RELATIVE NAKAYAMA-ZARISKI DECOMPOSITION AND MINIMAL MODELS OF GENERALIZED PAIRS

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Abstract. In this note, we prove some basic properties on the relative Nakayama-Zariski decomposition. We apply them to the study of generalized lc pairs. We prove the existence of log minimal models or Mori fiber spaces for (relative) generalized lc pairs, an analogue of a result of Hashizume-Hu. We also show that, for any generalized lc pair $(X,B+A,\mathbf{M})/Z$ such that $K_X+B+A+\mathbf{M}_X\sim_{\mathbb{R},Z}0$ and $B\geq 0, A\geq 0$, $(X,B,\mathbf{M})/Z$ has either a log minimal model or a Mori fiber space. This is an analogue of a result of Birkar/Hacon-Xu and Hashizume in the category of generalized pairs.

Contents

1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

The theory of *generalized pairs* (*g-pairs* for short) was introduced by Birkar and Zhang in [BZ16] to tackle the effective Iitaka fibration conjecture. The structure of g-pairs naturally appears in the canonical bundle formula and sub-adjunction formulas [Kaw98, FM00]. This theory has been used in an essential way in the proof of the Borisov-Alexeev-Borisov conjecture [Bir19, Bir21]. We refer the reader to [Bir20] for a more detailed introduction to the theory of g-pairs.

Recently, there are some progress towards the minimal model program theory for generalized pairs. In particular, in [HL21], Hacon and the first author proved the cone theorem, contraction theorem, and the existence of flips for $\mathbb{Q}$-factorial glc g-pairs. However, some related results on the termination of flips and the existence of log minimal models and good minimal models for generalized pairs remain unknown. For example, we have the following results:

Theorem 1.1. *(1) ([HH20, Theorem 1.5]) Let $(X,B)/Z$ be a pair and $A\geq 0$ an ample/Z $\mathbb{R}$-divisor such that $(X,\Delta:=B+A)$ is lc. Then $(X,\Delta)/Z$ has a good minimal model or a Mori fiber space.*

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(2) *([Has19, Theorem 1.1]; see [Bir12, HX13] for the $\mathbb{Q}$-coefficient case) Let $(X,B)/Z$ be a pair and $A\geq 0$ an $\mathbb{R}$-divisor such that $(X,B+A)$ is glc and $K_X+B+A\sim_{\mathbb{R},Z}0$. Then:*
   (a) *$(X,B)/Z$ has either a Mori fiber space or a log minimal model $(Y,B_Y)/Z$.*
   (b) *If K_Y+B_Y is nef$/Z$, then K_Y+B_Y is semi-ample$/Z$.*
   (c) *If (X,B) is $\mathbb{Q}$-factorial dlt, then any (K_X+B)-MMP$/Z$ with scaling of an ample$/Z$ $\mathbb{R}$-divisor terminates.*

In this paper, we further investigate the minimal model program for generalized pairs. We prove the following results, which can be considered as analogues of Theorem 1.1:

Theorem 1.2. *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair and $A\geq 0$ an ample$/U$ $\mathbb{R}$-divisor such that $(X,\Delta:=B+A,\mathbf{M})$ is glc. Then*
(1) *$(X,\Delta,\mathbf{M})/U$ has a log minimal model or a Mori fiber space, and*
(2) *if $\mathbf{M}_X$ is $\mathbb{R}$-Cartier, then $(X,\Delta,\mathbf{M})/U$ has a good minimal model or a Mori fiber space.*

Theorem 1.3. *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair, $X\rightarrow Z$ a projective morphism$/U$ between normal quasi-projective varieties, and $A\geq 0$ an $\mathbb{R}$-divisor such that $(X,B+A,\mathbf{M})$ is glc and $K_X+B+A+\mathbf{M}_X\sim_{\mathbb{R},Z}0$. Then*
(1) *$(X,B,\mathbf{M})/Z$ has a log minimal model or a Mori fiber space, and*
(2) *if $(X,B,\mathbf{M})$ is $\mathbb{Q}$-factorial gdlt, then any $(K_X+B+\mathbf{M}_X)$-MMP$/Z$ with scaling of an ample$/Z$ $\mathbb{R}$-divisor terminates.*

We expect Theorems 1.2 and 1.3 to play important roles in the minimal model program theory for generalized lc pairs.

Note that when $\mathbf{M}=\mathbf{0}$, Theorem 1.2 is exactly Theorem 1.1(1) and Theorem 1.3 is exactly Theorem 1.1(2.a, 2.c). For technical reasons, at the moment, we cannot remove the "$\mathbf{M}_X$ is $\mathbb{R}$-Cartier" assumption in Theorem 1.2(2).

We still expect the analogue of Theorem 1.1(2.b) to be true. That is, we expect that any log minimal model of $(X,B,\mathbf{M})/Z$ is a good minimal model under the setting of Theorem 1.3. This is because such $K_X+B+\mathbf{M}_X$ is log abundant$/U$ with respect to $(X,B,\mathbf{M})$ by Theorem 8.4 below. However, the following example shows that the question is very subtle as "log abundant" does not imply semi-ampleness in general for glc g-pairs:

Example 1.4. Let C_0 be a nodal cubic in $\mathbb{P}^2$ and l the hyperplane class on $\mathbb{P}^2$. Let $P_1,P_2,...,P_{12}$ be twelve distinct points on C_0 which are different from the nodal point. Let

$$\mu: X=\mathrm{Bl}_{\{P_1,...,P_{12}\}}\rightarrow\mathbb{P}^2$$

to be the blow-up of $\mathbb{P}^2$ at the chosen points with the exceptional divisor $E=\sum_{i=1}^{12}E_i$, where E_i is the prime exceptional divisor over P_i for each i. Let $H:=\mu^*l$ and $C:=\mu^{-1}_*C_0$. Then $C\cong C_0$, $C\in|3H-E|$, and $K_X+C=\mu^*(K_{\mathbb{P}^2}+C_0)=0$.

We consider the big divisor $M=4H-E\sim H+C$. Since H is semi-ample and $M\cdot C=0$, M is nef. Notice that $\mathcal{O}_C(M)=\mathcal{O}_{C_0}(4l-\sum_{i=1}^{12}P_i)$ and $\mathrm{Pic}^0(C)\cong\mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplication group of $\mathbb{C}^*$.

(1) Suppose that $P_1,...,P_{12}$ are in general position such that $\mathcal{O}_C(M)$ is a non-torsion in $\mathrm{Pic}^0(C)$. Then M can never be semi-ample since $M|_C$ is not. However, the normalization C^n of C is $\mathbb{P}^1$, so $M|_{C^n}$ is semi-ample. This gives a glc pair $(X,C,\mathbf{M}:=\overline{M})$ such that both M and K_X+C+M are nef and log abundant with respect to $(X,C,\mathbf{M})$, but K_X+C+M is not semi-ample. One can further take the blow-up of the nodal point and take the crepant pullback to make each lc center normal.
(2) Suppose that $P_1,...,P_{12}$ are the intersection points of C_0 with a general quartic curve $Q_0\in|4l|$. Let Q be the birational transform of Q_0 on X. Then $M\sim Q\sim H+C$ is semi-ample and defines a projective birational contraction $f:X\rightarrow Y$ which contracts exactly the nodal curve C. Let $M'=H-3E_1$, then $\mathcal{O}_C(M')=\mathcal{O}_{C_0}(l-3P_1)$ is a non-torsion since Q_0 is general. Therefore M' is not $\mathbb{Q}$-linearly equivalent to 0 over Y (which

also implies that $f(M')$ is not $\mathbb{Q}$-Cartier). This gives a glc pair $(X, C, \mathbf{M}' := \overline{M'})/Y$ such that both M' and $K_X + C + M'$ are log abundant and numerically trivial over Y but $K_X + C + M'$ is not semi-ample over Y.

We refer the reader to [BH22] for some other interesting examples on the failure of positivity results for generalized pairs.

To prove our main theorems, the central idea is to combine the methods in [Has22] (some originated in [Has20a, Has20b, HH20]) and [HL21]. In particular, we need to generalize many results in [Has22] for projective varieties X to normal quasi-projective varieties X equipped with projective morphisms $\pi : X \to U$. Despite their similarities, a major difficulty is the use of Nakayama-Zariski decomposition [Nak04, III. §1], which is usually applied to projective varieties only. It is important to remark that the relative Nakayama-Zariski decomposition [Nak04, III. §4] does not always behave as good as the global Nakayama-Zariski decomposition (cf. [Les16]), and we are lack of references for even the most basic properties of them. In this note, we will study the behavior and basic properties on the relative Nakayama-Zariski decomposition. We refer the reader to [LT21] for further applications of the relative Nakayama-Zariski decomposition on the minimal model theory for generalized pairs.

Idea of the proof. It is important to notice that, Theorems 1.2 and 1.3 both have some "$\boldsymbol{b}$-log abundant" conditions:

(1) In Theorem 1.2, possibly replacing $(X, B, \mathbf{M})$ with $(X, B, \mathbf{M} + \frac{1}{2}\bar{A})$ and A with $\frac{1}{2}A$, we may assume that $\mathbf{M}$ is $\boldsymbol{b}$-log abundant with respect to $(X, B, \mathbf{M})$.
(2) In Theorem 1.3, $K_X + B + A + \mathbf{M}_X$ is automatically $\boldsymbol{b}$-log abundant$/Z$ as it is $\mathbb{R}$-linearly trivial over Z.

Therefore, one important goal of this paper is to study the minimal model program for g-pairs $(X, B, \mathbf{M})$ with $\boldsymbol{b}$-log abundant nef part $\mathbf{M}$ or with log abundant $K_X + B + \mathbf{M}_X$. Despite the technicality, the condition "$\boldsymbol{b}$-log abundant" is actually a very natural condition as it is preserved under adjunction. The key idea to study the minimal model program for such g-pairs is the following:

- By applying the Iitaka fibration and the generalized canonical bundle formula, we reduce the questions to the cases when either $\kappa_\iota(X/U, K_X+B+\mathbf{M}_X) = 0$ or when $\kappa_\iota(X/U, K_X+B+\mathbf{M}_X) = \dim X$ (see Section 4).
- When the invariant Iitaka dimension is 0, by abundance, the minimal model program behaves well (cf. Lemma 4.1). So we can reduce the question to the case when $K_X + B + \mathbf{M}_X$ is big$/U$.
- If $(X, B, \mathbf{M})$ is gklt then we can apply [BZ16, Lemma 4.4(2)]. Otherwise, by induction on dimension, we can apply the the special termination results near $\mathrm{Ngklt}(X, B, \mathbf{M})$.

Structure of the paper. In section 2, we introduce some preliminary results. In particular, we will recall some results on the the minimal model program for generalized pairs that are already included in [HL21, Version 2, Version 3] (but may not appear in the published version). In section 3, we study the basic behavior of the relative Nakayama-Zariski decomposition. In section 4, we use the Iitaka fibration and the generalized canonical bundle formula to simplify the question. In section 5, we prove a special termination result for generalized pairs. In section 6,7,8, we use the relative Nakayama-Zariski decomposition to prove analogues of most results in [Has22, Section 3] (section 6), [Has22, Theorem 3.14] (section 7), and [Has20b, Theorem 4.1] (section 8) respectively. In section 9, we prove Theorems 1.2 and 1.3.

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Some parts of this note has overlap with results in [HL21, Version 2 or Version 3]. Since these results are not expected to be published in the final version of [HL21] due to the simplification of the proofs of the main theorems of [HL21], for the reader's convenience, we include some of the results of [HL21, Version 2 or Version 3] in this paper and provide detailed proofs. The authors would like to thank Christopher D. Hacon for granting the text overlap.

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2. Preliminaries

We adopt the same notation as in [KM98, BCHM10]. For g-pairs, we adopt the same notation as in [HL21], which is the same as [Has20b, Has22] except that we use "$a(E,X,B,\mathbf{M})$ instead of "$a(E,X,B+\mathbf{M}_X)$" to represent log discrepancies. This is because $(X,B+\mathbf{M}_X)$ is a sub-pair and the log discrepancies of this sub-pair may be different with the log discrepancies of the generalized pair $(X,B,\mathbf{M})$.

2.1. Iitaka dimensions.

Lemma 2.1 ([HL21, Version 2, Lemma 2.9], cf. [Nak04, V. 2.6(5) Remark])**.** *Let X be a normal projective variety and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\kappa_\sigma(D)\geq 0$. Then D is pseudo-effective.*

Proof. By definition, there exists a Cartier divisor A on X such that $\sigma(D;A)\geq 0$. In particular, there exists a sequence of strictly increasing positive integers m_i, such that $\dim H^0(X,\lfloor m_iD\rfloor+A)>0$, hence $\lfloor m_iD\rfloor+A$ is effective for any i. Thus m_iD+A is effective for any i, hence $D+\frac{1}{m_i}A$ is effective for any i. Thus D is the limit of the effective $\mathbb{R}$-divisors $D+\frac{1}{m_i}A$, hence D is pseudo-effective. $\square$

Lemma 2.2 (cf. [HL21, Version 2, Lemma 2.10])**.** *Let $\pi: X\to U$ be a projective morphism from a normal variety to a variety, and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then:*

(1) D is big$/U$ if and only if $\kappa_\sigma(X/U,D)=\dim X-\dim U$.

(2) Let D_1,D_2 be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X. Suppose that $D_1\sim_{\mathbb{R},U}E_1\geq 0$ and $D_2\sim_{\mathbb{R},U}E_2\geq 0$ for some $\mathbb{R}$-divisors E_1,E_2 such that $\operatorname{Supp}E_1=\operatorname{Supp}E_2$. Then $\kappa_\sigma(X/U,D_1)=\kappa_\sigma(X/U,D_2)$ and $\kappa_\iota(X/U,D_1)=\kappa_\iota(X/U,D_2)$.

(3) Let $f:Y\to X$ be a surjective birational morphism and D_Y an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y such that $D_Y=f^*D+E$ for some f-exceptional $\mathbb{R}$-divisor $E\geq 0$. Then $\kappa_\sigma(Y/U,D_Y)=\kappa_\sigma(X/U,D)$ and $\kappa_\iota(Y/U,D_Y)=\kappa_\iota(X/U,D)$.

(4) Let $g:Z\to X$ be a surjective morphism from a normal variety such that Z is projective over U. Then $\kappa_\sigma(Z/U,f^*D)=\kappa_\sigma(X/U,D)$ and $\kappa_\iota(Z/U,f^*D)=\kappa_\iota(X/U,D)$.

(5) Let $\bar D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D\equiv_U\bar D$. Then $\kappa_\sigma(X/U,D)=\kappa_\sigma(X/U,\bar D)$.

(6) Let $\phi: X\dashrightarrow X'$ be a partial D-MMP$/U$ and let $D':=\phi_*D$. Then $\kappa_\sigma(X/U,D)=\kappa_\sigma(X'/U,D')$ and $\kappa_\iota(X/U,D)=\kappa_\iota(X'/U,D')$

Proof. For (1)-(5), let F be a very general fiber of the Stein factorization of π. Possibly replacing X with F, U with $\{pt\}$, and $D,D_1,D_2,\bar D$ with $D|_F,D_1|_F,D_2|_F,\bar D|_F$ respectively, we may assume that X is projective and $U=\{pt\}$. (2) follows from [HH20, Remark 2.8(1)] and (3)(4) follow from [HH20, Remark 2.8(2)].

To prove (1)(5), let $h:\tilde X\to X$ be a resolution of X. By (4), we may replace X with $\tilde X$, D with h^*D, and $\bar D$ with $h^*\bar D$, and assume that X is smooth.

If D is big, then $\kappa_\sigma(D)=\dim X$ by definition. If $\kappa_\sigma(D)=\dim X$, then D is pseudo-effective by Lemma 2.1, hence D is big by [Nak04, V. 2.7(3) Proposition]. This gives (1).

To prove (5), notice that D is pseudo-effective if and only if $\bar{D}$ is pseudo-effective. If D is not pseudo-effective, then $\kappa_\sigma(D) = \kappa_\sigma(\bar{D}) = -\infty$ by Lemma 2.1. If D is pseudo-effective, then (5) follows from from [Nak04, V. 2.7(1) Proposition].

To prove (6), let $p : W \to X$ and $q : W \to X'$ be a common resolution such that $q = \phi \circ p$, then $p^*D = q^*D' + F$ for some $F \geq 0$ that is q-exceptional. By (3), we have

$$\kappa_\sigma(X/U, D) = \kappa_\sigma(W/U, p^*D) = \kappa_\sigma(W/U, q^*D' + F) = \kappa_\sigma(X'/U, D')$$

and

$$\kappa_\iota(X/U, D) = \kappa_\iota(W/U, p^*D) = \kappa_\iota(W/U, q^*D' + F) = \kappa_\iota(X'/U, D').$$

□

2.2. **Preliminaries on the MMP for generalized pairs.** This subsection includes results in [HL21, Version 2, Version 3] that are not included in its final version.

Lemma 2.3 ([HL21, Version 3, Lemma 2.20], cf. [HL18, Proposition 3.8])**.** *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial glc g-pair such that X is klt and $K_X + B + \mathbf{M}_X \equiv_U D_1 - D_2$ (resp. $\sim_{\mathbb{R},U} D_1 - D_2$) where $D_1 \geq 0$, $D_2 \geq 0$ have no common components. Suppose that D_1 is very exceptional over U. Then any $(K_X + B + \mathbf{M}_X)$-MMP/U with scaling of an ample/U $\mathbb{R}$-divisor either terminates with a Mori fiber space or contracts D_1 after finitely many steps. Moreover, if $D_2 = 0$, then this MMP terminates with a model Y such that $K_Y + B_Y + \mathbf{M}_Y \equiv_U 0$ (resp. $\sim_{\mathbb{R},U} 0$), where B_Y is the strict transform of B on Y.*

Proof. The numerical equivalence part of the lemma is exactly [HL18, Proposition 3.8]. Thus we can assume that $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} D_1 - D_2$, $D_2 = 0$, and we only need to show that the MMP terminates with a model Y such that $K_Y + B_Y + \mathbf{M}_Y \sim_{\mathbb{R},U} 0$, where B_Y is the strict transform of B on Y.

By the numerical equivalence part of the lemma, the MMP contracts D_1 after finitely many steps. We may let $\phi : X \dashrightarrow Y$ be the birational map corresponding to this partial MMP. Since $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} D_1$ and ϕ contracts D_1, we have $K_Y + B_Y + \mathbf{M}_Y \sim_{\mathbb{R},U} 0$, where B_Y is the strict transform of B on Y, and the lemma is proved. □

Lemma 2.4 ([HL21, Version 3, Lemma 2.25])**.** *Let $X \to U$ be a projective morphism such that X is normal quasi-projective. Let D, A be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X and let $\phi : X \dashrightarrow X'$ be a partial D-MMP/U. Then there exists a positive real number t_0, such that for any $t \in (0, t_0]$, ϕ is also a partial $(D + tA)$-MMP/U. Note that A is not necessarily effective.*

Proof. We let

$$X := X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X'$$

be this partial MMP, and D_i, A_i the strict transforms of D and A on X_i respectively. Let $X_i \to Z_i$ be the D_i-negative extremal contraction of a D_i-negative extremal ray R_i in this MMP for each i, then $D_i \cdot R_i < 0$ for each i. Thus there exists a positive real number t_0, such that $(D_i + t_0 A_i) \cdot R_i < 0$ for each i. In particular, $(D_i + tA_i) \cdot R_i < 0$ for any i and any $t \in (0, t_0]$. Thus ϕ is a partial $(D + tA)$-MMP/U for any $t \in (0, t_0]$. □

Lemma 2.5 ([HL21, Version 2, Lemma 2.37])**.** *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair such that X is klt, and $A \geq 0$ an ample/U $\mathbb{R}$-divisor on X such that $(X, B + A, \mathbf{M})$ is glc and $K_X + B + A + \mathbf{M}_X$ is nef/U. Let*

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \ldots$$

be a $(K_X + B + \mathbf{M}_X)$-MMP/U with scaling of A, and A_i the strict transform of A on X_i for each i. Then there exists a positive integer n and a positive real number ϵ_0, such that $K_{X_j} + B_j + \epsilon A_j + \mathbf{M}_{X_j}$ is movable/U for any $\epsilon \in (0, \epsilon_0)$ and $j \geq n$. In particular, $K_{X_j} + B_j + \mathbf{M}_{X_j}$ is a limit of movable/U $\mathbb{R}$-divisors.

Proof. Let λ_i be the i-th scaling number of this MMP for each i, i.e.

$$\lambda_i := \inf\{t \mid t \geq 0, K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}.$$

We may assume that this MMP does not terminate. By [HL21, Theorem 2.8] (=[HL21, Version 3,Theorem 2.24]), we have $\lim_{i\to+\infty} \lambda_i = 0$.

Let n be the minimal positive integer such that $X_i \dashrightarrow X_{i+1}$ is a flip for any $i \geq n$. For any i, $X \dashrightarrow X_i$ is a $(K_X + B + tA + \mathbf{M}_X)$-MMP$/U$ with scaling of $(1-t)A$ for any $t \in [\lambda_i, \lambda_{i-1})$. Since X is $\mathbb{Q}$-factorial klt, there exists $\Delta_t \sim_{\mathbb{R},U} B + tA + \mathbf{M}_X$ such that (X, Δ_t) is klt and Δ_t is big for any $t \in [0, 1]$. Thus $K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i}$ is semi-ample$/U$ for any i and any $t \in [\lambda_i, \lambda_{i-1})$. Let $\epsilon_0 := \lambda_n$, then for any $\epsilon \in (0, \epsilon_0)$, there exists $i \geq n$ such that $\epsilon \in [\lambda_i, \lambda_{i-1})$, and $K_{X_i} + B_i + \epsilon A_i + \mathbf{M}_{X_i}$ is semi-ample$/U$. Since $X_i \dashrightarrow X_j$ is small for any $i, j \geq n$, $K_{X_j} + B_j + \epsilon A_j + \mathbf{M}_{X_j}$ is movable$/U$ for any $j \geq n$ and $\epsilon \in (0, \epsilon_0)$, and $K_{X_j} + B_j + \mathbf{M}_{X_j}$ is a limit of movable$/U$ $\mathbb{R}$-divisors. □

Lemma 2.6 ([HL21, Version 2, Lemma 2.38])**.** *Let $X \to U$ be a projective morphism such that X is quasi-projective. Assume that D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that D is a limit of movable$/U$ $\mathbb{R}$-divisors on X, and let $\phi : X \dashrightarrow X'$ be a partial D-MMP$/U$. Then ϕ only contains flips.*

Proof. Since D is a limit of movable$/U$ $\mathbb{R}$-divisors, D is pseudo-effective$/U$, so ϕ only contains flips and divisorial contractions.

If ϕ contains a divisorial contraction, let $\psi : X_1 \to X_1'$ be the first divisorial contraction in ϕ. Let D_1 be the strict transform of D on X_1, then since $X \dashrightarrow X_1$ only contains flips, D_1 is a limit of movable$/U$ $\mathbb{R}$-divisors on X_1. Let $D_1' := \psi_* D_1$, then

$$D_1 = \psi^* D_1' + F$$

for some $F \geq 0$ that is exceptional over X_1'.

Since D_1 is a limit of movable$/U$ divisors, D_1 is also a limit of movable$/X_1'$ divisors. Thus for any very general ψ-exceptional curve C, $D_1 \cdot C \geq 0$. By the general negativity lemma [Bir12, Lemma 3.3], $-F \geq 0$. Thus $F = 0$, and ψ cannot be a D_1-negative extremal contraction, a contradiction. Thus ϕ only contains flips. □

Lemma 2.7 ([HL21, Version 2, Lemma 2.39])**.** *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair. Let $U^0 \subset U$ be a non-empty open subset such that $(X^0 := X \times_U U^0, B^0 := B \times_U U^0, \mathbf{M}^0 := \mathbf{M} \times_U U^0)/U^0$ has a log minimal model. Then any $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor terminates over U^0 with a log minimal model of $(X^0, B^0, \mathbf{M}^0)/U^0$.*

Proof. We run a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor $A \geq 0$, such that $(X, B + A, \mathbf{M})$ is glc and $K_X + B + A + \mathbf{M}_X$ is nef$/U$. If this MMP terminates then we are done. Otherwise, we may assume that this MMP does not terminate. Let

$$(X, B, \mathbf{M}) := (X_1, B_1, \mathbf{M}) \dashrightarrow (X_2, B_2, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

be this MMP. Let A_i be the strict transform of A on X_i for each i and let λ_i the i-th scaling number of this MMP for each i, i.e.

$$\lambda_i := \inf\{t \mid t \geq 0, K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}.$$

By [HL18, Remark 3.21] $\lambda_i \geq \lambda_{i+1}$ for each i, and $\lim_{i\to+\infty} \lambda_i = 0$. We let $X_i^0 := X_i \times_U U^0$, $B_i^0 := B_i \times_U U^0$, and $A_i^0 := A_i \times_U U^0$. Let

$$\mathcal{N} := \{i \in \mathbb{N}^+ \mid X_i^0 \dashrightarrow X_{i+1}^0 \text{ is not the identity map over } U^0\}.$$

There are two cases:

Case 1. $\mathcal{N}$ is a finite set. In this case, let $n := \max\{j \mid j \in \mathcal{N}\} + 1$. Then $X_i^0 \dashrightarrow X_{i+1}^0$ is the identity map over U^0 for any $i \geq n$. In this case, $K_{X_i^0} + B_i^0 + \lambda_i A_i^0 + \mathbf{M}_{X_i^0}^0$ is nef$/U^0$ for any

$i \geq n$, hence $K_{X_n^0} + B_n^0 + \lambda_i A_n^0 + \mathbf{M}^0_{X_n^0}$ is nef$/U^0$ for any $i \geq n$. Since $\lambda = 0$, $K_{X_n^0} + B_n^0 + \mathbf{M}^0_{X_n^0}$ is nef$/U^0$. Thus $(X_n^0, B_n^0, \mathbf{M}^0)/U^0$ is a log minimal model of $(X^0, B^0, \mathbf{M}^0)/U^0$.

