# SARKISOV PROGRAM FOR LC PAIRS

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Abstract. We establish the Sarkisov program for lc pairs. As applications and related results, we prove a result on the finiteness of models for lc pairs, and show that lc Fano varieties are Mori dream spaces. We also establish the lc generalized pair version of the forestated results.

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## 1. Introduction

In this paper we work over the field of complex numbers $\mathbb{C}$.

**Theorem 1.1** (Lc Sarkisov program)**.** *Assume that*

*(1)* $(W, B_W)/Z$ *is an lc pair such that* $K_W + B_W$ *is not pseudo-effective*$/Z$*,*
*(2)* $\rho_X : W \dashrightarrow X$ *and* $\rho_Y : W \dashrightarrow Y$ *are two* $(K_W + B_W)$*-MMP*$/Z$*,* $B_X := (\rho_X)_* B_W$*, and* $B_Y := (\rho_Y)_* B_W$*, and*
*(3)* $\phi_X : X \to S_X$ *is a* $(K_X + B_X)$*-Mori fiber space*$/Z$ *and* $\phi_Y : Y \to S_Y$ *is a* $(K_Y + B_Y)$*-Mori fiber space*$/Z$*.*

$$\begin{tikzcd} & W \arrow[dl, dashed, "\rho_X"'] \arrow[dr, dashed, "\rho_Y"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\ S_X & & S_Y \end{tikzcd}$$

*Thent he induced birational map* $f : X \dashrightarrow Y$ *is given by a finite sequence of Sarkisov links*$/Z$*, i.e.* $f$ *can be written as* $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$*, where each* $X_i \dashrightarrow X_{i+1}$ *is a Sarkisov link*$/Z$*,*

**Theorem 1.2** (lc Sarkisov program for generalized pairs)**.** *Assume that*

*(1)* $(W, B_W, \mathbf{M})/Z$ *is an lc generalized pair such that* $K_W + B_W + \mathbf{M}_W$ *is not pseudo-effective*$/Z$*,*
*(2)* $\rho_X : W \dashrightarrow X$ *and* $\rho_Y : W \dashrightarrow Y$ *are two* $(K_W + B_W + \mathbf{M}_W)$*-MMP*$/Z$*,* $B_X := (\rho_X)_* B_W$*, and* $B_Y := (\rho_Y)_* B_W$*, and*
*(3)* $\phi_X : X \to S_X$ *is a* $(K_X + B_X + \mathbf{M}_X)$*-Mori fiber space*$/Z$ *and* $\phi_Y : Y \to S_Y$ *is a* $(K_Y + B_Y + \mathbf{M}_Y)$*-Mori fiber space*$/Z$*.*

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$$\begin{tikzcd} & W \arrow[dl, dashed, "\rho_X"'] \arrow[dr, dashed, "\rho_Y"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\ S_X & & S_Y \end{tikzcd}$$

*Thent he induced birational map $f : X \dashrightarrow Y$ is given by a finite sequence of Sarkisov links/$Z$, i.e. $f$ can be written as $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$, where each $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link/$Z$,*

**Theorem 1.3** (Generalized lc Fano varieties are Mori dream spaces)**.** *Let $(X, B, \mathbf{M})/Z$ be an lc generalized pair such that $-(K_X + B + \mathbf{M}_X)$ is ample/$Z$. Then $X$ is a Mori dream space/$Z$. In particular, for any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ on $X$, we may run a $D$-MMP/$Z$ which termiantes with either a good minimal model/$Z$ or a Mori fiber space/$Z$.*

**Theorem 1.4** (Finiteness of weak log canonical models for lc generalized pairs)**.** *Let $X \to Z$ be a projective morphism between normal quasi-projective varieties, $\mathbf{M}$ an NQC/$Z$ $\boldsymbol{b}$-divisor on $X$, and $A \geq 0$ an ample $\mathbb{R}$-divisor on $X$. Let $\mathcal{V} \subset \mathrm{Weil}_{\mathbb{R}}(X)$ a finite dimensional rational subspace and $\mathcal{C} \subset \mathcal{L}_A(\mathcal{V})$ a rational polytope such that $(X, B, \mathbf{M})$ is lc for any $B \in \mathcal{C}$. Then there exists an integer $k \geq 0$ and birational maps/$Z$ $\phi_i : X \dashrightarrow Y_i$ for each $1 \leq i \leq k$, such that*

*(1) $\phi_i$ does not extract any divisor,*
*(2) for every $B \in \mathcal{C}$, there exists $i$ such that $(Y_i, (\phi_i)_*B, \mathbf{M})/Z$ is a weak lc model of $(X, B, \mathbf{M})/Z$, and*
*(3) for any $B \in \mathcal{C}$ and any log minimal model $(Y, B_Y, \mathbf{M})/Z$ of $(X, B, \mathbf{M})/Z$ with induced birational map $\phi : X \dashrightarrow Y$, if*
   - *there exists an ample $\mathbb{R}$-divisor $A_Y \geq 0$ and an $\mathbb{R}$-divisor $\Delta_Y \geq 0$ on $Y$, such that $B_Y \sim_{\mathbb{R},Z} A_Y + \Delta_Y$ and $(Y, \Delta_Y + A_Y)$ is lc,*

   *then there exists $j$, such that $\psi := \phi_j \circ \phi^{-1} : Y \to Y_j$ is an isomorphism.*

We remark that the existence of the ample $\mathbb{R}$-divisors $A$ and $A_Y$ in Theorem 1.4 are crucial by considering the following example:

**Example 1.5** ([Gon09])**.** Let $S$ be a K3 surface with infinitely many $(-2)$-curves, $X_0$ the projective cone of $S$, and $\phi : X \to X_0$ the blow-up of the vertex.

