dashrightarrow X_i$ such that*

1. $\{\mathcal{A}_i = \mathcal{A}_{A, f_i}\}$ is a partition of $\mathcal{E}_A(V)$. \mathcal{A}_i is a finite union of interiors of rational polytopes. If f_i is birational then $\mathcal{C}_i = \mathcal{C}_{A, f_i}$ is a rational polytope;
2. If i, j are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ then there is a contraction $f_{ij} : X_i \rightarrow X_j$ and $f_j = f_{ij} \circ f_i$;
3. Suppose in addition V spans $\text{NS}(W)$. Pick i such that a connected components \mathcal{C} of \mathcal{C}_i intersects the interior of $\mathcal{L}_A(V)$, TFAE:
 - (a) \mathcal{C} spans V ;
 - (b) If $D \in \mathcal{A}_i \cap \mathcal{C}$ then f_i is a log terminal model of $K_W + D$;
 - (c) f_i is birational and X_i is \mathbb{Q} -factorial.
4. Suppose in addition V spans $\text{NS}(W)$. If i, j are two indices such that \mathcal{C}_i spans V and D is a general point of $\mathcal{A}_j \cap \mathcal{C}_i$ which is also a point of interior of $\mathcal{L}_A(V)$, then \mathcal{C}_i and $\overline{\text{NE}}(X_i/X_j)^* \times \mathbb{R}^k$ for some $k \leq 0$. Furthermore $\rho(X_i/X_j)$ equals the difference in the dimensions of \mathcal{C}_i and $\mathcal{C}_j \cap \mathcal{C}_i$.

Proof. 1. is proved in [4].

2. Pick a divisor $D \in \mathcal{A}_j \cap \mathcal{C}_i$ and $D' \in \mathcal{A}_i$ such that

$$D_t = D + t(D' - D) \in \mathcal{A}_i$$

for $t \in (0, 1]$. By finiteness of log terminal models, we may find a positive constant $\delta > 0$ and a birational contraction $f : W \dashrightarrow X$ which is a log terminal model of $K_W + D_t$ for $t \in (0, \delta]$. Replacing $D' = D_1$ by D_δ we may assume $\delta' = 1$. If we set

$$B_t = f_* D_t,$$

then $K_X + \Delta_t$ is klt and nef, and f is $K_W + D_t$ non-positive for $t \in [0, 1]$. As D_t is big the base point free theorem implies that $K_X + B_t$ is semiample and so there is an induced contraction morphism $g_i : X \rightarrow X_i$ together with ample divisors $H_{1/2}$ and H_1 such that

$$K_X + B_{1/2} = g_i^* H_{1/2}, K_X + B_1 = g_i^* H_1$$

If we set

$$H_t = (2t - 1)H_1 + 2(1 - t)H_{1/2}$$

then

$$\begin{aligned} K_X + B_t &= (2t - 1)(K_X + B_1) + 2(1 - t)H_{1/2} \\ &= (2t - 1)g_i^* H_1 + 2(1 - t)g_i^* H_{1/2} \\ &= g_i^* H_t \end{aligned}$$

for all $t \in [0, 1]$. As $K_X + B_0$ is semiample, it follows that H_0 is semiample and the associated contraction $f_{i,j} : X_i \rightarrow X_j$ is the required morphism.

3. Suppose that \mathcal{C} spans V . Pick D in the interior of $\mathcal{C} \cap \mathcal{A}_i$. Let $f : W \dashrightarrow X$ be a log terminal model of $(K_W + D)$, then $f = f_j$ for some index $1 \leq j \leq k$ and that $D \in \mathcal{C}_j$. But then $\mathcal{A}_i \cap \mathcal{A}_j \neq \emptyset$ so that $i = j$. If f_i is a log terminal model of $K_W + D$ then f_i is birational and X is \mathbb{Q} -factorial. Finally suppose that f_i is birational and X_i is \mathbb{Q} -factorial. Fix $D \in \mathcal{A}_i$. Pick any divisor $G \in V$ such that $-G$ is ample and $K_{X_i} + f_{i*}(D + G)$ is ample and $D + G \in \mathcal{L}_A(V)$. Then f_i is $(K_W + D + G)$ -negative and so $D + G \in \mathcal{A}_i$. But \mathcal{C}_i spans V , this implies (3).
4. Let $f = f_i$ and $X = X_i$. As \mathcal{C}_i spans V , (3) implies that f is birational and X is \mathbb{Q} -factorial so that f is a \mathbb{Q} -factorial weak log canonical model of $K_W + D$. Suppose that E_1, E_2, \dots, E_k are the divisors contracted by f . Pick $F_i \in V$ numerically equivalent to E_i . If we let $E_0 = \sum_{i=1}^k E_i$ and $F_0 = \sum_{i=1}^k F_i$, then E_0 and F_0 are numerically equivalent. As D belongs to interior of $\mathcal{L}_A(V)$ we may find $\delta > 0$ such that $K_W + D + \delta F_0$ and $K_W + D + \delta B_0$ are both klt. Then f is $(K_W + D + \delta E_0)$ -negative and so f is a log terminal model of $(K_W + D + \delta E_0)$ and f_j is the ample model of $K_W + D + \delta B_0$. In particular $D + \delta F_0 \in \mathcal{A}_j \cap \mathcal{C}_i$. As we are supposing that D is general in $\mathcal{A}_j \cap \mathcal{C}_i$, in fact f must be a log terminal model of $K_W + D$, and f is $(K_W + D)$ -negative.

