Problems in Algebraic Geometry

Fall 2020

Problem 1. Let A be a ring. Let U and V be quasi-compact open subsets of $\mathrm{Spec}(A)$. Show that $U \cap V$ is quasi-compact.

Problem 2. An open subset of $\mathrm{Spec}(A)$ is called *principal* if it is of the form $D(f)$ for some $f \in A$.

(a) Find an open subset of $\mathrm{Spec}(\mathbb{Z}[X])$ that is not principal.

(b) Let A be a Dedekind domain whose ideal class group is torsion (e.g. A is the ring of integers of a number field). Show that every open subset of $\mathrm{Spec}(A)$ is principal.

Problem 3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. We let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ denote the presheaf on X carrying an open subset $U \subseteq X$ to $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf on X.

Problem 4. Let X be a topological space, U an open subset, and $j \colon U \to X$ the inclusion map.

(a) (Extension by the empty set) Let $\mathcal{F}$ be a sheaf of sets on U. Show that the presheaf on X

$$j_!^{\mathrm{Set}}\mathcal{F} \colon V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ \emptyset & V \not\subseteq U \end{cases}$$

is a sheaf. Compute the stalks of $j_!^{\mathrm{Set}}\mathcal{F}$.

(b) (Extension by zero) Let $\mathcal{F}$ be a sheaf of abelian groups on U. Let $j_!\mathcal{F}$ be the sheafification of the presheaf on X

$$j_!^{\mathrm{psh}}\mathcal{F} \colon V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & V \not\subseteq U. \end{cases}$$

Compute the stalks of $j_!\mathcal{F}$. Deduce that $j_! \colon \mathrm{Shv}(U, \mathrm{Ab}) \to \mathrm{Shv}(X, \mathrm{Ab})$ is an exact functor. Find an example for which $j_!^{\mathrm{psh}}\mathcal{F}$ is not a sheaf.

(*Remark.* $j_!^{\mathrm{Set}}$ is a left adjoint of $j^{-1} \colon \mathrm{Shv}(X, \mathrm{Set}) \to \mathrm{Shv}(U, \mathrm{Set})$ and $j_!$ is a left adjoint of $\mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Shv}(U, \mathrm{Ab})$.)

Problem 5.

(a) Show that a ring homomorphism $\phi\colon A \to B$ is a monomorphism if and only if ϕ is an injection. (*Hint.* Consider ring homomorphisms $\mathbb{Z}[X] \to A$ or the diagonal $\Delta_\phi\colon B \to B \times_A B$).

(b) Let $f\colon Y \to X$ be an epimorphism of schemes. Show that $f^\flat_X\colon \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is an injection and $f(Y)$ intersects with every nonempty closed subset Z of X. (*Hint* for the second assertion. Consider the scheme obtained by gluing two copies of X along $X\backslash Z$.)

(c) Use (b) to give an example of an injective ring homomorphism $\phi\colon A \to B$ such that $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is *not* an epimorphism of schemes.

Problem 6. We say that a continuous map $f\colon Y \to X$ is *dominant* if $f(Y)$ is dense in X. We say that a morphism $f\colon Y \to X$ of schemes is *scheme-theoretically dominant* if $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a monomorphism. (You may either admit the fact that a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(X, \mathrm{Ring})$ is a monomorphism if and only if $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an injection for every open subset U of X, or take this as a definition.)

(a) Show that a ring homomorphism $\phi\colon A \to B$ is an injection if and only if $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is scheme-theoretically dominant.

(b) Show that a scheme-theoretically dominant morphism $f\colon Y \to X$ is dominant. Show moreover that the converse holds for X reduced.

(c) Show that a scheme-theoretically dominant morphism that is surjective is an epimorphism of schemes. Deduce that any surjective morphism of schemes $f\colon Y \to X$ with X reduced is an epimorphism.

Problem 7.

(a) Show that a ring homomorphism $\phi\colon A \to B$ is an epimorphism if and only if $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a monomorphism of schemes.

(b) Let X be a scheme. Let $X' = \coprod_{x\in X} \mathrm{Spec}(\kappa(x))$, where $\kappa(x)$ denotes the residue field of $\mathcal{O}_{X,x}$ and let $f\colon X' \to X$ be the canonical morphism sending $x' = \mathrm{Spec}(\kappa(x))$ to x with $f^\sharp_{x'}\colon \mathcal{O}_{X,x} \to \kappa(x)$ given by the projection. Show that f is a monomorphism of schemes.

(c) Use (b) and Problem 6(c) to give an example of a morphism of affine schemes that is a monomorphism of schemes, an epimorphism of schemes, and a bijection, but not an isomorphism of schemes.

Problem 8. Let $\mathcal{P}$ be an infinite set and let $A \subseteq \prod_{p\in\mathcal{P}} \mathbb{F}_2$ be the subring consisting of $a = (a_p)$ such that $\mathrm{supp}(a) := \{p \mid a_p \neq 0\}$ is a finite or cofinite subset of $\mathcal{P}$. (Recall that a *cofinite* subset is the complement of a finite subset.) Let $\mathfrak{m}_p$ be the kernel of the projection $A \to \mathbb{F}_2$ sending a to a_p and let $\mathfrak{m}_\infty = \bigoplus_{p\in\mathcal{P}} \mathbb{F}_2 \subseteq A$.

