§ Associated prime ideals.

Def: A prime ideal $\mathfrak{p}$ is said to be associated to M if $\exists$ non-zero $m \in M$ with
$$\mathfrak{p} = \mathrm{Ann}_A(m).$$

$\mathrm{Ass}(M) := \{\text{associated primes of } M\}$.

The primes which are minimal in $\mathrm{Ass}(M)$ are called minimal primes; the others are called embedded primes.

Ex: $\mathrm{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$, More generally, if q is primaire, then
$$\mathrm{Ass}(A/q) = \{\sqrt{q}\}.$$

Pf: let $\bar{a} \in A/q$, $\bar{a} \neq 0$ s.t. $\mathrm{Ann}(\bar{a}) = \mathfrak{p}$, this means $\mathfrak{p} = (q:a)$.

but since $a \notin q$, we have $(q:a) \subseteq \sqrt{q}$., ie $\mathfrak{p} \subseteq \sqrt{q}$.

Since on the other hand, $q \subseteq \mathfrak{p}$, so we have $\mathfrak{p} = \sqrt{q}$. $\square$

<u>Note:</u> we actually have $\forall\, m \in A/\mathfrak{p}$, non-zero, $\mathrm{Ann}_A(m) = \mathfrak{p}$.
prime ideal

Lemma: (i) $\mathfrak{p} \in \mathrm{Ass}(M)$. iff $\exists$ an injection $A/\mathfrak{p} \hookrightarrow M$.

(ii) $0 \to M' \to M \to M'' \to 0$. then
$$\mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

(iii) $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$.

Pf: (i) Assume $\mathfrak{p} = \mathrm{Ann}(m)$, then $\begin{array}{l} A \to M \\ 1 \mapsto m. \end{array}$ gives $A/\mathfrak{p} \hookrightarrow M$

Conversely. Suppose $A/\mathfrak{p} \hookrightarrow M$. let $m \in M$ be the image of 1, then $\mathfrak{p} = \mathrm{Ann}(m)$. $\square$

(ii). If $A/\mathfrak{p} \hookrightarrow M'$. then also $A/\mathfrak{p} \hookrightarrow M$. $\Rightarrow$ first inclusion.

- take $\mathfrak{p} \in \mathrm{Ass}(M)$. so $A/\mathfrak{p} \hookrightarrow M$. let N be its image.

If $N \cap M' = 0$, then $N \hookrightarrow M \twoheadrightarrow M''$ is again injective. $\Rightarrow \mathfrak{p} \in \mathrm{Ass}(M'')$.

or. $N \cap M' \neq 0$. take $m \in N \cap M'$. then $\mathrm{Ann}_A(m) = \mathfrak{p}$ b/c $N \simeq A/\mathfrak{p}$.

$\Rightarrow A/\mathfrak{p} \hookrightarrow M'$. (sending 1 to m). $\square$

(iii) Say $\mathfrak{p} = \mathrm{Ann}_A(m)$. Then $m/1 \in M_\mathfrak{p}$ is non-zero. (there is no $r \in A \setminus \mathfrak{p}$ which annihilates m.).

$\Rightarrow M_\mathfrak{p} \neq 0$. ie $\mathfrak{p} \in \mathrm{Supp}(M)$. $\square$

From now on, Consider the case: A is noeth and M is finite A-mod.

Lemma: If A is noeth. and $M \neq 0$, then $\mathrm{Ass}(M) \neq \phi$.

Pf: Consider the set $\{\mathfrak{a} = \mathrm{Ann}_A(m), \text{ for some } m \neq 0\}$.

this is non-empty. hence. $\exists$ max element as A is noeth.

Let $\mathfrak{p}$ = max element; $= \mathrm{Ann}(m)$. show that $\mathfrak{p}$ is prime.

Let x, y., s.t $xy \in \mathfrak{p}$, but $y \notin \mathfrak{p}$. (want to show $x \in \mathfrak{p}$)

Then $xy \cdot m = 0$ but $y \cdot m \neq 0$.,

Consider $\mathrm{Ann}(ym)$.; clearly $\mathfrak{p} = \mathrm{Ann}(m) \subseteq \mathrm{Ann}(ym)$. so equality by maximality.

Clearly, $x \in \mathrm{Ann}(ym)$. so. also $x \in \mathfrak{p}$. $\Rightarrow$ the claim. $\square$

Rk: The proof implies: $\forall m \in M$, $m \neq 0$, $\exists \mathfrak{p} \in \mathrm{Ass}(M)$. s.t. $\mathrm{Ann}(m) \subseteq \mathfrak{p}$.

hence $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} = \bigcup_{m \in M} \mathrm{Ann}(m)$. ← called zero-divisors of M.

Lemma. Let A = noeth, M = finite A-mod. Then there exists a chain of submodules $0 = M_0 \subset M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$.

with $M_i / M_{i-1} \simeq A/\mathfrak{p}_i$. for some prime ideal $\mathfrak{p}_i$. $(i = 1, \dots, n)$.

For any such chain; we have

$$\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subset \mathrm{Supp}(M).$$

Pf: Take $\mathfrak{p}_1 \in \mathrm{Ass}(M)$. then $\exists A/\mathfrak{p}_1 \hookrightarrow M$. , put $M_1 = A/\mathfrak{p}_1$ (image)

Now consider M/M_1; if $M_1 \neq M$, we can continue this argument.
In this way we get an ascending chain of submodules of M, and since M is noeth. it must be stable. $\Rightarrow$ existence of such chain.
Then it is clear that $\mathrm{Ass}(M) \subseteq \bigcup_i \mathrm{Ass}(M_i/M_{i-1}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.
Since $\mathfrak{p}_i \in \mathrm{Supp}(A/\mathfrak{p}_i)$. $\Rightarrow \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \mathrm{Supp}(M)$.

Cor: A = noeth, M = finite A-mod. then $\mathrm{Ass}(M)$ is finite set.

Pf: direct consequence of the previous Lemma.

Lemma: A = noeth. M = finite A-mod. S = multip subset of A.

Then $\mathrm{Ass}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M), \mathfrak{p} \cap S = \phi\}$.

Pf: $\supseteq$ Let $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\mathfrak{p} \cap S = \phi$.

then $\exists$ injection $A/\mathfrak{p} \hookrightarrow M$. it induces $S^{-1}(A/\mathfrak{p}) \hookrightarrow S^{-1}M$.

$$S^{-1}(A/\mathfrak{p}) \cong S^{-1}A/S^{-1}\mathfrak{p}$$ $\leftarrow$ proper prime ideal

so $S^{-1}\mathfrak{p} \in \mathrm{Ass}(S^{-1}M)$

$\subseteq$: Assume $S^{-1}\mathfrak{p} \in \mathrm{Ass}(S^{-1}M)$. Then $\exists\, m \in M$, $r \in S$. st.

$$S^{-1}\mathfrak{p} = \mathrm{Ann}(m/r)$$

write $\mathfrak{p} = \langle x_1, \dots, x_n \rangle$.

Fix i. then $x_i \cdot m/r = 0$. (in $S^{-1}M$). $\Rightarrow \exists\, s_i \in S$. st. $s_i x_i \cdot m = 0$

Set $s = \prod_i s_i$. $\Rightarrow s \cdot x_i \cdot m = 0$, $\forall i$. i.e. $x_i \in \mathrm{Ann}(s \cdot m)$, $\forall i$

$\Leftrightarrow \mathfrak{p} \subseteq \mathrm{Ann}(sm)$.

Conversely, take $b \in \mathrm{Ann}(sm)$; $\Rightarrow b\, sm/sr = 0$. So. $b/1 \in \mathrm{Ann}(m/r) = S^{-1}\mathfrak{p}$

$\Rightarrow b \in \mathfrak{p}$. ($\mathfrak{p} \cap S = \phi$)

$\Rightarrow \mathrm{Ann}(sm) \subseteq \mathfrak{p}$

Thus $\mathfrak{p} \in \mathrm{Ass}(M)$. $\square$

Thm: Let A = noeth ring, M = finite A-mod. (recall $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$).

Then $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$. and

$$\{\text{minimal elements of } \mathrm{Ass}(M)\} = \{\text{minimal elements of } \mathrm{Supp}(M)\}.$$

Pf: The inclusion is already shown; left to show the last statement.

Observe that: if $X_1 \subseteq X$ are ordered sets.

To show $\{\text{min elements of } X\} = \{\text{min elements of } X\}$.

it suffices to show $\{\text{min elements of } X\} \subseteq X_1$.

Let $\mathfrak{p} \in \mathrm{Supp}(M)$. so $M_\mathfrak{p} \neq 0$. $\Rightarrow \mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) \neq \emptyset$

Lemma $\Rightarrow \mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{S^{-1}\mathfrak{p}' \mid \mathfrak{p}' \subseteq \mathfrak{p},\ \mathfrak{p}' \in \mathrm{Ass}(M)\}. \subseteq \mathrm{Supp}(M_\mathfrak{p})$

Recall $\mathrm{Ann}(S^{-1}M) = S^{-1}(\mathrm{Ann}(M))$, so

$$\mathrm{Supp}(M_\mathfrak{p}) = \{S^{-1}\mathfrak{p}' \mid \mathfrak{p}' \subseteq \mathfrak{p},\ \mathfrak{p}' \in \mathrm{Supp}(M)\}$$

if $\mathfrak{p}$ is minimal in $\mathrm{Supp}(M)$., then $\mathrm{Supp}(M_\mathfrak{p}) = \{S^{-1}\mathfrak{p}\}$ one element

but $\mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) \neq \emptyset$.! So we must have.

$$\mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}\} = \mathrm{Supp}(M_\mathfrak{p}).$$

ie. $\mathfrak{p} \in \mathrm{Ass}(M)$. by lemma. ($\mathfrak{p}$ is automatically minimal). $\square$

Primary decomposition of modules.

Def: $M = A$-module. $N \subseteq M$. submodule.

N is said primary if: $\forall a \in A$. $m \in M$. s.t. $am \in N$,

then. either $m \in N$, or $a^n M \subseteq N$. for some. $n \geq 1$.

[$\Leftrightarrow$ z.div.$(M/N) = \sqrt{\mathrm{Ann}_A(M/N)}$ ← radical.

if $\bar{m} \in M/N$, $a \in A$. st. $a \cdot \bar{m} = 0$, ie $\begin{cases} a \cdot m \in N; \\ m \notin N. \end{cases}$ $\Rightarrow a^n M \subseteq N$
$\ne 0$.

ie $a^n \in \mathrm{Ann}(M/N)$.]

Prop: Assume A is noeth, M = finite A-mod, $N \subseteq M$. submodule.

Then N is primary iff $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}\}$. (has only one element).

In this case, we. have. $\mathfrak{p} = \sqrt{\mathrm{Ann}_A(M/N)}$. and $\mathrm{Ann}_A(M/N)$ is primary ideal.

Pf: If $X_1 \subseteq X$ are two. set of prime ideals. ordered by inclusion.

St. • $\{\text{min elements of } X_1\} = \{\text{min elements of } X\}$.

• $\bigcup_{\mathfrak{p} \in X_1} \mathfrak{p} = \bigcap_{\min \mathfrak{p} \in X} \mathfrak{p}$.

Then. we have. $\bigcup_{\mathfrak{p} \in X_1} \mathfrak{p} = \bigcap_{\substack{\mathfrak{p} \in X_1 \\ \min.}} \mathfrak{p}$ $\Rightarrow$ for any $\mathfrak{p}_1 \neq \mathfrak{p}_2$ in X_1, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$.

this is impossible unless.

$X_1 = \{\mathfrak{p}\}$. has single element.

Apply this to $X_1 = \mathrm{Ass}(M/N)$, $X = \mathrm{Supp}(M/N)$.

N primary $\Leftrightarrow$ z-div$(M/N) = \sqrt{\mathrm{Ann}(M/N)}$.

$\Leftrightarrow \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/N)} \mathfrak{p} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Supp}(M/N) \\ \min.}} \mathfrak{p}$ $\Longleftrightarrow$ $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$

Finally prove. that $\mathrm{Ann}(M/N)$ is primary ideal. with radical. $\mathfrak{p}$:

Let $xy \in \mathrm{Ann}(M/N)$, and assume. $x \notin \mathrm{Ann}(M/N)$. then $\exists\ m \in M$. st. $xm \notin N$.

Since. $y \cdot xm \in N$, and N is primary. we have. $y^n \cdot M \subseteq N$. for some n, ie $y \in \sqrt{\mathrm{Ann}(M/N)}$. □

Def. We say N is $\mathfrak{p}$-primary if $\mathrm{Ass}_A(M/N) = \{\mathfrak{p}\}$.

Prop: If $N_1, N_2 \subseteq M$ are $\mathfrak{p}$-primary submodules, then so is $N_1 \cap N_2$.

Pf: We have $M/N_1 \cap N_2 \hookrightarrow M/N_1 \oplus M/N_2$.

So. $\mathrm{Ass}(M/N_1 \cap N_2) \subseteq \mathrm{Ass}(M/N_1) \cup \mathrm{Ass}(M/N_2) = \{\mathfrak{p}\}$.

but since $M/N_1 \cap N_2 \neq 0$. So. $\mathrm{Ass}(M/N_1 \cap N_2) \neq \emptyset$. $\Rightarrow$ $= \{\mathfrak{p}\}$. $\square$

Def: A submodule N of M is called irreducible if

$$N = N_1 \cap N_2 \text{ (where } N_1, N_2 \subseteq M) \Rightarrow N = N_1 \text{ or } N = N_2.$$

Prop. Irreducible submodules of M are primary.

Pf: We can assume $N = 0$. (and irreducible).

Suppose that (0) is not primary, then M has at least 2 associated primes $\mathfrak{p}_1, \mathfrak{p}_2$.

$\Rightarrow \exists$ two submodules N_1, N_2 of M st. $N_i \simeq A/\mathfrak{p}_i$.

If $m \in N_1 \cap N_2$, $m \neq 0$, then $\mathrm{Ann}_A(m) = \mathfrak{p}_1 = \mathfrak{p}_2$, which is impossible.

thus $N_1 \cap N_2 = (0)$ and (0) is not irreducible.

Thm: A = Noeth, M = finite A-module, $N \subseteq M$.

Then N has a minimal (or irredundant) primary decomposition.

$$N = Q_1 \cap \cdots \cap Q_r, \quad \text{with } Q_i \text{ being } \mathfrak{p}_i\text{-primary}.$$

Satisfying.

(1). $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are distinct.

(2). $\forall i$, $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$.

Moreover, the $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are uniquely determined, $= \mathrm{Ass}(M/N)$.

also, if $\mathfrak{p}_i$ is minimal in $\mathrm{Ass}(M)$, then Q_i is uniquely determined.

Noetherian modules and artin modules.

Def: we say a module M is noetherian if one of the following holds:

(a). a.c.c. holds for submodules of M

(b). max. condition holds for non-empty family of submod of M

(c) every submod of M is f.g.

Rem: by (c). A is noeth ring iff A is a noeth. A-mod.

Def M is called artinian if one of the following holds

(a). d.c.c. holds for submod of M

(b). minimal condition holds, i.e every non-empty family of submod of M has a minimal element.

(no analogous (c)!)

A is called artinian ring if it is an Artinian A-module.

Ex: (i) $\mathbb{Z}$ satisfies a.c.c. but not d.c.c.

$$(p) \supseteq (p^2) \supseteq \cdots \supseteq (p^n) \supseteq \cdots$$

(ii) $M := \{x \in \mathbb{Q}/\mathbb{Z} : \operatorname{ord}(x) \text{ is power of } p.\}$. e.g. $p^{-n}+\mathbb{Z}$, $n>0$

$\quad \cup\!\!\!\mid$
$p^{-n}\mathbb{Z}/\mathbb{Z}$

$\Rightarrow \quad 0 \subseteq p^{-1}\mathbb{Z}/\mathbb{Z} \subset p^{-2}\mathbb{Z}/\mathbb{Z} \subseteq \cdots,$ ie a.c.c. does not hold.

but d.c.c holds. (as $p^{-n}\mathbb{Z}/\mathbb{Z}$ are the only submod of M).

Note: $\mathbb{Q}/\mathbb{Z}$ does not satisfies d.c.c.!! (neither a.c.c.).

(iii) $k[x_1, x_2, \cdots]$ poly ring with inf. many indeterminates does not satisfy a.c.c. or d.c.c. $\square$

Prop: Let $0 \to M' \to M \to M'' \to 0$. exact sequence

Then M is noeth (resp. artin.) $\Leftrightarrow$ M', M'' are noeth (resp. artin.),

Cor: If M_i $(1 \le i \le n)$ are noeth (resp. artin), then so is $\bigoplus_{i=1}^{n} M_i$.

Prop: If A is noeth (resp. artin), M is f.g. A-mod. then M is noeth module. (resp. artin).

Pf: as M is a quot of A^n for some $n \geq 0$.

Prop: M has a composition series $\Leftrightarrow$ M is both noetherian and artinian.

pf: $\Rightarrow$, all chains of submodules of M have a bounded length $\Rightarrow$, satisfies a.c.c., d.c.c.

$\Leftarrow$ Construct a composition series of M as follows.

Since $M = M_0$ satisfies maximum condition, it has a maximal submodule $M_1 \subsetneq M_0$

$\rightsquigarrow$ Similarly get a chain $M_0 \supsetneq M_1 \supsetneq \cdots$; with M_i/M_{i+1} being simple.

Since M is artinian, this is finite chain, which is a composition series of M. □

Rem: We can't apply Zorn's lemma to show any non-zero module M has a maximal submodule!

However, if M is finite A-module, then it does has maximal submodules.

Pf: let $\Sigma = \{$proper submodules of $M\} \neq \phi$. (as $(0) \in \Sigma$).

if $M_1 \subseteq M_2 \subseteq \cdots$ is an ascending chain in Σ.

Check that $\bigcup_i M_i \subsetneq M$ is again proper submodule.

Say M is generated by $m_1, \ldots, m_r$. if $\bigcup_i M_i = M$, then $\exists i$ large enough.

s.t. $m_1, \ldots, m_r \in M_i$. so $M = M_i$, Contradiction. □

Prop: let k = field, V = k-vector space (or k-mod.), TFAE:

(i) V has finite dim. ($\Leftrightarrow$ V has finite length).

(ii) V is noeth.

(iii) V is artinian.

Pf: Easy. (ii) $\Rightarrow$ (i) if $\dim V$ is infinite.

then $\exists x_1, \ldots, x_n, \ldots$ linearly independent, so we can construct

$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_n \subsetneq \cdots$ ascending chain of infinite length.

with $U_n = \bigoplus_{i=1}^{n} k x_i$; Contradiction.

(iii) $\Rightarrow$ (i). Set $V_0 = V$, $V_n = \bigoplus_{i \geq n} k x_i$, get descending chain of infinite length. □

Cor: let A be a ring. assume that $(0) = $ product $m_1 \cdots m_n$ of max ideals. (not necessarily distinct).

Then A is noeth. iff A is artinian.

Pf: Consider the chain $A \supseteq m_1 \supseteq m_1 m_2 \supseteq \cdots \supseteq m_1 \cdots m_n = (0)$.

s.t. each graded piece $m_1 \cdots m_i / m_1 \cdots m_{i+1}$ is a vector space over A/m_{i+1} which is a field.

We have seen that for vector spaces over field, noeth $\Leftrightarrow$ artinian.

hence. A is noeth. $\Leftrightarrow$ A is artinian. $\square$

Artinian rings

Prop: In an Artin ring A, every prime ideal is maximal.

Pf: let p be prime ideal. Then $B := A/p$ is artin and a domain.

Need to show that B is a field

Let $x \in B$, $x \neq 0$. Consider descending chain $(x) \supseteq (x^2) \supseteq \cdots \supseteq (x^n) \supseteq \cdots$

By d.c.c. we have $(x^n) = (x^{n+1})$ for some $n > 0$, ie. $x^n = x^{n+1} \cdot y$ for some $y \in B$

But B is a domain $\Rightarrow$ $1 = xy$, ie. x is invertible. $\Rightarrow$ B is a field. $\square$

Rem: will see later this means that k.dim (Artin ring) $= 0$.

Prop. In Artin ring A, it has only finite number of max ideals.

ie. Artin ring is semi-local.

Pf: Consider the set of all finite intersection $m_1 \cap \cdots \cap m_r$, where m_i are max ideals.

This set has a minimal element, say $m_1 \cap \cdots \cap m_n$.

Thus for any m max, $m \cap (m_1 \cap \cdots \cap m_n) = m_1 \cap \cdots \cap m_n$

ie $m_1 \cap \cdots \cap m_n \subseteq m$.

$\Rightarrow$ $m_i \subseteq m$ for some i, $\Rightarrow$ $m_i = m$ (as both are max ideals)

$\Rightarrow$ $\{m_1, \ldots, m_n\} = $ all max ideals of A. $\square$

Prop: In an Artin ring A, nilradical = Jacobson radical, and it is nilpotent.

Pf: The first one is clear, as prime ideal = max ideal in A.