Case 2. $\mathcal{N}$ is not a finite set. We may write $\mathcal{N} = \{n_i\}_{i=1}^{+\infty}$ such that $n_i < n_{i+1}$ for each i, then we get a sequence of induced birational maps

$$(X^0, B^0, \mathbf{M}^0) = (X^0_{n_1}, B^0_{n_1}, \mathbf{M}^0) \dashrightarrow (X^0_{n_2}, B^0_{n_2}, \mathbf{M}^0) \dashrightarrow \cdots \dashrightarrow (X^0_{n_i}, B^0_{n_i}, \mathbf{M}^0) \dashrightarrow \dots,$$

which is a sequence of the $(K_{X^0} + B^0 + \mathbf{M}^0_{X^0})$-MMP$/U^0$ with scaling of $A^0 := A \times_U U^0$. Since $(X^0, B^0, \mathbf{M}^0)/U^0$ has a log minimal model, by [HL21, Theorem 2.8] (=[HL21, Version 3,Theorem 2.24]), this MMP terminates, a contradiction. $\square$

Lemma 2.8 ([HL21, Version 2, Lemma 2.40]). *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair, X is klt ,and the induced morphism $\pi : X \to U$ is a contraction. Let F be a very general fiber of π, and $(F, B_F, \mathbf{M}^F)$ the projective generalized pair given by the adjunction*

$$K_F + B_F + \mathbf{M}^F_F := (K_X + B + \mathbf{M}_X)|_F.$$

Assume that $(F, B_F, \mathbf{M}^F)$ has a log minimal model. Then any $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor terminates along F with a log minimal model of $(F, B_F, \mathbf{M}^F)$.

Proof. We may assume that $F = \pi^{-1}(z)$ for some very general point $z \in U$. We run a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor $A \geq 0$, such that $(X, B + A, \mathbf{M})$ is glc and $K_X + B + A + \mathbf{M}_X$ is nef$/U$. If this MMP terminates then we are done. Otherwise, we may assume that this MMP does not terminate. Let

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

be this MMP. Let A_i be the strict transform of A on X_i for each i and let λ_i the i-th scaling number of this MMP for each i, i.e.

$$\lambda_i := \inf\{t \mid t \geq 0, K_{X_i} + B_i + tA_i + \mathbf{M}_{X_i} \text{ is nef}/U\}.$$

By [HL18, Remark 3.21, Theorem 4.1], $\lambda_i \geq \lambda_{i+1}$ for each i, and $\lambda := \lim_{i \to +\infty} \lambda_i = 0$.

Let $\pi_i : X_i \to U$ be the induced morphism for each i. Since z is a very general point, we may let $F_i := \pi_i^{-1}(z)$, $(F_i, B_{F_i}, \mathbf{M}^F)$ the projective generalized pair given by the adjunction

$$K_{F_i} + B_{F_i} + \mathbf{M}^F_{F_i} := (K_{X_i} + B_i + \mathbf{M}_{X_i})|_{F_i},$$

and $A_{F_i} := A_i|_{F_i}$. Note that the nef part $\mathbf{M}^F$ does not depend on i by the construction of generalized adjunction [BZ16, Definition 4.7]. Let

$$\mathcal{N} := \{i \in \mathbb{N} \mid F_i \dashrightarrow F_{i+1} \text{ is not the identity map}\}.$$

There are two cases:

Case 1. $\mathcal{N}$ is a finite set. In this case, let $n := \max\{j \mid j \in \mathcal{N}\} + 1$. Then $F_i \dashrightarrow F_{i+1}$ is the identity map for any $i \geq n$. In this case, $K_{F_i} + B_{F_i} + \lambda_i A_{F_i} + \mathbf{M}^F_{F_i}$ is nef for any $i \geq n$, hence $K_{F_n} + B_{F_n} + \lambda_i A_{F_n} + \mathbf{M}^F_{F_n}$ is nef for any $i \geq n$. Since $\lambda = 0$, $K_{F_n} + B_{F_n} + \mathbf{M}^F_{F_n}$ is nef. Thus $(F_n, B_{F_n}, \mathbf{M}^F)$ is a log minimal model of $(F, B_F, \mathbf{M}^F)$.

Case 2. $\mathcal{N}$ is not a finite set. We may write $\mathcal{N} = \{n_i\}_{i=1}^{+\infty}$ such that $n_i < n_{i+1}$ for each i, then we get a sequence of induced birational maps

$$(F, B_F, \mathbf{M}^F) = (F_{n_1}, B_{F_{n_1}}, \mathbf{M}^0) \dashrightarrow (F_{n_2}, B_{F_{n_2}}, \mathbf{M}^0) \dashrightarrow \cdots \dashrightarrow (F_{n_i}, B_{F_{n_i}}, \mathbf{M}^0) \dashrightarrow \dots,$$

which is a sequence of the $(K_F + B_F + \mathbf{M}^F_F)$-MMP with scaling of $A_F := A|_F$. Since $(F, B_F, \mathbf{M}^F)$ has a log minimal model, by [HL21, Theorem 2.8] (=[HL21, Version 3,Theorem 2.24]), this MMP terminates, a contradiction. $\square$

Lemma 2.9 ([HL21, Version 2, Lemma 2.43]). *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair such that X is klt. Let $H\geq 0$ be an $\mathbb{R}$-divisor on X such that $(X,B+H,\mathbf{M})$ is glc and $K_X+B+H+\mathbf{M}_X$ is nef$/U$. Assume that $(X,B+\mu H,\mathbf{M})/U$ has a log minimal model for any $\mu\in(0,1]$. Then we can construct a $(K_X+B+\mathbf{M}_X)$-MMP$/U$ with scaling of H:*

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots.$$

Let H_i be the strict transform of H on X_i for each i, and let

$$\lambda_i:=\inf\{t\mid t\geq 0, K_{X_i}+B_i+tH_i+\mathbf{M}_{X_i}\text{ is nef}/U\}$$

be the i-th scaling number of this MMP for each i. Then this MMP

(1) either terminates after finitely many steps, or
(2) does not terminate and $\lim_{i\to+\infty}\lambda_i=0$.

Proof. If $\lambda_0=0$ then there is nothing left to prove. So we may assume that $\lambda_0>0$. By [HL18, Lemma 3.17], we may pick $\lambda_0'\in(0,\lambda_0)$ such that any sequence of the $(K_X+B+\lambda_0'H+\mathbf{M}_X)$-MMP$/U$ is $(K_X+B+\lambda_0H+\mathbf{M}_X)$-trivial.

By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]), we may run a $(K_X+B+\lambda_0'H+\mathbf{M}_X)$-MMP$/U$ with scaling of a general ample$/U$ divisor, which terminates with a log minimal model. We let

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_{k_1},B_{k_1},\mathbf{M})$$

be this sequence of the MMP$/U$. Then this sequence consists of finitely many steps of a $(K_X+B+\mathbf{M}_X)$-MMP$/U$ with scaling of H, with scaling numbers $\lambda_0=\lambda_1=\cdots=\lambda_{k_1-1}$. Since

$$K_{X_{k_1}}+B_{k_1}+\lambda_1'H_{k_1}+\mathbf{M}_{X_{k_1}}$$

is nef$/U$, we have $\lambda_{k_1}\leq\lambda_1'<\lambda_1$.

We may replace $(X,B,\mathbf{M})/U$ with $(X_{k_1},B_{k_1},\mathbf{M})/U$ and continue this process. If this MMP does not terminate, then we may let $\lambda:=\lim_{i\to+\infty}\lambda_i$. By our construction, $\lambda\neq\lambda_i$ for any i, and the lemma follows from [HL18, Remark 3.21, Theorem 4.1]. □

Definition 2.10 ([HL21, Version 3, Definition 3.1]). Let $(X,B,\mathbf{M})/U$ be a glc g-pair and $(W,B_W,\mathbf{M})$ a log smooth model of $(X,B,\mathbf{M})$. If any exceptional$/X$ prime divisor D on W such that $a(D,X,B,\mathbf{M})>0$ is a component of $\{B_W\}$, then $(W,B_W,\mathbf{M})$ is called a *proper log smooth model* of $(X,B,\mathbf{M})$.

Lemma 2.11 ([HL21, Version 3, Lemma 3.6]). *Let $(X,B,\mathbf{M})/U$ be a glc g-pair and $h:W\to X$ a log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W. Then $(X,B,\mathbf{M})$ has a proper log smooth model $(W,B_W,\mathbf{M})$ for some $\mathbb{R}$-divisor B_W on W.*

Proof. Assume that

$$K_W+h^{-1}_*B+\Gamma+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X),$$

then Γ is h-exceptional. Let $E=\operatorname{Exc}(h)$ be the reduced h-exceptional divisor. Then there exists a real number $\epsilon\in(0,1)$, such that for any component D of E, if $\operatorname{mult}_D\Gamma<1$, then $\operatorname{mult}_D\Gamma<1-\epsilon$. We let

$$B_W:=h^{-1}_*B+\epsilon\Gamma^{=1}+(1-\epsilon)E,$$

then $(W,B_W,\mathbf{M})$ is a proper log smooth model of $(X,B,\mathbf{M})$. □

Lemma 2.12 ([HL21, Version 3, Lemma 3.7]). *Let $(X,B,\mathbf{M})/U$ be a glc g-pair and $(W,B_W,\mathbf{M})$ a proper log smooth model of $(X,B,\mathbf{M})$ with induced morphism $h:W\to X$. Assume that*

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E,$$

then:

(1) $\operatorname{Supp}B_W=\operatorname{Supp}h^{-1}_*B\cup\operatorname{Exc}(h)$.
(2) For any prime divisor D on W that is exceptional over X, D is a component E if and only if $a(D,X,B,\mathbf{M})>0$.

(3) *Any glc place of* $(W, B_W, \mathbf{M})$ *is a glc place of* $(X, B, \mathbf{M})$. *In particular, the image of any glc center of* $(W, B_W, \mathbf{M})$ *on* X *is a glc center of* $(X, B, \mathbf{M})$.

Proof. First we prove (1). By construction, $\operatorname{Supp} B_W \subset \operatorname{Supp} h_*^{-1}B \cup \operatorname{Exc}(h)$ and $\operatorname{Supp} h_*^{-1}B \subset \operatorname{Supp} B_W$. Let D be a component of $\operatorname{Exc}(h)$. If $a(D, X, B, \mathbf{M}) = 0$, then since $E \geq 0$, D is a component of B_W. If $a(D, X, B, \mathbf{M}) > 0$, by Definition 2.10, E is a component of $\{B_W\}$, hence a component of B_W. Thus $\operatorname{Exc}(h) \subset \operatorname{Supp} B_W$, and we have (1).

We prove (2). Let D be a prime divisor on W. If $a(D, X, B, \mathbf{M}) > 0$, then D is a component of E by the definition of log smooth models. If $a(D, X, B, \mathbf{M}) = 0$, then

$$0 = a(D, X, B, \mathbf{M}) = a(D, W, B_W - E, \mathbf{M}) \geq a(D, W, B_W, \mathbf{M}) \geq 0,$$

which implies that $a(D, W, B_W - E, \mathbf{M}) = a(D, W, B_W, \mathbf{M})$, hence $\operatorname{mult}_D E = 0$. Thus we have (2).

We prove (3). Let D be a glc place of $(W, B_W, \mathbf{M})$. Then the center of D on W is a stratum of $\lfloor B_W \rfloor$. If $\operatorname{center}_W D \subset \operatorname{Supp} E$, then since $B_W + E$ is simple normal crossing, there exists a prime divisor F that is a component of $\lfloor B_W \rfloor$ such that $\operatorname{center}_W D \subset F$ and F is a component of E. By (2), $a(F, X, B, \mathbf{M}) > 0$. By Definition 2.10, F is a component of $\{B_W\}$, so F cannot be a component of $\lfloor B_W \rfloor$, a contradiction. Thus $\operatorname{center}_W D \not\subset \operatorname{Supp} E$. Therefore, any glc place of $(W, B_W, \mathbf{M})$ is a glc place of of $(W, B_W - E, \mathbf{M})$, hence a glc place of $(X, B, \mathbf{M})$, and we have (3). $\square$

Lemma 2.13 ([HL21, Version 3, Lemma 3.19])**.** *Let* $(X, B, \mathbf{M})/U$ *and* $(Y, B_Y, \mathbf{M})/U$ *be two* $\mathbb{Q}$*-factorial NQC gdlt g-pairs, and* $f : Y \to X$ *a projective birational morphism such that*

$$K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X) + E$$

for some $E \geq 0$ *that is exceptional over* X. *Assume that*

(1) $\mathbf{M}$ *descends to* Y,
(2) $(Y, B_Y + \operatorname{Exc}(f))$ *is log smooth, and*
(3) $(Y, B_Y, \mathbf{M})/U$ *has a weak glc model.*

Then $(X, B, \mathbf{M})/U$ *has a weak glc model.*

Proof. By our assumption, $K_Y + B_Y + \mathbf{M}_Y$ and $K_X + B + \mathbf{M}_X$ are pseudo-effective/U. Since $(X, B, \mathbf{M})$ is gdlt, we may pick an ample/U $\mathbb{R}$-divisor $A \geq 0$ on X such that $(X, B + A, \mathbf{M})$ is glc, and $K_X + B + A + \mathbf{M}_X$ and $A + \lfloor B \rfloor$ are ample. Since $(X, B, \mathbf{M})$ is gdlt, $(X, \{B\}, \mathbf{M})$ is gklt, so we may pick an ample/U $\mathbb{R}$-divisor $0 \leq A' \sim_{\mathbb{R},U} A + \lfloor B \rfloor$ such that $(X, \Delta := \{B\} + A', \mathbf{M})$ is gklt and f is a log resolution of $(X, B + A)$. Since $\Delta \sim_{\mathbb{R},U} B + A$, $K_X + \Delta + \mathbf{M}_X$ is big/U. We may write

$$K_Y + \Gamma + \mathbf{M}_Y = f^*(K_X + \Delta + \mathbf{M}_X) + F$$

for some $\Gamma \geq 0$, $F \geq 0$ such that $\Gamma \wedge F = 0$. By our construction, $\mathbf{M}$ descends to Y, $(Y, B_Y + \operatorname{Exc}(f))$ is log smooth, (Y, Γ) is log smooth, $(Y, \Gamma, \mathbf{M})$ is gklt, and $K_Y + \Gamma + \mathbf{M}_Y$ is big/U. We let

$$\Delta_t := t\Delta + (1-t)B \sim_{\mathbb{R},U} B + tA$$

and

$$\Gamma_t := t\Gamma + (1-t)B_Y$$

for any real number t. Then $(X, \Delta_t, \mathbf{M})$ and $(Y, \Gamma_t, \mathbf{M})$ are gklt for any $t \in (0, 1]$, and $K_X + \Delta_t + \mathbf{M}_X$ and $K_Y + \Gamma_t + \mathbf{M}_Y$ are big/U for any $t \in (0, 1]$.

Since $(Y, B_Y, \mathbf{M})/U$ has a weak glc model, by [HL21, Lemma 3.8] (=[HL21, Version 3, Lemma 3.15]), $(Y, B_Y, \mathbf{M})/U$ has a log minimal model. Since Y is klt, by [HL21, Theorem 2.8] (=[HL21, Version 3,Theorem 2.24]), we may run a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP/$U$ with scaling of a general ample/U divisor H, which terminates with a log minimal model $(Y', B_{Y'}, \mathbf{M})/U$ with induced birational map $\phi : Y \dashrightarrow Y'$.

We let Γ_t' be the strict transform of Γ_t on Y' for any t. By Lemmas 2.4 and [HL18, Lemma 3.17], there exists $t_0 \in (0, 1)$, such that

- ϕ is also a $(K_Y+\Gamma_{t_0}+\mathbf{M}_Y)$-MMP$/U$, and
- for any $t\in(0,t_0]$, any partial $(K_{Y'}+\Gamma'_t+\mathbf{M}_{Y'})$-MMP$/U$ is $(K_{Y'}+B_{Y'}+\mathbf{M}_{Y'})$-trivial.

Thus $(Y',\Gamma'_{t_0},\mathbf{M})$ is gklt and $K_{Y'}+\Gamma'_{t_0}+\mathbf{M}_{Y'}$ is big$/U$. By [BZ16, Theorem 4.4(2)], we may run a $(K_{Y'}+\Gamma'_{t_0}+\mathbf{M}_{Y'})$-MMP$/U$, which terminates with a log minimal model $(Y'',\Gamma''_{t_0},\mathbf{M})/U$ of $(Y',\Gamma'_{t_0},\mathbf{M})/U$. We let Γ''_t be the strict transform of Γ_t on Y'' for any t and $B_{Y''}$ the strict transform of B_Y on Y''. Since the induced birational map $\phi':Y'\dashrightarrow Y''$ is $(K_{Y'}+B_{Y'}+\mathbf{M}_{Y'})$-trivial, $K_{Y''}+B_{Y''}+\mathbf{M}_{Y''}$ is nef$/U$. Moreover, the induced map $\phi'\circ\phi:Y\dashrightarrow Y''$ does not extract any divisor, and is both $(K_Y+B_Y+\mathbf{M}_Y)$-non-positive and $(K_Y+\Gamma_{t_0}+\mathbf{M}_Y)$-non-positive. Thus $(Y'',B_{Y''},\mathbf{M})/U$ is a weak glc model of $(Y,B_Y,\mathbf{M})/U$ and $(Y'',\Gamma''_{t_0},\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_{t_0},\mathbf{M})/U$, hence $(Y'',\Gamma''_t,\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_t,\mathbf{M})/U$ for any $t\in[0,t_0]$.

By [HL18, Lemma 3.5], we can run a $(K_X+B+\mathbf{M}_X)$-MMP$/U$ with scaling of A:

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots.$$

Let $A_i,\Delta_i,\Delta_{t,i}$ be the strict transforms of A,Δ,Δ_t on X_i for any t,i respectively, and let

$$\lambda_i:=\inf\{t\mid t\geq 0, K_{X_i}+B_i+tA_i+\mathbf{M}_{X_i}\text{ is nef}/U\}$$

be the scaling numbers. If this MMP terminates, then there is nothing left to prove as we already get a log minimal model for $(X,B,\mathbf{M})/U$. Thus we may assume that this MMP does not terminate. By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]), $\lim_{i\to+\infty}\lambda_i=0$.

In particular, there exists a positive integer n such that $\lambda_n<\lambda_{n-1}\leq t_0$. Since $\Delta_{t,i}\sim_{\mathbb{R},U}B_i+tA_i$ for any t, $(X_n,\Delta_{\lambda_{n-1},n},\mathbf{M})/U$ is a weak glc model of $(X,\Delta_{\lambda_{n-1}},\mathbf{M})/U$ and $(X_n,\Delta_{\lambda_n,n},\mathbf{M})/U$ is a weak glc model of $(X,\Delta_{\lambda_n},\mathbf{M})/U$. Since

$$K_Y+\Gamma_t+\mathbf{M}_Y=f^*(K_X+\Delta_t+\mathbf{M}_X)+tF+(1-t)E$$

for any t, by [HL21, Lemma 3.10(1)] (=[HL21, Version 3, Lemma 3.17]), $(X_n,\Delta_{\lambda_{n-1},n},\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_{\lambda_{n-1}},\mathbf{M})/U$ and $(X_n,\Delta_{\lambda_n,n},\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_{\lambda_n},\mathbf{M})/U$. By our construction, $(Y'',\Gamma''_{\lambda_{n-1}},\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_{\lambda_{n-1}},\mathbf{M})/U$ and $(Y'',\Gamma''_{\lambda_n},\mathbf{M})/U$ is a weak glc model of $(Y,\Gamma_{\lambda_n},\mathbf{M})/U$.

We let $p:W\to X_n$ and $q:W\to Y''$ be a resolution of indeterminacy.

$$\begin{tikzcd}
Y \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & Y' \arrow[r, dashed, "\phi'"] & Y'' & & W \arrow[ll, "q"'] \arrow[d, "p"] \\
X \arrow[r, dashed] & X_2 \arrow[r, dashed] & \dots \arrow[rr, dashed] & & X_n
\end{tikzcd}$$

By [HL21, Lemma 3.5(1)] (=[HL21, Version 3, Lemma 3.9(1)]),

$$p^*(K_{X_n}+\Delta_{\lambda_{n-1},n}+\mathbf{M}_{X_n})=q^*(K_{Y''}+\Gamma''_{\lambda_{n-1}}+\mathbf{M}_{Y''}).$$

and

$$p^*(K_{X_n}+\Delta_{\lambda_n,n}+\mathbf{M}_{X_n})=q^*(K_{Y''}+\Gamma''_{\lambda_n}+\mathbf{M}_{Y''}).$$

Thus

$$p^*(K_{X_n}+t\Delta_i+(1-t)B_i+\mathbf{M}_{X_n})=q^*(K_{Y''}+t\Gamma''_1+(1-t)B_{Y''}+\mathbf{M}_{Y''})$$

when $t\in\{\lambda_{n-1},\lambda_n\}$. Since $\lambda_{n-1}\neq\lambda_n$, we have

$$p^*(K_{X_n}+t\Delta_i+(1-t)B_i+\mathbf{M}_{X_n})=q^*(K_{Y''}+t\Gamma''_1+(1-t)B_{Y''}+\mathbf{M}_{Y''})$$

for any t. In particular,

$$p^*(K_{X_n}+B_n+\mathbf{M}_{X_n})=q^*(K_{Y''}+B_{Y''}+\mathbf{M}_{Y''})$$

is nef$/U$, hence $K_{X_n}+B_n+\mathbf{M}_{X_n}$ is nef$/U$, and $\lambda_n=0$, a contradiction. $\square$

Theorem 2.14 ([HL21, Version 3, Theorem 3.14])**.** *Let $(X,B,\mathbf{M})/U$ and $(Y,B_Y,\mathbf{M})/U$ be two NQC glc g-pairs and let $f:Y\to X$ be a projective birational morphism such that*

$$K_Y+B_Y+\mathbf{M}_Y=f^*(K_X+B+\mathbf{M}_X)+E$$

for some $E\geq 0$ that is exceptional over X. Then $(X,B,\mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model) if and only if $(Y,B_Y,\mathbf{M})/U$ has a weak glc model (resp. log minimal model, good minimal model).

Proof of Theorem 2.14. First we prove the weak glc model case. By [HL21, Lemma 3.10(1)] (=[HL21, Version 3, Lemma 3.17]), we only need to prove that if $(Y,B_Y,\mathbf{M})/U$ has a weak glc model, then $(X,B,\mathbf{M})/U$ has a weak glc model. Let $g:\bar{X}\to X$ be a gdlt modification of $(X,B,\mathbf{M})$ such that

$$K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}=g^*(K_X+B+\mathbf{M}_X),$$

and let $p:W\to Y$ and $q:W\to\bar{X}$ be a resolution of indeterminacy, such that $\mathbf{M}$ descends to W, p is a log resolution of $(Y,\operatorname{Supp}(B_Y+E))$, and q is a log resolution of $(\bar{X},\operatorname{Supp}\bar{B})$. By Lemma 2.11, we may find a proper log smooth model $(W,B_W,\mathbf{M})$ of $(Y,B_Y,\mathbf{M})$. We have

$$K_W+B_W+\mathbf{M}_W=p^*(K_Y+B_Y+\mathbf{M}_Y)+F=(p\circ f)^*(K_X+B+\mathbf{M}_X)+p^*E+F$$

for some p-exceptional $\mathbb{R}$-divisor $F\geq 0$.