(a) Let $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$ be the one-point compactification of the discrete set $\mathcal{P}$. (In other words, the open subsets of $\mathcal{P}^*$ are the cofinite subsets of $\mathcal{P}^*$ and all the subsets of $\mathcal{P}$.) Show that $\mathfrak{m}_p$ and $\mathfrak{m}_\infty$ are maximal ideals of A and the map $\mathcal{P}^* \to \operatorname{Spec}(A)$ sending p to $\mathfrak{m}_p$ and ∞ to $\mathfrak{m}_\infty$ is a homeomorphism.

(b) Show that $A/\mathfrak{m}_\infty \simeq \mathbb{F}_2$.

Problem 9. Let X be a nonempty quasi-compact scheme. Show that X has a closed point. (*Remark.* In fact, every nonempty quasi-compact T_0 space has a closed point.)

Problem 10.

(a) Let X be a quasi-compact scheme. Let $A = \mathcal{O}_X(X)$ and $f \in A$. Show that the restriction map $A \to \mathcal{O}_X(X_f)$ factors through an injective homomorphism $\phi\colon A_f \to \mathcal{O}_X(X_f)$.

(b) Let X be a scheme admitting a finite cover $\{U_i\}$ by open affines such that each intersection $U_i \cap U_j$ is quasi-compact. Show that $\phi\colon A_f \to \mathcal{O}_X(X_f)$ is an isomorphism.

(c) Let X be a scheme such that there exist $f_1, \dots, f_n \in \mathcal{O}_X(X) = A$ with $\sum_{i=1}^n f_i A = A$ and X_{f_i} affine for all i. Show that X is affine.

Problem 11. Let $f\colon Y \to X$ be a morphism of schemes.

(a) Show that if f is locally of finite type, $U \simeq \operatorname{Spec}(A)$ is an affine open of X and $V \simeq \operatorname{Spec}(B)$ is an affine open of $f^{-1}(U)$, then B is a finitely-generated A-algebra.

(b) Show that if f is quasi-compact and U is a quasi-compact open subset of X, then $f^{-1}(U)$ is quasi-compact.

(c) Show that if f is affine and U is an affine open of X, then $f^{-1}(U)$ is an affine open of Y.

(d) Show that if f is finite and $U \simeq \operatorname{Spec}(A)$ is an affine open of X, then $f^{-1}(U) \simeq \operatorname{Spec}(B)$ with B a finite A-algebra.

Problem 12. (a) Let A be a Noetherian local ring of dimension ≥ 1. Show that the maximal ideal $\mathfrak{m}$ is the union of prime ideals of A of height 1. (*Hint.* Use Krull's principal ideal theorem. The weaker assertion with height 1 replaced by height ≤ 1 suffices for (b).)

(b) Let A be a Noetherian ring of dimension ≥ 2. Deduce from (a) that there are infinitely many prime ideals of A of height 1. (*Hint.* Use the prime avoidance lemma.)

(c) Deduce from (b) that every locally Noetherian scheme of dimension ≥ 2 has infinitely many points.

Problem 13. (a) Show that a morphism $f\colon Y \to X$ in a category admitting fiber products is a monomorphism if and only if the first projection $Y \times_X Y \to Y$ is an isomorphism. Show moreover that monomorphisms are stable under base change.

(b) Let k be a field. Use (a) to show that a ring homomorphism $\phi\colon k \to B$ is an epimorphism if and only if $B = 0$ or ϕ is an isomorphism. Deduce that a morphism of schemes $f\colon Y \to \operatorname{Spec}(k)$ is a monomorphism if and only if $Y = \emptyset$ or f is an isomorphism.

(c) Let $f\colon Y \to X$ be a monomorphism of schemes. Show that f is an injection and for every point $y \in Y$, the extension of residue fields $\kappa(y)/\kappa(f(y))$ is trivial.

Problem 14. Given a scheme X and a field K, we let $X(K)$ denote $\operatorname{Hom}(\operatorname{Spec}(K), X)$.

(a) Let $\phi\colon K \to L$ be a field embedding. Show that the induced map $X(\phi)\colon X(K) \to X(L)$ is an injection. (*Hint.* Use Problem 6 or the identification of $X(K)$ with $\{(x, \iota) \mid x \in X, \iota\colon \kappa(x) \to K\}$.)

(b) Show that a morphism of schemes $f\colon X \to Y$ is surjective if and only if for every field K, there exists a field extension L/K such that $f(L)\colon X(L) \to Y(L)$ is a surjection. (*Hint* for the "only if" part. One can start by showing that for every K and every $y \in Y(K)$, there exists a field embedding $\phi_y\colon K \to L_y$ such that $Y(\phi_y)(y) \in Y(L_y)$ belongs to the image of $f(L_y)$. A more direct proof is also possible.)