Let $J(A)$ = Jacobson radical.

$$J \supseteq J^2 \supseteq \cdots \overset{\text{d.c.c}}{\Rightarrow} J^k = J^{k+1} = \cdots = \mathfrak{a}, \text{ for some } k > 0.$$

Suppose $\mathfrak{a} \neq 0$, and let $\Sigma = \{ \mathfrak{b} \text{ ideal} : \mathfrak{a}\mathfrak{b} \neq 0 \}$.

Then Σ is not empty b/c $\mathfrak{a} \in \Sigma$.

Let $\mathfrak{c}$ be a minimal element of Σ. Then $\exists\, x \in \mathfrak{c}$. Such that $x \cdot \mathfrak{a} \neq 0$. $\Rightarrow (x) \in \Sigma$.

Clearly $(x) \subseteq \mathfrak{c}$. So by minimality of $\mathfrak{c} \Rightarrow \mathfrak{c} = (x)$ is principal ideal.

But $(x\mathfrak{a}) \cdot \mathfrak{a} = x \cdot (\mathfrak{a}^2) = x\mathfrak{a} \neq 0$, so $x\mathfrak{a} \in \Sigma$ too.

and as $x\mathfrak{a} \subseteq (x)$, we must have $x\mathfrak{a} = (x)$ by minimality. (again)

Hence $x = x \cdot y$ for some $y \in \mathfrak{a}$., and therefore,

$$x = xy = xy^2 = \cdots = xy^n = \cdots$$

But $y \in \mathfrak{a} = J^k \subseteq J = \mathrm{nil}(A)$. y is nilpotent, so $x = 0$.!

This contradicts to the choice of x. $\Rightarrow \mathfrak{a} = 0$ $\square$

Rem: If we know A is Noetherian, then $J^k = \underbrace{J^{k+1}}_{= J \cdot J^k} \Rightarrow J^k = 0$

(J^k f.g.)

$\Rightarrow$ result is clear $\square$

However, we will use this prop to prove that A is Noetherian.

Thm: A is Artinian iff $\begin{cases} A \text{ is noetherian} \\ + \\ \dim A = 0. \end{cases}$

Def: The (Krull) dimension of A is

$$\dim(A) := \sup \{ n : \exists \text{ chain } P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ prime ideals} \}.$$

it is an integer ≥ 0. or $+\infty$.

Ex: A field has dim 0; $\mathbb{Z}$ has dim 1 : $(0) \subsetneq (p)$.

Pf: $\Rightarrow$. prime ideals in A are maximal $\Rightarrow \dim A = 0$.

Let $m_1, \dots, m_n$ be the max ideals of A; they are coprime to each other.

so $Jac(A) = \bigcap_i m_i = \prod_{i=1}^{n} m_i$, but it is nilpotent. so, $\exists k$. s.t

$$(\prod_{i=1}^{n} m_i)^k = 0. \quad \Rightarrow A \text{ is noetherian.}$$

$\Leftarrow$ Since (0) has a primary decomposition. A has only a finite number of minimal prime ideals. but $(**)$ $\Rightarrow$ these are max ideals.

Hence $nil(A) = \bigcap_i m_i$. say.

But in a Noetherian ring, $Nil(A)$ is nilpotent; so.

$$(\prod_{i=1}^{n} m_i)^k \subseteq (\bigcap_{i=1}^{n} m_i)^k = 0, \text{ for } k \gg 0.$$

hence A is also Artinian. $\square$

Thm: (Structure Theorem for Artin rings).

A is Artin ring is uniquely isom to. a finite direct product of Artin local rings.

Pf: Let $m_1, \dots, m_n$ be all max ideals of A. and.

take k large enough. st. $\prod_i m_i^k = 0$

Since m_i, m_j are coprime $\forall (i, j)$

m_i^k, m_j^k are also coprime (ie. if $x+y=1$, then $1 = (x+y)^{2k} \in m_i^k + m_j^k$)

$\Rightarrow$ $A \to \prod_{i=1}^{n} A/m_i^k$ is an isom. by Chinese Rem. Theorem.

with each A/m_i^k being local ring, because only max ideal is m_i/m_i^k.

Now if $A \simeq \prod_i A_i$, for A_i Artin, the factors A_i are uniquely determined by A (up to isom). See. [AM]. $\square$

§ Valuation rings

Let A be a domain, $K = \mathrm{Frac}(A)$.

Def: A is a valuation ring of K if $\forall x \neq 0$, either $x \in A$, or $x \in A^{-1}$.

Prop: Let A be a valuation ring of K.

(i) A is a local ring.

(ii) If A' is a ring s.t. $A \subseteq A' \subseteq K$, then A' is also valuation ring of K.

(iii) A is integrally closed (in K).

Pf: (i) let $m = \{x \in A : x \text{ not invertible in } A\}$.

So $x \in m$ iff $x = 0$, or x^{-1} (in K) does not lie in A.

Need to show m is an ideal of A.

* if $a \in A$, $x \in m$, then $ax \in m$? if not, ax is invertible in A.

ie $(ax)^{-1} \in A$.

$\Rightarrow x^{-1} = a \cdot (ax)^{-1} \in A$. contradiction.

* if $x, y \in m$, non-zero, need $x + y \in m$.

Since A is valuation ring, either $xy^{-1} \in A$, or $x^{-1}y \in A$.

$\Downarrow$ $\qquad\qquad\qquad\qquad \Downarrow$

$x+y = (1+x^{-1}y) \cdot x \in Am \subseteq m$.

$x+y = (1+xy^{-1})y \in A \cdot m \subseteq m$.

$\Rightarrow$ m is indeed an ideal of A.

(ii). this follows from the definition.

(iii). Let $x \in K$ be integral over A. Then $\exists$

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0. \qquad a_i \in A.$$

Since A is valuation ring. $\Rightarrow$ either $x \in A$. $\Rightarrow$ OK

or $x^{-1} \in A$, $\Rightarrow x = -(a_1 + \cdots + a_n x^{1-n}) \in A$.

$\square$

Question: given a field K, does it always contain valuation rings?

Def: (A, m), (A', m') are local rings. $A \subseteq A'$.

We say A' dominates A, if $m' \cap A = m$.

Let $\Sigma = \{$ local subrings of K $(A, m)\}$ ordered by domination.

Then Σ is non-empty, and if any chain has an upper bound (taking their union)

$\Rightarrow$ Σ has max elements.

Thm: any max element in Σ is a valuation ring.

In particular, K always contains valuation subrings.

Lemma. Let A be a domain, $\mathfrak{a}$ an ideal, $K = \mathrm{Frac}(A)$. $x \in K^\times$.

Then either $1 \notin \mathfrak{a} \cdot A[x]$ or $1 \notin \mathfrak{a} A[1/x]$.

Pf: Assume $1 \in \mathfrak{a} A[x]$ and $1 \in \mathfrak{a} \cdot A[1/x]$.

Then there are equations.

$(*)$ $1 = a_0 + \cdots + a_n x^n$, $(**)$ $1 = b_0 + \cdots + b_m \cdot \frac{1}{x^m}$, $a_i, b_j \in \mathfrak{a}$.

Assume n, m minimal. and $\underline{n \geq m}$

$(*) \cdot (1-b_0)$ gives. $(1-b_0) = (1-b_0) a_0 + \cdots + (1-b_0) a_n x^n$.

$(**) \cdot a_n x^n$ gives. $(1-b_0) \cdot a_n x^n = b_1 a_n x^{n-1} + \cdots + b_m a_n x^{n-m}$, $(n \geq m)$.

$\Rightarrow$ $(1-b_0) = (1-b_0) a_0 + \cdots + (1-b_0) a_{n-1} x^{n-1} + b_1 a_n x^{n-1} + \cdots + b_m a_n x^{n-m}$

this gives an equation. $1 = c_0 + \cdots + c_{n-1} x^{n-1}$. of degree $n-1$.

which contradicts the minimality of n.

If $m > n$, then write. $(1-a_0) \frac{1}{x^n} = a_1 x^{1-n} + \cdots + a_n$

$\begin{cases} (1-a_0) = (1-a_0) b_0 + \cdots + (1-a_0) b_m \frac{1}{x^m} \end{cases}$

$\Rightarrow$ get an equation contradicting to the minimality of m $\square$

Pf of Thm: let (A, m) be a max element in Σ.

need to show if $x \in K$, $x \neq 0$, then either $x \in A$ or $x^{-1} \in A$.

We may assume. $m \cdot A[x] \neq A[x]$. by Lemma (otherwise. have. $m A[\frac{1}{x}] \neq A[\frac{1}{x}]$.).

then $\exists$ max ideal m' of $A[x]$. s.t. $m A[x] \subseteq m'$.

Consider $A[x]_{m'} \subseteq K$. local subring., it dominates A $\Rightarrow$ by maximality of A, $A[x]_{m'} = A$

$\Rightarrow x \in A$. $\square$

Cor: let A be a subring of a field K.

Then the integral closure $\bar{A}$ of A (in K) is the intersection of all the valuation rings of K which contain A.

Pf: let $A \subseteq B$. B = valuation ring. then B is integrally closed.

if $x \in \bar{A}$, then x is integral over A. hence also integral over B. hence $x \in B$.

$\Rightarrow \bar{A} \subseteq B$.

Conversely. let $x \notin \bar{A}$. Then $x \notin A[x^{-1}] =: A'$ (otherwise we would get an equation of integral dependence of x)

Hence x^{-1} is not invertible in A', so it is contained in some max ideal m' of A'.

Consider $(A'_{m'}, m'A'_{m'})$ which is a local subring of K.;

it is dominated by some valuation ring (B, n).

Since $x^{-1} \in m'$, we have $x^{-1} \in n$. (not invertible in B) $\Rightarrow x \notin B$. $\square$

§. Discrete valuation rings

K = field. A discrete valuation on K is a surjective map.

$$v: K^\times \to \mathbb{Z}$$

s.t. (1) $v(xy) = v(x) + v(y)$

(2). $v(x+y) \geq \min(v(x), v(y))$., Set $v(0) = +\infty$

The set $\{v(x) \geq 0 : x \in K\}$ is a ring. called the valuation ring of v.

Ex: (i) $K = \mathbb{Q}$, fix prime number p.

if $x \in \mathbb{Q}$, $x = p^a m/n$, with $(m,p) = (n,p) = 1$, then $v(x) := a$.

$\rightsquigarrow$ p-adic valuation v_p, and the valuation ring is $\mathbb{Z}_{(p)}$. (local ring!)

(ii) $K = k(x)$., take $f \in k[x]$ irred.,

$\rightsquigarrow$ similarly a valuation v_f.

Def: An integral domain A is a discrete valuation ring if there is a discrete valuation v on $K = \mathrm{Frac}(A)$ s.t. A = valuation ring of v; ie $= \{x \in K \mid v(x) \geq 0\}$

In particular, A must be a local ring, with

$$m = \{x : v(x) > 0\}.$$

and any element in A with $v(-) = 0$, is a unit in A.

Fact: if $v(x) = v(y)$, for $x, y \in A$, then $(x) = (y)$.

Given $\mathfrak{a} \subseteq A$ ideal. can define $v(\mathfrak{a}) := \min\{v(x) : x \in \mathfrak{a}\}$. $\Rightarrow$ each $\mathfrak{a}$ is principal.

Fact $\Rightarrow$ given $k \geq 0$. $\exists$ unique ideal m_k s.t. $v(m_k) = k$.

$m_1 = m \supseteq m_2 \supseteq m_3 \supseteq \cdots$ (a single chain).

Since each m_k is principal; in particular A is Noetherian and PID.

Moreover, since $v: K^\times \to \mathbb{Z}$ is surjective. $\exists x \in m$. s.t. $v(x) = 1$.

$\Rightarrow$ $m = (x)$, and $m_k = (x^k)$.

So m is the unique nonzero prime ideal of A

$\Rightarrow$ A is noeth. local domain of dim 1.

Def: An element $x \in A$ with $\nu(x)=1$ is called a uniformizer, or prime element.

Any element $a \in K$. ($a \neq 0$). can be written as $a = ux^n$ for $u \in A$ unit, $n \in \mathbb{Z}$.

Prop: Let A be a Noeth. local domain of dim 1., m = max ideal.

TFAE: (i) A is d.v.r.

(ii) A is integrally closed;

(iii) m is principal ideal;

(iv) Every non-zero ideal in A is a power of m.

Pf: (i) $\Rightarrow$ (ii) Ok, as valuation rings are integrally closed.

(ii) $\Rightarrow$ (iii) Let $a \in m$, $a \neq 0$.

Since m is the unique prime ideal containing (a). we have $\sqrt{(a)} = m$.

and (a) is a m-primary ideal.

Then $\exists\, n > 0$. Such that $m^n \subseteq (a)$. and $m^{n-1} \nsubseteq (a)$.

Choose $b \in m^{n-1}$ and $b \notin (a)$. and let $x = a/b \in K$., where $K = \mathrm{Frac}(A)$.

We have

- $x^{-1} m \subseteq A$. because. $\frac{b}{a} \cdot m \subseteq \frac{m^n}{a} \subseteq A$.
- $x^{-1} \notin A$. (since $b \notin (a)$) hence x^{-1} is not integral over A by (ii)

$\Rightarrow x^{-1} m \nsubseteq m$.

Thus. $x^{-1} m = A$, and $m = (x)$. is principal.

(iii) $\Rightarrow$ (iv). Let $\mathfrak{a} \subseteq A$ non-zero, and proper, ie. $\mathfrak{a} \subseteq m$.

$\exists\, r$ s.t. $\mathfrak{a} \subseteq m^r$. but $\mathfrak{a} \nsubseteq m^{r+1}$.

So $\exists\, y \in \mathfrak{a}$, $\begin{cases} y = ax^r \text{ for some } a \in A. \\ y \notin (x^{r+1}). \end{cases}$ (here $m = (x)$).

$\Rightarrow a \notin (x)$, but A is local ring with $m = (x)$. so. a is unit in A

$\Rightarrow x^r \in \mathfrak{a} \Rightarrow m^r = (x^r) \subseteq \mathfrak{a}$. so equality $\mathfrak{a} = m^r$. $\square$.

(iv). $\Rightarrow$ (i). We have $m^k \neq m^{k+1}$, $\forall k \geq 1$: indeed. if have equality $m^k = m^{k+1}$

then Nakayama $\Rightarrow m^k = 0 \quad \Rightarrow m = 0$ (A is domain).

In particular. $m \neq m^2$. hence $\exists\, x \in m$, $x \notin m^2$.

But $(x) = m^r$ for some r, we must have $r=1$. ie $m = (x)$. is principal.

Let $a \in A$, non-zero, then $(a) = m^k = (x^k)$. for unique value of k

Define $v(a) = k$, and extends to $K^\times$ by defining.

$$v(ab^{-1}) = v(a) - v(b).$$

we obtain a well-defined discrete valuation. s.t. A = valuation ring. □

§ Dedekind domain ← Rk: in some books, Dedekind domain include "fields".

Def: A Dedekind domain is a Noetherian domain., integrally closed. of dim 1.

Thm: Let A be a Noeth domain of dim 1. TFAE:

(i) A is integrally closed (i.e. Dedekind)

(ii) Every local ring $A_\mathfrak{p}$ $(\mathfrak{p} \neq 0)$ is a d.v.r.

(iii). Every non-zero primary ideal is a power of prime ideal. (in fact $= \mathfrak{p}^k$ with $\mathfrak{p} = \sqrt{\mathfrak{q}}$).

Pf: (i) $\Leftrightarrow$ (ii) A integ closed iff $A_\mathfrak{p}$. is integ. closed for $\forall \mathfrak{p}$ max ideal., i.e non-zero prime ideal as A has dim 1.

iff $A_\mathfrak{p}$ is d.v.r.

(ii) $\Rightarrow$ (iii). $\mathfrak{q} \subseteq A$ be primary., with. $\mathfrak{p} = \sqrt{\mathfrak{q}}$. then in $A_\mathfrak{p}$. $\mathfrak{q}A_\mathfrak{p} = \mathfrak{p}^k A_\mathfrak{p}$ some k.

$\overset{\text{(AM). Prop 4.8}}{\Longrightarrow}$. $\mathfrak{q} = \mathfrak{p}^k$ in A. ($\mathfrak{p}^k$ also primary)

(iii) $\Rightarrow$ (ii). Let $\mathfrak{a} \overset{\neq 0}{\subseteq} A_\mathfrak{p}$ be an ideal.

Since $A_\mathfrak{p}$ is Noeth., local. of dim 1, $\mathfrak{p}A_\mathfrak{p}$ is the unique prime containing $\mathfrak{a}$

$\Rightarrow$ $\mathfrak{a}$ is primary ideal in $A_\mathfrak{p}$.

Let $\mathfrak{q} = \mathfrak{a} \cap A = \mathfrak{a}^c$, then. $\mathfrak{q} \subseteq \mathfrak{p}$ and is primary ideal, and $\mathfrak{a} = \mathfrak{q}^e$.

by (iii). $\mathfrak{q} = \mathfrak{p}'^k$ for some $\mathfrak{p}'$ and some k; we must have $\mathfrak{p}' = \mathfrak{p}$, (otherwise. $\mathfrak{q}^e = A_\mathfrak{p}$)

So $\mathfrak{a} = \mathfrak{q}^e = (\mathfrak{p}A_\mathfrak{p})^k$. $\Rightarrow$ $A_\mathfrak{p}$ is d.v.r.

Thm: In a Dedekind domain, every non-zero. ideal has a unique factorization as a finite product of prime ideals.

Lemma. Let A be a noeth domain of dim 1.

Then every non-zero ideal $\mathfrak{a}$ in A can be. uniquely expressed as a product of primary ideals whose radical are all distinct.

Pf: A is noeth. so $\mathfrak{a}$ has a minimal primary decomp. $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$, $p_i = r(q_i)$

Since $\dim A = 1$, and A is domain. $\Rightarrow$ each p_i is max.

and p_i, p_j are pairwise coprime.

$\Rightarrow (p_i^{n_i})$ also pairwise coprime, $n_i > 0$

for n_i large, $p_i^{n_i} \subseteq q_i$. so (q_i) also pairwise coprime

So. $\prod_i q_i = \bigcap_i q_i = \mathfrak{a}$.

Uniqueness: if $\mathfrak{a} = \prod_i q_i$ with $r(q_i)$ distinct., then as above we get (q_i) coprime

and so $\mathfrak{a} = \bigcap_i q_i$ is a minimal primary decomposition

($q_i \not\supseteq \bigcap_{j \neq i} q_j$. otherwise $\underset{q_i}{=} r(q_i) \supseteq \bigcap_{j\neq i} r(q_j) = \bigcap_{j \neq i} p_j$. $\Rightarrow p_i \supseteq p_j$ for some $j \neq i$)

$\Rightarrow q_i$ are uniquely determined by $\mathfrak{a}$. $\square$

Because in Dedekind domain, primary ideals are powers of prime ideals.

So Thm follows from Lemma. $\square$

Hence in a Dedekind domain A, for any $0 \neq a \in A$, (a) decomposes uniquely as. finite product.

$$(a) = \prod_i p_i^{r_i}, \quad p_i \text{ prime ideal } (\neq 0).$$

$\rightsquigarrow$ can define. p_i-adic valuation. $v_{p_i}(a) := r_i$.

Ex: PID are Dedekind..

The most important examples of Dedekind domains are:

Thm: The ring of integers in an algebraic number field K is Dedekind.

Pf: $K/\mathbb{Q}$ is separable, so $A := \{\text{integers of } K\}$ is f.g. as $\mathbb{Z}$-mod.

Also A is integrally closed.

To see. A is of dim 1.: let $p \neq 0$ be prime.

$p \cap \mathbb{Z}$ is also prime, non-zero., hence $p \cap \mathbb{Z}$ is max. $= (p)$.

$\Rightarrow p$ is also maximal.

Fractional ideals

A = integral domain, $K = \mathrm{Frac}(A)$.

$I \subseteq K$ submodule (over A) is a fractional ideal of A if $x \cdot I \subseteq A$ for some $x \neq 0$ in A.

Ex: "ordinary" ideals are fractional. (take $x=1$)

- $u \in K$, generates a frac ideal. (u); called principal fractional ideal.

- Every f.g. A-submod of K is a fractional ideal.

  say $I = \sum_{i=1}^{n} A \cdot x_i$, $x_i \in K$., write $x_i = \frac{y_i}{z}$, $z \in A$. $y_i \in A$. (the same z)

  thus $z \cdot I \subseteq A$.

  Conversely, if A is noeth. then every fractional ideal is f.g.

Def: An A-submod I of K is an invertible ideal if there exists a submodule J of K.

such that $IJ = A$.

Note: J is then uniquely determined: $J = (A:I). = \{x \in K : xI \subseteq A\}$.

Pf: $J \subseteq (A:I)$ ok. as $IJ = A$.

Conversely. $(A:I) = (A:I) \cdot IJ \subseteq A \cdot J = J$. $\square$

$\Rightarrow$ we call J the inverse of I.

Lemma: invertible ideals are f.g. hence are fractional ideals.

Pf: $\exists\, x_i \in I$, $y_i \in J$, s.t. $\sum x_i y_i = 1$.

So $\forall\, x \in I$, $x = \sum_i x_i (x y_i) \in (x_1, \dots, x_n)$.

