SARKISOV PROGRAM FOR GLC GENERALIZED PAIRS

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Abstract. We prove the Sarkisov program for lc generalized pairs

Contents

1. Introduction

Theorem 1.1. *Let (Z,Φ) be a pair such that $K_X+\Phi$ is not pseudo-effective. Assume that $\phi : X \to S$ and $\psi : Y \to T$ are two Mori fiber spaces which are obtained by running two different $(K_Z+\Phi)$-MMPs. Then the induced birational map $\sigma : X \dashrightarrow Y$ is a composition of Sarkisov links.*

Theorem 1.2. *Assume that*

- *$W \to Z$ is a contraction between normal quasi-projective varieties,*
- *$(W, B_W + M_W)$ is glc NQC g-pair with associated nef/Z $\boldsymbol{b}$-divisor M, such that $K_W + B_W + M_W$ is not pseudo-effective/Z,*
- *$\rho_X : W \dashrightarrow X$ and $\rho_Y : W \dashrightarrow Y$ are two $(K_W + B_W + M_W)$-MMP/Z such that $(\rho_X)_*(K_W+B_W+M_W) = K_X+B_X+M_X$ and $(\rho_Y)_*(K_W+B_W+M_W) = K_X + B_Y + M_Y$,*
- *$\phi_X : X \to S_X$ is a $(K_X + B_X + M_X)$-Mori fiber space/Z and $\phi_Y : X \to S_Y$ is a $(K_Y + B_Y + M_Y)$-Mori fiber space/Z.*

$$\begin{tikzcd}
& W \arrow[ld, dashed, "\rho_X"'] \arrow[rd, dashed, "\rho_Y"] & \\
X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\
S_X & & S_Y
\end{tikzcd}$$

Then

(1) the induced birational map $f : X \dashrightarrow Y$ is given by a finite sequence of Sarkisov links/Z, i.e. f can be written as $X_0 \dashrightarrow X_1 \cdots \dashrightarrow X_n \cong Y$, where each $X_i \dashrightarrow X_{i+1}$ is a Sarkisov link/Z, and

Date: April 30, 2023.

TODO: NOT ϵ-lc

2. Notation and Conventions

We adopt the standard notation and definitions in [1] and [KM98], and will freely use them.

Definition 2.1 ($\boldsymbol{b}$-divisors)**.** Let X be a normal quasi-projective variety. A $\boldsymbol{b}$-$\mathbb{R}$ Cartier $\boldsymbol{b}$-divisor ($\boldsymbol{b}$-divisor for short) over X is the choice of a projective birational morphism $Y \to X$ from a normal quasi-projective variety Y and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on Y up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal quasi-projective variety and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M' defines the same $\boldsymbol{b}$-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullback of M and M' coincide. If there is a choice of birational morphism $Y \to X$ such that the corresponding $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M is a prime divisor, the $\boldsymbol{b}$-divisor is called *prime*.

Let E be a prime $\boldsymbol{b}$-divisor over X. The *center* of E on X is the closure of its image on X, and is denoted by $c_X(E)$. If $c_X(E)$ is not a divisor, E is called *exceptional*$/X$. If $c_X(E)$ is a divisor, we say that E is *on* X. For any $\boldsymbol{b}$-divisor $M = \sum a_iE_i$ over X, where E_i are prime $\boldsymbol{b}$-divisors over X, we define $M_X := \sum a_ic_X(E_i)$ to be the $\mathbb{R}$-divisor where the sum is taken over all the prime $\boldsymbol{b}$-divisors E_i which are on X. If all the E_i are on X, we say that M is *on* X.

Definition 2.2 (Multiplicities)**.** Let X be a normal quasi-projective variety, E a prime divisor on X and D an $\mathbb{R}$-divisor on X. We define $\operatorname{mult}_E D$ to be the multiplicity of E along D. Let F be a prime $\boldsymbol{b}$-divisor over X, B an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and $\phi : Y \to X$ a birational morphism such that F is on Y. We define $\operatorname{mult}_F D := \operatorname{mult}_F \phi^*D$.

