Lectures on Algebra II

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Contents

Planning

1. Part I: category and functors

   3 weeks, following Zheng's notes

   Contents: category, functor, transformation; additive and abelian categories; diagram and complexes; chain homotopy

2. Part II: derived functors

   4 weeks, following Weibel.

   Contents: δ-functors and derived functors; Ext^n and Yoneda extensions; flatness and Tor.

3. Part III: spectral sequence

   3 weeks, following Weibel.

   Contents: spectral sequences and examples.

References:
Weizhe Zheng: Lecture notes, 2017
Weible: An introduction to homological lgebra.
Hilton, Stammbach: A course in Homological algebra, GTM 4.
Cartan-Eilenberg, Homological algebra.

Examination

Chapter 1

Basics

1.1 Lecture 1 (2019-02-25)

1.1.1 Categories and functors

Definition 1.1.1. *A category $\mathcal{C}$ consists of a set of objects* $\mathrm{Ob}(\mathcal{C})$*, and a set of morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*, for all $X, Y \in \mathcal{C}$ such that*

- *(associativity axiom) for $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, there exists a composition map* $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ *such that (associativity)*

$$f(gh) = (fg)h$$

- *(unit axiom) for $X \in \mathrm{Ob}(\mathcal{C})$, there exists an identity morphism* $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ *such that*
$$\mathrm{Id}_X \circ f = f, \quad g \circ \mathrm{Id}_X = g$$
for $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$.

The first example to keep in mind is the category **SET** of sets, with morphisms being maps between sets. But Russell's paradox shows that... Therefore, we shall only consider small sets, i.e. a set containing itself as an element does not belong to $\mathrm{Ob}(\mathbf{Set})$.

Definition 1.1.2. *A category $\mathcal{C}$ is called* locally small *if* Hom *sets are small; called* small *if both* Hom *sets and* $\mathrm{Ob}(\mathcal{C})$ *are small.*

We will only consider *locally small* categories.

Example 1.1.3. *(1)* **Set***: the category of (small) sets.*

(2) **Top***: the category of topological spaces, with morphisms being continuous maps.*

(3) **Grp***: the category of groups, with morphisms being homomorphisms of groups.*

*(4) R-***Mod***: the category of left R-modules. When $R = \mathbb{Z}$, this is the category of abelian groups.*

(5) Any partially ordered set $(S, \leq)$ can be regarded as a category by

$$\mathrm{Hom}(x, y) = \begin{cases} \{*\} & x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$

(6) Any monoid M can be regarded as a category with one object $*$ and $\mathrm{End}(*) = M$*. If you take $*$ to be a* large *set (i.e. which is not small), this gives an example of locally small, but not small, category.*

(7) Given any ring R, we can form a category $\mathbf{Mat}(R)$ *by taking objects A_n indexed by the set of natural numbers (including zero) and letting the* ***hom-set*** *of morphisms from A_n to A_m be the set of $m \times n$ matrices over R, and where composition is given by matrix multiplication.*

(8) **Cat***: the category of small categories, whose objects are all small categories and whose morphisms are functors between categories.*

Definition 1.1.4. *A morphism $f : X \to Y$ in $\mathcal{C}$ is called an isomorphism if there exists a morphism $g : Y \to X$ such that $gf = \mathrm{Id}_X$ and $fg = \mathrm{Id}_Y$. The morphism g is then unique and is called the* inverse *of f, denoted by f^{-1}.*

An isomorphism in **Set** is a set bijection; an isomorphism in **Top** is a homeomorphism (e.g. $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ is not an isomorphism).

Definition 1.1.5. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$.*

- *f is called a* momomorphism *if for any pair of morphisms $(g_1, g_2) : W \to X$ such that $fg_1 = fg_2$, we have $g_1 = g_2$; equivalently, f is a monomorphism if and only if $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ is an injection.*

- *f is called a* epimorphism *if for any pair of morphisms $(h_1, h_2) : Y \to Z$ such that $h_1 f = h_2 f$ we have $h_1 = h_2$; equivalently, f is an epimorphism if and only if the map $\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an injection.*

Remark 1.1.6. *An isomorphism is necessarily a monomorphism and an epimorphism; but the converse need not be true. See the example $\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ in* **Top**.