Let D be a component of p^*E+F. Then $a(D,W,B_W,\mathbf{M})<a(D,X,B,\mathbf{M})$ and D is exceptional over X. If D is not exceptional over $\bar{X}$, then $a(D,W,B_W,\mathbf{M})<a(D,X,B,\mathbf{M})=0$, which is not possible. Thus p^*E+F is exceptional over $\bar{X}$.

By [HL21, Lemma 3.10(1)] (=[HL21, Version 3, Lemma 3.17]), $(W,B_W,\mathbf{M})/U$ has a weak glc model. Since p^*E+F is exceptional over $\bar{X}$, $\mathbf{M}$ descends to W, (W,B_W+p^*E+F) is log smooth, by Lemma 2.13, we have that $(\bar{X},\bar{B},\mathbf{M})/U$ has a weak glc model. By [HL21, Lemma 3.11] (=[HL21, Version 3, Lemma 3.13]), $(X,B,\mathbf{M})/U$ has a weak glc model, and we have proven the weak glc model case.

Now we prove the general case. By [HL21, Lemma 3.10(2)] (=[HL21, Version 3, Lemma 3.18]), we only need to prove that if $(Y,B_Y,\mathbf{M})/U$ has a weak glc (resp. log minimal model, good minimal model), then $(X,B,\mathbf{M})/U$ has a weak glc (resp. log minimal model, good minimal model). The weak glc case has just been proven, and the log minimal model case follows from the weak glc model case and [HL21, Lemma 3.8] (=[HL21, Version 3, Lemma 3.15]). Assume that $(Y,B_Y,\mathbf{M})/U$ has a good minimal model. By the log minimal model case, we may assume that $(X',B',\mathbf{M})/U$ is a log minimal model of $(X,B,\mathbf{M})/U$. By [HL21, Lemma 3.10(1)] (=[HL21, Version 3, Lemma 3.17]), $(X',B',\mathbf{M})/U$ is also a weak glc model of $(Y,B_Y,\mathbf{M})/U$. By [HL21, Lemma 3.5(2)] (=[HL21, Version 3, Lemma 3.9(2)]), $K_{X'}+B'+\mathbf{M}_{X'}$ is semi-ample$/U$, hence $(X',B',\mathbf{M})/U$ is a good minimal model of $(X,B,\mathbf{M})/U$, and the proof is concluded. □

Theorem 2.15 ([HL21, Version 2, Theorem 4.11])**.** *Let (X,B) be a dlt pair and $\pi:X\to U$ a projective surjective morphism over a normal variety U. Then there exists a commutative diagram of projective morphisms*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ V \arrow[r, "\varphi"] & U \end{tikzcd}$$

such that

(1) *f,φ are birational morphisms, π' is an equidimensional contraction, Y only has $\mathbb{Q}$-factorial toroidal singularities, and V is smooth, and*
(2) *there exist two $\mathbb{R}$-divisors B_Y and E on Y, such that*
   (a) *$K_Y+B_Y=f^*(K_X+B)+E$,*
   (b) *$B_Y\geq 0$, $E\geq 0$, and $B_Y\wedge E=0$,*

(c) (Y, B_Y) *is lc quasi-smooth, and any lc center of* (Y, B_Y) *on* X *is an lc center of* (X, B).

Proof. This result follows from [AK00], see also [Hu20, Theorem B.6], [Kaw15, Theorem 2] and [Has19, Step 2 of Proof of Lemma 3.2]. $\square$

Theorem 2.16 ([HL21, Version 2, Theorem 5.1])**.** *Let* $(X, B, \mathbf{M})/U$ *be an NQC glc g-pair such that* U *is quasi-projective, and let* $\pi : X \to V$ *be a surjective morphism over* U*. Assume that* $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},V} 0$*, then there exists an NQC glc g-pair* $(V, B_V, \mathbf{M}^V)/U$*, such that*

(1) $K_X + B + \mathbf{M}_X \sim_{\mathbb{R}} \pi^*(K_V + B_V + \mathbf{M}^V_V)$*,*
(2) any glc center of $(V, B_V, \mathbf{M}^V)$ *is the image of a glc center of* $(X, B, \mathbf{M})$ *in* V*, and*
(3) if all glc centers of $(X, B, \mathbf{M})$ *dominate* V*, then* $(V, B_V, \mathbf{M}^V)$ *is gklt.*

Proof. By the theory of Shokurov-type rational polytopes (cf. [HL18, Proposition 3.16]) and the theory of uniform rational polytopes (cf. [HLS19, Lemma 5.3], [Che20, Therem 1.4]), we may assume that $(X, B, \mathbf{M})/U$ is a $\mathbb{Q}$-g-pair.

Step 1. In this step, we prove the case when $X \to V$ is a generically finite morphism. Within this step, we assume that $X \to V$ is a generically finite morphism.

By [HL20, Theorem 4.5, (4.3),(4.4)], there exists a glc $\mathbb{Q}$-g-pair $(V, B_V, \mathbf{M}^V)/U$, such that $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} \pi^*(K_V + B_V + \mathbf{M}^V_V)$, and B_V and $\mathbf{M}^V$ are defined in the following way:

Let V^0 be the smooth locus of V, $X^0 := X \times_V V^0$, and $\pi|_{X^0} : X^0 \to V^0$ the restriction of π. Then we have the Hurwitz formula

$$K_{X^0} = (\pi|_{X^0})^* K_{V^0} + R^0$$

where R^0 is the effective ramification divisor of $f|_{X^0}$. Let R be the closure of R^0 in X, and let $B_V := \frac{1}{\deg \pi}\pi_*(R + B)$. For any proper birational morphism $\mu : V' \to V$, let X' be the main component of $X \times_V V'$ with induced birational map $\pi' : X' \to V'$. We let $\mathbf{M}^V_{V'} = \frac{1}{\deg \pi}\pi'_*\mathbf{M}_{X'}$.

(1) follows immediately.

Since $(V, B, \mathbf{M}^V)/U$ is a g-pair, for any prime divisor E over V, there exists a birational morphism $h_V : \tilde{V} \to V$ such that $\mathbf{M}^V$ descends to $\tilde{V}$ and E is on $\tilde{V}$. We let $h : \tilde{X} \to X$ be a birational morphism such that h descends to $\tilde{X}$, $\mathbf{M}$ descends to $\tilde{X}$, and the induced map $\tilde{\pi} : \tilde{X} \to \tilde{V}$ is a morphism.

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"'] & \tilde{X} \arrow[l, "h"'] \arrow[d, "\tilde{\pi}"] \\
V' \arrow[r, "\mu"] & V & \tilde{V} \arrow[l, "h_V"']
\end{tikzcd}$$

There are two cases:

Case 1. E is exceptional over V. In this case, we let $F \subset \tilde{\pi}^{-1}(E)$ be a prime divisor, and let $r \leq \deg f$ be the ramification index of $\tilde{\pi}$ along F. Near the generic point of F, we have

$$K_{\tilde{X}} = h^*(K_X+B+\mathbf{M}_X)+(a(F,X,B,\mathbf{M})-1)F \sim_{\mathbb{Q}} h^*\pi^*(K_V+B_V+\mathbf{M}_V)+(a(F,X,B,\mathbf{M})-1)F$$

and

$$\begin{aligned} K_{\tilde{X}} &= \tilde{\pi}^* K_{\tilde{Z}} + (r-1)F = \tilde{\pi}^* h_V^*(K_V + B_V + \mathbf{M}_V) + r(a(E, V, B_V, \mathbf{M}^V) - 1)F + (r-1)F \\ &= h^*\pi^*(K_V + B_V + \mathbf{M}_V) + (ra(E, V, B_V, \mathbf{M}^V) - 1)F. \end{aligned}$$

Let $\tilde{X} \to \bar{X} \to V$ be the Stein factorization of $\pi \circ h = h_V \circ \tilde{\pi}$. Since E is exceptional over V, F is exceptional over $\bar{X}$. By the negativity lemma, we have

$$a(F, X, B, \mathbf{M}) - 1 = ra(E, V, B_V, \mathbf{M}^V) - 1,$$

hence $a(F,X,B,\mathbf{M})\geq 0$ if and only if $a(E,V,B_V,\mathbf{M}^V)\geq 0$ and $a(F,X,B,\mathbf{M})>0$ if and only if $a(E,V,B_V,\mathbf{M}^V)>0$. Moreover, since $F\subset\tilde{\pi}^{-1}(E)$, if E is a glc place of $(V,B_V,\mathbf{M}^V)$, then F is a glc place of $(X,B,\mathbf{M})$ and $\operatorname{center}_V E$ is contained in the image of $\operatorname{center}_X F$ in V.

Case 2. E is not exceptional over V. In this case, if E is not a component of B_V, then $a(E,V,B_V,\mathbf{M}^V)=1>0$. If E is a component of B_V, then we may let $B_1,\dots,B_m\subset\pi^{-1}(E)$ be the prime divisors on X lying over V and let d_i be the degree of the induced morphism $\pi|_{B_i}:B_i\to E$. By our construction of B_V,

$$a(E,V,B_V,\mathbf{M}^V)=1-\operatorname{mult}_E B_V=1-\frac{\sum_{i=1}^m d_i\operatorname{mult}_{B_i}B}{\deg\pi}.$$

Since $\sum_{i=1}^m d_i\leq\deg\pi$, $a(E,V,B_V,\mathbf{M}^V)\geq 0$ if $\operatorname{mult}_{B_i}B\leq 1$ for each i, and $a(E,V,B_V,\mathbf{M}^V)>0$ if $\operatorname{mult}_{B_i}B<1$ for each i. Moreover, since $B_i\subset\pi^{-1}(E)$ for each i, if E is a glc place of $(V,B_V,\mathbf{M}^V)$, then B_i is a glc place of $(X,B,\mathbf{M})$ for some i and E is contained in the image of B_i in V.

By our discussions above, we finish the proof in the case when $X\to V$ is a generically finite morphism.

Step 2. In this step, we prove the case when $X\to V$ is a contraction. Within this step, we assume that $X\to V$ is a contraction.

By [FS20, Theorem 2.20], there exists a glc $\mathbb{Q}$-g-pair $(V,B_V,\mathbf{M}^V)/U$, such that $K_X+B+\mathbf{M}_X\sim_{\mathbb{Q}}\pi^*(K_V+B_V+\mathbf{M}^V_V)$. Moreover, for any birational morphism $h_V:\tilde{V}\to V$, we have an $\mathbb{R}$-divisor $B_{\tilde{V}}$ satisfies that $K_{\tilde{V}}+B_{\tilde{V}}+\mathbf{M}^V_{\tilde{V}}=h_V^*(K_V+B_V+\mathbf{M}^V_V)$ and defined in the following way: let $\tilde{X}$ be the main component of $X\times_V\tilde{V}$, and $h:\tilde{X}\to X$ and $\tilde{\pi}:\tilde{X}\to\tilde{V}$ the induced morphisms. Let $K_{\tilde{X}}+\tilde{B}+\mathbf{M}_{\tilde{X}}:=h^*(K_X+B+\mathbf{M}_X)$. For any prime divisor E on $\tilde{V}$, $\operatorname{mult}_E B_{\tilde{V}}=1-t_E$, where

$$t_E:=\sup\{s\mid(\tilde{X},\tilde{B}+s\tilde{\pi}^*E,\mathbf{M})\text{ is glc over the generic point of }E\}.$$

Note that E may not be $\mathbb{Q}$-Cartier but $\tilde{\pi}^*E$ is always defined over the generic point of E.

(1) follows immediately.

If E is a glc place of $(V,B_V,\mathbf{M}^V)$ on $\tilde{V}$, then $t_E=0$, hence $\tilde{\pi}^*E$ contains a glc center F of $(\tilde{X},\tilde{B},\mathbf{M})$ over the generic point of E. We have $F\subset\operatorname{Supp}\tilde{\pi}^*E$ and $\tilde{\pi}(F)\subset E$, hence $\tilde{\pi}(F)=E$. Thus E is the image of a glc center of $(\tilde{X},\tilde{B},\mathbf{M})$ on $\tilde{V}$, hence $\operatorname{center}_V E$ is the image of a glc center of $(X,B,\mathbf{M})$ in V.

By our discussions above, we finish the proof in the case when $X\to V$ is a contraction.

Step 3. In this step we prove the general case.

We let $X\xrightarrow{f}Y\xrightarrow{g}V$ be the Stein factorization of π. Then $K_X+B+\mathbf{M}_X\sim_{\mathbb{Q},Y}0$, $f:X\to Y$ is a contraction and $g:Y\to V$ is a finite morphism. By Step 2, $K_X+B+\mathbf{M}_X\sim_{\mathbb{Q}}f^*(K_Y+B_Y+\mathbf{M}^Y_Y)$ for some glc $\mathbb{Q}$-g-pair $(Y,B_Y,\mathbf{M}^Y)/U$ such that any glc center of $(Y,B_Y,\mathbf{M}^Y)$ is the image of a glc center of $(X,B,\mathbf{M})$ in Y. Moreover, $K_Y+B_Y+\mathbf{M}^Y_Y\sim_{\mathbb{Q},V}0$. By Step 1, $K_Y+B_Y+\mathbf{M}^Y_Y\sim_{\mathbb{Q}}g^*(K_V+B_V+\mathbf{M}^V_V)$ for some glc g-pair $(V,B_V,\mathbf{M}^V)/U$ such that any glc center of $(V,B_V,\mathbf{M}^V)$ is the image of a glc center of $(Y,B_Y,\mathbf{M}^Y)$ in V, hence the image of a glc center of $(X,B,\mathbf{M})$ in V. We immediately get (1)(2) and (3) follows from (2). $\square$

Lemma 2.17 ([HL21, Version 2, Lemma 8.2])**.** *Let $f:X\to Y$ and $g:Y\to Z$ be two contractions between normal quasi-projective varieties such that general fibers of $Y\to Z$ are smooth and Y is $\mathbb{Q}$-Gorenstein. Let (X,B) be a pair that is lc over a non-empty open subset of Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D-(K_{X/Y}+B)$ is nef$/Z$. Then for any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor Q on Y,*

$$\kappa_\sigma(X/Z,D+f^*Q)\geq\kappa_\sigma(X/Y,D)+\kappa(Y/Z,Q).$$

Proof. Let $z\in Z$ be a very general point and let $X_z:=(g\circ f)^{-1}(z), Y_z:=g^{-1}(z)$ be the fibers of X and Y over z respectively. We have an induced contraction $f_z: X_z\to Y_z$. Let F be a very general fiber of f_z, then F is also a very general fiber of f.

First assume that $\dim Y>\dim Z$. By our assumption, Y_z is smooth, $(X_z, B|_{X_z})$ is lc over a non-empty open subset of Y_z, and

$$D|_{X_z}-(K_{X_z/Y_z}+B|_{X_z})=(D-(K_{X/Y}+B))|_{X_z}$$

is nef. By [Fuj19, (3.3)],

$$\begin{aligned}\kappa_\sigma(X/Z,D+f^*Q)&=\kappa_\sigma(X_z,D|_{X_z}+f_z^*Q|_{Y_z})\geq\kappa_\sigma(X_z/Y_z,D|_{X_z})+\kappa(Y_z,Q|_{Y_z})\\&=\kappa_\sigma(F,D|_F)+\kappa(Y/Z,Q)=\kappa_\sigma(X/Y,D)+\kappa(Y/Z,Q).\end{aligned}$$

Now assume that $\dim Y=\dim Z$. If $\dim X=\dim Y$ then there is nothing left to prove, so we may assume that $\dim X>\dim Y$. In this case, $f^*Q|_{X_z}=0$, so we have

$$\begin{aligned}\kappa_\sigma(X/Z,D+f^*Q)&=\kappa_\sigma(X_z,D|_{X_z}+f^*Q|_{X_z})=\kappa_\sigma(X_z,D|_{X_z})=\kappa_\sigma(X/Z,D)\\&\geq\kappa_\sigma(X/Y,D)=\kappa_\sigma(X/Y,D)+\kappa(Y/Z,Q).\end{aligned}$$

□

Lemma 2.18 ([HL21, Version 2, Lemma 8.3])**.** *Let $(X,B,\mathbf{M})/U$ be a glc g-pair such that $K_X+B+\mathbf{M}_X\equiv_U G$ for some $\mathbb{R}$-divisor $G\geq 0$, such that U is quasi-projective and G is abundant over U. Let $X\dashrightarrow V$ be the Iitaka fibration over U associated to G, and $(W,B_W,\mathbf{M})$ a log smooth model of $(X,B,\mathbf{M})$ such that the induced map $\psi: W\to V$ is a morphism over U. Then*

(1) $\kappa_\sigma(W/U,K_W+B_W+\mathbf{M}_W)=\dim V-\dim U$, and
(2) $\kappa_\sigma(W/V,K_W+B_W+\mathbf{M}_W)=0$.

Proof. Let $h_V:\bar V\to V$ be a resolution of V. By Lemmas 2.2(3) and 2.11 possibly replacing $(W,B_W,\mathbf{M})/U$ with a higher model, we may assume that the induced map $\bar\psi: W\to\bar V$ is a morphism. Since $(W,B_W,\mathbf{M})$ a log smooth model of $(X,B,\mathbf{M})$, we have

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E$$

where $h: W\to X$ is the induced morphism, $\mathbf{M}$ descends to W, and $E\geq 0$.

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "\bar\psi"'] \arrow[dr, "\psi"] & X \arrow[d, dashed] \\ \bar V \arrow[r, "h_V"] \arrow[dr] & V \arrow[d] \\ & U \end{tikzcd}$$

Since $G\geq 0$ is abundant over U, by [Cho08, Proposition 2.2.2(1)],

$$\dim V-\dim U=\kappa(X/U,G)=\kappa_\iota(X/U,G)=\kappa_\sigma(X/U,G)\geq 0.$$

Since $X\dashrightarrow V$ is the Iitaka fibration associated to G over U, there exists an ample$/U$ $\mathbb{R}$-divisor A on V and an $\mathbb{R}$-divisor $F\geq 0$ on W such that $h^*G=\psi^*A+F$ for some h-exceptional $\mathbb{R}$-divisor $F\geq 0$ on W. Then for any real number k, we have

$$K_W+B_W+\mathbf{M}_W+k\psi^*A\equiv_U(1+k)\psi^*A+E+F.$$

By Lemma 2.2(2)(3)(5), for any $k\geq 0$ we have

$$\begin{aligned}\kappa_\sigma(W/U,K_W+B_W+\mathbf{M}_W+k\psi^*A)&=\kappa_\sigma(W/U,(1+k)\psi^*A+E+F)=\kappa_\sigma(W/U,\psi^*A+E+F)\\&=\kappa_\sigma(W/U,K_W+B_W+\mathbf{M}_W)=\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)\\&=\kappa_\sigma(X/U,G)=\kappa(X/U,G)=\dim V-\dim U\end{aligned}$$

for any non-negative real number k. In particular, we get (1). Since A is ample$/U$, h_V^*A is big$/U$, and we may pick a sufficiently large positive integer k such that $K_{\bar{V}} + kh_V^*A$ is big$/U$.

Since $(W, B_W, \mathbf{M})$ is a log smooth model of $(X, B, \mathbf{M})$, (W, B_W) is lc. Since $\bar{V}$ is smooth, any very general fiber of the induced morphism $\bar{V} \to U$ is smooth. Let $D := K_W + B_W + \mathbf{M}_W - \bar{\psi}^* K_{\bar{V}}$ and $Q := K_{\bar{V}} + kh_V^*A$, then $D - (K_{W/\bar{V}} + B_W) = \mathbf{M}_W$ is nef$/U$. By Lemma 2.17,

$$\begin{aligned}\dim V - \dim U &= \kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W + k\psi^*A) = \kappa_\sigma(W/U, K_W + B_W + \mathbf{M}_W + k\bar{\psi}^* h_V^* A)\\ &= \kappa_\sigma(W/U, D + \bar{\psi}^* Q) \geq \kappa_\sigma(W/\bar{V}, D) + \kappa(\bar{V}/U, Q)\\ &= \kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W - \bar{\psi}^* K_{\bar{V}}) + \kappa(\bar{V}/U, K_{\bar{V}} + kh_V^*A)\\ &= \kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W) + (\dim V - \dim U).\end{aligned}$$

Thus $\kappa_\sigma(W/\bar{V}, K_W + B_W + \mathbf{M}_W) \leq 0$, hence $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) \leq 0$. Since $K_W + B_W + \mathbf{M}_W \equiv_U h^*G + E \geq 0$, $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) \geq 0$. Thus $\kappa_\sigma(W/V, K_W + B_W + \mathbf{M}_W) = 0$, and we get (2). □

Lemma 2.19 ([HL21, Version 2, Lemma 8.4])**.** *Let $(X, B, \mathbf{M})/U$ be a glc g-pair. Then there exists a proper log smooth model $(W, B_W = B_W^h + B_W^v, \mathbf{M})$ of $(X, B, \mathbf{M})$, such that*

(1) $B_W^h \geq 0$ and B_W^v is reduced,
(2) B_W^v is vertical over U, and
(3) for any real number $t \in (0, 1]$, all glc centers of $(W, B_W - tB_W^v, \mathbf{M})$ dominate U.

Proof. By Lemma 2.11, possibly replacing $(X, B, \mathbf{M})$ with a proper log smooth model, we may assume that $(X, \operatorname{Supp} B)$ is log smooth and $\mathbf{M}$ descends to X. By [Has18, Lemma 2.10], there exists a proper log smooth model $(W, B_W = B_W^h + B_W^v)$ of (X, B), such that

- $B_W^h \geq 0$ and B_W^v is reduced,
- B_W^v is vertical over U, and
- for any real number $t \in (0, 1]$, all lc centers of $(W, B_W - tB_W^v)$ dominate U.

Since $\mathbf{M}$ descends to X, $(W, B_W, \mathbf{M})$ is a proper log smooth model of $(X, B, \mathbf{M})$, and for any real number $t \in (0, 1]$, any glc center of $(W, B_W - tB_W^v, \mathbf{M})$ is an lc center of $(W, B_W - tB_W^v)$ and dominates U. Thus $(W, B_W = B_W^h + B_W^v, \mathbf{M})$ satisfies our requirements. □

3. Relative Nakayama-Zariski decomposition

Definition 3.1. Let $\pi : X \to U$ be a projective morphism from a normal quasi-projective variety to a variety, A an ample$/U$ $\mathbb{R}$-divisor on X, D a pseudo-effective$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, and P a prime divisor on X. For any big$/U$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B, we define

$$\sigma_P(X/U, B) := \inf\{\operatorname{mult}_P B' \mid 0 \leq B' \sim_{\mathbb{R},U} B\}.$$

We define

$$\sigma_P(X/U, D) := \lim_{\epsilon \to 0^+} \sigma_P(X/U, D + \epsilon A),$$

where we allow $+\infty$ be a limit as well. We let

$$N_\sigma(X/U, D) := \sum_{C \text{ is a prime divisor on } X} \sigma_C(X/U, D)$$

and

$$P_\sigma(X/U, D) := D - N_\sigma(X/U, D).$$

Definition 3.1 is the same as the one adopted in [HX13, HMX18]. The following lemma shows that relative Nakayama-Zariski decomposition defined in Definition 3.1 is the same as the σ-decomposition defined in [Nak04, III. §4.a]:

Lemma 3.2. *Notation as in Definition 3.1. If X is smooth, then $\sigma_P(X/U, D)$ is the same as $\sigma_P(D, X/U)$, where the latter is the value defined as in Nakayama's original relative σ-decomposition* [Nak04, III. §4.a]*.*

Proof. By definition, we only need to deal with the case when D is big. We may pick an affine open subset U^0 of U such that P intersects $X^0 := X \times_U U^0$. Let $P^0 := P \times_U U^0$ and $D^0 := D \times_U U^0$. Then

$$\sigma_P(X/U, D) = \sigma_{P^0}(X^0/U^0, D^0).$$

Possibly replacing $(X/U, D)$ and P with $(X^0/U^0, D^0)$ and P^0 respectively, we may assume that U is affine. Thus for any Cartier divisor Q on U, there exists a principle divisor Q' on U such that $Q' = Q$ in a neighborhood of the generic point of $\pi(P)$. In particular, we have

$$\sigma_P(X/U, D) = \inf\{\operatorname{mult}_{P^0} B' \mid 0 \leq B' \sim_{\mathbb{R}} B^0\}.$$

For any Cartier divisor F on X, let

$$m_F := \inf\{+\infty, \operatorname{mult}_P F' \mid 0 \leq F' \sim F\}.$$

If $m_F < +\infty$, then by definition,

$$m_F = \max\{m \mid m \in \mathbb{N}, H^0(X, F - mP) \hookrightarrow H^0(X, F) \text{ is isomorphic}\}.$$

Moreover, since U is affine and $H^0(X, \mathcal{O}_X(F)) = H^0(U, \pi_*\mathcal{O}_X(F))$, if $m_F < +\infty$, then

$$m_F = \max\{m \mid m \in \mathbb{N}, \pi_*\mathcal{O}_X(F - mP) \hookrightarrow \pi_*\mathcal{O}_X(F) \text{ is isomorphic}\}.$$

The lemma follows by the construction in [Nak04, III. §4.a]. □

Lemma 3.3. *Let $\pi : X \to U$ be a projective morphism from a normal quasi-projective variety to a variety and D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $f : Y \to X$ be a birational morphism. Then:*

(1) For any prime divisor P on X, we have

$$\sigma_P(X/U, D) = \sigma_{f_*^{-1}P}(Y/U, f^*D).$$

(2) For any exceptional/X $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $E \geq 0$ and prime divisor P on Y,

$$\sigma_P(Y/U, f^*D + E) = \sigma_P(Y/U, f^*D) + \operatorname{mult}_P E.$$

(3) For any exceptional/X $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $E \geq 0$, $N_\sigma(X/U, D)$ is well-defined if $N_\sigma(Y/U, f^*D + E)$ is well-defined. Moreover, $N_\sigma(X/U, D) = f_*N_\sigma(Y/U, f^*D)$ and $P_\sigma(X/U, D) = f_*P_\sigma(Y/U, f^*D)$.