$\Rightarrow I = (x_1, \dots, x_n)$ f.g. $\square$

Conversely, non-zero principal fractional ideals are invertible: $(u)^{-1} = (u^{-1})$.

Prop. Let I be a fractional ideal. TFAE:

(i) I is invertible.

(ii) I is f.g. and $\forall\, \mathfrak{p}$ prime ideal. $I_{\mathfrak{p}}$ is invertible in $A_{\mathfrak{p}}$.

(iii). I is f.g. and $\forall\, \mathfrak{m}$ max ideal, $I_{\mathfrak{m}}$ is invertible in $A_{\mathfrak{m}}$.

ie. being invertible is a local property.

Pf: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) ✓

(iii) $\Rightarrow$ (i). let $\mathfrak{a} = I \cdot (A : I)$, which is an integral ideal ie. $\subseteq A$.

$\forall$ m max ideal. $\mathfrak{a}_m = I_m \cdot (A_m : I_m) = A_m$ ← b/c I_m is invertible.

(recall: $S^{-1}(M:N) = (S^{-1}M : S^{-1}N)$ if N is f.g).

$\Rightarrow \mathfrak{a} = A$. (look at $\mathfrak{a} \hookrightarrow A$ which is an isom when localized at m, $\forall m$).

□

Prop: Let A be a local domain. Then A is d.v.r. iff every non-zero fractional ideal of A is invertible.

Pf: $\Rightarrow$ easy

$\Leftarrow$ Every non-zero integral ideal is invertible, hence is f.g. $\Rightarrow$ A is noeth.

Prove that every non-zero integral ideal is a power of m (the max ideal)

(implies A is d.v.r, since if $\mathfrak{p} \neq 0$ is prime, $\mathfrak{p} = m^k$, $\Rightarrow k=1$, $\mathfrak{p} = m$ ie A has dim 1)

Consider $\Sigma = \{0 \neq \mathfrak{a} \subsetneq A \mid \mathfrak{a}$ is not a power of $m\}$.

If $\Sigma \neq \phi$, then $\exists$ max element (as A is noeth.), say $\mathfrak{a}$.

Then $\mathfrak{a} \neq m$, so $\mathfrak{a} \subsetneq m$.

let m^{-1} be the inverse of m. (in K). then $m^{-1}\mathfrak{a} \subset m^{-1}m = A$.

(is a proper ideal of A).

Since $1 \in m^{-1}$, $\mathfrak{a} \subseteq m^{-1}\mathfrak{a}$.

If $m^{-1}\mathfrak{a} = \mathfrak{a}$, then $\mathfrak{a} = m\mathfrak{a}$, and NAK $\Rightarrow \mathfrak{a} = 0$, impossible.

So $\mathfrak{a} \subsetneq m^{-1}\mathfrak{a}$. and maximality of $\mathfrak{a}$ $\Rightarrow m^{-1}\mathfrak{a} = m^k$, $\Rightarrow \mathfrak{a} = m^{k+1}$.

again contradiction. □

<u>Thm</u>: Let A be an integral domain. Then A is Dedekind domain iff every non-zero fractional ideal of A is invertible.

Pf: $\Rightarrow$ Let I be a fractional ideal. Since A is noeth, I is finitely generated.

For $\mathfrak{p} \neq 0$ prime ideal. $I_\mathfrak{p}$ is a fractional ideal in $A_\mathfrak{p}$ (d.v.r.)

hence $I_\mathfrak{p}$ is invertible and I is also invertible

$\Leftarrow$ Let $\mathfrak{a} \subseteq A$ be an ideal. Since it is invertible, it is f.g. hence A is noeth.

We prove that $A_\mathfrak{p}$ is d.v.r. for each $\mathfrak{p} \neq 0$.

Equivalently, we prove that each non-zero fractional ideal in $A_\mathfrak{p}$ is invertible.

it suffices to prove this for non-zero integral ideals in $A_\mathfrak{p}$.

Let $\mathfrak{b} \subseteq A_\mathfrak{p}$, $\mathfrak{b} \neq 0$, and let $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b} \cap A$. $\Rightarrow \mathfrak{b} = \mathfrak{a}^e$

(always have $\mathfrak{b} = \mathfrak{b}^{ce}$).

Since $\mathfrak{a}$ is invertible, so $\mathfrak{b}$ is also invertible. $\square$

Cor: If A is a Dedekind domain, the non-zero fractional ideals of A form a group wrt. multiplication.

It is called "the group of ideals" of A.

$\rightsquigarrow$ related to Algebraic Number theory. / class group.

Graded rings and modules.

A graded ring is a ring A, with a family of subgps $\{A_n\}_{n\geq 0}$ s.t.

$$A = \bigoplus_{n\geq 0} A_n, \qquad A_m A_n \subseteq A_{m+n}.$$

Thus A_0 is a subring of A, and A_n is A_0-mod. Let $A_+ = \bigoplus_{n>0} A_n$, an ideal of A.

Ex: $A = k[x_1, \dots, x_n]$, A_n = set of homogeneous poly of degree n.

Let A be a graded ring, a graded A-mod is

$$M = \bigoplus_{n\geq 0} M_n, \quad \text{s.t.} \quad A_m \cdot M_n \subseteq M_{m+n}.$$

So each M_n is an A_0-mod.

$x \in M$ is called homog. if $x \in M_n$ for some n. ($n = \deg$ of x).

Any element $y \in M$ can be written uniquely as finite sum $\sum_n y_n$. $y_n \in M_n$.

A homomorphism of graded A-mod is $f: M \to N$, s.t. $f(M_n) \subseteq N_n$. $\forall n \geq 0$.

Prop: TFAE for a graded ring A:

(i) A is Noeth.

(ii) A_0 is Noeth and A is f.g. as an A_0-alg.

Pf: (i) $\Rightarrow$ (ii), $A_0 \simeq A/A_+$ is quot ring of A, so A_0 is also noeth.

Now A_+ is an ideal of A. so it is f.g. say by $x_1, \dots, x_s$.

We may assume they are homogeneous, up to replacing them by their homog components.

say x_i has degree k_i, $(k_i > 0)$.

Let A' = subring of A generated by $x_1, \dots, x_s$ over A_0.

We prove that $A_n \subseteq A'$, $\forall n \geq 0$. by induction.

For $n = 0$, clear.

Let $n > 0$, and $y \in A_n$. we may write $y = \sum_{i=1}^{s} a_i x_i$, $a_i \in A_{n-k_i}$ (set $A_m = 0$, $m < 0$)

Since $k_i > 0$, by inductive hyp. $a_i \in A' \Rightarrow y \in A'$. $\Rightarrow A = A'$ $\square$

(ii) $\Rightarrow$ (i) by Hilbert basis Thm.

Let A be a ring. a filtration of A is a family $\{A_n\}$. of add. subgps

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

with. $A_m \cdot A_n \subseteq A_{m+n}$. we call A a filtered ring.

A filtered module. M: over A is:

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

s.t. $A_n M_m \subseteq M_{n+m}$.

Ex: $\mathfrak{a} \subseteq A$ ideal, the $\mathfrak{a}$-adic filtration of A (resp. M) is:

$$A_n = \mathfrak{a}^n, \quad (M_n = \mathfrak{a}^n M.).$$

A filtration on M is called $\mathfrak{a}$-filtration if $\mathfrak{a} M_n \subseteq M_{n+1}$.

If $\mathfrak{a} M_n = M_{n+1}$ for $n \gg 0$, then it is called stable $\mathfrak{a}$-filtration.

Filtered rings/modules to graded rings/modules

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if A is filtered, define.

$$gr(A) := \bigoplus_{n \geq 0} \underbrace{gr_n(A)}_{} = A_n / A_{n+1}$$

multiplication in $gr(A)$, $a \in gr_n(A)$. $b \in gr_m(A)$.

$$(a + A_{n+1}) \cdot (b + A_{m+1}) = (ab + A_{m+n+1}).$$

Similarly. if M is a filtered mod. define

$$gr(M) = \bigoplus_{n \geq 0} gr_n(M) := M_n / M_{n+1}$$

it is a graded module over $gr(A)$.

Let A = ring, $\mathfrak{a}$ = ideal. M, = A-mod. with. $\mathfrak{a}$-filtration.

$\Rightarrow$ $M^* := gr(M)$. is. module over $A^* = \bigoplus_{n \geq 0} \mathfrak{a}^n$ (so $\mathfrak{a}^0 = A$).

Prop: Assume A is noeth and M is f.g. over A.

Suppose $\{M_n\}$ is an $\mathfrak{a}$-filtration. TFAE:

(i) $\{M_n\}$ is a stable $\mathfrak{a}$-filtration.

(ii). M^* is f.g. A^*-module.

Pf: Let $Q_n = \bigoplus_{i\geq 0}^{n} M_i$ and define

$Q_n^* = \bigoplus_{i\geq 0} \mathfrak{a}^i Q_n$ be the submod. of M^* generated by Q_n.

Explicitly, $Q_n^* = M_0 \oplus \cdots \oplus M_{n-1} \oplus M_n \oplus \mathfrak{a} M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots\cdots$

Since Q_n is f.g. over A, Q_n^* is f.g. over A^*

We have $Q_n^* \subseteq Q_{n+1}^* \subseteq \cdots$ ascending chain, and $\bigcup_{n\geq 0} Q_n^* = M^*$.

Thus M^* is f.g. A^*-mod. iff $M^* = Q_m^*$ for some m.

iff $M_{m+k} = \mathfrak{a}^k M_m$ $\forall k \geq 1$.

i.e. the filtration $\{M_n\}$ is stable $\square$

Induced filtration: if $M' \subseteq M$ is a submod, then $\{M_n \cap M'\}$ is a filtration on M'.

$\{(M_n + M')/M'\}$ is a filtration on M/M'.

Lemma (Artin-Rees Lemma):

Let A = noeth ring, $\mathfrak{a} \subseteq A$ ideal.

M = f.g. A-mod. $\{M_n\}$ = $\mathfrak{a}$-stable filtration.

$M' \subseteq M$ submod.

Then the filtration $\{M_n' := M_n \cap M'\}$ is also $\mathfrak{a}$-stable.

Pf: Set $A^* = \bigoplus_{n\geq 0} \mathfrak{a}^n$, $M^* = \bigoplus M_n$, $M'^* = \bigoplus_n M_n'$.

Since A is noeth., $\mathfrak{a}$ is f.g., so A^* is a f.g. A-algebra.

$\Rightarrow$ A^* is also noetherian ring.

Since $\{M_n\}$ is stable, M^* is f.g. A^*-module,

$\Rightarrow M'^*$ is also f.g. A^*-mod. $\Rightarrow$ $\{M_n'\}$ is stable. $\square$

Cor: There exists $k\geq 0$ s.t.

$$(\mathfrak{a}^{n+k} M) \cap M' = \mathfrak{a}^n \left((\mathfrak{a}^k M) \cap M' \right) \quad \forall n \geq 0$$

Cor (Krull intersection Thm)

For $\mathfrak{a} \subseteq A$ noeth, $\mathfrak{b} = \bigcap_{n\geq 0} \mathfrak{a}^n$, Then $\exists x \in \mathfrak{a}$ s.t. $(1+x)\cdot\mathfrak{b} = 0$

Pf: $\exists\, k \geq 0$. s.t. $\mathfrak{a}^n (\mathfrak{a}^k \cap \mathfrak{b}) = \mathfrak{a}^{n+k} \cap \mathfrak{b}$. $\forall n \geq 0$

take $n=1$, gives $\mathfrak{a} \cdot (\mathfrak{a}^k \cap \mathfrak{b}) = \mathfrak{a}^{k+1} \cap \mathfrak{b}$.

but $\mathfrak{b} \subseteq \mathfrak{a}^k$, $\mathfrak{b} \subseteq \mathfrak{a}^{k+1}$, $\Rightarrow \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b}$.

determinant trick gives the result. $\square$

Topological groups

Def A topological abelian group G is both a top space and an abelian gp. s.t. the two structures are compatible in the sense that

$$G \times G \to G \qquad\qquad G \to G$$
$$(x, y) \mapsto x+y \qquad\qquad x \mapsto -x.$$

are continuous

Some facts:

- $g \in G$, $T_g : G \to G$, $x \mapsto x+g$. is a homeomorphism.

  b/c. T_g and its inverse T_{-g} are both continuous.

  $\Rightarrow$ if $0 \in U$ is open, then $g+U$ is an open nbhd of g.

- If $\{0\}$ is closed in G, then G is Hausdorff.

  Pf: the diag. is closed in $G \times G$, being the inverse image of $\{0\}$. under $G \times G \to G$, $(x, y) \mapsto x-y$.

Lemma: Let $H =$ intersection of all nbhds of 0 in G.

(i) H is a subgp.

(ii). H is the closure of $\{0\}$.

(iii) G/H is Hausdorff.

(iv). G is Hausdorff iff $H = 0$.

Pf: (i). let $x_1, x_2 \in H$, need to show. $x+y \in H$

Fix U. The map $G \times G \to G$ is continuous. so $\forall U$ open nbhd of 0.

$(x, y) \mapsto x+y$

$\Rightarrow \exists U_1, U_2$ open s.t. $U_1 \times U_2 \to U$.

then $x_1 \in U_1$, $x_2 \in U_2$, so $x+y \in U$.

(ii). $x \in H$ iff $0 \in x - U$ for all nbhd U of 0.

iff $x \in \overline{\{0\}}$.

(iii). by (ii). any coset of H is closed. thus points are closed in G/H.

$\Rightarrow G/H$ is Hausdorff.

(iv). clear. $\square$

Now we consider the following situation:

M = A-module with $\{M_n\}$ a filtration of M.

Then M has a topology: open nbhds of any $m \in M$, is $\{m + M_n\}_{n \geq 0}$.

More precisely, open sets are: arbitrary union of the sets $m + M_n$.

Note that: if $U = (m + M_n) \cap (m' + M_{n'}) \neq \phi$, and if $n \geq n'$, then $U = m + M_n$.

b/c. if $m'' \in U$. then $m + M_n = m'' + M_n \subseteq m'' + M_{n'} = m' + M_{n'}$.

The addition map $M \times M \to M$ is continuous. as

$(m + M_n) + (m' + M_n) \subseteq (m + m') + M_n$.

$\leadsto M$ is a top group.

Let $\mathfrak{a} \subseteq A$ ideal. and give A the $\mathfrak{a}$-adic topology.

If $\{M_n\}$ is $\mathfrak{a}$-filtration. then the multiplication $(x, m) \mapsto xm$ is also continuous.

If $\{M_n\}$ is stable. then it induces the same topo as $\mathfrak{a}$-adic topology. as

$M_n \supseteq \mathfrak{a}^n M \supseteq \mathfrak{a}^n M_{n'} = M_{n+n'}$, for $n \gg 0$.

In particular, any two stable $\mathfrak{a}$-filtrations induce the same topology.

Let $N \subseteq M$ be a submodule, the closure of N, $\bar{N}$ equals $\bigcap_{n \geq 0} (N + M_n)$

(because $m \notin \bar{N}$ means $\exists n \geq 0$. s.t $(m + M_n) \cap N = \phi$ or equiv. $m \notin N + M_n$.)

In particular, M_n is closed, and $\{0\}$ is closed iff $\bigcap_{n\geq 0} M_n = \{0\}$.

- M is Hausdorff, iff $\{0\}$ is closed. (i.e. $\bigcap_n M_n = \{0\}$, also say the filtration is separated)
- M is discrete iff $\{0\}$ is open.

A sequence $(m_n)_{n\geq 0}$ is called Cauchy if given n_0, $\exists n_1$ s.t.

$$m_n - m_{n'} \in M_{n_0} \quad \forall n, n' \geq n_1.$$

Rk: This is equiv to $m_n - m_{n+1} \in M_{n_0}$ $(\forall n \geq n_1)$ as M_{n_0} is a subgroup:

$$m_n - m_{n'} = (m_n - m_{n+1}) + (m_{n+1} - m_{n+2}) + \cdots + (m_{n'-1} - m_{n'}).$$

An $m \in M$ is called a limit of (m_n) if given n_0, $\exists n_1$ s.t. $m - m_n \in M_{n_0}$ $\forall n \geq n_1$

If every Cauchy sequence has a limit, then M is called complete.

$\{$Cauchy sequences$\}$ form a module, with $(m_n) + (m_n') = (m_n + m_n')$

$$a \cdot (m_n) = (a m_n).$$

$\{$Cauchy sequences with limit $0\}$ form a submodule. (Note: limit is unique if exists)

Def: $\hat{M} := \{$Cauchy sequences $(m_n)\} / \{$Cauchy sequences with limit $0\}$.

<u>Another definition of completion</u>

Prop: Let $M = A$-mod., (M_n) a filtration.

Then $\hat{M} \simeq \varprojlim M/M_n$.

Pf: Suppose (x_k) is a Cauchy sequence in M. Then the image of (x_k) in M/M_n is constant for k large enough, equal to ξ_n, say.

It is clear that $\xi_{n+1} \to \xi_n$ under the projection.

$$M/M_{n+1} \xrightarrow{\theta_{n+1}} M/M_n.$$

Thus a Cauchy sequence (x_k) gives a sequence (ξ_n) in M/M_n s.t.

$$\theta_{n+1}(\xi_{n+1}) = \xi_n,$$

i.e. an element in the inverse limit $\varprojlim M/M_n$.

Moreover, if $(x_k) \sim (x_k')$, then the define the same sequence in (M/M_n).

Conversely, given $(\xi_n) \in \varprojlim M/M_n$, we take x_n = any element in the coset $\xi_n + M_n$, then (x_n) is a Cauchy sequence. $\square$

Recall: if $0 \to \{A_n\} \to \{B_n\} \to \{C_n\}$ is an exact sequence of inverse systems, then $0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$ is exact.

If, moreover, $\{A_n\}$ is a surjective system (i.e. Mittag-Leffler) then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0 \text{ is exact.}$$

Prop: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Let (M_n) be a filtration of M, which induces a filtration of M' and of M'', say M_n', M_n'', resp.

Then we have a short exact sequence

$$0 \to \varprojlim M'/M_n' \to \varprojlim M/M_n \to \varprojlim M''/M_n'' \to 0.$$

Prop: Let A = noeth. ring, $\mathfrak{a} \subseteq A$ ideal.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of f.g. A-mod. Then the sequence of $\mathfrak{a}$-adic completions

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$$

is exact.

Pf: By the above Prop, it suffices to prove that the $\mathfrak{a}$-adic completion $\hat{M}'$ coincides with the completion with respect to the filtration (M_n'), where $M_n' = \mathfrak{a}^n M \cap M'$.

but this follows from Artin-Rees Lemma. $\square$

Prop: For any ring A, $\mathfrak{a} \subseteq A$ ideal, If M is f.g. then $\hat{A} \otimes_A M \to \hat{M}$ is surjective.
If moreover, A is noeth, then $\hat{A} \otimes_A M \to \hat{M}$ is an isom.

Pf: Let $F \to M \to 0$ be surjective, with $F \simeq A^n$, then have

$$\begin{tikzcd} \hat{A} \otimes_A F \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \hat{F} \arrow[d, two heads] \\ \hat{A} \otimes_A M \arrow[r] & \hat{M} \end{tikzcd}$$

(isom for finite free module)

$\Rightarrow \hat{A} \otimes_A M \to \hat{M}$ is surjective.

If A is noeth, then M is of finite presentation: $0 \to N \to F \to M \to 0$.
with N also f.g. A-mod, $\Rightarrow$

$$\begin{tikzcd} & \hat{A} \otimes_A N \arrow[r] \arrow[d, two heads] & \hat{A} \otimes_A F \arrow[r] \arrow[d, "\wr"] & \hat{A} \otimes_A M \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r] & \hat{M} \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow$ Conclude by snake lemma. $\square$

Prop: If A is noeth, $\mathfrak{a} \subseteq A$ ideal, $\hat{A} = \mathfrak{a}$-adic completion,
then $\hat{A}$ is a flat A-algebra.

Pf: Equivalently, need to show if $0 \to M' \to M$, then $0 \to \hat{A} \otimes_A M' \to \hat{A} \otimes_A M$.
By a general property (Prop 2.19, [A-M]), may assume M' is f.g. A-mod.
Then we have

$$\begin{tikzcd} & \hat{A} \otimes_A M' \arrow[r] \arrow[d, "\wr"] & \hat{A} \otimes_A M \arrow[d] \\ 0 \arrow[r] & \hat{M'} \arrow[r, "\text{inj}"] & \hat{M} \end{tikzcd}$$

(Prop.)

So the top horizontal map is also injective. $\square$

Rk: the functor $M \mapsto \hat{M}$ is not exact for non-finitely-generated modules.
|| Only $M \mapsto \hat{A} \otimes_A M$ is exact and it coincides with $\hat{M}$ for f.g. A-mod M.

Prop: A = noetherian ring, $\hat{A} = \mathfrak{a}$-adic completion of A.

