SARKISOV PROGRAM FOR GENERALIZED PAIRS

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ABSTRACT. In this paper we show that any two birational Mori fiber spaces of $\mathbb{Q}$-factorial gklt g-pairs are connected by a finite sequence of Sarkisov links.

CONTENTS

1. INTRODUCTION

In this paper we work over the field of complex numbers $\mathbb{C}$.

The minimal model program is an attempt to classify higher dimensional projective varieties. Let X be a smooth complex projective variety. There are two cases: if K_X is pseudo-effective, i.e. K_X belongs to the closure of the big cone of divisors on X, then it is expected that X has a minimal model $X \dashrightarrow X_{min}$. In particular, $K_{X_{min}}$ has terminal singularities, $K_{X_{min}}$ is $\mathbb{Q}$-Cartier and nef. The existence of such minimal models is known when K_X is big, (cf. [1]).

Given smooth projective variety X such that K_X is pseudo-effective, it is well-known that X may have more than one minimal model. Nevertheless, the birational map connecting these minimal models are well-studied. For example, Kawamata (cf. [11]) shows the following: for any smooth projective variety X and any minimal models X_{min} and X'_{min} of X, the induced birational map $X_{min} \dashrightarrow X'_{min}$ is a composition of flops. In particular, X_{min} and X'_{min} are isomorphic in codimension 1.

If K_X is not pseudo-effective, by [1], we may also run a K_X-MMP. It is not known whether this K_X-MMP terminates. However, by [1, Corollary 1.3.2], if this K_X-MMP is a MMP with scaling of some ample divisor, then

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after a sequence of flips and divisorial contractions, it terminates with a Mori fiber space $X_m \to S$. It is also well-known that the Mori fiber spaces associated to X may not be unique.

Question 1.1. *Let X be a smooth projective variety such that K_X is not pseudo-effective, and $X_m \to S$ and $X'_m \to S'$ two Mori fiber spaces associated to X, i.e. there are two K_X-MMPs $X \dashrightarrow X_m$ and $X \dashrightarrow X'_m$ which terminates with $X_m \to S$ and $X'_m \to S'$ respectively.*
   What can we say on the relationship between these two Mori fiber spaces?

The *Sarkisov program*, which was first introduced by Sarkisov in [14] and [15] in order to study the conic bundles of threefolds and was further developed and generalized by Corti ([7]), Bruno-Matsuki ([5]) in order to study the minimal model program of klt pairs, gives us some good understanding on the question above. More precisely, the goal of the Sarkisov program is to decompose the natural birational map $X_m \dashrightarrow X'_m$ as in Question 1.1 into four different types of birational maps, which are called the *Sarkisov links*. This kind of decomposition is particularly useful when calculating the birational automorphism group of Mori fiber spaces of Fano varieties.

Based on the minimal model program established in [1], in [10], Hacon and McKernan have established the Sarkisov program for any klt pairs, i.e. they show the existence of such decomposition for any Sarkisov related pairs:

Theorem 1.2 ([10, Theorem 1.3])**.** *Let (Z, Φ) be a $\mathbb{Q}$-factorial klt pair such that $K_X + \Phi$ is not pseudo-effective. Assume that $\phi : X \to S$ and $\psi : Y \to T$ are two Mori fiber spaces which are obtained by running two different $(K_Z + \Phi)$-MMPs. Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of Sarkisov links.*

The concept of *generalized pairs* was introduced in [6], and it has become clear that the study of birational geometry in this category is important. For example, the most important application of generalized pairs appears in the proof of the BAB conjecture and the existence of n-complements ([2], [4]). For simplicity, in this paper, a generalized pair is also called a *g-pair*: see Definition 2.7 for more details.

It is a natural question to ask whether the theory of Sarkisov program holds for generalized pairs. The intuition of this question comes from the famous McKernan-Shokurov conjecture (cf. [2, Conjecture 1.2],[3, Conjecture 1.7]):

Conjecture 1.3 (McKernan-Shokurov conjecture)**.** *Let $d > 0$ be an integer and $\epsilon > 0$ a real number. Then there is a real number $\delta > 0$ depending only on d and ϵ satisfying the following. Assume that (X, B) is a pair and $X \to Z$ is a contraction, such that*

  - *(X, B) is ϵ-lc of dimension d,*
  - *$K_X + B \sim_{\mathbb{R}, Z} 0$, and*
  - *$-K_X$ is big$/Z$,*

*then the discriminant **b**-divisor $\mathbf{B}_Z$ has coefficients in $[-\infty, 1 - \delta]$.*

In the conjecture above, it was predicted that $(Z, B_Z + M_Z)$ given by the canonical bundle formula has the structure of a generalized pair. The case when B is a $\mathbb{Q}$-divisor is well-known, but when B is an $\mathbb{R}$-divisor, this is only very recently proved by Han and Liu (cf. [9, Corollary 1.2]). Nevertheless, it is now natural to generalize Conjecture 1.3 to the category of generalized pairs, which is only possible after the recent works on the generalized canonical bundle formula by Filipazzi (cf. [8, Theroem 1.4]) for generalized $\mathbb{Q}$-pairs and Han-Liu (cf. [9, Corollary 1.2]) for any case. Therefore, we have the following conjecture, whose $\mathbb{Q}$-pair version is stated as in [3, Conjecture 2.4]:

Conjecture 1.4. *Let $d > 0$ be an integer and $\epsilon > 0$ a real number. Then there is a real number $\delta > 0$ depending only on d and ϵ satisfying the following. Assume that $f : X \to Z$ is a contraction and $(X, B + M_X)$ is a generalized pair/Z with associated nef/Z **b**-divisor M, such that*

- *$(X, B + M_X)$ is generalized ϵ-lc of dimension d,*
- *$K_X + B + M_X \sim_{\mathbb{R}, Z} 0$, and*
- *$-K_X$ is big/Z,*

then the generalized pair given by the sub-adjunction

$$K_X + B + M_X \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

is generalized δ-lc.

The most interesting cases of Conjecture 1.3 and Conjecture 1.4 appear when $X \to Z$ has a Mori fiber space structure. Indeed, Conjecture 1.3 was originally proposed by McKernan when $B = 0$ and $X \to Z$ is a K_X-Mori fiber space. In this case, we may use the Sarkisov program to attack the conjectures above in the following way:
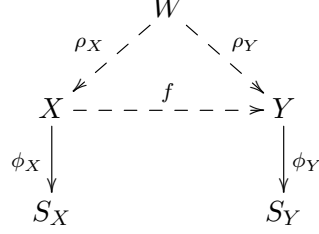
(1) We prove Conjecture 1.3 (or Conjecture 1.4) for Mori fiber spaces with good properties, e.g. when $\dim Z = 0$, or in the case of [3, Theorem 1.8, 1.9].
(2) We show that if one Mori fiber space satisfies Conjecture 1.3 (or Conjecture 1.4), then another Mori fiber space associated to a Sarkisov link also satisfies Conjecture 1.3 (or Conjecture 1.4).
(3) By applying the Sarkisov program, we show that all Mori fiber spaces that are birational to a Mori fiber spaces with good properties satisfy Conjecture 1.3 (or Conjecture 1.4)

In this paper, we deal with a small part of the outline above, that is, we extend the theory of the Sarkisov program to the theory of generalized pairs:

Theorem 1.5. *Assume that*

- *$W \to Z$ is a contraction between normal quasi-projective varieties,*
- *$(W, B_W + M_W)$ is a $\mathbb{Q}$-factorial gklt g-pair with associated nef/Z **b**-divisor M, such that $K_W + B_W + M_W$ is not pseudo-effective/Z,*

- $\rho_X : W \dashrightarrow X$ and $\rho_Y : W \dashrightarrow Y$ are two $(K_W + B_W + M_W)$-MMP/Z such that $(\rho_X)_*(K_W + B_W + M_W) = K_X + B_X + M_X$ and $(\rho_Y)_*(K_W + B_W + M_W) = K_X + B_Y + M_Y$,
- $\phi_X : X \to S_X$ is a $(K_X + B_X + M_X)$-Mori fiber space/Z and $\phi_Y : X \to S_Y$ is a $(K_Y + B_Y + M_Y)$-Mori fiber space/Z.

$$
\begin{array}{ccc}
 & W & \\
\rho_X \swarrow & & \searrow \rho_Y \\
X \dashrightarrow\!\!\!\!\overset{f}{\dashrightarrow}\!\!\!\!\dashrightarrow & & Y \\
\phi_X \downarrow & & \downarrow \phi_Y \\
S_X & & S_Y
\end{array}
$$

Then

(1) the induced birational map $f : X \dashrightarrow Y$ is given by a finite sequence of Sarkisov links/Z, i.e. f can be written as $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$, where each $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link/Z, and

(2) for any real number $\epsilon > 0$, if $(W, B_W + M_W)$ is generalized ϵ-lc, then $(X_i, B_i + M_{X_i})$ is generalized ϵ-lc for every i, where each B_i is the birational transform of B_W on X_i.