Proof. Let $g = \pi \circ f$ and A (resp. A') be an ample/U divisor on X (resp. Y). Fix a real number $a > 0$ such that $aA' + f^*A$ is ample/U. For any prime divisor P on Y, since f^*A is semi-ample, we have

$$\begin{aligned}\sigma_P(X/U, f^*D) &= \lim_{\epsilon \to 0^+} \sigma_P(X/U, f^*D + \epsilon(aA' + f^*A)) \leq \lim_{\epsilon \to 0^+} \sigma_P(X/U, f^*D + \epsilon f^*A)\\ &\leq \sigma_P(X/U, f^*D),\end{aligned}$$

hence $\lim_{\epsilon \to 0^+} \sigma_P(Y/U, f^*D + \epsilon f^*A) = \sigma_P(Y/U, f^*D)$.

We prove (1). Since $\pi_*\mathcal{O}_X(F) = g_*\mathcal{O}_Y(f^*F)$ for any Cartier divisor F on X. Then by definition we have $\sigma_P(X/U, D + \epsilon A) = \sigma_{f_*^{-1}P}(Y/U, f^*D + \epsilon f^*A)$ for any $\epsilon > 0$. Thus we have

$$\sigma_P(X/U, D) = \lim_{\epsilon \to 0^+} \sigma_{f_*^{-1}P}(Y/U, f^*D + \epsilon f^*A) = \sigma_{f_*^{-1}P}(Y/U, f^*D)$$

which is (1).

We prove (2). Since $\lim_{\epsilon \to 0^+} \sigma_P(Y/U, f^*D + \epsilon f^*A) = \sigma_P(Y/U, f^*D)$, we may assume that D is a big/U. (2) follows from the fact that $\pi_*\mathcal{O}_Y(f^*F + E) = g_*\mathcal{O}_X(F)$ for any Cartier divisor F on X and any exceptional/X divisor $E \geq 0$.

(3) is an immediate consequence of (1) and (2). □

Lemma 3.4. *Let $\pi : X \to U$ be a projective morphism from a normal quasi-projective variety to a variety and D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there are only finitely many prime divisors P on X such that $\sigma_P(X/U, D) \neq 0$. In particular, if $\sigma_P(X/U, D) < +\infty$ for any prime divisor P on X, then $N_\sigma(X/U, D)$ and $P_\sigma(X/U, D)$ are well-defined.*

Proof. We show that there are at most $\dim N^1(X/U)_{\mathbb{R}}$ prime divisors P on X such that $\sigma_P(X/U,D)\neq 0$. By Lemma 3.3 we may assume that X is smooth. Let $P_1,P_2,...,P_l$ be distinct prime divisors of X such that $\sigma_{P_i}(X/U,D)>0$ for each i. If $l\leq \dim N^1(X/U)_{\mathbb{R}}$ then we are done. Otherwise, by Lemma 3.2 and [Nak04, III, Lemma 4.2(2)],

$$\sigma_{P_i}(X/U,\sum_{j=1}^{l}x_jP_j)=x_i$$

for any $x_1,x_2,...,x_l\in\mathbb{R}_{\geq 0}$, and possibly reordering indices, we have

$$\sum_{i=1}^{s}x_iP_i\equiv_U\sum_{j=s+1}^{l}x_jP_j\in \mathrm{N}^1(X/U)$$

for some $1\leq s\leq l$. By Lemma 3.2 and [Nak04, III, Lemma 4.2(2)] again,

$$x_1=\sigma_{P_1}(X/U,\sum_{i=1}^{s}x_iP_i)=\sigma_{P_1}(X/U,\sum_{j=s+1}^{l}x_jP_j)=0,$$

a contradiction. □

Definition 3.5. Let $\pi:X\to U$ be a projective morphism from a normal quasi-projective variety to a variety, D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, and P a prime divisor over X. Let $f:Y\to X$ be a birational morphism such that P descends to Y. We define

$$\sigma_P(X/U,D):=\sigma_P(Y/U,f^*D).$$

By Lemma 3.3, $\sigma_P(X/U,D)$ is well-defined and is independent of the choice of Y.

Lemma 3.6. *Let $\pi:X\to U$ be a projective morphism from a normal quasi-projective variety to a variety, D,D' two pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and P a prime divisor over X. Then*

1. $\sigma_P(X/U,D+D')\leq\sigma_P(X/U,D)+\sigma_P(X/U,D')$.
2. *If* $D'\geq 0$, *then* $\lim_{\epsilon\to 0^+}\sigma_P(X/U,D+\epsilon D')=\sigma_P(X/U,D)$.
3. *If D is a limit of movable/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, then* $P_\sigma(X/U,D)=D$ *and* $N_\sigma(X/U,D)=0$.
4. *If $N_\sigma(X/U,D)$ is well-defined, then* $\operatorname{Supp}N_\sigma(X/U,D)$ *conincides with the divisorial part of* $\mathbf{B}_-(D/U)$.
5. *If* $0\leq D'\leq N_\sigma(X/U,D)$, *then* $P_\sigma(X/U,D-D')=P_\sigma(X/U,D)$.
6. *If* $D'\geq 0$ *and* $\operatorname{Supp}D'\subset\operatorname{Supp}N_\sigma(X/U,D)$, *then* $P_\sigma(X/U,D+D')=P_\sigma(X/U,D)$.

Proof. By Lemma 3.3, possibly replacing X with a resolution, we may assume that X is smooth and P is a prime divisor on X. Let A be an ample/U divisor on X.

(1) follows from the fact that $\sigma_P(X/U,D+D'+\epsilon A)\leq\sigma_P(X/U,D+\frac{\epsilon}{2}A)+\sigma_P(X/U,D'+\frac{\epsilon}{2}A)$.

There exists $a>0$ such that $A-aD'$ is ample/U. Thus by (1),

$$\sigma_P(X/U,D)+\sigma_P(X/U,a\epsilon D')\geq\sigma_P(X/U,D+a\epsilon D')\geq\sigma_P(X/U,D+\epsilon A),$$

and (2) follows after taking $\epsilon\to 0^+$.

For (3), if this is not true, then we have $\sigma_P(X/U,D)>0$ for some P. By definition, there exist an $\epsilon>0$ such that $\sigma_P(X/U,D+\epsilon A)>0$. Assume $D=\lim_i D_i$, where D_i is a movable divisor for each $i\geq 1$. Then $\epsilon A-(D_i-D)$ is ample for any $i\gg 0$. Thus $0=\sigma_P(X/U,D_i)=\sigma_P(X/U,D_i+\epsilon A-(D_i-D))=\sigma_P(X/U,D+\epsilon A)>0$, which is a contradiction.

(4) follows from the definition of $\mathbf{B}_-(D/U)$.

(5) and (6) follows from Lemmma 3.2 and [Nak04, III, Lemma 4.2]. □

Lemma 3.7 (cf. [Has20a, Remark 2.4])**.** *Let $\pi:X\to U$ be a projective morphism from a normal quasi-projective variety to a variety, D a pseudo-effective/U $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X*

such that $N_\sigma(X/U,D)$ is well-defined, and $D'\geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Then there exists $t_0>0$, such that $\operatorname{Supp} N_\sigma(X/U,D+tD')$ *is independent of t for any $t\in(0,t_0]$.*

Proof. Since $N_\sigma(X/U,D)$ is well-defined and $D'\geq N_\sigma(X/U,D')$ is well-defined, by Lemma 3.6(1), we may let $D_1,\dots,D_k$ be the irreducible components of $\operatorname{Supp} N_\sigma(X/U,D)\cup\operatorname{Supp} D'$. Let

$$J_i:=\{s\mid s\in(0,1],\sigma_{D_i}(X/U,D+sD')=0\}$$

for any $1\leq i\leq k$. For each i, we define

$$s_i=\begin{cases}\text{some number in } J_i & J_i\neq\emptyset,\inf J_i=0\\ \frac{1}{2}\inf J_i & J_i\neq\emptyset,\inf J_i>0\\ 1 & J_i=\emptyset.\end{cases}$$

Let $t_0:=\min_{1\leq i\leq k}\{s_i\}$. Then by Lemma 3.6(1), for any $1\leq i\leq k$,

- if $J_i\neq\emptyset$ and $\inf J_i=0$, then $\sigma_{D_i}(X/U,D+tD')=0$ for any $t\in(0,t_0]$, and
- $\sigma_{D_i}(X/U,D+tD')>0$ for any $t\in(0,t_0]$ otherwise.

Thus t_0 satisfies our requirement. $\square$

Lemma 3.8. *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair, $D:=K_X+B+\mathbf{M}_X$, and P a prime divisor over X. Then*

(1) $\sigma_P(X/U,D)<+\infty$. In particular, $N_\sigma(X/U,D)$ is well-defined and $N_\sigma(X/U,f^*D)$ is well-defined for any birational morphism $f:\tilde X\to X$, and

(2) if $(X,B,\mathbf{M})$ is $\mathbb{Q}$-factorial gdlt, then for any partial D-MMP/U $\phi:X\dashrightarrow\bar X$,

(a) the divisors contracted by ϕ are contained in $\operatorname{Supp} N_\sigma(X/U,D)$, *and*

(b) let $\bar B$ be the strict transform of B on $\bar X$. If $K_{\bar X}+\bar B+\mathbf{M}_{\bar X}$ is a limit of movable/U $\mathbb{R}$-divisors, then $\operatorname{Supp} N_\sigma(X/U,D)$ *is the set of all ϕ-exceptional divisors.*

Proof. First we show that $\sigma_P(X/U,D)<+\infty$ for any prime divisor P on X and also prove (2). Let $(Y,B_Y,\mathbf{M})$ be a gdlt model of $(X,B,\mathbf{M})$. By Lemma 3.3, we may replace $(X,B,\mathbf{M})$ with $(Y,B_Y,\mathbf{M})$ and assume that $(X,B,\mathbf{M})$ is $\mathbb{Q}$-factorial gdlt. By Lemma 2.5, we may run a partial $(K_X+B+\mathbf{M}_X)$-MMP/U $\psi:X\dashrightarrow X'$, such that

- ψ is the composition of ϕ and a partial $(K_{\bar X}+\bar B+\mathbf{M}_{\bar X})$-MMP/$U$ with scaling of an ample/U $\mathbb{R}$-divisor, where $\bar B$ is the strict transform of B on $\bar X$, and
- $D':=K_{X'}+B'+\mathbf{M}_{X'}$ is a limit of movable/U $\mathbb{R}$-divisors, where B' is the strict transform of B on X'.

Let $p:W\to X$ and $q:W\to X'$ be a resolution of indeterminacy, then

$$p^*D=q^*D'+E$$

for some $E\geq 0$ that is exceptional/X'. Moreover, q^*D' is a limit of movable/U $\mathbb{R}$-divisors. By Lemmas 3.6(3) and 3.3(2)(3),

$$N_\sigma(X/U,D)=p_*(N_\sigma(W/U,q^*D'+E))=p_*(E+N_\sigma(W/U,q^*D'))=p_*E$$

is well-defined. Thus $\sigma_P(X/U,D)<+\infty$ for any prime divisor P on X, and we also get (2.b). Since the divisors contracted by ψ are contained in $\operatorname{Supp} p_*E=\operatorname{Supp} N_\sigma(X/U,D)$, the divisors contracted by ϕ are contained in $\operatorname{Supp} N_\sigma(X/U,D)$, and we get (2.a).

For any prime divisor P over X, let $(\tilde B,\tilde B,\mathbf{M})/U$ be a log smooth model of $(X,B,\mathbf{M})$. Then $\sigma_P(\tilde X/U,K_{\tilde X}+\tilde B+\mathbf{M}_{\tilde X})<+\infty$ and (1) follows from Lemma 3.3. $\square$

Lemma 3.9. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair such that $K_X+B+\mathbf{M}_X$ is pseudo-effective/U. Let $\phi:X\dashrightarrow X'$ be a birational map/U which does not extract any divisor and B' the strict transform of B on X', such that*

(1) $K_{X'}+B'+\mathbf{M}_{X'}$ is nef/U, and

(2) ϕ only contract divisors contained in $\operatorname{Supp} N_\sigma(X/U,K_X+B+\mathbf{M}_X)$,

then $(X',B',\mathbf{M})/U$ is a log minimal model of $(X,B,\mathbf{M})/U$.

Proof. Let $p: W \to X$ and $q: W \to X'$ be a resolution of indeterminacy of ϕ, such that

$$p^*(K_X + B + \mathbf{M}_X) + E = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F$$

where $E \geq 0, F \geq 0$, and $E \wedge F = 0$. Then E and F are q-exceptional. By Lemmas 3.6(3) and 3.3(2)(3), $F = N_\sigma(W/U, q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F)$.

We may write $E = E_1 + E_2$ such that E_1 is p-exceptional and every component of E_2 is not p-exceptional. For any component D of E_1, by Lemma 3.3(2), $D \subset \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E)$. Thus $N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E) = N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E_2)$. For any component D of E_2, since p_*D is exceptional$/X'$, p_*D is contained in $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X)$. By Lemmas 3.3(3) and 3.6(6), D is a component of $N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E)$. Therefore,

$$\operatorname{Supp} E \subset \operatorname{Supp} N_\sigma(W/U, p^*(K_X + B + \mathbf{M}_X) + E) = \operatorname{Supp} F,$$

hence $E = 0$. By Lemma 3.8(2.b), ϕ contracts all components of $\operatorname{Supp} N_\sigma(X/U, K_X + B + \mathbf{M}_X) = \operatorname{Supp} p_*F$, and the lemma follows. $\square$

4. Reduction via Iitaka fibration

This section is similar to [HL21, Version 2, Section 4].

Lemma 4.1 (cf. [HL21, Version 2, Lemma 4.9])**.** *Let $(X, B, \mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC glc g-pair with X klt and $\pi: X \to U$ the induced morphism, such that*

(1) π is an equidimensional contraction,
(2) U is quasi-projective and $\mathbb{Q}$-factorial, and
(3) $\kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) = \kappa_\iota(X/U, K_X + B + \mathbf{M}_X) = 0$.

Let $A \geq 0$ be an ample$/U$ $\mathbb{R}$-divisor on X such that $(X, B + A, \mathbf{M})$ is glc and $K_X + B + A + \mathbf{M}_X$ is nef$/U$, and run a $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of A. Then this MMP terminates with a good minimal model $(X', B', \mathbf{M})/U$ of $(X, B, \mathbf{M})/U$. Moreover, $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R},U} 0$.

Proof. If $\dim X = \dim U$, then since π is an equidimensional contraction, π is the identity map, and there is nothing left to prove. In the following, we assume that $\dim X > \dim U$.

Since $\kappa_\iota(X/U, K_X + B + \mathbf{M}_X) = 0$, $K_X + B + \mathbf{M}_X \sim_{\mathbb{R},U} E \geq 0$ for some $\mathbb{R}$-divisor E on X. We may write $E = E^h + E^v$, such that $E^h \geq 0, E^v \geq 0$, each component of E^h is horizontal over U, and E^v is vertical over U. Since π is equidimensional, the image of any component of E^v on U is a divisor. Since U is $\mathbb{Q}$-factorial, for any prime divisor P on U, we may define

$$\nu_P := \sup\{\nu \mid \nu \geq 0, E^v - \nu\pi^*P \geq 0\}.$$

Then $\nu_P > 0$ for only finitely many prime divisors P on U. Possibly replacing E^v with $E^v - \pi^*(\sum_P \nu_P P)$, we may assume that E^v is very exceptional over U.

Let F be a very general fiber of π, and $(F, B_F, \mathbf{M}^F)$ the projective g-pair induced by the adjunction to the fiber

$$K_F + B_F + \mathbf{M}^F_F := (K_X + B + \mathbf{M}_X)|_F.$$

Then $\kappa_\sigma(K_F + B_F + \mathbf{M}^F_F) = \kappa_\iota(K_F + B_F + \mathbf{M}^F_F) = 0$. By [Has22, Lemma 3.10], $(F, B_F, \mathbf{M}^F)$ has a good minimal model $(F', B_{F'}, \mathbf{M}^F)$ such that $K_{F'} + B_{F'} + \mathbf{M}^F_{F'} \sim_{\mathbb{R}} 0$. By Lemma 2.8, any $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of A terminates along F with a log minimal model of $(F, B_F, \mathbf{M}^F)$. In particular, let

$$(X, B, \mathbf{M}) := (X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \cdots \dashrightarrow (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

be our $(K_X + B + \mathbf{M}_X)$-MMP$/U$ with scaling of A, and let A_i, E^h_i, E^v_i, F_i be the strict transforms of A, E^h, E^v, F on X_i respectively, then there exists a positive integer n such that

$$E^h_n|_{F_n} = (E^h_n + E^v_n)|_{F_n} \sim_{\mathbb{R}} (K_{X_n} + B_n + \mathbf{M}_{X_n})|_{F_n},$$

and the projective g-pair $(F_n, B_{F_n}, \mathbf{M}^F)$ given by the adjunction

$$K_{F_n} + B_{F_n} + \mathbf{M}^F_{F_n} := (K_{X_n} + B_n + \mathbf{M}_{X_n})|_{F_n}$$

is a log minimal model of $(F, B_F, \mathbf{M}^F)$.

By [HL21, Lemma 3.5(1)] (=[HL21, Version 3, Lemma 3.9(1)]), $K_{F_n}+B_{F_n}+\mathbf{M}^F_{F_n}\sim_{\mathbb{R}} 0$. Thus $E^h_n|_{F_n}\sim_{\mathbb{R}} 0$. Since $E^h_n\geq 0$ is horizontal over U, $E^h_n=0$, and $K_{X_n}+B_n+\mathbf{M}_{X_n}\sim_{\mathbb{R},U} E^v_n$. Since this MMP$/U$ is also a (E^h+E^v)-MMP$/U$ and E^v_n is very exceptional over U, by Lemma 2.3, this MMP terminates with a log minimal model $(X',B',\mathbf{M})/U=(X_m,B_m,\mathbf{M})/U$ of $(X,B,\mathbf{M})/U$ for some positive integer m, such that $K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R},U} 0$. In particular, $(X',B',\mathbf{M})/U$ is a good minimal model of $(X,B,\mathbf{M})/U$. □

Theorem 4.2 (cf. [HL21, Version 2, Theorem 4.1])**.** *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair and $\pi: X\to V$ a contraction over U such that V is quasi-projective. Assume that $\kappa_\sigma(X/V,K_X+B+\mathbf{M}_X)=\kappa_\iota(X/V,K_X+B+\mathbf{M}_X)=0$. Then there exists a $\mathbb{Q}$-factorial NQC gdlt g-pair $(X',B',\mathbf{M})/U$, a contraction $\pi': X'\to V'$ over U, and a birational projective morphism $\varphi: V'\to V$ over U satisfying the following:*

$$\begin{tikzcd}
X' \arrow[rr, dashed] \arrow[d, "\pi'"'] & & X \arrow[d, "\pi"] \\
V' \arrow[rr, "\varphi"] \arrow[dr] & & V \arrow[dl] \\
& U &
\end{tikzcd}$$

(1) X' is birational to X and V' is smooth,
(2) $K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R},V'} 0$.
(3) $(X,B,\mathbf{M})/U$ has a good minimal model if and only if $(X',B',\mathbf{M})/U$ has a good minimal model.
(4) Any weak glc model of $(X,B,\mathbf{M})/U$ is a weak glc model of $(X',B',\mathbf{M})/U$, and any weak glc model of $(X',B',\mathbf{M})/U$ is a weak glc model of $(X,B,\mathbf{M})/U$.
(5) If all glc centers of $(X,B,\mathbf{M})$ dominate V, then all glc centers of $(X',B',\mathbf{M})$ dominate V'.
(6) $\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)=\kappa_\sigma(X'/U,K_{X'}+B'+\mathbf{M}_{X'})$ and $\kappa_\iota(X/U,K_X+B+\mathbf{M}_X)=\kappa_\iota(X'/U,K_{X'}+B'+\mathbf{M}_{X'})$

Proof. Let $h: W\to X$ be a log resolution of $(X,\operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W. By Lemma 2.11, $(X,B,\mathbf{M})$ has a proper log smooth model $(W,B_W,\mathbf{M})$ for some $\mathbb{R}$-divisor B_W on W. By Lemmas 2.2(3) and 2.12(3), Theorem 2.14, and [HL21, Lemmas 3.6, 3.10] ((=[HL21, Version 3, Lemmas 3.10, 3.17]), we may replace $(X,B,\mathbf{M})$ with $(W,B_W,\mathbf{M})$, and assume that (X,B) is log smooth dlt and $\mathbf{M}$ descends to X.

By Theorem 2.15, there exists a commutative diagram of projective morphisms

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi"] \\
V' \arrow[r, "\varphi"] & V
\end{tikzcd}$$

such that

- f,φ are birational morphisms, π_Y is an equidimensional contraction, Y only has $\mathbb{Q}$-factorial toroidal singularities, and V' is smooth, and
- there exist two $\mathbb{R}$-divisors B_Y and E on Y, such that
  - $K_Y+B_Y+\mathbf{M}_Y=f^*(K_X+B+\mathbf{M}_X)+E$,
  - $B_Y\geq 0$, $E\geq 0$, and $B_Y\wedge E=0$,
  - (Y,B_Y) is lc quasi-smooth, and any glc center of $(Y,B_Y,\mathbf{M})$ on X is a glc center of $(X,B,\mathbf{M})$.

In particular, $(Y, B_Y, \mathbf{M})$ is $\mathbb{Q}$-factorial NQC glc and Y is klt. Since φ is birational, by Lemma 2.2(3),

$$\kappa_\sigma(Y/V', K_Y + B_Y + \mathbf{M}_Y) = \kappa_\sigma(Y/V, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\sigma(X/V, K_X + B + \mathbf{M}_X) = 0$$

and

$$\kappa_\iota(Y/V', K_Y + B_Y + \mathbf{M}_Y) = \kappa_\iota(Y/V, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\iota(X/V, K_X + B + \mathbf{M}_X) = 0.$$

By Lemma 4.1, we may run a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP$/V'$ with scaling of a general ample$/V'$ divisor A on Y, which terminates with a good minimal model $(X', B', \mathbf{M})/V'$ of $(Y, B_Y, \mathbf{M})/V'$ such that $K_{X'} + B' + \mathbf{M}_{X'} \sim_{\mathbb{R},V'} 0$. Let $\pi' : X' \to V'$ be the induced contraction.

$$\begin{tikzcd}
X' \arrow[dr, "\pi'"'] & Y \arrow[l, dashed] \arrow[r, "f"] \arrow[d, "\pi_Y"] & X \arrow[d, "\pi"] \\
& V' \arrow[r, "\varphi"] & V
\end{tikzcd}$$

We show that $(X', B', \mathbf{M})/U, \pi', \varphi$ satisfy our requirements. (1)(2) follow from our construction.

Let $p : W' \to Y$ and $q : W' \to X'$ be a resolution of indeterminacy of the induce map $Y \dashrightarrow X'$ such that p is a log resolution of (Y, B_Y). Then we have

$$p^*(K_Y + B_Y + \mathbf{M}_Y) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F$$

for some $F \geq 0$ that is exceptional over X'. Let $B_{W'} := p_*^{-1} B_Y + \operatorname{Exc}(p)$, then $(W', B_{W'}, \mathbf{M})$ is a log smooth model of $(Y, B_Y, \mathbf{M})$ and $(X', B', \mathbf{M})$.

