SARKISOV PROGRAM FOR LC PAIRS

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Abstract. We establish the Sarkisov program for lc pairs. As applications and related results, we prove a result on the finiteness of models for lc pairs, and show that lc Fano varieties are Mori dream spaces. We also establish the lc generalized pair version of the forestated results.

Contents

1. Introduction

In this paper we work over the field of complex numbers $\mathbb{C}$.

Theorem 1.1 (Lc Sarkisov program)**.** *Assume that*

(1) $(W, B_W)/Z$ *is an lc pair such that* $K_W + B_W$ *is not pseudo-effective/*Z*,*
(2) $\rho_X : W \dashrightarrow X$ *and* $\rho_Y : W \dashrightarrow Y$ *are two* $(K_W + B_W)$*-MMP/*Z*,* $B_X := (\rho_X)_* B_W$*, and* $B_Y := (\rho_Y)_* B_W$*, and*
(3) $\phi_X : X \to S_X$ *is a* $(K_X + B_X)$*-Mori fiber space/*Z *and* $\phi_Y : Y \to S_Y$ *is a* $(K_Y + B_Y)$*-Mori fiber space/*Z*.*

$$\begin{tikzcd} & W \arrow[dl, dashed, "\rho_X"'] \arrow[dr, dashed, "\rho_Y"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\ S_X & & S_Y \end{tikzcd}$$

Thent he induced birational map $f : X \dashrightarrow Y$ *is given by a finite sequence of Sarkisov links/*Z*, i.e.* f *can be written as* $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$*, where each* $X_i \dashrightarrow X_{i+1}$ *is a Sarkisov link/*Z*,*

Theorem 1.2 (lc Sarkisov program for generalized pairs)**.** *Assume that*

(1) $(W, B_W, \mathbf{M})/Z$ *is an lc generalized pair such that* $K_W + B_W + \mathbf{M}_W$ *is not pseudo-effective/*Z*,*

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(2) $\rho_X : W \dashrightarrow X$ *and* $\rho_Y : W \dashrightarrow Y$ *are two* $(K_W + B_W + \mathbf{M}_W)$*-MMP/*$Z$*,* $B_X := (\rho_X)_* B_W$*, and* $B_Y := (\rho_Y)_* B_W$*, and*

(3) $\phi_X : X \to S_X$ *is a* $(K_X + B_X + \mathbf{M}_X)$*-Mori fiber space/*$Z$ *and* $\phi_Y : Y \to S_Y$ *is a* $(K_Y + B_Y + \mathbf{M}_Y)$*-Mori fiber space/*$Z$*.*

$$\begin{tikzcd} & W \arrow[dl, dashed, "\rho_X"'] \arrow[dr, dashed, "\rho_Y"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\ S_X & & S_Y \end{tikzcd}$$

Thent he induced birational map $f : X \dashrightarrow Y$ *is given by a finite sequence of Sarkisov links/*Z*, i.e.* f *can be written as* $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$*, where each* $X_i \dashrightarrow X_{i+1}$ *is a Sarkisov link/*Z*,*

Definition 1.3 (Sarkisov links)**.** Assume that

- $X_1 \to Z$ and $X_2 \to Z$ are two contractions,
- $(X_1, B_1 + M_{X_1})$ and $(X_2, B_2 + M_{X_2})$ are two gklt g-pairs with the same associated nef/Z $\boldsymbol{b}$-divisor M,
- $\phi_1 : X_1 \to S_1$ is a $(K_{X_1} + B_1 + M_{X_1})$-Mori fiber space/$Z$ and $\phi_2 : X_2 \to S_2$ is a $(K_{X_2} + B_2 + M_{X_2})$-Mori fiber space/$Z$,
- there are two birational morphisms $W \to X_1$ and $W \to X_2$, and an effective $\mathbb{R}$-divisor B_W on W, such that B_1 and B_2 are the pushforwards of B_W on X_1 and X_2 respectively, and
- $f : X_1 \dashrightarrow X_2$ is the induced birational map/Z.

Is this necessary?

Then

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/*$Z$ *of type I*, or a *Sarkisov link/*Z *of type I*, if there exists an extraction $g : V \to X_1$, a sequence of flips $V \dashrightarrow X_2$ over Z, and an extremal contraction $S_2 \to S_1$, such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[rr, dashed] \arrow[d, "g"'] & & X_2 \arrow[d, "\phi_2"] \\ X_1 \arrow[urr, dashed, "f"'] \arrow[dr, "\phi_1"'] & & S_2 \arrow[dl] \\ & S_1 & \end{tikzcd}$$

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/*$Z$ *of type II*, or a *Sarkisov link/*Z *of type II*, if there exists an extraction $g : V \to X_1$, a sequence of flips $V \dashrightarrow U$ over Z, and a divisorial contraction $U \to X_2$, such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d, "g"'] & U \arrow[d] \\ X_1 \arrow[r, dashed, "f"'] \arrow[d, "\phi_1"'] & X_2 \arrow[d, "\phi_2"] \\ S_1 \arrow[r, "\cong"] & S_2 \end{tikzcd}$$

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/*$Z$ *of type III*, or a *Sarkisov link/*Z *of type III*, if there exists a sequence of flips $X_1 \dashrightarrow U$ over Z, a divisorial contraction $U \to X_2$ and an extremal contraction $S_1 \to S_2$, such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[rr, dashed] \arrow[d, "\phi_1"'] \arrow[drr, dashed, "f"] & & U \arrow[d] \\ S_1 \arrow[dr] & & X_2 \arrow[dl, "\phi_2"] \\ & S_2 & \end{tikzcd}$$

- f is called a $(K_{X_1}+B_1+M_{X_1})$-*Sarkisov link/Z of type IV*, or a *Sarkisov link/Z of type IV*, if f is a sequence of flips/Z, and there are two extremal contractions $S_1 \to T$ and $S_2 \to T$ over Z, such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[rr, dashed, "f"] \arrow[d, "\phi_1"'] & & X_2 \arrow[d, "\phi_2"] \\ S_1 \arrow[dr] & & S_2 \arrow[dl] \\ & T & \end{tikzcd}$$

- f is called a $(K_{X_1}+B_1+M_{X_1})$-*Sarkisov link/Z*, or a *Sarkisov link/Z*, if it is a Sarkisov link/Z of one of the four types above. We remark that we allow f to be the identity map.