There are three additional ingredients in Theorem 1.5 comparing to Theorem 1.2. Firstly, we consider *gklt g-pairs* instead of klt pairs. Secondly, we indeed construct a *relative Sarkisov program* since all the birational maps are over Z. Finally, we show that the generalized pairs constructed in the Sarkisov program do not have worse singularities comparing to $(W, B_W + M_W)$. We remark that the last two ingredients are also unknown for the klt pair case previously, and the third ingredient will be important when we apply Theorem 1.5 to attack Conjecture 1.4 as in the outline above.

It is worth to mention that our proof of Theorem 1.5 is quite close to the original ideas of the Sarkisov program in dimension 3. Comparing to the proof of Hacon and McKernan in [10], our proof has not used many detailed results on combinatorics, but also eseentially use the finiteness of weak log canonical models (Theorem 3.6) and the uniqueness of log canonical models. We refer the readers to the proof of Theorem 5.1 for more details.

Indeed, the Sarkisov links constructed in our paper can be described very precisely, and can be viewed as a "Sarkisov program with double scaling". We refer the readers to Section 4 for more details.

2. Notation and conventions

We adopt the standard notation and definitions in [16] and [12], and will freely use them.

Definition 2.1 (**b**-divisors). Let X be a normal quasi-projective variety. A **b**-$\mathbb{R}$ Cartier **b**-divisor (**b**-divisor for short) over X is the choice of a projective birational morphism $Y \to X$ from a normal quasi-projective variety Y and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on Y up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal quasi-projective variety and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M' defines the same **b**-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullback of M and M' coincide. If there is a choice of birational morphism $Y \to X$ such that the corresponding $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M is a prime divisor, the **b**-divisor is called *prime*.

Let E be a prime **b**-divisor over X. The *center* of E on X is the closure of its image on X, and is denoted by $c_X(E)$. If $c_X(E)$ is not a divisor, E is called *exceptional*$/X$. If $c_X(E)$ is a divisor, we say that E is *on* X. For any **b**-divisor $M = \sum a_i E_i$ over X, where E_i are prime **b**-divisors over X, we define $M_X := \sum a_i c_X(E_i)$ to be the $\mathbb{R}$-divisor where the sum is taken over all the prime **b**-divisors E_i which are on X. If all the E_i are on X, we say that M is *on* X.

Definition 2.2 (Multiplicities). Let X be a normal quasi-projective variety, E a prime divisor on X and D an $\mathbb{R}$-divisor on X. We define $\operatorname{mult}_E D$ to be the multiplicity of E along D. Let F be a prime **b**-divisor over X, B an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $\phi : Y \to X$ a birational morphism such that F is on Y. We define $\operatorname{mult}_F D := \operatorname{mult}_F \phi^* D$.

Definition 2.3 (Pairs). A *pair* (X, B) consists of a normal quasi-projective variety X and an effective $\mathbb{R}$-divisor B on X such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $\phi : W \to X$ be any log resolution of (X, B) and let

$$K_W + B_W := \phi^*(K_X + B).$$

The *log discrepancy* of a prime divisor D on W with respect to (X, B) is $1 - \operatorname{mult}_D B_W$ and is denoted by $a(D, X, B)$. For any real number $\epsilon \geq 0$, we say that (X, B) is *lc* (resp. *klt*, *ϵ-lc*) if $a(D, X, B) \geq 0$ (resp. $> 0, \geq \epsilon$) for every log resolution $\phi : W \to X$ as above and every prime divisor D on W.

We say that (X, B) is $\mathbb{Q}$-factorial if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

For any prime **b**-divisor E over X, let $Y \to X$ be a birational morphism such that E_Y is a prime divisor. The *log discrepancy* of E with respect to (X, B) is $a(E_Y, X, B)$. We say that (X, B) is *terminal* if $a(E, X, B) > 1$ for every exceptional$/X$ prime **b**-divisor E.

Definition 2.4. Let $f : X \dashrightarrow Y$ a birational map between normal quasi-projective varieties, $p : W \to X$ and $q : W \to Y$ a resolution of indeterminacy of f, and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D_Y := f_* D$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. f is called D-*non-positive* (resp. D-*negative*), if
  - f does not extract any divisor, and
  - $p^* D = q^* D_Y + E$, where $E \geq 0$ is exceptional/Y (resp. $E \geq 0$ is exceptional/Y, and $\operatorname{Supp} p_* E$ contains all f-exceptional divisors).

Definition 2.5. Let X be a normal quasi-projective variety. We define $\operatorname{Weil}_\mathbb{R}(X)$ to be the $\mathbb{R}$-vector space spanned by all Weil divisors on X. Let $\mathcal{V}$ be a finite dimensional subspace of $\operatorname{Weil}_\mathbb{R}(X)$ and $A \in \mathcal{V}$ an $\mathbb{R}$-divisor. We define

$$\mathcal{L}_A(\mathcal{V}) := \{B \mid (X, B) \text{ is lc}, B = A + B', B' \geq 0, B' \in \mathcal{V}\} \subset \operatorname{Weil}_\mathbb{R}(X).$$

By [1, Lemma 3.7.2], if $\mathcal{V}$ is a rational subspace, then $\mathcal{L}_A(\mathcal{V})$ is a rational polytope.

Definition 2.6. A *contraction* is a projective morphism $f : X \to Z$ between normal quasi-projective varieties such that $f_* \mathcal{O}_X = \mathcal{O}_Z$.

For any $\boldsymbol{b}$-divisor M over X, M is called nef/Z if there is a projective morphism $Y \to X$ such that M is on Y and M_Y is nef/Z.

Definition 2.7. A *generalized pair* (*g-pair* for short) consists of a normal quasi-projective variety X, an effective $\mathbb{R}$-divisor B on X, a contraction $X \to Z$, and a $\boldsymbol{b}$-divisor M over X such that M is nef/Z. If there is no confusion, we usually say that $(X, B + M_X)$ is a generalized pair/Z. M is called the *associated nef/Z $\boldsymbol{b}$-divisor* of the generalized pair $(X, B + M_X)$. If Z is not important, we may omit Z and say that $(X, B + M_X)$ is a generalized pair.

Let $(X, B + M_X)$ be a generalized pair/Z with associated nef/Z $\boldsymbol{b}$-divisor M. Let $\phi : W \to X$ be a log resolution of (X, B) such that $M_W = M$ (i.e. M is the choice of M_W and the morphism ϕ) and

$$K_W + B_W + M_W := \phi^*(K_X + B + M_X).$$

The *generalized log discrepancy* of a prime divisor D on W with respect to $(X, B + M_X)$ is $1 - \operatorname{mult}_D B_W$ and is denoted by $a(D, X, B + M_X)$. For any prime $\boldsymbol{b}$-divisor E over X, let $Y \to X$ be a birational morphism such that E_Y is a prime divisor. The *generalized log discrepancy* of E with respect to $(X, B + M_X)$ is $a(E_Y, X, B + M_X)$. For any real number $\epsilon \geq 0$, we say that
  - $(X, B + M_X)$ is *glc* (resp. *gklt*, ϵ-*glc*) if $a(E, X, B) \geq 0$ (resp. > 0, $\geq \epsilon$) for every prime $\boldsymbol{b}$-divisor E over X,
  - $(X, B + M_X)$ is *g-terminal* if $a(E, X, B) > 1$ for every exceptional/X prime $\boldsymbol{b}$-divisor E,
  - $(X, B + M_X)$ is *gdlt* if $a(D, X, B) > 0$ for some log resolution $\phi : W \to X$ as above and every prime divisor D on W that is exceptional/X,
  - $(X, B + M_X)$ is $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

A *generalized terminalization* of a glc g-pair $(X, B + M_X)$ is a birational morphism $f : Y \to X$ satisfying the following.

- $K_Y + B_Y + M_Y = f^*(K_X + B + M_X)$,
- $(Y, B_Y + M_Y)$ is $\mathbb{Q}$-factorial g-terminal,
- f only extracts prime **b**-divisors E over X such that $0 \le a(E, X, B + M) \le 1$.

Definition 2.8. Assume that

- $X \to Z$ and $Y \to Z$ are two contractions,
- $(X, B + M_X)$ and $(Y, B_Y + M_Y)$ are two g-pairs/Z with the same associated nef/Z **b**-divisor M, and
- $f : X \dashrightarrow Y$ is a birational map/Z,

such that

- f does not extract any divisor, and
- $a(E, X, B + M_X) \le a(E, Y, B_Y + M_Y)$ for every prime **b**-divisor E over X,

then we may write $(X, B + M_X) \ge (Y, B_Y + M_Y)$.