Pick $\epsilon > 0$ such that if $G \in V$ and $\|G - D\| < \epsilon$ then G belongs to the interior of $\mathcal{L}_A(V)$ and f is $(K_W + G)$ -negative. Then $G \in \mathcal{C}_i$ simply means $K_X + H = f_*(K_W + G)$ is nef. Let V_X be the affine supspace of $\text{WDiv}_{\mathbb{R}}(X)$ given by pushing forward the elements of V and let

$$\mathcal{N} = \{H \in V_X : K_X + H \text{ is nef}\}.$$

Given $(a_1, \dots, a_k) \in \mathbb{R}^k$ and let $F = \sum a_i F_i$ and $E = \sum a_i E_i$. If $\|F\| < \epsilon$ then $K_X + H \in \mathcal{N}$ if and only if $K_X + H + f_* F \in \mathcal{N}$. In particular \mathcal{C}_i is locally isomorphic to $\mathcal{N} \times \mathbb{R}^k$.

But since f_j is the ample model of $K_W + D$, in fact we can choose ϵ sufficiently small such that $K_X + H$ is nef if and only if $K_X + H$ is nef over X_j . There is a surjective affine linear

map from V_X to the space of Weil divisor on X modulo numerical equivalence over X_j and this induces an isomorphism

$$\mathcal{N} \cong \overline{\text{NE}}(X/X_j)^* \times \mathbb{R}^l,$$

in a neighbourhood of f_*D .

Note that $K_X + f_*D$ is numerical trivial over X_j . As f_*D is big and $K_X + f_*D$ is klt we may find an ample \mathbb{Q} -divisor A' and a divisor $B' \geq 0$ such that

$$K_X + A' + B' \sim_{\mathbb{R}} K_X + f_*D$$

is klt. But then

$$-(K_X + B') \sim_{\mathbb{R}} -(K_X + H) + A'$$

is ample over X_j . Hence $f_{ij} : X \rightarrow X_j$ is a Fano fibration and so by cone theorem

$$\rho(X_i/X_j) = \dim \mathcal{N}$$

This is (4). □

Lemma 5.1.2. [8, Corollary 3.4] *If V spans $\text{NS}(W)$, then there is a Zariski dense open subset U of the Grassmannian $G(r, V)$ of real affine subspace of dimension r such that any $[V'] \in U$ defined on rational numbers satisfy (1-4) of 5.1.1*

Proof. Let $U \subset G(r, V)$ be the set of real affine subspace V' of V of dimension r , which contain any sub no face of any \mathcal{C}_i or $\mathcal{L}(V)$. In particular, the interior of $\mathcal{L}_A(V')$ is contained in the interior of $\mathcal{L}_A(V)$. Clearly that any $V' \in U$ satisfies (1-4) of 5.1.1. □

From now on in this subsection, we always assume that V has dimension 2 and satisfies 5.1.1.

