# Sarkisov program

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## 1 Introduction

The purpose of this article is to show that two different log Mori fibre spaces as outputs of a klt pair can be linked by composition of Sarkisov links.

## 1.1 Motivation and Main theorem

The **Minimal model program (MMP)** aims to classify varieties up to birational equivalent classed, by finding a minimal model all or Mori fibre space. Let $(X, B)$ be a (klt or lc) pair, and assume we can run $(K_X + B)$-MMP on it. Note that the varieties appear in the program are called **results** of the MMP, and the varieties where the MMP ends are called the **output** of the MMP.

1. If $\kappa(X, B) \geqslant 0$, then we expected that MMP ends with a **minimal model**, i.e. a birational map $X \dashrightarrow Y$ such that $(K_Y + B_Y)$ is nef;
2. If $\kappa(X, B) = -\infty$, then we expected that MMP ends with a log Mori fibre space, i.e. a birational map $X \dashrightarrow Y$ and a contraction $Y \to S$ such that $\dim Y < \dim X$ and $-(K_Y + B_Y)$ is relative ample.

However, for each case the output may not be unique.

For the first case, it is shown that two different minimal model can be linked by flops:

**Theorem 1.1.1.** *[9, Theorem 1] Let $(W, B_W)$ be a $\mathbb{Q}$-factorial terminal pair, and $(X, B), (Y, D)$ are two minimal models of $(W, B_W)$. Then the birational map $X \dashrightarrow Y$ may be factored as sequence of $(K_X + B)$ flops.*

For the second case, it is shown that:

**Theorem 1.1.2.** *Let $f : (X, B) \to S$ and , $f' : (X', B') \to S'$ be two MMP related $\mathbb{Q}$-factorial klt log Mori fibre spaces with induced induced birational map $\Phi$:*

$$\begin{tikzcd}
(X,B) \arrow[r, dotted, "\Phi"] \arrow[d, "f"'] & (X',B') \arrow[d, "f'"] \\
S & S'
\end{tikzcd}$$

*Then $\Phi$ can be decomposed into sequence of Sarkisov links.*

**Definition 1.1.3.** *The following four types of birational maps $X \dashrightarrow X_1$ are called Sarkisov links:*

***I***:
$$\begin{tikzcd}
Z \arrow[r, dotted] \arrow[d, "p"'] & X_1 \arrow[d, "f_1"] \\
X \arrow[d, "f"'] & S_1 \arrow[dl, "t"] \\
S &
\end{tikzcd}$$

***II***:
$$\begin{tikzcd}
Z \arrow[r, dotted] \arrow[d, "p"'] & Z' \arrow[d, "q"] \\
X \arrow[d, "f"'] & X_1 \arrow[d, "f_1"] \\
S \arrow[r, "\sim"] & S_1
\end{tikzcd}$$

***III***:
$$\begin{tikzcd}
X \arrow[r, dotted] \arrow[d, "f"'] & Z \arrow[d, "q"] \\
S \arrow[dr, "s"'] & X_1 \arrow[d, "f_1"] \\
 & S_1
\end{tikzcd}$$

***IV***:
$$\begin{tikzcd}
X \arrow[rr, dotted] \arrow[d, "f"'] & & X_1 \arrow[d, "f_1"] \\
S \arrow[dr, "s"'] & & S_1 \arrow[dl, "t"] \\
 & T &
\end{tikzcd}$$

*Here, all $f : (X, B) \to S$ and $f_1 : (X_1, B_1) \to S_1$ are log Mori fibre space, and all $p, q$ are divisorial contractions, and all dash arrows are composition of flips (or flops in section 5).*

The Sarkisov program has its origin in the birational classification of ruled surfaces [12]. Reid explained the original idea of Sarkisov [11]. The complete proof of Sarkisov program for terminal threefolds is given in by Corti [6]. Andrea Bruno and Kenji Matsuki generalized it to klt pairs of dimension 3. In fact, they showed that the Sarkisov program works for klt pairs of all dimensions, but in higher dimension it may not terminate. Due to the finiteness of weak log canonical model [4], Hacon gives another Sarkisov program called double scaling [7] which terminates in all dimensions. Liu Jihao generalized it to generalized pairs [10].

As another application of [4], Hacon and M$^{\text{c}}$kern gave another proof [8] and is quite different.

## 1.2 Using MMP

Assume $f : (X, B) \to S'$ and $f' : (X', B') \to S'$ are two Mori fibre spaces as outputs of $(K_W + B_W)$-MMP on $W$. The Sarkisov program constructs each Sarkisov link $X_i \dashrightarrow X_{i+1}$ inductively. For each $X_i$ we shall find some $W_i$ such that $X_i$ and $X_{i+1}$ are two Mori fibre spaces as outputs of certain MMP on $W_i$. Moreover, $W_i \dashrightarrow X_{i+1}$ is a 2-tay game. More precisely, there are two cases:

1. Find a contraction $g : X_i \to T_i$ such that $\rho(X_i/T_i) = 2$ and factor though $f_i : X_i \to S_i$, then we run MMP on $X_i$ over $T_i$, and obtains a Sarkisov link of type III or IV. Here $W_i = X_i$;

2. Find a divisorial contraction $p : Z_i \to X_i$, and therefore $\rho(Z_i/S_i) = 2$. Then we run MMP on $Z_i$ over $S_i$, and obtains a Sarkisov link of type I or II. Here $W_i = Z_i$.

In [6], original proof;

In [7], double scaling;

## 1.3 Using polytope

## 1.4 Structure of the article

# 2 Preliminary

In this article, all varieties are over complex number $\mathbb{C}$.

## 2.1 Models

**Definition 2.1.1.** *[8, 2.Notation and Conventions] A rational map $f : X \to S$ is called a **rational contraction** if there is a resolution $p : W \dashrightarrow X$ and $q : W \dashrightarrow Y$ of $f$ such that $p$ and $q$ are contraction morphisms and $p$ is birational. $f$ is called a **birational contraction** if $q$ is in addition birational and every $p$ -exceptional divisor is $q$ -exceptional. If in addition $f^{-1}$ is also a **birational contraction**, then $f$ is called a **small birational map**.*

**Definition 2.1.2.** *[4, Definition 3.6.1]Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties, and $p : W \to X$ and $q : W \to Y$ be a resolution of indeterminacy of $fl$. Let $D$ be a $\mathbb{R}$-Cartier divisor on $X$ such $D_Y = f_*D$ is also $\mathbb{R}$-Cartier. Then $f$ is called $D$-**non-positive** ($D$-**negative**) if*

- *$f$ does not extract any divisor;*
- *$E = p^*D - q^*D_Y$ is effecive and exceptional over $Y$ (and $\operatorname{Supp} p_*E$ contains all $f$-exceptional divisors).*

**Definition 2.1.3.** *[7, 13.2.Notation and conventions] Let $f : X \dashrightarrow Y$ be a rational map of normal quasi-projective varieties over $S$, and $D$ be a $\mathbb{R}$-Cartier $\mathbb{R}$-divisor divisor on $X$ with $f_*D = D_Y$. Then $f$ is called $D$-**trivial** if $D$ is pull back of a $\mathbb{R}$-Cartier divisor on $S$.*

Recall the definitions of models in [4]

**Definition 2.1.4.** *[4, Definition 3.6.5] Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties and let $D$ be an $\mathbb{R}$-Cartier divisor on $X$. Let $f : X \dashrightarrow Y$ be a birational map over $U$, then $Z$ is an* ***semiample model*** *for $D$ over $U$ if $f$ is $K_X + D$-non-positive and $K_Y + f_*D$ is semiample over $U$.*

*Let $g : X \dashrightarrow Z$ be a rational map over $U$, then $Z$ is an* ***ample model*** *for $D$ over $U$ if there is a an ample divisor over $U$ on $Z$ such that if $p : W \to X$ and $q : W \to Z$ resolves $g$, then $q$ is a contraction morphism and we may write $p^*D \sim_{\mathbb{R},U} q^*H + E$, where $E \geqslant 0$ and for any $B \in |p^*D/U|_{\mathbb{R}}$, then $B \geqslant E$.*

**Definition 2.1.5.** *[4, Definition 3.6.7] Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties, if $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then define:*

1. *$Y$ is* ***weak log canonical model*** *for $K_X + D$ over $U$ if $f$ is $K_X + D$-non-positive and $K_Y + f_*D$ is nef over $U$;*
2. *$Y$ is* ***log canonical model*** *for $K_X + D$ over $U$ if $f$ is $K_X + D$-non-positive and $K_Y + f_*D$ is ample over $U$;*
3. *$Y$ is* ***log terminal model*** *for $K_X + D$ over $U$ if $f$ is $K_X + D$-negative and $K_Y + f_*D$ is dlt and nef over $U$ and $Y$ is $\mathbb{Q}$-factorial.*

**Lemma 2.1.6.** *[4, Lemma 3.6.6] Let $\pi : X \to U$ ve a projective morphism of normal quasi-projective varieties and let $D$ be an $\mathbb{R}$-Cartier divisor on $X$.*

1. *If $g_i : X \dashrightarrow X_i, i = 1, 2$ are two ample models of $D$ over $U$, then there is an isomorphism $h : X_1 \to X_2$ and $g_2 = h \circ g_1$.*
2. *If $f : X \dashrightarrow Y$ is a semiample model of $D$ over $U$, then the ample model $g : X \dashrightarrow Z$ of $D$ over $U$ exits and $g = h \circ f$, where $h : Y \to Z$ is a contraction and $f_*D \sim_{\mathbb{R},U} h^*H$.*
3. *If $f : X \dashrightarrow Y$ is a birational map over $U$, then $f$ is the ample model of $D$ over $U$ if and only if $f$ is semiample model of $D$ over $U$ and $f_*D$ is ample over $U$.*

By above lemma there is another definition of log canonical models:

**Definition 2.1.7.** *Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties and $K_X + D$ is log canonical and $f : X \dashrightarrow Y$ is a birational map extracts no divisors, then $Y$ is* ***log canonical model*** *if it is the ample model.*

Furthermore, for big boundary, we have

**Lemma 2.1.8.** *[4, Lemma 3.9.3]Let $\pi : (X, D) \to U$ be a projective morphism of normal quasi-projective varieties. Suppose $(X, B)$ is a klt pair and $B$ is big over $U$. If $f : X \dashrightarrow Y$ is a weak log canonical model over $U$ then*

- *$f$ is a semiample model over $U$;*
- *the ample model $g : X \dashrightarrow Z$ over $U$ exits;*
- *there is a contraction $h : Y \to Z$ such that $K_Y + f_*B \sim_{\mathbb{R},U} h^*H$ for some ample $\mathbb{R}$-divisor $H$ on $Z$ over $U$.*

**Definition 2.1.9.** *[4, Definition 1.1.4] Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let $V$ be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ defined over rational numbers. Define*

$$\mathcal{L}(V) = \{D \in V : K_X + D \text{ is log canonical }\}$$
$$\mathcal{N}_\pi(V) = \{D \in \mathcal{L} : K_X + D \text{ is nef over } U\}$$

*Moreover, if fix an $\mathbb{R}$-divisor $A \geqslant 0$, and then define*

$$V_A = \{D = A + B : B \in V\}$$
$$\mathcal{L}_A(V) = \{D = A + B \in V_A : K_X + D \text{ is log canonical and } B \geqslant 0\}$$
$$\mathcal{E}_{A,\pi}(V) = \{D = A + B \in \mathcal{L}_A(V) : K_X + D \text{ is pseudo effective over } U\}$$
$$\mathcal{N}_{A,\pi}(V) = \{D \in \mathcal{L}_A(V) : K_X + D \text{ is nef over } U\}$$

*Given a birational contraction $f : X \dashrightarrow Y$, define*

$$\mathcal{W}_{A,f}(V) = \{D \in \mathcal{E}_A(V) : f \text{ is weak log model of } (X, D) \text{ over } U\}$$

*Given a rational contraciton $g : X \dashrightarrow Z$ over $U$, define*

$$\mathcal{A}_{A,g}(V) = \{D \in \mathcal{E}_A(V) : g \text{ is ample model of } (X, D) \text{ over } U\}$$

*In addition, let $\mathcal{C}_{A,g}(V)$ denote the closure of $\mathcal{A}_{A,g}(V)$*

By [4, Lemma 3.7.2], if $V$ is a rational subspace, then $\mathcal{L}_A(V)$ is a rational polytope.