Definition 1.1.7. *The opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.*

1.1.2 Functors

Definition 1.1.8. *A (covariant) functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories consists of*

- *a map $X \mapsto F(X)$ for any $X \in \mathrm{Ob}(\mathcal{C}_1)$*
- *for $X, Y \in \mathrm{Ob}(\mathcal{C}_1)$, a map $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(FX, FY)$: $f \mapsto Ff$*

such that

- *if $g \circ f$ is defined in $\mathcal{C}_1$, then so is $F(g)F(f)$ and $F(gf) = F(g)F(f)$ (covariance)*
- *$F(1_X) = 1_{FX}$.*

Similarly we can define a contravariant functor with the only difference being $F(gf) = F(f)F(g)$.

Example 1.1.9. *Let R-***Mod** *denote the category of left R-modules, and $M \in R$-***Mod***. Then*

$$\mathrm{Hom}_R(M, -) : R\text{-}\mathbf{Mod} \to \mathbf{Ab}, \quad N \mapsto \mathrm{Hom}_R(M, N)$$

gives a covariant functor from R-Mod to the category Ab *of abelian groups. Similarly, $\mathrm{Hom}_R(-, M)$ defines a contravariant functor.*

*(2) The $\mathrm{Hom}_R(A, -)$ functor on R-***Mod***: if $B \to C$ is a morphism, then we have*

$$\mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, C).$$

1.1.3 Transformations

Definition 1.1.10. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A* natural transformation *$\alpha : F \to G$ consists of morphisms $\alpha_X : FX \to GX$ for all objects X of $\mathcal{C}$, such that for every morphism $f : X \to Y$, the following diagram commutes*

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"] & GY. \end{tikzcd}$$

Remark 1.1.11. *If for every object X of $\mathcal{C}$, α_X is an isomorphism, then α is said to be a natural isomorphism of functors F and G.*

Example 1.1.12. *(1) Let $U : R\text{-}\mathbf{Mod} \to \mathbf{Set}$ be the forgetful functor sending any R-module M to the underlying set.*

*(2) If $A \to A'$ is a morphism in R-**Mod**, and F_A denotes the functor $\mathrm{Hom}_R(A, -)$, then $\alpha : F_{A'} \to F_A$ is a natural transformation in the sense that, for any morphism $B \to C$ there is a commutative diagram:*

$$\begin{tikzcd} F_{A'}(B) \arrow[r] \arrow[d] & F_{A'}(C) \arrow[d] \\ F_A(B) \arrow[r] & F_A(C). \end{tikzcd}$$

Given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations

$$\alpha : F \to G, \quad \beta : G \to H$$

we have the composite natural transformation $\beta\alpha : F \to H$. Functors $\mathcal{C} \to \mathcal{D}$ and natural transformations form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 1.1.13. *An equivalence of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ such that there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{Id}_{\mathcal{C}} \cong GF$ and $FG \cong \mathrm{Id}_{\mathcal{D}}$. The functors F and G are called* quasi-inverses *of each other.*

Remark 1.1.14. *A related concept is "isomorphism of categories": two categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ which are mutually inverse to each other, i.e. $FG = \mathbf{1}_{\mathcal{D}}$ (the identity functor on $\mathcal{D}$) and $GF = \mathbf{1}_{\mathcal{C}}$. This means that both the objects and the morphisms of $\mathcal{C}$ and $\mathcal{D}$ stand in a one-to-one correspondence to each other.*

Example*: For a field k, the category of k-linear group representations of G is isomorphic to the category of left modules over kG.*

Consider the category $\mathcal{C} = \mathbf{Vect}_k$ of finite-dimensional vector spaces over a field k, and the category $\mathcal{D} = \mathrm{Mat}(k)$ of all matrices. Then $\mathcal{C}$ and $\mathcal{D}$ are equivalent: The functor F which maps the object A_n of $\mathcal{D}$ to the vector space k^n and the matrices in $\mathcal{D}$ to the corresponding linear maps is full, faithful and essentially surjective. Note that this is *not* an isomorphism of categories.