Lemma 5.1.3. [8, Lemma 3.5] *Let $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ be two rational contractions such that $\mathcal{C}_{A,f}$ is dimension 2 and $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$ is dimension 1. Assume $\rho(X) \geq \rho(Y)$ and \mathcal{O} is not contained in the boundary of $\mathcal{L}_A(V)$. Let D be an interior point of \mathcal{O} and $B = f_*D$. Then there is a rational contraction $\pi : X \dashrightarrow Y$ and $g = \pi \circ f$ such that either*

1. $\rho(X) = \rho(Y) + 1$ and π is $(K_X + B)$ -trivial, and either

(a) π is birational and \mathcal{O} is not contained in the boundary of $\mathcal{E}_A(V)$, and either

i. π is a divisorial contraction and $\mathcal{O} \neq \mathcal{C}_{A,g}$, or

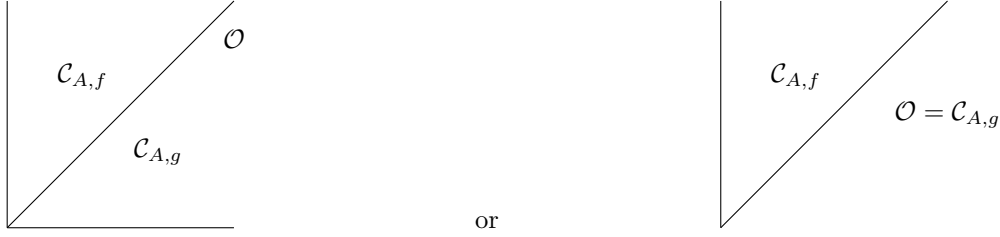
ii. π is a small contraction and $\mathcal{O} = \mathcal{C}_{A,g}$

or

(b) π is a Mori fibre space, and $\mathcal{O} = \mathcal{C}_{A,g}$ is contained in the boundary of $\mathcal{E}_A(V)$

or

2. $\rho(X) = \rho(Y)$, and π is a $(K_X + B)$ -flop and $\mathcal{O} \neq \mathcal{C}_{A,g}$ is not contained in the boundary of $\mathcal{E}_A(V)$.



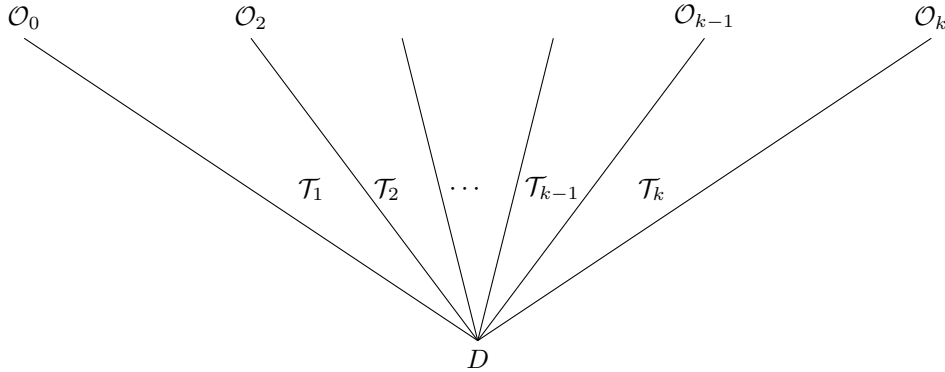
Proof. By assumption f is birational and X is \mathbb{Q} -factorial. Let $h : W \dashrightarrow S$ be the ample model corresponding to $K_W + D$. Since D is not a point of the boundary of $\mathcal{L}_A(V)$, if D belongs to the boundary of \mathcal{E}_A then $K_W + D$ is not big and so h is not birational. As \mathcal{O} is a subset of both $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ there are morphisms $p : X \rightarrow S$ and $q : Y \rightarrow S$ of relative Picard number at most one. There are therefore only two cases

1. $\rho(X) = \rho(Y) + 1$, or
2. $\rho(X) = \rho(Y)$

Suppose we are in the first case, then q is the identity and $\pi : X \rightarrow Y$ is a contraction morphism such that $g = p \circ f$. Suppose that π is birational, then h is birational and \mathcal{O} is not contained in the boundary of $\mathcal{E}_A(V)$. If π is divisorial then Y is \mathbb{Q} -factorial and so $\mathcal{O} \neq \mathcal{C}_{A,g}$. If π is a small contraction then π is not \mathbb{Q} -factorial and so $\mathcal{C}_{A,g} = \mathcal{O}$ is one dimensional. If π is a Mori fibre space then \mathcal{O} is contained in the boundary of $\mathcal{E}_A(V)$ and $\mathcal{O} = \mathcal{C}_{A,g}$.