(i). $\hat{\mathfrak{a}} = \hat{A}\mathfrak{a} = \hat{A} \otimes_A \mathfrak{a}$

(ii). $(\mathfrak{a}^n)^\wedge = (\hat{\mathfrak{a}})^n$

(iii) $\mathfrak{a}^n / \mathfrak{a}^{n+1} \simeq \hat{\mathfrak{a}}^n / \hat{\mathfrak{a}}^{n+1}$

(iv). $\hat{\mathfrak{a}}$ is contained in the Jacobson ideal of $\hat{A}$.

Pf (iii). We have. $A/\mathfrak{a}^n \simeq \hat{A}/\hat{\mathfrak{a}}^n$. So taking quotient. gives the result.

(iv). for any $x \in \hat{\mathfrak{a}}$, $1 + x + x^2 + \cdots$ converges in. $\hat{A}$.

So $1-x$ is a unit.

This implies that $\hat{\mathfrak{a}} \subseteq \mathrm{Jac}(\hat{A})$. $\square$

Cor: $\mathrm{gr}_{\mathfrak{a}}(A) \simeq \mathrm{gr}_{\hat{\mathfrak{a}}}(\hat{A})$.

Cor: Let A = noeth local ring. $\mathfrak{m}$ = max ideal., $\hat{A}$ = $\mathfrak{m}$-adic comp.

Then. $\hat{A}$ is also a local ring. with max ideal. $\hat{\mathfrak{m}}$.

Pf: $A/\mathfrak{m} \simeq \hat{A}/\hat{\mathfrak{m}}$ is a field, so. $\hat{\mathfrak{m}}$ is max ideal.

Since. $\hat{\mathfrak{m}} \subseteq \mathrm{Jac}(\hat{A}) = \bigcap$ max ideal, so. $\hat{\mathfrak{m}}$ is the unique max ideal. $\square$

We have a natural map: $M \to \hat{M} = \varprojlim M/M_n$, continuous

$m \mapsto (m + M_n)$.

the kernel is just $\bigcap_{n \geq 0} M_n$, the closure of $\{0\}$.

In the case $M_n = \mathfrak{a}^n M$, (A = noeth ring. and M = finite A-mod).

by Krull intersection thm, $m \in \bigcap_{n \geq 0} M_n$ iff $\exists x \in \mathfrak{a}$, s.t. $(1+x) \cdot m = 0$.

In particular, the map is injective. if $\mathfrak{a} \subseteq \mathrm{Jac}(A)$.

Ex: • $A = \mathbb{Z}$, $\mathfrak{a} = (p)$. p = prime.

Then. $\hat{A} = \mathbb{Z}_p$ the p-adic integers., Its elements are infinite series

$$\sum_{n \geq 0}^{\infty} a_n p^n, \quad 0 \leq a_n \leq p-1.$$

• $A = k[x_1, \ldots, x_n]$, its completion at $\mathfrak{a} = (x_1, \cdots, x_n)$ is $k[[x_1, \cdots, x_n]]$.

which is also noeth.

Thm 1: If A is a noeth ring, $\mathfrak{a} \subseteq A$ ideal, then $\hat{A}$ is also noeth. ring.

We will prove more generally.

Thm 2: Let A = ring. $\mathfrak{a}$ = ideal, M = A-mod., (M_n) = $\mathfrak{a}$-filtration.

Suppose A is complete in the $\mathfrak{a}$-adic topology and M is Hausdorff, i.e. $\bigcap_n M_n = (0)$

If $G(M)$ is f.g. $G(A)$-module, then M is a f.g. A-module.

Proof: Say $G(M) = \sum_{i=1}^{s} G(A)\cdot \sigma(u_i)$, $\quad u_i \in M_{k_i} - M_{k_i+1}$, where $\sigma(u_i) = u_i + M_{k_i+1} \in M_{k_i}/M_{k_i+1}$ called the principal part of u_i.

then $\forall n$. $\quad G(M)_n = \sum_{i=1}^{s} G(A)_{n-k_i}\cdot\sigma(u_i)$

$$M_n = \sum_{i=1}^{s} A_{n-k_i}\cdot u_i + M_{n+1}$$

if $u \in M_n$, then $u = \sum_{i=1}^{s} a_{n-k_i} u_i + u_{n+1}$.

Similarly, $u_{n+1} = \sum a_{n+1-k_i} u_i + u_{n+2}$.

Inductively, we obtain. $u - \sum_{i=1}^{s}\left(\sum_{t=1}^{q} a_{n+t-k_i}\right) u_i \in M_{n+q+1}$, $\forall q \geq 1$.

Since A is complete, we may define. $a_i = \sum_{t=1}^{\infty} a_{n+t-k_i}$, for $i=1,\dots,s$.

then $a_i \in A_{n-k_i}$, and $u - \sum_{i=1}^{s} a_i u_i \in M_{n+q+1}$, $\forall q$.

Since M is Hausdorff, $u - \sum_{i=1}^{s} a_i u_i = 0$, i.e. $u \in \sum_i A\cdot u_i$. $\square$.

Cor 3: With the same hypotheses of Thm 2, if $G(M)$ is a noeth. $G(A)$-mod. then M is a noeth A-module.

Pf: Show every submodule M' of M is f.g. A-mod.

Let $M_n' = M' \cap M_n$, then (M_n') is an $\mathfrak{a}$-filtration of M'. and the embedding $M_n' \hookrightarrow M_n$ induces (when passing to quotients)

$$M_n'/M_{n+1}' \hookrightarrow M_n/M_{n+1}$$

(because $M_n' \cap M_{n+1} = M_{n+1}'$)

hence induces an embedding $G(M') \hookrightarrow G(M)$, i.e. $G(M')$ is a submod of $G(M)$

Since $G(M)$ is noeth, $G(M')$ is f.g. $G(A)$-module.

Moreover. (M_n') is Hausdorff, i.e. $\bigcap_n M_n' = 0$.

So, by Thm 2. M' is f.g. A-module. $\square$

Proof of Thm 1: A is noeth $\Rightarrow$ $G(A)$ is noeth. where $G(A) = \bigoplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$.

(i.e. it is a noeth. $G(A)$-module)

By Cor 3, A is an A-module, i.e. A is noeth ring. $\square$.

Cor: If A is noeth, then $A[[x_1, \dots, x_n]]$ (power series ring) is noeth.

Pf: it is the completion of $A[x_1, \dots, x_n]$. for the $\mathfrak{a}$-topo. $\mathfrak{a} := (x_1, \dots, x_n)$.

Thm: Let A be a semi-local ring with max ideals $m_1, \dots, m_r$, and

$$\mathfrak{a} := \mathrm{Jac}(A) = \bigcap_{i=1}^{r} m_i = \prod_{i=1}^{r} m_i$$

Then $\hat{A}$ ($\mathfrak{a}$-adic. completion) is isom to. $\hat{A}_{m_1} \times \cdots \times \hat{A}_{m_r}$.

Pf: $\forall (i,j)$. m_i^n, m_j^n are coprime, So. Chinese Remainder Thm.

$$A/I^n = A/(m_1 \cdots m_r)^n \simeq A/m_1^n \times \cdots \times A/m_r^n$$

taking inverse limit. gives. $\hat{A} \simeq \varprojlim A/m_1^n \times \cdots \times \varprojlim A/m_r^n \simeq \hat{A}_{m_1} \times \cdots \times \hat{A}_{m_r}$ $\square$

§ Hilbert function.

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a Noetherian graded ring.

Thus. A_0 is Noeth., and A is generated (as A_0-alg) by $x_1, \dots, x_s$. which are homogeneous. of degree $k_1, \dots, k_s$. (>0).

Let M be a f.g. graded A-module. Then M is generated by f.m. homoge. elements. say $m_1, \dots, m_t$. of degree $r_1, \dots, r_t$.

$\forall n>0$
$\Rightarrow$ M_n is f.g. as A_0-mod., by. $g_j(x).m_j$, where $g_j(x)$ is a monomial in x_i of total degree $n-r_j$.

Assume. A_0 is artinian. (Then A_0 is also noeth).

Since each M_n is f.g. A_0-module. it is artin. and noeth. so has finite length.

The Hilbert series of M is the power series. (or Poincaré series).

$$H(M,t) := \sum_{n=0}^{\infty} \ell(M_n)t^n \in \mathbb{Z}[[t]].$$

Thm. (Hilbert-Serre). $H(M,t)$ is a rational function in t of the form

$$f(t) \Big/ \prod_{i=1}^{s} (1-t^{k_i}). \qquad \text{where } f(t) \in \mathbb{Z}[t]$$

Pf. Do induction on S, := the number of generators of A over A_0.

Start with $s=0$, ie $A_n=0$, $\forall n>0$. $\Rightarrow$. $M_n=0$ for $n>>0$

So $H(M,t)$ is a poly. in this case.

Now suppose $s>0$, and the result is true for $s-1$.

Multip. by x_s gives an A-mod homom $M_n \to M_{n+k_s}$.

$$\Rightarrow \quad 0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0. \quad , \quad \forall n \geq 0.$$

Let $K = \bigoplus_{n\geq 0} K_n$. $L = \bigoplus_{n\geq 0} L_{n+k_s}$, these are both. f.g A-mod. b/c. K is a submod of M, and L is a quot of M.

Since they are annihilated by x_s $\Rightarrow$ they are A/x_s-mod. which is a graded ring. and can be generated by $s-1$ elements.

Apply $\ell(\cdot)$ to it. get

$$\ell(K_n) - \ell(M_n) + \ell(M_{n+k_s}) - \ell(L_{n+k_s}) = 0.$$

multip by t^{n+k_s}, and summing over n. we get.

$$\underbrace{\sum_{n\geq 0} t^{n+k_s}\ell(K_n)}_{= t^{k_s}H(K,t)} - \underbrace{\sum_{n\geq 0} t^{n+k_s}\ell(M_n)}_{= t^{k_s}H(M,t)} + \underbrace{\sum_{n\geq 0} t^{n+k_s}\ell(M_{n+k_s})}_{= (H(M,t)-\text{poly})} - \underbrace{\sum_{n\geq 0} t^{n+k_s}\ell(L_{n+k_s})}_{= (H(L,t)-\text{poly})} = 0$$

$$\Rightarrow (1-t^{k_s})H(M,t) = H(L,t) - t^{k_s}H(K,t) + \underbrace{g(t)}_{\text{some poly.}}$$

Apply induction. $\Rightarrow$ result. $\square$

The order of the pole of $H(M,t)$ at $t=1$, is denoted by $d(M)$.
It measures the size of M. In particular, $d(A)$ is defined.

Ex: • Let $A = k[x_1, \dots, x_s]$ with $\deg x_i = 1$.
then A_n is free k-mod with basis $x_1^{m_1} \cdots x_s^{m_s}$, with $\sum m_i = n$.
So $\ell(A_n) = \dim_k(A_n) = \binom{s+n-1}{s-1}$.

$$\Rightarrow H(A,t) = \sum_{n\geq 0}\binom{s+n-1}{s-1}t^n = \frac{1}{(1-t)^s}, \quad \Rightarrow d(A) = s.$$

• If $A = A_0 = k$, then $M_n = 0$ for $n >> 0$.
then $H(M,t) \in \mathbb{Z}[t]$. $\Rightarrow d(M) = 0$. (ie. small size).

Cor. If each $k_i = 1$, then for $n >> 0$, $\ell(M_n)$ is a polynomial in n of degree $d-1$. called Hilbert polynomial.

Pf: We have $\ell(M_n) = $ coeff of t^n in $f(t)\cdot(1-t)^{-s}$.
We may write $f(t) = f'(t)(1-t)^r$. So may assume.

$$H(M,t) = f(t)/(1-t)^d. \qquad \text{where } d = d(M) = r-s.$$

Suppose $f(t) = \sum_{k=0}^{N} a_k t^k$. Since.

$$(1-t)^{-d} = \sum_{n=0}^{\infty}\binom{d+n-1}{d-1}t^n$$

One have. $$\ell(M_n) = \sum_{k=0}^{N} a_k \binom{d+n-k-1}{d-1} \qquad \text{for all } n \geq N.$$

RHS is a poly in n. with leading term $\left(\sum_{k=0}^{N} a_k\right) n^{d-1}/(d-1)! \neq 0$; as $f(1) \neq 0$. $\square$

§ Hilbert-Samuel function.

Def: A polynomial-like function is a function $f: \mathbb{N} \to \mathbb{Q}$ s.t.

$$f(n) = g(n). \quad \text{for } n \gg 0.$$

where $g \in \mathbb{Q}[x]$. is a polynomial. ($\Rightarrow$ g is unique)

We define the degree of f to be $\deg(g)$. and leading coeff of f to be the leading coeff of g.

Lemma: f is poly-like function of degree r. iff Δf is poly-like of degree $r-1$.

where $\Delta f(k) = f(k+1) - f(k)$. called the difference of f. $(\deg 0 := -1)$.

Pf: $\Rightarrow$. clear.

$\Leftarrow$ use the following fact: $g = \Delta(f) \Rightarrow \sum_{k=a}^{b} g(k) = f(b+1) - f(a)$. ◻

Now we consider filtered modules. let $M_0 = M \supseteq M_1 \supseteq \cdots \supseteq \cdots$ be a filtration

assume each $\ell(M/M_n)$ is finite, we may consider the Hilbert-Samuel function

$$n \longmapsto \ell(M/M_n)$$

and the Hilbert-Samuel series :

$$P(M,t) = \sum_{n \geq 0} \ell(M/M_n) t^n.$$

Thm (Samuel) let A = noeth, $\mathfrak{a} \subseteq A$ ideal. M = f.g. A-mod. with stable $\mathfrak{a}$-filt. (M_n).

Assume $A/\mathfrak{a}$ is artinian ring.

Then $\ell(M/M_n) < \infty$, $\forall n \geq 0$, and

$$P(M,t) = H(G(M),t) \cdot t/(1-t).$$

Pf: each M_n/M_{n+1} is finitely gen. $A/\mathfrak{a}$-mod, so has finite length.

Thus $\ell(M/M_n)$ has finite length $\forall n$.

We have $\ell(M_n/M_{n+1}) = \ell(M/M_{n+1}) - \ell(M/M_n)$

So. $$\sum_n \ell(M_n/M_{n+1}) t^n = \sum_n \ell(M/M_{n+1}) t^n - \sum_n \ell(M/M_n) t^n$$

$$H(G(M),t) = (t^{-1}-1) P(M,t) = P(M,t) \cdot (1-t)/t.$$ ◻

Cor: with the conditions of Thm. Assume $\mathfrak{a}$ can be generated by s elements.
Then for all suff large n, $\ell(M/M_n)$ is a poly. $g(n)$ of degree $\le s$.
(we call it a polynomial like function).
Moreover, the degree and leading coeff of $g(n)$ depends only on M and $\mathfrak{a}$, not on the filtration (M_n).

Pf: If $x_1, \dots, x_s$ generate $\mathfrak{a}$, then the image $\bar{x}_i$ of x_i in $\mathfrak{a}/\mathfrak{a}^2$, generate $G(A)$ as an $A/\mathfrak{a}$-alg. with $\deg \bar{x}_i = 1$.
$\Rightarrow$ $\ell(M_n/M_{n+1}) = f(n)$ for large n, where $f(n)$ is a poly in n of degree $\le s-1$.
Since $\ell(M/M_{n+1}) - \ell(M/M_n) = f(n)$, it follows that $\ell(M/M_n)$ is a poly of deg $\le s$, for large n.
Let $(\tilde{M}_n)$ be another stable $\mathfrak{a}$-filtration of M, and $\tilde{\ell}_n = \ell(M/\tilde{M}_n)$.
The two filtrations are equivalent, i.e. $\exists n_0$ s.t.

$$M_{n+n_0} \subseteq \tilde{M}_n, \qquad \tilde{M}_{n+n_0} \subseteq M_n. \quad \forall n \ge 0.$$

thus $$M/M_{n+n_0} \twoheadrightarrow M/\tilde{M}_n, \qquad M/\tilde{M}_{n+n_0} \twoheadrightarrow M/M_n.$$

$$\ell(M/M_{n+n_0}) \ge \ell(M/\tilde{M}_n). \qquad \ell(M/\tilde{M}_{n+n_0}) \ge \ell(M/M_n).$$

This implies $\lim_{n} \ell_n/\tilde{\ell}_n = 1$. thus ℓ_n, $\tilde{\ell}_n$ have the same degree and leading coeff. □

We will be of particular interest in the case A is local, $\mathfrak{m}$ = max ideal.

Def: An ideal $\mathfrak{a}$ of A is said to be an ideal of definition if

$$\mathfrak{m}^r \subseteq \mathfrak{a} \subseteq \mathfrak{m}. \quad \text{for some } r \ge 1.$$

Equivalently, • $\mathfrak{a}$ is a $\mathfrak{m}$-primary ideal.
• or. $A/\mathfrak{a}$ is artinian ring.

Prop: (i) Let q be a $\mathfrak{m}$-primary ideal. For all large n, $\ell(A/q^n)$ is a poly. of degree $\le s$. where s is the least number of generators of q.
(ii). $\deg \ell(A/q^n)$ does not depend on the choice of q.

Pf: (i). is a special case of Thm. (Samuel)

(ii). use the fact. $m^n \supseteq q^n \supseteq m^{rn}$, if $m^r \subseteq q \subseteq m$.

$\Rightarrow \quad \ell(A/m^n) \leq \ell(A/q^n) \leq \ell(A/m^{rn})$.

let $n \to \infty$, and notice that these are poly-like functions, they must have the same deg. □

We write $\chi_q^M(n) = \ell(M/q^n M)$, and if $M=A$, $\chi_q(n) = \ell(A/q^n)$: called char. poly.

the degree of χ_q^M is denoted by $d(M)$., of χ_q is denoted by $d(A)$.

Key observation: $d(A) \leq \min\limits_{q} \min\{s : \text{number of generators of } q\}$.

Prop: Let A = noeth, q = m-primary ideal and

$$0 \to M' \to M \to M'' \to 0.$$

an exact sequence of f.g. A-modules. Then $d(M) = \max\{d(M'), d(M'')\}$.

[Moreover, $\chi_q^{M'}(n) - \chi_q^M(n) + \chi_q^{M''}(n)$

is a poly-like function of degree $\leq d(M) - 1$ $(= \deg \chi_q^M(n) - 1)$.

Pf: let $M'_n = M' \cap q^n M$. then (M'_n) is stable q-filtration by Artin-Rees lemma, and we have

$$0 \to M'/M'_n \to M/q^n M \to M''/q^n M'' \to 0.$$

This implies $\chi_q^M(n) = \chi_q^{M''}(n) + \ell(M'/M'_n)$.

but $\ell(M'/M'_n)$ and $\chi_q^{M'}(n)$ have the same degree and leading coeff., the result follows. □

Cor: If $x \in A$ is non-zero-divisor for M. ie $x: M \to M$ is injective.

Then $d(M/xM) \leq d(M) - 1$.

Pf: Apply Prop to the sequence

$$0 \to M \xrightarrow{\times x} M \to M/xM \to 0.$$

Dimension theory of noeth. local rings.

Let A be a noeth local ring, m = max ideal.
We have the following three numbers attached to (A, m):

- $\dim(A)$: the Krull dim of A; the max length of chains of prime ideals in A.
- $d(A)$: degree of char. polynomial $\chi_m(n) = \ell(A/m^n)$.
- $\delta(A)$:= least number of generators of an m-primary ideal of A.

Main Thm: $\dim(A) = d(A) = \delta(A)$.

We shall prove $\underbrace{\delta(A) \geq d(A)}_{\text{already seen}} \geq \dim(A) \geq \delta(A)$.

$\boxed{\text{Prop. 1. + 2 + 3} \Rightarrow \text{Main Thm.}}$

Prop 1: (i) let $x \in m$, then $\delta(A) \leq \delta(A/xA) + 1$.

(ii) if $x \in m$ is non-zero divisor, then $d(A/x) \leq d(A) - 1$.

(iii) if $x \in m$, and $x \notin p_i$, for any minimal prime ideal p_i of A s.t. $\dim A/p_i = \dim A$. then $\dim(A/x) \leq \dim(A) - 1$.

Pf: (i) if $\bar{q} \subseteq A/xA$ is $\bar{m}$-primary ideal. $(\Rightarrow \bar{m}^n \subseteq \bar{q})$

Let q be the preimage of $\bar{q}$, then $m^n \subseteq q$. so q is m-primary.

If $\bar{q}$ can be generated by r elements, then q can be generated by $r+1$ elements. □

(ii) It is a consequence of Cor. in last page.

(iii). Let $\bar{p}_0 \subsetneq \cdots \subsetneq \bar{p}_r$ be a chain of prime ideals in A/xA.

and $p_0 \subsetneq \cdots \subsetneq p_r$ be the preimage in A.

In particular, $x \in p_0$. We have two cases.

- if p_0 is minimal in A, then by assump. $\dim A/p_0 < \dim(A)$, $\Rightarrow r < \dim(A)$
- if p_0 is not minimal in A, then $\exists p' \subsetneq p_0$ s.t. $\dim(A) \geq r+1$.

again gives $r \leq \dim(A) - 1$.

This is true for any chain in A/xA, we get $\dim(A/xA) \leq \dim(A) - 1$. □

Prop 2: $d(A) \geqslant \dim(A)$.

