WEEK 5: 2021-06-22, 2021-06-24

[AM]: Atiyah-Macdonald, Introduction to commutative algebra.

Exercise 1. Let A be a noetherian ring, and $\mathfrak{a}, \mathfrak{b}$ be ideals of A. If A is complete for the $\mathfrak{a}$-adic topology (i.e. $A \cong \varprojlim_n A/\mathfrak{a}^n$) and the $\mathfrak{b}$-adic topology, then A is also complete for the $(\mathfrak{a}+\mathfrak{b})$-adic topology.

Exercise 2. Let k be a field and f be a homogeneous polynomial of degree s in $k[X_1, \ldots, X_n]$ (viewed as a graded ring in the standard way). Compute the Hilbert polynomial of $A = k[X_1, \ldots, X_n]/(f)$.

Exercise 3. [AM] Page 125, Exercise 1.

Exercise 4. Prove the following generalization of Krull's principal ideal theorem. Let A be a noetherian ring

(1) If $\mathfrak{a} \subset A$ is an ideal generated by n elements, then every minimal prime ideal containing $\mathfrak{a}$ has height $\leq n$.
(2) Conversely, if $\mathrm{ht}(\mathfrak{p}) \leq n$ (where $\mathfrak{p}$ is a prime ideal of A), then $\mathfrak{p}$ contains some ideal $\mathfrak{a}$ which can be generated by n elements such that $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{a}$. (Note that $\mathfrak{a}$ need not be primary).

Exercise 5. (optional) [AM] Page 126, Exercise 4.