(c) Show that a morphism of schemes $f\colon X \to Y$ is radiciel if and only if the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is surjective. Deduce that every radiciel morphism is separated.

Problem 15.

(a) Let g and h be morphisms of schemes $X \to W$ and let E be their equalizer. Show that the morphism $E \to X$ is an immersion whose image is contained in the set-theoretic equalizer $E' = \{x \in X \mid g(x) = h(x)\}$.

(b) Deduce the following improvement of Problem 6(c): a scheme-theoretically dominant morphism $f\colon Y \to X$ such that $f(Y)$ intersects with every nonempty closed subset Z of X is an epimorphism.

(c) Let A be a local domain of dimension ≥ 2 with fraction field K and residue field k. Use (b) and Problem 7(b) to show that $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is a monomorphism of schemes and an epimorphism of schemes, but not a surjection.

Problem 16. (a) Let $i\colon Z \to X$ be a closed immersion of schemes. Let

$$W = \underline{\mathrm{Spec}}(\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X).$$

Show that the canonical morphism $X \coprod X \simeq \underline{\mathrm{Spec}}(\mathcal{O}_X \times \mathcal{O}_X) \to W$ is finite surjective. Describe the underlying topological space of W.

(*Remark.* This construction and its generalizations are called *pinching*.)

(b) Let $f\colon Y \to X$ be a quasi-compact morphism of schemes. Show that the ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$ is quasi-coherent and the closed subscheme Z of X defined by $\mathcal{I}$ is the smallest closed subscheme of X through which f factors. We call Z the *scheme-theoretic image* of f.

(c) Deduce that a quasi-compact morphism of schemes $f\colon Y \to X$ is an epimorphism if and only if f is scheme-theoretically dominant and $f(Y)$ intersects with every nonempty closed subset of X. (See also Problems 5(b) and 15(b).)

Problem 17. Let k be an algebraically closed field. In each of the following cases, compute the normalization $f\colon X^\nu \to X$ of X. Describe all fibers of f that are not geometrically irreducible or geometrically reduced. Is f a universal homeomorphism?

(a) $X = \mathrm{Spec}(k[x,y]/(y^7 - x^{2020}))$;

(b) $X = \mathrm{Spec}(k[x,y,z]/(xy^2 - z^2))$. (*Hint.* The answers depend on whether $\mathrm{char}(k) = 2$.)

Problem 18. (a) Show that an injective and closed morphism of schemes is affine.

(b) Deduce that an injective and universally closed morphism of schemes is integral.

Problem 19. (a) Show that a scheme X is separated if and only if there exists an affine open cover $\{U_i\}$ of X such that $U_i \cap U_j$ is affine and the canonical homomorphism

$$\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

is surjective for all i, j.

(b) Let R be a graded ring. Show that, for all $f, g \in R_+$ homogeneous, the canonical homomorphism $R_{(f)} \otimes R_{(g)} \to R_{(fg)}$ is surjective. Deduce that $\mathrm{Proj}(R)$ is separated.

Problem 20. Let R be a graded ring.

(a) Show that for any prime ideal $\mathfrak{p}$ of R, $\bigoplus_{d\geq 0}(\mathfrak{p} \cap R_d)$ is a homogeneous prime ideal of R. Deduce that any minimal prime ideal of R is homogeneous.

(b) Let T be the set of maximal points of $\mathrm{Spec}(R)$. Show that $T \cap \mathrm{Proj}(R)$ is the set of maximal points of $\mathrm{Proj}(R)$.

(c) Show that $\mathrm{Proj}(R)$ is normal if R is an integrally closed domain.

Problem 21. (a) Let A be a Noetherian ring and $\mathfrak{b}$ an ideal of A. We say that an ideal $\mathfrak{a}$ of A is $\mathfrak{b}$-*saturated* if $(\mathfrak{a} : \mathfrak{b}) = \mathfrak{a}$, where $(\mathfrak{a} : \mathfrak{b}) := \{x \in A \mid \mathfrak{b}x \subseteq \mathfrak{a}\}$. For any ideal $\mathfrak{a}$ of A, show that the sequence of ideals $(\mathfrak{a} : \mathfrak{b}^n)$, $n \geq 0$ is stationary and $(\mathfrak{a} :^\infty \mathfrak{b}) := \bigcup_{n\geq 0}(\mathfrak{a} : \mathfrak{b}^n)$ is the smallest $\mathfrak{b}$-saturated ideal containing $\mathfrak{a}$.

(*Remark.* We have $(\mathfrak{a} :^\infty \mathfrak{b})/\mathfrak{a} \simeq \Gamma_{V(\mathfrak{b})}(\mathrm{Spec}(A), \widetilde{A/\mathfrak{a}})$, where Γ_Z denotes the set of global sections supported in a closed subset Z.)