Theorem 1.4 (Generalized lc Fano varieties are Mori dream spaces)**.** *Let $(X,B,\mathbf{M})/Z$ be an lc generalized pair such that $-(K_X+B+\mathbf{M}_X)$ is ample/Z. Then X is a Mori dream space/Z. In particular, for any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X, we may run a D-MMP/Z which termiantes with either a good minimal model/Z or a Mori fiber space/Z.*

Theorem 1.5 (Finiteness of weak log canonical models for lc generalized pairs)**.** *Let $X \to Z$ be a projective morphism between normal quasi-projective varieties, $\mathbf{M}$ an NQC/Z $\boldsymbol{b}$-divisor on X, and $A \geq 0$ an ample $\mathbb{R}$-divisor on X. Let $\mathcal{V} \subset \mathrm{Weil}_{\mathbb{R}}(X)$ a finite dimensional rational subspace and $\mathcal{C} \subset \mathcal{L}_A(\mathcal{V})$ a rational polytope such that $(X,B,\mathbf{M})$ is lc for any $B \in \mathcal{C}$. Then there exists an integer $k \geq 0$ and birational maps/Z $\phi_i : X \dashrightarrow Y_i$ for each $1 \leq i \leq k$, such that*

(1) ϕ_i does not extract any divisor,
(2) for every $B \in \mathcal{C}$, there exists i such that $(Y_i,(\phi_i)_*B,\mathbf{M})/Z$ is a weak lc model of $(X,B,\mathbf{M})/Z$, and
(3) for any $B \in \mathcal{C}$ and any log minimal model $(Y,B_Y,\mathbf{M})/Z$ of $(X,B,\mathbf{M})/Z$ with induced birational map $\phi : X \dashrightarrow Y$, if
  - *there exists an ample $\mathbb{R}$-divisor $A_Y \geq 0$ and an $\mathbb{R}$-divisor $\Delta_Y \geq 0$ on Y, such that $B_Y \sim_{\mathbb{R},Z} A_Y + \Delta_Y$ and $(Y,\Delta_Y + A_Y)$ is lc,*

  then there exists j, such that $\psi := \phi_j \circ \phi^{-1} : Y \to Y_j$ is an isomorphism.

We remark that the existence of the ample $\mathbb{R}$-divisors A and A_Y in Theorem 1.5 are crucial by considering the following example:

Example 1.6 ([Gon09])**.** Let S be a K3 surface with infinitely many (-2)-curves, X_0 the projective cone of S, and $\phi : X \to X_0$ the blow-up of the vertex.

Let H_0 be a general and sufficiently ample divisor on X_0, E the ϕ-exceptional prime divisor, and $H := \phi^{-1}_*H_0$. Then $K_X + E + H = \phi^*(K_{X_0}+H_0)$ and

K_X+E+H is big and nef. By [Gon09, Example 0.3], there are infinitely many log minimal models of $(X,E+H)$. Therefore, there are infinitely many log minimal models of (X_0,H_0). However, it is easy to see that the only log minimal model of (X_0,H_0) which does not extract any divisor is (X_0,H_0) itself.

Now we let Y_0 be the cone of X_0 and let Y be the main component of $Y\times_{X_0} X$. Then the induced morphism $\phi_Y: Y\to Y_0$ is small. Let H_{Y_0} be a general and sufficiently ample divisor on Y_0 and let $H_Y:=(\phi_Y)^{-1}_*H_{Y_0}$. By the same arguments as in [Gon09, Example 0.3], there are infinitely many log minimal models of (Y,H_Y). Therefore, there are infinitely many log minimal models of (Y_0,H_{Y_0}) which does not extract any divisor.

However, except (Y_0,H_{Y_0}) itself, no log minimal modeo of (Y_0,H_{Y_0}) satisfies the additional condition as in Theorem 1.5(3). In particular, they cannot be achieved by running a $(K_{Y_0}+H_{Y_0})$-MMP.

2. Preliminaries

We will work over the field of complex numbers $\mathbb{C}$. Throughout the paper, we will mainly work with normal quasi-projective varieties to ensure consistency with the references. However, most results should also hold for normal varieties that are not necessarily quasi-projective. Similarly, most results in our paper should hold for any algebraically closed field of characteristic zero. We will adopt the standard notations and definitions in [KM98, BCHM10] and use them freely. For generalized pairs, we will follow the notations and definitions in [HL21]. We emphasize that, throughout this paper, generalized pairs are always assumed to be NQC.

Definition 2.1 ($\boldsymbol{b}$-divisors)**.** Let X be a normal quasi-projective variety. We call Y a *birational model* over X if there exists a projective birational morphism $Y\to X$.

Let $X\dashrightarrow X'$ be a birational map. For any valuation ν over X, we define $\nu_{X'}$ to be the center of ν on X'. A $\boldsymbol{b}$*-divisor* $\mathbf{D}$ over X is a formal sum $\mathbf{D}=\sum_\nu r_\nu\nu$ where ν are valuations over X and $r_\nu\in\mathbb{R}$, such that ν_X is not a divisor except for finitely many ν. If in addition, $r_\nu\in\mathbb{Q}$ for every ν, then $\mathbf{D}$ is called a $\mathbb{Q}$-$\boldsymbol{b}$*-divisor*. The *trace* of $\mathbf{D}$ on X' is the $\mathbb{R}$-divisor

$$\mathbf{D}_{X'}:=\sum_{\nu_{i,X'}\text{ is a divisor}} r_i\nu_{i,X'}.$$

If $\mathbf{D}_{X'}$ is $\mathbb{R}$-Cartier and $\mathbf{D}_Y$ is the pullback of $\mathbf{D}_{X'}$ on Y for any birational model Y of X', we say that $\mathbf{D}$ *descends* to X', and also say that $\mathbf{D}$ is the *closure* of $\mathbf{D}_{X'}$, and write $\mathbf{D}=\overline{\mathbf{D}_{X'}}$.

Let $X\to U$ be a projective morphism and assume that $\mathbf{D}$ is a $\boldsymbol{b}$-divisor over X such that $\mathbf{D}$ descends to some birational model Y over X. If $\mathbf{D}_Y$ is nef$/U$, then we say that $\mathbf{D}$ is *nef*$/U$. If $\mathbf{D}_Y$ is a Cartier divisor, then we say that $\mathbf{D}$ is $\boldsymbol{b}$*-Cartier*. If $\mathbf{D}_Y$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we say that $\mathbf{D}$ is $\mathbb{Q}$-$\boldsymbol{b}$*-Cartier*. If $\mathbf{D}$ can be written as an $\mathbb{R}_{\geq 0}$-linear combination of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors, then we say that $\mathbf{D}$ is *NQC*$/U$.

We let $\mathbf{0}$ be the $\boldsymbol{b}$-divisor $\bar{0}$.

Definition 2.2 (Generalized pairs)**.** A *generalized sub-pair* (*g-sub-pair* for short) $(X,B,\mathbf{M})/U$ consists of a normal quasi-projective variety X associated with a projective morphism $X\to U$, an $\mathbb{R}$-divisor B on X, and an NQC$/U$ $\boldsymbol{b}$-divisor $\mathbf{M}$ over X, such that $K_X+B+\mathbf{M}_X$ is $\mathbb{R}$-Cartier. If B is a $\mathbb{Q}$-divisor and $\mathbf{M}$ is a $\mathbb{Q}$-$\boldsymbol{b}$-divisor, then we say that $(X,B,\mathbf{M})/U$ is a $\mathbb{Q}$-*g-sub-pair*.