**Lemma 2.1.10.** *[4, Theorem E] Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and $A$ be an general divisor relatively ample over $U$, and $V \subset \mathrm{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a klt pair $(X, \Delta_0)$. Then there are finitely many birational maps $f_i : X \dashrightarrow X_i$ such that if $f : X \dashrightarrow Y$ is a weak log canonical model of $K_X + D$ over $U$ for some $D \in \mathcal{L}_A(V)$, then there is an isomorphism $h_i : X_i \to Y$ and $f = h_i \circ f_i$.*

**Theorem 2.1.11.** *[4, Corollary 1.1.5]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and $A$ be an general divisor relatively ample over $U$, and $V \subset \mathrm{Div}_{\mathbb{R}}(X)$ be a finite dimensional rational subspace. Suppose that there is a divisor $\Delta_0 \in V$ such that $(X, \Delta_0)$ is klt. Let $A$ be a general ample $\mathbb{Q}$-divisor over $U$ which has no components common with any element of $V$.*

1. *There are finitely many birational maps $f_i : X \dashrightarrow X_i$ over $U$ such that*
$$\mathcal{E}_{A,\pi}(V) = \bigcup_i \mathcal{W}_i$$
*where $\mathcal{W}_i = \mathcal{W}_{A,f_i}(V)$ is a rational polytope. Moreover, if $f : X \dashrightarrow Y$ is a log terminal model of $K_X + D$ over $U$ for some $D \in \mathcal{E}_A(V)$, then $f = f_i$ for some $i$.*

2. *There are finitely many birational maps $g_j : X \dashrightarrow Z_j$ over $U$ such that*
$$\mathcal{E}_{A,\pi}(V) = \coprod_j \mathcal{A}_j$$
*$\{\mathcal{A}_j = \mathcal{A}_{A,g_j}\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If $f_i$ is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;*

3. *For every $f_i$ there is a $g_j$ and a morphism $h_{ij} : Y_i \to Z_j$ such that $\mathcal{W}_i \subset \overline{\mathcal{A}_j}$.*

*In particular $\mathcal{E}_{A,\pi}$ is a rational polytope and $\overline{\mathcal{A}_j}$ is a finite union of raional polytopes.*

## 2.2 MMP

**Definition 2.2.1.** *Let $(X,B)$ be a pair and let $f : Y \to X$ be a log resolution of $(X,B)$. Suppose*

$$K_Y + C = f^*(K_X + B),$$

*then the discrepancy of exceptional divisor $E_i$ over $X$ is*

$$a(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$

*Moreover, let*

$$\operatorname{discrep}(X,B) := \inf\{a(E;X,B) : E \text{ is an exceptional divisor over } X\}$$

*and*

$$\operatorname{totdiscrep}(X,B) := \inf\{a(E;X,B) : E \text{ is a divisor over } X\}.$$

**Theorem 2.2.2.** *[4, Corollary 1.4.2]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let $(X,B)$ be a $\mathbb{Q}$-factorial klt pair where $K_X + B$ is $\mathbb{R}$-Cartier and $B$ is $\pi$-big. Let $C \geqslant 0$ be an $\mathbb{R}$-divisor. If $K_X + B + C$ is klt and $\pi$-nef, then we may run $(K_X + B)$-MMP over $U$ with scaling of $C$ and terminates.*

**Theorem 2.2.3.** *[4, Corollary 1.3.3]Let $\pi : X \to U$ be a projective morphism of normal quasi-projective varieties, and let $(X,B)$ be a $\mathbb{Q}$-factorial klt pair where $K_X+B$ is $\mathbb{R}$-Cartier. If $K_X+B+C$ is not $\pi$-peseudo-effective, then we may run $f : X \dashrightarrow Y$ a $(K_X + B)$-MMP over $U$ and end with a Mori fibre space $g : Y \to Z$.*

**Corollary 2.2.4.** *[7, Corollary 13.7] and [4, Corollary 1.4.3]: Let $(X,B)$ be a a klt pair and $\mathfrak{C}$ be any set of exceptional divisors such that contains only exceptional divisors $E$ of discrepancy $a(E;X,B) \leqslant 0$. Then there is a birational morphism $f : Z \to X$ and a $\mathbb{Q}$-divisor $B_Z$ such that:*

1. *$(Z, B_Z)$ is klt;*
2. *$E$ is a $f$-exceptional divisor if and only if $E \in \mathfrak{C}$;*
3. *$B_Z = \sum -a(E;X,B)$ and $f_*B_Z = B$ and $K_Z + B_Z = f^*(K_X + B)$.*

*In particular, if we take $\mathfrak{C}$ containing all such divisors, then $Z$ is called **terminalization** of $X$; if take $\mathfrak{C}$ containing only one such divisor, then $f : Z \to X$ is called a **divisorial extraction**.*

**Definition 2.2.5.** *[5, Definition 3.3] Two or more pairs $\{(X_i, B_i)\}$ are called **MMP-related** if they are results of $(K + B)$-MMP from a log smooth pair $(W, B_W)$.*

**Lemma 2.2.6.** *[5, Proposition 3.4] Let $\{(X_l, B_l)\}$ be a finite set of $\mathbb{Q}$-factorial klt pairs such that birational to other, then TFAE:*

1. *They are MMP-related;*
2. *There is a nonsingular pair $(W, B_W)$ with snc boundary, and projective birational morphisms $f_l : W \to X_l$ dominating each $X_l$, such that $f_{l*}B_W = B_l$ and*

$$K_W + B_W = f_l^*(K_{X_l} + B_l) + \sum_{exceptional} a_{li} E_{li}$$

*with $a_{li} > 0$ for all $f_i$-exceptional divisors;*

3. *For any two pairs $(X, B = \sum_i b_i B_i), (X', B' = \sum_j b'_j B'_j)$ in the set, $a(B_i; X', B') \geqslant -b_i$ and strict inequality holds if and only if $B_i$ exceptional over $X'$, and $a(B'_j; X, B) \geqslant -b'_j$ and strict inequality holds if and only if $B'_j$ exceptional over $X$.*

Let $K = K(X)$ be the function field, and let $\Sigma = \{\nu\}$ be the set of discrete valutions of the field .

**Definition 2.2.7.** *[5, Definition 3.5] Let $\theta : \Sigma \to [0,1)_{\mathbb{Q}}$ be a function. Then we can construct a collection $\mathcal{C}_\theta$ of pairs associated to $\theta$, consists of klt pairs $(X, B = \sum a_i B_i)$ satisfying*

1. *$a_i = \theta(B_i)$;*
2. *$a(E; X, B) > -\theta(E)$ for all $E$ exceptional over $X$.*

For example, if we take $\theta \equiv 0$ constant, the $\mathcal{C}_\theta$ is the collection of all terminal varieties $Y$ without boundary birational to $X$. Furthermore, we can define the corresponding discrepancy:

**Definition 2.2.8** ($\theta$-discrepancy)**.** *Let $(X, B)$ be a pair in the category $\mathcal{C}_\theta$ for some function $\theta$ and let $f : Y \to X$ be a log resolution of $(X, B)$. Suppose*

$$K_Y + B_Y + C = f^*(K_X + B)$$

*where $B_Y = (f^{-1})_* B + \sum_{E_i \text{ exceptional}} \theta(E_i) E_i$, then the $\theta$-discrepancy of exceptional divisor $E_i$ over $X$ is*

$$a_\theta(E_i; X, B) = -\operatorname{mult}_{E_i} C.$$

*Or equivalently, we have*

$$a_\theta(E_i; X, B) = a(E_i; X, B) + \theta(E_i).$$

*A pair $(X, B)$ is called $\theta$-canonical($\theta$-terminal) if $a_\theta(E; X, B) \geqslant 0$($a_\theta(E; X, B) > 0$ ) for all exceptional divisors $E$ over $X$. Note that $\theta$-canonical pair is not always in $\mathcal{C}_\theta$.*

# 3 Original proof

## 3.1 Prepare

First we fix a category:

**Proposition 3.1.1.** *[5, Lemma 3.6] Let $f : (X, B) \to S, f' : (X', B') \to S'$ be two $\mathbb{Q}$-factorial log Mori fibre spaces with only klt singularities and MMP-related, inducing a birational map $\Phi$:*

$$\begin{tikzcd}
(X, B) \arrow[r, dashed, "\Phi"] \arrow[d, "f"'] & (X', B') \arrow[d, "f'"] \\
S & S'
\end{tikzcd}$$

*Suppose $B = \sum_i b_i B_i + \sum_j d_j D_j$ and $B' = \sum_j d'_j D_j + \sum_k b'_k B'_k$, where $B_i$ are divisors on $X$ but not on $X'$, $B'_k$ are divisors on $X'$ but not on $X$, and $D_j$ are divisors on both $X$ and $X'$. By Lemma 2.2.6, $d_j = d'_j$. Take a rational number $\epsilon < 1$ such that $\epsilon > -\operatorname{totdiscrep}(X, B), -\operatorname{totdiscrep}(X', B')$, and take the function $\theta : \{\nu\} \to [0,1)_{\mathbb{Q}}$ as following:*

- *$\theta(B_i) = b_i, \theta(D_j) = d_j, \theta(B'_k) = b'_k$;*
- *$\theta(E) = \epsilon$ if $E$ is exceptional over both $X$ and $X'$;*
- *$\theta(D) = 0$ if $D$ is a divisor on both $X$ and $X'$, but not a component of $B$ or $B'$.*