(b) For any primary ideal $\mathfrak{q}$ of A, show that

$$(\mathfrak{q} :^\infty \mathfrak{b}) = \begin{cases} \mathfrak{q} & \sqrt{\mathfrak{q}} \not\supseteq \mathfrak{b}, \\ A & \sqrt{\mathfrak{q}} \supseteq \mathfrak{b}. \end{cases}$$

Deduce that $(\sqrt{\mathfrak{a}} :^\infty \mathfrak{b}) = \bigcap_{\mathfrak{p}\in V(\mathfrak{a})\backslash V(\mathfrak{b})} \mathfrak{p}$.

(c) Let R be a graded ring. For any subset $Y \subseteq \mathrm{Proj}(R)$, let $I(Y) = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$. Show that $V_+(I(Y)) = \overline{Y}$ is the closure of Y in $\mathrm{Proj}(R)$.

(d) Assume that R is Noetherian. For any homogeneous ideal $\mathfrak{a}$ of R, show that $I(V_+(\mathfrak{a})) = (\sqrt{\mathfrak{a}} :^\infty R_+)$. Deduce that the maps V_+ and I induce a one-to-one order-reversing correspondence between R_+-saturated radical homogeneous ideals of R and closed subsets of $\mathrm{Proj}(R)$.

Problem 22. Let A be a ring and let $a, b \geq 1$ be integers. Show that the weighted projective line $\mathbb{P}_A(a, b)$ is canonically isomorphic to $\mathbb{P}^1_A$.

Problem 23. Let A be a ring and let $d \geq 2$ be an integer. Let $I \subseteq R = A[x_0, \dots, x_d]$ denote the homogeneous ideal of the d-uple embedding $\mathbb{P}^1_A \hookrightarrow \mathbb{P}^d_A$.

(a) Show that $I \cap R_2$ is a free A-module of rank $\binom{d}{2}$. Deduce that I cannot be generated by less than $\binom{d}{2}$ elements unless $A = 0$.

(b) Show that I is generated by $I \cap R_2$. (*Hint.* Show that $I \cap R_n$ is generated by $x_{i_0} \cdots x_{i_n} - x_{j_0} \cdots x_{j_n}$ with $i_0 + \cdots + i_n = j_0 + \cdots + j_n$. Proceed by induction on n to show that such elements are generated by $I \cap R_2$.)

(c) Assume $d = 3$. Let $J = (x_1^2 - x_0x_2, x_2^3 - x_0x_3^2) \subseteq I$. Check that $\sqrt{J} = \sqrt{I}$.

Problem 24. We say that a scheme X is *locally integral* if $\mathcal{O}_{X,x}$ is a domain for every $x \in X$. Show that the irreducible components of a locally integral scheme are disjoint. Deduce that a locally integral scheme with finitely many irreducible components is a finite coproduct of integral schemes.

Problem 25. Let k be a field.

(a) Let A be a finitely generated k-algebra that is a domain. Assume that $A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$ of height 1. Show that the integral closure of A is $\bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through height 1 prime ideals. (*Remark.* The assumption that A is a finitely generated k-algebra can be weakened to A being a universally catenary Japanese Noetherian domain. The universal catenarity cannot be dropped. See [EGA IV, Exemple 5.6.11].)

(b) Let R be a finitely generated graded k-algebra that is a domain generated by R_1 over R_0. Assume that R_+ has height ≥ 2 and $X = \mathrm{Proj}(R)$ is normal. Show that the canonical map $R \to \Gamma_*(\mathcal{O}_X) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ identifies $\Gamma_*(\mathcal{O}_X)$ with the integral closure of R.

Problem 26. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Show that for any affine open U of X, $X_s \cap U$ is affine.

Problem 27. Let A be a ring. For an A-module M, we let $\mathbb{P}_A(M)$ denote $\mathrm{Proj}(\mathrm{Sym}_A(M))$. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of A-modules.

(a) Show that g induces a closed immersion $\mathbb{P}_A(g)\colon \mathbb{P}_A(M'') \to \mathbb{P}_A(M)$ and f induces an affine morphism $\mathbb{P}_A(g)\colon \mathbb{P}_A(M) \backslash \mathrm{im}(\mathbb{P}_A(g)) \to \mathbb{P}_A(M')$.

(b) Assume that the exact sequence splits. Show that $\mathbb{P}_A(g)$ can be identified with the projection $\mathbb{V}(\mathcal{O}_Y(-1) \otimes_A M'') \to Y$. Here $Y := \mathbb{P}_A(M')$, and, for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F}))$.

Problem 28. Let $f\colon X \to Y$ be a morphism of schemes with X quasi-compact. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on X and $\mathcal{M}$ an invertible sheaf on Y.

(a) Show that $X = \bigcup_{s \in S_{+,\mathrm{homog}}} X_s$ if and only if $\mathcal{L}^{\otimes n}$ is globally generated for some $n \geq 1$. Here $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. In this case we say that $\mathcal{L}$ is *semiample*.

(b) Show that if $\mathcal{L}$ is ample and $\mathcal{L}'$ is semiample, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is ample.

(c) Show that if $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then for $n \gg 0$, $\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M}^{\otimes n}$ is ample.

(d) Show that if $\mathcal{L}$ is f-very ample and $\mathcal{L}'$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is f-very ample.