Remark 1.1.15. *Given a pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, if two of F, G and GF are equivalences of categories, then so is the third one.*

Definition 1.1.16. *A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful (resp. full, resp. fully faithful) if the map*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

is an injection (resp. surjection, resp. bijection) for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.

Definition 1.1.17. *A functor $F : \mathcal{C} \to \mathcal{D}$ is* essentially surjective *if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an isomorphism $FX \cong Y$.*

Proposition 1.1.18. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

This is a quite useful and commonly applied criterion, because one does not have to explicitly construct the "inverse" G and the natural isomorphisms between FG, GF and the identity functors.

Proof. (*Sketch*) Step 1. Using the axiom of choice, show that any category $\mathcal{C}$ admits a full subcategory $\mathcal{C}_0$ such that the inclusion functor $\mathcal{C}_0 \to \mathcal{C}$ is an equivalence of categories and isomorphism objects in $\mathcal{C}_0$ are equal. For example, the category $\mathbf{Mat}(k)$ is equal to $\mathcal{C}_0$, if $\mathcal{C} = \mathbf{Vect}_k$.

Step 2. The composition:

$$\mathcal{C}_0 \cong \mathcal{C} \xrightarrow{F} \mathcal{D} \cong \mathcal{D}_0$$

which is fully faithful and essentially surjective, hence is an isomorphism of categories, in particular an equivalence of categories. Hence $\mathcal{C} \to \mathcal{D}$ is also an equivalence. □

1.1.4 Yoneda lemma and representable functors

Given $X \in \mathrm{Ob}(\mathcal{C})$, consider the functor $\mathrm{h}_X : \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Lemma 1.1.19. *(Yoneda) For every (contravariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, the map*

$$\phi : \mathrm{Nat}(\mathrm{h}_X, F) \to F(X), \quad \alpha \mapsto \alpha_X(\mathrm{Id}_X)$$

is a bijection.

Proof. We construct the inverse $\psi : F(X) \to \mathrm{Nat}(\mathrm{h}_X, F)$ by

$$\psi(x)_Y(f) = F(f)(x),$$

here $f \in h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and $F(f) : F(X) \to F(Y)$ is provided by F. We have

$$(\phi\psi)(x) = \psi(x)_X(\mathrm{Id}_X) = F(\mathrm{Id}_X)(x) = x$$

and since $h_X(f)$

$$(\psi\phi)(\alpha)_Y(f) = F(f)(\phi(\alpha)) = F(f)(\alpha_X(\mathrm{Id}_X)) \overset{(*)}{=} \alpha_Y(\mathrm{h}_X(f)(\mathrm{Id}_X)) = \alpha_Y(f).$$

where $(*)$ comes from the natural diagram

$$\begin{tikzcd} h_X(X) \arrow[r, "h_X(f)"] \arrow[d, "\alpha_X"] & h_X(Y) \arrow[d, "\alpha_Y"] \\ F(X) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

and $h_X(f)$ sends Id_X to f. □

Note that h_X is functorial in X, in the sense we get a functor $\mathrm{h} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$.

Corollary 1.1.20. *The functor* h *is fully faithful.*

Proof. The map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Nat}(\mathrm{h}_X, \mathrm{h}_Y)$ given by $\mathrm{h}_{\mathcal{C}}$ coincides with the bijection ψ constructed in the proof of the lemma for $F = \mathrm{h}_Y$. □

The Yoneda lemma suggests that instead of studying the (locally small) category $\mathcal{C}$, one should study the category of all functors of $\mathcal{C}$ into **Set**. **Set** is a category we (maybe) understand well, and a functor of $\mathcal{C}$ into **Set** can be seen as a "representation" of $\mathcal{C}$ in terms of known structures. The original category $\mathcal{C}$ is contained in this functor category, but new objects also appear in it, which were absent and "hidden" in $\mathcal{C}$. This approach is analogue to (and in fact generalizes) the method of studying a ring by investigating the modules over that ring.