If $\mathbf{M}=\mathbf{0}$, a g-sub-pair $(X,B,\mathbf{M})/U$ is called a *sub-pair* and is denoted by (X,B) or $(X,B)/U$.

If $U=\{pt\}$, we usually drop U and say that $(X,B,\mathbf{M})$ is a *projective*.

A g-sub-pair (resp. NQC g-sub-pair, $\mathbb{Q}$-g-sub-pair) $(X,B,\mathbf{M})/U$ is called a *g-pair* (resp. *NQC g-pair*, *$\mathbb{Q}$-g-pair*) if $B\geq 0$. A sub-pair (X,B) is called a *pair* if $B\geq 0$.

Notation 2.3. In the previous definition, if U is not important, we may also drop U. This usually happens when we emphasize the structures of $(X,B,\mathbf{M})$ that are independent of the choice of U, such as the singularities of $(X,B,\mathbf{M})$. See Definition 2.4 below.

Definition 2.4 (Singularities of generalized pairs)**.** Let $(X,B,\mathbf{M})/U$ be a g-(sub-)pair. For any prime divisor E and $\mathbb{R}$-divisor D on X, we define $\operatorname{mult}_E D$ to be the *multiplicity* of E along D. Let $h:W\to X$ be any log resolution of $(X,\operatorname{Supp}B)$ such that $\mathbf{M}$ descends to W, and let

$$K_W+B_W+\mathbf{M}_W:=h^*(K_X+B+\mathbf{M}_X).$$

The *log discrepancy* of a prime divisor D on W with respect to $(X,B,\mathbf{M})$ is $1-\operatorname{mult}_D B_W$ and it is denoted by $a(D,X,B,\mathbf{M})$.

We say that $(X,B,\mathbf{M})$ is *(sub-)lc* (resp. *(sub-)klt*) if $a(D,X,B,\mathbf{M})\geq 0$ (resp. >0) for every log resolution $h:W\to X$ as above and every prime divisor D on W.

We say that $(X,B,\mathbf{M})$ is *dlt* if $(X,B,\mathbf{M})$ is lc, and there exists a closed subset $V\subset X$, such that

(1) $X\backslash V$ is smooth and $B_{X\backslash V}$ is simple normal crossing, and
(2) for any prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$, $\operatorname{center}_X E\not\subset V$ and $\operatorname{center}_X E\backslash V$ is an lc center of $(X\backslash V,B|_{X\backslash V})$.

If $\mathbf{M}=\mathbf{0}$ and $(X,B,\mathbf{M})$ is (sub-)lc (resp, (sub-)klt, dlt), we say that (X,B) is (sub-)lc (resp. (sub-)klt, dlt). We remark that the definition of dlt for g-pairs coincides with the definitions in all literature thanks to [Has22, Theorem 6.1].

Suppose that $(X,B,\mathbf{M})$ is sub-lc. A *lc place* of $(X,B,\mathbf{M})$ is a prime divisor E over X such that $a(E,X,B,\mathbf{M})=0$. A *lc center* of $(X,B,\mathbf{M})$ is the center of a lc place of $(X,B,\mathbf{M})$ on X. The *non-klt locus* $\operatorname{Nklt}(X,B,\mathbf{M})$ of $(X,B,\mathbf{M})$ is the union of all lc centers of $(X,B,\mathbf{M})$. If $\mathbf{M}=\mathbf{0}$, a lc place (resp. a lc center, the non-klt locus) of $(X,B,\mathbf{M})$ will be called an lc place (resp. an lc center, the non-klt locus) of (X,B), and we will denote $\operatorname{Nklt}(X,B,\mathbf{M})$ by $\operatorname{Nklt}(X,B)$.

We note that the definitions above are independent of the choice of U.

Definition 2.5. Assume that

- $X\to Z$ and $Y\to Z$ are two contractions,
- $(X,B,\mathbf{M})$ and $(Y,B_Y,\mathbf{M})$ are two g-pairs$/Z$ with the same associated nef$/Z$ ***b***-divisor $\mathbf{M}$, and
- $f:X\dashrightarrow Y$ is a birational map$/Z$,

such that

- f does not extract any divisor, and
- $a(E,X,B+\mathbf{M})\leq a(E,Y,B_Y,\mathbf{M})$ for every prime ***b***-divisor E over X,

then we may write $(X,B,\mathbf{M})\geq(Y,B_Y,\mathbf{M})$.

Proposition 2.6. *Let $W\to Z$ and $X\to Z$ be two contractions, $f:W\dashrightarrow X$ a birational map$/Z$, and $(W,B_W,\mathbf{M})$ and $(X,B,\mathbf{M})$ two g-pairs$/Z$. Assume that*

- *K_X+B+M_X is nef$/Z$,*
- *f does not extract any divisor,*

- *for any prime divisor $D \subset W$, $a(D, X, B, \mathbf{M}) \geq a(D, W, B_W, \mathbf{M})$, and*
- *$(W, B_W, \mathbf{M})$ is lc,*

then

(1) *$a(E, X, B) \geq a(E, W, B_W\mathbf{M})$ for any prime **b**-divisor E over X. In other words, $(W, B_W, \mathbf{M}) \geq (X, B, \mathbf{M})$.*

(2) *$(X, B, \mathbf{M})$ is lc.*

Proof. Let $p : V \to W$ and $q : V \to X$ be any resolution of indeterminacy of f

$$\begin{tikzcd} & V \arrow[dl, "p"'] \arrow[dr, "q"] & \\ W \arrow[rr, dashed, "f"] & & X \end{tikzcd}$$

such that

$$p^*(K_W + B_W + M_W) = q^*(K_X + B + M_X) + E_V,$$

then $p_*E_V = \sum_{E\subset W}(a(E, X, B, \mathbf{M}) - a(E, W, B_W, \mathbf{M}))E \geq 0$. Since $K_X + B + M_X$ is nef$/Z$, $-E_V$ is nef$/W$. By the negativity lemma, $E_V \geq 0$, which implies (1). Since $(W, B_W, \mathbf{M})$ is lc, $a(E, W, B_W\mathbf{M}) \geqslant 0$, and (2) follows from (1). $\square$

TODO: cone theorem

Theorem 2.7 (Cone and contraction theorems for generalized lc pairs)**.** *Let $(X, B, \mathbf{M})/U$ be an NQC lc g-pair and $\pi : X \to Z$ the associated morphism. Let $\{R_j\}_{j\in\Lambda}$ be the set of $(K_X + B + \mathbf{M}_X)$-negative extremal rays in $\overline{NE}(X/U)$ that are rational. Then:*

(1)
$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X\geq 0} + \sum_{j\in\Lambda} R_j.$$
In particular, any $(K_X + B + \mathbf{M}_X)$-negative extremal ray in $\overline{NE}(X/U)$ is rational.