Pf: By induction on $d = d(A)$.

If $d=0$, then $\ell(A/m^n)$ is constant for all large n, hence $m^n = m^{n+1}$.

NAK $\Rightarrow m^n = 0$, ie A is artin ring, so $\dim(A) = 0$.

Suppose $d>0$ and let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$ be any chain of prime ideals.

let $x \in P_1 \setminus P_0$, and $A' := A/P_0$. x' = image of x in A' (non-zero).

Note that A' is a domain, so x' is non-zero-divisor in A.

$\Rightarrow \quad d(A'/(x')) \leq d(A') - 1$.

Also, if m' is the max ideal of A', then $A/m^n \twoheadrightarrow A'/m'^n$.

So $\ell(A/m^n) \geq \ell(A'/m'^n)$. and $d(A) \geq d(A')$.

Consequently, $d(A'/x') \leq d-1$

By induction hypo. $\dim(A'/x') \leq d(A'/x') \leq d-1$.

But the image of $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_r$ in A'/x' form a chain of length $r-1$.

So. $r-1 \leq \dim(A'/x') \leq d-1$. , ie $r \leq d$.

This being true for any chain, we deduce. $\dim(A) \leq d$. $\square$.

Prop 3: $\dim(A) \geq \delta(A)$.

Pf: if $\dim(A) = 0$, then A is artin, and (0) is m-primary.

Assume. $\dim(A) > 0$, then $\exists x \in m$, $x \notin P_i$. $\forall P_i$ minimal prime ideal of A.

by induction. $\delta(A) \leq \delta(A/xA) + 1 \leq \dim(A/xA) + 1 \leq \dim(A)$. $\square$

Cor: $\dim(A)$ is finite.

Cor: $\dim(A) \leq \dim_k(m/m^2)$.

Pf: By NAK, if $x_1, \ldots, x_r \in m$. are such that. $\bar{x}_i$ form a basis of. m/m^2,

then $x_1, \ldots, x_r$. generate m, $\Rightarrow$. $\delta(A) \leq r$. $\square$

Def: A is called regular local ring. if $\dim(A) = \dim_k(m/m^2)$.

Thm: Let A = noeth local ring. m = max ideal
Let $x \in m$ be a non-zero-divisor, then $\dim(A/xA) = \dim(A) - 1$.

Pf: We must have $\dim A > 0$. otherwise A is artin. and m is the only prime ideal
So. m consists of zero-divisors (or. nilpotent. elements), a contradiction.

We have $\dim(A/xA) \le \dim(A) - 1$ (already shown).
(Lemma): $\delta(A) \le \delta(A/xA) + 1$. So $\delta(A/xA) \ge \delta(A) - 1$.
Since $\begin{cases} \delta(A) = \dim(A) \\ \delta(A/xA) = \dim(A/xA) \end{cases} \Rightarrow$ result. $\square$

Def: height of a prime ideal. if $\sup\{n \mid \exists \text{ chain. } p_0 = p \supsetneq p_1 \supsetneq \cdots \supsetneq p_n\}$.
$\Rightarrow ht(p) = \dim(A_p)$ and $\dim(A) = \sup_p\{ht(p)\}$,

Rem: If A is not local, then this is not true. in general.
Namely, there exists noetherian ring. A which has infinite Krull dimension.
Nagata has constructed such an example, as follows: (See [A-M], P126, Ex4).
Let $A = k[x_1, \cdots, x_n, \cdots]$ be poly ring in countably many indeterminates
Let $m_1, m_2, \cdots$ be increasing sequence in $\mathbb{N}$. s.t.
$$m_{i+1} - m_i > m_i - m_{i-1}. \quad \forall i > 1.$$
Let $P_i = (x_{m_i+1}, \cdots, x_{m_{i+1}})$, and $S = A \setminus \bigcup_i P_i$.
Each P_i is prime ideal, So S is multiplicative.
Each $S^{-1}P_i$ has height equal to $m_{i+1} - m_i$, hence. $\dim(S^{-1}A) = \infty$.

Need to. check that: A is a noetherian ring.
Lemma ([A-M]. Ex 7.9). Let A be a ring such that.
(1). for each max ideal m of A, A_m is noeth.
(2). $\forall\, 0 \ne x \in A$, the set of max ideals of A which contain x is finite.
Then A is noetherian.

Cor: (Krull's principal ideal thm).

let A = noeth ring. $x \in A$. which is not zero-divisor. nor a unit.

Then every minimal prime ideal $\mathfrak{p}$ of (x) has height 1.

Pf: first note that in $A_\mathfrak{p}$, $(x)\cdot A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$-primary., because. $\mathfrak{p}A_\mathfrak{p}$ is a min. ideal containing $(x)A_\mathfrak{p}$ and it is also max ideal.

Main Thm $\Rightarrow$. $\dim(A_\mathfrak{p}) = ht(\mathfrak{p}) \leq 1$.

If height $\mathfrak{p} = 0$, then $\mathfrak{p}$ is a prime ideal associated to (0).

hence every element of $\mathfrak{p}$ is a zero-divisor., contradiction. $\square$

§. Dimension theory (II).

Integral extensions

Prop: If $A \hookrightarrow B$ is an integral extension.

then $\dim(A) = \dim(B)$. (with convention $\infty = \infty$ if $\dim A$ or $\dim B = \infty$)

Pf: This is a consequence of Going-up theorem: given

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \quad \text{in } A$$

and $\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ in B. s.t. $\mathfrak{q}_i \cap A = \mathfrak{p}_i$. $\forall\ 1 \le i \le m$.

then we may extend the chain to $\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m \subsetneq \cdots \subsetneq \mathfrak{q}_n$ s.t

$$\mathfrak{q}_i \cap A = \mathfrak{p}_i \quad \forall\ 1 \le i \le n.$$

Also. for any $\mathfrak{p} \in \operatorname{Spec} A$, there always exists $\mathfrak{q} \in \operatorname{Spec} B$, s.t. $\mathfrak{q} \cap A = \mathfrak{p}$.

So $\dim(A) \le \dim(B)$.

On the other hand, for $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec} B$, if $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$, then $\mathfrak{q}_1 \cap A \neq \mathfrak{q}_2 \cap A$,

hence $\dim(B) \le \dim(A)$. $\square$

Thm: Let A = noeth ring. $B = A[X]$. Then $\dim(B) = \dim(A) + 1$.

We need some Lemmas. for $\mathfrak{p} \in A$, $\mathfrak{p}^e = \mathfrak{p}[X] \subseteq B$.;

this is a prime ideal. because $B/\mathfrak{p}^e \simeq (A/\mathfrak{p})[X]$ is a domain.

Moreover. $\mathfrak{p}^{ec} = \mathfrak{p}$. (easy to check).

Lemma: If $\mathfrak{q} \subsetneq \mathfrak{q}'$ are prime ideals, assume $\mathfrak{q}^c = \mathfrak{q}'^c = \mathfrak{p}$ (prime ideal in A)

Then. $\mathfrak{q} = \mathfrak{p}^e$

Pf: $A/\mathfrak{p} \hookrightarrow B/\mathfrak{p}^e$, so we may assume $\mathfrak{p} = 0$, in particular. A is domain.

If $S \subseteq A$ is multiplicative, then. (by direct check).

$$(S^{-1}A)[X] \simeq S^{-1}(A[X]).$$

Taking $S = A \setminus \{0\}$, we obtain. $(K := \operatorname{Frac}(A))$

$$K[X] = S^{-1}B.$$

Taking $S = A\setminus\{0\}$, we obtain. $(K := \mathrm{Frac}(A))$

$$K[x] = S^{-1}B.$$

Since $K[x]$ is PID, of Krull-dim 1, so $S^{-1}B$ has dim 1.

We have. $S^{-1}q \subsetneq S^{-1}q'$, $\Rightarrow S^{-1}q = (0)$. equiv. $q = (0) = (0)[x]$ $\square$

Lemma 2: let $\mathfrak{a} \subseteq A$. if $\mathfrak{p}$ is a min prime ideal of $\mathfrak{a}$,

then $\mathfrak{p}^e$ is a min prime ideal of $\mathfrak{a}^e$.

Pf: clearly, $\mathfrak{a}^e \subseteq \mathfrak{p}^e$. if $\mathfrak{p}^e$ is not minimal, then $\exists q$. s.t $\mathfrak{a}^e \subseteq q \subseteq \mathfrak{p}^e$.

but then $q \cap A = \mathfrak{p}^e \cap A = \mathfrak{p}$, so Lemma 1 $\Rightarrow q = \mathfrak{p}^e$. $\square$

Lemma 3: $ht(\mathfrak{p}) = ht(\mathfrak{p}^e)$.

Pf: let $n = ht(\mathfrak{p})$. by generalized Krull principal ideal thm.

$\mathfrak{p}$ contains some ideal $\mathfrak{a}$. which is generated by n elements.

but then $\mathfrak{p}^e$, is also generated by n elements (in B). So. $ht(\mathfrak{p}^e) \le n$

On the other hand, we clearly have. $ht(\mathfrak{p}^e) \ge ht(\mathfrak{p})$. because for any chain.

$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$, we have a chain in B:

$$\mathfrak{p}_0^e \subsetneq \cdots \subsetneq \mathfrak{p}_n^e = \mathfrak{p}^e. \quad \square$$

Now we can prove. Thm.

Pf of Thm: Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a chain of prime ideals in A.

we obtain a chain. $\mathfrak{p}_0^e \subsetneq \cdots \subsetneq \mathfrak{p}_n^e \subsetneq \mathfrak{p}_n^e + (x)$., of length $n+1$.

So. $\dim(B) \geq \dim(A) + 1$.

Now consider a chain $q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n$ in B; need to show $\dim A \geq n-1$.

Set $\mathfrak{p}_i = q_i^c$. If $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$, $\forall i$, then $\dim A \geq n$., hence $\dim A \geq n-1$.

Otherwise, there exists i s.t. $\mathfrak{p}_i = \mathfrak{p}_{i+1}$.

Let j be the largest index i with this property.

ie. $q_j \subsetneq q_{j+1}$ but $\mathfrak{p}_j = \mathfrak{p}_{j+1}$.

Lemma 1 $\Rightarrow$ $q_j = P_j^e$. So $ht(q_j) = ht(P_j)$ by lemma 3 $\Rightarrow ht(P_j) \geq j$

By the choice of j. $P_{j+1} \subsetneq P_{j+2} \subsetneq \cdots \subsetneq P_n$ distinct.

So we obtain $\dim(A) \geq ht(P_j) + (n-j-1) \geq j + (n-j-1) = n-1$.

This proves $\dim(A) \geq \dim(B) - 1$. $\square$

Cor: A = noeth, $\dim(A[X_1, \dots, X_n]) = \dim(A) + n$.

If $A = k$ is a field, then $\dim(k[X_1, \dots, X_n]) = n$.

Thm: If A is finitely gen. alg over a field, and is a domain.

then $\dim(A) = \mathrm{tr.deg}_k(\mathrm{Frac}(A))$.

Pf: By Noether normalization theorem, $\exists\, t_1, \dots, t_d \in A$. s.t.

A is integral over $k[t_1, \dots, t_d]$.

So $\dim(A) = d$, the same as $k[t_1, \dots, t_d]$.

On the other hand $\mathrm{Frac}(A)$ is alg. ext over $k(t_1, \dots, t_d)$, so has tr.deg. d. $\square$

Thm: Let k = field, A = finite generated k-alg. Assume A is domain.

Then $\forall$ saturated chain of prime ideals has length $\dim(A)$.

Pf: by induction on $\dim(A) = d$.

If $d = 0$, then A is itself artin and domain, so is a field.

Assume $d \geq 1$. first by Noether normalization, $\exists\, t_1, \dots, t_d$. s.t. (here $d = \dim(A)$!)

A is finite over $B := k[t_1, \dots, t_d]$.

Let $(0) = P_0 \subsetneq \cdots \subsetneq P_n$ be a saturated chain in A, and let $q_i := P_i^c$ in B.

then $0 = q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_n$

Note that B is UFD, let $b \in q_1$ be an irreducible element, $\Rightarrow (b)$ is prime ideal.

but $(b) \subseteq q_1$, by $\underbrace{\text{going-down}}_{(B \text{ is integ closed})}$. $\exists\, P_1' \subseteq P_1$, but $P_1' \neq 0$, and the chain is saturated

$(b) \subseteq q_1$ $\Rightarrow P_1' = P_1$ and thus $(b) = q_1$

$\Rightarrow B/(b) \hookrightarrow A/\mathfrak{p}_1$, and is integral extension.

Since $\dim B/(b) = d-1$, we have $\dim(A/\mathfrak{p}_1) = d-1$.

Of course $A/\mathfrak{p}_1$ is f.g. k-alg and is a domain, with $\dim(A/\mathfrak{p}_1) \leq d-1$.

and $(0) = \mathfrak{p}_1/\mathfrak{p}_1 \subsetneq \mathfrak{p}_2/\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n/\mathfrak{p}_1$ is a saturated chain of length $n-1$.

by induction. $n-1 = \dim(A/\mathfrak{p}_1) = d-1$, so $n = d$. $\square$

Cor: let A = f.g. k-alg, $\mathfrak{p} \subseteq A$ prime ideal.

Then $ht(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p})$.

Pf: Note that $ht(\mathfrak{p}) + \dim A/\mathfrak{p}$ is the max length of chain of prime ideals which contain $\mathfrak{p}$.

There exists such chain which is saturated. So by Thm. this length is $\dim(A)$. $\square$

Regular local rings

<u>Thm</u>: let A be a noeth. local ring of dim d, m = max ideal $k = A/m$.

TFAE:

(i) A is regular local ring, i.e. $\dim_k(m/m^2) = d$.

(ii) m can be generated by d elements.

(iii) $G_m(A) \simeq k[t_1, \ldots, t_d]$, is poly ring.

Pf: (i) $\Rightarrow$ (ii) by NAK.

(ii) $\Rightarrow$ (iii). let $m = (x_1, \ldots, x_d)$. then $G_m(A)$ can be generated by d elements as k-alg.

So there exists a surjection. $k[t_1, \ldots, t_d] \to G_m(A)$.

$t_i \longmapsto \bar{x}_i$.

but as abstract ring. $G_m(A) \simeq k[t_1, \ldots, t_d]$, so the above map must be an isom

(otherwise, if $f \in \ker$, then $\dim(k[t_1, \ldots, t_d]/(f)) \leq d-1$, a contradiction. $\square$

Prop: let A be a noeth local ring. Then A is regular if and only if $\hat{A}$ is regular.

Pf: we know that $\hat{A}$ is noeth local ring and $\dim(\hat{A}) = \dim(A)$.

Since $G_{\hat{m}}(\hat{A}) = G_m(A)$ is $\simeq k[t_1, \ldots, t_d]$, $\hat{A}$ is regular by Thm. $\square$

Prop: let A be a ring. $\mathfrak{a} \subseteq A$ ideal. s.t $\bigcap_{n\geq 0} \mathfrak{a}^n = 0$.

Suppose that $G_{\mathfrak{a}}(A)$ is an integral domain, then so is A.

Pf: let $x, y \in A$. non-zero. Since $\bigcap_{n\geq 0} \mathfrak{a}^n = 0$, $\exists\, r, s > 0$. s.t.

$$x \in \mathfrak{a}^r,\ x \notin \mathfrak{a}^{r+1}$$
$$y \in \mathfrak{a}^s,\ y \notin \mathfrak{a}^{s+1}$$

$\Rightarrow \bar{x} \in \mathfrak{a}^r/\mathfrak{a}^{r+1}$, $\bar{y} \in \mathfrak{a}^s/\mathfrak{a}^{s+1}$.

Since $G(A)$ is domain, $\bar{x}\bar{y} = \overline{xy} \in \mathfrak{a}^{r+s}/\mathfrak{a}^{r+s+1}$ is non-zero.

$\Rightarrow xy \neq 0$ in A. $\square$

Cor: A regular local ring is an integral domain.

Prop: If A is a local ring and $G_{\mathfrak{m}}(A)$ is an integrally closed integ domain.

Then A is also integrally closed.

In particular, a regular local ring is integrally closed.

Def: A is integ domain, $s \in \operatorname{Frac}(A)$ is said to be almost integral over A.

if $\exists\, c \in A$. $c \neq 0$. s.t. $cs^n \in A$, $\forall n \geq 0$.

Lemma: (i) If s is integral, then s is almost integral.

(ii) If A is noeth, the converse is also true.

Pf: (i) s is integral iff $A[s]$ is f.g. A-mod.

So $\exists\, c \in A$ s.t. $c \cdot A[s] \subseteq A$. in particular. $cs^n \in A$, $\forall n$.

(ii). Conversely, if s is almost integral, then. $A[s]$ is contained in $c^{-1}A$., which is f.g. A-mod. Since A is noeth, $A[s]$ is also f.g, so s is integ. $\square$

Pf of Thm.: let $a, b \in A$, $b \neq 0$, s.t. $\frac{a}{b}$ is integral over A., need to show $a \in (b)$, may assume $b \notin \mathfrak{m}$.

We will prove by induction on n that

$(a) \in \mathfrak{m}^n + (b)$., $\forall n > 0$ (conclude by krull intersection thm).

For $n=0$, OK. So we assume $n > 0$.

By induction, $a = \tilde{a} + r \cdot b$, with $\tilde{a} \in \mathfrak{m}^{n-1}$; we may assume. $\tilde{a} \notin \mathfrak{m}^n$.

$\Rightarrow \frac{a}{b} = \frac{\tilde{a}}{b} + r, \quad r \in A, \quad$ so. $\frac{\tilde{a}}{b}$ is also integral over A; $\Rightarrow$ almost integral.

hence $\exists\, c \in A$, $c \neq 0$, s.t. $c\tilde{a}^n \in (b^n)$. for all $n \geq 1$.

Since $gr(A)$ is a domain, we must have. $gr(c) \cdot gr(\tilde{a})^n = (gr(b))^n \cdot \quad \forall n.$

ie $\dfrac{gr(\tilde{a})}{gr(b)}$ is almost integral over $gr(A)$.

but since $gr(A)$ is noetherian, we deduce that. $\dfrac{gr(\tilde{a})}{gr(b)}$ is integral over $gr(A)$.

hence lies in $gr(A)$ (it is integ closed)

$\Rightarrow \exists\, s \in A$, s.t. $gr(\tilde{a}) = gr(s) \cdot gr(b)$., with $\deg s = \deg \tilde{a} - \deg b$.

ie. $\tilde{a} \in sb + m^n$ $\square$

Dimension of modules

Def: A = ring, M = finite A-module. define the dimension of M to be:

$$\dim(M) := \dim A/\mathrm{ann}(M)$$

We only consider the case A is noetherian local, then

$$\dim(M) = \sup\{\dim A/\mathfrak{p},\ \mathfrak{p} \in \mathrm{Ass}(M)\} = \sup\{\dim A/\mathfrak{p} : \mathfrak{p} \in \mathrm{Supp}(M)\}.$$

they have the same minimal elements.

If $M=0$, we take $\dim M = -1$.

Given M, we consider the following quantities:

- $\dim(M)$ also called Krull dimension of M.
- $d(M)$: degree of Hilbert-Samuel poly of M
- $\delta(M)$: called Chevalley dimension: the smallest $n \geq 0$ s.t. $\exists a_1, \dots, a_n$ s.t.
$$\ell(M/(a_1, \dots, a_n)M) < \infty.$$
(take $\delta(M) = -1$ if $M = 0$)

Thm: $\dim(M) = d(M) = \delta(M)$.

Pf: Exercise.

Def. A system of parameters for M is a set $\{a_1, \dots, a_n\}$ of elements of $\mathfrak{m}$ such that $M/(a_1, \dots, a_n)M$ has finite length.

By main thm, system of parameters for M always exist.

Lemma: if $(a_1, \dots, a_n)$ is system of parameters for M, iff for any i,

$$\dim M/(a_1, \dots, a_i)M = n - i, \quad \text{where } n = \dim(M).$$

Pf: $\Rightarrow$. by induction, it suffices to prove that $\dim M/a_1M = n-1$.

First, $\dim(M/a_1M) \geq n-1$: this is because $\delta(M) \leq \delta(M/a_1M) + 1$.
(Similar to the case of noeth local rings).

i.e. mod. by any element, the dim. decreases at most by 1.

Now, $\dim(M/(a_1,\dots,a_n)M)=0$ (because it has finite length), we must have at each step, we have equality: $\dim(M/(a_1,\dots,a_i)M)=n-i$.

$\Leftarrow$ clear. $\square$

§. Regular sequences

Let A be a ring, M = A-module, let $a_1,\dots,a_n\in A$.

We say it is an M-sequence, or regular sequence for M, if

- $\forall i=1,\dots,n$, a_i is not a zero-divisor of $M/(a_1,\dots,a_{i-1})M$.
- and $M/(a_1,\dots,a_n)M\neq 0$.

Ex: $A=k[X_1,\dots,X_n]$. then $X_1,\dots,X_n$ is an A-sequence.

Ex: $A=k[X_1,X_2,X_3,X_4]/(X_1X_2-X_3X_4)$.