(e) Show that if f is locally of finite type and $\mathcal{L}$ is ample, then there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is f-very ample for all $n \geq n_0$. (*Hint.* Use (d).)

Problem 29. (a) Let $f\colon X \to S$ be a separated morphism of schemes. Show that every section s of f is a closed immersion.

(b) Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $f\colon \mathbb{V}(\mathcal{E}) \to S$ and let $s\colon S \to \mathbb{V}(\mathcal{E})$ be the *zero section* of f, namely the section induced by $0\colon \mathcal{E} \to \mathcal{O}_S$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{V}(\mathcal{E})}$ be ideal sheaf corresponding to s. Show that $s^*\mathcal{I} \simeq \mathcal{E}$.

Problem 30. Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $P = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_S)$. Let Z_P and 0_P denote the closed subschemes defined respectively by the closed immersions $\mathbb{P}(\mathcal{E}) \to P$ and $\mathbb{P}(\mathcal{O}) \to P$ given by the projections $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{E}$ and $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{O}_S$. We call Z_P the *infinity locus* and 0_P the *zero section* of $P \to S$.

(a) Show that Z_P is an effective Cartier divisor of P and that $P \backslash Z_P$ can be identified with $\mathbb{V}(\mathcal{E})$. We call P the *projective closure* of $\mathbb{V}(\mathcal{E})$.

(b) Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})})$. Let Z_X and 0_X denote respectively the infinity locus and zero sections of $X \to \mathbb{P}(\mathcal{E})$. Construct an S-morphism $\pi\colon X \to P$ identifying X with the blowing up of P at 0_P such that $\pi^{-1}(0_P) = 0_X$ and $\pi^{-1}(Z_P) = Z_X$ as subschemes of X. Describe π in terms of the functors $(\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set}$ that X and P represent.

(c) Deduce that $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq \mathrm{Bl}_{0_P}(\mathbb{V}(\mathcal{E}))$ and $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)) \simeq P \backslash 0_P$. (For the last isomorphism, see also Problem 27(b).)

Problem 31. Let k be a field of characteristic $\neq 2$ and let $S = \mathrm{Spec}(k[x,y]/(y^2 - x^4))$. (The point $V(x,y)$ is called a *tacnode*.) Find blowings up $S' \to S$ and $S'' \to S'$ such that S'' is normal. (*Remark.* In fact one blowing up suffices.)

Problem 32. Show that, in a triangulated category, the direct sum of two distinguished triangles is a distinguished triangle. (*Hint.* Let $T_i\colon X_i \xrightarrow{f_i} Y_i \to Z_i \to X_i[1]$, $i = 1, 2$ be distinguished triangles. Extend $f_1 \oplus f_2$ to a distinguished triangle T and construct a morphism from $T_1 \oplus T_2$ to T.)

Problem 33. Let $\mathcal{D}$ be a triangulated category.

(a) Show that for objects X and Y in $\mathcal{D}$, the triangle $X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$, where i and p are the canonical morphisms, is a distinguished triangle. (*Hint.* Use Problem 32.)

(b) Conversely, show that every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{D}$ with $h = 0$ is isomorphic to the distinguished triangle in (1).

Problem 34. Let $\mathcal{A}$ be an abelian category. For every $L \in D(\mathcal{A})$ and $n \in \mathbb{Z}$, construct a distinguished triangle $\tau^{\leq n}L \to L \to \tau^{\geq n+1}L \xrightarrow{h} (\tau^{\leq n}L)[1]$ in $D(\mathcal{A})$. Show that $H^i h = 0$ for all i. Give an example with h nonzero in $D(\mathcal{A})$.

Problem 35. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories admitting an F-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$. We say that $X \in \mathcal{A}$ is *F-acyclic* if $R^n FX = 0$ for all $n \geq 1$. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ spanned by F-acyclic objects.

(a) Show that $\mathcal{I}$ is F-injective.

In the rest of this problem, assume that there exists $N > 0$ such that $R^N FX = 0$ for all $X \in \mathcal{A}$.

(b) Show that $R^n FX = 0$ for all $X \in \mathcal{A}$ and $n \geq N$.

(c) Show that for every exact sequence $X_{N-1} \to \cdots \to X_1 \to Y \to 0$ in $\mathcal{A}$ with $R^j FX_i = 0$ for all $j \geq i$, Y is F-acyclic.

(d) Deduce that for every $L \in C(\mathcal{I})$ acyclic, FL is acyclic.

Problem 36 (Serre)**.** Let X be a quasi-compact scheme. Assume that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$. Proceed in the following steps to show that X is affine.

(a) Show that for every closed point $x \in X$, there exists $f \in \mathcal{O}_X(X)$ such that $x \in X_f$ and X_f is affine. (*Hint.* Choose an affine open neighborhood U of x and consider the short exact sequence $0 \to \mathcal{I}_{Z'} \to \mathcal{I}_Z \to \mathcal{I}_Z/\mathcal{I}_{Z'} \to 0$, where $Z = X \backslash U$ and $Z' = Z \cup \{x\}$ are equipped with the reduced closed subscheme structures.)