Definition 2.9 (Sarkisov links). Assume that

- $X_1 \to Z$ and $X_2 \to Z$ are two contractions,
- $(X_1, B_1 + M_{X_1})$ and $(X_2, B_2 + M_{X_2})$ are two gklt g-pairs with the same associated nef/Z **b**-divisor M,
- $\phi_1 : X_1 \to S_1$ is a $(K_{X_1} + B_1 + M_{X_1})$-Mori fiber space/$Z$ and $\phi_2 : X_2 \to S_2$ is a $(K_{X_2} + B_2 + M_{X_2})$-Mori fiber space/$Z$,
- there are two birational morphisms $W \to X_1$ and $W \to X_2$, and an effective $\mathbb{R}$-divisor B_W on W, such that B_1 and B_2 are the pushforwards of B_W on X_1 and X_2 respectively, and
- $f : X_1 \dashrightarrow X_2$ is the induced birational map/Z.

Then

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/$Z$ of type I*, or a *Sarkisov link/Z of type I*, if there exists an extraction $g : V \to X_1$, a sequence of flips $V \dashrightarrow X_2$ over Z, and an extremal contraction $S_2 \to S_1$, such that the following diagram commutes:

$$
\begin{array}{ccc}
V & \dashrightarrow & X_2 \\
\downarrow{\scriptstyle g} & {\scriptstyle f} & \downarrow{\scriptstyle \phi_2} \\
X_1 & & S_2 \\
& \searrow{\scriptstyle \phi_1} \quad \swarrow & \\
& S_1 &
\end{array}
$$

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/$Z$ of type II*, or a *Sarkisov link/Z of type II*, if there exists an extraction $g : V \to X_1$, a sequence of flips $V \dashrightarrow U$ over Z, and a divisorial contraction $U \to X_2$, such that the following diagram commutes:

$$V \dashrightarrow U$$

$$g \downarrow \qquad \downarrow$$

$$X_1 \underset{f}{\dashrightarrow} X_2$$

$$\phi_1 \downarrow \qquad \downarrow \phi_2$$

$$S_1 \overset{\cong}{\longrightarrow} S_2$$

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/Z of type III*, or a *Sarkisov link/Z of type III*, if there exists a sequence of flips $X_1 \dashrightarrow U$ over Z, a divisorial contraction $U \to X_2$ and an extremal contraction $S_1 \to S_2$, such that the following diagram commutes:

$$X_1 \dashrightarrow U$$

- f is called a $(K_{X_1} + B_1 + M_{X_1})$-*Sarkisov link/Z of type IV*, or a *Sarkisov link/Z of type IV*, if f is a sequence of flips/Z, and there are two extremal contractions $S_1 \to T$ and $S_2 \to T$ over Z, such that the following diagram commutes:

$$X_1 \overset{f}{\dashrightarrow} X_2$$

- f is called a $(K_{X_1}+B_1+M_{X_1})$-*Sarkisov link/Z*, or a *Sarkisov link/Z*, if it is a Sarkisov link/Z of one of the four types above. We remark that we allow f to be the identity map.

3. Preliminaries

Lemma 3.1 ([6, Lemma 4.5 and 4.6])**.** *Let $m \geq 0$ be an integer, $(X, B+M_X)$ a gklt g-pair, $E_1, \ldots, E_m$ exceptional/X **b**-divisors, such that $a(E_i, X, B + M_X) \leq 1$ for each i. Then there is an extraction $f : Y \to X$ satisfying the following.*

- *f extracts exactly $E_1, \ldots, E_m$,*
- *Y is $\mathbb{Q}$-factorial, and*
- *if $m = 1$ and X is $\mathbb{Q}$-factorial, then f is a divisorial contraction.*

Corollary 3.2. *Let $(X, B + M_X)$ be a gklt g-pair, then there is a generalized terminalization of $(X, B + M_X)$. More precisely, there is an extraction $f : Y \to X$, such that Y is $\mathbb{Q}$-factorial, and f extracts exactly all the exceptional/X **b**-divisors E such that $a(E, X, B + M_X) \leq 1$. In particular, let $K_Y + B_Y + M_Y := f^*(K_X + B + M_X)$, then $(Y, B_Y + M_Y)$ is g-terminal.*

Proof. By Lemma 3.1, we only need to show that there are finitely many prime **b**-divisor E over X such that $a(E, X, B + M_X) \leq 1$.

Let $g : W \to X$ be a log resolution of (X, B) such that $M = M_W$. Let $K_W + B_W + M_W := g^*(K_X + B + M_X)$. Since $(X, B + M_X)$ is gklt, all the coefficients of B_W are $\leq 1 - c$ for some real number $c > 0$. Suppose that $B_W = B_W^+ - B_W^-$, where $B_W^+, B_W^- \geq 0$ and $B_W^+ \wedge B_W^- = 0$. Then (W, B_W^+) is klt. Let $h : U \to W$ be a terminalization of (W, B_W^+), then all the prime **b**-divisors E such that $a(E, X, B + M_X) \leq 1$ are on U. Therefore there are finitely many prime **b**-divisors E over X such that $a(E, X, B + M_X) \leq 1$, and the corollary follows. $\qquad\square$

Proposition 3.3. *Let $W \to Z$ and $X \to Z$ be two contractions, $f : W \dashrightarrow X$ a birational map/Z, and $(W, B_W + M_W)$ and $(X, B + M_X)$ two g-pairs/Z. Assume that*

- *$K_X + B + M_X$ is nef/Z,*
- *f does not extract any divisor,*
- *for any prime divisor $D \subset W$, $a(D, X, B + M_X) \geq a(D, W, B_W + M_W)$, and*
- *$(W, B_W + M_W)$ is g-terminal,*

then

(1) *$a(E, X, B + M_X) \geq a(E, W, B_W + M_W)$ for any prime **b**-divisor E over X. In other words, $(W, B + M_W) \geq (X, B + M_X)$.*
(2) *$(X, B + M_X)$ is gklt,*
(3) *there is a generalized terminalization $g : Y \to X$ of $(X, B + M_X)$,*
(4) *the induced birational map $W \dashrightarrow Y$ does not extract any divisor,*
(5) *for any exceptional/X **b**-divisor E such that $a(E, X, B + M_X) \leq 1$, E is on W.*

Proof. Let $p : V \to W$ and $q : V \to X$ be any resolution of indeterminacy of f such that

$$p^*(K_W + B_W + M_W) = q^*(K_X + B + M_X) + E_V,$$

then $p_* E_V = \sum_{E \subset W}(a(E, X, B + M_X) - a(E, W, B_W + M_W))E \geq 0$. Since $K_X + B + M_X$ is nef/Z, $-E_V$ is nef/W. By the negativity lemma, $E_V \geq 0$, which implies (1). Since $(W, B_W + M_W)$ is g-terminal, $\lfloor B_W \rfloor = 0$, and (2) follows from (1). (3) follows from (2) and Corollary 3.2.

Suppose that E is an exceptional/W prime **b**-divisor. Since $(W, B_W + M_W)$ is g-terminal, $a(E, W, B_W + M_W) > 1$. Since f does not extract any divisor, E is exceptional/X. By construction of generalized terminalization, we deduce (4). (5) follows from (4). $\qquad\square$

Theorem 3.4 (MMP for generalized pairs, [6, Lemma 4.4(1)(2)]). *Let $X \to Z$ be a contraction, $(X, B + M_X)$ a $\mathbb{Q}$-factorial glc g-pair/Z such that $(X, 0)$ is klt, and A a general ample/Z $\mathbb{R}$-divisor on X. Then there is a $(K_X + B + M_X)$-MMP/Z with scaling of A satisfying the following.*

    (1) *If $K_X + B + M_X$ is not pseudo-effective/Z, then the $(K_X + B + M_X)$-MMP/Z with scaling of A terminates with a Mori fiber space/Z.*

    (2) *If*

        • *$K_X + B + M_X$ is pseudo-effective/Z,*

        • *$(X, B + M_X)$ is gklt, and*

        • *there are real numbers $\alpha, \beta \geq 0$ such that $K_X + (1 + \alpha)B + (1 + \beta)M_X$ is big/Z,*

    the $(K_X + B + M_X)$-MMP/Z with scaling of A terminates with a log model/Z $f : X \dashrightarrow Y$, such that $K_Y + B_Y + M_Y := f_*(K_X + B + M_X)$ is semi-ample/Z.