Definition 2.3. Let $f : X \dashrightarrow Y$ a birational map between normal quasi-projective varieties, $p : W \to X$ and $q : W \to Y$ a resolution of indeterminacy of f, and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D_Y := f_*D$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. f is called *D-non-positive* (resp. *D-negative*), if

- f does not extract any divisor, and
- $p^*D = q^*D_Y + E$, where $E \geq 0$ is exceptional$/Y$ (resp. $E \geq 0$ is exceptional$/Y$, and $\operatorname{Supp} p_*E$ contains all f-exceptional divisors).

Definition 2.4. Let X be a normal quasi-projective variety. We define $\mathrm{WDiv}_{\mathbb{R}}(X)$ to be the $\mathbb{R}$-vector space spanned by all Weil divisors on X. Let $\mathcal{V}$ be a finite dimensional subspace of $\mathrm{Weil}_{\mathbb{R}}(X)$ and $A \in \mathcal{V}$ an $\mathbb{R}$-divisor. We define

$$\mathcal{L}_A(\mathcal{V}) := \{B \mid (X, B) \text{ is lc}, B = A + B', B' \geq 0, B' \in \mathcal{V}\} \subset \mathrm{WDiv}_{\mathbb{R}}(X)$$

By [BCHM10, Lemma 3.7.2], if $\mathcal{V}$ is a rational subspace, then $\mathcal{L}_A(\mathcal{V})$ is a rational polytope.

Definition 2.5. A *contraction* is a projective morphism $f : X \to Z$ between normal quasi-projective varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$.

For any $\boldsymbol{b}$-divisor M over X, M is called nef$/Z$ if there is a projective morphism $Y \to X$ such that M is on Y and M_Y is nef$/Z$.

Definition 2.6. A *generalized pair* (*g-pair* for short) consists of a normal quasi-projective variety X, an effective $\mathbb{R}$-divisor B on X, a contraction $X \to Z$, and a $\boldsymbol{b}$-divisor M over X such that M is nef$/Z$. If there is no confusion, we usually say that $(X, B+M_X)$ is a generalized pair$/Z$. M is called the *associated nef$/Z$ $\boldsymbol{b}$-divisor* of the generalized pair $(X, B + M_X)$. If Z is not important, we may omit Z and say that $(X, B + M_X)$ is a generalized pair.

Let $(X, B + M_X)$ be a generalized pair/Z with associated nef/Z $\boldsymbol{b}$-divisor M. Let $\phi : W \to X$ be a log resolution of (X, B) such that $M_W = M$ (i.e. M is the choice of M_W and the morphism ϕ) and

$$K_W + B_W + M_W := \phi^*(K_X + B + M_X).$$

The *generalized log discrepancy* of a prime divisor D on W with respect to $(X, B + M_X)$ is $1 - \operatorname{mult}_D B_W$ and is denoted by $a(D, X, B + M_X)$. For any prime $\boldsymbol{b}$-divisor E over X, let $Y \to X$ be a birational morphism such that E_Y is a prime divisor. The *generalized log discrepancy* of E with respect to $(X, B + M_X)$ is $a(E_Y, X, B + M_X)$. For any real number $\epsilon \geq 0$, we say that

- $(X, B + M_X)$ is *glc* (resp. *gklt*, *ϵ-glc*) if $a(E, X, B) \geq 0$ (resp. > 0, $\geq \epsilon$) for every prime $\boldsymbol{b}$-divisor E over X,
- $(X, B + M_X)$ is *g-terminal* if $a(E, X, B) > 1$ for every exceptional/X prime $\boldsymbol{b}$-divisor E,
- $(X, B + M_X)$ is *gdlt* if there exists an open subset $U \subseteq X$ such that $(U, B|_U)$ is a log smooth pair, and if $a(E, X, B + M) = 0$ for some prime $\boldsymbol{b}$-divisor E over X, then $c_X(E) \cap U \neq \emptyset$ and $c_X(E) \cap U$ is an lc center of $(U, B|_U)$,
- $(X, B + M_X)$ is *$\mathbb{Q}$-factorial* if every $\mathbb{Q}$-divisor on X is $\mathbb{Q}$-Cartier.