(2) *Each R_j is spanned by a rational curve C_j such that $\pi(C_j) = \{pt\}$ and*
$$0 < -(K_X + B + \mathbf{M}_X) \cdot C_j \leq 2 \dim X.$$

(3) *For any ample$/U$ $\mathbb{R}$-divisor A on X,*
$$\Lambda_A := \{j \in \Lambda \mid R_j \subset \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A<0}\}$$
is a finite set. In particular, $\{R_j\}_{j\in\Lambda}$ is countable, and is a discrete subset in $\overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A<0}$. Moreover, we may write
$$\overline{NE}(X/U) = \overline{NE}(X/U)_{K_X+B+\mathbf{M}_X+A\geq 0} + \sum_{j\in\Lambda_A} R_j.$$

(4) *Assume that $\mathbf{M}_X$ is $\mathbb{R}$-Cartier. Let R be a $(K_X + B + \mathbf{M}_X)$-negative extremal ray in $\overline{NE}(X/U)$. Then R is a rational extremal ray. In particular, there exists a projective morphism $\mathrm{cont}_R : X \to Y$ over U satisfying the following.*
   (a) *For any integral curve C such that $\pi(C)$ is a point, $\mathrm{cont}_R(C)$ is a point if and only if $[C] \in R$.*
   (b) *$\mathcal{O}_Y \cong (\mathrm{cont}_R)_*\mathcal{O}_X$. In other words, cont_R is a contraction.*
   (c) *Let L be a line bundle on X such that $L \cdot R = 0$. Then there exists a line bundle L_Y on Y such that $L \cong f^*L_Y$.*

Is this true?

Theorem 2.8 (contraction extremal faces)**.** *contraction extremal faces*

Theorem 2.9 (MMP for lc gpairs, [TX23, Theorem 4.2])**.** *Let $(X/Z, B, \mathbf{M})$ be an NQC lc g-pair. Assume that $(X, B, \mathbf{M})$ has a minimal model in the sense of Birkar-Shokurov over Z or that $K_X + B + M_X$ is not pseudo-effective over Z. Let A be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X which is ample over Z such that $\big(X/Z, (B + A), \mathbf{M}\big)$ is lc and $K_X + B + A + M_X$ is nef over Z. Then there exists a $(K_X + B + M_X)$-MMP over Z with scaling of A that terminates. In particular, $(X, B, \mathbf{M})$ has a minimal model or a Mori fiber space over Z.*

Remark 2.1. See [LT22, Proposition A.3 and Theorem 1.3] for $\mathbb{Q}$-factorial case.

Theorem 2.10 (MMP for lc gpairs, [TX23, Theorem 4.4])**.** *Let $\big(X/Z, (B + A), \mathbf{M}\big)$ be an NQC lc g-pair, where A is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor which is ample over Z. If the divisor $K_X + B + A + M_X$ is pseudo-effective over Z, then there exists a $(K_X + B + A + M_X)$-MMP over Z which terminates with a good minimal model of $\big(X, (B + A), \mathbf{M}\big)$ over Z.*

TODO: need to determine B_W

Lemma 2.11. *Let $X \to Z$ be a contraction, $(X, B, \mathbf{M}_X)$ a ($\mathbb{Q}$-factorial) lc g-pair/Z, and $f : X \dashrightarrow Y$ a $(K_X + B + M_X)$-non-positive map/Z such that $f_*(K_X + B + M_X) = K_Y + B_Y + M_Y$. Then there is*

- *a resolution of indeterminacy $p : W \to X$ and $q : W \to Y$, and*
- *a ($\mathbb{Q}$-factorial) lc pair $(W, B_W, \mathbf{M})$,*

such that

(1) q is $(K_W + B_W + M_W)$-non-positive and $q_*(K_W + B_W + M_W) = K_Y + B_Y + M_Y$,
(2) $(W, B_W, \mathbf{M}) \geq (Y, B_Y, \mathbf{M}_Y)$,
(3) $M = M_W$.

Theorem 2.12 (extract a divisor, [LX22b, Theorem 1.7])**.** *Let $(X, B, \mathbf{M})$ be a lc g-pair, and E a prime divisor that is exceptional over X such that $a(E, X, B, \mathbf{M}) \in [0, 1)$. Then there exists a birational morphism $f : Z \to X$ which extracts E such that $-E$ is ample over X.*

3. Finiteness of models

4. Double scaling

In this section we construct a special type of Sarkisov program, called the "Sarkisov program with double scaling". As the notation is complicated and technical, we first illustrate our ideas.

Replacing W by further resolution, we may assume that ρ_X and ρ_Y are morphisms:

$$\begin{tikzcd}
& W \arrow[dl, "\rho_X"'] \arrow[dr, "\rho_Y"] & \\
X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\
S_X & & S_Y
\end{tikzcd}$$

and $\mathbf{M}$ decends to W, and $(W, B_W, \mathbf{M}) \geqslant (X, B_X, \mathbf{M}), (Y, B_Y, \mathbf{M})$. Here $\phi_X : X \to S_X$ is a $(K_X + B_X, \mathbf{M})$-Mori fiber space/Z and $\phi_Y : Y \to S_Y$ is a $(K_Y + B_Y, Mm)$-Mori fiber space/Z.

We need to study the difference and similarity between $\phi_X : X \to S_X$ and $\phi_Y : Y \to S_Y$. A common strategy in birational geometry is to study the ample divisors on X and Y. This works well in our setting, as $-(K_X+B_X+M_X)$ is ample over S_X and $-(K_Y+B_Y,Mm)$ is ample over S_Y. Therefore, we may pick general ample/Z $\mathbb{R}$-divisors L_X and H_Y on X and Y respectively, such that

- $L_X\sim_{\mathbb{R},Z}-(K_X+B_X+M_X)+\phi_X^*A_{S_X}$ and
- $H_Y\sim_{\mathbb{R},Z}-(K_Y+B_Y+M_Y)+\phi_Y^*A_{S_Y}$,

for some general ample $\mathbb{R}$-divisors A_{S_X} and A_{S_Y} on S_X and S_Y respectively. In particular, $L_W:=\rho_X^*L_X$ and $H_W:=\rho_Y^*H_Y$ are big and nef/Z, and we may define $H_X:=(\rho_X)_*H_W$ and $L_Y:=(\rho_Y)_*L_W$. We have

- $K_X+B_X+L_X+0H_Y+M_X\sim_{\mathbb{R},S_X}0$, and
- $K_Y+B_Y+0L_Y+H_Y+M_Y\sim_{\mathbb{R},S_Y}0$.