*Then the collection $\mathcal{C}_\theta$ satisfies*

1. *$(X, B)$ and $(X', B')$ belongs to $\mathcal{C}_\theta$;*
2. *For any finitely many klt pairs $\{(X_l, B_l)\}$ in $\mathcal{C}_\theta$, there is an object $(Z, B_Z) \in \mathcal{C}_\theta$ and projective birational morphisms $Z \to X_l$ dominating each $X_l$ as a process of $(K_Z + B_Z)$-MMP over $X_l$ (thus over* $\operatorname{Spec}\mathbb{C}$*);*
3. *Any $(K + B)$-MMP starting from an object in $\mathcal{C}_\theta$ stays inside of $\mathcal{C}_\theta$, and so does any $(K + B + cH)$-MMP where $H$ is base point free and $c \in \mathbb{Q}_{>0}$.*

**Remark 3.1.2.** *Let $\delta = 1 - \epsilon$, then all pairs in $\mathcal{C}_\theta$ is $\delta$-lc.*

With notations and assumptions in Proposition 3.1.1, we shall define the Sarkisov degree. We take a very ample divisor $A'$ on $S'$ and a sufficiently large and divisible integer $\mu' > 1$ such that

$$\mathcal{H}' = |-\mu'(K_{X'} + B') + f'^*A'|$$

is a very ample complete linear system on $X'$ over $\operatorname{Spec}\mathbb{C}$. Let $(W, B_W)$ be a common log resolution of $X$ and $X'$ in $\mathcal{C}_\theta$ with projective birational morphism $\sigma : W \to X$, $\sigma' : W \to X'$ and $\sigma_* B_W = B, \sigma'_* B_W = B'$. Let $\mathcal{H}_W := \sigma'^*\mathcal{H}'$ and then $\mathcal{H} := (\Phi^{-1})_*\mathcal{H}' = \sigma_*\mathcal{H}_W$. Furthermore, if $\mathcal{H}$ is not base point free, then

$$\sigma^*\mathcal{H} = \mathcal{H}_W + F$$

where $F = \sum f_l F_l \geqslant 0$ is the fixed part. Take a general member $H'$ of the linear system $\mathcal{H}'$ such that $H_W := \sigma'^*H' = (\sigma'^{-1})_*H' \in \mathcal{H}_W$, and let $H := (\Phi^{-1})_*H' = \sigma_*H_W$, then $H$ if $f$-ample and $\sigma^*H = H_W + F$. By taking further resolution, we may assume $H_W$ is smooth and crosses normally with exceptional locus of $\sigma$ and $\sigma'$.

Now we can define the Sarkisov degree in $\mathcal{C}_\theta$ with respect to $H'$ (or $\mathcal{H}'$ ):

**Definition 3.1.3.** *[5, Definition 3.8] Sarkisov degree of $(X, B)$ with respect to $H$ (or $\mathcal{H}$) in $\mathcal{C}_\theta$ is a triple $(\mu, \lambda, e)$ ordered lexicographically:*

- ***Nef threshold** $\mu$: Let $C \subset X$ be a curve contracted by $f$, then*

$$\mu := -\frac{H.C}{(K_X + B).C}$$

  *i.e. $K_X + B + \frac{1}{\mu}H \equiv_S 0$;*

- ***$\theta$-canonical threshold** $c$ **and** $\lambda$: $\lambda = 0$ if $\mathcal{H}$ is base point free; otherwise,*

$$c := \frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geqslant 0, E \text{ exceptionl over } X\}$$

- ***Number of*** $(K_X+B_X+\frac{1}{\mu}H)$***-crepant divisors***: *Let $e=0$ if $\mathcal{H}$ is base point free (and hence $\lambda=0$), otherwise*

$$e=\#\{E; E \textit{ is } \sigma\textit{-exceptional and } a_\theta(E;X,B+\frac{1}{\lambda}H)=0\}$$

**Remark 3.1.4.** *1. The Sarkisov degree is dependent on the choice of $A', H'$ and $\theta$.*

2. *Take a common log resolution $(W,B_W)\in\mathcal{C}_\theta$ with $B_W=\sum\theta(E)E$ and projective birational morphisms $\sigma: W\to X$, $\sigma': W\to X'$. Since $\sigma^*\mathcal{H}=\mathcal{H}_W+\sum f_lF_l$, we have ramification formula:*

$$K_W+B_W+tH_W=\sigma^*(K_X+B+tH)+\sum(a_l-tf_l)E_l$$

   *where $\sum a_lE_l$ is effective and supported on $\operatorname{Exc}\sigma$. Then $\lambda:=\max\{\frac{f_l}{a_l}\}$. If $\mathcal{H}$ is base point free, then $\sum f_lF_l=0$ and $\lambda=0$.*

3. *$e$ is the number of components in $\sum(a_l-cf_l)E_l$ with coefficient $0$ in the formula*

$$K_W+B_W+\frac{1}{\lambda}H_W=\sigma^*(K_X+B+\frac{1}{\lambda}H)+\sum(a_l-\lambda f_l)E_l.$$

   *Such prime divisors $E_1\dots E_e$ are called $(K_X+B_X+\frac{1}{\lambda}H)$-$\theta$-crepant.*

We also need some extraction map in this category:

**Lemma 3.1.5.** *Using the notation in the definition of Sarkisov degree, then there is a contraction $f: Z\to X$ such that*

- *$(Z,B_Z)\in\mathcal{C}_\theta$ and $(Z,B_Z+\frac{1}{\lambda}H_Z)$ is $\theta$-terminal and $\mathbb{Q}$-factorial;*
- *$\rho(Z)=\rho(X)+1$;*
- *$f$ is $(K_X+B_X+\frac{1}{\lambda}H_X)$-crepant, that is*

$$K_Z+B_Z+\frac{1}{\lambda}H_Z=f^*(K_X+B+\frac{1}{\lambda}H).$$

*Proof.* We follow the proof in [5, Proposition 1.6] but for klt pair case. Let $(W,B_W)\in\mathcal{C}_\theta$ and $\sigma: W\to X, \sigma': W\to X'$ be the common resolution as in Definition 3.1.3 ,and suppose $E_1,\dots,E_e$ are $(K_X+B_X+\frac{1}{\mu}H)$-$\theta$-crepant divisors after renumbering. Then we have

$$K_W+B_W+\frac{1}{\lambda}H_W=\sigma^*(K_X+B+\frac{1}{\lambda}H)+\sum_{l=1}^{e}0\cdot E_l+\sum_{l>e}(a_l-\frac{1}{\lambda}f_l)E_l.$$

We run $(K_W+B_W+\frac{1}{\lambda}H_W)$-MMP on $W$ over $X$ with scaling of some ample divisor, then the MMP ends with a minimal model $p:(Y,B_Y+\frac{1}{\lambda}H_Y)\to X$ of $(W,B_W+\frac{1}{\lambda}H_W)$ over $X$ and the exceptional locus is exactly $\cup_{i=1}^{e}E_i$ and $p$ is crepant:

$$K_Y+B_Y+\frac{1}{\lambda}H_Y=p^*(K_X+B_X+\frac{1}{\lambda}H_X).$$

Then we run $(K_Y+B_Y)$-MMP on $Y$ over $X$ with scaling of some ample divisor. This ends with the minimal model $(X,B)$ of $(Y,B_Y)$ over $X$, and the last contraction in the MMP is $f:Z\to X$ as required. ∎

## 3.2 Flowchart for the Log Sarkisov program

We follow [5, Flowchart for the Sarkisov program] in this subsection.

If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is nef, the two Mori fibre spaces are isomorphic (shown in next subsection by proposition 3.3.5 ) and we stop here. Otherwise:

**Claim 3.2.1.** *1. If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef, then there is a contraction $f : X \to T$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type III or IV; ...*

*2. If $\lambda > \mu$, then there is a divisorial extraction $p : Z \to X$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type I or II.*

*Proof.* 1. By assumption, $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef. Suppose $f$ is the contraction with respect to a $(K_X + B)$-negative extremal ray $R = \overline{\mathrm{NE}}(X/S)$, then $(K_X + B + \frac{1}{\mu}H).R = 0$ by definition of $\mu$. There is an extremal ray $P \subset \overline{\mathrm{NE}}(X)$ such that $(K_X + B + \frac{1}{\mu}H).P < 0$ and $F := P + R$ is an extremal face (Check [6, 5.4.2] for details). Take $0 < \delta \ll 1$ such that $(K_X + B + (\frac{1}{\mu} - \delta)H).P < 0$, then $(K_X + B + (\frac{1}{\mu} - \delta)H).R < 0$ since $H$ is $f$-ample, and $F$ is a $(K_X + B + (\frac{1}{\mu} - \delta)H)$-negative extremal face. Since $(X, B + (\frac{1}{\mu} - \delta)H)$ is klt, there is a contraction $g : X \to T$ with respect to $F$ factorizing through $f : X \to S$. Since $(X, B + \frac{1}{\mu}H)$ is klt, and $\rho(X/T) = 2$, we can run $(K_X + B + \frac{1}{\mu}H)$-MMP on $X$ with scaling of some ample divisor. Since $B + \frac{1}{\mu}H$ is relatively big, the MMP terminates. There are following cases:

(a) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$, and then followed by a Mori fibre space $(X_1, B_1 + \frac{1}{\mu}H_1) \to S_1$. Then $S_1 \cong T$ and this is a link of type III.

(b) After finitely many flips $X \dashrightarrow X_1$, first non-flip contraction is a Mori fibre space $f_1 : X_1 \to S_1$. This is a link of type IV.

(c) After finitely many flips $X \dashrightarrow Z$, first non-flip contraction is a divisorial contraction $p : Z \to X_1$ with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where $e > 0$ and $E = \mathrm{Exc}\, p$ and $g_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to T$ is a log minimal model of $(X, B + \frac{1}{\mu}H)$ over $T$ . In fact the only ray of $\overline{\mathrm{NE}}(X_1/T)$ is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and hence is $(K_{X_1} + B_1)$-negative, therefore $(X_1, B_1)/T$ is a log Mori fibre space. Take $S_1 = T$, then this is a link of type III:

(d) After finitely many flips $X \dashrightarrow X_1$, $(K_X + B + \frac{1}{\mu}H)$-MMP ends with a log minimal model $(X_1, B_1 + \frac{1}{\mu}H_1)$ over $T$. Then there is an extremal ray $R$ of $\overline{\mathrm{NE}}(X_1/T)$, which is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and $(K_{X_1} + B_1)$-negative. Let $f_1 : X_1 \to S_1$ be the contraction with respect to $R$. This is a link of type IV. In fact, $X \dashrightarrow S_1$ is the ample model of $K_X + B + \frac{1}{\mu}H$.