Then X_1, X_2, X_3+X_4 is regular sequence, because:

$A/X_1\cong\left(k[X_3,X_4]/(X_3X_4)\right)[X_2]$, it suffices to check that X_3+X_4 is regular for $k[X_3,X_4]/(X_3X_4)$.

if $(X_3+X_4)\cdot f=X_3X_4\cdot g$ in $k[X_3,X_4]$, then $X_3X_4\mid f$.

But X_1, X_2, X_3 is not a regular sequence. $\square$

Attention: A permutation of a regular sequence need not be regular sequence.

e.g. $A=k[X,Y,Z]$. Then $X, Y(1-X), Z(1-X)$ is regular sequence.

but $Y(1-X), Z(1-X), X$ is not, because $Z(1-X)$ is zero-divisor of $A/Y(1-X)$.

However, we have the following result.

Thm: let M = Noeth. A-mod, assume $a_1,\dots,a_n\in \mathrm{Jac}(A)$.

Then any permutation of the a_i is also a regular sequence.

Pf: Since S_n is generated by transposition, it suffices to treat the transpositions.

In fact, it suffices to treat the case $n=2$: if a_1, a_2 is regular sequence for M, then a_2, a_1 is also regular sequence.

(i) first show a_2 is M-regular:

Consider $0 \to K \to M \xrightarrow{\times a_2} M \to M/a_2M \to 0$

let $z \in K$. then $a_2 \cdot z = 0$., Since a_2 is non-zero-divisor of M/a_1M. we have. $z \in a_1 M$.

write. $z = a_1 \cdot z'$. then. $a_1 a_2 z' = 0$.

Since a_1 is M-regular, $a_2 z' = 0$, ie $z' \in K$. $\Rightarrow K = a_1 K$.

By assumption, M is Noeth, so K is. f.g. By. NAK, $K = 0$. ie. a_2 is M-regular.

(ii). Show that a_1 is M/a_2M-regular,

say $\bar{z} \in M/a_2M$. s.t. $a_1 \cdot \bar{z} = 0$. ie. $a_1 z \in a_2 M$.

write $a_1 \cdot z = a_2 \cdot z'$. Since a_2 is M/a_1M-regular, we get $z' \in a_1 M$.

$\Rightarrow a_1 z = a_2 z' = a_2 \cdot a_1 z''$. for some $z'' \in M$.

Since a_1 is M-regular $\Rightarrow z = a_2 z''$. ie. $\bar{z} = 0$. □

Lem: let A be a ring, M = A-mod. $(a_1, \dots, a_n)$. M-sequence.

Then an exact sequence $N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} M \to 0$. induces an exact sequence

(*) $N_2/(a)N_2 \xrightarrow{\bar{\varphi}_2} N_1/(a)N_1 \xrightarrow{\bar{\varphi}_1} N_0/(a)N_0 \xrightarrow{\bar{\varphi}_0} M/(a)M \to 0$.

Pf: By induction, it is enough to consider the case $n=1$. Write $a = a_1$

the sequence. is obtained by tensoring with A/a, and it suffices to check the exactness at N_1/aN_1. If $\bar{\varphi}_1(\bar{y}) = 0$, then $\varphi_1(y) = az$ for some $z \in N_0$ and. $a\,\varphi_0(z) = 0$

Since a is M-regular, we get $\varphi_0(z) = 0$., hence there is $y' \in N_1$ with $z = \varphi_1(y')$.

This implies $\varphi_1(y - ay') = 0$. So. $y - ay' \in \varphi_2(N_2)$. and $\bar{y} \in \bar{\varphi}_2(N_2)$. as desired. □

Next result generalizes the above. Lem, but requires a stronger hypothesis.

Prop: let A be a ring and.

$N_\cdot$: $\to N_m \to N_{m-1} \to \dots \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} N_{-1} \to 0$

an exact sequence of A-modules. If $(a_1, \dots, a_n)$ is N_i-regular sequence. $\forall i$.

Then $N. \otimes_A A/(a)$ is again exact.

Pf: We may assume $n=1$.

Note that since a_1 is regular for N_i, it is also regular for $\mathrm{Im}(\varphi_{i+1}) \subseteq N_i$.

So we may apply Lemma to. $N_{i+3} \to N_{i+2} \to N_{i+1} \to \mathrm{Im}(\varphi_{i+1}) \to 0$ □

Next, we will show that any two regular sequences for M have the same length.

Lemma: let M, N be A-modules. Set $\mathfrak{a} = \mathrm{Ann}(N)$.

(i) if $\mathfrak{a}$ contains an M-regular element, then $\mathrm{Hom}_A(N, M) = 0$.

(ii) Conversely, if A is noeth and M, N are finite generated, then $\mathrm{Hom}_A(N, M) = 0$ implies that $\mathfrak{a}$ contains an M-regular element.

Pf: (i) Clear: if $\varphi: N \to M$, $\underset{0}{\underset{\shortparallel}{\varphi(an)}} = a \cdot \varphi(n)$, for $a \in \mathfrak{a}$, $n \in N$.

but a is regular for M. So $\varphi(n) = 0$.

(ii) Let's first look at a special case $N = A/\mathfrak{a}$, ie $\mathrm{Hom}_A(A/\mathfrak{a}, M) = 0$ implies $\mathfrak{a}$ contains an M-regular element.

Suppose not, then $\mathfrak{a}$ is contained in $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$ hence $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p}$.

← finite set.

but by definition, $\exists\, z \in M$ s.t. $\mathfrak{p} = \mathrm{Ann}(z)$. So $\exists$ embedding $A/\mathfrak{p} \hookrightarrow M$.

$\Rightarrow$ composition $A/\mathfrak{a} \twoheadrightarrow A/\mathfrak{p} \hookrightarrow M$ is a nonzero element in $\mathrm{Hom}_A(A/\mathfrak{a}, M)$, contradiction

Now we look at the general case.

Suppose not, ie $\mathfrak{a}$ consists of zero-divisors of M.

Since A is noeth and M is finite, $\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$ $\Rightarrow \mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p}$ as above.

By assumption, $\mathfrak{p} \in \mathrm{Supp}(N) = V(\mathrm{Ann}(N))$. ie $N_\mathfrak{p} \neq 0$.

We claim that $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$,

Note that $N_\mathfrak{p} \otimes_{A_\mathfrak{p}} k_\mathfrak{p}$ is isom to finite direct sum of $k_\mathfrak{p}$, So $\exists\, N_\mathfrak{p} \otimes_A k_\mathfrak{p} \twoheadrightarrow k_\mathfrak{p}$,

it suffices to show $\mathrm{Hom}_{A_\mathfrak{p}}(k_\mathfrak{p}, M_\mathfrak{p}) \neq 0$, but the existence of embedding

$$A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \hookrightarrow M_\mathfrak{p}$$

equivalent to $\mathfrak{p}A_\mathfrak{p} \in \mathrm{Ass}(M_\mathfrak{p})$; $\Leftrightarrow \mathfrak{p} \in \mathrm{Ass}(M)$, which is true.

→ See Lecture 4-1: $\mathrm{Ass}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(M), \mathfrak{p} \cap S = \emptyset\}$

Recall that we have an isom (for N f.g. A-mod)

$$S^{-1}\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Which implies that $\mathrm{Hom}_A(N, M) \neq 0$.

See Matsumura, p.52. Thm 7.11, + Cor.

□

Prop. Let M, N be A-modules, $a_1, \dots, a_n$ be M-sequence, and assume $a_i \in \mathrm{Ann}(N)$.

Then $\mathrm{Ext}_A^n(N, M) \simeq \mathrm{Hom}_A(N, M/(a_1, \dots, a_n)M)$.

Pf: The exact sequence $0 \to M \to M \to M/a_1M \to 0$ induces

$$\mathrm{Ext}_A^{n-1}(N, M) \to \mathrm{Ext}_A^{n-1}(N, M/a_1M) \xrightarrow{\delta} \mathrm{Ext}_A^n(N, M) \xrightarrow{a_1} \mathrm{Ext}_A^n(N, M).$$

The map $\times a_1 : \mathrm{Ext}_A^n(N, M) \to \mathrm{Ext}_A^n(N, M)$ coincides the one induced functorially from $N \xrightarrow{a_1} N$

So it is identically zero. ie δ is surjective.

On the other hand, by induction, $\mathrm{Ext}_A^{n-1}(N, M) \simeq \mathrm{Hom}_A(N, M/(a_1, \dots, a_{n-1})M)$

$= 0$ by Lemma (i). because $a_n \in \mathrm{Ann}(N)$ is a regular element for $M/(a_1, \dots, a_{n-1})M$.

$\Rightarrow \delta$ is an isomorphism and $\mathrm{Ext}_A^{n-1}(N, M/a_1M) \simeq \mathrm{Ext}_A^n(N, M)$.

Continue this argument gives the result. $\square$

Thm: Let A = noeth, M = f.g. A-mod, $\mathfrak{a} \subseteq A$ ideal s.t. $\mathfrak{a}M \neq M$.

Then any two maximal M-sequences contained in $\mathfrak{a}$ have the same length, and equal to $\min\{ n \mid \mathrm{Ext}_A^n(A/\mathfrak{a}, M) \neq 0 \}$.

Pf: Let $\{a_1, \dots, a_n\}$ be an M-sequence contained in $\mathfrak{a}$, then take $N = A/\mathfrak{a}$,

$$\mathrm{Ext}_A^n(A/\mathfrak{a}, M) \simeq \mathrm{Hom}_A(A/\mathfrak{a}, M/(a_1, \dots, a_n)M).$$

By Lemma, (Since A is noeth. and M, N are f.g). we have

$\mathrm{Hom}_A(A/\mathfrak{a}, M/(a_1, \dots, a_n)M) = 0$ iff $\exists\, a_{n+1} \in \mathfrak{a}$ an regular element for $M/(a_1 \dots a_n)M$.

iff $(a_1, \dots, a_n)$ is not "maximal". $\square$

Def: With the above conditions, the common length of maximal M-seq. in $\mathfrak{a}$ is called the grade of $\mathfrak{a}$ on M. denoted by

$$\mathrm{depth}(\mathfrak{a}, M).$$

If $\mathfrak{a}M = M$, we make the convention that $\operatorname{depth}(\mathfrak{a}, M) = +\infty$.

This is equiv. to $\operatorname{Ext}_A^i(A/\mathfrak{a}, M) = 0$ for all i.

It is consistent: when $\mathfrak{a}M = M$, we have $\operatorname{Ext}_A^i(A/\mathfrak{a}, M) = 0$, because

$$\operatorname{Supp} \operatorname{Ext}_A^i(A/\mathfrak{a}, M) \subseteq \operatorname{Supp}(A/\mathfrak{a}) \cap \operatorname{Supp}(M) = V(\mathfrak{a}) \cap \{\mathfrak{p} \mid M_\mathfrak{p} \neq 0\} = \phi.$$

as $\mathfrak{a}M_\mathfrak{p} = M_\mathfrak{p} \Rightarrow M_\mathfrak{p} = 0$ or $\mathfrak{a} \not\subseteq \mathfrak{p}$.

Prop. Let A = noeth. M = finite A-mod., $\underline{x}$ = M-sequence in $\mathfrak{a}$ (where $\mathfrak{a} \subseteq A$ ideal, $\mathfrak{a}M \neq M$).

Then $\operatorname{depth}(\mathfrak{a}, M/\underline{x}M) = \operatorname{grade}(\mathfrak{a}, M) - r$. (here r = length of $\underline{x}$).

Pf: $\operatorname{depth}(\mathfrak{a}, M) = \min\{n \mid \operatorname{Ext}_A^n(A/\mathfrak{a}, M) \neq 0\}$.

$\operatorname{depth}(\mathfrak{a}, M/\underline{x}M) = \min\{n' \mid \operatorname{Ext}_A^{n'}(A/\mathfrak{a}, M/\underline{x}M) \neq 0\}$.

we conclude using $\operatorname{Ext}_A^{n+r}(A/\mathfrak{a}, M) \simeq \operatorname{Ext}_A^n(A/\mathfrak{a}, M/\underline{x}M)$ $\square$

Special case: $(A, \mathfrak{m})$ is noeth local ring, then

$$\operatorname{depth} M := \operatorname{depth}(\mathfrak{m}, M).$$

called the depth of M.

Thm: Let $(A, \mathfrak{m}, k)$ be a noeth local ring, M: finite A-mod. (non-zero).

Then $\operatorname{depth}(M) = \min\{i : \operatorname{Ext}_A^i(k, M) \neq 0\}$.

Prop: With the above notation, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finite A-mod.

Then (a) $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(M'), \operatorname{depth}(M'')\}$.

(b) $\operatorname{depth}(M') \geq \min\{\operatorname{depth}(M), \operatorname{depth}(M'') + 1\}$

(c). $\operatorname{depth}(M'') \geq \min\{\operatorname{depth}(M') - 1, \operatorname{depth}(M)\}$.

Pf: Exercise.

Cor: Assume, in $0 \to M' \to M \to M'' \to 0$, that

- $\operatorname{depth} M \geq \operatorname{depth} M'' + 1$.

then $\operatorname{depth} M' = \operatorname{depth} M'' + 1$.

Pf: by (b). we have. depth $M' \geq$ depth $M''+1$. (together with assumption).

If we had. depth $M' >$ depth $M''+1$, then.

$$\min\{\text{depth } M'-1, \text{ depth } M\} > \text{depth } M''$$

which contradicts. (c). $\square$

Prop: depth $(M) \leq \dim(M)$.

Pf: if a is M-regular, then $\dim(M/aM) = \dim(M) - 1$ $\Rightarrow$ result. $\square$

This bound can be refined as follows:

Prop: $\text{depth}(M) \leq \dim A/\mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(M)$.

Rk: $\dim M = \dim A/_{\text{Ann}(M)}$.
$= \max\{\dim A/_{\mathfrak{p}} : \mathfrak{p} \in \text{Ass}(M)\}$

Pf: we use induction on depth (M). If depth $(M)=0$, obvious.

If depth$(M) > 0$, then $\exists\, x \in \mathfrak{m}$. which is M-regular element..

Fix $\mathfrak{p} \in \text{Ass}(M)$. and need to prove $\text{depth}(M) \leq \dim(A/\mathfrak{p})$

by induction. we have. $\forall\, q \in \text{Ass}(M/xM)$. $\text{depth}(M/xM) \leq \dim A/q$. $(*)$

So it suffices to prove the following claim: for (the fixed) $\mathfrak{p} \in \text{Ass}(M)$. $\exists\, q \in \text{Ass}(M/xM)$, $\mathfrak{p} \subsetneq q$.

(then $\Rightarrow$ $\text{depth}(M) = \text{depth}(M/xM) + 1 \underset{(*)}{\leq} \dim A/q + 1 \leq \dim A/\mathfrak{p}$.)

Now prove the claim: it suffices to prove that $\mathfrak{p}$ consists of zero-divisors of M/xM.

because this implies $\mathfrak{p} \subseteq \bigcup_{q \in \text{Ass}(M/xM)} q$, hence $\mathfrak{p} \subseteq q$. for some $q \in \text{Ass}(M/xM)$

we automatically have $\mathfrak{p} \neq q$, because: $\mathfrak{p} \notin \text{Supp}(M/xM)$: $M_{\mathfrak{p}}/xM_{\mathfrak{p}} = 0$, $(x \notin \mathfrak{p})$

Since $\mathfrak{p} \in \text{Ass}(M)$. there exist $m \in M$. s.t. $\mathfrak{p} = \text{Ann}(m)$.

If $m \in xM$, say $m = xm'$, then since x is M-regular, we have

$$\text{Ann}(m') = \text{Ann}(m) = \mathfrak{p}.$$

So we may replace m'., If $m' \in xM$, then. further get m'', with $\text{Ann}(m'') = \mathfrak{p}$.

Since $\bigcap_{n \geq 0} x^n M = 0$ by Krull intersection theorem. we finally obtain $m \in M \setminus xM$. with. $\text{Ann}(m) = \mathfrak{p}$.

then we deduce $\bar{m} \in M/xM$ is nonzero, and $\mathfrak{p} \subseteq \text{Ann}_A(\bar{m})$. proving the claim. $\square$

§ Cohen-Macaulay rings and modules.

Def: let A be a noeth local ring. A finite A-mod M is a Cohen-Macaulay module. if

$$\operatorname{depth} M = \dim M.$$

If A is itself a CM-mod., it is called a Cohen-Macaulay ring.

A max CM mod is a CM mod such that $\dim M = \dim R$.

Def: if A is arbitrary noeth ring, then M is CM mod if M_m is CM for all max ideal m.

(Convention: zero mod. is CM module.)

Thm!: let (A, m) be a noeth local ring, $M \neq 0$. a CM A-mod.

(i) $\dim A/p = \operatorname{depth}(M)$ for all $p \in \operatorname{Ass}(M)$. (In particular, no embedded prime).

(ii). $\operatorname{depth}(\mathfrak{a}, M) = \dim M - \dim M/\mathfrak{a}M$ for all $\mathfrak{a} \subseteq m$.

(iii) $\underline{x} = (x_1, \dots, x_r)$ is an M-sequence iff $\dim M/(\underline{x})M = \dim M - r$.

(iv) $\underline{x}$ is an M-sequence iff it is part of a system of parameters of M.

Pf: (i). Always have $\operatorname{depth}(M) \leq \dim(A/p)$. $\forall p \in \operatorname{Ass}(M)$.

but $\dim(A/p) \leq (\dim M)$. because. $\forall p \in \operatorname{Ass}(M)$. $\exists A/p \hookrightarrow M$.

M is CM. $\Rightarrow$ $\operatorname{depth}(M) = \dim(M)$ $\Rightarrow$ the result.

(ii) • If $\operatorname{depth}(\mathfrak{a}, M) = 0$, we need to prove. $\dim M/\mathfrak{a}M = \dim M$. obviously. $\leq$. holds..

By definition, $\dim M/\mathfrak{a}M. = \dim A/\operatorname{Ann}(M/\mathfrak{a}M)$ (*)

Since $\operatorname{depth}(\mathfrak{a}, M) = 0$. $\exists p \in \operatorname{Ass}(M)$ with $\mathfrak{a} \subseteq p$.

Recall: $\sqrt{\operatorname{Ann}(M/\mathfrak{a}M)} = \sqrt{\operatorname{Ann}(M) + \mathfrak{a}}$, $\subseteq p$ (because $\operatorname{Ass}(M) \subseteq \operatorname{Supp}(M) = V(\operatorname{Ann}(M))$).

In particular, $A/\operatorname{Ann}(M/\mathfrak{a}M) \twoheadrightarrow A/p$. So by (*). $\dim M/\mathfrak{a}M \geq \dim A/p \underset{(i)}{=} \dim M$.

• If $\operatorname{depth}(\mathfrak{a}, M) > 0$, then choose $x \in \mathfrak{a}$. regular on M.

One has $\operatorname{depth}(\mathfrak{a}, M/xM) = \operatorname{depth}(\mathfrak{a}, M) - 1$.

and $\dim(M/xM) = \dim(M) - 1$.

Noting that $x \in \mathfrak{a}$ implies $M/\mathfrak{a}M$ remains the same when replace. M by M/xM.

$\Rightarrow$ result by induction. $\square$

(iii) If $\underline{x}$ is a regular sequence on M, then clearly $\dim M/(\underline{x}) = \dim(M) - r$.

Conversely, if $\dim M/(\underline{x}) = \dim(M) - r$, then $\dim M/x_1 = \dim M - 1$.

We first prove that this implies that x_1 is M-regular.

By (ii), letting $\mathfrak{a} = (x_1)$, we have $\operatorname{depth}(\mathfrak{a}, M) = 1$, ie $\exists\, y \in \mathfrak{a}$, which is M-regular.

writing $y = ax_1$, this means that $M \xrightarrow{\times y} M$ is injective.

but this map is equal to the composition $M \xrightarrow{x_1} M \xrightarrow{a} M$. So x_1 is also regular.

Theorem (a) below shows that M/x_1M is also CM, So we conclude by induction.

(iv) both the two statements are equivalent to: $\dim M/\underline{x}M = \dim M - r$ $\square$

Thm2: Let A = noeth local ring, M = finite A-mod.

(a) Suppose that $\underline{x}$ is an M-sequence. Then M is CM-mod. iff $M/\underline{x}M$ is also CM (over A or $A/\underline{x}$).

(b) Suppose M is CM, then $S^{-1}M$ is also CM for any $S \subseteq A$ multiplicative subset.

In particular, $M_\mathfrak{p}$ is CM for any $\mathfrak{p} \in \operatorname{Spec} A$.

Pf: (a). the result is clear, because $\dim M/\underline{x} = \dim M - r$. if $\underline{x}$ has length r.

and $\operatorname{depth} M/\underline{x}M = \operatorname{depth} M - r$.

(b). let q be a max ideal of $S^{-1}A$, then it is equal to $S^{-1}\mathfrak{p}$, for some $\mathfrak{p} \in \operatorname{Spec} A$.

and $(S^{-1}A)_q \simeq (S^{-1}A)_{S^{-1}\mathfrak{p}} \simeq A_\mathfrak{p}$. $\mathfrak{p} \cap S = \emptyset$

$(S^{-1}M)_q \simeq M_\mathfrak{p}$.