Corollary 1.1.21. *(i) Let $f : X \to Y$ be a morphism such that $\mathrm{h}_f : \mathrm{h}_X \to \mathrm{h}_Y$ is a natural isomorphism. Then f is an isomorphism.*

(ii) Let X, Y be objects such that $\mathrm{h}_X \cong \mathrm{h}_Y$. Then $X \cong Y$.

Definition 1.1.22. *We say that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is represented by an object X of $\mathcal{C}$ if there exists a natural isomorphism $F \cong \mathrm{h}_X$.*

Example 1.1.23. *Given a group G, a permutation of G is any* bijective *function from G to G; and the set of all such functions forms a group under function composition, called the* permuation group *on G, and written as* $\mathrm{Perm}(G)$. *Caylay's theorem says that every group G is isomorphic to a subgroup of* $\mathrm{Perm}(G)$.

Yondea's lemma generalizes Caylay's theorem in the following sense. Given the group G, let $\mathcal{C}$ be the category with a single object $*$ such that $G = \mathrm{Hom}_{\mathcal{C}}(*, *)$. Then

1. *A functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ consists of a set X and a group homomorphism $G \to \mathrm{Perm}(X)$, i.e. G-sets (a set X with a group action of G).*

2. *A natural transformation between two such functors is an equivariant map between G-sets. For exmaple, $h_* = \mathrm{Hom}_{\mathcal{C}}(-, *)$ is G, viewed as a G-set by right multiplication, and $\mathrm{Nat}(h_*, h_*)$ is the set of equivariant maps $G \to G$.*

3. *The Yoneda lemma with $F = \mathrm{Hom}_{\mathcal{C}}(*, -)$ states that*

$$\mathrm{Nat}(\mathrm{h}_*, \mathrm{h}_*) \to \mathrm{Hom}_{\mathcal{C}}(*, *) = G$$

is a bijection. That is, any equivariant map $G \to G$ has the form $h \mapsto gh$ for some $g \in G$. In fact, this is even an isomorphism of groups.

4. *The LHS is clearly a subgroup of* $\mathrm{Perm}(G)$.

Definition 1.1.24. *We say that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is represented by an object X of $\mathcal{C}$ if there exists a natural isomorphism $F \cong \mathrm{h}_X$.*

If we take $\mathcal{C}$ to be the category associated to a group G, then there is only one functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ which is representable, namely, h_*. It corresponds the G-set whose underlying set is G with right multiplication action.

1.2 Lecture 2 (2019-02-27)

1.2.1 Initial, final objects

Definition 1.2.1. *Let $\mathcal{C}$ be a category. An object X of $\mathcal{C}$ is called an* initial *object if, for every object Y of $\mathcal{C}$, there exists precisely one morphism $X \to Y$. An object Y is called a* final *(or* terminal*) object if, for every object X of $\mathcal{C}$, there exists precisely one morphism $X \to Y$; or equivalently an initial object in $\mathcal{C}^{\mathrm{op}}$.*

Initial or final objects, if exist, are unique, but up to unique isomorphism.