(b) Use Problem 9 to deduce that there exist $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ with $X = \bigcup_{i=1}^n X_{f_i}$ and X_{f_i} affine.

(c) Show that $f_1, \ldots, f_n$ generate the unit ideal in $\mathcal{O}_X(X)$. Conclude by Problem 10(c).

Problem 37. Let $F\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be a triangulated functor carrying $D^{\geq 0}(\mathcal{A})$ into $D^{\geq 0}(\mathcal{B})$. Let $X \in D^{\geq 0}(\mathcal{A})$. Prove the existence of an isomorphism $H^0FH^0X \simeq H^0FX$ and an exact sequence

$$0 \to H^1FH^0X \to H^1FX \to H^0FH^1X \to H^2FH^0X \to H^2FX.$$

(*Hint.* Use the distinguished triangle $H^0X \to X \to \tau^{\geq 1}X \to (H^0X)[1]$.)

Problem 38. Let $\mathcal{G}$ be a sheaf of groups on a topological space X. A sheaf $\mathcal{F}$ of sets on X equipped with a (left) action of $\mathcal{G}$ is called a *$\mathcal{G}$-torsor* if

- For every open subset U of X and every pair of sections $s, t \in \mathcal{F}(U)$, there exists a unique $g \in \mathcal{G}(U)$ such that $gs = t$.
- $\mathcal{F}_x \neq \emptyset$ for all $x \in X$.

A morphism of $\mathcal{G}$-torsors is a morphism of sheaves preserving the $\mathcal{G}$-action.

(a) Show that every morphism of $\mathcal{G}$-torsors is an isomorphism. Let $\mathrm{Tors}(\mathcal{G})$ denote the set of isomorphism classes of $\mathcal{G}$-torsors.

(b) In the case with $\mathcal{G}$ abelian, establish a bijection between $\mathrm{Tors}(\mathcal{G})$ and $H^1(X, \mathcal{G})$. For every open cover $\mathcal{U}$ of X, describe the collection of $\mathcal{G}$-torsors corresponding to the image of the map $H^1(\mathcal{U}, \mathcal{G}) \to H^1(X, \mathcal{G})$.

(c) Let $\mathcal{O}_X$ be a sheaf of rings on X. Let $\mathrm{Loc}_n(\mathcal{O}_X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$ modules of rank n. Establish a bijection between $\mathrm{Loc}_n(\mathcal{O}_X)$ and $\mathrm{Tors}(\mathrm{GL}_n(\mathcal{O}_X))$, where $\mathrm{GL}_n(\mathcal{O}_X)$ denotes the sheaf of groups $U \mapsto \mathrm{GL}_n(\mathcal{O}_X(U))$. (*Hint.* For a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank n, consider $\mathcal{I}som_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F})$.)

(d) Establish a group isomorphism $\mathrm{Pic}(X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ denotes the abelian sheaf $U \mapsto \mathcal{O}_X(U)^\times$.

Problem 39. Let X be a quasi-compact quasi-separated topological space such that quasi-compact open subsets form a basis. The goal of this problem is to show that $H^q(X, -)$ commutes with filtered colimit: for every filtered system $(\mathcal{F}_i)$ of abelian sheaves on X, the canonical map

$$\operatorname*{colim}_i H^q(X, \mathcal{F}_i) \to H^q(X, \operatorname*{colim}_i \mathcal{F}_i)$$

is an isomorphism.

(a) Let Cov denote the collection of finite quasi-compact open covers of open subsets of X. Show that the full subcategory $\mathcal{J}$ of $\mathrm{Shv}(X)$ consisting of $\mathcal{G}$ satisfying $\check{H}^p(\mathcal{U}, \mathcal{G}) = 0$ for all $\mathcal{U} \in \mathrm{Cov}$ and $p \geq 1$ is $\Gamma(X, -)$-injective.

(b) Let $\mathcal{G}_i$ be a filtered system of flabby sheaves. Show that $\operatorname{colim}_i \mathcal{G}_i \in \mathcal{J}$.

(c) Conclude by induction on q. (*Hint.* Choose a functorial monomorphism $\mathcal{F}_i \to \mathcal{G}_i$ with $\mathcal{G}_i$ flabby.)

Problem 40. (a) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf of $\mathcal{O}_X$ such that $\mathcal{I}^n = 0$. Assume that the closed subscheme $X_0 = (X, \mathcal{O}_X/\mathcal{I})$ of X defined by $\mathcal{I}$ is affine. Show that X is affine. (*Hint.* Show that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ using the filtration $(\mathcal{I}^m\mathcal{F})_{0\leq m\leq n}$.)

(b) Deduce that a Noetherian scheme X such that X_{red} is affine is affine.

Remark. (Yin Hang) The Noetherian assumption can be removed by applying Problem 10(c) and a limit argument.

(c) Show that a reduced scheme X admitting a finite cover by affine closed subschemes is affine.

Problem 41. Let $f\colon X \to Y$ be a finite surjective morphism of Noetherian schemes with X affine. Show that Y is affine. You may follows the following steps.