Remark 2.1. If (X, B, M) is gdlt g-pair, then X is klt.

A *generalized terminalization* of a glc g-pair $(X, B+M_X)$ is a birational morphism $f : Y \to X$ satisfying the following.

- $K_Y + B_Y + M_Y = f^*(K_X + B + M_X)$,
- $(Y, B_Y + M_Y)$ is $\mathbb{Q}$-factorial g-terminal,
- f only extracts prime $\boldsymbol{b}$-divisors E over X such that $0 \leq a(E, X, B+M) \leq 1$.

Definition 2.7. Assume that

- $X \to Z$ and $Y \to Z$ are two contractions,
- $(X, B + M_X)$ and $(Y, B_Y + M_Y)$ are two g-pairs/Z with the same associated nef/Z $\boldsymbol{b}$-divisor M, and
- $f : X \dashrightarrow Y$ is a birational map/Z,

such that

- f does not extract any divisor, and
- $a(E, X, B + M_X) \leq a(E, Y, B_Y + M_Y)$ for every prime $\boldsymbol{b}$-divisor E over X,

then we may write $(X, B + M_X) \geq (Y, B_Y + M_Y)$.

TODO: maybe only consider the divisor E on Y.

3. Preliminaries

Lemma 3.1 (dlt modification, [HL22, Proposition 3.10])**.** *Let $(X, B+M)$ be an lc g-pair with data $W \xrightarrow{f} X \to Z$ and M_W. Then, after possibly replacing f with a higher model, there exist a $\mathbb{Q}$-factorial dlt g-pair $(X', B' + M')$ with data $W \xrightarrow{g} X' \to Z$ and M_W, and a projective birational morphism $h\colon X' \to X$ such that*

$$K_{X'} + B' + M' \sim_{\mathbb{R}} h^*(K_X + B + M) \quad \text{and} \quad B' = h_*^{-1}B + E,$$

where E is the sum of all h-exceptional prime divisors on X'. The g-pair $(X', B' + M')$ is called a dlt blow-up *of $(X, B + M)$.*

Theorem 3.2 (contraction extremal faces)**.** *contraction faces*

Theorem 3.3 (MMP for glc gpairs, [TX23, Theorem 4.4])**.** *Let $\big(X/Z,(B+A)+M\big)$ be an NQC lc g-pair, where A is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor which is ample over Z. If the divisor $K_X+B+A+M$ is pseudo-effective over Z, then there exists a $(K_X+B+A+M)$-MMP over Z which terminates with a good minimal model of $\big(X,(B+A)+M\big)$ over Z.*

Theorem 3.4 (extract a divisor, [LX22b, Theorem 1.7])**.** *Let $(X,B,\mathbf{M})$ be a glc g-pair, and E a prime divisor that is exceptional over X such that $a(E,X,B,\mathbf{M})\in[0,1)$. Then there exists a birational morphism $f:Z\to X$ which extracts E such that $-E$ is ample over X.*

Remark 3.1. Furthermore, we have

$$K_Z+f_*^{-1}B+(1-a)E_Z+M_Z=f^*(K_X+B+M_X)$$

4. Double scaling

In this section we construct a special type of Sarkisov program, called the “Sarkisov program with double scaling”. As the notation is complicated and technical, we first illustrate our ideas.

First, recall the typical structure of the Sarkisov program as in Theorem 1.2. Possibly replacing W, we may assume that ρ_X and ρ_Y are morphisms:

$$\begin{tikzcd} & W \arrow[dl, "\rho_X"'] \arrow[dr, "\rho_Y"] & \\ X \arrow[rr, dashed, "f"] \arrow[d, "\phi_X"'] & & Y \arrow[d, "\phi_Y"] \\ S_X & & S_Y \end{tikzcd}$$

Here $\phi_X:X\to S_X$ is a $(K_X+B_X+M_X)$-Mori fiber space$/Z$ and $\phi_Y:Y\to S_Y$ is a $(K_Y+B_Y+M_Y)$-Mori fiber space$/Z$.