Lemma 3.5. *Let $X \to Z$ be a contraction, $(X, B + M_X)$ a $\mathbb{Q}$-factorial gklt g-pair/Z, and $f : X \dashrightarrow Y$ a $(K_X + B + M_X)$-non-positive map/Z such that $f_*(K_X + B + M_X) = K_Y + B_Y + M_Y$. Then there is*

    • *a resolution of indeterminacy $p : W \to X$ and $q : W \to Y$, and*

    • *a $\mathbb{Q}$-factorial g-terminal pair $(W, B_W + M_W)$,*

such that

    (1) *q is $(K_W + B_W + M_W)$-non-positive and $q_*(K_W + B_W + M_W) = K_Y + B_Y + M_Y$,*

    (2) *$(W, B_W + M_W) \geq (Y, B_Y + M_Y)$,*

    (3) *for any real number $\epsilon > 0$, if $(X, B + M_X)$ is generalized ϵ-lc, then $(W, B_W + M_W)$ is generalized ϵ-lc, and*

    (4) *$M = M_W$.*

Proof. Let $p : W \to X$ and $q : W \to Y$ be a resolution of indeterminacy, such that p is a log resolution of (W, B_W) and $M = M_W$. Let $K_W + B'_W + M_W := p^*(K_X + B + M_X)$, then we may write $B'_W = B_{1,W} + B_{2,W} - B_{3,W}$, such that

    • $B_{1,W}$ is the strict transform of B on W,

    • $B_{2,W}, B_{3,W}$ are exceptional/X, and

    • $B_{2,W}, B_{3,W} \geq 0$ and $B_{2,W} \wedge B_{3,W} = 0$.

Possibly blowing up more, we may assume that all the irreducible components of $B_{1,W}$ and $B_{2,W}$ do not intersect. Since $(X, B + M_X)$ is gklt, $(W, B_W := B_{1,W} + B_{2,W})$ is terminal. Since W is smooth, $(W, B_W + M_W)$ is $\mathbb{Q}$-factorial g-terminal.

    Since p is $(K_W + B_W + M_W)$-non-positive, $K_X + B + M_X = p_*(K_W + B_W + M_W)$ and f is $(K_X + B + M_X)$-non-positive, we have that $q = f \circ p$ is $(K_W + B_W + M_W)$-non-positive, which implies (1). (2) follows from (1) and (3)(4) follow from the construction of $(W, B_W + M_W)$. $\qquad\square$

Theorem 3.6 (Finiteness of weak log canonical models). *Let*

- $X \to Z$ be a projective morphism between normal quasi-projective varieties,
- A a general ample/Z $\mathbb{Q}$-divisor on X,
- $\mathcal{V} \subset \mathrm{Weil}_\mathbb{R}(X)$ a finite dimensional rational subspace, and
- $\mathcal{C} \subset \mathcal{L}_A(\mathcal{V})$ a rational polytope such that (X, B) is klt for any $B \in \mathcal{C}$,

then there exists an integer $k \geq 0$ and birational maps/Z $\phi_i : X \dashrightarrow Y_i$ for each $1 \leq i \leq k$, such that for every $B \in \mathcal{C}$,

(1) there exists an integer $1 \leq i \leq k$ such that ϕ_i is a weak log canonical model/Z of (X, B),

(2) for any log terminal model $\phi : X \dashrightarrow Y$, there exists an integer $1 \leq j \leq k$ such that $\phi_j \circ \phi^{-1} : Y \to Y_j$ is an isomorphism.

Proof. The theorem follows from [1, Theorem C] and [1, Theorem E]. $\qquad\square$

Theorem 3.7. *Let $X \to Z$ be a contraction, $(X, B + M_X)$ a gklt g-pair/Z, and $F \subset \overline{NE}(X/Z)$ a $(K_X + B + M_X)$-negative extremal face. Then there is a unique contraction $f : X \to Y$ over Z, such that any irreducible curve $C \subset X$ is mapped to a point by f if and only if $[C] \in F$.*

Proof. Since $(X, B + M_X)$ a gklt g-pair/Z, there exists an ample/Z $\mathbb{R}$-divisor A on X and $\Delta \geq 0$ on X, such that

- F is also a $(K_X + B + M_X + A)$-negative extremal face,
- $\Delta \sim_\mathbb{R} B + M_X + A$, and
- (X, Δ) is klt.

The existence of f follows from the usual cone theorem, c.f. [12, Theorem 3.25], [13, Theorem 3-2-1]. $\qquad\square$

4. SARKISOV PROGRAM WITH DOUBLE SCALING

In this section we construct a special type of Sarkisov program, called the "Sarkisov program with double scaling". As the notation is complicated and technical, we first illustrate our ideas.

Idea 4.1. First, recall the typical structure of the Sarkisov program as in Theorem 1.5. Possibly replacing W, we may assume that ρ_X and ρ_Y are morphisms:

Here $\phi_X : X \to S_X$ is a $(K_X + B_X + M_X)$-Mori fiber space/Z and $\phi_Y : Y \to S_Y$ is a $(K_Y + B_Y + M_Y)$-Mori fiber space/Z.

We need to study the difference and similarity between $\phi_X : X \to S_X$ and $\phi_Y : Y \to S_Y$. A common strategy in birational geometry is to study

the ample divisors on X and Y. This works well in our setting, as $-(K_X + B_X + M_X)$ is ample over S_X and $-(K_Y + B_Y + M_Y)$ is ample over S_Y. Therefore, we may pick general ample/Z $\mathbb{R}$-divisors L_X and H_Y on X and Y respectively, such that

- $L_X \sim_{\mathbb{R},Z} -(K_X + B_X + M_X) + \phi_X^* A_{S_X}$ and
- $H_Y \sim_{\mathbb{R},Z} -(K_Y + B_Y + M_Y) + \phi_Y^* A_{S_Y}$,

for some general ample $\mathbb{R}$-divisors A_{S_X} and A_{S_Y} on S_X and S_Y respectively. In particular, $L_W := \rho_X^* L_X$ and $H_W := \rho_Y^* H_Y$ are big and nef/Z, and we may define $H_X := (\rho_X)_* H_W$ and $L_Y := (\rho_Y)_* L_W$. We have

- $K_X + B_X + L_X + 0 H_Y + M_X \sim_{\mathbb{R},S_X} 0$, and
- $K_Y + B_Y + 0 L_Y + H_Y + M_Y \sim_{\mathbb{R},S_Y} 0$.

One key observation is the following: pick any big/Z $\mathbb{R}$-divisor D_W on W and let D_Y be its image on Y. If we assume that $(K_W + B_W + M_W + H_W)$ is pseudo-effective, then $(Y, B_Y + M_Y + H_Y + \epsilon D_Y)$ is a log canonical model of $(W, B_W + M_W + H_W + \epsilon D_W)$ over Z for an $0 < \epsilon \ll 1$. We would like to use this fact and construct the Sarkisov program by using the uniqueness of the log canonical model, and we make it by using the idea for "double scaling" in the following way:

(1) First, we let $l := 1$ and $h := 0$. Starting with X, we decrease l and increase h linearly, making sure that $K_X + B_X + l L_X + h H_X + M_X \sim_{\mathbb{R},S_X} 0$ still holds.

(2) Next, we "aggressively" reduce l to 0. In this case, if $h = 1$, then by picking a general big/Z $\mathbb{R}$-divisor D_X on X and let D_Y be the birational transform of D_X on Y, we deduce that $X = Y$ by the uniqueness of the log canonical model.

(3) However, it is possible that $h < 1$ or $h > 1$. Nevertheless,
   - if $h < 1$, we observe that $K_X + B_X + h H_X + M_X$ is not nef over Z but $K_X + B_X + L_X + M_X$ is nef over Z, and
   - if $h > 1$, we observe that $(W, B_W + h H_W + M_W) \geq (X, B_W + h H_W + M_W)$ does not hold, while $(W, B_W + L_W + M_W) \geq (X, B_W + L_W + M_W)$ holds.

   Therefore, instead of directly reducing l to 0, we may reduce l "moderately" until reaching a threshold when either case above appears.

(4) Finally, it appears that when l and h reach the threshold, one of the four types of Sarkisov links will naturally appear.

The writing of the rest of this section is based on the idea above, which contains three parts.

In the first part, we construct a Sarkisov link with a double scaling structure by using the threshold which appears in Idea 4.1(3). All four different types of Sarkisov links may appear in this part.

In the second part, we study the behavior of some invariants for every Sarkisov link constructed in the first part.

In the last part, we use the invariants constructed in the second part to show that any sequence of the Sarkisov links constructed in the first part terminates.