2. By assumption, $\lambda > \mu$. Take an extraction $p : (Z, B_Z, H_Z) \to (X, B, H)$ as in Lemma 3.1.5. That is, $(Z, B_Z)$ is $\theta$-terminal and $p^*(K_X + B + \frac{1}{\lambda}H) = K_Z + B_Z + \frac{1}{\lambda}H_Z$ where

$B_Z = \sum \theta(E_\nu)E_\nu$ and $E = \operatorname{Exc} p$ is a prime divisor on $Z$. Then we run $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$-MMP on $Z$ over $S$ with scaling of some ample divisor. Since $Z$ is covered by $(K_Z+B_Z+\frac{1}{\lambda}H_Z)$-negative curves, $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$ is not relatively pseudo-effective. Hence this ends with a Mori fibre space by Theorem 2.2.3. There are two cases:

(a) After finitely many flips $Z \dashrightarrow Z'$, the first non-flip contraction is a divisorial contraction $q : Z' \to X_1$. Then $X_1 \to S$ is a log Mori fibre space of $(X, B)$ and $(X, B + \frac{1}{\lambda}H)$. Let $S_1 = S$ and this is a link of type II.

(b) After finitely many flips $Z \dashrightarrow X_1$, first non-flip contraction is a fibering contraction $f_1 : X_1 \to S_1$. Since $(K_{X_1} + B_1 + \frac{1}{\lambda}H_1)$ is $f_1$-negative and $H_1$ is $f_1$- ample, $(K_{X_1} + B_1)$ is $f_1$-negative, and $(X_1, B_1)/Y$ is a log Mori fibre space. Take $S_1 = Y$ and this is a link of type I.

□

**Remark 3.2.2.** *1. (a) For case 1a and 1b, since* $K_{X_1} + B_1 + \frac{1}{\mu}H_1$ *is* $f_1$*-negative, we have* $\mu_1 < \mu$*.*

*(b) For case 1c and 1d, Since* $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$ *is trivial on the ray* $R = \overline{\mathrm{NE}}(X_1/S_1)$ *for both cases, we have* $\mu_1 = \mu$*. Notice that* $(X_1, B_1 + \frac{1}{\mu}H_1)$ *stays* $\theta$*-canonical, we have* $\lambda_1 \leqslant \mu = \mu_1$*, thus next link goes back to case 1. Furthermore, for case 1c we have* $\rho(X_1) = \rho(X) - 1$*.*

*2. For case 2:*

*(a) For both case 2a and 2b, we have* $\mu_1 \leqslant \mu$ *with equality holds if and only if*

- *either* $\dim S_i < \dim S_{i+1}$
- *or* $\dim S_i = \dim S_{i+1}$ *and the link is square.*

*(b) We have* $\lambda_1 \leqslant \lambda$ *and if* $\lambda_1 = \lambda$*, then* $e_1 < e$*.*

## 3.3 Termination

We need following theorems:

**Theorem 3.3.1.** *[3, Theorem 1.1] Let* $d$ *be a natural number and* $\delta$ *be a positivity real number, then the projective varieties* $X$ *such that*

- $(X, B)$ *is a* $\delta$*-lc pair of dimension* $d$ *for some boundary* $B$*, and*
- $-(K_X + B)$ *nef and big,*

*form a bounded family.*

**Lemma 3.3.2.** *[2, Lemma 2.24] Let* $\mathcal{P}$ *be a bounded set of couples. Then there is a natural number* $I$ *depending only on* $\mathcal{P}$ *satisfying the following: Assume* $X$ *is projective with klt singularities and* $M \geqslant 0$ *an integral divisor on* $X$ *so that* $(X, \operatorname{Supp} M) \in \mathcal{P}$*, then* $IK_X$ *and* $IM$ *are cartier.*

**Corollary 3.3.3.** *The nef threshold* $\mu$ *with respect to* $\theta$ *is discrete.*

*Proof.* Notice that all pairs in $\mathcal{C}_\theta$ are $\delta$-lc, then the general fibre of $(F_i, B_{F_i})$ of $(X_i, B_i) \to S_i$ is also $\delta$-lc with $\dim F_i \leqslant \dim X_i$. Thus they form a bounded family by Theorem 3.3.1. Take the integral $I$ in Lemma 3.3.2, then $I(K_{F_i} + B_{F_i})$ is Cartier. Take a rational curve $C_{F_i}$ in $\overline{\mathrm{NE}}(F_i)$, then

$$0 < -I(K_{F_i} + B_F).C_{F_i} \leqslant 2I \dim F_i$$

Notice that $\mu = \frac{IH_{F_i}.C_{F_i}}{-I(K_{F_i}+B_{F_i}).C_{F_i}}$, where $H_{F_i}.C_{F_i}$ and $-I(K_{F_i} + B_{F_i}).C_{F_i}$ are integers, thus

$$\mu \in \frac{1}{(2I \dim F_i)!}\mathbb{N}.$$

□

We prove the termination by contraction. Otherwise, if there is an infinite sequence, i.e. there are infinitely many $X_i$ and birational maps obtained from the program:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X'$$

Then we have:

- Since $\mu' \leqslant \mu_{i+1} \leqslant \mu_i$, and as is shown in 3.3.3 that $\{\mu_i\}$ is discreteness, there is an integer $N$ such that $\mu_i = \mu_N$ for all $i > N$. In fact, we may assume $N = 0$ and $\mu_i = \mu_0 = \mu$ for all $i$;
- Notice that for case 1a and 1b, we have $\mu_{i+1} < \mu_i$, thus there is no such links in the infinite sequence. If there is a link as case 1c or 1d, then $\mu_{i+1} = \mu_i = \mu$ and $\lambda_{i+1} \leqslant \mu$, thus next link must be case 1c or 1d again, and all links following must be case 1c or 1d. For case 1c we have $\rho(X_{i+1}) = \rho(X_i) - 1$, therefore there are only finitely many such links, and all links after are case 1d;
- Each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is obtained by $(K+B+\frac{1}{\mu}H)$-MMP with scaling of a $\mathbb{Q}$-divisor $C_i$. But we can choose $C_{i+1}$ to be the strict transform of $C_i$ in $X_{i+1}$, then the whole sequence is $(K + B + \frac{1}{\mu}H)$-MMP with scaling of a $\mathbb{Q}$-divisor $C_0$, and this ends. Therefore there are no links of case 1c or 1d, and i.e. all links are of case 2.
- For case 2, recall that $\mu_{i+1} = \mu_i$ implies that

$$\begin{aligned} &\text{either } \dim S_i < \dim S_{i+1} \\ &\text{or } \dim S_i = \dim S_{i+1} \text{ and the link is square} \end{aligned}$$

  and notice that $\dim S_i < \dim X$, hence we may assume $\dim S_i = \dim S_0$ (Note that $\dim S_0 \neq 0$, otherwise all $X_i$ are isomorphic, which is absurd).

We are left to show that there is no infinite sequence with stationary $\mu_i$ and $\dim S_i$. Since for case 2, $\lambda_{i+1} \leqslant \lambda_i$ and $\lambda_{i+1} = \lambda_i$ implies $e_{i+1} < e_i$, furthermore $\frac{1}{\lambda_i} \leqslant \frac{1}{\mu_0}$ , we have

$$c := \lim_i \frac{1}{\lambda_i} > \frac{1}{\lambda_i} = c_i$$

We prove it in servel steps:

**Step 1**: Claim that $(X_i, B_i + cH_i)$ and $(Z_i, B_i + cH_i)$ are log canonical for all $i \gg 0$. Otherwise, let

$$\alpha_i = \mathrm{lct}(X_i, B_i; H_i)$$

then there are infinitely many $i$ such that $c > \alpha_i$. By definition of $\lambda_i$, we have $\alpha_i > c_i$. Notice that $c_i$ accumulates from below to $c$ and never equals, there are infinitely many $\alpha_i$, which contradicts to acc conditation of lct:

**Theorem 3.3.4.** *[1, Theorem 1.1] Fix a positive integral $n$, $I \subset [0, 1]$ and a subset $J$ of positive real numbers. If $I, J$ satisfy the DCC, then $\mathrm{LCT}_n(I, J)$ satisfies ACC.*

The same argument applies to $(Z_i, B_i + cH_i)$. Therefore, we may assume all pairs are log canonical.

**Step 2**: For each link there are flips

$$\begin{tikzcd}
 & Z_i = Z_i^0 \arrow[dl, "p_i^0"'] \arrow[dr, "q_i^0"] \arrow[rr, dotted] & & Z_i^1 \arrow[dl, "p_i^1"'] \arrow[dr, "q_i^1"] & & \cdots & & Z_i^k \arrow[dl, "p_i^k"'] \arrow[dr, "q_i^k"] & & \cdots \\
X_i = X_i^0 & & X_i^1 & & X_i^2 & \cdots & {} & & .
\end{tikzcd}$$

This step we will show that such 2-ray game of $(K + B + c_i H)$-MMP on $Z_i$ is also a 2-ray game of $(K + B + cH)$-MMP.

Let $P^k = \overline{\mathrm{NE}}(Z^k/X^k)$ and $Q^k = \overline{\mathrm{NE}}(Z^k/X^{k+1})$, then $P^k$ is $(K_{Z^k} + B_{Z^k} + c_0 H_{Z^k})$-positive and $(K_{Z^k} + B_{Z^k} + c_0 H_{Z^k})$-negative. Need to show this also holds for each $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$. Prove this by induction on $k$.

Since $c > c_i$, we have

$$K_Z + B_Z + cH_Z = p^*(K_X + B + cH) - aE \; (a > 0)$$

By negativity lemma, there is a curve $C_Z$ on $Z$ mapping to a point on $X$, and $E.C_Z < 0$, thus we have $(K_Z + B_Z + cH_Z).P^0 > 0$, where $P^0 = \mathbb{R}_{\geqslant 0}[C_Z] = \overline{\mathrm{NE}}(Z/X)$. Suppose $(K_{Z^k} + B_{Z^k} + cH_{Z^k}).P^k > 0$, then $(K_{Z^k} + B_{Z^k} + cH_{Z^k})$ is not nef over $S$. In particular, $P^k$ is positive, and the other extremal ray $Q^k$ is negative. This implies step 2. Furthermore, by decreasing of canonical divisor, we have

$$a(\nu; X_i, B_i + cH_i) \leqslant a(\nu; X, B + cH)$$

and strictly inequality holds if and only if $X_l \dashrightarrow X_{l+1}$ is not an isomorphism at center of $\nu$ on $X_l$ for some $l < i$

**Step 3**: Claim that $(X_i, B_i + cH_i)$ is klt for all $i \gg 0$. Otherwise, if there are infinitely many $i$ such that $(X_i, B_i + cH_i)$ is not klt, since they are all log canonical, this is equivalent to say there infinitely many $i$ and $\nu_i$ such that

$$-1 = a(\nu_i; X_i, B_i + cH_i) \geqslant a(\nu_i; X_0, B_0 + cH_0) \geqslant -1$$

Therefore $a(\nu; X_i, B_i + cH_i) = -1$ and $X_0 \dashrightarrow X_i$ isomorphism at the center $z(\nu_i, X)$. Thus the local $\theta$-canonical threholds are same

$$ct_\theta(\nu_i; X, B; H) = ct_\theta(\nu_i; X_i, B_i; H_i)$$

On the other hand, by definition

$$c_i \leqslant ct_\theta(\nu_i; X_i, B_i; H_i)$$

and since $(X, B + cH)$ is not klt along $z(\nu_i, X)$, it is not $\theta$-canonical, thus

$$ct_\theta(\nu_i; X_i, B_i; H_i) < c$$

Therefore

$$c_i \leqslant ct_\theta(\nu_i; X, B; H) < c$$

But the set $\{ct_\theta(x; X, B; H); x \in X\}$ is finite, a contradiction! We may assume $(X_i, B_i + cH_i)$ are all klt.