We need to study the difference and similarity between $\phi_X:X\to S_X$ and $\phi_Y:Y\to S_Y$. A common strategy in birational geometry is to study the ample divisors on X and Y. This works well in our setting, as $-(K_X+B_X+M_X)$ is ample over S_X and $-(K_Y+B_Y+M_Y)$ is ample over S_Y. Therefore, we may pick general ample$/Z$ $\mathbb{R}$-divisors L_X and H_Y on X and Y respectively, such that

- $L_X\sim_{\mathbb{R},Z}-(K_X+B_X+M_X)+\phi_X^*A_{S_X}$ and
- $H_Y\sim_{\mathbb{R},Z}-(K_Y+B_Y+M_Y)+\phi_Y^*A_{S_Y}$,

for some general ample $\mathbb{R}$-divisors A_{S_X} and A_{S_Y} on S_X and S_Y respectively. In particular, $L_W:=\rho_X^*L_X$ and $H_W:=\rho_Y^*H_Y$ are big and nef$/Z$, and we may define $H_X:=(\rho_X)_*H_W$ and $L_Y:=(\rho_Y)_*L_W$. We have

- $K_X+B_X+L_X+0H_Y+M_X\sim_{\mathbb{R},S_X}0$, and
- $K_Y+B_Y+0L_Y+H_Y+M_Y\sim_{\mathbb{R},S_Y}0$.

4.1. **Construct a Sarkisov link.**

Construction 4.1 (Setting)**.** This setting will be used in the rest of this section. We assume that

- $X\to Z$ is a contraction,
- $\rho:W\dashrightarrow X$ is a birational map,
- (W,B_W+M_W) is a g-pair with associated nef$/Z$ ***b***-divisor M,
- L_W and H_W are two general big and nef$/Z$ $\mathbb{R}$-divisors on W,
- $(X,B+M_X)$ is a g-pair,
- $\phi:X\to S$ is a (K_X+B+M_X)-Mori fiber space$/Z$,

- Σ is a ϕ-vertical curve,
- L and H are two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, and
- $0 < l \leq 1$ and $0 \leq h \leq 1$ are two real numbers,

such that

(1) $(W, B_W + 2(L_W + H_W) + M_W)$ is $\mathbb{Q}$-factorial g-terminal,
(2) Maybe $(W, B_W + 2(L_W + H_W) + M_W)$ is glc and log smooth,
(3) $K_W + B_W + H_W + M_W$ is pseudo-effective$/Z$,
(4) $(X, B + M_X)$ is glc,
(5) $(W, B_W + lL_W + hH_W + M_W) \geq (X, B + lL + hH + M_X)$. In particular, ρ does not extract any divisor,
(6) B, L and H are the birational transforms of B_W, L_W and H_W on X respectively,
(7) $K_X + B + lL + hH + M_X \sim_{\mathbb{R},S} 0$, and
(8) $K_X + B + lL + hH + M_X$ is nef$/Z$.

We illustrate this setting in the following diagram:

$$\begin{tikzcd}
W \arrow[d, dashed, "\rho"'] & \supset & B_W \arrow[d, dashed] & lL_W \arrow[d, dashed] & hH_W \arrow[d, dashed] & M_W \arrow[d, dashed] & \\
X \arrow[d, "\phi"] & \supset & B & lL & hH & M_X & \Sigma : \phi\text{-vertical} \\
S & & & & & &
\end{tikzcd}$$

Definition 4.2 (Auxiliary constants and divisors)**.** Assumptions and notations as Construction 4.1,

(1) we define
$$r := \frac{H \cdot \Sigma}{L \cdot \Sigma}.$$
(2) For any real number t, we define
$$B_W(t) := B_W + lL_W + hH_W + t(H_W - rL_W),$$
and
$$B(t) := B + lL + hH + t(H - rL).$$
(3) We define Γ to be the set of all real number t satisfying the following:
  (a) $0 \leq t \leq \frac{l}{r}$,
  (b) for any prime divisor $E \subset W$,
  $$a(E, W, B_W(t) + M_W) \leq a(E, X, B(t) + M_X),$$
  and
  (c) $K_X + B(t) + M_X$ is nef$/Z$.
(4) We define $s := \sup\{t \mid t \in \Gamma\}$.
(5) We define $l_Y := l - rs$ and $h_Y := h + s$.