4.1. **Constructing a Sarkisov link.**

Construction 4.2 (Setting). This setting will be used in the rest of this section. We assume that

- $X \to Z$ is a contraction,
- $\rho : W \dashrightarrow X$ is a birational map,
- $(W, B_W + M_W)$ is a g-pair with associated nef/Z $\boldsymbol{b}$-divisor M,
- L_W and H_W are two general big and nef/Z $\mathbb{R}$-divisors on W,
- $(X, B + M_X)$ is a g-pair,
- $\phi : X \to S$ is a $(K_X + B + M_X)$-Mori fiber space/Z,
- Σ is a ϕ-vertical curve,
- L and H are two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and
- $0 < l \le 1$ and $0 \le h \le 1$ are two real numbers,

such that

(1) $(W, B_W + 2(L_W + H_W) + M_W)$ is $\mathbb{Q}$-factorial g-terminal,
(2) $K_W + B_W + H_W + M_W$ is pseudo-effective/Z,
(3) $(X, B + M_X)$ is $\mathbb{Q}$-factorial gklt,
(4) $(W, B_W + lL_W + hH_W + M_W) \ge (X, B + lL + hH + M_X)$. In particular, ρ does not extract any divisor,
(5) B, L and H are the birational transforms of B_W, L_W and H_W on X respectively,
(6) $K_X + B + lL + hH + M_X \sim_{\mathbb{R},S} 0$, and
(7) $K_X + B + lL + hH + M_X$ is nef/Z.

We illustrate this setting in the following diagram:

$$
\begin{array}{cccccc}
W & \supset & B_W & lL_W & hH_W & M_W \\
\Big\downarrow{\scriptstyle\rho} & & \Big\downarrow & \Big\downarrow & \Big\downarrow & \Big\downarrow \\
X & \supset & B & lL & hH & M_X \qquad \Sigma : \phi\text{-vertical} \\
\Big\downarrow{\scriptstyle\phi} & & & & & \\
S & & & & &
\end{array}
$$

Definition 4.3 (Auxiliary constants and divisors). Assumptions and notations as Construction 4.2,

(1) we define
$$ r := \frac{H \cdot \Sigma}{L \cdot \Sigma}. $$

(2) For any real number t, we define
$$ B_W(t) := B_W + lL_W + hH_W + t(H_W - rL_W), $$

and
$$B(t) := B + lL + hH + t(H - rL).$$

(3) We define Γ to be the set of all real number t satisfying the following:
   (a) $0 \leq t \leq \frac{l}{r}$,
   (b) for any prime divisor $E \subset W$,
   $$a(E, W, B_W(t) + M_W) \leq a(E, X, B(t) + M_X),$$
   and
   (c) $K_X + B(t) + M_X$ is nef/Z.
(4) We define $s := \sup\{t \mid t \in \Gamma\}$.
(5) We define $l_Y := l - rs$ and $h_Y := h + s$.

Lemma 4.4. *Assumptions and notations as Construction 4.2 and Definition 4.3, then*

   (1) $r > 0$ is well-defined,
   (2) either $\Gamma = \{0\}$, or Γ is a closed interval,
   (3) Γ is non-empty and $s \in \Gamma$,
   (4) $l_Y = l$ if and only if $h_Y = h$, and
   (5) $\Gamma \subset [0, 1 - h]$. In particular, $h_Y \leq 1$.

Proof. Since L_W and H_W are general big and nef/Z divisors on W, L and H are big/Z, hence ample/S. Thus $H \cdot \Sigma > 0$ and $L \cdot \Sigma > 0$, hence $r > 0$ is well-defined. This is (1).

By Definition 4.3(3), $0 \in \Gamma$ and Γ is closed and connected, which implies (2). (3) follows from (2) and the definition of s. (4) follows from (1) and the definitions of l_Y and h_Y.

Assume that (5) does not hold. By (2), there exists $t_0 \in \Gamma$ such that $1 < h + t_0 < 2$. By Construction 4.2(1), $(W, B_W(t_0) + M_W)$ is g-terminal. By Proposition 3.3(1) and the definition of Γ, $(W, B_W(t_0) + M_W) \geq (X, B(t_0) + M_X)$. Therefore $(X, B(t_0) + M_X)$ is gklt. Since $(K_X + B(t_0) + M_X) \cdot \Sigma = 0$ and H is big/Z,

$$(K_X + B + (l - t_0 r)L + H + M_X) \cdot \Sigma = ((K_X + B(t_0) + M_X) - (h + t_0 - 1)H) \cdot \Sigma < 0.$$

Thus ϕ is a $(K_X + B + (l - t_0 r)L + H + M_X)$-Mori fiber space/$Z$. In particular, $K_X + B + H + M_X$ is not pseudo-effective/Z. Since ρ does not extract any divisor, $K_W + B_W + H_W + M_W$ is not pseudo-effective/Z, which contradicts Construction 4.2(2). $\qquad\square$

Construction 4.5. Assumptions and notations as Construction 4.2 and Definition 4.3. Then there are three possibilities for s:

Case 1 $s = \frac{l}{r}$. In particular, $l_Y = 0$.

Case 2 $- s < \frac{l}{r}$. In particular, $l_Y > 0$, and
   $-$ there exists $0 < \epsilon \ll 1$ and a prime divisor $E \subset W$, such that
   $a(E, W, B_W(s + \epsilon) + M_W) > a(E, X, B(s + \epsilon) + M_X)$.

Case 3 $- s < \frac{l}{r}$. In particular, $l_Y > 0$, and
   $-$ there exists $0 < \epsilon \ll 1$, such that

$* \ a(E, W, B_W(s + \epsilon) + M_W) \leq a(E, X, B(s + \epsilon) + M_X)$ for any prime divisor $E \subset W$, and

$* \ K_X + B(s + \epsilon) + M_X$ is not nef/Z.

Theorem 4.6 (Sarkisov link with double scaling). *Assumptions and notations as Construction 4.2 and Definition 4.3. The there exist*

- *a birational map/Z $\rho_Y : W \dashrightarrow Y$ which does not extract any divisor,*
- *three $\mathbb{R}$-divisors B_Y, L_Y and H_Y on Y,*
- *a $(K_Y + B_Y + M_Y)$-Mori fiber space/Z $\phi_Y : Y \to S_Y$, and*
- *a Sarkisov link/Z $f : X \dashrightarrow Y$,*

such that

(1) *$(Y, B_Y + M_Y)$ is a $\mathbb{Q}$-factorial gklt g-pair/Z,*
(2) *$(W, B_W + l_Y L_W + h_Y H_W + M_W) \geq (Y, B_Y + l_Y L_Y + h_Y H_Y + M_Y)$. In particular, ρ_Y does not extract any divisor,*
(3) *B_Y, L_Y and H_Y are the birational transforms of B_W, L_W and H_W on Y respectively,*
(4) *$K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y \sim_{\mathbb{R}, S_Y} 0$,*
(5) *$K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y$ is nef/Z,*
(6) *for any ϕ_Y-vertical curve Σ_Y on Y, and $r = \frac{H \cdot \Sigma}{L \cdot \Sigma} \geq \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y} > 0$.*

Proof. We prove the Theorem by considering the three different cases in Construction 4.5 separately.

Case 1. In this case, we finish the proof by letting $\rho_Y := \rho, Y := X, B_Y := B, L_Y := L, H_Y := H, M_Y := M_X, \phi_Y := \phi_X, S_Y := S$, and $f := \mathrm{id}_X$.

Case 2. In this case, $a(E, W, B_W(s) + M_W) = a(E, X, B(s) + M_X)$, and E is exceptional/X. Since $E \subset W$,

$$a(E, X, B(s + \epsilon) + M_X) < a(E, W, B_W(s + \epsilon) + M_W) \leq 1.$$

By Lemma 3.1, there is an extraction $g : V \to X$ of E such that V is $\mathbb{Q}$-factorial. By Proposition 3.3(4), the induced birational map $W \dashrightarrow V$ does not extract any divisor. We let B_V, L_V, H_V be the birational transforms of B_W, L_W and H_W on V respectively, then we have

$$K_V + B_V + (l_Y - r\epsilon)L_V + (h_Y + \epsilon)H_V + M_V$$
$$= g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X).$$

Moreover, since $a(E, X, B(s + \epsilon) + M_X) < 1$, $\mathrm{mult}_E(B_V + (l_Y - r\epsilon)L_V + (h_Y + \epsilon)H_V) > 0$. Thus we may pick a sufficiently small positive real number $0 < \delta \ll \epsilon$, such that $(V, \Delta_V + M_V)$ is gklt, where

$$K_V + \Delta_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta)L + (h_Y + \epsilon)H + M_X).$$

We may run a $(K_V + \Delta_V + M_V)$-MMP/S $\psi : V \dashrightarrow Y$ which terminates with a Mori fiber space/S $\phi_Y : Y \to S_Y$ by Theorem 3.4. Since $\rho(V/S) = \rho(V/X) + \rho(X/S) = 2$ and $1 = \rho(Y/S_Y) \leq \rho(V/S_Y) \leq \rho(V/S)$, there are two possibilities:

Case 2.1. $\rho(V/Y) = 0$. In this case ψ is a sequence of flips, and we get a Sarkisov link/Z $f : X \dashrightarrow Y$ of type I. Let B_Y, L_Y and H_Y be the birational transforms of B_V, L_V and H_V on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0 < \delta' \ll \delta$ and let

$$K_V + \Delta'_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X),$$

then ψ is also a $(K_V + \Delta'_V + M_V)$-MMP/S. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned}
&g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \\
&= (K_Y + \Delta'_Y + M_Y) \cdot \Sigma_Y < 0
\end{aligned}$$

Let $\delta' \to 0$, then we have

$$g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \leq 0.$$

Since $g^*(K_X + B + l_Y L + h_Y H + M_X) \sim_{\mathbb{R},S} 0$, we deduce that

$$g^*(H - rL) \cdot \Sigma_V \leq 0.$$

Moreover, by our assumptions, $g^*(H - rL) = g_*^{-1}(H - rL) + eE$ for some real number $e > 0$, and $\Sigma_V \not\subset E$. Thus

$$\begin{aligned}
(H_Y - rL_Y) \cdot \Sigma_Y = g_*^{-1}(H - rL) \cdot \Sigma_V &= (g^*(H - rL) - eE) \cdot \Sigma_V \\
&\leq g^*(H - rL) \cdot \Sigma_V \leq 0,
\end{aligned}$$

which implies (6), and the theorem follows in this case.