(a) In the case where X and Y are integral, show that there exists a coherent sheaf $\mathcal{M}$ on X and a morphism of $\mathcal{O}_Y$-modules $\alpha\colon \mathcal{O}_Y^r \to f_*\mathcal{M}$ with $r > 0$ which is an isomorphism at the generic point η_Y of Y.

(b) Deduce in the case of (a) that for every coherent sheaf $\mathcal{F}$ on Y, there exists a coherent sheaf $\mathcal{G}$ on X and a morphism of $\mathcal{O}_Y$-modules $f_*\mathcal{G} \to \mathcal{F}^r$ that is an isomorphism at η_Y. (*Hint.* Apply $\mathcal{H}om(-,\mathcal{F})$ to α.)

(c) Use Problem 40 to reduce to the integral case. Conclude by Serre's criterion (Problem 36) and Noetherian induction on Y.

Remark. This result is due to Chevalley in the case of schemes of finite type over a field. It holds in fact more generally without the Noetherian assumption, generalizing Problem 40.

Problem 42. Let X be a scheme proper over a field k. Assume that X is geometrically connected and geometrically reduced over k. Show that the canonical map $k \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

Problem 43. Let S be a scheme and let X and Y be schemes over S.

(a) Assume that X is integral and Y is of finite type over S. Let $s \in S$ be a point and let $x \in X$ and $y \in Y$ be points above s. Let $\phi\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a homomorphism of $\mathcal{O}_{S,s}$-algebras. Show that there exists an open neighborhood $U \subseteq X$ of x and a morphism $f\colon U \to Y$ over S such that $f(x) = y$ and $f_x^\sharp = \phi$.

(b) Assume that X is Noetherian normal of dimension 1 and Y is proper over S. Let $U \subseteq X$ be a dense open subset. Show that every S-morphism $U \to Y$ extends uniquely to an S-morphism $X \to Y$:

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

(c) Deduce from Chow's lemma that a normal scheme of dimension 1 and proper over k is projective over k. (*Remark.* This holds in fact without the normality assumption.)

Problem 44. Let $X \to S$ and $Y \to S$ be morphisms of schemes and let $p\colon X \times_S Y \to X$ and $q\colon X \times_S Y \to Y$ be the projections. Show that the canonical morphism $p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S}$ is an isomorphism.

Problem 45. Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Consider the condition $(*)$: the sequence

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

is exact and locally splits.

(a) Show that if f is formally smooth, then $(*)$ holds.

(b) Show that if $(*)$ holds and gf is formally smooth, then f is formally smooth.

Problem 46. (a) Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Show that if B is regular, then so is A. (*Hint.* By theorems of Serre and Auslander, a Noetherian local ring A is regular if and only if A has finite weak dimension, namely there exists an integer d such that $\mathrm{Tor}_n^A(M, N) = 0$ for all A-modules M, N and all $n > d$.)

(b) Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes, locally of finite presentation. Show that if f is flat and surjective and gf is smooth, then g is smooth.

Problem 47. (a) Let A be a ring and let $R = A[x_0, \dots, x_n]/I$, where I is a finitely generated graded ideal. Show that $X = \mathrm{Proj}(R)$ is smooth over A if and only if $\mathrm{Spec}(R)\backslash V(R_+)$ is smooth over A. (*Hint.* Identify the latter with $\mathbb{V}(\mathcal{O}_X(1))\backslash 0_X$, where 0_X denotes the zero section.)

(b) Let $n \geq 1$ and $d \geq 3$ be integers and let k be a field of characteristic $p \mid d$. Show that Gabber's hypersurface $X = \mathrm{Proj}(k[x_0, \dots, x_n]/(f))$ in $\mathbb{P}^n$, where $f = x_0^d + \sum_{i=0}^{n-1} x_i x_{i+1}^{d-1}$, is smooth over k.

Problem 48. Let k be an infinite field. Let X be a variety over k admitting a dominant rational map $\mathbb{P}_k^n \dashrightarrow X$ over k (such a variety said to be *unirational*). Show that $X(k)$ is dense in X.

Problem 49. Let $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Show that we have isomorphisms

$$\begin{aligned} Rf_*R\mathcal{H}om_X(Lf^*N,M) &\simeq R\mathcal{H}om_Y(N,Rf_*M),\\ R\mathrm{Hom}_X(Lf^*N,M) &\simeq R\mathrm{Hom}_Y(N,Rf_*M), \end{aligned}$$

functorial in $M \in D(X)$ and $N \in D(Y)$.

Problem 50. (a) Let $f_i\colon X_i \to S$, $i = 1,2$ be quasi-compact quasi-separated morphisms of schemes. Let $X := X_1 \times_S X_2$ and $f := f_1 \times_S f_2\colon X \to S$. Assume that f_1 is flat. Prove the Künneth formula

$$Rf_{1*}M_1 \otimes^L Rf_{2*}M_2 \simeq Rf_*(M_1 \boxtimes^L_S M_2)$$

for $M_i \in D_{\mathrm{qcoh}}(X_i)$. Here $M_1 \boxtimes^L_S M_2 := Lp_1^*M_1 \otimes^L_{\mathcal{O}_X} Lp_2^*M_2$ and $p_i\colon X \to X_i$ is the projection. (You may admit the fact that the flat base change theorem extends to D_{qcoh}.)