Remark 4.1. We can run MMP on X.

Lemma 4.3. *Assumptions and notations as Construction 4.1 and Definition 4.2, then*

(1) $r > 0$ is well-defined,
(2) either $\Gamma = \{0\}$, or Γ is a closed interval,
(3) Γ is non-empty and $s \in \Gamma$,
(4) $l_Y = l$ if and only if $h_Y = h$, and
(5) $\Gamma \subset [0, 1-h]$. In particular, $h_Y \leq 1$.

TODO: proof $h_Y \leq 1$

Proof. Since L_W and H_W are general big and nef$/Z$ divisors on W, L and H are big$/Z$, hence ample$/S$. Thus $H \cdot \Sigma > 0$ and $L \cdot \Sigma > 0$, hence $r > 0$ is well-defined. This is (1).

By Definition 4.2(3), $0 \in \Gamma$ and Γ is closed and connected, which implies (2). (3) follows from (2) and the definition of s. (4) follows from (1) and the definitions of l_Y and h_Y.

Assume that (5) does not hold. By (2), there exists $t_0 \in \Gamma$ such that $1 < h+t_0 < 2$. By Construction of terminalization, $(W, B_W(t_0) + M_W)$ is g-terminal.

By Proposition terminalization and the definition of Γ, $(W, B_W(t_0) + M_W) \geq (X, B(t_0) + M_X)$. Therefore $(X, B(t_0) + M_X)$ is gklt.

This is not necessary? Yes.

Since $(K_X + B(t_0) + M_X) \cdot \Sigma = 0$ and H is big$/Z$,

$$(K_X + B + (l - t_0 r)L + H + M_X) \cdot \Sigma = ((K_X + B(t_0) + M_X) - (h + t_0 - 1)H) \cdot \Sigma < 0.$$

Thus ϕ is a $(K_X + B + (l - t_0 r)L + H + M_X)$-Mori fiber space$/Z$. In particular, $K_X + B + H + M_X$ is not pseudo-effective$/Z$. Since ρ does not extract any divisor, $K_W + B_W + H_W + M_W$ is not pseudo-effective$/Z$, which contradicts Construction 4.1(2). $\square$

Construction 4.4. Assumptions and notations as Construction 4.1 and Definition 4.2. Then there are three possibilities for s:

Case 1 $s = \frac{l}{r}$. In particular, $l_Y = 0$.

Case 2
- $s < \frac{l}{r}$. In particular, $l_Y > 0$, and
- there exists $0 < \epsilon \ll 1$ and a prime divisor $E \subset W$, such that $a(E, W, B_W(s+\epsilon) + M_W) > a(E, X, B(s+\epsilon) + M_X)$.

Case 3
- $s < \frac{l}{r}$. In particular, $l_Y > 0$, and
- there exists $0 < \epsilon \ll 1$, such that
  - $a(E, W, B_W(s+\epsilon) + M_W) \leq a(E, X, B(s+\epsilon) + M_X)$ for any prime divisor $E \subset W$, and
  - $K_X + B(s+\epsilon) + M_X$ is not nef$/Z$.

TODO: replace terminal singularity

Theorem 4.5 (Sarkisov link with double scaling)**.** *Assumptions and notations as Construction 4.1 and Definition 4.2. The there exist*