Now suppose we are in the second case. Since $\rho(X/S) = \rho(Y/S) = 1$, we know that p, q are not divisorial contractions as \mathcal{O} is one dimensional and p, q are not Mori fibre spaces as \mathcal{O} is cannot be contained in the boundary of $\mathcal{E}_A(V)$. Hence p, q are small and the the rest is clear. \square

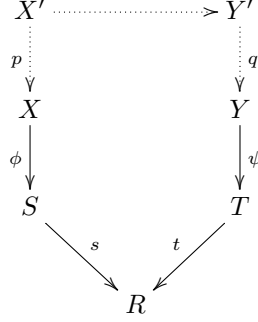
Let $D = A + B$ be a point of boundary of $\mathcal{E}_A(V)$ in the interior of $\mathcal{L}_A(V)$. Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be the polytopes \mathcal{C}_i of dimension 2 containing D . Let \mathcal{O}_0 and \mathcal{O}_k be the intersection of \mathcal{T}_0 and \mathcal{T}_k with boundary of $\mathcal{E}_A(V)$, and let $\mathcal{O}_i = \mathcal{O}_i \cap \mathcal{O}_{i+1}$. Let $f_i : W \rightarrow X_i$ be the rational contraction associated to \mathcal{T}_i and $g_i : W \rightarrow S_i$ be the rational contraction associated to \mathcal{O}_i .



Set $f = f_1 : W \dashrightarrow X, g = f_k : W \dashrightarrow Y$ and $\phi : X \rightarrow S = S_0, \psi : Y \rightarrow T = S_k$ and $X' = X_2, Y' = X_{k-1}$ and let $W \dashrightarrow R$ be the ample model of D . Then

Theorem 5.1.4. [8, Theorem 3.7] Suppose B_W is a divisor such that $K_Z + B_W$ is klt and $D - B_W$ is ample. Then ϕ and ψ are Mori fibre spaces as outputs of $(K_Z + B_W)$ -MMP and connected by a Sarkisov link if D is contained in more than two polytopes.

Proof. WMA $k \geq 3$ and we have



Note that $\rho(X_i/R) \leq 2$ and $\rho(X/S) = \rho(Y/T) = 1$. Thus

1. s is identity and p is a divisorial contraction (extraction), or
2. s is a contraction and p is a flop.

The same holds for q and t . And the map $X' \rightarrow Y'$ is clear the composition of flops. This gives 4 types of links. \square

5.2 Construction of Sarkisov links

Lemma 5.2.1. [8, Lemma 3.6] *Let $f : W \dashrightarrow X$ be a birational contraction between \mathbb{Q} -factorial varieties. Suppose (W, D) and $(W, D + A)$ are both klt. If f is ample model of $(W, D + A)$ and A is ample, then f is result of running $(K_W + D)$ -MMP.*

This lemma guarantee that every variety in the Sarkisov links constructed later is a MMP result of (W, B_W) . We need a special resolution W and an affine subspace $V \subset \text{WDiv}(W)$ such that we can find two Mori fibre spaces X/S and Y/T and vertexs connecting them. The following lemma shows the desired affine subspace exists.

Lemma 5.2.2. [8, Lemma 4.1] *Let $\phi : X \rightarrow S$ and $\psi : Y \rightarrow T$ be two MMP related Mori fibre space corresponding to two klt projective varieties (X, B_X) and (Y, B_Y) . Then we may find a smooth projective variety W , two biratinal morphism $f : W \rightarrow X$ and $g : W \rightarrow Y$, a klt pair (W, B_W) , an ample \mathbb{Q} -divisor A on W and a two dimensional rational affine subspace V of $\text{WDiv}_{\mathbb{R}}(W)$ such that*

1. *If $D \in \mathcal{L}_A(V)$ then $D - B_W$ is ample;*
2. *$\mathcal{A}_{A, \phi \circ f}$ and $\mathcal{A}_{A, \psi \circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$;*
3. *V satisfy 5.1.1;*
4. *$\mathcal{C}_{A, f}$ and $\mathcal{C}_{A, g}$ are two dimensional;*
5. *$\mathcal{C}_{A, \phi \circ f}$ and $\mathcal{C}_{A, \psi \circ g}$ are one dimensional.*

Proof. By assumption there is a \mathbb{Q} -factorial klt pair (W, B_W) such that $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ are both outcomes of $(K_W + B_W)$ -MMP. Let $p' : W' \rightarrow W$ be any log resolution such that resolves the indeterminacy of f and g , then we may write

$$K_{W'} + B_{W'} = p'^*(K_W + B_W) + E'$$

where $E' \geq 0$ and $B_{W'} \geq 0$ have no common components, and E' is exceptional and $p'_* B_{W'} = B_W$. Pick a divisor $-F$ which is ample over W with support equal to the full exceptional locus such that $K_{W'} + B_{W'} + F$ is klt. As p' is $(K_{W'} + B_{W'} + F)$ -negative and $(K_W + B_W)$ is klt and W is \mathbb{Q} -factorial, the $(K_{W'} + B_{W'} + F)$ -MMP over W terminates with the pair (W, B_W) . Replacing (W, B_W) by $(W', B_{W'} + F)$ we may assume that (W, B_W) is log smooth and f, g are morphisms.