Let $H_0$ be a general and sufficiently ample divisor on $X_0$, $E$ the $\phi$-exceptional prime divisor, and $H := \phi_*^{-1}H_0$. Then $K_X + E + H = \phi^*(K_{X_0} + H_0)$ and $K_X + E + H$ is big and nef. By [Gon09, Example 0.3], there are infinitely many log minimal models of $(X, E + H)$. Therefore, there are infinitely many log minimal models of $(X_0, H_0)$. However, it is easy to see that the only log minimal model of $(X_0, H_0)$ which does not extract any divisor is $(X_0, H_0)$ itself.

Now we let $Y_0$ be the cone of $X_0$ and let $Y$ be the main component of $Y \times_{X_0} X$. Then the induced morphism $\phi_Y : Y \to Y_0$ is small. Let $H_{Y_0}$ be a general and sufficiently ample divisor on $Y_0$ and let $H_Y := (\phi_Y)_*^{-1}H_{Y_0}$. By the same arguments as in [Gon09, Example 0.3], there are infinitely many log minimal models of $(Y, H_Y)$. Therefore, there are infinitely many log minimal models of $(Y_0, H_{Y_0})$ which does not extract any divisor.

However, except $(Y_0, H_{Y_0})$ itself, no log minimal modeo of $(Y_0, H_{Y_0})$ satisfies the additional condition as in Theorem 1.4(3). In particular, they cannot be achieved by running a $(K_{Y_0}+H_{Y_0})$-MMP.

## 2. Preliminaries

We will work over the field of complex numbers $\mathbb{C}$. Throughout the paper, we will mainly work with normal quasi-projective varieties to ensure consistency with the references. However, most results should also hold for normal varieties that are not necessarily quasi-projective. Similarly, most results in our paper should hold for any algebraically closed field of characteristic zero. We will adopt the standard notations and definitions in [KM98, BCHM10] and use them freely. For generalized pairs, we will follow the notations and definitions in [HL21]. We emphasize that, throughout this paper, generalized pairs are always assumed to be NQC.

**Definition 2.1** ($\boldsymbol{b}$-divisors)**.** Let $X$ be a normal quasi-projective variety. We call $Y$ a *birational model* over $X$ if there exists a projective birational morphism $Y \to X$.

Let $X \dashrightarrow X'$ be a birational map. For any valuation $\nu$ over $X$, we define $\nu_{X'}$ to be the center of $\nu$ on $X'$. A $\boldsymbol{b}$*-divisor* $\mathbf{D}$ over $X$ is a formal sum $\mathbf{D} = \sum_\nu r_\nu \nu$ where $\nu$ are valuations over $X$ and $r_\nu \in \mathbb{R}$, such that $\nu_X$ is not a divisor except for finitely many $\nu$. If in addition, $r_\nu \in \mathbb{Q}$ for every $\nu$, then $\mathbf{D}$ is called a $\mathbb{Q}$-$\boldsymbol{b}$*-divisor*. The *trace* of $\mathbf{D}$ on $X'$ is the $\mathbb{R}$-divisor

$$\mathbf{D}_{X'} := \sum_{\nu_{i,X'} \text{ is a divisor}} r_i \nu_{i,X'}.$$

If $\mathbf{D}_{X'}$ is $\mathbb{R}$-Cartier and $\mathbf{D}_Y$ is the pullback of $\mathbf{D}_{X'}$ on $Y$ for any birational model $Y$ of $X'$, we say that $\mathbf{D}$ *descends* to $X'$, and also say that $\mathbf{D}$ is the *closure* of $\mathbf{D}_{X'}$, and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Let $X \to U$ be a projective morphism and assume that $\mathbf{D}$ is a $\boldsymbol{b}$-divisor over $X$ such that $\mathbf{D}$ descends to some birational model $Y$ over $X$. If $\mathbf{D}_Y$ is nef$/U$, then we say that $\mathbf{D}$ is *nef*$/U$. If $\mathbf{D}_Y$ is a Cartier divisor, then we say that $\mathbf{D}$ is $\boldsymbol{b}$*-Cartier*. If $\mathbf{D}_Y$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we say that $\mathbf{D}$ is $\mathbb{Q}$-$\boldsymbol{b}$*-Cartier*. If $\mathbf{D}$ can be written as an $\mathbb{R}_{\geq 0}$-linear combination of nef$/U$ $\boldsymbol{b}$-Cartier $\boldsymbol{b}$-divisors, then we say that $\mathbf{D}$ is *NQC*$/U$.