- *a birational map$/Z$ $\rho_Y : W \dashrightarrow Y$ which does not extract any divisor,*
- *three $\mathbb{R}$-divisors B_Y, L_Y and H_Y on Y,*
- *a $(K_Y + B_Y + M_Y)$-Mori fiber space$/Z$ $\phi_Y : Y \to S_Y$, and*
- *a Sarkisov link$/Z$ $f : X \dashrightarrow Y$,*

such that

1. *$(Y, B_Y + M_Y)$ is a $\mathbb{Q}$-factorial gklt g-pair$/Z$,*
2. *$(W, B_W + l_Y L_W + h_Y H_W + M_W) \geq (Y, B_Y + l_Y L_Y + h_Y H_Y + M_Y)$. In particular, ρ_Y does not extract any divisor,*
3. *B_Y, L_Y and H_Y are the birational transforms of B_W, L_W and H_W on Y respectively,*
4. *$K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y \sim_{\mathbb{R}, S_Y} 0$,*
5. *$K_Y + B_Y + l_Y L_Y + h_Y H_Y + M_Y$ is nef$/Z$,*
6. *for any ϕ_Y-vertical curve Σ_Y on Y, and $r = \frac{H\cdot\Sigma}{L\cdot\Sigma} \geq \frac{H_Y\cdot\Sigma_Y}{L_Y\cdot\Sigma_Y} > 0$.*

Proof. We prove the Theorem by considering the three different cases in Construction 4.4 separately.

Case 1. In this case, we finish the proof by letting $\rho_Y := \rho, Y := X, B_Y := B, L_Y := L, H_Y := H, M_Y := M_X, \phi_Y := \phi_X, S_Y := S$, and $f := \mathrm{id}_X$.

Case 2. In this case, $a(E, W, B_W(s) + M_W) = a(E, X, B(s) + M_X)$, and E is exceptional$/X$. Since $E \subset W$,

$$a(E, X, B(s+\epsilon) + M_X) < a(E, W, B_W(s+\epsilon) + M_W) \leq 1.$$

TODO: replace the terminalization

By Lemma 3.4, there is an extraction $g : V \to X$ of E such that V is $\mathbb{Q}$-factorial. By Proposition terminal(4), the induced birational map $W \dashrightarrow V$ does not extract any divisor. We let B_V, L_V, H_V be the birational transforms of B_W, L_W and H_W on V respectively, then we have

$$\begin{aligned}&K_V + B_V + (l_Y - r\epsilon)L_V + (h_Y + \epsilon)H_V + M_V\\=&g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X).\end{aligned}$$

Moreover, since $a(E, X, B(s+\epsilon)+M_X) < 1$, $\mathrm{mult}_E(B_V + (l_Y - r\epsilon)L_V + (h_Y + \epsilon)H_V) > 0$. Thus we may pick a sufficiently small positive real number $0 < \delta \ll \epsilon$, such that $(V, \Delta_V + M_V)$ is gklt, where

$$K_V + \Delta_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta)L + (h_Y + \epsilon)H + M_X).$$

We may run a $(K_V + \Delta_V + M_V)$-MMP$/S$ $\psi : V \dashrightarrow Y$ which terminates with a Mori fiber space$/S$ $\phi_Y : Y \to S_Y$ by Theorem 3.3. Since $\rho(V/S) = \rho(V/X) + \rho(X/S) = 2$ and $1 = \rho(Y/S_Y) \leq \rho(V/S_Y) \leq \rho(V/S)$, there are two possibilities:

Case 2.1. $\rho(V/Y) = 0$. In this case ψ is a sequence of flips, and we get a Sarkisov link$/Z$ $f : X \dashrightarrow Y$ of type I. Let B_Y, L_Y and H_Y be the birational transforms of B_V, L_V and H_V on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0 < \delta' \ll \delta$ and let

$$K_V + \Delta'_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X),$$

then ψ is also a $(K_V + \Delta'_V + M_V)$-MMP$/S$. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned}&g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V\\=&(K_Y + \Delta'_Y + M_Y) \cdot \Sigma_Y < 0\end{aligned}$$

Let $\delta' \to 0$, then we have

$$g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \leq 0.$$

Since $g^*(K_X + B + l_Y L + h_Y H + M_X) \sim_{\mathbb{R},S} 0$, we deduce that

$$g^*(H - rL) \cdot \Sigma_V \leq 0.$$

Moreover, by our assumptions, $g^*(H - rL) = g_*^{-1}(H - rL) + eE$ for some real number $e > 0$, and $\Sigma_V \not\subset E$. Thus

$$\begin{aligned}(H_Y - rL_Y) \cdot \Sigma_Y &= g_*^{-1}(H - rL) \cdot \Sigma_V = (g^*(H - rL) - eE) \cdot \Sigma_V\\&\leq g^*(H - rL) \cdot \Sigma_V \leq 0,\end{aligned}$$

which implies (6), and the theorem follows in this case.