Pick general ample \mathbb{Q} -divisors A, H_1, H_2, \dots, H_k on W such that H_1, \dots, H_k generate the Neron-Severi group of W . Let

$$H = A + H_1 + \dots + H_k$$

Pick sufficiently ample divisor A_S on S and A_T on T such that

$$-(K_X + B_X) + \phi^* A_S \text{ and } -(K_Y + B_Y) + \psi^* A_T$$

are both ample. Pick a rational number $0 < \delta < 1$ such that

$$-(K_X + B_X + \delta f_* H) + \phi^* A_S \text{ and } -(K_Y + B_Y + \delta g_* H) + \psi^* A_T$$

are both ample and $(K_W + B_W + \delta H)$ is both f and g negative. Replacing H by δH we may assume that $\delta = 1$. Now pick a \mathbb{Q} -divisor $B_0 \leq B_W$ such that $A + (B_0 - B_W)$, $-(K_X + f_* B_0 + f_* H) + \phi^* A_S$ and $-(K_Y + g_* B_0 + g_* H) + \psi^* A_T$ are all ample and $(K_W + B_0 + H)$ is both f and g negative.

Pick general ample \mathbb{Q} -divisors $F_1 \geq 0$ and $G_1 \geq 0$ such that

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_* B_0 + f_* H) + \phi^* A_S \text{ and } G_1 \sim_{\mathbb{Q}} -(K_Y + g_* B_0 + g_* H) + \psi^* A_T$$

and

$$K_W + B_0 + H + F + G$$

is klt, where $F = f^* F_1$ and $G = g^* G_1$.

Let V_0 be the affine subspace of $\text{WDiv}_{\mathbb{R}}(W)$ which is the translate by B_0 of the vector subspace spanned by H_1, \dots, H_k, F, G . Suppose that $D = A + B \in \mathcal{L}_A(V_0)$. Then

$$D - B_W = (A + B_0 - B_W) + (B - B_0)$$

is ample, as $B - B_0$ is nef by definition of V_0 . Note the

$$B_0 + F + H \in \mathcal{A}_{A, \phi \circ f}(V_0), B_0 + G + H \in \mathcal{A}_{\psi \circ g}(V_0)$$

and f , respectively g , is a weak log canonical model of $K_W + B_0 + F + H$, respectively $K_W + B_0 + G + H$. Thus theorem 5.1.1 implies that V_0 satisfies (1-4) of 5.1.1.

Since H_1, \dots, H_k generated the Neron-Severi group of W we may find constants h_1, \dots, h_k such that $G \equiv \sum_{i=1}^k h_i H_i$. Then there is $0 < \delta \ll 1$ such that $B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \in \mathcal{L}_A(V_0)$ and

$$B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \equiv B_0 + F + H.$$

Thus $\mathcal{A}_{A, \phi \circ f}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. Similarly $\mathcal{A}_{A, \psi \circ g}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. In particular $\mathcal{A}_{A, \phi \circ f}$ and $\mathcal{A}_{A, \psi \circ g}$ span affine hyperplanes of V_0 , since $\rho(X) = \rho(Y) = 1$.

Let V_1 be the translate by B_0 of two dimensional vector space spanned by $F + H - A$ and $F + G - A$. Let V be a small general perturbation of V_1 , which is defined over rationals. This is the affine subspace we need. \square

Then we can prove the main theorem

Proof of 1.1.2. Let (W, B_W) , A and V as in the lemma 5.2.2. Pick $D_0 \in \mathcal{A}_{A, \phi \circ f}$ and $D_1 \in \mathcal{C}_{A, g}$ belonging to the interior of $\mathcal{L}_A(V)$. As V is two dimensional, removing D_0 and D_1 divides the boundary of $\mathcal{E}_A(V)$ into two parts. The part which consists entirely of divisors which are not big is contained in the interior of $\mathcal{L}_A(V)$. Consider tracing this boundary from D_0 to D_1 . Then there are finitely many $2 \leq i \leq N$ points D_i which are contained in more than two polytopes $\mathcal{C}_{A, f_i}(V)$. By lemma 5.1.4, each point D_i gives a Sarkisov link. And the birational map $X \dashrightarrow Y$ is composition of such links. \square

6 Application

References

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