So we are reduced prove $M_\mathfrak{p}$ is CM for $\mathfrak{p} \in \operatorname{Spec} A$.

We may assume $M_\mathfrak{p} \neq 0$, and do induction on $\operatorname{depth} M_\mathfrak{p}$.

If $\operatorname{depth} M_\mathfrak{p} = 0$, then $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p})$, so $\mathfrak{p} \in \operatorname{Ass}(M)$. (because $\operatorname{Ass}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}(M), S \cap \mathfrak{p} = \emptyset\}$)

Since M is CM, $\mathfrak{p}$ is minimal in $\operatorname{Ass}(M)$, So also minimal in $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$,

Recall that $\operatorname{Ann}(S^{-1}M) = S^{-1}(\operatorname{Ann}(M))$.

$\Rightarrow$ $\mathfrak{p}A_\mathfrak{p}$ is minimal prime in $\operatorname{Supp}(M_\mathfrak{p})$, ie $\dim A_\mathfrak{p}/\operatorname{Ann}(M_\mathfrak{p}) = 0$, $\Rightarrow \dim M_\mathfrak{p} = 0$.

If $\operatorname{depth} M_\mathfrak{p} > 0$, the same argument shows that $\mathfrak{p}$ can not be contained in any $q \in \operatorname{Ass}(M)$.

So $\mathfrak{p}$ contains an M-regular element x, and by induction $(M/xM)_\mathfrak{p}$ is CM.

ie $M_\mathfrak{p}/xM_\mathfrak{p}$ is CM. hence $M_\mathfrak{p}$ is itself CM. by (i). $\square$

Def: for $\mathfrak{a} \subsetneq A$ ideal; $ht(\mathfrak{a}) := \min\{ht(p),\ p \in V(\mathfrak{a})\}$.

Thm 3. Let A = CM local ring. then for any $\mathfrak{a} \subseteq m$.

$$ht(\mathfrak{a}) + \dim A/\mathfrak{a} = \dim A.$$

Pf: By definition, $ht(\mathfrak{a}) = \min\{\dim A_p,\ p \in V(\mathfrak{a})\}$

By Lemma, $\operatorname{depth}(\mathfrak{a}, A) = \min\{\operatorname{depth} A_p,\ p \in V(\mathfrak{a})\}$.

Since A_p is CM ring, we have $\dim A_p = \operatorname{depth} A_p$, $\forall p$

hence. $ht(\mathfrak{a}) = \operatorname{depth}(\mathfrak{a}, A)$.

So we conclude using Thm 1, (ii).

Lemma: $\operatorname{depth}(\mathfrak{a}, M) = \min\{\operatorname{depth} M_p,\ p \in V(\mathfrak{a})\}$.

Pf: clearly, by definition. $\operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{depth}(p, M) \quad \forall p \in V(\mathfrak{a})$

Moreover, we have. $\operatorname{grade}(p, M) \leq \operatorname{depth} M_p$, this is because.

$$M_p \simeq A_p \otimes_A M, \quad \text{and } A_p \text{ is flat over } A.$$

So if $x: M \to M$ is injective, then $\frac{x}{1}: M_p \to M_p$ is also injective.

hence an M-sequence $x_1, \dots, x_r \in p$. gives. $\frac{x_1}{1}, \dots, \frac{x_r}{1} \in pA_p$. which is M_p-sequence.

This implies: $\operatorname{depth}(\mathfrak{a}, M) \leq \min\{\operatorname{depth} M_p,\ p \in V(\mathfrak{a})\}$.

Left to prove that. $\exists\, p \in V(\mathfrak{a})$ s.t. $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{depth} M_p$. $\quad (*)$

let $x_1, \dots, x_r \in \mathfrak{a}$. be a max M-sequence, then. $\mathfrak{a}$ consists of zero-divisors for $M/\underline{x}M$, so $\exists\, p \in \operatorname{Ass}(M/\underline{x}M)$. s.t. $\mathfrak{a} \subseteq p$.

It suffices to show. equality $(*)$ for this p, ie. show $\operatorname{depth} M_p = r$.

but since $pA_p \in \operatorname{Ass}(M/\underline{x}M)_p$, $\frac{x_1}{1}, \dots, \frac{x_r}{1} \in pA_p$ is a max M_p-sequence.

$\Rightarrow$ OK $\quad \square$

We say an ideal $\mathfrak{a} \subseteq A$ is unmixed if $\mathfrak{a}$ has no embedded prime ideal.

Thm 4. A noeth ring A is Cohen-Macaulay. if and only if every ideal $\mathfrak{a}$. generated by $ht(\mathfrak{a})$ elements is unmixed.

Pf: $\Rightarrow$ Suppose that A is CM, and $\mathfrak{a} = (a_1, \dots, a_r)$ is an ideal of height r.

We assume that $\mathfrak{p}$ is an embedded prime. ideal of $\mathfrak{a}$. for a contradiction.

Localizing at $\mathfrak{p}$. we may assume A is a CM local ring.

By. Thm 3, $\dim M/\mathfrak{a} = \dim M - r$.

Then by Thm 1. (iii). $a_1, \dots, a_r$ is an A-sequence.,

hence $A/\mathfrak{a}$ is also a CM local ring. by Thm 2.

But then $\mathfrak{a}$ does not have. embedded ass prime, a contradiction,

$\Leftarrow$ Suppose that the unmixedness condition holds for A. (again we may assume A is local)

If $\mathfrak{a} \subsetneq A$, with. $ht(\mathfrak{a}) = r$., then we can choose $a_1, \dots, a_r \in \mathfrak{a}$. s.t.

$$ht(a_1, \dots, a_i) = i. \text{ for } 1 \le i \le r.$$

by the lemma below.

Clearly. a_{i+1} is not contained in a minimal prime ideal of $(x_1, \dots, x_i)$.

So by. unmixedness, a_{i+1} is an $A/(a_1, \dots, a_i)$-regular element.

So $a_1, \dots, a_r$ is an A-sequence, in particular, $\text{grade}(\mathfrak{a}) = \text{height}(\mathfrak{a})$

Specialize to $\mathfrak{a} = \mathfrak{m}$, gives. $\underset{\text{depth}(\mathfrak{m}, A)}{\overset{\shortparallel}{\text{depth}.A}} = \underset{\text{height}(\mathfrak{m})}{\overset{\shortparallel}{\dim A}}$. , so A is CM. $\square$

Lemma: Let A = noeth ring. $\mathfrak{a} \subsetneq A$ ideal.

If $ht(\mathfrak{a}) = r$, then $\exists (a_1, \dots, a_r) \in \mathfrak{a}$. such that

$$ht(a_1, \dots, a_i) = i, \quad \forall\, 1 \le i \le r.$$

Pf. may assume $r \ge 1$.

(i). let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime ideals of A.

if $a_1 \notin \bigcup_j \mathfrak{p}_j$, then (a_1) has height ≥ 1, and by Krull's principal ideal thm.

$ht(a_1) = 1$.

Since $ht(\mathfrak{a}) \geq 1$, $\mathfrak{a} \not\subseteq \bigcup_j P_j$, so we may take such element a_1.

(ii). By induction, assume $a_1, \ldots, a_{i-1}$ already constructed, and let $P_1, \ldots, P_s$ be the minimal prime ideals which contain $(a_1, \ldots, a_{i-1})$, and have height exactly $i-1$. Since $ht(\mathfrak{a}) = r \geq i > i-1$, we have $\mathfrak{a} \not\subseteq \bigcup_j P_j$.
Choose $a_i \in \mathfrak{a} \setminus \bigcup_j P_j$, and let q be any prime containing $(a_1, \ldots, a_i)$.
Then q contains some minimal prime ideal p of $(a_1, \ldots, a_{i-1})$.

- If $p = P_j$ for some j, then $ht(q) \geq ht(p) + 1 = i$. (hence equality by Krull's thm).
- If $p \neq P_j$, $\forall j$, then $ht(p) \geq i$, so $ht(q) \geq i$. (hence equality).

$\Rightarrow$ Conclude by induction.

Rk: Macaulay proved that an ideal $\mathfrak{a} = (a_1, \ldots, a_r)$ of height r is unmixed when A = poly ring over a field, and Cohen proved this for regular local ring.

$\rightsquigarrow$ name "Cohen-Macaulay".

Thm: Let (A, m) be a noeth local ring and $\hat{A}$ its m-adic completion.
Then (i) $\operatorname{depth}(A) = \operatorname{depth}(\hat{A})$

(ii). A is CM $\Leftrightarrow$ $\hat{A}$ is CM.

Pf (i). This follows from $\operatorname{Ext}^i_A(A/m, A) \otimes \hat{A} \simeq \operatorname{Ext}^i_{\hat{A}}(\hat{A}/m\hat{A}, \hat{A})$, $\forall i$. (flat base change)

(ii). follows from (i) and the fact $\dim(A) = \dim(\hat{A})$. $\square$

§. Homological dimension.

Let A be a ring, M = A-module.

Def: (1) The projective dimension $pd(M)$ is the minimum integer n. (if it exists). such that there exists a resolution of M by proj modules

$$0 \to P_n \to \cdots\cdots \to P_1 \to P_0 \to M \to 0.$$

(2) The injective dimension $id(M)$ is the minimum integer n (if it exists) such that there exists a resolution of M. by injective modules

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0.$$

(3). The flat dimension $fd(M)$ is the minimum integer n. (if it exists) such that. there exists a resolution of M by projective modules

$$0 \to F_n \to F_{n-1} \to \cdots\cdots \to F_0 \to M \to 0.$$

If no finite resolution exists, we set $pd(M)$, $id(M)$, or $fd(M)$ equal to ∞.

Rk: Proj modules are flat, so $fd(M) \leq pd(M)$.

In general, we need not have equality: over $A = \mathbb{Z}$, $fd(\mathbb{Q}) = 0$, but $pd(\mathbb{Q}) = 1$.

Lemma: TFAE:

(i) $pd(M) \leq d$

(ii) $Ext^n_A(M,N) = 0$ for all $n > d$. and all A-mod. N.

(iii) $Ext^{d+1}_A(M,N) = 0$. for all A-mod N.

(iv) If $0 \to M_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0$ is any resolution with P_i projective. then the syzygy M_d is also projective.

Pf: (i) $\Rightarrow$ (ii). by definition of Ext^n.

(ii) $\Rightarrow$ (iii) trivial.

(iii) $\Rightarrow$ (iv). check that $Ext^1_A(M_d, N) \simeq Ext^{d+1}_A(M,N) = 0$. $\forall N$, So M_d is proj.

(iv) $\Rightarrow$ (i) clear. $\square$

Similarly, we have the following results.

Lemma: TFAE:

(i). $id(N) \leq d$

(ii). $Ext_A^n(M,N)=0$ for all $n>d$, and all A-mod M.

(iii) $Ext_A^{d+1}(M,N)=0$. for all M.

(iv) If $0 \to N \to I^0 \to I^1 \to \cdots \to I^{d-1} \to N^d \to 0$. is any resolution with I^i injective. then N^d is also injective.

Lemma: TFAE:

(i). $fd(M) \leq d$.

(ii). $Tor_n^A(M,N)=0$. $\forall n>d$, $\forall N$.

(iii) $Tor_{d+1}^A(M,N)=0$, $\forall N$.

(iv). If $0 \to M_d \to F_{d-1} \to \cdots \to F_0 \to M \to 0$. is any resolution with F_i flat, then M_d is also flat.

Ex: (1) As $\mathbb{Z}$-mod:

- $M=\mathbb{Z}$, $pd=0$, $id=1$, $fd=0$.
- $M=\mathbb{Z}/n\mathbb{Z}$, $pd=1$. $id=1$, $fd=1$.
- $M=\mathbb{Q}$, $pd=1$. $id=0$, $fd=0$.

(2). As $\mathbb{Z}/n\mathbb{Z}$-modules:

- $M=\mathbb{Z}/n\mathbb{Z}$: $pd=0$, $id=0$. $fd=0$. (ie M= free A-mod.)
- $M=\mathbb{Z}/d\mathbb{Z}$. with $n=dd'$, $d'>1$, then $pd=\infty$, $id=1$, $fd=\infty$.

b/c. $\cdots \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times d'} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times d} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times d'} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\times d} \mathbb{Z}/n\mathbb{Z} \to M \to 0$. has infinite length.

showing that $Ext^n_{\mathbb{Z}/n\mathbb{Z}}(M, \mathbb{Z}/n\mathbb{Z}) \neq 0$. $\forall n \geq 0$.

Lemma: N is injective A-module iff $Ext_A^1(A/\mathfrak{a}, N)=0$. for all ideal $\mathfrak{a} \subseteq A$.

Pf: use Baer's criterion.

Thm: The following numbers are the same:

(1). $\sup\{pd(M), M \in A\text{-mod}\}$. (1)' $\sup\{pd(A/\mathfrak{a}) : \mathfrak{a} \subseteq A \text{ ideal}\}$.

(2) $\sup\{id(N) : N \in A\text{-mod}\}$

If A is noeth, they also equal to:

(3). $\sup\{fd(M) : \text{finite } M \in A\text{-mod}\}$ (Note: also $= \sup\{fd(M) : M \in A\text{-mod}\}$.).

Pf: (1) = (2), because they both equal to $\sup\{d : Ext^d(M,N) \neq 0 \text{ for some } A\text{-mod } M, N\}$.

(1) $\geq$ (1)' trivial.

(1)' $\geq$ (2) Assume (1)' is finite, $= d$, For $N \in A$-mod, consider a resolution.

$$0 \to N \to I^0 \to \cdots \to I^{d-1} \to N' \to 0, \text{ with } I^i \text{ injective}.$$

We have $0 = Ext_A^{d+1}(A/\mathfrak{a}, N) = Ext_A^1(A/\mathfrak{a}, N')$, So by Lemma, N' is injective, $\Rightarrow id(N) \leq d$,

We may assume (2) $= d < \infty$ and prove that (2) $\leq$ (3). ie. $pd(M) \leq d$.

Since A is noeth, there is a resolution

$$0 \to M_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0.$$

in which P_i are f.g. proj. mod. and M_d is f.p.

Since $fd(M) \leq d$, M_d is flat A-mod,

but this implies that M_d is projective, so $pd(M) \leq d$. $\square$

§ Depth and projective dimension.

From now on. assume (A, m, k) is noeth local ring.

Then we may take P_i to be f.g. A-mod, and thus free A-mod.

Then we can construct a free resolution of M. as follows.

First choose $x_1, \ldots, x_m$ a minimal system of generators of M.

we know that m is equal to $\dim_k M \otimes_A k$ by NAK.

Set $\beta_0 = m$, let $d_0 : A^{\beta_0} \to M \to 0$. Sending e_i to x_i.

Let $M_1 = Ker(d_0)$, it is also f.g, so we can construct

$$d_1 : A^{\beta_1} \twoheadrightarrow M_1, \text{ where } \beta_1 := \dim_k M_1 \otimes_A k.$$

In this way, we get a minimal free resolution:

$$F_\cdot \quad \to \cdots \to A^{\beta_1} \to A^{\beta_0} \to M \to 0$$

The number $\beta_i(M) = \beta_i$ is called the i-th Betti number of M.

Prop. Let (A, m, k) be a noeth local ring, M = finite A-mod, and

$$F_\cdot : \to F_n \to F_{n-1} \to \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0.$$

a free resolution. Then TFAE:

(i) $F_\cdot$ is minimal

(ii). $d_i(F_i) \subseteq mF_{i-1}$ for all $i \geq 1$ $\quad (\Leftrightarrow d_i \otimes_A k = 0.\ \forall i \geq 1)$

(iii) $\operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^A(M, k) \quad \forall i \geq 0.$

(iv) $\operatorname{rank} F_i = \dim_k \operatorname{Ext}_A^i(A, k). \quad \forall i \geq 0.$

Proof: (i) $\Leftrightarrow$ (ii) by NAK.

(ii) $\Leftrightarrow$ (iii) as $d_i \otimes_A k = 0. \quad \forall i \geq 1$

(ii) $\Leftrightarrow$ (iv). as $\operatorname{Hom}(F_i, k) \to \operatorname{Hom}(F_{i-1}, k)$ is zero map. $\square$

Cor: Let (A, m, k) be noeth local ring, and M = finite A-module.

Then $\beta_i(M) = \dim_k \operatorname{Tor}_i^A(M, k). \quad \forall i \geq 0$ and

$$pd(M) = \sup\{i : \operatorname{Tor}_i^A(M, k) \neq 0\}.$$

<u>Thm</u>: (Auslander-Buchsbaum). Let (A, m, k) be noeth local ring. $M \neq 0$, finite A-mod.

If $pd(M) < \infty$, then

$$pd(M) + \operatorname{depth}(M) = \operatorname{depth}(A)$$

Lemma: Let $\varphi : F \to G$ be a morphism of finite free A-modules

Let M = A-mod. S.t. $m \in \operatorname{Ass}(M)$.

Suppose that $\varphi \otimes_A M$ is injective.

Then (1) $\varphi \otimes_A k$ is also injective.

(2). φ is injective and $\varphi(F)$ is a free direct summand of G.

Pf: (1) Since $m \in \mathrm{Ass}(M)$. $\exists$ embedding. $k \simeq A/m \hookrightarrow M$.

Since F is free, $F \otimes k \hookrightarrow F \otimes M$ is again injective. thus

$$\begin{tikzcd} F \otimes k \arrow[r, "inj"] \arrow[d, "\varphi \otimes k"'] \arrow[dr, phantom, "2"] & F \otimes M \arrow[d, "\varphi \otimes M.\ (\text{inj by. assumption})"] \\ G \otimes k \arrow[r] & G \otimes M. \end{tikzcd}$$

we deduce. $\varphi \otimes k$ is injective.

(2) First prove that $\mathrm{Coker}\,\varphi$ is a finite. free A-mod.

Consider $0 \to \mathrm{Im}\,\varphi \to G \to \mathrm{Coker}\,\varphi \to 0$. $(*)$

Apply $- \otimes_A k$, get. $0 \to \mathrm{Tor}_1^A(\mathrm{Coker}\,\varphi, k) \to \mathrm{Im}\,\varphi \otimes k \xrightarrow{f} G \otimes k \to \mathrm{Coker}\,\varphi \otimes k \to 0$

Note: f is injective, because the composition is injective. (proved in step (1)).

$$F \otimes k \xrightarrow{g} \mathrm{Im}\,\varphi \otimes k \xrightarrow{f} G \otimes k$$

and g is surjective. (in fact g is an isom!)

This implies. $\mathrm{Tor}_1^A(\mathrm{Coker}\,\varphi, k) = 0$, thus. $\mathrm{Coker}\,\varphi$ is finite. flat A-mod.

Since A is noeth local, $\mathrm{Coker}\,\varphi$ is in fact free A-mod.

Thus. the sequence $(*)$ splits and. $\mathrm{Im}\,\varphi$ is a direct summand of G, hence projective.
($\Rightarrow$ free)

Finally prove that φ is injective.

Consider $0 \to \mathrm{Tor}_1^A(\mathrm{Im}\,\varphi, k) \to \mathrm{Ker}\,\varphi \otimes k \to F \otimes k \xrightarrow{g} \mathrm{Im}\,\varphi \otimes k \to 0$

As g is an isom, and $\mathrm{Tor}_1^A(\mathrm{Im}\,\varphi, k) = 0$. (by (2). $\mathrm{Im}\,\varphi$ is free).

we deduce. $\mathrm{Ker}\,\varphi \otimes k = 0$, hence $\mathrm{Ker}\,\varphi = 0$. by. NAK. $\square$

Lemma 2: Let (A, m, k) be a noeth local ring, M = finite A-mod.

If $x \in m$ is A-regular and M-regular, then.

$$\mathrm{pd}_A(M) = \mathrm{pd}_{A/x}(M/xM).$$

Pf: choose a minimal resolution $F_\cdot$ of M. Since. x is A-regular. $F_\cdot \otimes A/x$ is exact.

hence is a minimal free resolution of M/xM over A/xM.

This implies the result. $\square$

Pf of Thm: let first $\operatorname{depth}(A)=0$. By hypothesis, M has a (finite) minimal free resolution

$$F_\bullet : 0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with $n = \operatorname{proj\,dim}(M)$. Assume $n \geq 1$. (for a contradiction).

Since $\operatorname{depth}(A)=0$, we have $\mathfrak{m} \in \operatorname{Ass}(A)$.

But $\varphi_n = \varphi_n \otimes A$ is injective; So by Lemma 1, $\varphi_n \otimes k$ is injective. So that $F_\bullet$ is not a minimal resolution. Contradiction.

Hence $n=0$. We also have $\operatorname{depth} M = 0$. So the result holds in this case.

Now assume $\operatorname{depth}(A) > 0$.

Case 1: Assume $\operatorname{depth}(M)=0$. ($\Rightarrow$ M is not projective A-mod).

Let M_1 be a first syzygy, ie choose a (minimal) surjection $F_0 \to M \to 0$ with M_1 being the kernel: $0 \to M_1 \to F_0 \to M \to 0$.