Example 1.2.2.
1. *In* **Set***, the initial object $\emptyset$, and every one-element set (i.e. singleton) $\{*\}$ is a final object.*
2. *In* **Ring***, $\mathbb{Z}$ is an initial object, and $\{0\}$ is a final object (the ring with $0 = 1$).*
3. *In R-***Mod***, $\{0\}$ is both an initial and final object.*
4. *In* **Cat***, the empty category* **0** *(with no objects and no morphism) is an initial object and* **1** *(with a single object with a single identity morphism) as final object.*

Definition 1.2.3. *If an object is both initial and final, it is called a zero object and often denoted by 0.*

Remark 1.2.4. *If $\mathcal{C}$ admits a zero object, then for every pair of objects X, Y, there exists a unique morphism $X \to Y$ given by*

$$X \to 0 \to Y.$$

It is called the zero morphism *and also denoted by 0.*

1.2.2 Products, coproducts

Definition 1.2.5. *Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of (X_i) is an object P of $\mathcal{C}$ equipped with morphisms $p_i : P \to X_i$, $i \in I$, satisfying the following universal property: for each object Q of $\mathcal{C}$ with morphisms $q_i : Q \to X_i$, there exists a unique morphism $q : Q \to P$ such that $q_i = p_i q$.*

A coproduct *is an object U of $\mathcal{C}$ equipped with morphisms $u_i : X_i \to U$ satisfying the following universal property: for each object V of $\mathcal{C}$ equipped with morphisms $v_i : X_i \to V$, there exists a unique morphism $v : U \to V$ such tha $v_i = v u_i$.*

The product (resp. coproduct) of (X_i), if exists, is unique up to unique isomorphism. Notation: $\prod_{i\in I} X_i$ for product and $\coprod_{i\in I} X_i$ for coproduct.

Properties:

- A product in $\mathcal{C}$ is a coproduct in $\mathcal{C}^{\mathrm{op}}$ and vice versa.
- For $I = \emptyset$, a product is a final object and a coproduct is an initial object.
- the universal property can be summarized as follows”

$$\mathrm{Hom}_{\mathcal{C}}(Q, \prod_{i\in I} X_i) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Q, X_i).$$

We say $\mathcal{C}$ admits finite (resp. small) products if $\mathcal{C}$ admits limits indexed by all finite (resp. small) set I.

Example 1.2.6. *In* **Set***, $\prod_{i\in I} X_i$ is the usual product of X_i and $\coprod_i X_i$ is the disjoint union of X_i.*

*In R-***Mod***, $\prod_{i\in I} X_i$ is the direct product and $\coprod_{i\in I} X_i$ is the direct sum of X_i.*

1.2.3 Pullback, pushout

Given $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Z$ and $g : Y \to Z$. A pullback of f and g consists of an object P of $\mathcal{C}$ and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ for which the diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes; moreover, the pullback must be universal with respect to the diagram. That is, for any other such triple (Q, q_1, q_2) for which the following diagram commutes,

$$\begin{tikzcd} Q \arrow[dr, dotted, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

there must exist a unique $u : Q \to P$ (called a *mediating morphism*) such that

$$p_2 \circ u = q_2, \qquad p_1 \circ u = q_1.$$

As usual, a pullback, if it exists, is unique up to isomorphism. We denote commonly $X \times_Z Y$ for the pullback.

Example 1.2.7. *In* **Set**, *the pullback exists and is given by*

$$X \times_Z Y = \{(x, y) \in X \times Y | f(x) = g(y)\}. \tag{1.1}$$

It is also fibered product.

In **Top**, *$X \times_Z Y$ is the subspace given by (1.1) endowed with the subspace topology.*

In **Grp**, *$X \times_Z Y$ is again the subset defined by (1.1) which has a natural group structure.*

Dually, we may define the pushout of morphisms f and g where

$$f : Z \to X, \qquad g : Z \to Y,$$

by reversing all arrows in the above construction for pullback. It is commonly denoted by $X \sqcup_Z Y$.