Since $K_Y + B_Y + \mathbf{M}_Y = f^*(K_X + B + \mathbf{M}_X) + E$, by Theorem 2.14, $(X, B, \mathbf{M})/U$ has a good minimal model if and only if $(Y, B_Y, \mathbf{M})/U$ has a good minimal model, if and only if $(W', B_{W'}, \mathbf{M})/U$ has a good minimal model, if and only if $(X', B', \mathbf{M})/U$ has a good minimal model, hence (3).

By [HL21, Lemmas 3.6, 3.10(1)] (=[HL21, Version 3, Lemmas 3.10, 3.17]), a g-pair $(X'', B'', \mathbf{M})/U$ is a weak glc model of $(X, B, \mathbf{M})/U$ if and only if $(X'', B'', \mathbf{M})/U$ is a weak glc model of $(W', B_{W'}, \mathbf{M})/U$, if and only if $(X'', B'', \mathbf{M})/U$ is a weak glc model of $(X', B', \mathbf{M})/U$, hence (4).

Let D be a glc place of $(X', B', \mathbf{M})$. Since $Y \dashrightarrow X'$ is a $(K_Y + B_Y + \mathbf{M}_Y)$-MMP$/V'$, D is a glc place of $(Y, B_Y, \mathbf{M})$, hence a glc place of $(X, B, \mathbf{M})$. Thus if all glc centers of $(X, B, \mathbf{M})$ dominate V, then all glc centers of $(X', B', \mathbf{M})$ dominate V, hence all glc centers of $(X', B', \mathbf{M})$ dominate V' as φ is birational, and we have (5).

Finally, by Lemma 2.2(3),

$$\begin{aligned}
\kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) &= \kappa_\sigma(Y/U, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\sigma(W'/U, p^*(K_Y + B_Y + \mathbf{M}_Y)) \\
&= \kappa_\sigma(W'/U, q^*(K_{X'} + B' + \mathbf{M}_{X'})) = \kappa_\sigma(X'/U, K_{X'} + B' + \mathbf{M}_{X'})
\end{aligned}$$

and

$$\begin{aligned}
\kappa_\iota(X/U, K_X + B + \mathbf{M}_X) &= \kappa_\iota(Y/U, K_Y + B_Y + \mathbf{M}_Y) = \kappa_\iota(W'/U, p^*(K_Y + B_Y + \mathbf{M}_Y)) \\
&= \kappa_\iota(W'/U, q^*(K_{X'} + B' + \mathbf{M}_{X'})) = \kappa_\iota(X'/U, K_{X'} + B' + \mathbf{M}_{X'}),
\end{aligned}$$

and we get (6). ☐

Proposition 4.3. *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair and $\pi : X \to V$ a contraction over U, such that*

- *V is normal quasi-projective,*
- *$\kappa_\sigma(X/V, K_X + B + \mathbf{M}_X) = \kappa_\iota(X/V, K_X + B + \mathbf{M}_X) = 0$ and $\kappa_\sigma(X/U, K_X + B + \mathbf{M}_X) = \dim V - \dim U$, and*
- *all glc centers of $(X, B, \mathbf{M})$ dominate V.*

Then:

(1) $(X, B, \mathbf{M})/U$ has a good minimal model, and

(2) *Let $(\bar{X},\bar{B},\mathbf{M})/U$ be a good minimal model of $(X,B,\mathbf{M})/U$ and $\bar{X}\to\bar{V}$ is the contraction over U induced by $K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}$. Then all glc centers of $(\bar{X},\bar{B},\mathbf{M})$ dominate $\bar{V}$.*

Proof. By Theorem 4.2, there exists a $\mathbb{Q}$-factorial NQC gdlt g-pair $(X',B',\mathbf{M})/U$, a contraction $\pi':X'\to V'$ over U, and a birational projective morphism $\varphi:V'\to V$ over U, such that

- X' is birational to X and V' is smooth,
- $K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R},V'}0$. In particular, $\kappa_\sigma(X'/V',K_{X'}+B'+\mathbf{M}_{X'})=0$ by Lemma 2.2(5),
- $(X,B,\mathbf{M})/U$ has a good minimal model if and only if $(X',B',\mathbf{M})/U$ has a good minimal model,
- any weak glc model of $(X,B,\mathbf{M})/U$ is a weak glc model of $(X',B',\mathbf{M})/U$, and any weak glc model of $(X',B',\mathbf{M})/U$ is a weak glc model of $(X,B,\mathbf{M})/U$,
- all glc centers of $(X',B',\mathbf{M})$ dominate V', and
- $\kappa_\sigma(X'/U,K_{X'}+B'+\mathbf{M}_{X'})=\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)=\dim V-\dim U=\dim V'-\dim U$.

Claim 4.4. *Assume that $(X',B',\mathbf{M})/U$ has a good minimal model $(\bar{X}',\bar{B}',\mathbf{M})/U$, $\bar{X}'\to\bar{V}'$ is the contraction over U induced by $K_{\bar{X}'}+\bar{B}'+\mathbf{M}_{\bar{X}'}$, and all glc centers of $(\bar{X}',\bar{B}',\mathbf{M})$ dominate $\bar{V}'$. Then Proposition 4.3(2) holds for $(X,B,\mathbf{M})/U$.*

Proof. Let $(\bar{X},\bar{B},\mathbf{M})/U$ be a good minimal model of $(X,B,\mathbf{M})/U$. Then $(\bar{X},\bar{B},\mathbf{M})/U$ is a weak glc model of $(X',B',\mathbf{M})/U$. Since $(\bar{X}',\bar{B}',\mathbf{M})/U$ is also a weak glc model of $(X',B',\mathbf{M})/U$, by [HL21, Lemma 3.5(1)] (=[HL21, Version 3, Lemma 3.9(1)]), we may take a resolution of indeterminacy $p:W\to\bar{X}$ and $q:W\to\bar{X}'$ of the induced birational map $\bar{X}\dashrightarrow\bar{X}'$, such that

$$p^*(K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}})=q^*(K_{\bar{X}'}+\bar{B}'+\mathbf{M}_{\bar{X}'}).$$

Then $K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}$ is semi-ample$/U$, and if we let $\bar{X}\to\bar{V}$ be the contraction over U induced by $K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}$, then $\bar{V}=\bar{V}'$. Since all glc centers of $(\bar{X}',\bar{B}',\mathbf{M})$ dominate $\bar{V}'=\bar{V}$, all glc centers of $(\bar{X},\bar{B},\mathbf{M})$ dominate $\bar{V}$, and the claim is proved. $\square$

Proof of Proposition 4.3 continued. By Claim 4.4, we may replace $(X,B,\mathbf{M}),V$ and π with $(X',B',\mathbf{M}),V'$ and π' respectively, and assume that V is smooth and $K_X+B+\mathbf{M}_X\sim_{\mathbb{R},V}0$. By Theorem 2.16, there exists an NQC gklt g-pair $(V,B_V,\mathbf{M}^V)/U$ such that

$$K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}\pi^*(K_V+B_V+\mathbf{M}^V_V).$$

By Lemma 2.2(4)(5), we have

$$\kappa_\sigma(V/U,K_V+B_V+\mathbf{M}^V_V)=\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)=\dim V-\dim U.$$

By Lemma 2.2(1), $K_V+B_V+\mathbf{M}^V_V$ is big$/U$. By [BZ16, Lemma 4.4(2)], we may run a $(K_V+B_V+\mathbf{M}^V_V)$-MMP$/U$ with scaling of some general ample$/U$ divisor A, which terminates with a good minimal model $(\widehat{V},B_{\widehat{V}},\mathbf{M}^V)/U$ of $(V,B_V,\mathbf{M}^V)/U$. Let $\phi:V\dashrightarrow\widehat{V}$ be the induced morphism, and let $g:\tilde{V}\to V$ and $\widehat{g}:\tilde{V}\to\widehat{V}$ be a common resolution such that $\widehat{g}=\phi\circ g$. Let $h:W\to X$ be a log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W and the induced map $\pi_W:W\to\tilde{V}$ is a morphism. By Lemma 2.11, there exists a proper log smooth model $(W,B_W,\mathbf{M})$ of $(X,B,\mathbf{M})$. In particular,

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E$$

for some h-exceptional $\mathbb{R}$-divisor $E\geq 0$. Assume that

$$g^*(K_V+B_V+\mathbf{M}^V_V)=\widehat{g}^*(K_{\widehat{V}}+B_{\widehat{V}}+\mathbf{M}^V_{\widehat{V}})+F.$$

Then

$$\begin{aligned}K_W+B_W+\mathbf{M}_W&=h^*(K_X+B+\mathbf{M}_X)+E\sim_{\mathbb{R}}(\pi\circ h)^*(K_V+B_V+\mathbf{M}^V_V)+E\\&=\pi_W^*g^*(K_V+B_V+\mathbf{M}^V_V)+E=\pi_W^*\widehat{g}^*(K_{\widehat{V}}+B_{\widehat{V}}+\mathbf{M}^V_{\widehat{V}})+\pi_W^*F+E.\end{aligned}$$

Since E is exceptional over X, E is very exceptional over V. Since ϕ is a birational contraction, E is very exceptional over $\widehat{V}$. Since F is exceptional over $\widehat{V}$, π_W^*F is very exceptional over $\widehat{V}$. Thus π_W^*F+E is very exceptional over $\widehat{V}$. In particular,

$$K_W+B_W+\mathbf{M}_W\sim_{\mathbb{R},\widehat{V}}\pi_W^*F+E$$

is very exceptional over $\widehat{V}$. By Lemma 2.3, we may run a $(K_W+B_W+\mathbf{M}_W)$-MMP$/\widehat{V}$ with scaling of a general ample$/\widehat{V}$ divisor which terminates with a good minimal model $(\widehat{W},B_{\widehat{W}},\mathbf{M})/\widehat{V}$ such that $K_{\widehat{W}}+B_{\widehat{W}}+\mathbf{M}_{\widehat{W}}\sim_{\mathbb{R},\widehat{V}}0$ and the induced birational map $W\dashrightarrow\widehat{W}$ exactly contracts $\mathrm{Supp}(\pi_W^*F+E)$. In particular, let $\pi_{\widehat{W}}:\widehat{W}\to\widehat{V}$ be the induced morphism, then

$$K_{\widehat{W}}+B_{\widehat{W}}+\mathbf{M}_{\widehat{W}}\sim_{\mathbb{R}}\pi_{\widehat{W}}^*(K_{\widehat{V}}+B_{\widehat{V}}+\mathbf{M}^V_{\widehat{V}}).$$

Since $(\widehat{V},B_{\widehat{V}},\mathbf{M}^V)/U$ is a good minimal model of $(V,B_V,\mathbf{M}^V)/U$, $K_{\widehat{V}}+B_{\widehat{V}}+\mathbf{M}^V_{\widehat{V}}$ is semi-ample$/U$, hence $K_{\widehat{W}}+B_{\widehat{W}}+\mathbf{M}_{\widehat{W}}$ is semi-ample$/U$. Thus $(\widehat{W},B_{\widehat{W}},\mathbf{M})/U$ is a good minimal model of $(W,B_W,\mathbf{M})/U$. By [HL21, Lemma 3.6] (=[HL21, Version 3, Lemma 3.10]), $(\widehat{W},B_{\widehat{W}},\mathbf{M})/U$ is a good minimal model of $(X,B,\mathbf{M})/U$, which implies (1).

Let $(\bar{X},\bar{B},\mathbf{M})/U$ be a good minimal model of $(X,B,\mathbf{M})/U$. By [HL21, Lemma 3.5(1)] (=[HL21, Version 3, Lemma 3.9(1)]), there exists a resolution $f:Z\to\bar{X}$ and $\widehat{f}:Z\to\widehat{W}$ of indeterminacy of the induced birational map $\bar{X}\dashrightarrow\widehat{W}$, such that

$$f^*(K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}})=\widehat{f}^*(K_{\widehat{W}}+B_{\widehat{W}}+\mathbf{M}_{\widehat{W}}).$$

In particular, any glc place of $(\bar{X},\bar{B},\mathbf{M})$ is a glc place of $(\widehat{W},B_{\widehat{W}},\mathbf{M})$, hence a glc place of $(W,B_W,\mathbf{M})$, and hence a glc place of $(X,B,\mathbf{M})$ by Lemma 2.12 as $(W,B_W,\mathbf{M})$ is a proper log smooth model of $(X,B,\mathbf{M})$. In particular, any glc place of $(\bar{X},\bar{B},\mathbf{M})$ dominates V. Moreover, the contraction $\bar{X}\to\bar{V}$ induced by $K_{\bar{X}}+\bar{B}+\mathbf{M}_{\bar{X}}$ factors through $\widehat{V}$, and the induced morphism $\widehat{V}\to\bar{V}$ is birational as $K_{\widehat{V}}+B_{\widehat{V}}+\mathbf{M}^V_{\widehat{V}}$ is big$/U$. In particular, the induced map $V\dashrightarrow\bar{V}$ is birational. Thus all glc places of $(\bar{X},\bar{B},\mathbf{M})$ dominate $\bar{V}$, hence all glc centers of $(\bar{X},\bar{B},\mathbf{M})$ dominate $\bar{V}$, which implies (2).

$$\begin{tikzcd}
Z \arrow[rr, bend left, "\widehat{f}"] \arrow[d, "f"'] & W \arrow[r, dashed] \arrow[d, "h"] \arrow[dr, "\pi_W"] & \widehat{W} \arrow[dd, bend left=60, "\pi_{\widehat{W}}"] \\
\bar{X} \arrow[d] & X \arrow[l, dashed] \arrow[d, "\pi"'] & \tilde{V} \arrow[dl, "g"'] \arrow[d, "\widehat{g}"] \\
\bar{V} & V \arrow[l, dashed] \arrow[r, dashed, "\phi"] & \widehat{V} \arrow[ll, bend left]
\end{tikzcd}$$

$\square$

5. Special termination

This section is adopted from [HL21, Version 2, Section 6].

Definition 5.1 ([HL21, Definition 6.1])**.** Let $\mathcal{I}\subset[0,1]$ and $\mathcal{I}'\subset[0,+\infty)$ be two sets. We define

$$\mathbb{S}(\mathcal{I},\mathcal{I}'):=\{1-\frac{1}{m}+\sum_j\frac{r_jb_j}{m}+\sum_i\frac{s_i\mu_i}{m}\mid m\in\mathbb{N}^+,r_i,s_i\in\mathbb{N},b_j\in\mathcal{I},\mu_j\in\mathcal{I}'\}\cap(0,1].$$

Proposition 5.2 ([HL18, Proposition 2.8])**.** *Let $\mathcal{I}\subset[0,1]$ and $\mathcal{I}'\subset[0,+\infty)$ be two sets. Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair such that $B\in\mathcal{I}$ and $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where $\mu_i\in\mathcal{I}'$ for each i and each $\mathbf{M}_i$ is nef/U $\boldsymbol{b}$-Cartier. Then for any glc center S of $(X,B,\mathbf{M})$, the g-pair $(S,B_S,\mathbf{M}^S)/U$ given by the adjunction*

$$K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S$$

is gdlt, and $B_S\in\mathbb{S}(\mathcal{I},\mathcal{I}')$.

Definition 5.3 (Difficulty, [HL18, Definition 4.3])**.** Let $\mathcal{I}$ and $\mathcal{I}'$ be two finite sets of non-negative real numbers. Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair such that $B\in\mathcal{I}$ and $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where $\mu_i\in\mathcal{I}'$ for each i and each $\mathbf{M}_i$ is nef/U $\boldsymbol{b}$-Cartier. For any glc center S of $(X,B,\mathbf{M})$ of dimension ≥ 1, let $(S,B_S,\mathbf{M}^S)$ be the g-pair given by the generalized adjunction

$$K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S,$$

then we define

$$\begin{aligned}d_{\mathcal{I},\mathcal{I}'}(S,B_S,\mathbf{M}^S):=&\sum_{\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')}\#\{E\mid a(E,B_S,\mathbf{M}^S)<1-\alpha,\operatorname{center}_S E\not\subset\lfloor B_S\rfloor\}\\&+\sum_{\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')}\#\{E\mid a(E,B_S,\mathbf{M}^S)\leq 1-\alpha,\operatorname{center}_S E\not\subset\lfloor B_S\rfloor\}.\end{aligned}$$

Lemma 5.4. *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair and let*

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots$$

be a $(K_X+B+\mathbf{M}_X)$-MMP/U. Let $\phi_{i,j}:X_i\dashrightarrow X_j$ be the induced birational maps for each i. For any $i\geq 0$ and any glc center S_i of $(X_i,B_i,\mathbf{M})$ of dimension ≥ 1, we let $(S_i,B_{S_i},\mathbf{M}^{S_i})/U$ be the generalized pair given by the adjunction

$$K_{S_i}+B_{S_i}+\mathbf{M}^{S_i}_{S_i}:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Then we have the following.

1. *For any $i\gg 0$, $j\geq i$, and any glc center S_i of $(X_i,B_i,\mathbf{M})$, $\phi_{i,j}$ induces an isomorphism near the generic point of S_i. In particular, for any $i,j\gg 0$ and any glc center S_i of $(X_i,B_i,\mathbf{M})$, we may let $S_{i,j}$ be the strict transform of S_i on X_j.*
2. *Fix $i\gg 0$ and a glc center S_i of $(X_i,B_i,\mathbf{M})$ such that $\phi_{i,j}$ induces an isomorphism for every glc center of $(S_i,B_{S_i},\mathbf{M}^{S_i})/U$ for any $j\geq i$. Then*
   (a) *$\phi_{j,k}|_{S_{i,j}}:S_{i,j}\dashrightarrow S_{i,k}$ is an isomorphism in codimension 1 for any $j,k\gg i$, and*
   (b) *$B_{S_{i,j}}$ is the strict transform of $B_{S_{i,k}}$ for any $j,k\gg i$.*
3. *Suppose that this $(K_X+B+\mathbf{M}_X)$-MMP/U is a MMP with scaling of an $\mathbb{R}$-divisor $A\geq 0$ on X. Let*

$$\lambda_j:=\inf\{t\mid t\geq 0, K_{X_j}+B_j+tA_j+\mathbf{M}_{X_j}\text{ is nef}/U\}$$

be the scaling numbers, where A_j is the strict transform of A on X_j for each j. Fix $i\gg 0$ and a glc center S_i of $(X_i,B_i,\mathbf{M})$ such that $\phi_{j,k}|_{S_{i,j}}:S_{i,j}\dashrightarrow S_{i,k}$ is an isomorphism in codimension 1 and $B_{S_{i,j}}$ is the strict transform of $B_{S_{i,k}}$ for any $k,j\geq i$. Let T be the normalization of the image of S_i on U, $(S_i',B_{S_i'},\mathbf{M}^{S_i})$ a gdlt model of $(S_i,B_{S_i},\mathbf{M}^{S_i})$, and $A_{S_i'}$ the pullback of A_i on S_i'. Then this $(K_X+B+\mathbf{M}_X)$-MMP/U induces the following commutative diagram/T

$$\begin{array}{ccccccc}(S_i',B_{S_i'},\mathbf{M}^{S_i}) & \dashrightarrow & (S_{i,i+1}',B_{S_{i,i+1}'},\mathbf{M}^{S_i}) & \dashrightarrow\dots\dashrightarrow & (S_{i,j}',B_{S_{i,j}'},\mathbf{M}^{S_i}) & \dashrightarrow & \dots\\ \downarrow & & \downarrow & & \downarrow & & \\ (S_i,B_{S_i},\mathbf{M}^{S_i}) & \dashrightarrow & (S_{i,i+1},B_{S_{i,i+1}},\mathbf{M}^{S_i}) & \dashrightarrow\dots\dashrightarrow & (S_{i,j},B_{S_{i,j}},\mathbf{M}^{S_i}) & \dashrightarrow & \dots\end{array}$$

such that

(a)

$$(S'_i,B_{S'_i},\mathbf{M}^{S_i}) \dashrightarrow (S'_{i,i+1},B_{S'_{i,i+1}},\mathbf{M}^{S_i}) \dashrightarrow \cdots \dashrightarrow (S'_{i,j},B_{S'_{i,j}},\mathbf{M}^{S_i}) \dashrightarrow \ldots$$

is a $(K_{S'_i}+B_{S'_i}+\mathbf{M}^{S_i}_{S'_i})$-MMP/$T$ with scaling of $A_{S'_i}$. Note that it is possible that $(S'_{i,j},B_{S'_{i,j}},\mathbf{M}^{S_i}) \dashrightarrow (S'_{i,j+1},B_{S'_{i,j+1}},\mathbf{M}^{S_i})$ is the identity morphism or a composition of several steps of the $(K_{S'_{i,j}}+B_{S'_{i,j}}+\mathbf{M}^{S_i}_{S'_{i,j}})$-MMP/$T$ for some j,

(b) for any $j\geq i$, $(S'_{i,j},B_{S'_{i,j}},\mathbf{M}^{S_i})$ is a gdlt model of $(S_{i,j},B_{S_{i,j}},\mathbf{M}^{S_i})$, and

(c) let

$$\mu_j:=\inf\{t\mid t\geq 0, K_{S'_{i,j}}+B_{S'_{i,j}}+tA_{S'_{i,j}}+\mathbf{M}^{S_i}_{S'_{i,j}} \text{ is nef}/T\}$$

for each $j\geq i$, where $A_{S'_{i,j}}$ is the pullback of A_j on $S'_{i,j}$. Then $\mu_j\leq\lambda_j$ for each $j\geq i$.

Proof. Let $\mathcal{I}\subset[0,1]$ be a finite set such that $B\in\mathcal{I}$, and let $\mathcal{I}'\subset[0,+\infty)$ be a finite set such that $\mathbf{M}=\sum\mu_i\mathbf{M}_i$, where each $\mathbf{M}_i$ is nef/U $\boldsymbol{b}$-Cartier and each $\mu_i\in\mathcal{I}'$. Let $\phi_i:=\phi_{i,i+1}$ for each i.

We may assume that the MMP does not terminate, otherwise there is nothing left to prove. Possibly replacing X with X_i for $i\gg 0$, we may assume that each ϕ_i is a flip. Since the number of glc centers of $(X,B,\mathbf{M})$ is finite, possibly replacing X with X_i for $i\gg 0$, we may assume that the flipping locus of ϕ_i does not contain any glc centers. This proves (1).

We prove (2). We let $S:=S_i$. By (1), we may let $S_j:=S_{i,j}$ for any $j\geq i$. Possibly replacing X we X_i, we may assume that $i=0$. By [HL18, Proposition 2.8], for any j, the g-pair $(S_j,B_{S_j},\mathbf{M}^S)$ given by the adjunction

$$K_{S_j}+B_{S_j}+\mathbf{M}^S_{S_j}:=(K_{X_j}+B_j+\mathbf{M}_{X_j})|_{S_j}$$

is gdlt, and $B_{S_j}\in\mathbb{S}(\mathcal{I},\mathcal{I}')$. By assumption, $\phi_{j,k}$ induces an isomorphism on $\lfloor B_{S_j}\rfloor$ for any j,k. Thus for any j and any prime divisor E over S_j, $\operatorname{center}_{S_j}E\subset\lfloor B_{S_j}\rfloor$ if and only if $\operatorname{center}_{S_{j+1}}E\subset\lfloor B_{S_{j+1}}\rfloor$. By the negativity lemma, $a(E,S_j,B_{S_j},\mathbf{M}^S)\leq a(E,S_{j+1},B_{S_{j+1}},\mathbf{M}^S)$ for each j and any prime divisor E over S_j. Thus

$$d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)\geq d_{\mathcal{I},\mathcal{I}'}(S_{j+1},B_{S_{j+1}},\mathbf{M}^S)$$

for each j. Moreover, for any j such that S_j and S_{j+1} are not isomorphic in codimension 1, if there exists a prime divisor E on S_{j+1} that is exceptional over S_j, then

$$1-\alpha=a(E,S_{j+1},B_{S_{j+1}},\mathbf{M}^S)>a(E,S_j,B_{S_j},\mathbf{M}^S)$$

for some $\alpha\in\mathbb{S}(\mathcal{I},\mathcal{I}')$, and hence

$$d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)>d_{\mathcal{I},\mathcal{I}'}(S_{j+1},B_{S_{j+1}},\mathbf{M}^S).$$

By [HL18, Remark 4.4], $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)<+\infty$. Thus possibly replacing X with X_j for some $j\gg 0$, we may assume that $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)=d_{\mathcal{I},\mathcal{I}'}(S_k,B_{S_k},\mathbf{M}^S)$ for any j,k. Thus $S_j\dashrightarrow S_{j+1}$ does not extract any divisor for any j. In particular, $\rho(S_{j+1})\leq\rho(S_j)$, and $\rho(S_{j+1})<\rho(S_j)$ if $S_j\dashrightarrow S_{j+1}$ contracts a divisor. Thus possibly replacing X with X_j for some $i\gg 0$, we may assume that S_j and S_{j+1} are isomorphic in codimension 1 for each j, which implies (2.a). Since $d_{\mathcal{I},\mathcal{I}'}(S_j,B_{S_j},\mathbf{M}^S)=d_{\mathcal{I},\mathcal{I}'}(S_k,B_{S_k},\mathbf{M}^S)$ for any j,k, (2.b) follows from (2.a).