4.1. **Construct a Sarkisov link.**

Construction 4.1 (Setting)**.** This setting will be used in the rest of this section. We assume that

- $X\to Z$ is a contraction,
- $\rho:W\dashrightarrow X$ is a birational map,
- $(W,B_W,\mathbf{M})$ is a g-pair with associated nef/Z $\boldsymbol{b}$-divisor which decends to W,
- L_W and H_W are two general big and nef/Z $\mathbb{R}$-divisors on W,
- $(X,B,\mathbf{M})$ is a g-pair,
- $\phi:X\to S$ is a (K_X+B+M_X)-Mori fiber space/Z,
- Σ is a ϕ-vertical curve,
- L and H are two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and
- $0<l\le 1$ and $0\le h\le 1$ are two real numbers,

such that

(1) $(W,B_W+2(L_W+H_W)+M_W)$ is $\mathbb{Q}$-factorial lc and log smooth,
(2) $K_W+B_W+H_W+M_W$ is pseudo-effective/Z,
(3) $(X,B,\mathbf{M})$ is lc,
(4) $(W,B_W+lL_W+hH_W,\mathbf{M})\ge(X,B+lL+hH,\mathbf{M})$. In particular, ρ does not extract any divisor,
(5) B,L and H are the birational transforms of B_W,L_W and H_W on X respectively,
(6) $K_X+B+lL+hH+M_X\sim_{\mathbb{R},S}0$, and
(7) $K_X+B+lL+hH+M_X$ is nef/Z.

We illustrate this setting in the following diagram:

$$\begin{tikzcd}
W \arrow[d, dashed, "\rho"'] & \supset & B_W \arrow[d, dashed] & lL_W \arrow[d, dashed] & hH_W \arrow[d, dashed] & M_W \arrow[d, dashed] & \\
X \arrow[d, "\phi"] & \supset & B & lL & hH & M_X & \Sigma:\phi\text{-vertical} \\
S & & & & & &
\end{tikzcd}$$

Definition 4.2 (Auxiliary constants and divisors)**.** Assumptions and notations as Construction 4.1,

(1) we define
$$r:=\frac{H\cdot\Sigma}{L\cdot\Sigma}.$$
(2) For any real number t, we define
$$D_W(t):=B_W+lL_W+hH_W+t(H_W-rL_W),$$

and

$$D(t) := B + lL + hH + t(H - rL).$$

(3) We define Γ to be the set of all real number t satisfying the following:
  (a) $0 \leq t \leq \frac{l}{r}$,
  (b) for any prime divisor $E \subset W$,

$$a(E, W, D_W(t), \mathbf{M}) \leq a(E, X, D(t), \mathbf{M}),$$

  and
  (c) $K_X + D(t) + M_X$ is nef$/Z$.

(4) We define $s := \sup\{t \mid t \in \Gamma\}$.
(5) We define $l_Y := l - rs$ and $h_Y := h + s$.

Lemma 4.3. *Assumptions and notations as Construction 4.1 and Definition 4.2, then*

(1) $r > 0$ is well-defined,
(2) either $\Gamma = \{0\}$, or Γ is a closed interval,
(3) Γ is non-empty and $s \in \Gamma$,
(4) $l_Y = l$ if and only if $h_Y = h$, and
(5) $\Gamma \subset [0, 1-h]$. In particular, $h_Y \leq 1$.

Proof. (1) Since L_W and H_W are general big and nef$/Z$ divisors on W, L and H are big$/Z$, hence ample$/S$. Thus $H \cdot \Sigma > 0$ and $L \cdot \Sigma > 0$, hence $r > 0$ is well-defined.

(2) By Definition 4.2(3), $0 \in \Gamma$ and Γ is closed and connected, which implies (2).

(3) This follows from (2) and the definition of s.

(4) This follows from (1) and the definitions of l_Y and h_Y.

(5) Assume that (5) does not hold. By (2), there exists $t_0 \in \Gamma$ such that $1 < h + t_0 < 2$. By Construction 4.1(1), $(W, D_W(t_0), \mathbf{M})$ is lc.

Since $(K_X + D(t_0) + M_X) \cdot \Sigma = 0$ and H is big$/Z$,

$$(K_X + B + (l - t_0 r)L + H + M_X) \cdot \Sigma = ((K_X + D(t_0) + M_X) - (h + t_0 - 1)H) \cdot \Sigma < 0.$$

Thus ϕ is a $(K_X + B + (l - t_0 r)L + H + M_X)$-Mori fiber space$/Z$. In particular, $K_X + B + H + M_X$ is not pseudo-effective$/Z$. Since ρ does not extract any divisor, $K_W + B_W + H_W + M_W$ is not pseudo-effective$/Z$, which contradicts Construction 4.1(2).

□

Construction 4.4. Assumptions and notations as Construction 4.1 and Definition 4.2. Then there are three possibilities for s:

Case 1 $s = \frac{l}{r}$. In particular, $l_Y = 0$.

Case 2
- $s < \frac{l}{r}$. In particular, $l_Y > 0$, and
- there exists $0 < \epsilon_0 \ll 1$ and a prime divisor $E \subset W$, such that $a(E, W, D_W(s + \epsilon), \mathbf{M}) > a(E, X, D(s + \epsilon), \mathbf{M})$ for all $0 < \epsilon < \epsilon_0$.

Case 3
- $s < \frac{l}{r}$. In particular, $l_Y > 0$, and
- there exists $0 < \epsilon_0 \ll 1$, such that for all $0 < \epsilon < \epsilon_0$
  - $a(E, W, D_W(s + \epsilon), \mathbf{M}) \leq a(E, X, D(s + \epsilon), \mathbf{M})$ for any prime divisor $E \subset W$, and
  - $K_X + D(s + \epsilon), \mathbf{M})$ is not nef$/Z$.

Theorem 4.5 (Sarkisov link with double scaling)**.** *Assumptions and notations as Construction 4.1 and Definition 4.2. The there exist*

- *a birational map$/Z$ $\rho_Y : W \dashrightarrow Y$ which does not extract any divisor,*

- *three* $\mathbb{R}$*-divisors* B_Y, L_Y *and* H_Y *on* Y*,*
- *a* $(K_Y+B_Y+M_Y)$*-Mori fiber space/*Z $\phi_Y : Y \to S_Y$*, and*
- *a Sarkisov link/*Z $f : X \dashrightarrow Y$*,*

If Y is not $\mathbb{Q}$-
factorial, th
how to show
and H_Y are
Cartier divi

such that

(1) $(Y, B_Y, \mathbf{M})$ *is a* $\mathbb{Q}$*-factorial lc g-pair/*Z*,*
(2) $(W, B_W + l_Y L_W + h_Y H_W, \mathbf{M}) \geq (Y, B_Y + l_Y L_Y + h_Y H_Y, \mathbf{M})$*. In particular,* ρ_Y *does not extract any divisor,*
(3) B_Y, L_Y *and* H_Y *are the birational transforms of* B_W, L_W *and* H_W *on* Y *respectively,*
(4) $K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y \sim_{\mathbb{R}, S_Y} 0$*,*
(5) $K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y$ *is nef/*Z*,*
(6) *for any* ϕ_Y*-vertical curve* Σ_Y *on* Y*, and* $r = \frac{H\cdot\Sigma}{L\cdot\Sigma} \geq \frac{H_Y\cdot\Sigma_Y}{L_Y\cdot\Sigma_Y} > 0$.