**Step** 4: Note that $E_i = \mathrm{Exc}(p_i)$ are all distinct. Otherwise, assume $E_i = E_j$ for some $i < j$, then $Z_i$ and $Z_j$ are isomorphic in a neighborhood of $E_i$ and $E_j$, thus

$$a(E_i; X_i, B_i + cH_i) = a(E_j; X_j, B_j + cH_j)$$

However, since $E_i = E_j$ is not a divisor on $X_j$, there is $k < j$ such that $E_j$ is contracted by $Z'_k \to X_{k+1}$, therefore $X_k \dashrightarrow X_{k+1}$ is not isomorphic at $E_j$, hence

$$a(E_i; X_i, B_i{+}cH_i) \leqslant a(E_j; X_k, B_k{+}cH_k) < a(E_j; X_{k+1}, B_{k+1}{+}cH_{k+1}) \leqslant a(E_j; X_j, B_j{+}cH_j)$$

which is a contradiction.

Since $(X, B + cH)$ is klt, then there are only finitely many $E_i$ with $a(E_i, X, B + cH) < 0$. But there are in fact infinitely many

$$a(E_i; X, B + cH) \leqslant a(E_i; X_i, B_i + cH_i) < -\theta(E) \leqslant 0,$$

a contradiction!

At last we have the Noether-Fano-Iskovskikh criterion to show when they are isomorphic:

**Theorem 3.3.5.** *(Noether-Fano-Iskovskikh Criterion): Notations as in the definition of Sarkisov degree, then*

1. $\mu \geqslant \mu'$;
2. *If* $\mu \geqslant \lambda$ *and* $(K_X + B + \frac{1}{\mu}H)$ *is nef, then* $\Phi$ *is an isomorphisms of Mori fibre spaces, i.e., we have commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\sim", "\Phi"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "\sim"] & S' \end{tikzcd}$$

*Proof.* We follow the proof in [7],[10] and [6]:

1. Only need to show $(K_X + B + \frac{1}{\mu'}H)$ is $f$-nef. Let $\sigma : W \to X$ and $\sigma' : W \to X'$ be the common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W =& \sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ =& \sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here $\{G_i\}, \{E_j\}$ are $\sigma$-exceptional divisors, and $\{E_j\}\{G'_k\}$ are $\sigma'$-exceptional divisors. Since $H_W = \sigma'^* H'$, $g'_k > 0$ (or there are no such $G'_k$). Then take a general curve $C \subset X$ contracted by $f$, such that its strict transform $\tilde{C}$ on $W$ is disjoint from $G_i, E_j$, and is not contained in $G'_k$. Then we have:

$$\begin{aligned} C.\left(K_X + B + \frac{1}{\mu'}H\right) =& \tilde{C}.\sigma^*\left(K_X + B + \frac{1}{\mu'}H\right) \\ =& \tilde{C}.\left(\sigma^*\left(K_W + B_W + \frac{1}{\mu'}H\right) + \sum g_i G_i + \sum e_j E_j\right) \\ =& \tilde{C}.\left(K_W + B_W + \frac{1}{\mu'}H_W\right) \\ =& \tilde{C}.\left(\sigma'^*\left(K_{X'} + B' + \frac{1}{\mu'}H'\right) + \sum e'_j E_j + \sum g'_k G'_k\right) \\ =& \tilde{C}.\sigma'^* f'^* A' + C.\left(\sum g'_k G'_k\right) \\ \geqslant& 0 \end{aligned}.$$

This implies $(K_X + B + \frac{1}{\mu'}H)$ is $f$-nef and $\mu \geqslant \mu'$;

2. First we show that $\mu = \mu'$. By 1, we only need to show $(K_{X'} + B' + \frac{1}{\mu}H')$ is $f'$-nef. Indeed, same as 1, we can take a curve $C'$ on $X'$ contracted by $f'$, such that its strict transform $\tilde{C}'$ on $W$ is disjoint from $G'_i, E_j$, and is not contained in $G'_k$ and $C'.\left(K_{X'} + B' + \frac{1}{\mu}H'\right) \geqslant 0$.

   Then we show there are isomorphic. Take a very ample divisor $D$ on $X$ and let $D'$ be its strict transform on $X'$. $D'$ is $f'$-ample, thus there exists $0 < d \ll 1$ such that the following holds:

   - $K_X + B + \frac{1}{\mu}H + dD$ is ample;
   - $K_{X'} + B' + \frac{1}{\mu}H' + dD'$ is ample.

   Therefore $X$ and $X'$ are both log canonical models of $(W, B_W + \frac{1}{\mu}H_W + dD_W)$, hence $X \cong X'$. Furthermore, $f$ and $f'$ are contractions of same numerical class of curves, thus two log Mori fibre spaces are isomorphic.

□

# 4 Double scaling

This section we follows [7, 13.The Sarkisov program] and [10].

## 4.1 Prepare

Let $(W, B_W)$ be a $\mathbb{Q}$-factorial klt pair and $f : (X, B) \to S$ and $f' : (X', B') \to S'$ be two different log Mori fibre spaces as outputs of $(K_W + B_W)$-MMP. To modify the beginning setting, we need more conventions and lemmas:

**Definition 4.1.1.** *Let $f : X \dashrightarrow Y$ be a birational map of normal quasi-projective varieties. If*

- *$f$ does not extract divisors;*
- *$a(E; X, B_X) \leqslant a(E; Y, B_Y)$ for all divisors $E$ over $X$.*

*then we denote $(X, B) \geqslant (Y, B_Y)$.*

In particular, for terminal pairs, we have following lemma:

**Lemma 4.1.2.** *[7, Lemma 13.8] Let $f : W \dashrightarrow X$ be a birational map where $(W, B_W)$ is terminal. If*

- *$f$ does not extract divisors;*
- *$K_X + B$ is nef, where $B = f_*B_W$;*
- *$a(E; X, B) \geqslant a(E; W, B_W)$ for all divisors $E \subset W$,*

*then*

- *$(W, B_W) \geqslant (X, B)$.*
- *$(X, B)$ is klt*
- *If $Z \to X$ is a divisorial extraction of a divisor $E$ with $a(E; X, B) \leqslant 0$, then $E$ is a divisor on $W$ ;*
- *If $Z \to X$ is terminalization of $(X, B)$, then $W \dashrightarrow Z$ extracts no divisors.*

Conversely, start from a klt pair and non-positive map, we have

**Lemma 4.1.3.** *[10, Lemma 3.5] Let $\sigma : (W, B_W) \dashrightarrow (X, B)$ be a $K_W + B_W$-non-positive birational map such that $\sigma_*(K_W + B_W) = K_X + B$ and $(W, B_W)$ is a $\mathbb{Q}$-factorial klt pair. Then there is a resolution of indeterminacy $\pi : \tilde{W} \to W$ and $\tilde{\sigma} : \tilde{W} \to X$ such that*

- *$(\tilde{W}, B_{\tilde{W}})$ is $\mathbb{Q}$-factorial terminal and $\tilde{\sigma}_* B_{\tilde{W}} = B$,*
- *$\tilde{\sigma}$ is $(K_{\tilde{W}} + B_{\tilde{W}})$-non-positive and $(\tilde{W}, B_{\tilde{W}}) \geqslant (X, B)$.*

By Lemma 4.1.3, we replace $(W, B_W)$ by its log resolution such that $(W, B_W)$ is terminal and $\sigma : W \to X$ and $\sigma' : W \to X'$ are $(K_W + B_W)$-non-positive morphisms, and $(W, B_W) \geqslant (X, B), (X', B')$.

Take very general ample $\mathbb{Q}$-divisors $A$ and $A'$ on $S$ and $S'$ such that $G \sim_{\mathbb{Q}} -(K_X + B) + f^*A$ and $H \sim_{\mathbb{Q}} -(K_{X'} + B') + f'^*A'$ are two ample $\mathbb{Q}$-divisors. Moreover, we may assume $G$ and $H$ satisfying $G_W := \sigma^*G = \sigma_*^{-1}G$ and $H_W := \sigma'^*H = \sigma'^{-1}_*H$. Therefore $\sigma_*(K_W + B_W + G_W) = K_X + B + G$ is nef, and Lemma 4.1.2 holds. Furthermore, we may assume $(W, B_W + gG_W + hH_W)$ is log smooth and terminal for all $0 \leqslant g, h \leqslant 2$ by taking furthermore blowing up if necessary. Then we have:

**Theorem 4.1.4** (Sarkisov program with double scaling)**.** *Notations as above, there is a finite sequence of Sarkisov links*

$$\begin{tikzcd}
X = X_0 \arrow[r, dashed] \arrow[d, "f=f_0"'] & X_1 \arrow[r, dashed] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"'] & \cdots \arrow[r, dashed] & X_N = X' \arrow[d, "f_N"'] \\
S = S_0 & S_1 & S_2 & & S_N = S'
\end{tikzcd}$$

*and rational numbers*

$$\begin{aligned} 1 = g_0 \geqslant g_1 \geqslant \cdots \geqslant g_N &= 0 \\ 0 = h_0 \leqslant h_1 \leqslant \cdots \leqslant h_N &= 1 \end{aligned}$$

*such that*

1. *For each $i$, $\sigma_i : W \dashrightarrow X_i$ is $(K_W + B_W + g_iG_W + h_iH_W)$-non-positive, and $(K_{X_i} + B_i + g_iG_i + h_iH_i) = \sigma_{i*}(K_W + B_W + g_iG_W + h_iH_W)$ is nef and is relatively trivial over $S_i$;*
2. *$(W, B_W + g_iG_W + h_iH_W) \geqslant (X_i, B_i + g_iG_i + h_iH_i)$;*
3. *each Sarkisov link $X_i \dashrightarrow X_{i+1}$ is given by a sequence of $(K_{X_i} + B_i + g_iG_i + h_iH_i)$-trivial maps.*
4. *The last link $X_N \to S_N$ is isomorphic to $X' \to S'$*

## 4.2 Construct Sarkisov links

This subsection we construct the links inductively. Suppose we have $\sigma_i : W \dashrightarrow X_i$ as in Theorem 4.1.4, that is

- $f_i : (X_i, B_i) \to S_i$ is a log Mori fibre space and $\sigma_{i*}B_W = B_i$;
- $\sigma_i : W \dashrightarrow X_i$ is $(K_W + g_iG_i + h_iH_W)$-non-positive birational map, and $(K_{X_i} + B_i + g_iG + h_iH) = \sigma_{i*}(K_W + B_W + g_iG_W + h_iH_W)$ is nef and is relatively trivial over $S_i$;
- $(W, B_W + g_iG_W + h_iH_W) \geqslant (X_i, B_i + g_iG_i + h_iH_i)$;
- $0 \leqslant g_i, h_i \leqslant 1$ are rational numbers.