We prove (3). Since $i\gg 0$, possibly replacing X with X_i, we may assume that $i=0$ and ϕ_j is a flip for every j. We let $S:=S_0, S':=S'_0, S_j:=S_{0,j}$, and $S'_j:=S'_{0,j}$ for every j. We let $X_j\to Z_j\leftarrow X_{j+1}$ be each flip and let T_j be the normalization of the image of S_j on Z_j for each j. Then we have an induced birational map $S_j\dashrightarrow S_{j+1}$ for each j.

Since ϕ_0 is a $(K_{X_0}+B_0+\mathbf{M}_{X_0})$-flip/$U$, $X_1\to Z_0$ is $(K_{X_1}+B_1+\mathbf{M}_{X_1})$-positive and $K_{S_1}+B_{S_1}+\mathbf{M}^S_{S_1}$ is ample/T_0. In particular, $(S_1,B_{S_1},\mathbf{M}^S)/T_0$ is a weak glc model of $(S_0,B_{S_0},\mathbf{M}^S)$. By [HL21, Lemmas 3.5, 3.8] (=[HL21, Version 3, Lemmas 3.9, 3.15]) and Theorem 2.14, we may run a $(K_{S_0'}+B_{S_0'}+\mathbf{M}^S_{S_0'})$-MMP$/T_0$ with scaling of an ample$/T_0$ divisor, which terminates with a good minimal model of $(S_0', B_{S_0'}, \mathbf{M}^S)/T_0$. By [HL21, Lemma 3.5] (=[HL21, Version 3, Lemma 3.9]), $(S_0', B_{S_0'}, \mathbf{M}^S)$ is a gdlt model of $(S_1, B_{S_1}, \mathbf{M}^S)$. Since

$$K_{S_0'}+B_{S_0'}+\lambda_0 A_{S_0'}+\mathbf{M}^S_{S_0'}\equiv_{T_0} 0,$$

this MMP is also a $(K_{S_0'}+B_{S_0'}+\mathbf{M}^S_{S_0'})$-MMP$/T_0$ with scaling of $\lambda_0 A_{S_0'}$. We may replace $(S_0, B_{S_0}, \mathbf{M}^S)/T$ with $(S_1, B_{S_1}, \mathbf{M}^S)/T$ and continue this process. This gives us the desired $(K_{S_0'}+B_{S_0'}+\mathbf{M}^S_{S_0'})$-MMP$/T$ with scaling of $A_{S_0'}$, which gives the commutative diagram, and proves (3.a) and (3.b). For each j, since $K_{S_j'}+B_{S_j'}+\lambda_j A_{S_j'}+\mathbf{M}^S_{S_j'}\equiv_{T_j} 0$, $K_{S_j'}+B_{S_j'}+\lambda_j A_{S_j'}+\mathbf{M}^S_{S_j'}$ is nef, hence $\mu_j\le\lambda_j$, and we get (3.c). □

6. Apply Nakayama-Zariski decomposition

This similar is similar to [Has22, Section 3, before Theorem 3.14].

Lemma 6.1 (cf. [Has22, Lemma 3.5])**.** *Let $(X,B,\mathbf{M})/U$ and $(X',B',\mathbf{M})/U$ be NQC gdlt g-pairs with a birational map $\phi: X \dashrightarrow X'$ over U such that $\phi_*\mathbf{M}=\mathbf{M}$. Let S and S' be glc centers of $(X,B,\mathbf{M})$ and $(X',B',\mathbf{M})$ respectively, such that ϕ is an isomorphism near the generic point of S, and $\phi|_S: S\dashrightarrow S'$ defines a birational map$/U$. Suppose that*

1. *$K_X+B+\mathbf{M}_X$ is pseudo-effective$/U$,*
2. *for any prime divisor D' on X', $a(D',X',B',\mathbf{M})\le a(D',X,B,\mathbf{M})$, and*
3. *for every prime divisor P over X such that $a(P,X,B,\mathbf{M})<1$ and $\operatorname{center}_X(P)\cap S\neq\emptyset$, then $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$.*

Let $(S,B_S,\mathbf{M}^S)/U$ and $(S',B_{S'},\mathbf{M}^S)/U$ be the g-pairs induced by the adjunctions $K_S+B_S+\mathbf{M}^S_S:=(K_X+B+\mathbf{M}_X)|_S$ and $K_{S'}+B_{S'}+\mathbf{M}^S_{S'}:=(K_{X'}+B'+\mathbf{M}_{X'})|_{S'}$. Then $a(Q,S',B_{S'},\mathbf{M}^S)\le a(Q,S,B_S,\mathbf{M}^S)$ for all prime divisors Q on S'.

Proof. We follow [Has22, Proof of Lemma 3.5]. Let $p:W\to X$ and $q:W\to X'$ be a resolution of indeterminacy such that $\mathbf{M}$ descends to W, and let S_W be the strict transform of S on W. Then $p_S:=p|_{S_W}$ and and $q_S:=q|_{S_W}$ is a common resolution of $\phi|_S$. We may write

$$p^*(K_X+B+\mathbf{M}_X)=q^*(K_{X'}+B'+\mathbf{M}_{X'})+G_+-G_-$$

such that $G_+\ge 0, G_-\ge 0$, and $G_+\wedge G_-=0$. By (2), G_+ is q-exceptional. Since $K_X+B+\mathbf{M}_X$ is pseudo-effective$/U$, $K_{X'}+B'+\mathbf{M}_{X'}$ is pseudo-effective$/U$. Since S and S' are glc centers of $(X,B,\mathbf{M})$ and $(X',B',\mathbf{M})$ respectively, S_W is not contained in $\operatorname{Supp}G_+$ and $\operatorname{Supp}G_-$.

We can write $G_+=G_0+G_1$, where all components of G_0 intersect S_W and $G_1|_{S_W}=0$. For any component E of $\operatorname{Supp}G_0$, we have

$$\begin{aligned}
&\sigma_E(X/U,K_X+B+\mathbf{M}_X)\\
=&\sigma_E(W/U,p^*(K_X+B+\mathbf{M}_X)) && \text{(Definition 3.5)}\\
=&\sigma_E(W/U,p^*(K_X+B+\mathbf{M}_X))+\operatorname{mult}_E(G_-) && (G_-\wedge G_+=0, E\subset\operatorname{Supp}G_+)\\
\ge&\sigma_E(W/U,p^*(K_X+B+\mathbf{M}_X)+G_-) && \text{(Lemma 3.6(1))}\\
=&\sigma_E(W/U,q^*(K_{X'}+B'+\mathbf{M}_{X'})+G_+) &&\\
=&\sigma_E(W/U,q^*(K_{X'}+B'+\mathbf{M}_{X'}))+\operatorname{mult}_E(G_+)>0 && (\text{Lemma 3.3(2)}, E\subset\operatorname{Supp}G_+).
\end{aligned}$$

By (3), $a(E,X,B,\mathbf{M})\ge 1$. Since $E\subset\operatorname{Supp}G_+$, $a(E,X',B',\mathbf{M})>a(E,X,B,\mathbf{M})\ge 1$.

Claim 6.2. *$E|_{S_W}$ is exceptional over S'.*

Grant the claim for the time being. We have $G_0|_{S_W}$ is exceptional over S', hence for any prime divisor Q on S', $\operatorname{mult}_Q(G_0|_{S_W}) = 0$. Since we have $G_1|_{\bar{S}} = 0$ and

$$\begin{aligned}&p_S^*(K_S + B_S + \mathbf{M}_S^S) - q_S^*(K_{S'} + B_{S'} + \mathbf{M}_{S'}^S)\\ =&p_S^*((K_X + B + \mathbf{M}_X)|_S) - q_S^*((K_{X'} + B' + \mathbf{M}_{X'})|_{S'}) = (G_0 + G_1)|_{S_W} - G_-|_{S_W},\end{aligned}$$

we have

$$a(Q, S', B_{S'}, \mathbf{M}) - a(Q, S, B_S, \mathbf{M}^S) = \operatorname{mult}_Q((G_0 + G_1)|_{S_W} - G_-|_{S_W}) = \operatorname{mult}_Q(-G_-|_{S_W}) \leq 0$$

for any prime divisor Q on S', and the lemma follows. □

Proof of Claim 6.2. We may write

$$K_W + B_W + \mathbf{M}_W = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + G'$$

such that $B_W \geq 0, G' \geq 0$, G' is q-exceptional, and $B_W \wedge G' = 0$. By our construction, $(B_W - S_W)|_{S_W}$ and $G'|_{S_W}$ does not have any common component. By [BZ16, Definition 4.7],

$$0 \leq B_S = (q_S)_*((B_W - S_W - G')|_{S_W}) = (q_S)_*(B_W - S_W) - (q_S)_*G'|_{S_W},$$

hence $(q_S)_*G'|_{S_W} = 0$. Thus $G'|_{S_W}$ is exceptional over S'. Since $a(E', X', B', \mathbf{M}) > 1$, E is a component of G', hence $E|_{S_W}$ is exceptional over S', and the claim follows. □

Lemma 6.3 (cf. [Has22, Lemma 3.6], [HMX18, Lemma 5.3])**.** *Let $(X, B_1, \mathbf{M})/U$ and $(X, B_2, \mathbf{M})/U$ be $\mathbb{Q}$-factorial NQC gdlt g-pairs, such that $K_X + B_1 + \mathbf{M}_X$ is pseudo-effective/U and $0 \leq B_1 - B_2 \leq N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$. Then $(X, B_1, \mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(X, B_2, \mathbf{M})/U$ has a log minimal model (resp. good minimal model).*

Proof. $N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$ is well-defined by Lemma 3.4(1).

First we assume that $(X, B_1, \mathbf{M})/U$ has a log minimal model (resp. good minimal model). By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]), we may run a $(K_X + B_1 + \mathbf{M}_X)$-MMP/$U$ which terminates with a log minimal model (resp. good minimal model) $(X', B', \mathbf{M})/U$ with induced birational map $\phi : X \dashrightarrow X'$ over U. By Lemma 3.8(2.b), f contracts every component of $N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$. Thus B' is also the strict transform of B on X'.

Let $p : W \to X$ and $q : W \to X'$ be a resolution of indeterminacy of ϕ, and let

$$p^*(K_X + B_1 + \mathbf{M}_X) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + E,$$

then by Lemmas 3.6(3) and 3.3(2)(3), $p_*E = N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$. Suppose that

$$p^*(K_X + B_2 + \mathbf{M}_X) = q^*(K_{X'} + B' + \mathbf{M}_{X'}) + F,$$

then

$$F = E - p^*(B_1 - B_2) \geq E - p^*N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X) = E - p^*p_*E.$$

Since $E - p^*p_*E$ is p-exceptional, $p_*F \geq 0$. By the negativity lemma, $F \geq 0$. Thus $(X', B', \mathbf{M})/U$ is a weak glc model of $(X, B_2, \mathbf{M})/U$. By [HL21, Lemmas 3.5(2), 3.8] (=[HL21, Version 3, Lemmas 3.9(2), 3.15]), $(X, B_2, \mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Now we assume that $(X, B_2, \mathbf{M})/U$ has a log minimal model (resp. good minimal model). By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]), we may run a $(K_X + B_2 + \mathbf{M}_X)$-MMP/$U$ which terminates with a log minimal model (resp. good minimal model) $(X', B', \mathbf{M})/U$ with induced birational map $\phi : X \dashrightarrow X'$ over U. Let $C := B_1 - B_2$, then ϕ is also a $(K_X + B_2 + \epsilon C + \mathbf{M}_X)$-MMP/$U$ for any $0 < \epsilon \ll 1$. Let C' be the strict transform of C on X'. By Lemma 3.6(5) and [HL18, Lemma 3.17], we may pick $0 < \epsilon \ll 1$, such that

- $\operatorname{Supp} N_\sigma(X/U, K_X + B_2 + \epsilon C + \mathbf{M}_X) = \operatorname{Supp} N_\sigma(X/U, K_X + B_1 + \mathbf{M}_X)$, and
- any partial $(K_{X'} + B' + \epsilon C' + \mathbf{M}_{X'})$-MMP/$U$ is $(K_{X'} + B' + \mathbf{M}_{X'})$-trivial/$U$.

We run a $(K_{X'}+B'+\epsilon C'+\mathbf{M}_{X'})$-MMP$/U$ with scaling of an ample$/U$ $\mathbb{R}$-divisor. By Lemma 2.5, after finitely many steps we get a birational map $\psi: X' \dashrightarrow X''$ such that $K_{X''}+B''+\epsilon C''+\mathbf{M}_{X''}$ is a limit of movable$/U$ $\mathbb{R}$-divisors, where B'' and C'' are the strict transforms of B' and C' on X'' respectively. By Lemma 3.8(2.b), the set of $(\psi\circ\phi)$-exceptional divisors is exactly $\operatorname{Supp} N_\sigma(X/U, K_X+B_2+\epsilon C+\mathbf{M}_X)=\operatorname{Supp} N_\sigma(X/U, K_X+B_1+\mathbf{M}_X)$. Thus $C''=0$, B'' is also the strict transform of B_1 on X'', and $K_{X''}+B''+\mathbf{M}_{X''}$ is nef$/U$ (resp. semi-ample$/U$). By Lemma 3.9, $(X'',B'',\mathbf{M})/U$ is a log minimal model of $(X,B_1,\mathbf{M})/U$. The lemma follows from [HL21, Lemma 3.5(2)] (=[HL21, Version 3, Lemma 3.9(2)]). □

Lemma 6.4 (cf. [Has22, Lemma 3.6])**.** *Let $(X,B,\mathbf{M})/U$ and $(Y,B_Y,\mathbf{M})/U$ be NQC glc g-pairs and $f: Y\to X$ a birational morphism, such that*

(1) $K_X+B+\mathbf{M}_X$ is pseudo-effective$/U$, and

(2) for any prime divisor D on Y,

$$0\leq a(D,Y,B_Y,\mathbf{M})-a(D,X,B,\mathbf{M})\leq \sigma_D(X/U,K_X+B+\mathbf{M}_X).$$

Then $(X,B,\mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(Y,B_Y,\mathbf{M})/U$ has a log minimal model (resp. good minimal model).

Proof. Let $g: W\to Y$ be a log resolution of $(Y,\operatorname{Supp} B_Y)$ such that $\mathbf{M}$ descends to W and $h:=f\circ g$ is a log resolution of $(X,\operatorname{Supp} B)$. Let $B_W:=h^{-1}_*B+\operatorname{Supp}\operatorname{Exc}(h)$ and $B'_W:=g^{-1}_*B_Y+\operatorname{Supp}\operatorname{Exc}(g)$, then we have

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E$$

for some $E_W\geq 0$ that is exceptional$/X$. By Lemma 3.3(1)(2),

$$\sigma_P(W/U,K_W+B_W+\mathbf{M}_W)=\sigma_P(X/U,K_X+B+\mathbf{M}_X)+\operatorname{mult}_P E$$

for any prime divisor P on W.

Claim 6.5. *For any prime divisor P on W,*

$$0\leq a(P,W,B'_W,\mathbf{M})-a(P,W,B_W,\mathbf{M})\leq \sigma_P(W/U,K_W+B_W+\mathbf{M}_W).$$

Grant Claim 6.5 for the time being. By Claim 6.5 and Theorem 2.14, possibly replacing $(X,B,\mathbf{M})/U$ and $(Y,B_Y,\mathbf{M})/U$ with $(W,B_W,\mathbf{M})/U$ and $(W,B'_W,\mathbf{M})/U$ respectively, we may assume that $(X,B,\mathbf{M})$ and $(Y,B_Y,\mathbf{M})$ are $\mathbb{Q}$-factorial gdlt and $X=Y$. The lemma follows from Lemma 6.3. □

Proof of Claim 6.5. For any prime divisor P on W, one of the following cases holds:

Case 1. P is not exceptional over X. In this case,

$$a(P,W,B'_W,\mathbf{M})-a(P,W,B_W,\mathbf{M})=a(P,Y,B_Y,\mathbf{M})-a(P,X,B,\mathbf{M})$$

and the claim follows.

Case 2. P is exceptional over X but not exceptional over Y. In this case, $a(P,W,B_W,\mathbf{M})=0$, $a(P,W,B'_W,\mathbf{M})=a(P,Y,B_Y,\mathbf{M})$, and $a(P,X,B,\mathbf{M})=\operatorname{mult}_P E$, so

$$0\leq a(P,Y,B_Y,\mathbf{M})=a(P,W,B'_W,\mathbf{M})-a(P,W,B_W,\mathbf{M})=a(P,Y,B_Y,\mathbf{M})-a(P,X,B,\mathbf{M}),$$

and

$$\begin{aligned} a(P,Y,B_Y,\mathbf{M}) &\leq \sigma_P(X/U,K_X+B+\mathbf{M}_X)+a(P,X,B,\mathbf{M})\\ &=\sigma_P(X/U,K_X+B+\mathbf{M}_X)+\operatorname{mult}_P E=\sigma_P(W/U,K_W+B_W+\mathbf{M}_W)\end{aligned}$$

and the claim follows.

Case 3. P is exceptional over Y. In this case, $a(P,W,B_W,\mathbf{M})=a(P,W,B'_W,\mathbf{M})=0$, and the claim follows. □

Lemma 6.6 (cf. [Has22, Lemma 3.8]). *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair with induced morphism $\pi: X\to U$ such that U is quasi-projective. Let S be a subvariety of X, and*
$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow (X_1,B_1,\mathbf{M})\dashrightarrow\dots\dashrightarrow (X_n,B_n,\mathbf{M})\dashrightarrow\dots$$
a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of an ample/U $\mathbb{R}$-divisor $A\geq 0$. Let
$$\lambda_i:=\inf\{t\mid t\geq 0, K_{X_i}+B_i+\mathbf{M}_{X_i}+tA_i \text{ is nef}/U\}$$
be the scaling numbers, where A_i is the strict transform of A on X_i. Suppose that

- *each step of this MMP is an isomorphism on a neighborhood of S, and*
- $\lim_{i\to+\infty}\lambda_i=0$,

then

(1) *for any π-ample $\mathbb{R}$-divisor H on X and any closed point $x\in S$, there exists an $\mathbb{R}$-divisor E such that $0\leq E\sim_{\mathbb{R},U} K_X+B+\mathbf{M}_X+H$ and $x\notin \operatorname{Supp} E$, and*
(2) *for any prime divisor P over X such that $\operatorname{center}_X P\cap S\neq\emptyset$, $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$.*

Proof. (1) follows from [Has22, Lemma 3.8] and (2) follows from (1) and Lemma 3.6(4). □

Lemma 6.7 ([Has22, Lemma 3.9]). *Let $(X,B,\mathbf{M})/U$ and $(X',B',\mathbf{M})/U$ be two NQC glc g-pairs and $\phi: X\dashrightarrow X'$ a birational map such that $\phi_*\mathbf{M}=\mathbf{M}$. Suppose that*

- $a(P,X,B,\mathbf{M})\leq a(P,X',B',\mathbf{M})$ *for any prime divisor P on X, and*
- $a(P',X',B',\mathbf{M})\leq a(P',X,B,\mathbf{M})$ *for any prime divisor P' on X',*

then

(1) *$K_X+B+\mathbf{M}_X$ is abundant/U if and only if $K_{X'}+B'+\mathbf{M}_{X'}$ is abundant/U, and*
(2) *$(X,B,\mathbf{M})/U$ has a log minimal model (resp. good minimal model) if and only if $(X',B',\mathbf{M})/U$ has a log minimal model (resp. good minimal model).*

Proof. Let $p:W\to X$ and $q:W\to X'$ be a resolution of indeterminacy such that $\mathbf{M}$ descends to W, p is a log resolution of $(X,\operatorname{Supp} B)$, and q is a log resolution of $(X',\operatorname{Supp} B')$. Let
$$B_W:=\sum_{D\text{ is a prime divisor on }W}\max\{1-a(D,X,B,\mathbf{M}),1-a(D,X',B',\mathbf{M}),0\}D.$$
Then $(W,B_W,\mathbf{M})$ is glc and (W,B_W) is log smooth. By construction, there exists a p-exceptional $\mathbb{R}$-divisor $E\geq 0$ and a q-exceptional $\mathbb{R}$-divisor $F\geq 0$, such that
$$E+p^*(K_X+B+\mathbf{M}_X)=K_W+B_W+\mathbf{M}_W=q^*(K_{X'}+B'+\mathbf{M}_{X'})+F.$$
(1) follows from Lemma 2.2(3) and (2) follows from Theorem 2.14. □

7. A special log minimal model

The purpose of this section is to prove Theorem 7.1, an analogue of [Has22, Theorem 3.14].

Theorem 7.1 (cf. [Has22, Theorem 3.14]). *Let $(X,B,\mathbf{M})/U$ be an NQC gdlt g-pair such that*

- *$K_X+B+\mathbf{M}_X$ is pseudo-effective/U and abundant/U,*
- *for any glc center S of $(X,B,\mathbf{M})$, $(K_X+B+\mathbf{M}_X)|_S$ is nef/U, and*
- *for any prime divisor P over X such that $a(P,X,B,\mathbf{M})<1$ and $\operatorname{center}_X P\cap \operatorname{Ngklt}(X,B,\mathbf{M})\neq\emptyset$, $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$.*

Then $(X,B,\mathbf{M})/U$ has a log minimal model.

Proof. **Step 1**. In this step, we show that we may replace $(X,B,\mathbf{M})$ with a $\mathbb{Q}$-factorial gdlt model and find two $\mathbb{R}$-divisors $G\geq 0,H\geq 0$, and a real number $1>t_0>0$, such that

(I) $K_X+B+\mathbf{M}_X\sim_{\mathbb{R},U} G+H$,
(II) $\operatorname{Supp} G\subset \operatorname{Supp}\lfloor B\rfloor$, and
(III) for any $t\in(0,t_0]$,
  (III.1) $(X,B+tH,\mathbf{M})/U$ is gdlt, $\operatorname{Supp} N_\sigma(X/U,K_X+B+tH+\mathbf{M}_X)$ is well-defined and does not depend on t, and

(III.2) $(X,B-tG,\mathbf{M})/U$ has a good minimal model.

Since $K_X+B+\mathbf{M}_X$ is pseudo-effective/U and abundant/U, $K_X+B+\mathbf{M}_X\sim_{\mathbb{R},U}D\geq 0$ for some $\mathbb{R}$-divisor D. Let $X\dashrightarrow V$ be the Iitaka fibration/U associated to D, then $\dim V-\dim U=\kappa_\sigma(X/U,K_X+B+\mathbf{M}_X)$. Let $h:W\to X$ be a log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W and the induced map $\psi:W\dashrightarrow V$ is a morphism. Then we may write

$$K_W+B_W+\mathbf{M}_W=h^*(K_X+B+\mathbf{M}_X)+E$$

such that $B_W\geq 0,E\geq 0$, and $B_W\wedge E=0$. By Lemma 2.18,

(i) $\kappa_\sigma(W/U,K_W+B_W+\mathbf{M}_W)=\dim V-\dim U$ and $\kappa_\sigma(W/V,K_W+B_W+\mathbf{M}_W)=0$.

Thus by construction $K_W+B_W+\mathbf{M}_W$ is $\mathbb{R}$-linearly equivalent/U to the sum of an effective $\mathbb{R}$-divisor and the pullback of an ample/U $\mathbb{R}$-divisor on V. In particular, we may find $0\leq D_W\sim_{\mathbb{R},U}K_W+B_W+\mathbf{M}_W$ such that $\operatorname{Supp}D_W$ contains all glc centers of $(W,B_W,\mathbf{M})$ that are vertical over V.

Let $(\bar X,\bar B,\mathbf{M})$ be a proper log smooth model of $(W,B_W,\mathbf{M})$ with induced morphism $g:\bar X\to W$, such that g is a log resolution of (W,B_W+D_W), and

$$K_{\bar X}+\bar B+\mathbf{M}_{\bar X}=g^*(K_W+B_W+\mathbf{M}_W)+\bar E$$

for some $\bar E\geq 0$. By Lemma 2.19, possibly replacing $(\bar X,\bar B,\mathbf{M})$ with a higher model, we may assume that $\bar B=\bar B^h+\bar B^v$, such that

(ii) $\bar B^h\geq 0$ and $\bar B^v$ is reduced,
(iii) $\bar B^v$ is vertical over V, and
(iv) for any $t\in(0,1]$, all glc centers of $(\bar X,\bar B-t\bar B^v,\mathbf{M})$ dominate V.