For any general ϕ_Y-vertical curve Σ_Y, ψ is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_V be the birational transform of Σ_Y on V. Pick any $0 < \delta' \ll \delta$ and let

$$K_V + \Delta'_V + M_V := g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X),$$

then ψ is also a $(K_V + \Delta'_V + M_V)$-MMP$/S$. Let Δ'_Y be the birational transform of Δ'_V on Y. Then

$$\begin{aligned} &g^*(K_X + B + (l_Y - r\epsilon - \delta')L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \\ =&(K_Y + \Delta'_Y + M_Y) \cdot \Sigma_Y < 0 \end{aligned}$$

Let $\delta' \to 0$, then we have

$$g^*(K_X + B + (l_Y - r\epsilon)L + (h_Y + \epsilon)H + M_X) \cdot \Sigma_V \le 0.$$

Since $g^*(K_X + B + l_Y L + h_Y H + M_X) \sim_{\mathbb{R},S} 0$, we deduce that

$$g^*(H - rL) \cdot \Sigma_V \le 0.$$

Moreover, by our assumptions, $g^*(H - rL) = g_*^{-1}(H - rL) + eE$ for some real number $e > 0$, and $\Sigma_V \not\subset E$. Thus

$$\begin{aligned} (H_Y - rL_Y) \cdot \Sigma_Y = g_*^{-1}(H - rL) \cdot \Sigma_V &= (g^*(H - rL) - eE) \cdot \Sigma_V \\ &\le g^*(H - rL) \cdot \Sigma_V \le 0, \end{aligned}$$

which implies (6), and the theorem follows in this case.

Case 3. In this case, there exists a $(K_X+B(s+\epsilon)+M_X)$-negative extremal ray $[C]$ on X. Since $(K_X+B(s+\epsilon)+M_X)\cdot\Sigma = 0$, $[C] \neq [\Sigma]$. Let $P \subset \overline{NE}(X/Z)$ be the extremal face over Z defined by all $(K_X + B(s + \epsilon) + M_X)$-non-positive irreducible curves. Then $P \neq [\Sigma]$, and hence there exists an extremal ray $[\Pi]$ such that $[\Sigma]$ and $[\Pi]$ span a two-dimensional face of P. By our construction, $(K_X + B(s + \epsilon) + M_X) \cdot \Pi < 0$. Now for $0 < \delta \ll 1$, we have

$$(K_X + B + (l_Y - r\epsilon - \delta)L_X + (h_Y + \epsilon)H_X + M_X) \cdot \Sigma < 0$$

and

$$(K_X + B + (l_Y - r\epsilon - \delta)L_X + (h_Y + \epsilon)H_X + M_X) \cdot \Pi < 0.$$

By Theorem 3.2, there exists a contraction $\pi : X \to T$ of the extremal face of $\overline{NE}(X/Z)$ spanned by $[\Sigma]$ and $[\Pi]$. Then π factors through S, and $K_X + B(s) + M_X \sim_{\mathbb{R},T} 0$.