Then we need to show that there is a Sarkisov link $X_i \dashrightarrow X_{i+1}$ satisfying Theorem 4.1.4. Similarly with Sarkisov degree, we have following notations:

**Definition 4.2.1.** *Let $C_i$ be a general $f_i$-vertical curve on $X_i$, then*

- $r_i := \frac{H_i.C_i}{G_i.C_i}$ *;*
- *Let $\Gamma$ be the set of $t \in [0, \frac{g_i}{r_i}]$ such that*
  1. $(W, B_W + g_iG_W + h_iH_W + t(H_W - r_iG_W)) \geqslant (X_i, B_i + g_iG_i + h_iH_i + t\,(H_i - r_iG_i))$
  2. $K_{X_i} + B_i + g_iG + h_iH + t(H_i - r_iG_i)$ *is nef;*

  *Let $s_i = \max\Gamma$;*

- *Let $D_{W,i} = B_W + g_iG_W + h_iH_W$ and $D_i = B_i + g_iG_i + h_iH_i$. Let $D_{W,i}(t) = B_W + g_iG_W + h_iH_W + t(G_W - r_iH_W)$ and $D_i(t) = B_i + g_iG_i + h_iH_i + t\,(G_i - r_iH_i))$. Let $g_{i+1} = g_i - r_is_i$ and $h_{i+1} = h_i + s_i$. Note that $D_{W,i+1} = D_{W,i}(s_i)$.*

Then we have (check [10, Lemma 4.4] for details)

1. $r_i > 0$;
2. either $\Gamma = \{0\}$ or is a closed interval;
3. $g_{i+1} = g_i \Leftrightarrow h_{i+1} = h_i \Leftrightarrow s_i = 0$;

**Construct links**.: If $s_i = \frac{g_i}{r_i}$, then $g_{i+1} = 0$. Let $N = i+1$ and let $f_N : X_N = X_i \to S_N = S_i$, then $X_N \to S_N$ is isomorphic to $f' : X' \to S'$ (see Proposition 4.3.2) and we stop. Otherwise, if $s_i < \frac{g_i}{r_i}$, then we construct the Sarkisov link $X_i \dashrightarrow X_{i+1}$ in following cases:

1. Suppose $s_i$ is not the threshold of condition 1 of $\Gamma$. That is, there exists $0 < \epsilon \ll 1$, such that for any divisor $E$ on $W$, we have

$$a(E; X_i, D_i(s_i + \epsilon)) \geqslant a(E; W, D_{W,i}(s_i + \epsilon))$$

   and $K_{X_i} + D_i(s_i+\epsilon)$ is not nef. Then there is a 2-dimensional $(K_{X_i} + D_i(s_i+\epsilon) - \delta G_i)$-negative extremal face $F$ for some $0 < \delta \ll \epsilon$, spaned by $R = \mathbb{R}_{\geqslant 0}[C_i]$ and another extremal ray $P$. Hence there is a contraction $X_i \to T_i$ corresponding to $F$ factoring through $f_i$. Then we run $(K_{X_i} + D_i(s_i+\epsilon))$-MMP on $X_i$ with scaling over $T_i$. After finitely many flips, we either have a $(K_{X_i} + D_i(s_i+\epsilon))$ minimal model, a divisorial contraction, or a Mori fibre space over $T_i$:

   (a) After finitely many flips $X_i \dashrightarrow X_{i+1}$ there is a log Mori fibre space $X_{i+1} \to S_{i+1}$, and this is a link of type III.

   (b) After finitely many flips $X_i \dashrightarrow Z_i$ there is a divisorial contraction $Z_i \to X_{i+1}$, then let $S_{i+1} = T_i$ and $X_{i+1} \to S_{i+1}$ is a log Mori fibre space and this is a link of type IV.

   (c) After finitely many flips $X_i \dashrightarrow X_{i+1}$, the contraction $X_{i+1} \to T_i$ is a log minimal model of $(X_i, D_i\,(s_i+\epsilon))$ over $T_i$. Let $C'$ be the strict transform of $C_i$ on $X_{i+1}$, then $(K_{X_{i+1}} + D_{i+1}(\epsilon)).C' = 0$ and $(K_{X_{i+1}} + B_{i+1}).C' < 0$, therefore there is a contraction $X_{i+1} \to S_{i+1}$ which is a log Mori fibre space. And this is a link of type IV.

2. Suppose $s_i$ is the threshold of condition 1 of $\Gamma$, that is, there exists $0 < \epsilon \ll 1$ and a $\sigma_i$-exceptional divisor $E_i$ on $W$ such that

$$a(E_i; X_i, D_i(s_i+\epsilon)) < a(E_i; W, D_{W,i}(s_i+\epsilon)).$$

   In this case, we have

$$a(E_i; X_i, D_i(s_i)) = a(E_i; W, D_{W,i}(s_i)) = -\operatorname{mult}_{E_i}(D_{W,i}(s_i)) \leqslant 0.$$

   Let $p_i : Z_i \to X_i$ be the divisorial extraction of the divisor $E_i$ as in Corollary 2.2.4, and suppose $K_{Z_i} + D_{Z_i}(s_i) = K_{Z_i} + B_{Z_i} + g_{i+1}G_{Z_i} + h_{i+1}H_{Z_i} = p_i^*\,(K_{X_i} + D_i\,(s_i+\epsilon))$. Take a sufficiently small $\delta$ such that $0 < \delta \ll \epsilon \ll 1$ and

$$K_{Z_i} + \Delta_i = p_i^*(K_{X_i} + D_i(s_i+\epsilon) - \delta G_i)$$

is klt. Then we run $(K_{Z_i}+\Delta_i)$-MMP on $Z_i$ over $S_i$. Since $Z_i$ is covered by $(K_{Z_i}+\Delta_i)$-negative curves, it follows that $(K_{Z_i}+\Delta_i)$ is not peseudo-effective over $S_i$, and this MMP ends with a log Mori fibre space. Moreover, this is a MMP for $p_i^*(K_{X_i}+D_i(s_i+\epsilon)-\delta' G_i)$ for all $0<\delta'\leqslant\delta$. After finitely many flips, we either have a $(K_{Z_i}+\Delta_i)$ log Mori fibre space or a $(K_{Z_i}+\Delta_i)$ divisorial contraction.

(a) After finitely many flips $Z_i \dashrightarrow X_{i+1}$ there is a Mori fibre space $X_{i+1}\to S_{i+1}$, and this is a link of type I. In this case we have $\rho(X_{i+1})=\rho(X_i)+1$.

(b) After finitely many flips $Z_i \dashrightarrow Z'_{i+1}$ there is a divisorial contraction $q_i : Z'_{i+1}\to X_{i+1}$, and then a logMori fibre space $X_{i+1}\to S_i =: S_{i+1}$. This is a link of type II.

**Claim 4.2.2.** *By [7, Lemma 13.14-17] and [10, Lemma 4.2] , we have:*

1. *$r_i \leqslant r_{i+1}$. Moreover, in case 1a, we have $r_i < r_{i+1}$.*

2. *Since the birational map $X_i \dashrightarrow X_{i+1}$ is over $T_i$ (over $S_i$) and $(K_{X_i}+D_i(s_i))$ is numerically trivial over $T_i$ (over $S_i$) in case 1 (case 2), it follows that $a(E;X_i,D_i(s_i)) = a(E;X_{i+1},D_{i+1})$ for any divisors $E$ over $W$ and so the inequality*

$$a(E;X_{i+1},D_{i+1}) \geqslant a(E;W,D_{W,i+1})$$

3. *In case 1, for any divisor $E\subset W$, we have $a(E;X_i,D_i(s_i+\epsilon)) \leqslant a(E;X_{i+1},D_{i+1}(\epsilon))$ for all $0<\epsilon\ll 1$. Moreover, since $X_i \ncong X_{i+1}$, there is a divisor $F$ over $W$ such that $a(F;X_i,D_i(s_i+\epsilon)) < a(F;X_{i+1},D_{i+1}(\epsilon))$.*

4. *In case 2, for any divisor $E\subset W$, we have $a(E;X_i,D_i(s_i+\epsilon)-\delta G_i) \leqslant a(E;X_{i+1},D_{i+1}(\epsilon)-\delta G_{i+1})$ for all $0<\epsilon\ll 1$. Moreover, since $X_i \ncong X_{i+1}$, there is a divisor $F$ over $W$ such that $a(F;X_i,D_i(s_i+\epsilon)-\delta G_i) < a(F;X_{i+1},D_{i+1}(\epsilon)-\delta G_{i+1})$.*

5. *$h_i\leqslant 1$, and $h_i=1$ if and only if $g_i=0$;*

## 4.3 Termination

**Lemma 4.3.1.** *[7, Lemma 13.18 and Lemma 13.19] (or [10, Lemma 4.9]) Suppose we construct a sequence of Sarkisov links:*

$$\begin{array}{ccccccccc} X=X_0 & \dashrightarrow & X_1 & \dashrightarrow & X_2 & \longrightarrow \cdots \dashrightarrow & X_i & \longrightarrow & \cdots, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ S=S_0 & & S_1 & & S_2 & & S_i & & \end{array}$$

*then*

1. *there are only finitely many possibilities of $f_i : X_i \to S_i$ up to isomorphism;*

2. *the Sarkisov program with double scalling of $(G_W,H_W)$ terminates. That is, there exits an integer $N>0$ such that $g_N=0$.*

*Proof.* 1. This essentially follows from the finiteness of weak log canonical model (Theorem 2.1.10). We construct the subspace $V$ of $\mathrm{WDiv}_{\mathbb{R}}(W)$ as following:

(a) If $h_k > 0$ for some $k$: Since $H_W$ is nef and big, take an ample $\mathbb{Q}$-divisor $A_W$ and an effective $\mathbb{Q}$-divisor $C_W$ such that $H_W \sim_{\mathbb{Q}} A_W + C_W$. Let $V$ be the affine space spaned by components of $B_W, G_W, H_W, C_W$, then for $i > k$:

$$B_W + g_i G_W + h_i H_W \sim_{\mathbb{Q}} h_k A_W + B_W + g_i G_W + (h_i - h_k) H_W + h_k C_W =: \Delta_i \in \mathcal{L}_{h_k A_W}(V)$$

(b) If $h_k = 0$ for all $k$, then $h_i \equiv 0$ and $g_i \equiv 1$. Since $G_W$ is nef and big, take an ample $\mathbb{Q}$-divisor $A_W$ and an effective $\mathbb{Q}$-divisor $C_W$ such that $G_W \sim_{\mathbb{Q}} A_W + C_W$. Let $V$ be the affine space spaned by components of $B_W, C_W$, then

$$B_W + G_W \sim_{\mathbb{Q}} A_W + B_W + C_W =: \Delta_i \in \mathcal{L}_{A_W}(V)$$

Then all $X_i$ are weak log canonical models of $(W, \Delta_i)$. By finiteness of weak log canonical models, there are finitely many $\sigma_i : W \dashrightarrow X_i$ up to isomorpism. Then we shall show that for $\sigma_i : W \dashrightarrow X_i$ there are finitely many log Mori fibre spaces in the sequence up to isomorpism. Indeed, we may assume that there is a $k$ such that $X_i \cong X_k$ for all $i > k$, and $f_i$ is the contraction corresponding to an extremal ray $R_i \subset \overline{\mathrm{NE}}(X_k)$. Then we have $(K_{X_k} + B_k).R_i < 0$ and $(K_{X_k} + B_k + g_i G_k + h_i H_k).R_i = 0$. Furthermore, $H_k$ and $G_k$ are relatively ample over $S_i$ for all $i > k$.