Let $\bar D:=g^*D_W+\bar E$, then $(\bar X,\bar B+\bar D)$ is log smooth and $\bar D\sim_{\mathbb{R},U}K_{\bar X}+\bar B+\mathbf{M}_{\bar X}$. By Lemma 2.12, $\operatorname{Supp}\bar B^v\subset\operatorname{Supp}\bar D$. Thus we may way write $\bar D=\bar G+\bar H$, such that

(v) $K_{\bar X}+\bar B+\mathbf{M}_{\bar X}\sim_{\mathbb{R},U}\bar G+\bar H$,
(vi) $\operatorname{Supp}\bar B^v\subset\operatorname{Supp}\bar G\subset\operatorname{Supp}\lfloor\bar B\rfloor$, and
(vii) no component of $\bar H$ is contained in $\lfloor\bar B\rfloor$ and $(\bar X,\bar B+\bar H)$ is log smooth.

We fix a real number $t_1\in(0,1)$ such that $\bar B-t_0\bar G\geq 0$. For any $t\in(0,t_1]$, by (ii)(iii)(iv)(vi), any glc center of $(\bar X,\bar B-t\bar G,\mathbf{M})$ dominates V. By (i)(v) and Lemma 2.2(2), $\kappa_\sigma(\bar X/U,K_{\bar X}+\bar B-t\bar G+\mathbf{M}_{\bar X})=\dim V-\dim U$ and $\kappa_\sigma(\bar X/V,K_{\bar X}+\bar B-t\bar G+\mathbf{M}_{\bar X})=\kappa_\iota(\bar X/V,K_{\bar X}+\bar B-t\bar G+\mathbf{M}_{\bar X})=0$. By Proposition 4.3,

(viii) $(\bar X,\bar B-t\bar G,\mathbf{M})/U$ has a good minimal model for any $t\in(0,t_1]$.

Since $(\bar X,\bar B,\mathbf{M})$ is a log smooth model of $(X,B,\mathbf{M})$, we may run a $(K_{\bar X}+\bar B+\mathbf{M}_{\bar X})$-MMP/$X$ which terminates with a gdlt model $(Y,B_Y,\mathbf{M})$ of $(X,B,\mathbf{M})$ with induced morphism $f:Y\to X$ and birational map $\phi:\bar X\dashrightarrow Y$. Let G_Y and H_Y be the strict transforms of $\bar G$ and $\bar H$ on Y respectively, then $K_Y+B_Y+\mathbf{M}_Y\sim_{\mathbb{R},U}G_Y+H_Y$. By (vii) and Lemma 2.4, there exists $0<t_2<t_1$ such that $(\bar Y,\bar B+t_2\bar H,\mathbf{M})$ is gdlt and ϕ is a $(K_{\bar X}+\bar B+t\bar H+\mathbf{M}_{\bar X})$-MMP/$X$ as well as a $(K_{\bar X}+\bar B-t\bar G+\mathbf{M}_{\bar X})$-MMP/$X$ for any $t\in(0,t_2]$. Then $(Y,B_Y+t_2H_Y,\mathbf{M})$ is gdlt, and by (viii) and [HL21, Theorem 2.8, Lemma 3.5(2)] (=[HL21, Version 3, Theorem 2.24, Lemma 3.9(2)]), $(Y,B_Y-tG,\mathbf{M}))/U$ has a good minimal model for any $t\in(0,t_2]$. By Lemma 3.7, we may pick $0<t_0<t_2$ such that $\operatorname{Supp}N_\sigma(Y/U,K_Y+B_Y+tH_Y+\mathbf{M}_Y)$ is well-defined and does not depend on t for any $t\in(0,t_0]$.

We may replace $(X,B,\mathbf{M})$ with $(Y,B_Y,\mathbf{M})$ and let $G:=G_Y$ and $H:=H_Y$, and assume that $(X,B,\mathbf{M}),G,H$ and t_0 satisfy (I)(II)(III). In the following, we forget all other auxiliary varieties and divisors constructed in this step.

Step 2. For any $t\in(0,t_0]$, by (III.2), $(X,B-\frac{t}{1+t}G,\mathbf{M})/U$ has a good minimal model. Since

$$K_X+B+tH+\mathbf{M}_X\sim_{\mathbb{R},U}(1+t)(K_X+B-\frac{t}{1+t}G+\mathbf{M}_X),$$

by (III.1) and [HL21, Theorem 2.8, Lemmas 3.5(2), 4.3] (=[HL21, Version 3, Theorem 2.24, Lemma 3.9(2), 4.2]), we may run a $(K_X+B+tH+\mathbf{M}_X)$-MMP/U $\phi_t:X\dashrightarrow X_t$ which terminates with a good minimal model $(X_t, B_t + tH_t, \mathbf{M})/U$ of $(X, B + tM, \mathbf{M})/U$. By (III.1) and Lemma 3.8(2.b), divisors contracted by ϕ_t is $\operatorname{Supp} N_\sigma(X/U, K_X + B + t_0H + \mathbf{M}_X)$ and is independent of t. We let $X_0 := X_{t_0}, B_0 := B_{t_0}$, and $H_0 := H_{t_0}$. Then X_0 and X_t are isomorphic in codimension 1, and $K_{X_0} + B_0 + \mathbf{M}_{X_0}$ is a limit of movable/U $\mathbb{R}$-divisors. By the negativity lemma, $(X_t, B_t + tH_t, \mathbf{M})/U$ is a good minimal model of $(X_0, B_0 + tH_0, \mathbf{M})/U$ for any $t \in (0, t_0]$.

Claim 7.2. *We may run a $(K_{X_0} + B_0 + \mathbf{M}_{X_0})$-MMP/$U$ with scaling of H_0, such that we either get a log minimal model of $(X_0, B_0, \mathbf{M})/U$ or a sequence of flips*

$$(X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \dots (X_i, B_i, \mathbf{M}) \dashrightarrow \dots$$

with scaling numbers

$$\lambda_i := \inf\{t \mid t \geq 0, K_{X_i} + B_i + tH_i + \mathbf{M}_{X_i} \text{ is nef}/U\}$$

where H_i is the strict transform of H on X_i, such that

1. *either the MMP/U terminates, or $\lim_{i\to+\infty} \lambda_i = 0$,*
2. *for any $i \geq 1$ and $\lambda \in [\lambda_i, \lambda_{i-1}]$, $(X_i, B_i + \lambda H_i, \mathbf{M})/U$ is a good minimal model of both $(X, B + \lambda H, \mathbf{M})$ and $(X_0, B_0 + \lambda H_0, \mathbf{M})/U$, and*
3. *the MMP only contracts sub-varieties of $\operatorname{Supp}\lfloor B_0\rfloor$.*

Proof. Since $K_{X_0} + B_0 + \mathbf{M}_{X_0}$ is a limit of movable/U $\mathbb{R}$-divisors, by Lemma 2.6, any $(K_{X_0} + B_0 + \mathbf{M}_{X_0})$-MMP/$U$ only contains flips. (1) follows from Lemma 2.9. For any $i \geq 1$ and $\lambda \in [\lambda_i, \lambda_{i-1}]$, $(X_i, B_i + \lambda H_i, \mathbf{M})$ is gdlt and $K_{X_i} + B_i + \lambda H_i + \mathbf{M}_{X_i}$ is nef/U. Since the induced birational maps $X_0 \dashrightarrow X_\lambda$ and $X_i \to X_\lambda$ are both small, by Lemma 3.9 and [HL21, Lemma 3.5(2)] (=[HL21, Version 3, Lemma 3.9(2)]), we get (2).

Let $X_i \to Z_i \leftarrow X_{i+1}$ be the i-th step of the MMP where $X_i \to Z_i$ the flipping contraction. Then for any flipping curve C_i of $X_i \to Z_i$, we have $(K_{X_i} + B_i + \mathbf{M}_{X_i}) \cdot C_i < 0$ and $H_i \cdot C_i > 0$. Let G_i be the strict transform of G on X_i, then $0 > (K_{X_i} + B_i + \mathbf{M}_{X_i} - H_i) \cdot C_i = G_i \cdot C_i$. Thus $C_i \subset \operatorname{Supp} G_i$. Since $\operatorname{Supp} G \subset \operatorname{Supp}\lfloor B\rfloor$, $\operatorname{Supp} G_i \subset \operatorname{Supp}\lfloor B_i\rfloor$, and we get (3). $\square$

Claim 7.3. *Let*

$$(X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \dots (X_i, B_i, \mathbf{M}) \dashrightarrow \dots,$$

λ_i, and H_i be the MMP/U, the scaling numbers, and the strict transform of H on X_i for each i as in Claim 7.2 respectively. If the MMP/U terminates, then Theorem 7.1 holds.

Proof. Let $\lambda_{-1} := t_0$. If the MMP/U terminates, then $\lambda_{l-1} > \lambda_l = 0$ for some $l \in \mathbb{N}$. By Claim 7.2(2), for any $t \in (0, \lambda_{l-1}]$, $K_{X_l} + B_l + tH_l + \mathbf{M}_{X_l}$ is nef/U, and $a(P, X, B + tH, \mathbf{M}) \leq a(P, X_l, B_l + tH_l, \mathbf{M})$ for any prime divisor P on X that is exceptional/X_l. Let $t \to 0$, we have that $K_{X_l} + B_l + \mathbf{M}_{X_l}$ is nef/U and $a(P, X, B, \mathbf{M}) \leq a(P, X_l, B_l, \mathbf{M})$ for any prime divisor P on X that is exceptional/X_l. Thus $(X_l, B_l, \mathbf{M})/U$ is a weak glc model of $(X, B, \mathbf{M})/U$. The Claim follows from [HL21, Lemma 3.8] (=[HL21, Version 3, Lemma 3.15]). $\square$

Proof of Theorem 7.1 continued. In the following, we let

$$(X_0, B_0, \mathbf{M}) \dashrightarrow (X_1, B_1, \mathbf{M}) \dashrightarrow \dots (X_i, B_i, \mathbf{M}) \dashrightarrow \dots,$$

λ_i, and H_i be the MMP/U, the scaling numbers, and the strict transform of H on X_i for each i as in Claim 7.2 respectively.

Step 3. For every i and glc center S_i of $(X_i, B_i, \mathbf{M})$, we let $(S_i, B_{S_i}, \mathbf{M}^{S_i})$ be the g-pair induced by the adjunction

$$K_{S_i} + B_{S_i} + \mathbf{M}^{S_i}_{S_i} := (K_{X_i} + B_i + \mathbf{M}_{X_i})|_{S_i},$$

and let $H_{S_i} := H_i|_{S_i}$ for each i. For every glc center of $(X, B, \mathbf{M})$ we let $(S, B_S, \mathbf{M}^S)$ be the g-pair induced by the adjunction

$$K_S + B_S + \mathbf{M}^S_S := (K_X + B + \mathbf{M}_X)|_S$$

and let $H_S := H|_S$. By Lemma 5.4, [HL18, Remark 3.21, Theorem 4.1], Claim 7.2(1), and induction on the dimension of glc centers, we only need to show that for any $m \gg 0$ and any glc center S_m of $(X_m, B_m, \mathbf{M})$ of dimension ≥ 1, $(S_m, B_{S_m}, \mathbf{M}^S)/U$ has a log minimal model. For each i, we let S_i, S be the strict transforms of S_m on X_i, X respectively. We let $\phi^S_{i,j} : S_i \dashrightarrow S_j$ and $\phi^S_i : S \dashrightarrow S_i$ be the induced birational maps. By Lemma 5.4 and induction on the dimension of glc centers, possibly replacing m, we may assume that $\phi_{m,i}$ is small for any $i \geq m$.

Step 4. We prove the following claim.

Claim 7.4. *There exists a $\mathbb{Q}$-factorial glc g-pair $(T, B_T, \mathbf{M}^S)/U$ and a birational morphism $\psi : T \to S_m$ satisfying the following:*

(1) For any prime divisor D on S such that $a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S)$, D is on T and is a ψ-exceptional.

(2)
$$B_T = \sum_{D \text{ is a prime divisor on } T} (1 - a(D, S, B_S, \mathbf{M}^S))D.$$

(3) For any $i \geq m$ and any prime divisor Q over S,
$$a(Q, S, B_S + \lambda_i H_S, \mathbf{M}^S) \leq a(Q, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S).$$

(4) For any prime divisor Q' over S_m,
$$a(Q', S_m, B_{S_m}, \mathbf{M}^S) \leq a(Q', T, B_T, \mathbf{M}^S).$$

Proof. By Claim 7.2(2), $(X_i, B_i + \lambda_i H_i, \mathbf{M})/U$ is a good minimal model of $(X, B + \lambda_i H, \mathbf{M})$, hence (3) holds.

Since ϕ_i does not extract any divisor, $a(P, X_i, B_i, \mathbf{M}) \leq a(P, X, B, \mathbf{M})$ for any prime divisor P on X_i. Since $\sigma_P(X/U, K_X + B + \mathbf{M}_X) = 0$ for any prime divisor P over X such that $a(P, X, B, \mathbf{M}) < 1$ and $\operatorname{center}_X P \cap \operatorname{Ngklt}(X, B, \mathbf{M}) \neq \emptyset$, by Lemma 6.1 and since $\phi_{m,i}$ is small for any $i \geq m$, $a(D, S_i, B_{S_i}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S)$ for any prime divisor D on S_m and $i \geq m$. Thus $a(D, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S)$ for any prime divisor D on S_m and $i \geq m$. By (3),

$$\begin{aligned} a(D, S, B_S + \lambda_i H_S, \mathbf{M}^S) &\leq a(D, S_i, B_{S_i} + \lambda_i H_{S_i}, \mathbf{M}^S) \\ &= a(D, S_m, B_{S_m} + \lambda_i H_{S_m}, \mathbf{M}^S) \leq a(D, S, B_S, \mathbf{M}^S). \end{aligned}$$

for any $i \geq m$.

Let $i \to +\infty$, we have
$$a(D, S, B_S, \mathbf{M}^S) = a(D, S_m, B_{S_m}, \mathbf{M}^S)$$
for any prime divisor D on S_m. We define
$$\mathcal{C} := \{D \mid D \text{ is a prime divisor on } S, a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S)\},$$
then any element of $\mathcal{C}$ is exceptional over S_m. Thus for any $D \in \mathcal{C}$, by (3),
$$\begin{aligned} a(D, S, B_S + \lambda_m H_S, \mathbf{M}^S) &\leq a(D, S_m, B_{S_m} + \lambda_m H_{S_m}, \mathbf{M}^S) \\ &\leq a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S) \leq 1. \end{aligned}$$

Since any element of $\mathcal{C}$ is a prime divisor on S, any element of $\mathcal{C}$ is a component of H_S. Thus $\mathcal{C}$ is a finite set, and for every $D \in \mathcal{C}$, since $\lambda_m < t_0$,
$$\begin{aligned} 0 \leq & a(D, S, B_S + t_0 H_S, \mathbf{M}^S) < a(D, S, B_S + \lambda_m H_S, \mathbf{M}^S) \\ \leq & a(D, S_m, B_{S_m} + \lambda_m H_{S_m}, \mathbf{M}^S) \leq a(D, S_m, B_{S_m}, \mathbf{M}^S) < a(D, S, B_S, \mathbf{M}^S) \leq 1. \end{aligned}$$

Thus $0 < a(D, S_m, B_{S_m}, \mathbf{M}^S) < 1$ for any $D \in \mathcal{C}$. By [Has22, Lemma 3.4], there exists a birational morphism $\psi : T \to S_m$ from a $\mathbb{Q}$-factorial variety T which extracts exactly divisors contained in $\mathcal{C}$. (1) follows immediately from the construction of $\mathcal{C}$. Since $(S, B_S, \mathbf{M}^S)$ is glc, there are only finitely many divisors D on T such that $a(D,S,B_S,\mathbf{M}^S)\neq 1$, hence $B_T\geq 0$ is well-defined, and we get (2).

For any prime divisor D on T, if D is ψ-exceptional, then

$$a(D,S_m,B_{S_m},\mathbf{M}^S)<a(D,S,B_S,\mathbf{M}^S)\leq 1$$

as $D\in\mathcal{C}$, and if D is not ψ-exceptional, then center_{S_m} is a divisor, hence $a(D,S,B_S,\mathbf{M}^S)=a(D,S_m,B_{S_m},\mathbf{M}^S)\leq 1$. In either case,

$$a(D,S_m,B_{S_m},\mathbf{M}^S)\leq a(D,S,B_S,\mathbf{M}^S)\leq 1.$$

Since T is $\mathbb{Q}$-factorial, $K_T+B_T+\mathbf{M}^S_T$ is $\mathbb{R}$-Cartier, and

$$K_T+B_T+\mathbf{M}^S_T\leq\psi^*(K_{S_m}+B_{S_m}+\mathbf{M}^S_{S_m}).$$

Thus

$$0\leq a(Q',S_m,B_{S_m},\mathbf{M}^S)\leq a(Q',T,B_T,\mathbf{M}^S)$$

for any prime divisor Q' over S_m, and we get (4). In particular, $(T,B_T,\mathbf{M}^S)$ is glc, and the proof is concludes. $\square$

Proof of Theorem 7.1 continued. **Step 5**. In this step, we show that $(T,B_T,\mathbf{M}^S)/U$ has a log minimal model. By our assumption, $(S,B_S,\mathbf{M}^S)/U$ is a log minimal model of itself. We prove the following claim:

Claim 7.5. *For any prime divisor D over S,*

(1) if D is on S, then $a(D,S,B_S,\mathbf{M}^S)\leq a(D,T,B_T,\mathbf{M}^S)$, and
(2) if D is on T, then $a(D,T,B_T,\mathbf{M}^S)=a(D,S,B_S,\mathbf{M}^S)$.

Proof. By Claim 7.4(2), we only need to show that for any prime divisor D on S that is exceptional over T, $a(D,S,B_S,\mathbf{M}^S)\leq a(D,T,B_T,\mathbf{M}^S)$. By Claim 7.4(1)(4),

$$a(D,S,B_S,\mathbf{M}^S)\leq a(D,S_m,B_m,\mathbf{M}^S)\leq a(D,T,B_T,\mathbf{M}^S),$$

and we get (1). $\square$

Proof of Theorem 7.1 continued. By Lemma 6.7, $(T,B_T,\mathbf{M}^S)/U$ has a log minimal model.

Step 6. We conclude the proof in this step. Recall that we only need to show that $(S_m,B_{S_m},\mathbf{M}^S)/U$ has a good minimal model.

For any $i\geq m$, since $K_{X_i}+B_i+\lambda_iH_i+\mathbf{M}_{X_i}$ is nef$/U$, $K_{S_i}+B_{S_i}+\lambda_iH_{S_i}+\mathbf{M}^S_{S_i}=(K_{X_i}+B_i+\lambda_iH_i+\mathbf{M}_{X_i})|_{S_i}$ is nef$/U$. Since $\phi_{m,i}$ is small, by the negativity lemma, $(S_i,B_{S_i}+\lambda_iH_{S_i},\mathbf{M}^S)/U$ is a weak glc model of $(S_m,B_{S_m}+\lambda_iH_{S_m},\mathbf{M}^S)/U$. Let $h_m:W\to S_m$ and $h_i:W\to S_i$ be a resolution of indeterminacy of $\phi_{m,i}$. By Lemmas 3.3(2), 3.6(3), and 3.8(1) and the negativity lemma, for any prime divisor D on T, $\sigma_D(S_m/U,K_{S_m}+B_{S_m}+\lambda_iH_{S_m}+\mathbf{M}^S_{S_m})$ is well-defined, and

$$\begin{aligned}0&\leq a(D,S_i,B_{S_i}+\lambda_iH_{S_i},\mathbf{M}^S)-a(D,S_m,B_{S_m}+\lambda_iH_{S_m},\mathbf{M}^S)\\&=\sigma_D(S_m/U,K_{S_m}+B_{S_m}+\lambda_iH_{S_m}+\mathbf{M}^S_{S_m}).\end{aligned}$$

By Claim 7.4(3),

$$\sigma_D(S_m/U,K_{S_m}+B_{S_m}+\lambda_iH_{S_m}+\mathbf{M}^S_{S_m})\geq a(D,S,B_S+\lambda_iH_S,\mathbf{M}^S)-a(D,S_m,B_{S_m}+\lambda_iH_{S_m},\mathbf{M}^S).$$

By Claim 7.4(2), $a(D,S,B_S,\mathbf{M}^S)=a(D,T,B_T,\mathbf{M}^S)$. By Lemma 3.6(2) and Claims 7.2(1) and 7.4(4), for any prime divisor D on T,

$$\begin{aligned}
&\sigma_D(S_m/U,K_{S_m}+B_{S_m}+\mathbf{M}^S_{S_m})\\
=&\lim_{i\to+\infty}\sigma_D(S_m/U,K_{S_m}+B_{S_m}+\lambda_iH_{S_m}+\mathbf{M}^S_{S_m})\\
\geq&\lim_{i\to+\infty}(a(D,S,B_S+\lambda_iH_S,\mathbf{M}^S)-a(D,S_m,B_{S_m}+\lambda_iH_{S_m},\mathbf{M}^S))\\
=&a(D,S,B_S,\mathbf{M}^S)-a(D,S_m,B_{S_m},\mathbf{M}^S)\\
=&a(D,T,B_T,\mathbf{M}^T)-a(D,S_m,B_{S_m},\mathbf{M}^S)\geq 0.
\end{aligned}$$

Since $(T,B_T,\mathbf{M}^S)/U$ has a log minimal model, by Lemma 6.4, $(S_m,B_{S_m},\mathbf{M}^S)/U$ has a log minimal model, and we are done. $\square$

Theorem 7.6 (cf. [Has22, Theorem 3.15])**.** *Let $(X,B,\mathbf{M})/U$ be a $\mathbb{Q}$-factorial NQC gdlt g-pair and $A\geq 0$ an $\mathbb{R}$-divisor on X such that $K_X+B+\mathbf{M}_X+A$ is nef/U. The for any $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A*

$$(X,B,\mathbf{M})=:(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots,$$

with scaling numbers

$$\lambda_i:=\inf\{t\mid t\geq 0,K_{X_i}+B_i+tA_i+\mathbf{M}_{X_i}\text{ is nef}/U\}$$

where A_i is the strict transform of A on X_i, if $\lambda_i>0$ for each i and $\lim_{i\to+\infty}\lambda_i=0$, then there are only finitely many i such that $K_{X_i}+B_i+\mathbf{M}_{X_i}$ is log abundant/U with respect to $(X_i,B_i,\mathbf{M})$.

Proof. We apply induction on dimensions. Suppose that the theorem holds in dimension $\leq\dim X-1$ but the theorem does not hold. Then there exists a $(K_X+B+\mathbf{M}_X)$-MMP/U with scaling of A as in the statement of the theorem such that $K_{X_i}+B_i+\mathbf{M}_{X_i}$ is log abundant/U with respect to $(X_i,B_i,\mathbf{M})$ for infinitely many i. Let $\phi_{i,j}:X_i\dashrightarrow X_j$ be the induced birational map. Possibly replacing $(X,B,\mathbf{M})$ with $(X_m,B_m,\mathbf{M})$ for some $m\gg 0$, we may assume that $\phi_{i,j}$ are small for any i,j.

We prove the following claim.