Proof. We prove the Theorem by considering the three different cases in Construction 4.4 separately.

Case 1. In this case, we finish the proof by letting $\rho_Y := \rho, Y := X, B_Y := B, L_Y := L, H_Y := H, M_Y := M_X, \phi_Y := \phi_X, S_Y := S$, and $f := \mathrm{id}_X$.

Case 2. In this case, $a(E, W, D_W(s), \mathbf{M}) = a(E, X, D(s), \mathbf{M})$, and E is exceptional/X. Since $E \subset W$,

But $(X, D(s+\epsilon), \mathbf{M})$ may not be lc

$$a(E, X, D(s+\epsilon), \mathbf{M}) < a(E, W, D_W(s+\epsilon), \mathbf{M}) \leq 1.$$

We need $(X, D(s+\epsilon), \mathbf{M})$ to be lc. We only have $(X, D(s), \mathbf{M})$ lc.

By Lemma 2.12, there is an extraction $g : V \to X$ of E such that V is $\mathbb{Q}$-factorial. We let B_V, L_V, H_V be the birational transforms of B_W, L_W and H_W on V respectively, then we have

$$\begin{aligned}&K_V + B_V + (l_Y - r\epsilon)L_V + (h_Y+\epsilon)H_V + M_V\\ =&g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y+\epsilon)H + M_X).\end{aligned}$$

Moreover, since $a(E, X, D(s+\epsilon) + M_X) < 1$, $\mathrm{mult}_E(B_V + (l_Y - r\epsilon)L_V + (h_Y+\epsilon)H_V) > 0$. Thus we may pick a sufficiently small positive real number $0 < \delta \ll \epsilon$, such that $(V, \Delta_V + M_V)$ is lc, where

This may not lc

$$K_V + \Delta_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta)L + (h_Y+\epsilon)H + M_X).$$

We may run a $(K_V + \Delta_V + M_V)$-MMP/S $\psi : V \dashrightarrow Y$ which terminates with a Mori fiber space/S $\phi_Y : Y \to S_Y$ by Theorem 2.9. Since $\rho(V/S) = \rho(V/X) + \rho(X/S) = 2$ and $1 = \rho(Y/S_Y) \leq \rho(V/S_Y) \leq \rho(V/S)$, there are two possibilities:

Case 2.1. $\rho(V/Y) = 0$. In this case ψ is a sequence of flips, and we get a Sarkisov link/Z $f : X \dashrightarrow Y$ of type I. Let B_Y, L_Y and H_Y be the birational transforms of B_V, L_V and H_V on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0 < \delta' \ll \delta$ and let

$$K_V + \Delta'_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y+\epsilon)H + M_X),$$

then ψ is also a $(K_V + \Delta'_V + M_V)$-MMP/S. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned}&g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y+\epsilon)H + M_X)\cdot\Sigma_V\\ =&(K_Y + \Delta'_Y + M_Y)\cdot\Sigma_Y < 0\end{aligned}$$

Let $\delta' \to 0$, then we have

$$g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y+\epsilon)H + M_X)\cdot\Sigma_V \leq 0.$$

Since $g^*(K_X+B+l_YL+h_YH+M_X)\sim_{\mathbb{R},S}0$, we deduce that

$$g^*(H-rL)\cdot\Sigma_V\le 0.$$

Moreover, by our assumptions, $g^*(H-rL)=g^{-1}_*(H-rL)+eE$ for some real number $e>0$, and $\Sigma_V\not\subset E$. Thus

$$\begin{aligned}(H_Y-rL_Y)\cdot\Sigma_Y=g^{-1}_*(H-rL)\cdot\Sigma_V&=(g^*(H-rL)-eE)\cdot\Sigma_V\\ \le g^*(H-rL)\cdot\Sigma_V&\le 0,\end{aligned}$$

which implies (6), and the theorem follows in this case.

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0<\delta'\ll\delta$ and let

$$K_V+\Delta'_V+M_V:=g^*(K_X+B+(l_Y-r\epsilon-\delta')L+(h_Y+\epsilon)H+M_X),$$

then ψ is also a $(K_V+\Delta'_V+M_V)$-MMP$/S$. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned}&g^*(K_X+B+(l_Y-r\epsilon-\delta')L+(h_Y+\epsilon)H+M_X)\cdot\Sigma_V\\ =&(K_Y+\Delta'_Y+M_Y)\cdot\Sigma_Y<0\end{aligned}$$

Let $\delta'\to 0$, then we have

$$g^*(K_X+B+(l_Y-r\epsilon)L+(h_Y+\epsilon)H+M_X)\cdot\Sigma_V\le 0.$$

Since $g^*(K_X+B+l_YL+h_YH+M_X)\sim_{\mathbb{R},S}0$, we deduce that

$$g^*(H-rL)\cdot\Sigma_V\le 0.$$

Moreover, by our assumptions, $g^*(H-rL)=g^{-1}_*(H-rL)+eE$ for some real number $e>0$, and $\Sigma_V\not\subset E$. Thus

$$\begin{aligned}(H_Y-rL_Y)\cdot\Sigma_Y=g^{-1}_*(H-rL)\cdot\Sigma_V&=(g^*(H-rL)-eE)\cdot\Sigma_V\\ \le g^*(H-rL)\cdot\Sigma_V&\le 0,\end{aligned}$$

which implies (6), and the theorem follows in this case.

Case 3. In this case, there exists a $(K_X+D(s+\epsilon)+M_X)$-negative extremal ray $[C]$ on X. Since $(K_X+D(s+\epsilon)+M_X)\cdot\Sigma=0$, $[C]\neq[\Sigma]$. Let $P\subset\overline{NE}(X/Z)$ be the extremal face over Z defined by all $(K_X+D(s+\epsilon)+M_X)$-non-positive irreducible curves. Then $P\neq[\Sigma]$, and hence there exists an extremal ray $[\Pi]$ such that $[\Sigma]$ and $[\Pi]$ span a two-dimensional face of P. By our construction, $(K_X+D(s+\epsilon)+M_X)\cdot\Pi<0$. Now for $0<\delta\ll 1$, we have

$(X,D(s+\epsilon),\mathbf{M})$ may not be lc

$$(K_X+B+(l_Y-r\epsilon-\delta)L_X+(h_Y+\epsilon)H_X+M_X)\cdot\Sigma<0$$

and

$$(K_X+B+(l_Y-r\epsilon-\delta)L_X+(h_Y+\epsilon)H_X+M_X)\cdot\Pi<0.$$

By Theorem 2.8, there exists a contraction $\pi:X\to T$ of the extremal face of $\overline{NE}(X/Z)$ spanned by $[\Sigma]$ and $[\Pi]$. Then π factors through S, and $K_X+D(s)+M_X\sim_{\mathbb{R},T}0$.