(a) If $h_k > 0$: Since $H_k$ is big, we have $h_k H_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor $H_k$ and effective $\mathbb{Q}$-divisor $E_k$. Let $B'_k = B_k + (1-\epsilon) h_k H_k + \epsilon E_k$ for sufficiently small $\epsilon$ such that $(X_k, B'_k)$ is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \epsilon A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k} + B'_k + \epsilon A_k \geqslant 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_\alpha$$

All extremal rays $R_i$ corresponding to $f_i$ for $i > k$ are in the finite set $\{R_\alpha\}_{\alpha \in \Lambda}$, thus there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of $X_k$.

(b) If $h_i = 0$ for all $i$, and hence $g_i = 1$ for all $i$. Since $G_i$ is big, we have $G_k = A_k + E_k$ for some ample $\mathbb{Q}$-divisor $A_k$ and effective $\mathbb{Q}$-divisor $E_k$. Let $B'_k = B_k + (1-\epsilon) G_k + \frac{\epsilon}{2} E_k$ for sufficiently small $\epsilon$ such that $(X_k, B'_k)$ is klt, then $(K_{X_k} + B'_k).R_i < 0$ and $(K_{X_k} + B'_k + \frac{\epsilon}{2} A_k).R_i < 0$ for all $i > k$. By Cone theorem, we have

$$\overline{\mathrm{NE}}(X_k) = \overline{\mathrm{NE}}(X_k)_{K_{X_k} + B'_k + \frac{\epsilon}{2} A_k \geqslant 0} + \sum_{\alpha \in \Lambda \text{ finite set}} R_\alpha$$

Again, there are finitely many log Mori fibre spaces $f_i : X_i \to S_i$ of $X_k$.

2. Assume this sequence of links is infinite, then there exits an $i$ such that there are infiniely many $j > i$ such that $f_i : X_i \to S_i$ and $f_j : X_j \to S_j$ are isomorphic. Then we have $g_{i+1} = g_{j+1}$ and $h_{i+1} = h_{j+1}$. Since sequences of $h_k$ and $g_k$ are monotone, we have $h_{i+1} = h_k$ and $g_{i+1} = g_k$ for all $k > i$. Suppose $X_i \dashrightarrow X_{i+1}$ is a link in case 1 of the Construction in 4.2, then the next link is also in case 1, and all the links after are in case 1. Note that $X_i \cong X_j$ and therefore $\rho(X_i) = \rho(X_j)$, the links are all of type IV. But this contracts 3 of Claim 4.2.2. Therefore there are no link of type III or IV after $X_i$. In other words, the links after $X_i$ are all type I or II in case 2.

Since $\rho(X_i) = \rho(X_j)$ , $X_i$ and $X_j$ are linked by the Sarkisov links of type II. But this contracts 4 of Claim 4.2.2.

□

**Proposition 4.3.2.** *$X_N \to S_N$ is isomorphic to $X' \to S'$.*

*Proof.* By 2 of Theorem 3.3.5, we have $h_N = 1$ and they are isomorphic. □

# 5 Using the Polytope

In this section we follows [8].

## 5.1 Morphisms between models

Let $W$ be a smooth projective variety, and let $V$ be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective $\mathbb{Q}$-divisor $A$. In this subsection we describe the models $X_i$ of $(W, A+B)$ for some $B \in V$ and rational maps between $X_i$.

**Theorem 5.1.1.** *[8, Theorem 3.3] Let $W$ be a smooth projective variety, and $V$ be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ defined over rational numbers and fix an ample effective $\mathbb{Q}$-divisor $A$. Suppose that there is an element $D_0$ of $\mathcal{L}_A(V)$ such that $K_W + D_0$ is big and klt. Then there are finitely many rational contractions $f_i : W \dashrightarrow X_i$ such that*

1. *$\{\mathcal{A}_i = \mathcal{A}_{A,f_i}\}$ is a partition of $\mathcal{E}_A(V)$. $\mathcal{A}_i$ is a finite union of interiors of rational polytopes. If $f_i$ is birational then $\mathcal{C}_i = \mathcal{C}_{A,f_i}$ is a rational polytope;*
2. *If $i, j$ are two indices such that $\mathcal{A}_j \cap \mathcal{C}_i \neq \emptyset$ then there is a contraction $f_{ij} : X_i \to X_j$ and $f_j = f_{ij} \circ f_i$;*
3. *Suppose in addition $V$ spans Neron-Severi group of $W$. Pick $i$ such that a connected components $\mathcal{C}$ of $\mathcal{C}_i$ intersects the interior of $\mathcal{L}_A(V)$, the following are equivalent:*
   - *(a) $\mathcal{C}$ spans $V$;*
   - *(b) If $D \in \mathcal{A}_i \cap \mathcal{C}$ then $f_i$ is a log terminal model of $K_W + D$;*
   - *(c) $f_i$ is birational and $X_i$ is $\mathbb{Q}$-factorial.*
4. *Suppose in addition $V$ spans Neron-Severi group of $W$. If $i, j$ are two indices such that $\mathcal{C}_i$ spans $V$ and $D$ is a general point of $\mathcal{A}_j \cap \mathcal{C}_i$ which is also a point of interior of $\mathcal{L}_A(V)$, then $\mathcal{C}_i$ and $\overline{\mathrm{NE}}(X_i/X_j)^* \times \mathbb{R}^k$ for some $k \geqslant 0$. Furthermore $\rho(X_i/X_j)$ equals the differece in the dimensions of $\mathcal{C}_i$ and $\mathcal{C}_j \cap \mathcal{C}_i$.*

**Lemma 5.1.2.** *[8, Corollary 3.4]If $V$ spans Neron-Severi group of $W$, then there is a Zariski dense open subset $U$ of the Grassmannian $G(r, V)$ of real affine subspace of dimension $r$ such that any $[V'] \in U$ defined on ratinal numbers statify (1-4) of 5.1.1*

*Proof.* Let $U \subset G(r, V)$ be the set of real affine subspace $V'$ of $V$ of dimension $r$, which contion any sub no face of any $\mathcal{C}_i$ or $\mathcal{L}(V)$. In particular, the interior of $\mathcal{L}_A(V')$ is contained in the interior of $\mathcal{L}_A(V)$. Clearly that any $V' \in U$ satisfies (1-4) of 5.1.1. □

From now on in this subsection, we always assume that $V$ has dimension 2 and satisfies 5.1.1.

**Lemma 5.1.3.** *[8, Lemma 3.5] Let $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ be two rational contracions such that $\mathcal{C}_{A,f}$ is dimension 2 and $\mathcal{O} = \mathcal{C}_{A,f} \cap \mathcal{C}_{A,g}$ is dimension 1. Assume $\rho(X) \geqslant \rho(Y)$ and $\mathcal{O}$ is not contained in the boundary of $\mathcal{L}_A(V)$. Let $D$ be an interior point of $\mathcal{O}$ and $B = f_*D$. Then there is a rational contraction $\pi : X \dashrightarrow Y$ and $g = \pi \circ f$ such that either*

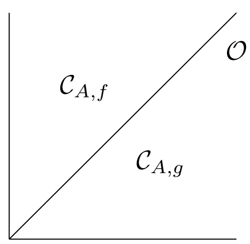
1. *$\rho(X) = \rho(Y) + 1$ and $\pi$ is $(K_X + B)$-trivial, and either*
   (a) *$\pi$ is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$, and either*
      i. *$\pi$ is a divisorial contraction and $\mathcal{O} \neq \mathcal{C}_{A,g}$, or*
      ii. *$\pi$ is a small contraction and $\mathcal{O} = \mathcal{C}_{A,g}$*

      *or*

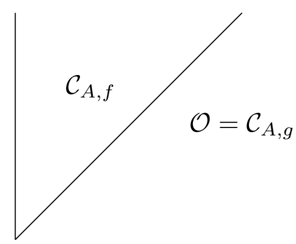
   (b) *$\pi$ is a Mori fibre space, and $\mathcal{O} = \mathcal{C}_{A,g}$ is contained in the boundary of $\mathcal{E}_A(V)$*

   *or*

2. *$\rho(X) = \rho(Y)$, and $\pi$ is a $(K_X + B)$-flop and $\mathcal{O} \neq \mathcal{C}_{A,g}$ is not contained in the boundary of $\mathcal{E}_A(V)$.*




or




*Proof.* By assumption $f$ is birational and $X$ is $\mathbb{Q}$-factorial. Let $h : W \dashrightarrow S$ be the ample model corresponding to $K_W + D$. Since $D$ is not a point of the boundary of $\mathcal{L}_A(V)$, if $D$ belongs to the boundary of $\mathcal{E}_A$ then $K_W + D$ is not big and so $h$ is not birational. As $\mathcal{O}$ is a subset of both $\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ there are morphisms $p : X \to S$ and $q : Y \to S$ of relative Picard number at most one. There are therefore only two cases

1. $\rho(X) = \rho(Y) + 1$, or
2. $\rho(X) = \rho(Y)$

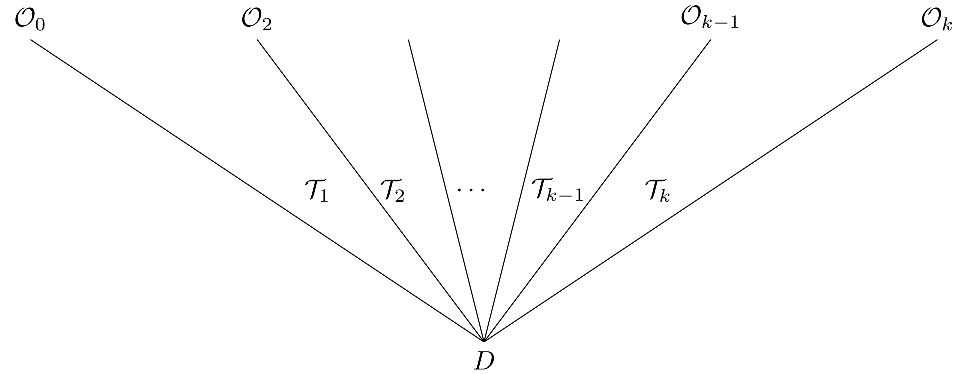
Suppose we are in the first case, then $q$ is the identity and $\pi : X \to Y$ is a contraction morphism such that $g = p \circ f$. Suppose that $\pi$ is birational, then $h$ is birational and $\mathcal{O}$ is not contained in the boundary of $\mathcal{E}_A(V)$. If $\pi$ is divisorial then $Y$ is $\mathbb{Q}$-factorial and so $\mathcal{O} \neq \mathcal{C}_{A,g}$. If $\pi$ is a small contraction then $\pi$ is not $\mathbb{Q}$-factorial and so $\mathcal{C}_{A,g} = \mathcal{O}$ is one dimensional. If $\pi$ is a Mori fibre space then $\mathcal{O}$ is contained in the boundary of $\mathcal{E}_A(V)$ and $\mathcal{O} = \mathcal{C}_{A,g}$.