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0 < \delta' \ll \delta$ and let

$$K_V + \Delta'_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X),$$

then ψ is also a $(K_V + \Delta'_V + M_V)$-MMP/S. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned}
&g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \\
&= (K_Y + \Delta'_Y + M_Y) \cdot \Sigma_Y < 0
\end{aligned}$$

Let $\delta' \to 0$, then we have

$$g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \leq 0.$$

Since $g^*(K_X + B + l_Y L + h_Y H + M_X) \sim_{\mathbb{R},S} 0$, we deduce that

$$g^*(H - rL) \cdot \Sigma_V \leq 0.$$

Moreover, by our assumptions, $g^*(H - rL) = g_*^{-1}(H - rL) + eE$ for some real number $e > 0$, and $\Sigma_V \not\subset E$. Thus

$$(H_Y - rL_Y) \cdot \Sigma_Y = g_*^{-1}(H - rL) \cdot \Sigma_V = (g^*(H - rL) - eE) \cdot \Sigma_V$$
$$\leq g^*(H - rL) \cdot \Sigma_V \leq 0,$$

which implies (6), and the theorem follows in this case.

Case 3. In this case, there exists a $(K_X + B(s+\epsilon) + M_X)$-negative extremal ray $[C]$ on X. Since $(K_X + B(s + \epsilon) + M_X) \cdot \Sigma = 0$, $[C] \neq [\Sigma]$. Let $P \subset \overline{NE}(X/Z)$ be the extremal face over Z defined by all $(K_X + B(s+\epsilon) + M_X)$-non-positive irreducible curves. Then $P \neq [\Sigma]$, and hence there exists an extremal ray $[\Pi]$ such that $[\Sigma]$ and $[\Pi]$ span a two-dimensional face of P. By our construction, $(K_X + B(s + \epsilon) + M_X) \cdot \Pi < 0$. Now for $0 < \delta \ll 1$, we have

$$(K_X + B + (l_Y - r\epsilon - \delta)L_X + (h_Y + \epsilon)H_X + M_X) \cdot \Sigma < 0$$

and

$$(K_X + B + (l_Y - r\epsilon - \delta)L_X + (h_Y + \epsilon)H_X + M_X) \cdot \Pi < 0.$$

By Theorem 3.7, there exists a contraction $\pi : X \to T$ of the extremal face of $\overline{NE}(X/Z)$ spanned by $[\Sigma]$ and $[\Pi]$. Then π factors through S, and $K_X + B(s) + M_X \sim_{\mathbb{R},T} 0$.

Since L, H are big$/Z$, L, H are big$/T$. Therefore, if $K_X + B(s+\epsilon) + M_X$ is pseudo-effective$/T$, then $K_X + (1+\alpha)B(s+\epsilon) + M_X$ is big$/T$. By Theorem 3.4, we may run a $(K_X + B(s + \epsilon) + M_X)$-MMP$/T$ with scaling of some ample$/T$ divisor, and this MMP$/T$ terminates. There are three cases:

Case 3.1. After a sequence of flips $f : X \dashrightarrow Y$, the MMP$/T$ terminates with a Mori fiber space$/T$ $\phi_Y : Y \to S_Y$. Therefore, f is a Sarkisov link$/Z$ of type IV. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since ϕ_Y is a $(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y)$-Mori fiber space$/T$,

$$-(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \cdot \Sigma_Y > 0,$$

which implies that

$$-(K_X + B(s + \epsilon) + M_X) \cdot \Sigma_X > 0.$$

Since $K_X + B(s) + M_X \sim_{\mathbb{R},T} 0$,

$$-(K_X + B(s) + M_X) \cdot \Sigma_X = 0,$$

which implies that

$$(H_Y - rL_Y) \cdot \Sigma_Y = (H - rL) \cdot \Sigma_X < 0.$$

Thus $r > \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.2. After a sequence of flips $X \dashrightarrow U$, we get a divisorial contraction/T: $U \to Y$. Therefore $\rho(Y/T) = 1$, which implies that the induced morphism $\phi_Y := Y \to T$ is a Mori fiber space, and the induced birational map $f : X \dashrightarrow Y$ is a Sarkisov link/Z of type III. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since $-(K_X + B(s + \epsilon) + M_X)$ is nef/T and $K_X + B(s) + M_X \sim_{\mathbb{R},T} 0$, we have

$$-(K_X + B(s + \epsilon) + M_X) \cdot \Sigma_X \geq 0 = -(K_X + B(s) + M_X) \cdot \Sigma_X,$$

which implies that

$$(H_Y - rL_Y) \cdot \Sigma_Y = (H - rL) \cdot \Sigma_X \leq 0.$$

Thus $r \geq \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.3. After a sequence of flips $f : X \dashrightarrow Y$, the MMP terminates with a minimal model Y over T. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively. Since Σ is a general ϕ-vertical curve, we may let Σ' be the birational transform of Σ on Y. Since $(K_X + B(s+\epsilon)+M_X) \cdot \Sigma = 0$ and $(K_X + B(s) + M_X) \cdot \Sigma = 0$, we have

$$(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \cdot \Sigma' = 0$$

and

$$(K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y) \cdot \Sigma' = 0$$

which implies that $(K_Y + B_Y + M_Y) \cdot \Sigma' < 0$ and $r = \frac{H_Y \cdot \Sigma'}{L_Y \cdot \Sigma'}$. Since Σ can be chosen to be any ϕ-vertical curve, by Theorem 3.7, there exists a contraction $\phi_Y : Y \to S_Y$ of $[\Sigma']$ such that ϕ_Y is a $(K_Y + B_Y + M_Y)$-Mori fiber space/T. Thus f is a Sarkisov link/Z of type IV. We finish the proof by letting $\rho_Y : W \dashrightarrow Y$ be the induced birational map. $\square$

4.2. **Behavior of invariants under a Sarkisov link with double scaling.**

Lemma 4.7. *Assumptions and notations as in Construction 4.2, Definition 4.3, and Theorem 4.6. Then*

(1) In **Case 2.1***, $\rho(Y) - \rho(X) = 1$.*
(2) In **Case 2.2***,*
  (a) $\rho(X) = \rho(Y)$,
  (b) there is a prime divisor F_0 over W, such that

$$a(F_0, X, B + (l_Y - r\epsilon - \delta)L + (h_Y + \epsilon)H + M_X)$$
$$<a(F_0, Y, B_Y + (l_Y - r\epsilon - \delta)L_Y + (h_Y + \epsilon)H_Y + M_Y),$$

  and

(c) for any prime divisor F over W,

$$a(F, X, B + (l_Y - r\epsilon - \delta)L + (h_Y + \epsilon)H + M_X)$$
$$\leq a(F, Y, B_Y + (l_Y - r\epsilon - \delta)L_Y + (h_Y + \epsilon)H_Y + M_Y).$$

*(3) In **Case 3**,*

$$a(F, Y, B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \geq a(F, W, B_W(s + \epsilon) + M_W).$$

*(4) In **Case 3.1**, $\frac{H \cdot \Sigma}{L \cdot \Sigma} > \frac{H_Y \cdot \Sigma_Y}{L_Y \cdot \Sigma_Y}$.*
*(5) In **Case 3.2**, $\rho(X) - \rho(Y) = 1$.*
*(6) In **Case 3.3**,*
    (a) $\rho(X) = \rho(Y)$,
    (b) there is a prime divisor F_0 over W, such that

$$a(F_0, Y, B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) > a(F_0, X, B(s + \epsilon) + M_X),$$

    and
    (c) for any prime divisor F over W,

$$a(F, Y, B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \geq a(F, X, B(s + \epsilon) + M_X).$$

Proof. (1)(4)(5) immediately follow from the proof of Theorem 4.6. (2) follows from the fact that in **Case 2.2**, the Sarkisov link/Z is constructed by running a $g^*(K_X + B + (l_Y - r\epsilon - \delta)L + (h_Y + \epsilon)H + M_X)$-MMP/$S$ and $X \not\cong Y$. (3)(6) follow from the fact that in **Case 3**, the Sarkisov link/Z is constructed by running a $(K_X + B(s + \epsilon) + M_X)$-MMP/$T$ and $X \not\cong Y$ in **Case 3.3**. $\square$

4.3. **Running the Sarkisov program with double scaling.**

Construction 4.8 (Sarkisov program with double scaling)**.** Assume that $W \to Z$ is a contraction and $(W, B_W + M_W)$ is a $\mathbb{Q}$-factorial gklt g-pair/Z with nef/Z **b**-divisor $M = M_W$, such that $K_W + B_W + M_W$ is not pseudo-effective/Z.