Since L,H are big$/Z$, L,H are big$/T$. Therefore, if $K_X+D(s+\epsilon)+M_X$ is pseudo-effective$/T$, then $K_X+(1+\alpha)D(s+\epsilon)+M_X$ is big$/T$. By Theorem 2.10, we may run a $(K_X+D(s+\epsilon)+M_X)$-MMP$/T$ with scaling of some ample$/T$ divisor, and this MMP$/T$ terminates. There are three cases:

Case 3.1. After a sequence of flips $f:X\dashrightarrow Y$, the MMP$/T$ terminates with a Mori fiber space$/T$ $\phi_Y:Y\to S_Y$. Therefore, f is a Sarkisov link$/Z$ of type IV. Let B_Y,L_Y,H_Y be the birational transforms of B,L and H on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since ϕ_Y is a $(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y)$-Mori fiber space$/T$,

$$-(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \cdot \Sigma_Y > 0,$$

which implies that

$$-(K_X + D(s + \epsilon) + M_X) \cdot \Sigma_X > 0.$$

Since $K_X + D(s) + M_X \sim_{\mathbb{R},T} 0$,

$$-(K_X + D(s) + M_X) \cdot \Sigma_X = 0,$$

which implies that

$$(H_Y - rL_Y) \cdot \Sigma_Y = (H - rL) \cdot \Sigma_X < 0.$$

Thus $r > \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.2. After a sequence of flips $X \dashrightarrow U$, we get a divisorial contraction$/T$: $U \to Y$. Therefore $\rho(Y/T) = 1$, which implies that the induced morphism $\phi_Y := Y \to T$ is a Mori fiber space, and the induced birational map $f : X \dashrightarrow Y$ is a Sarkisov link$/Z$ of type III. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since $-(K_X + D(s + \epsilon) + M_X)$ is nef$/T$ and $K_X + D(s) + M_X \sim_{\mathbb{R},T} 0$, we have

$$-(K_X + D(s + \epsilon) + M_X) \cdot \Sigma_X \geq 0 = -(K_X + D(s) + M_X) \cdot \Sigma_X,$$

which implies that

$$(H_Y - rL_Y) \cdot \Sigma_Y = (H - rL) \cdot \Sigma_X \leq 0.$$

Thus $r \geq \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.3. After a sequence of flips $f : X \dashrightarrow Y$, the MMP terminates with a minimal model Y over T. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively. Since Σ is a general ϕ-vertical curve, we may let Σ' be the birational transform of Σ on Y. Since $(K_X + D(s + \epsilon) + M_X) \cdot \Sigma = 0$ and $(K_X + D(s) + M_X) \cdot \Sigma = 0$, we have

$$(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \cdot \Sigma' = 0$$

and

$$(K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y) \cdot \Sigma' = 0$$

which implies that $(K_Y + B_Y + M_Y) \cdot \Sigma' < 0$ and $r = \frac{H_Y \cdot \Sigma'}{L_Y \cdot \Sigma'}$. Since Σ can be chosen to be any ϕ-vertical curve, by Theorem 2.8, there exists a contraction $\phi_Y : Y \to S_Y$ of $[\Sigma']$ such that ϕ_Y is a $(K_Y + B_Y + M_Y)$-Mori fiber space$/T$. Thus f is a Sarkisov link$/Z$ of type IV. We finish the proof by letting $\rho_Y : W \dashrightarrow Y$ be the induced birational map. $\square$

4.2. **Behavior of invariants under a Sarkisov lins.**

Lemma 4.6. *Assumptions and notations as in Construction 4.1, Definition 4.2, and Theorem 4.5. Then*

(1) In **Case 2.1**, $\rho(Y)-\rho(X)=1$.

(2) In **Case 2.2**,

  (a) $\rho(X)=\rho(Y)$,

  (b) there is a prime divisor F_0 *over* W, *such that*

$$\begin{aligned}&a(F_0,X,B+(l_Y-r\epsilon-\delta)L+(h_Y+\epsilon)H,\mathbf{M}_X)\\ <&a(F_0,Y,B_Y+(l_Y-r\epsilon-\delta)L_Y+(h_Y+\epsilon)H_Y,\mathbf{M}_Y),\end{aligned}$$

  and

  (c) for any prime divisor F *over* W,

$$\begin{aligned}&a(F,X,B+(l_Y-r\epsilon-\delta)L+(h_Y+\epsilon)H,\mathbf{M}_X)\\ \leq&a(F,Y,B_Y+(l_Y-r\epsilon-\delta)L_Y+(h_Y+\epsilon)H_Y,\mathbf{M}_Y).\end{aligned}$$

(3) In **Case 3**,

$$a(F,Y,B_Y+(l_Y-r\epsilon)L_Y+(h_Y+\epsilon)H_Y,\mathbf{M}_Y)\geq a(F,W,B_W(s+\epsilon),\mathbf{M}).$$

(4) In **Case 3.1**, $\frac{H\cdot\Sigma}{L\cdot\Sigma}>\frac{H_Y\cdot\Sigma_Y}{L_Y\cdot\Sigma_Y}$.

(5) In **Case 3.2**, $\rho(X)-\rho(Y)=1$.

(6) In **Case 3.3**,

  (a) $\rho(X)=\rho(Y)$,

  (b) there is a prime divisor F_0 *over* W, *such that*

$$a(F_0,Y,B_Y+(l_Y-r\epsilon)L_Y+(h_Y+\epsilon)H_Y,\mathbf{M}_Y)>a(F_0,X,B(s+\epsilon),\mathbf{M}_X),$$

  and

  (c) for any prime divisor F *over* W,

$$a(F,Y,B_Y+(l_Y-r\epsilon)L_Y+(h_Y+\epsilon)H_Y,\mathbf{M}_Y)\geq a(F,X,B(s+\epsilon),\mathbf{M}_X).$$

Proof. (1)(4)(5) immediately follow from the proof of Theorem 4.5. (2) follows from the fact that in **Case 2.2**, the Sarkisov link$/Z$ is constructed by running a $g^*(K_X+B+(l_Y-r\epsilon-\delta)L+(h_Y+\epsilon)H+M_X)$-MMP$/S$ and $X\not\cong Y$. (3)(6) follow from the fact that in **Case 3**, the Sarkisov link$/Z$ is constructed by running a $(K_X+B(s+\epsilon)+M_X)$-MMP$/T$ and $X\not\cong Y$ in **Case 3.3**. □

4.3. **Run the Sarkisov program with double scaling.**

Construction 4.7 (Sarkisov program with double scaling)**.** Assume that $W\to Z$ is a contraction and $(W,B_W,\mathbf{M})$ is a $\mathbb{Q}$-factorial lc g-pair$/Z$ with nef$/Z$ ***b***-divisor $M=M_W$, such that $K_W+B_W+M_W$ is not pseudo-effective$/Z$.

can we remove $\mathbb{Q}$-factoriality?