(b) Let X_1 and X_2 be proper smooth schemes over a field k. Express the Hodge numbers $h^{p,q}$ of $X := X_1 \times_{\mathrm{Spec}(k)} X_2$ in terms of those of X_1 and X_2.

Problem 51. Let A be a ring and let $P = \mathbb{P}^n_A$, where $n \geq 0$ is an integer.

(a) Show that $H^q(P,\Omega^p_{P/A}(m)) = 0$ unless one of the following holds:

(i) $0 \leq p = q \leq n$ and $m = 0$, in which case $H^p(P,\Omega^p_{P/A}) \simeq A$;

(ii) $q = 0$ and $m > p$;

(iii) $q = n$ and $m < p - n$.

(*Hint.* Use the exact sequence

$$0 \to \Omega^p_{P/A} \to \bigwedge^p(\mathcal{O}_P(-1)^{\oplus n+1}) \to \Omega^{p-1}_{P/A} \to 0.$$

The fact $H^q(P,\Omega^p_{P/A}(m)) = 0$ for $q > 0$ and $m > 0$ is called Bott vanishing.)

Assume in the sequel that $A = k$ is a field.

(b) Compute $\dim_k H^q(P,\Omega^p_{P/k}(m))$.

(c) Let $X \subseteq P$ be a hypersurface of degree d smooth over k. Show that the canonical map $H^q(P,\Omega^p_{P/k}(m)) \to H^q(X,\Omega^p_{X/k}(m))$ is an isomorphism for $p + q < n-1$ and $m < d$. Deduce that $H^q(X,\Omega^p_{X/k}(m)) = 0$ for $p+q > n-1$ and $m > 0$. (*Remark.* For k of characteristic zero, the last statement is a special case of the Kodaira vanishing theorem.)

Problem 52. Let k be an algebraically closed field and let X be a smooth projective curve over k of genus g. The *gonality* of X, denoted $\mathrm{gon}(X)$, is defined to be the least integer $d \geq 1$ such that there exists a morphism $X \to \mathbb{P}^1_k$ over k of degree d.

(a) Show that $\mathrm{gon}(X) = \min\{\deg(\mathcal{L}) \mid h^0(\mathcal{L}) \geq 2\}$.

(b) Show that $\mathrm{gon}(X) \leq g + 1$.

Problem 53. Let k be an algebraically closed field.

(a) Let X be a smooth projective curve of genus g over k. Let D be an effective divisor on X of degree $\geq 2g$. Show that D is rationally equivalent to an effective divisor D' on X disjoint from D. (*Hint.* Apply the Riemann-Roch theorem to D and $D - x$ for every x in the support of D.)

(b) Deduce that a curve C over k is either proper or affine. (*Hint.* Use Problem 41 to reduce to the case where C is smooth. Then apply (a) to an effective divisor whose support is precisely $\overline{C}\backslash C$. Here $\overline{C}$ denotes a smooth compactification of C.)

Problem 54. Let X be a nonempty scheme proper over a field k. The *arithmetic genus* of X is defined to be $g_a(X) := (-1)^{\dim(X)}(\chi(\mathcal{O}_X) - 1)$.

(a) Let X be a hypersurface of degree d in $\mathbb{P}^n_k$. Show that $g_a(X) = \binom{d-1}{n}$.

(b) Assume that k is algebraically closed. Let X be a proper curve over k. Show that $g_a(X) = g(X^\nu) + \sum_{x \in X} \dim_k(\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x})$, where X^ν denotes the normalization of X and $\mathcal{O}^\nu_{X,x}$ denotes the normalization of $\mathcal{O}_{X,x}$, and x runs through the singular points of X. Deduce that $g_a(X) = 0$ implies $X \simeq \mathbb{P}^1_k$.

Problem 55. Let k be a field, $R = k[x_0, \ldots, x_n]$, and $X = \mathrm{Proj}(R/I)$, where $I \subseteq R$ is the ideal generated by a regular sequence of $c \leq n$ homogeneous elements of positive degrees.

(a) Show that X has dimension $n - c$. We call X a *complete intersection* in $\mathbb{P}^n_k$. (*Remark.* In fact a complete intersection in $\mathbb{P}^n_k$ can be characterized as a scheme-theoretic intersection of dimension $n - c$ of c hypersurfaces in $\mathbb{P}^n_k$.)

(b) Assume that $n - c \geq 1$. Show that $H^0(\mathbb{P}^1_k, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$ is surjective and $H^i(X, \mathcal{O}(m)) = 0$ for all $m \in \mathbb{Z}$ and $0 < i < n - c$. Deduce that X is geometrically connected.

(c) Let X be a complete intersection of a hypersurface of degree d and a hypersurface of degree e in $\mathbb{P}^3_k$. Show that $g_a(X) = \frac{1}{2}ab(a + b - 4) + 1$.