Since L, H are big$/Z$, L, H are big$/T$. Therefore, if $K_X + B(s + \epsilon) + M_X$ is pseudo-effective$/T$, then $K_X + (1 + \alpha)B(s + \epsilon) + M_X$ is big$/T$. By Theorem 3.3, we may run a $(K_X + B(s + \epsilon) + M_X)$-MMP$/T$ with scaling of some ample$/T$ divisor, and this MMP$/T$ terminates. There are three cases:

Case 3.1. After a sequence of flips $f : X \dashrightarrow Y$, the MMP$/T$ terminates with a Mori fiber space$/T$ $\phi_Y : Y \to S_Y$. Therefore, f is a Sarkisov link$/Z$ of type IV. Let B_Y, L_Y, H_Y be the birational transforms of B, L and H on Y respectively and $\rho_Y : W \dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since ϕ_Y is a $(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y)$-Mori fiber space$/T$,

$$-(K_Y + B_Y + (l_Y - r\epsilon)L_Y + (h_Y + \epsilon)H_Y + M_Y) \cdot \Sigma_Y > 0,$$

which implies that

$$-(K_X + B(s + \epsilon) + M_X) \cdot \Sigma_X > 0.$$

Since $K_X+B(s)+M_X\sim_{\mathbb{R},T}0$,

$$-(K_X+B(s)+M_X)\cdot\Sigma_X=0,$$

which implies that

$$(H_Y-rL_Y)\cdot\Sigma_Y=(H-rL)\cdot\Sigma_X<0.$$

Thus $r>\frac{H_Y\cdot\Sigma_Y}{L_Y\cdot\Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.2. After a sequence of flips $X\dashrightarrow U$, we get a divisorial contraction$/T$: $U\to Y$. Therefore $\rho(Y/T)=1$, which implies that the induced morphism $\phi_Y:=Y\to T$ is a Mori fiber space, and the induced birational map $f:X\dashrightarrow Y$ is a Sarkisov link$/Z$ of type III. Let B_Y,L_Y,H_Y be the birational transforms of B,L and H on Y respectively and $\rho_Y:W\dashrightarrow Y$ the induced morphism. By our constructions, (1)-(5) are clear, and we only left to show (6).

For any general ϕ_Y-vertical curve Σ_Y, f is an isomorphism in a neighborhood of Σ_Y, and we may let Σ_X be the birational transform of Σ_Y on X. Since $-(K_X+B(s+\epsilon)+M_X)$ is nef$/T$ and $K_X+B(s)+M_X\sim_{\mathbb{R},T}0$, we have

$$-(K_X+B(s+\epsilon)+M_X)\cdot\Sigma_X\geq 0=-(K_X+B(s)+M_X)\cdot\Sigma_X,$$

which implies that

$$(H_Y-rL_Y)\cdot\Sigma_Y=(H-rL)\cdot\Sigma_X\leq 0.$$

Thus $r\geq\frac{H_Y\cdot\Sigma_Y}{L_Y\cdot\Sigma_Y}$, which implies (6), and the theorem follows in this case.

Case 3.3. After a sequence of flips $f:X\dashrightarrow Y$, the MMP terminates with a minimal model Y over T. Let B_Y,L_Y,H_Y be the birational transforms of B,L and H on Y respectively. Since Σ is a general ϕ-vertical curve, we may let Σ' be the birational transform of Σ on Y. Since $(K_X+B(s+\epsilon)+M_X)\cdot\Sigma=0$ and $(K_X+B(s)+M_X)\cdot\Sigma=0$, we have

$$(K_Y+B_Y+(l_Y-r\epsilon)L_Y+(h_Y+\epsilon)H_Y+M_Y)\cdot\Sigma'=0$$

and

$$(K_Y+B_Y+l_YL_Y+h_YH_Y+M_Y)\cdot\Sigma'=0$$

which implies that $(K_Y+B_Y+M_Y)\cdot\Sigma'<0$ and $r=\frac{H_Y\cdot\Sigma'}{L_Y\cdot\Sigma'}$. Since Σ can be chosen to be any ϕ-vertical curve, by Theorem 3.2, there exists a contraction $\phi_Y:Y\to S_Y$ of $[\Sigma']$ such that ϕ_Y is a $(K_Y+B_Y+M_Y)$-Mori fiber space$/T$. Thus f is a Sarkisov link$/Z$ of type IV. We finish the proof by letting $\rho_Y:W\dashrightarrow Y$ be the induced birational map. $\square$

4.2. Behavior of invariants under a Sarkisov lins.

4.3. Run the Sarkisov program with double scaling.
Construction
Lemma of termination.

5. Proof of main theorem

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