We let $\mathbf{0}$ be the $\boldsymbol{b}$-divisor $\bar{0}$.

**Definition 2.2** (Generalized pairs)**.** A *generalized sub-pair* (*g-sub-pair* for short) $(X, B, \mathbf{M})/U$ consists of a normal quasi-projective variety $X$ associated with a projective morphism $X \to U$, an $\mathbb{R}$-divisor $B$ on $X$, and an NQC$/U$ $\boldsymbol{b}$-divisor $\mathbf{M}$ over $X$, such that $K_X + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier. If $B$ is a $\mathbb{Q}$-divisor and $\mathbf{M}$ is a $\mathbb{Q}$-$\boldsymbol{b}$-divisor, then we say that $(X, B, \mathbf{M})/U$ is a $\mathbb{Q}$*-g-sub-pair*.

If $\mathbf{M} = \mathbf{0}$, a g-sub-pair $(X, B, \mathbf{M})/U$ is called a *sub-pair* and is denoted by $(X, B)$ or $(X, B)/U$.

If $U = \{pt\}$, we usually drop $U$ and say that $(X, B, \mathbf{M})$ is a *projective*.

A g-sub-pair (resp. NQC g-sub-pair, $\mathbb{Q}$-g-sub-pair) $(X, B, \mathbf{M})/U$ is called a *g-pair* (resp. *NQC g-pair*, $\mathbb{Q}$*-g-pair*) if $B \geq 0$. A sub-pair $(X, B)$ is called a *pair* if $B \geq 0$.

**Notation 2.3.** In the previous definition, if $U$ is not important, we may also drop $U$. This usually happens when we emphasize the structures of $(X, B, \mathbf{M})$ that are independent of the choice of $U$, such as the singularities of $(X, B, \mathbf{M})$. See Definition 2.4 below.

**Definition 2.4** (Singularities of generalized pairs)**.** Let $(X, B, \mathbf{M})/U$ be a g-(sub-)pair. For any prime divisor $E$ and $\mathbb{R}$-divisor $D$ on $X$, we define $\operatorname{mult}_E D$ to be the *multiplicity* of $E$ along $D$. Let $h : W \to X$ be any log resolution of $(X, \operatorname{Supp} B)$ such that $\mathbf{M}$ descends to $W$, and let

$$K_W + B_W + \mathbf{M}_W := h^*(K_X + B + \mathbf{M}_X).$$

The *log discrepancy* of a prime divisor $D$ on $W$ with respect to $(X, B, \mathbf{M})$ is $1 - \operatorname{mult}_D B_W$ and it is denoted by $a(D, X, B, \mathbf{M})$.

We say that $(X, B, \mathbf{M})$ is *(sub-)lc* (resp. *(sub-)klt*) if $a(D, X, B, \mathbf{M}) \geq 0$ (resp. $> 0$) for every log resolution $h : W \to X$ as above and every prime divisor $D$ on $W$.

We say that $(X, B, \mathbf{M})$ is *dlt* if $(X, B, \mathbf{M})$ is lc, and there exists a closed subset $V \subset X$, such that

(1) $X \backslash V$ is smooth and $B_{X \backslash V}$ is simple normal crossing, and
(2) for any prime divisor $E$ over $X$ such that $a(E, X, B, \mathbf{M}) = 0$, $\operatorname{center}_X E \not\subset V$ and $\operatorname{center}_X E \backslash V$ is an lc center of $(X \backslash V, B|_{X \backslash V})$.

If $\mathbf{M} = \mathbf{0}$ and $(X, B, \mathbf{M})$ is (sub-)lc (resp, (sub-)klt, dlt), we say that $(X, B)$ is (sub-)lc (resp. (sub-)klt, dlt). We remark that the definition of dlt for g-pairs coincides with the definitions in all literature thanks to [Has22, Theorem 6.1].

Suppose that $(X,B,\mathbf{M})$ is sub-lc. A *lc place* of $(X,B,\mathbf{M})$ is a prime divisor $E$ over $X$ such that $a(E,X,B,\mathbf{M})=0$. A *lc center* of $(X,B,\mathbf{M})$ is the center of a lc place of $(X,B,\mathbf{M})$ on $X$. The *non-klt locus* $\operatorname{Nklt}(X,B,\mathbf{M})$ of $(X,B,\mathbf{M})$ is the union of all lc centers of $(X,B,\mathbf{M})$. If $\mathbf{M}=\mathbf{0}$, a lc place (resp. a lc center, the non-klt locus) of $(X,B,\mathbf{M})$ will be called an lc place (resp. an lc center, the non-klt locus) of $(X,B)$, and we will denote $\operatorname{Nklt}(X,B,\mathbf{M})$ by $\operatorname{Nklt}(X,B)$.

We note that the definitions above are independent of the choice of $U$.

## 3. Finiteness of models

## 4. Proof of the main theorems

*Proof of Theorem 1.1.* It is a special case of Theorem 1.2. □

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