Let $\rho : W \dashrightarrow X$ be a $(K_W + B_W + M_W)$-non-positive map/Z and $\phi : X \to S$ a $(K_X + B + M_X)$-Mori fiber space/Z, where B is the birational transform of B_W on X. By Theorem 3.4, a special choice of ρ is when ρ is a $(K_W + B_W + M_W)$-MMP/Z. By Lemma 3.5, possibly taking a resolution of indeterminacy $p : W' \to W$ and $q : W' \to X$, we may assume that W is smooth and ρ is a morphism. Then ϕ is a $(K_X + B + M_X)$-Mori fiber space/Z. In particular, $-(K_X + B + M_X)$ is ample/S. Therefore, we may pick a general ample/Z $\mathbb{R}$-divisor A on S such that $-(K_X + B + M_X) + \phi^* A$ is ample/Z. We let L be a general element of $|-(K_X + B + M_X) + \phi^* A|_{\mathbb{R}/Z}$ and $L_W := \rho^* L = (\rho^{-1})_* L$. Then L_W is big and nef/Z, $K_X + B + L + M_X \sim_{\mathbb{R}, S} 0$ and $K_X + B + L + M_X$ is nef/Z.

Finally, we pick a general big and nef/Z $\mathbb{R}$-divisor H_W on W such that

- $(W, B_W + 2(L_W + H_W) + M_W)$ is $\mathbb{Q}$-factorial g-terminal, and
- $K_W + B_W + H_W + M_W$ is pseudo-effective/Z,

and pick a general ϕ-vertical curve Σ on X. We construct the *Sarkisov program/Z of $(X, B + M_X)$ with scaling of (L_W, H_W)* in the following way.

Step 1 We define $X_0 := X, B_0 := B, \rho_0 := \rho, \phi_0 := \phi$, $L_0 := L, H_0 := \rho_* H_W$, $r_0 := \frac{H_0 \cdot \Sigma}{L_0 \cdot \Sigma}$, $\Sigma_0 := \Sigma$, and $(l_0, h_0) := (1, 0)$.

Step 2 For any integer $i \geq 0$, suppose that we already have
- a $\mathbb{Q}$-factorial gklt g-pair $(X_i, B_i + M_{X_i})$,
- a birational map $\rho_i : W \dashrightarrow X_i$,
- a $(K_{X_i} + B_i + M_{X_i})$-Mori fiber space/Z $\phi_i : X_i \to S_i$,
- two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors L_i and H_i on X_i,
- two real number $0 < l_i \leq 1$ and $0 \leq h_i \leq 1$,
- a general ϕ_i-vertical curve Σ_i, and
- $r_i := \frac{H_i \cdot \Sigma_i}{L_i \cdot \Sigma_i} > 0$

such that
- $(W, B_W + l_i L_W + h_i H_W + M_W) \geq (X_i, B_i + l_i L_i + h_i H_i + M_{X_i})$,
- B_i, L_i and H_i are the birational transforms of B_i, L_i and H_i on X_i respectively,
- $K_{X_i} + B_i + l_i L_i + h_i H_i + M_{X_i} \sim_{\mathbb{R}, S_i} 0$, and
- $K_{X_i} + B_i + l_i L_i + h_i H_i + M_{X_i}$ is nef/Z,

then by Theorem 4.6, there exists
- a $\mathbb{Q}$-factorial gklt g-pair $(X_{i+1}, B_{i+1} + M_{X_{i+1}})$,
- a birational map $\rho_{i+1} : W \dashrightarrow X_{i+1}$,
- a $(K_{X_{i+1}} + B_{i+1} + M_{X_{i+1}})$-Mori fiber space/Z $\phi_{i+1} : X_{i+1} \to S_{i+1}$,
- two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors L_{i+1} and H_{i+1} on X_{i+1},
- two real number $0 \leq l_{i+1} \leq l_i$ and $h_i \leq h_{i+1} \leq 1$,
- a ϕ_{i+1}-vertical curve Σ_{i+1},
- a real number $r_{i+1} := \frac{H_{i+1} \cdot \Sigma_{i+1}}{L_{i+1} \cdot \Sigma_{i+1}}$, and
- a Sarkisov link/Z $f_i : X_i \dashrightarrow X_{i+1}$ as in **Case 1**, or **Case 2.1**, or **Case 2.2**, or **Case 3.1**, or **Case 3.2**, or **Case 3.3** of Theorem 4.6,
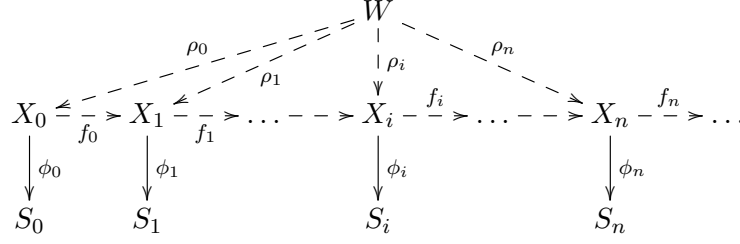
such that
- $(W, B_W + l_{i+1} L_W + h_{i+1} H_W + M_W) \geq (X_{i+1}, B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1} + M_{X_{i+1}})$,
- B_{i+1}, L_{i+1} and H_{i+1} are the birational transforms of B_i, L_i and H_i on X_{i+1} respectively,
- $K_{X_{i+1}} + B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1} + M_{X_{i+1}} \sim_{\mathbb{R}, S_{i+1}} 0$,
- $K_{X_{i+1}} + B_{i+1} + l_{i+1} L_{i+1} + h_{i+1} H_{i+1} + M_{X_{i+1}}$ is nef/Z, and
- $r_i \geq r_{i+1} > 0$.

Notice that the assumptions hold when $i = 0$.

Step 3 If $l_{i+1} = 0$, we stop and let $n := i + 1$. Otherwise, we replace i with $i + 1$ and return to **Step 2**.

The following diagram gives the birational maps and Mori fiber spaces in this construction:

Lemma 4.9. *Assumptions and notation as in Construction 4.8. Then*

(1) *there are only finitely many possibilities of X_i up to isomorphism, and*

(2) *the Sarkisov program of $(X, B + M_X)$ with scaling of (L_W, H_W) terminates, i.e. there exists an integer $n > 0$ such that $l_n = 0$.*

Proof. We construct two a ample$/Z$ $\mathbb{R}$-divisors Γ_W and N_W on W, boundary divisors $\Delta_{i,W}$ for every $i \gg 0$, and a finite dimensional affine subspace $\mathcal{V}$ of $\mathrm{Weil}_{\mathbb{R}}(W)$ in the following way:

- Assume that $h_k > 0$ for some integer $k \geq 0$. In this case, since H_W is big and nef$/Z$, we may write $H_W \sim_{\mathbb{R},Z} C_W + G_W$, such that C_W is ample$/Z$, G_W is effective, and $(W, B_W + L_W + H_W + C_W + G_W + M_W)$ is g-terminal. We let $\Gamma_W \sim_{\mathbb{R},Z} M_W + \frac{h_k}{2} C_W$ be a general ample$/Z$ $\mathbb{R}$-divisor on W such that

$$\left(W, B_W + l_i L_W + \frac{h_k}{2} C_W + h_i G_W + \Gamma_W + (h_i - h_k) C_W\right)$$

  is klt for every $i \geq k$. We define $N_W := \frac{h_k}{2} C_W$, $\Delta_{i,W} := B_W + l_i L_W + \frac{h_k}{2} C_W + h_i G_W + \Gamma_W + (h_i - h_k) C_W$ for every $i \geq 0$, and let $\mathcal{V}$ be the finite dimensional affine subspace of $\mathrm{Weil}_{\mathbb{R}}(W)$ generated by the irreducible components of B_W, L_W, G_W, Γ_W and C_W.