Now suppose we are in the second case. Since $\rho(X/S) = \rho(Y/S) = 1$, we know that $p, q$ are not divisoriacontractions as $\mathcal{O}$ is one dimensional and $p, q$ are not Mori fibre spaces as $\mathcal{O}$ is cannot be contained in the boundary of $\mathcal{E}_A(V)$. Hence $p, q$ are small and the the rest is clear. □

**Lemma 5.1.4.** *[8, Lemma 3.6] Let $f : W \dashrightarrow X$ be a birational contraction between $\mathbb{Q}$-factorial varieties. Suppose $(W, D)$ and $(W, D + A)$ are both klt. If $f$ is ample model of $(W, D + A)$ and $A$ is ample, then $f$ is result of running $(K_W + D)$-MMP.*

This lemma gunrantee that every variety in the Sarkisov links constructed later is a MMP result of $(W, B_W)$.

Let $D = A + B$ be a point of boundary of $\mathcal{E}_A(V)$ in the interior of $\mathcal{L}_A(V)$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be the polytopes $\mathcal{C}_i$ of dimension 2 containing $D$. Let $\mathcal{O}_0$ and $\mathcal{O}_k$ be the intersection of $\mathcal{T}_0$ and $\mathcal{T}_k$ with boundary of $\mathcal{E}_A(V)$, and let $\mathcal{O}_i = \mathcal{T}_i \cap \mathcal{T}_{i+1}$. Let $f_i : W \to X_i$ be the rational contraction associated to $\mathcal{T}_i$ and $g_i : W \to S_i$ be the rational contraction associated to $\mathcal{O}_i$.




Set $f = f_1 : W \dashrightarrow X, g = f_k : W \dashrightarrow Y$ and $\phi : X \to S = S_0, \psi : Y \to T = S_k$ and $X' = X_2, Y' = X_{k-1}$ and let $W \dashrightarrow R$ be the ample model of $D$. Then

**Theorem 5.1.5.** *[8, Theorem 3.7] Suppose $B_W$ is a divisor such that $K_Z + B_W$ is klt and $D - B_W$ is ample. Then $\phi$ and $\psi$ are Mori fibre spaces as outputs of $(K_Z + B_W)$-MMP and connected by a Sarkisov link if $D$ is contained in more than two polytopes.*

*Proof.* WMA $k \geqslant 3$ and we have

$$\begin{tikzcd}
X' \arrow[rr, dotted] \arrow[d, dotted, "p"'] & & Y' \arrow[d, dotted, "q"] \\
X \arrow[d, "\phi"'] & & Y \arrow[d, "\psi"] \\
S \arrow[dr, "s"] & & T \arrow[dl, "t"'] \\
& R &
\end{tikzcd}$$

Note that $\rho(X_i/R) \leqslant 2$ and $\rho(X/S) = \rho(Y/T) = 1$. Thus

1. $s$ is identity and $p$ is a divisorial contraction (extraction), or
2. $s$ is a contraction and $p$ is a flop.

The same holds for $q$ and $t$. And the map $X' \to Y'$ is clear the composition of flops. This gives 4 types of links. ☐

## 5.2 Construction of Sarkisov links

We need a special resolution $W$ and an affine subspace $V \subset \mathrm{WDiv}(W)$ such that we can find two Mori fibre spaces $X/S$ and $Y/T$ and vertexs connecting them. The following lemma shows the desired affine subspace exits.

**Lemma 5.2.1.** *[8, Lemma 4.1] Let $\phi : X \to S$ and $\psi : Y \to T$ be two MMP related Mori fibre space corresponding to two klt projective varieties $(X, B_X)$ and $(Y, B_Y)$. Then we may find a smooth projective variety $W$, two biratinal morphism $f : W \to X$ and $g : W \to Y$, a klt pair $(W, B_W)$, an ample $\mathbb{Q}$-divisor $A$ on $W$ and a two dimensional rational affine subspace $V$ of $\mathrm{WDiv}_{\mathbb{R}}(W)$ such that*

1. *If $D \in \mathcal{L}_A(V)$ then $D - B_W$ is ample;*
2. *$\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ are not contained in the boundary of $\mathcal{L}_A(V)$;*
3. *$V$ satisfy 5.1.1;*
4. *$\mathcal{C}_{A,f}$ and $\mathcal{C}_{A,g}$ are two dimensional;*
5. *$\mathcal{C}_{A,\phi\circ f}$ and $\mathcal{C}_{A,\psi\circ g}$ are one dimensional.*

*Proof.* By assumption there is a $\mathbb{Q}$-factorial klt pair $(W, B_W)$ such that $f : W \dashrightarrow X$ and $g : W \dashrightarrow Y$ are both outcomes of $(K_W + B_W)$-MMP. Let $p' : W' \to W$ be any log resolution such that resolves the indeterminacy of $f$ and $g$, then we may write

$$K_{W'} + B_{W'} = p'^*(K_W + B_W) + E'$$

where $E' \geqslant 0$ and $B_{W'} \geqslant 0$ have no common components, and $E'$ is exceptional and $p'_* B_{W'} = B_W$. Pick a divisor $-F$ which is ample over $W$ with support equal to the full exceptional locus such that $K_{W'} + B_{W'} + F$ is klt. As $p'$ is $(K_{W'} B_{W'} + F)$-negative and $(K_W + B_W)$ is klt and $W$ is $\mathbb{Q}$-factorial, the $(K_{W'} + B_{W'} + F)$-MMP over $W$ terminates with the pair $(W, B_W)$. Replacing $(W, B_W)$ by $(W', B_{W'} + F)$ we may assume that $(W, B_W)$ is log smooth and $f, g$ are morphisms.

Pick general ample $\mathbb{Q}$-divisors $A, H_1, H_2, \ldots, H_k$ on $W$ such that $H_1, \ldots, H_k$ generate te Neron-Severi group of $W$. Let $H = A + H_1 + \ldots + H_k$. Pick sufficiently ample divisor $A_S$ on $S$ and $A_T$ on $T$ such that

$$-(K_X + B_X) + \phi^* A_S \text{ and } -(K_Y + B_Y)\psi^* A_T$$

are both ample. Pick a rational number $0 < \delta < 1$ such that

$$-(K_X + B_X + \delta f_* H) + \phi^* A_S \text{ and } -(K_Y + B_Y + \delta g_* H) + \psi^* A_T$$

are both ample and $(K_W + B_W + \delta H)$ is both $f$ and $g$ negative. Replacing $H$ by $\delta H$ we may assume that $\delta = 1$. Now pick a $\mathbb{Q}$-divisor $B_0 \leqslant B_W$ such that $A + (B_0 - B_W), -(K_X + f_* B_0 + f_* H) + \phi^* A_S$ and $-(K_Y + g_* B_0 + f_* H) + \psi^* A_T$ are all ample and $(K_W + B_0 + H)$ is both $f$ and $g$ negative.

Pick general ample $\mathbb{Q}$-divisors $F_1 \geqslant 0$ and $G_1 \geqslant 0$ such that

$$F_1 \sim_{\mathbb{Q}} -(K_X + f_* B_0 + f_* H) + \phi^* A_S \text{ and } G_1 \sim_{\mathbb{Q}} -(K_Y + g_* B_0 + g_* H) + \psi^* A_T$$

and

$$K_W + B_0 + H + F + G$$

is klt, where $F = f^* F_1$ and $G = g^* G_1$.

Let $V_0$ be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(W)$ which is the translate by $B_0$ of the vector subspace spaned by $H_1, \dots, H_k, F, G$. Supppose that $D = A + B \in \mathcal{L}_A(V_0)$. Then

$$D - B_W = (A + B_0 - B_W) + (B - B_0)$$

is ample, as $B - B_0$ is nef by definition of $V_0$. Note the

$$B_0 + F + H \in \mathcal{A}_{A,\phi\circ f}(V_0), B_0 + G + H \in \mathcal{A}_{\psi\circ g}(V_0)$$

and $f$, respectively $g$, is a weak log canonical model of $K_W + B_0 + F + H$, respectively $K_W + B_0 + G + H$. Thus theorem 5.1.1 implies that $V_0$ satisfies (1-4) of 5.1.1.

Since $H_1, \dots, H_k$ generated the Neron-Severi group of $W$ we may find constants $h_1, \dots, h_k$ such that $G \equiv \sum_{i=1}^k h_i H_i$. Then there is $0 < \delta \ll 1$ such that $B_0 + F + \delta G + H - \delta(\sum_{i=1}^k h_i H_i) \in \mathcal{L}_A(V_0)$ and

$$B_0 + F + \delta G + H - \delta(\sum_i^k h_i H_i) \equiv B_0 + F + H.$$

Thus $\mathcal{A}_{A,\phi\circ f}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. Similarly $\mathcal{A}_{A,\psi\circ g}$ is not contained in the boundary of $\mathcal{L}_A(V_0)$. In particular $\mathcal{A}_{A,\phi\circ f}$ and $\mathcal{A}_{A,\psi\circ g}$ span affine hyperplanes of $V_0$, since $\rho(X) = \rho(Y) = 1$.

Let $V_1$ be the translate by $B_0$ of two dimensional vector space spaned by $F + H - A$ and $F + G - A$. Let $V$ be a small general perturbation of $V_1$, which is defined over rationals. This is the affine subspace we need. □

Then we can prove the main theorem

*Proof of 1.1.2.* Let $(W, B_W), A$ and $V$ as in the lemma 5.2.1. Pick $D_0 \in \mathcal{A}_{A,\phi\circ f}$ and $D_1 \in \mathcal{C}_{A,g}$ belonging to the interior of $\mathcal{L}_A(V)$. As $V$ is two dimensional, removing $D_0$ and $D_1$ divides the boundary of $\mathcal{E}_A(V)$ into two parts. The part which consists entirely of divisors which are not big is contained in the interior of $\mathcal{L}_A(V)$. Consider tracing this boundary from $D_0$ to $D_1$. Then there are finitely many $2 \leqslant i \leqslant N$ points $D_i$ which are contained in more than two polytopes $\mathcal{C}_{A,f_i}(V)$. By lemma 5.1.5, each point $D_i$ gives a Sarkisov link. And the birational map $X \dashrightarrow Y$ is composition of such links. □

# 6 Examples

# 7 Application

# References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