Let $\rho:W\dashrightarrow X$ be a $(K_W+B_W+M_W)$-non-positive map$/Z$ and $\phi:X\to S$ a (K_X+B+M_X)-Mori fiber space$/Z$, where B is the birational transform of B_W on X. By Theorem 2.10, a special choice of ρ is when ρ is a $(K_W+B_W+M_W)$-MMP$/Z$. By Lemma 2.11, possibly taking a resolution of indeterminacy $p:W'\to W$ and $q:W'\to X$, we may assume that W is smooth and ρ is a morphism. Then ϕ is a (K_X+B+M_X)-Mori fiber space$/Z$. In particular, $-(K_X+B+M_X)$ is ample$/S$. Therefore, we may pick a general ample$/Z$ $\mathbb{R}$-divisor A on S such that $-(K_X+B+M_X)+\phi^*A$ is ample$/Z$. We let L be a general element of $|-(K_X+B+M_X)+\phi^*A|_{\mathbb{R}/Z}$ and $L_W:=\rho^*L=(\rho^{-1})_*L$. Then L_W is big and nef$/Z$, $K_X+B+L+M_X\sim_{\mathbb{R},S}0$ and $K_X+B+L+M_X$ is nef$/Z$.

Finally, we pick a general big and nef$/Z$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H_W on W such that

- $(W, B_W + 2(L_W + H_W) + M_W)$ is $\mathbb{Q}$-factorial lc, and
- $K_W + B_W + H_W + M_W$ is pseudo-effective$/Z$,

and pick a general ϕ-vertical curve Σ on X. We construct the *Sarkisov program$/Z$ of* $(X, B + M_X)$ *with scaling of* (L_W, H_W) in the following way.

Step 1 We define $X_0 := X, B_0 := B, \rho_0 := \rho, \phi_0 := \phi$, $L_0 := L, H_0 := \rho_* H_W$, $r_0 := \frac{H_0 \cdot \Sigma}{L_0 \cdot \Sigma}$, $\Sigma_0 := \Sigma$, and $(l_0, h_0) := (1, 0)$.

Step 2 For any integer $i \geq 0$, suppose that we already have
- a $\mathbb{Q}$-factorial lc g-pair $(X_i, B_i, \mathbf{M})$,
- a birational map $\rho_i : W \dashrightarrow X_i$,
- a $(K_{X_i} + B_i + M_{X_i})$-Mori fiber space$/Z$ $\phi_i : X_i \to S_i$,
- two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors L_i and H_i on X_i,
- two real number $0 < l_i \leq 1$ and $0 \leq h_i \leq 1$,
- a general ϕ_i-vertical curve Σ_i, and
- $r_i := \frac{H_i \cdot \Sigma_i}{L_i \cdot \Sigma_i} > 0$

such that
- $(W, B_W + l_i L_W + h_i H_W, \mathbf{M}) \geq (X_i, B_i + l_i L_i + h_i H_i, \mathbf{M})$,
- B_i, L_i and H_i are the birational transforms of B_i, L_i and H_i on X_i respectively,
- $K_{X_i} + B_i + l_i L_i + h_i H_i + M_{X_i} \sim_{\mathbb{R}, S_i} 0$, and
- $K_{X_i} + B_i + l_i L_i + h_i H_i + M_{X_i}$ is nef$/Z$,

then by Theorem 4.5, there exists
- a $\mathbb{Q}$-factorial lc g-pair $(X_{i+1}, B_{i+1} + M_{X_{i+1}})$,
- a birational map $\rho_{i+1} : W \dashrightarrow X_{i+1}$,
- a $(K_{X_{i+1}} + B_{i+1} + M_{X_{i+1}})$-Mori fiber space$/Z$ $\phi_{i+1} : X_{i+1} \to S_{i+1}$,
- two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors L_{i+1} and H_{i+1} on X_{i+1},
- two real number $0 \leq l_{i+1} \leq l_i$ and $h_i \leq h_{i+1} \leq 1$,
- a ϕ_{i+1}-vertical curve Σ_{i+1},
- a real number $r_{i+1} := \frac{H_{i+1} \cdot \Sigma_{i+1}}{L_{i+1} \cdot \Sigma_{i+1}}$, and
- a Sarkisov link$/Z$ $f_i : X_i \dashrightarrow X_{i+1}$ as in **Case 1**, or **Case 2.1**, or **Case 2.2**, or **Case 3.1**, or **Case 3.2**, or **Case 3.3** of Theorem 4.5,

such that
- $(W, B_W + l_{i+1} L_W + h_{i+1} H_W, \mathbf{M}) \geq (X_{i+1}, B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1}, \mathbf{M})$,
- B_{i+1}, L_{i+1} and H_{i+1} are the birational transforms of B_i, L_i and H_i on X_{i+1} respectively,
- $K_{X_{i+1}} + B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1} + M_{X_{i+1}} \sim_{\mathbb{R}, S_{i+1}} 0$,
- $K_{X_{i+1}} + B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1} + M_{X_{i+1}}$ is nef$/Z$, and
- $r_i \geq r_{i+1} > 0$.

Notice that the assumptions hold when $i = 0$.

Step 3 If $l_{i+1} = 0$, we stop and let $n := i + 1$. Otherwise, we replace i with $i + 1$ and return to **Step 2**.

The following diagram gives the birational maps and Mori fiber spaces in this construction:

$$\begin{tikzcd}
 & & & W \arrow[dlll, dashed, "\rho_0"'] \arrow[dll, dashed, "\rho_1"] \arrow[d, dashed, "\rho_i"] \arrow[drr, dashed, "\rho_n"] & & & \\
X_0 \arrow[r, dashed, "f_0"'] \arrow[d, "\phi_0"] & X_1 \arrow[r, dashed, "f_1"'] \arrow[d, "\phi_1"] & \dots \arrow[r, dashed] & X_i \arrow[r, dashed, "f_i"] \arrow[d, "\phi_i"] & \dots \arrow[r, dashed] & X_n \arrow[r, dashed, "f_n"] \arrow[d, "\phi_n"] & \dots \\
S_0 & S_1 & & S_i & & S_n &
\end{tikzcd}$$

Lemma 4.8. *Assumptions and notation as in Construction 4.7. Then*

(1) there are only finitely many possibilities of X_i up to isomorphism, and
(2) the Sarkisov program of $(X, B + M_X)$ with scaling of (L_W, H_W) terminates, i.e. there exists an integer $n > 0$ such that $l_n = 0$.

TODO: In this proof, we need finiteness of gwlcm (generalized weak log minimal model)

5. Proof of the main theorems

Proof of Theorem 1.1. It is a special case of Theorem 1.2. □

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