- Assume that $h_i = 0$ for every integer $i \geq 0$. In this case, $l_i = 1$ for every $i \geq 0$. Since L_W is big and nef$/Z$, we may write $L_W \sim_{\mathbb{R},Z} D_W + J_W$, such that D_W is ample$/Z$, J_W is effective, and $(W, B_W + L_W + D_W + J_W + M_W)$ is g-terminal. We let $\Gamma_W \sim_{\mathbb{R},Z} M_W + \frac{1}{2} D_W$ be a general ample$/Z$ $\mathbb{R}$-divisor on W such that

$$\left(W, B_W + \frac{1}{2} D_W + J_W + \Gamma_W\right)$$

  is klt for every $i \geq k$. We define $N_W := \frac{1}{2} D_W$, $\Delta_{i,W} := B_W + \frac{1}{2} D_W + J_W + \Gamma_W)$ for every $i \geq 0$, and $\mathcal{V}$ be the finite dimensional affine subspace of $\mathrm{Weil}_{\mathbb{R}}(W)$ generated by the irreducible components of B_W, J_W and Γ_W.

Since $K_{X_i} + B_i + l_i L_i + h_i H_i + M_{X_i}$ is nef$/Z$ for every i and $K_W + B_W + l_i L_W + h_i H_W + M_W \sim_{\mathbb{R},Z} K_W + \Delta_{i,W}$, $\rho_i : W \dashrightarrow X_i$ is a weak log canonical model$/Z$ of $(W, \Delta_{i,W})$. Since $\Delta_{i,W} \in \mathcal{L}_{N_W}(\mathcal{V})$ for every i, we deduce (1) by applying Theorem 3.6.

Suppose that (2) does not hold. Then $l_i > 0$ for every $i > 0$. By (1), there exists a strictly increasing sequence $\{i_j\}_{j=1}^{+\infty}$, such that $X_{i_j} \cong X_{i_k}$ for any $j, k \geq 1$. By construction, we have $l_{i_j} = l_{i_k}$ and $h_{i_j} = h_{i,k}$ for every $j, k \geq 1$. Thus $l_i = l_{i_1}$ and $h_i = h_{i_1}$ for every $i \geq i_1$. In particular, f_i is a Sarkisov link$/Z$ of as in **Case 2** or **Case 3** of Construction 4.5 for every $i \geq i_1$.

Suppose that f_i is a Sarkisov link$/Z$ as in **Case 3** of Construction 4.5 for some $i \geq i_1$. By Lemma 4.7(3), f_{i+1} remains a Sarkisov link$/Z$ as in **Case 3** of Construction 4.5, which implies that f_j is a Sarkisov link$/Z$ as in **Case 3** of Theorem 4.6 for every $j \geq i$. By Theorem 4.6(6) and Lemma 4.7(4), f_j is not a Sarkisov link$/Z$ as in **Case 3.1** of Theorem 4.6 for any $j \geq i$. Since $\rho(X_j) > 0$ for every j, by Lemma 4.7(5)(6.a), for every $j \gg i$, f_j is a Sarkisov link$/Z$ as in **Case 3.3** of Theorem 4.6. This contradicts Lemma 4.7(6).

Therefore, we may assume that f_j is a Sarkisov link$/Z$ as in **Case 2** of Theorem 4.6 for every $j \geq i_1$. Since $\rho(X_j) \leq \rho(W)$ for every j, by Lemma 4.7(1)(2.a), for every $j \gg i_1$, f_j is a Sarkisov link$/Z$ as in **Case 2.2** of Theorem 4.6. This contradicts Lemma 4.7(2).   □

5. Proof of the main theorem

Theorem 5.1. *Assume that*

- *$W \to Z$ is a contraction between normal quasi-projective varieties,*
- *$(W, B_W + M_W)$ is a $\mathbb{Q}$-factorial gklt g-pair with associated nef$/Z$ **b**-divisor M, such that $K_W + B_W + M_W$ is not pseudo-effective$/Z$,*
- *$\rho_X : W \dashrightarrow X$ and $\rho_Y : W \dashrightarrow Y$ are two $(K_W + B_W + M_W)$-non-positive maps$/Z$ such that $(\rho_X)_*(K_W + B_W + M_W) = K_X + B_X + M_X$ and $(\rho_Y)_*(K_W + B_W + M_W) = K_X + B_Y + M_Y$,*
- *$\phi_X : X \to S_X$ is a $(K_X + B_X + M_X)$-Mori fiber space$/Z$ and $\phi_Y : X \to S_Y$ is a $(K_Y + B_Y + M_Y)$-Mori fiber space$/Z$.*
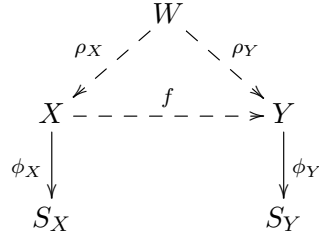
Then

(1) the induced birational map $f : X \dashrightarrow Y$ is given by a finite sequence of Sarkisov links$/Z$, i.e. f can be written as $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$, where each $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link,

(2) f is a Sarkisov program$/Z$ with double scaling, i.e., each $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link$/Z$ as in Construction 4.8,

(3) for any real number $\epsilon > 0$, if $(W, B_W + M_W)$ is generalized ϵ-lc, then $(X_i, B_i + M_{X_i})$ is generalized ϵ-lc for every i, where each B_i is the birational transform of B_W on X_i.

Proof. By Lemma 3.5, possibly replacing $(W, B_W + M_W)$ with a common log resolution of $(W, B_W + M_W)$, $(X, B_X + M_X)$ and $(Y, B_Y + M_Y)$, we may assume that $(W, B_W + M_W)$ is g-terminal, ρ_X and ρ_Y are morphisms, and $M = M_W$. By our assumptions, $-(K_X + B_X + M_X)$ is ϕ_X-ample and $-(K_Y + B_Y + M_Y)$ is ϕ_Y-ample. Therefore, there are

- two ample/Z $\mathbb{R}$-divisors A_{S_X} and A_{S_Y} on S_X and S_Y respectively,
- two general ample/Z $\mathbb{R}$-divisors L_X and H_Y on X and Y respectively, and
- two general big and nef/Z $\mathbb{R}$-divisors L_W and H_W on W,

such that

- $L_X \sim_{\mathbb{R},Z} -(K_X + B_X + M_X) + \phi_X^* A_{S_X}$,
- $H_Y \sim_{\mathbb{R},Z} -(K_Y + B_Y + M_Y) + \phi_Y^* A_{S_Y}$, and
- $(X, B_X + L_X + M_X)$ and $(Y, B_Y + H_Y + M_Y)$ are both gklt,
- $L_W = \rho_X^* L_X = (\rho_X^{-1})_* L_X$ and $H_W = \rho_X^* L_X = (\rho_Y^{-1})_* L_X$, and
- $(W, B_W + 2(H_W + L_W) + M_W)$ is g-terminal.

By Lemma 4.9, we may let $f' : X := X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n$ be a Sarkisov program of $(X, B_X + M_X)$ with scaling of (L_W, H_W) as in Construction 4.8.

We show that $f = f'$ and $h_n = 1$. By Lemma 4.4, $h_n \le 1$. If $h_n < 1$, then since $(\rho_Y)_*(K_W + B_W + h_n H_W + M_W) \sim_{\mathbb{R},S_Y} -(1 - h_n)H_Y$, ϕ_Y is a $(K_Y + B_Y + h_n H_Y + M_Y)$-Mori fiber space/Z, which implies that $K_W + B_W + h_n H_W + M_W$ is not pseudo-effective/Z. However, since $K_{X_n} + B_n + h_n H_n + M_{X_n}$ is nef/Z by construction, $K_W + B_W + h_n H_W + M_W$ is pseudo-effective/Z, a contradiction. Thus $h_n = 1$.

Let D_n be a general ample/Z $\mathbb{R}$-divisor on X_n and D_Y, D_W the birational transforms of D_n on Y and W respectively. Since D_Y is big/Z, D_Y is ample/S_Y. Pick $0 < \epsilon \ll 1$ and let $\Delta_W \sim_{\mathbb{R},Z} B_W + H_W + M_W + \epsilon D_n$ be an effective $\mathbb{R}$-divisor on W such that (W, Δ_W) is klt. Then $W \dashrightarrow X_n$ and $W \dashrightarrow Y$ are both log canonical models/Z of (W, Δ_W), which implies that $X_n \cong Y$.

Let Σ_n be a ϕ_n-vertical curve, then

$$\phi_Y^* A_{S_Y} \cdot \Sigma_n = (K_Y + B_Y + H_Y + M_Y) \cdot \Sigma_n = 0,$$

which implies that Σ_n is ϕ_Y-vertical. Thus ϕ_n and ϕ_Y define the same Mori fiber space and the theorem follows. $\qquad\square$

Proof of Theorem 1.5. Since a $(K_W + B_W + M_W)$-MMP/Z is a $(K_W + B_W + M_W)$-non-positive map/Z, Theorem 1.5 follows from Theorem 5.1. $\qquad\square$

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