Claim 7.7. *If there exists $m\gg 0$ such that $\phi_{m,i}|_S$ is an isomorphism for any glc center S of $(X_m,B_m,\mathbf{M})$ and $i\geq m$, then Theorem 7.6 holds.*

Proof. Possibly replacing $(X,B,\mathbf{M})$ we may assume that $\phi_{i,j}|_{\mathrm{Ngklt}(X_i,B_i,\mathbf{M})}$ is an isomorphism for any i,j and $K_X+B+\mathbf{M}_X$ is abundant/U. Since $\lim_{i\to+\infty}\lambda_i=0$ and $\phi_{i,j}$ are small for any i,j, $K_X+B+\mathbf{M}_X$ is a limit of movable/U $\mathbb{R}$-divisors, hence $K_X+B+\mathbf{M}_X$ is pseudo-effective/U. Notice that $(X_i,B_i,\mathbf{M})$ are all gdlt and $\mathbb{Q}$-factorial, let D be a component of $\lfloor B_i\rfloor$, then $\phi_{i,i+1}|_D$ being an isomorphism implies that the flip $\phi_{i,i+1}$ is an isomorphism near D. Therefore $\phi_{i,i+1}$ is an isomorphism on a neighborhood of $\lfloor B_i\rfloor$. By Lemma 6.6, $\mathbf{B}_-(K_X+B+\mathbf{M}_X/U)$ does not intersect $\operatorname{Supp}\lfloor B\rfloor$, and $\sigma_P(X/U,K_X+B+\mathbf{M}_X)=0$ for any prime divisor P over X such that $\operatorname{center}_X P\cap\operatorname{Supp}\lfloor B\rfloor\neq\emptyset$. In particular, $(K_X+B+\mathbf{M}_X)|_S$ is nef/U for any glc center S of $(X,B,\mathbf{M})$. By Theorem 7.1, $(X,B,\mathbf{M})/U$ has a log minimal model, but this contradicts [HL18, Theorem 4.1] so we are done. $\square$

Proof of Theorem 7.6 continued. We let $\phi_i:=\phi_{i,i+1}$ for every i and $X_i\to Z_i\leftarrow X_{i+1}$ the flip defined by ϕ_i. By Claim 7.7, we only need to show that for any glc center S of $(X,B,\mathbf{M})$, the MMP terminates along S after finitely many steps. By induction on the dimension of glc centers, we may assume that ϕ_i induces an isomorphism for every $\leq d-1$-dimensional glc centers and $i\gg 0$ where $d=\dim S$. We may let S_i be the strict transform of S on X_i and $(S_i,B_{S_i},\mathbf{M}^S)$ the g-pair given by the adjunction

$$K_{S_i}+B_{S_i}+\mathbf{M}^{S_i}_S:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Let $(S_i', B_{S_i'}, \mathbf{M}^S)$ be a gdlt model of $(S_i, B_{S_i}, \mathbf{M}^S)$. By Lemma 5.4, for $i \gg 0$, the $(K_X + B + \mathbf{M}_X)$-MMP/U with scaling of A induces a $(K_{S_i'} + B_{S_i'} + \mathbf{M}^{S_i'}_{S'})$-MMP/$T$ with scaling of $A_{S_i'}$ such that the limit of the scaling numbers is 0, where $A_{S_i'}$ is the pullback of A_i on S_i' and T is the normalization of the image of S_i in U. Since $K_{X_j} + B_j + \mathbf{M}_{X_j}$ is log abundant/U with respect to $(X_j, B_j, \mathbf{M})$ for infinitely many j, $K_{S_j'} + B_{S_j'} + \mathbf{M}^S_{S_j'}$ is log abundant/T with respect to $(S_j', B_{S_j'}, \mathbf{M}^S)$ for infinitely many j. By Theorem 7.6 in dimension $< \dim X$, the $(K_{S_j'} + B_{S_j'} + \mathbf{M}^{S_j'}_{S'})$-MMP/$T$ terminates, and the claim follows. $\square$

8. Log abundance under the MMP

This section is similar to [Has20b, Section 3 and Theorem 4.1].

Theorem 8.1 (cf. [Has20b, Theorem 3.5])**.** *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair and $\pi : X \to Z$ a projective morphism/U such that Z is normal quasi-projective. Let $C \geq 0$ be an $\mathbb{R}$-divisor on X, A_Z an ample/U $\mathbb{R}$-divisor on Z, and $0 \leq A \sim_{\mathbb{R},U} \pi^* A_Z$ and $\mathbb{R}$-divisor on X, such that*

(1) C does not contain any glc center of $(X, B, \mathbf{M})$,
(2) $K_X + B + C + \mathbf{M}_X \sim_{\mathbb{R},Z} 0$, and
(3) $(X, B + A, \mathbf{M})$ is glc.

Then $K_X + B + A + \mathbf{M}_X$ is abundant/U.

Proof. Possibly replacing π with the contraction in the Stein factorization of π, we may assume that π is a contraction. Possibly replacing $Z \to U$ with the Stein factorization of $Z \to U$, we may assume that $Z \to U$ is a contraction. Let F be a very general fiber of $X \to U$ and $F_Z := \pi(F)$, then F_Z is a very general fiber of $Z \to U$. Possibly replacing $(X, B, \mathbf{M}), A, C, Z, A_Z, \pi, U$ with $(F, B|_F, \mathbf{M}|_F), A|_F, C|_F, F_Z, A_Z|_{F_Z}, \pi|_F, \{pt\}$, we may assume that $U = \{pt\}$. The theorem follows from [Has20b, Theorem 3.5]. Note that we remove the $\mathbb{R}$-Cartier assumption of C as it is immediate from (2). $\square$

Lemma 8.2 (cf. [Has20b, Lemma 3.6])**.** *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair and $\pi : X \to Z$ a projective morphism/U such that Z is normal quasi-projective. Let $C \geq 0$ be an $\mathbb{R}$-divisor on X, A_Z an ample/U $\mathbb{R}$-divisor on Z, and $0 \leq A \sim_{\mathbb{R},U} \pi^* A_Z$ and $\mathbb{R}$-divisor on X, such that*

(1) C does not contain any glc center of $(X, B, \mathbf{M})$,
(2) $K_X + B + C + \mathbf{M}_X \sim_{\mathbb{R},Z} 0$, and
(3) $(X, B + A, \mathbf{M})$ is glc.

Let $h : W \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to W, and $B_W \geq 0$ an $\mathbb{R}$-divisor on W such that $(W, B_W + h^*A)$ is lc and $(K_W + B_W + \mathbf{M}_W - h^*(K_X + B + \mathbf{M}_X))^{\geq 0}$ is h-exceptional. Then $K_W + B_W + h^*A + \mathbf{M}_W$ is abundant/U.

Proof. Possibly replacing π with the contraction in the Stein factorization of π, we may assume that π is a contraction. Possibly replacing $Z \to U$ with the Stein factorization of $Z \to U$, we may assume that $Z \to U$ is a contraction. Let F_w be a very general fiber of $W \to U$, $F := h(F_W)$, and $F_Z := \pi(F)$, then F and F_Z are a very general fibers of $X \to U$ and $Z \to U$ respectively. Possibly replacing $(X, B, \mathbf{M}), A, C, Z, A_Z, \pi, U, W, h, B_W$ with $(F, B|_F, \mathbf{M}|_F), A|_F, C|_F, F_Z, A_Z|_{F_Z}, \pi|_F, \{pt\}, F_W, h|_{F_W}, B_W|_{F_W}$, we may assume that $U = \{pt\}$. The theorem follows from [Has20b, Theorem 3.5, Lemma 3.6]. Note that we remove the $\mathbb{R}$-Cartier assumption of C as it is immediate from (2). $\square$

Lemma 8.3 (cf. [HL18, 3.5])**.** *Let $(X, B, \mathbf{M})/U$ be an NQC glc g-pair, S a glc center of $(X, B, \mathbf{M})$, $(Y, B_Y, \mathbf{M})$ a gdlt model of $(X, B, \mathbf{M})$ with induced birational morphism $f : Y \to X$, and S_Y a component of $\lfloor B_Y \rfloor$ such that $f(S_Y) = S$. Let*

$$\phi : (X, B, \mathbf{M}) \dashrightarrow (X', B', \mathbf{M})$$

be a partial $(K_X+B+\mathbf{M}_X)$*-MMP/*U *and* S' *a glc center of* $(X',B',\mathbf{M})$*, such that* $\phi|_S: S \dashrightarrow S'$ *is a birational map. Then there exists a partial* $(K_Y+B_Y+\mathbf{M}_Y)$*-MMP/*U

$$\psi: (Y,B_Y,\mathbf{M}) \dashrightarrow (Y',B'_Y,\mathbf{M}),$$

such that

(1) $(Y',B'_Y,\mathbf{M})$ *is a gdlt model of* $(X',B',\mathbf{M})$*, and*
(2) *the strict transform of* S_Y *on* Y' *is a component of* $\lfloor B'_Y\rfloor$*.*

Proof. We only need to prove the lemma when ϕ is a divisorial contraction or a flip. If ϕ is a flip, then we let $X\to Z$ be the flipping contraction and let $X'\to Z$ be the flipped contraction. If ϕ is a divisorial contraction, then we let $Z=X'$. Then $(X',B',\mathbf{M})/Z$ is a log minimal model of $(X,B,\mathbf{M})/Z$ such that $K_{X'}+B'+\mathbf{M}_{X'}$ is ample$/Z$. By [HL21, Lemmas 3.5, 3.8] (=[HL21, Version 3, Lemmas 3.9, 3.15]) and Theorem 2.14, we may run a $(K_Y+B_Y+\mathbf{M}_Y)$-MMP$/Z$ with scaling of an ample$/Z$ divisor which terminates with a good minimal model $(Y',B'_Y,\mathbf{M})/Z$. By [HL21, Lemma 3.5] (=[HL21, Version 3, Lemma 3.9]), $(Y',B'_Y,\mathbf{M})$ is a gdlt model of $(X',B',\mathbf{M})$, and we get (1).

We let $p: W\to Y$ and $q: W\to Y'$ be a resolution of indeterminacy of the induced birational map $\phi_Y: Y\dashrightarrow Y'$. By [HL21, Lemma 3.4] (=[HL21, Version 3, Lemma 3.8]), $p^*(K_Y+B_Y+\mathbf{M}_Y)=q^*(K_{Y'}+B_{Y'}+\mathbf{M}_{Y'})+F$ where $F\geq 0$ is exceptional$/Y'$, and $\operatorname{Supp} p_*F$ contains all ϕ_Y-exceptional divisors. By (1), $a(S_Y,Y,B_{Y'},\mathbf{M})=0$, hence S_Y is not a component of $\operatorname{Supp} p_*F$, and we get (2). $\square$

Theorem 8.4 (cf. [Has20b, Theorem 4.1])**.** *Let* $(X,B,\mathbf{M})/U$ *be an NQC glc g-pair and* $\pi: X\to Z$ *a projective morphism/*U *such that* Z *is normal quasi-projective. Let* $C\geq 0$ *be an* $\mathbb{R}$*-divisor on* X*,* A_Z *an ample/*U $\mathbb{R}$*-divisor on* Z*, and* $0\leq A\sim_{\mathbb{R},U}\pi^*A_Z$ *and* $\mathbb{R}$*-divisor on* X*, such that*

(1) C *does not contain any glc center of* $(X,B,\mathbf{M})$*,*
(2) $K_X+B+C+\mathbf{M}_X\sim_{\mathbb{R},Z}0$*, and*
(3) $(X,\Delta:=B+A,\mathbf{M})$ *is glc and* $\operatorname{Ngklt}(X,B,\mathbf{M})=\operatorname{Ngklt}(X,\Delta,\mathbf{M})$*.*

Then for any $(K_X+\Delta+\mathbf{M}_X)$*-MMP/*$U$

$$(X,\Delta,\mathbf{M}):=(X_0,\Delta_0,\mathbf{M})\dashrightarrow(X_1,\Delta_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,\Delta_i,\mathbf{M})\dashrightarrow\dots,$$

$K_{X_i}+\Delta_i+\mathbf{M}_{X_i}$ *is log abundant/*U *with respect to* $(X_i,\Delta_i,\mathbf{M})$ *for every* i*.*

Proof. For each i, we let $\phi_i: X\dashrightarrow X_i$ be the induced birational map.

By Theorem 8.1, $K_X+B+A+\mathbf{M}_X$ is abundant$/U$. By Lemma 2.2(6), $K_{X_i}+\Delta_i+\mathbf{M}_{X_i}$ is abundant$/U$ for any i. Thus we only need to prove that $(K_{X_i}+\Delta_i+\mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ for any glc center S_i of $(X_i,\Delta_i,\mathbf{M})$.

Fix i and a glc center S_i of $(X_i,\Delta_i,\mathbf{M})$. Then there exists a glc center S of $(X,\Delta,\mathbf{M})$ such that $\phi_i|_S: S\dashrightarrow S_i$ is a birational map. Let $(X',B',\mathbf{M})$ be a gdlt model of $(X,B,\mathbf{M})$ with induced birational morhpism $f: X'\to X$, such that there exists a component S' of $\lfloor\Delta'\rfloor$ which dominates S. Let $C':=f^*C$, $A':=f^*A$, and $\Delta':=B'+A'$. Since $\operatorname{Ngklt}(X,B,\mathbf{M})=\operatorname{Ngklt}(X',\Delta',\mathbf{M})$, $(X',\Delta',\mathbf{M})$ is a gdlt model of $(X,\Delta,\mathbf{M})$. By Lemma 8.3, we may run a $(K_{X'}+\Delta'+\mathbf{M}_{X'})$-MMP$/U$ and get a gdlt model $(X'_i,\Delta'_i,\mathbf{M})$ of $(X_i,\Delta_i,\mathbf{M})$, such that the strict transform of S' on X'_i is a component of $\lfloor\Delta'_i\rfloor$. We let S'_i be the strict transform of S' on X'_i, then $(K_{X_i}+\Delta_i+\mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ if and only if $(K_{X'_i}+\Delta'_i+\mathbf{M}_{X'_i})|_{S'_i}$ is abundant$/U$. Moreover, C' does not contain any glc center of $(X',B',\mathbf{M})$, $K_{X'}+B'+C'+\mathbf{M}_{X'}\sim_{\mathbb{R},Z}0$, $(X',\Delta',\mathbf{M})$ is glc, and $\operatorname{Ngklt}(X',B',\mathbf{M})=\operatorname{Ngklt}(X',\Delta',\mathbf{M})$. Thus possibly replacing $(X,\Delta,\mathbf{M})\dashrightarrow(X_i,\Delta_i,\mathbf{M})$ with $(X',\Delta',\mathbf{M})\dashrightarrow(X'_i,\Delta',\mathbf{M})$ and A,B,C with A',B',C', we may assume that $(X,\Delta,\mathbf{M})$ is $\mathbb{Q}$-factorial gdlt, S is a component of $\lfloor\Delta\rfloor=\lfloor B\rfloor$, and S_i is a component of $\lfloor\Delta_i\rfloor=\lfloor B_i\rfloor$.

Let $(S,B_S,\mathbf{M}^S)/U$ and $(S_i,B_{S_i},\mathbf{M}^S)/$ be the gdlt g-pairs induced by the adjunction formulas

$$K_S+B_S+\mathbf{M}^S:=(K_X+B+\mathbf{M}_X)|_S$$

and

$$K_{S_i}+B_{S_i}+\mathbf{M}_{S_i}:=(K_{X_i}+B_i+\mathbf{M}_{X_i})|_{S_i}.$$

Let $p:W\to S$ and $q:W\to S_i$ be a resolution of indeterminacy such that $\mathbf{M}^S$ descends to W, p is a log resolution of $(S,\operatorname{Supp}B_S)$, and q is a log resolution of $(S_i,\operatorname{Supp}B_{S_i})$. Since A is semi-ample$/U$, possibly replacing A with a general member of $|A/U|_{\mathbb{R}}$ (i.e. write $A=\sum r_iA_i$ where $r_i\in(0,1)$ are real numbers and A_i are base-point-free$/U$ Cartier divisors. We replace A with $\sum r_iH_i$ where $H_i\in|A_i/U|$ are general members) and let $A_S:=A|_S$ and $A_{S_i}:=(\phi_i)_*A)|_{S_i}$, we may assume that

- $A_S\geq 0, A_{S_i}\geq 0$,
- $(S,\Delta_S:=B_S+A_S,\mathbf{M})$ and $(S_i,\Delta_{S_i}:=B_{S_i}+A_{S_i},\mathbf{M})$ are gdlt, and
- p is a log resolution of $(S,\operatorname{Supp}\Delta_S)$ and q is a log resolution of $(S_i,\operatorname{Supp}\Delta_{S_i})$.

Moreover, since A is general in $|A/U|_{\mathbb{R}}$, $p^*A_S=p^{-1}_*A_S$, hence $A_W:=p^*A\leq q^*A_{S_i}$. We may write

$$K_W+B_W+A_W+\mathbf{M}^S_W=q^*(K_{S_i}+\Delta_W+\mathbf{M}^S_{S_i})+E$$

and let $\Delta_W:=B_W+A_W$, such that $B_W\geq 0$, $E\geq 0$, and $\Delta_W\wedge E=0$. Then (W,Δ_W) is log smooth. We may write

$$K_W+B_W+\mathbf{M}^S_W=p^*(K_S+\Delta_S+\mathbf{M}^S_S)+G_+-G_-$$

where $G_+\geq 0, G_-\geq 0$, and $G_+\wedge G_-=0$.

For any component D of G_+,

$$a(D,S,\Delta_S,\mathbf{M}^S)>a(D,W,\Delta_W,\mathbf{M}^S)=\min\{a(D,S_i,\Delta_{S_i},\mathbf{M}^S),1\}.$$

Since ϕ_i is $(K_X+\Delta+\mathbf{M}_X)$-non-positive,

$$a(D,S_i,\Delta_{S_i},\mathbf{M}^S)\leq a(D,S,\Delta_S,\mathbf{M}^S).$$

Thus $a(D,S,\Delta_S,\mathbf{M}^S)>1$, hence D is p-exceptional, and G_+ is p-exceptional.

Let $\pi_S:=\pi|_S$ and $C_S:=C|_S$. Since C does not contain and glc center of $(X,B,\mathbf{M})$, S is not a component of C, hence $C_S\geq 0$. Then $(S,B_S,\mathbf{M}^S)/U$ is an NQC glc g-pair, $\pi_S:S\to Z$ is a projective morphism$/U$, Z is normal quasi-projective, $C_S\geq 0$ is an $\mathbb{R}$ divisor on X, $0\leq A_S\sim_{\mathbb{R},U}\pi_S^*A_Z$, such that

- C_S does not contain any glc center of $(S,B_S,\mathbf{M}^S)$,
- $K_S+B_S+C_S+\mathbf{M}^S_S\sim_{\mathbb{R},Z}0$,
- $(S,\Delta_S=B_S+A_S,\mathbf{M}^S)$ is glc and $\operatorname{Ngklt}(S,B_S,\mathbf{M}^S)=\operatorname{Ngklt}(S,\Delta_S,\mathbf{M}^S)$,
- $p:W\to S$ is a log resolution of $(S,\operatorname{Supp}B_S)$ such that $\mathbf{M}^S$ descends to W, $B_W\geq 0$ is an. $\mathbb{R}$-divisor on W such that $(W,B_W+p^*A_S)$ is lc and $(K_W+B_W+\mathbf{M}^S_W-p^*(K_S+B_S+\mathbf{M}^S_S))^{\geq 0}=G_+$ is p-exceptional.

By Lemma 8.2, $K_W+\Delta_W+\mathbf{M}^S_W$ is abundant$/U$. By Lemma 2.2(3), $K_{S_i}+\Delta_{S_i}+\mathbf{M}^S_{S_i}=(K_{X_i}+\Delta_i+\mathbf{M}_{X_i})|_{S_i}$ is abundant$/U$ and we are done. □

Theorem 8.5 (cf. [Has20b, Theroem 1.3])**.** *Let $(X,B,\mathbf{M})/U$ be an NQC glc g-pair and $A\geq 0$ an ample$/U$ $\mathbb{R}$-divisor such that $(X,\Delta:=B+A,\mathbf{M})$ is glc and* $\operatorname{Ngklt}(X,B,\mathbf{M})=\operatorname{Ngklt}(X,B+A,\mathbf{M})$*. Let $(Y,\Delta_Y,\mathbf{M})$ be a gdlt model of $(X,\Delta,\mathbf{M})$. The for any partial $(K_Y+\Delta_Y+\mathbf{M}_Y)$-MMP$/U$*

$$\phi:(Y,\Delta_Y,\mathbf{M})\dashrightarrow(Y',\Delta'_Y,\mathbf{M}),$$

$K_{Y'}+\Delta'_Y+\mathbf{M}_{Y'}$ is log abundant$/U$ with respect to $(Y',\Delta'_Y,\mathbf{M})$.

Proof. It follows from Theorem 8.4. □

9. Proof of the main theorems

Proof of Theorem 1.2. By [HL21, Lemma 4.3] (=[HL21, Version 3, Lemma 4.2]), possibly replacing A, we may assume that $\mathrm{Ngklt}(X,B,\mathbf{M})=\mathrm{Ngklt}(X,\Delta,\mathbf{M})$.

First we prove (1). Let $(Y,\Delta_Y,\mathbf{M})$ be a gdlt model of $(X,\Delta,\mathbf{M})$. By Theorem 2.14, we only need to show that $(Y,\Delta_Y,\mathbf{M})/U$ has a log minimal model. We run a $(K_Y+\Delta_Y+\mathbf{M}_Y)$-MMP$/U$

$$(Y,\Delta_Y,\mathbf{M}):=(Y_0,\Delta_0,\mathbf{M})\dashrightarrow(Y_1,\Delta_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(Y_i,\Delta_i,\mathbf{M})\dashrightarrow\dots$$

with scaling of a general ample$/U$ divisor $H\geq 0$, and let

$$\lambda_i:=\inf\{t\mid t\geq 0, K_{Y_i}+\Delta_i+\lambda_iH_i+\mathbf{M}_{Y_i}\text{ is nef}/U\}$$

be the scaling numbers. If $\lambda_i=0$ for some i then $(Y_i,\Delta_i,\mathbf{M})/U$ is a log minimal model of $(Y,\Delta,\mathbf{M})$ and we are done. Thus we may assume that $\lambda_i>0$ for each i. By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]), $\lim_{i\to+\infty}\lambda_i=0$. By Theorem 8.5, $K_{Y_i}+\Delta_i+\mathbf{M}_{Y_i}$ is log abundant$/U$ for each i, which contradicts Theorem 7.6.

Now we prove (2). We may pick $0<\epsilon\ll 1$ such that $\frac{1}{2}A+\epsilon\mathbf{M}_X$ is ample$/U$. Possibly replacing $(X,B,\mathbf{M})$ with $(X,B,(1-\epsilon)\mathbf{M})$ and A with a general member in $|A+\epsilon\mathbf{M}_X|_{\mathbb{R}}$, we may assume that $\mathrm{Nklt}(X,B)=\mathrm{Ngklt}(X,B,\mathbf{M})=\mathrm{Ngklt}(X,\Delta,\mathbf{M})$. By [HL21, Lemma 5.9] (=[HL21, Version 3, Lemma 5.18]), there exists a birational morphism $h:W\to X$ such that $\mathbf{M}$ descends to W and $\mathrm{Supp}(h^*\mathbf{M}_X-\mathbf{M}_W)=\mathrm{Exc}(h)$. We may pick an h-exceptional $\mathbb{R}$-divisor $E\geq 0$ such that $-E$ is ample$/X$. Let $K_W+B_W:=h^*(K_X+B)$. Since $\mathrm{Nklt}(X,B)=\mathrm{Ngklt}(X,B,\mathbf{M})$, there exists $0<\delta\ll 1$ such that $(W,B_W+\delta E)$ is sub-lc and $\frac{1}{2}h^*A-\delta E$ is ample$/U$. Thus $\mathbf{M}_W+\frac{1}{2}h^*A-\delta E$ is ample$/U$, and we may pick $0\leq H_W\sim_{\mathbb{R},U}\mathbf{M}_W+\frac{1}{2}h^*A-\delta E$ such that $(W,B_W+H_W+\delta E)$ is sub-lc. Let $\Delta':=h_*(B_W+H_W+\delta E)$, then (X,Δ') is lc and $\Delta'\sim_{\mathbb{R},U}B+\mathbf{M}_X+\frac{1}{2}A$. Possibly replacing A we may assume that $(X,\Delta'+\frac{1}{2}A)$ is lc. By [HH20, Theorem 1.5], $(X,\Delta'+\frac{1}{2}A)$ has a good minimal model. By [HL21, Lemma 4.3] (=[HL21, Version 3, Lemma 4.2]), we get (2). $\square$

Proof of Theorem 1.3. If $K_X+B+\mathbf{M}_X$ is not pseudo-effective$/Z$, then the theorem follows from [BZ16, Lemma 4.4(1)]. So we may assume that $K_X+B+\mathbf{M}_X$ is pseudo-effective$/Z$. By Theorem 2.14, we only need to prove (2), so we may assume that $(X,B,\mathbf{M})$ is $\mathbb{Q}$-factorial gdlt. We run a $(K_X+B+\mathbf{M}_X)$-MMP$/Z$ with scaling of an ample$/U$ $\mathbb{R}$-divisor $H\geq 0$:

$$(X,B,\mathbf{M}):=(X_0,B_0,\mathbf{M})\dashrightarrow(X_1,B_1,\mathbf{M})\dashrightarrow\cdots\dashrightarrow(X_i,B_i,\mathbf{M})\dashrightarrow\dots.$$

By Theorem 8.4 (U and Z in Theorem 8.4 both correspond to our Z, A_Z and A of Theorem 8.4 correspond to 0, and C corresponds to our A), $K_{X_i}+B_i+\mathbf{M}_{X_i}$ is log abundant$/Z$ with respect to $(X_i,B_i,\mathbf{M})$ for every i. By [HL21, Theorem 2.8] (=[HL21, Version 3, Theorem 2.24]) and Theorem 7.6, this MMP terminates with a log minimal model of $(X,B,\mathbf{M})/Z$. $\square$

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