Recall that, since $\operatorname{depth} F_0 = \operatorname{depth} A \geq \underbrace{\operatorname{depth} M}_{=0} + 1$, we have

$$\operatorname{depth} M_1 = \operatorname{depth} M + 1 = 1.$$

On the other hand, $\operatorname{pd}(M_1) = \operatorname{pd}(M) - 1$, So we are reduced to prove the formula for M_1

Case 2. assume $\operatorname{depth} M > 0$.

Then $\mathfrak{m} \notin \operatorname{Ass}(A)$. and $\mathfrak{m} \notin \operatorname{Ass}(M)$.

So $\mathfrak{m}$ contains an element x which is both A-regular and M-regular.

by Lemma 2. we have $\operatorname{pd}_A(M) = \operatorname{pd}_{A/x}(M/xM)$.

On the other hand, $\operatorname{depth}(A/x) = \operatorname{depth} A - 1$, $\operatorname{depth}_{A/xA}(M/xM) = \operatorname{depth}(M) - 1$

which gives the desired formula by induction on $\operatorname{depth}(A)$. $\square$

The following result shows that "$\operatorname{pd}(M) < +\infty$" is not always satisfied.

Lemma. If A is noeth. local and $\operatorname{depth}(A) = 0$. (ie every element of the max ideal is a zero-divisor in A), then for any finitely generated A-mod M,

$$\text{either } \operatorname{pd}(M) = 0 \quad \text{or} \quad \operatorname{pd}(M) = \infty.$$

Pf: This is a consequence of Auslander-Buchsbaum formula. Below is a direct proof.

If $0 < pd(M) < \infty$. then some syzygy of M is f.g. and $pd(-) = 1$.

So assume there exists M. with $pd(M) = 1$ and let

$$0 \to F_1 \xrightarrow{\varphi} F_0 \to M \to 0.$$

be a min. free resolution. (as A is local), $(\Rightarrow \varphi(F_1) \subseteq mF_0)$.

Since $m \in Ass(A)$, $\exists \underset{\neq 0}{a \in A}$ s.t $m = Ann(a)$; thus. $a \cdot \varphi(F_1) = 0$.

but φ is injective, we deduce. $a \cdot F_1 = 0$, $\Rightarrow a = 0$. because F_1 is free A-mod.

(contradiction). $\square$.

§ Koszul Complex

Let A be a ring, $x_1, \dots, x_r \in A$.

Def: $K_\bullet = K_\bullet(x_1, \dots, x_r)$, the Koszul complex associated to $A, (x_1, \dots, x_r)$, is defined to be:

- $K_0 = A$, $K_1 = \bigoplus_{i=1}^r A e_i$. free of rank r.
- $K_\ell = \Lambda^\ell \left(\bigoplus_{i=1}^r A e_i \right) \simeq \bigoplus_{i_1 < \dots < i_\ell} A e_{i_1} \wedge \dots \wedge e_{i_\ell}$, for $\ell = 1, \dots, r$.
- $K_\ell = 0$ for $\ell > r$.

The differential $d: K_\ell \to K_{\ell-1}$ is defined by:

$$d(e_{i_1} \wedge \dots \wedge e_{i_\ell}) = \sum_{m=1}^{\ell} (-1)^{m-1} x_{i_m} e_{i_1} \wedge \dots \wedge \widehat{e_{i_m}} \wedge \dots \wedge e_{i_\ell}$$

where "$\wedge$" indicates that the term e_{i_m} is removed.

It is direct to check that $d^2 = 0$, i.e. $K_\bullet$ is a complex.

Ex: $n=1$. $0 \to A \xrightarrow{x_1} A \to 0$

$n=2$. then $K_\bullet(x_1, x_2)$ is explicitly

$$\begin{tikzcd}[row sep=small] & & A \arrow[dr, "x_1"] & & \\ 0 \arrow[r] & A \arrow[ur, "-x_2"] \arrow[dr, "x_1"'] & \oplus & A \arrow[r] & 0. \\ & & A \arrow[ur, "x_2"'] & & \end{tikzcd}$$

So $H_0(K_\bullet) \simeq A/(x_1, x_2)$, $H_1(K_\bullet) \simeq \{(a_1, a_2) : x_1 a_1 + x_2 a_2 = 0\} / \{(-x_2 a, x_1 a) : a \in A\}$.

$H_2(K_\bullet) \simeq \{a \in A : x_1 a = x_2 a = 0\}$.

Lemma: $K_\bullet(x_1, \dots, x_r) \cong K_\bullet(x_1) \otimes \dots \otimes K(x_n)$.

Pf: By induction on r. If $r=1$, this is trivial.

If $K = K(x_1, \dots, x_{r-1})$. and $L = K(x_r)$, write.

$L: \quad 0 \to L_1 \to L_0 \to 0$, $L_1 = Af$. and $df = x_r$.

Then $(K\otimes L)_q = K_q\otimes L_0 \oplus K_{q-1}\otimes L_1$ and

$$d(e_{i_1}\wedge\cdots\wedge e_{i_q}\otimes 1) = d(e_{i_1}\wedge\cdots\wedge e_{i_q})\otimes 1$$

$$d(e_{i_1}\wedge\cdots\wedge e_{i_{q-1}}\otimes f) = d(e_{i_1}\wedge\cdots\wedge e_{i_{q-1}})\otimes f + (-1)^{q-1} e_{i_1}\wedge\cdots\wedge e_{i_{q-1}}\otimes x_r$$

Identify $e_{i_1}\wedge\cdots\wedge e_{i_q}\otimes 1$ with $e_{i_1}\wedge\cdots\wedge e_{i_q}$, and $f=e_r$, the result follows. □

For any A-mod M, $K_\cdot(\underline{x}, M) := K_\cdot(\underline{x})\otimes_A M$, and

$$H_*(\underline{x}, M) := H_*(K_\cdot(\underline{x}, M)).$$

e.g. $0 \to M \rightrightarrows \begin{matrix}M\\ M\end{matrix} \rightrightarrows M \to 0$

Let $\mathfrak{a} = (x_1,\dots,x_r)$ then clearly.

$$H_0(\underline{x}, M) = M/\mathfrak{a}M, \quad H_n(\underline{x}, M) = \{m\in M \mid \mathfrak{a}\cdot m = 0\}$$

For a chain complex C, define the shift $C[-1]$ to be the complex

$$(C[-1])_{q+1} = C_q \quad \text{(the same diff map).}$$

$$\Rightarrow H_q(C[-1]) = H_{q-1}(C)$$

Consider the chain complex $C\otimes K_\cdot(a)$, here $a\in A$, explicitly

$$\begin{tikzcd}
(C\otimes K_\cdot(a))_q = (C_q\otimes K_0(a)) \arrow[d] & \oplus & (C_{q-1}\otimes K_1(a)) \arrow[d] \arrow[dll, "(-1)^{q-1} id\otimes d"'] \\
(C\otimes K_\cdot(a))_{q-1} = (C_{q-1}\otimes K_0(a)) & \oplus & (C_{q-2}\otimes K_1(a))
\end{tikzcd}$$

Easy to check:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, equal] & C\otimes K_\cdot(a) \arrow[r, "p"] & C[-1] \arrow[r] \arrow[d, equal] & 0. \\
& C\otimes K_1(x) & & C\otimes K_0(x)[1]. &
\end{tikzcd}$$

where $i(c) = c\otimes e_0$, $p(c\otimes e_0) = 0$, $p(c\otimes e_1) = c$.

Example: If $C = (0\to C_1\to C_0\to 0)$. then.

$$\begin{tikzcd}
C\otimes K_\cdot(a): & 0 \arrow[r] & C_1\otimes K_1 \arrow[r] & \boxed{C_1\otimes K_0} \oplus C_0\otimes K_1 \arrow[r] & \boxed{C_0\otimes K_0} \arrow[r] & 0. \\
& & & c\otimes e_0 \quad\quad C \arrow[u, leftrightarrow] & & \\
& & & c \arrow[u] & &
\end{tikzcd}$$

This gives a long exact sequence.

$$\to H_q(C) \xrightarrow{i_*} H_q(C\otimes K.(a)) \xrightarrow{p_*} H_{q-1}(C) \xrightarrow{\partial} H_{q-1}(C) \to \cdots$$

Now we determine the map ∂.

Lemma: The connecting homom ∂ is multip. by $(-1)^{q-1}a$, and a annihilates $H_*(C\otimes K.(a))$.

Pf: Let $z\in C_{q-1}$ be a cycle. Then $z = p(z\otimes e_1)$ and

$$d(z\otimes e_1) = 0 + (-1)^{q-1}(z\otimes ae_0) = (-1)^{q-1}a\, i(z).$$

For the second statement, let $z = x\otimes e_1 + y\otimes e_0$ be a cycle in $(C\otimes K.(a))_q$.

then $d(z) = d(x)\otimes e_1 + (-1)^{q-1}ax\otimes e_0 + d(y)\otimes e_0 = 0$.

$\Rightarrow d(x) = 0$ and $d(y) = (-1)^q a\cdot x$.

$\Rightarrow az = ax\otimes e_1 + ay\otimes e_0 = (-1)^q d(y)\otimes e_1 + ay\otimes e_0$

$= (-1)^q d(y\otimes e_0)$, ie $a\cdot\bar z = 0$, where $\bar z\in H_q(-)$. $\square$

Cor: The ideal $(x_1,\dots,x_r)$ annihilates $H_i(\underline{x}, M)$.

Pf: By induction, taking $a = x_r$ and $C = K(x_1,\dots,x_{r-1})\otimes_A M$ in Lemma.

Thm: Let A = noeth ring, M = f.g. A-mod, $\underline{x} = (x_1,\dots,x_r)$, assume $\underline{x}A\neq A$.

(i) If $\underline{x}$ is an M-sequence, then

$$H_0(\underline{x}, M)\simeq M/\underline{x}M \text{ and } H_i(\underline{x}, M) = 0 \quad \forall i>0$$

(ii) Conversely, if A is local and $H_1(\underline{x}, M) = 0$, then $\underline{x}$ is an M-sequence.

($\Rightarrow H_i(\underline{x}, M) = 0 \quad \forall i\geq 1$).

Pf: Let $\underline{x}' = (x_1,\dots,x_{r-1})$, take $C = K.(\underline{x}')\otimes M$ and $a = x_r$ above.

We obtain the following:

$$(*)\quad H_1(\underline{x}', M) \xrightarrow{-x_r} H_1(\underline{x}', M) \to H_1(\underline{x}, M) \to H_0(\underline{x}', M) \xrightarrow{x_r} H_0(\underline{x}', M) \to H_0(\underline{x}, M) \to 0$$

(i). We do by induction on r. (thus $H_i(\underline{x}', M)=0$ for $i\geq 1$)

$\Rightarrow H_i(\underline{x}, M)=0$ for $i\geqslant 2$.

Since $H_0(\underline{x}', M)\simeq M/\underline{x}'M$ and since x_r is regular for $M/\underline{x}'M$.

we deduce $H_1(\underline{x}, M)=0$.

(ii). Since $H_1(\underline{x}, M)=0$, the map $H_1(\underline{x}', M)\to H_1(\underline{x}', M)$ is surjective.

hence $H_1(\underline{x}', M)=0$ by Nakayama.

By induction, this implies that, $\underline{x}'$ is an M-sequence.

Moreover, by (*) again, $x_n: H_0(\underline{x}', M)\to H_0(\underline{x}', M)$ is injective.

$\quad$ is $M/\underline{x}'M$.

thus $\underline{x}$ is an M-sequence. $\square$

Thm. Let $\mathfrak{a}=(x_1,\dots,x_n)\subseteq A$, and M = finite A-mod., $M/\mathfrak{a}M\neq 0$.

Let $s=\max\{i: H_i(\underline{x}, M)\neq 0\}$, then.

$$\operatorname{depth}(\mathfrak{a}, M)=n-s.$$

Pf: Let $(a_1,\dots,a_r)$ be a max M-sequence contained in $\mathfrak{a}$.

need to prove $H_i(\underline{x}, M)=0$ for $i>n-r$, and $H_{n-r}(\underline{x}, M)\neq 0$.

We prove this by induction on r.

If $r=0$. then all elements of $\mathfrak{a}$ are zero-divisors for M.

So $\mathfrak{a}\subseteq \mathfrak{p}$ for some $\mathfrak{p}\in \operatorname{Ass}(M)$.

By definition of $\operatorname{Ass}(M)$. $\mathfrak{a}\subseteq\mathfrak{p}=\operatorname{Ann}(m)$. so. $\exists\, m\in M$, $m\neq 0$, $\mathfrak{a}\cdot m=0$.

In particular, $H_n(\underline{x}, M)\neq 0$, proving the result.

Now let $r>0$, and set $M_1=M/a_1M$., so $0\to M\xrightarrow{a_1} M\to M_1\to 0$.

Tensoring with $K_\bullet(\underline{x})$. we obtain

$$0\to K_\bullet(\underline{x}, M)\to K_\bullet(\underline{x}, M)\to K_\bullet(\underline{x}, M_1)\to 0.$$

taking homology, we obtain

$$(*)\quad H_i(\underline{x}, M) \xrightarrow{a_1} H_i(\underline{x}, M) \to H_i(\underline{x}, M_1) \to H_{i-1}(\underline{x}, M)$$

Since $a_1 \in \mathfrak{a}$, and $\mathfrak{a}$ annihilates the gps. $H_i(\underline{x}, M)$, we obtain

$$0 \to H_i(\underline{x}, M) \to H_i(\underline{x}, M_1) \to H_{i-1}(\underline{x}, M) \to 0.$$

Since $(a_2, \ldots, a_r)$ is a max M_1-sequence contained in $\mathfrak{a}$.

the induction gives $\begin{cases} H_i(\underline{x}, M_1) = 0 \text{ for } i > n+1-r \\ H_{n+1-r}(\underline{x}, M_1) \neq 0. \end{cases}$

and the result follows. $\square$

From now on, we assume (A, m) is noeth local., and $x_1, \ldots, x_r \in m$.

The differential $d: K_\ell \to K_{\ell-1}$ takes values in $mK_{\ell-1}$, hence induces a morphism

$$\bar{d}: K_\ell / mK_\ell \to mK_{\ell-1} / m^2 K_{\ell-1}$$

Lemma: Assume that $\bar{x}_1, \ldots, \bar{x}_r \in m/m^2$ are k-linearly independent, then $\bar{d}$ is injective.

pf: do induction on ℓ.

For $\ell = 1$, we know $K_1 \simeq A^r \to K_0 = A$. sending $(a_1, \ldots, a_r)$ to $\sum_{i=1}^r x_i a_i$.

Since $\bar{x}_1, \ldots, \bar{x}_r$ are linearly independent over $A/m = k$, $\bar{d}$ is injective in this special case.

We write $I_\ell = \{ \underline{i} = (i_1, \ldots, i_\ell) \mid 1 \le i_1 < \cdots < i_\ell \le r \}$.

for $\underline{i} \in I_\ell$, write $e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$, and $S_{\underline{i}} = \{ i_1, \ldots, i_\ell \}$.

given $\underline{i}' \in I_{\ell-1}$ and given $j \notin S_{\underline{i}'}$, there exists exactly one $\underline{i} \in I_\ell$, s.t. $S_{\underline{i}} = S_{\underline{i}'} \cup \{j\}$.

Now let $v = \sum_{\underline{i} \in I_\ell} a_{\underline{i}} \cdot e_{\underline{i}} \in K_\ell$, by definition, $d(e_{\underline{i}}) = \sum_{m=1}^{\ell} (\pm 1)\, x_{i_m}\, e_{i_1} \wedge \cdots \wedge \hat{e}_{i_m} \wedge \cdots \wedge e_{i_\ell}$

So. $d(v) = \sum_{\underline{i}' \in I_{\ell-1}} f_{\underline{i}'} \cdot e_{\underline{i}'} \in K_{\ell-1}$

where $f_{\underline{i}'}$ has the form $\sum_{j \in S \setminus S_{\underline{i}'}} \pm x_j \cdot \underbrace{a_{\underline{i}' \cup \{j\}}}$ ← viewed as element in S_ℓ

If $\bar{d}(v)=0$, then $\bar{f}_{\underline{i}'}=0$ for any $\underline{i}'\in I_{\ell-1}$, and as in the case $\ell=1$, we have

$$\bar{a}_{\underline{i}'\cup\{j\}}=0 \text{ in } A/m \text{ for any } j\in S\setminus S_{\underline{i}'}$$

i.e. $v\in m\cdot K_\ell$ as required. $\square$

<u>Thm</u>: we have an injection $K_\bullet(x,k)\to \mathrm{Tor}_\bullet^A(k,k)$.

Pf: we have

$$\begin{tikzcd} K_\bullet: \arrow[r] & K_\ell \arrow[r, "d_\ell"] \arrow[d, "f_\ell"] & K_{\ell-1} \arrow[r] \arrow[d, "f_{\ell-1}"] & \cdots \arrow[r] & K_0 \arrow[r] \arrow[d] & 0 \\ F_\bullet: \arrow[r] & F_\ell \arrow[r, "\varphi_\ell"] & F_{\ell-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & 0 \end{tikzcd}$$

by induction: $\begin{matrix} K_0 \\ \downarrow \\ F_0 \end{matrix}$ is an isom., both isom to A, and mod m is isom.

For $n>0$,

$$\begin{tikzcd} K_\ell/mK_\ell \arrow[r, "\bar{d}_\ell"] \arrow[d, "\bar{f}_\ell"] & mK_{\ell-1}/m^2K_{\ell-1} \arrow[d] \\ F_\ell/mF_\ell \arrow[r, "\bar{\varphi}_\ell"] & mF_{\ell-1}/m^2F_{\ell-1} \end{tikzcd}$$

By lemma, $\bar{d}_\ell$ is injective.

by induction, $K_{\ell-1}$ is a direct summand of $F_{\ell-1}$, hence $mK_{\ell-1}/m^2K_{\ell-1}\to mF_{\ell-1}/m^2F_{\ell-1}$ is injective.

hence $\bar{f}_\ell$ is also injective, and implies that K_ℓ is a direct summand of F_ℓ.

Taking homology, we obtain the result. $\square$

Cor: $\dim_k \mathrm{Tor}_i^A(k,k)\geq\binom{n}{i}$, where $n=\dim_k m/m^2$. hence $\mathrm{pd}(k)\geq\dim_k m/m^2$.

Pf: We take $(\bar{x}_1,\cdots,\bar{x}_n)$ to be a basis of m/m^2,

then Thm above implies $\dim \mathrm{Tor}_i^A(k,k)\geq\dim H_i(x,k)=\binom{n}{i}$. $\square$

§ Regular local rings

Prop 1: Let A = noeth local. If $gl.dim(A) < \infty$, then $gl.dim(A) \leq \dim(A)$.

Pf: By Auslander–Buchsbaum Thm, for any finite A-mod. M.

$$pd(M) + depth(M) = depth(A) \leq \dim(A).$$

In particular, $pd(M) \leq \dim(A)$.

Since $gl.dim(A) = \sup\{pd(M) : M = \text{finite } A\text{-mod}\}$, this implies the result. □

Prop 2: Let A = noeth local, then $gl.dim(A) = pd(k)$. (allow $\infty = \infty$).

Pf: $\geq$: trivial

$\leq$: can assume $pd(k) = n < +\infty$.

Then $\exists$ $0 \to P_n \to \cdots \to P_0 \to k \to 0$ proj resolution

$\Rightarrow \forall$ finite M, $Tor_i^A(M, k) = 0$. $\forall\ i > n$

Recall $pd(M) = \sup\{i : Tor_i^A(M, k) \neq 0\}$, we see that $pd(M) \leq n$

This is true for any M, we obtain $gl.dim(A) = n$. □

Thm: TFAE:

(i) A is regular local ring.

(ii) $pd(k) < \infty$.

(iii) $gl.dim(A) < \infty$.

Moreover, if these conditions are satisfied, then $gl.dim(A) = \dim(A)$.

Pf: (i) $\Rightarrow$ (ii). by Koszul complex (below), choose $m = (x_1, \ldots, x_n)$, $n = \dim(A)$.

Claim: this is a regular sequence in A.

(first, since A is domain, x_1 is regular. Then again A/x_1 is regular local ring.

because $\dim(A/x_1) = \dim(A) - 1 = \underset{\underset{n-1}{\parallel}}{\dim}(m_{A/x_1}/m_{A/x_1}^2)$

So A/x_1 is a domain, and x_2 is regular for A/x_1.

Continuing this argument, we get the claim.)

By the claim, $K.(\underline{x})$ is a free resolution of k. (of length n). so $\mathrm{pd}(k) < \infty$.

(ii) $\Leftrightarrow$ (iii) : by Prop 2. above

(iii) $\Rightarrow$ (i). Know. $\mathrm{pd}(k) = \mathrm{gl.dim}(A) \leq \dim(A)$

Recall. $\dim(A) \leq \dim_k m/m^2$. and A is regular iff this is an equality.

By Cor. in § Koszul complex. $\mathrm{pd}_A(k) \geq \dim_k m/m^2$, the result follows. $\square$