Example 1.2.8. *In* **Set**, *$X \sqcup_Z Y$ is given by $X \coprod Y/\sim$ where the equivalence relation is given by $f(z) \sim g(z)$ where $z \in Z$.*

*In R-***Mod**, *$X \sqcup_Z Y$ is given by $X \oplus Y / \{(f(z), -g(z)) | z \in Z\}$, thought of as "direct sum with gluing".*

In **Grp**, *the pushout is called the free product with amalgamation. Precisely, given $f : H \to G$ and $g : H \to G'$, we consider the free product $G * G'$. Let N be the smallest normal subgroup of $G * H$ containing all elements $f(z)g(z)^{-1}$, for $z \in H$. Then the free product with amalgamation of G and G' is the quotient group $G * G'/N$.*

In **Cring** *(the category of commutative rings), the pushout is given by the* tensor product, *i.e. $R \sqcup_S R' = R \otimes_S R'$.*

1.2.4 Inverse limit, direct limit

Let I be a directed set[1], and $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ and suppose we have a family of morphisms $f_{ij} : X_j \to X_i$ for all $i \leq j$, called transition maps, with the following properties:

1. f_{ii} is the identity on X_i,

2. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

Such a pair (X_i, f_{ij}) an inverse system of objects and morphisms over I.

The *inverse limit* of this system is an object X in $\mathcal{C}$ together with morphisms $p_i : X \to X_i$ (called projections) satisfying $p_i = f_{ij} \circ p_j$ for all $i \leq j$. The pair (X, p_i) must be *universal* in the sense that for any other such pair (X', p_i') (i.e. $\psi_i : X' \to X_i$ with $\psi_i = f_{ij} \circ \psi_j$ for all $i \leq j$) there exists a unique morphism $u : X' \to X$ such that the diagram

$$\begin{tikzcd}
& X' \arrow[d, dotted, "u"] \arrow[ddl, "p'_j"'] \arrow[ddr, "p'_i"] & \\
& X \arrow[dl, "p_j"] \arrow[dr, "p_i"'] & \\
X_j \arrow[rr, "f_{ij}"'] & & X_i
\end{tikzcd} \tag{1.2}$$

commutes for all $i \leq j$, for which it suffices to show that $\psi_i = p_i \circ u$ for all i. The inverse limit is often denoted

$$X = \varprojlim X_i.$$

Example 1.2.9.
- *The ring of p-adic integers is the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$, with the morphisms being natural projections. The ring $R[[t]]$ of formal power series over a commutative ring R is the inverse limit of $R[t]/(t^n)$.*

- *In* **Set**, *every inverse system has an inverse limit, which can be constructed in an elementary manner as a subset of the product of the sets forming the inverse system. The inverse limit of any inverse system of non-empty finite sets is non-empty. This may be proved with Tychonoff's theorem.* [2] *See* `https://stacks.math.columbia.edu/tag/086J` *for a proof.*

We may view $\varprojlim$ as a functor $\mathcal{C}^I \to \mathcal{C}$ where $\mathcal{C} = \mathrm{Fun}(I, \mathcal{C})$. We will see later that it is left exact for an abelian category.

Dually, we may define the direct limit of a direct system, where we are given morphisms $f_{ij} : X_i \to X_j$ for $i \leq j$.

We saw that the above constructions are all characterized by some universal property. We next give a general construction, which works for a category $\mathcal{I}$.

[1] A directed set is a nonempty set A together with a pre-order, with the additional property that every pair of elements has an upper bound.

[2] Note that it is false if I is only partially ordered (but not directed). For example, if $I = \{1, 2, 2'\}$ with $2, 2' \geq 1$ but $2, 2'$ are not comparable, and if the images of $X_2 \to X_1$ and $X_{2'} \to X_1$ have **empty** intersection, then $\varprojlim X_i = \emptyset$.

1.2.5 Comma categories

Definition 1.2.10 (Lawvere, 1963)**.** *Suppose that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are categories, and F and G are functors:*

$$\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$$

We can form the comma category $(F \downarrow G)$ as follows:

1. *The objects are all triples (X, Y, h) with $X \in \mathrm{Ob}(\mathcal{C})$, $Y \in \mathrm{Ob}(\mathcal{D})$ and $h : F(X) \to G(Y)$ a morphism in $\mathcal{E}$.*

2. *The morphisms from (X, Y, h) to (X', Y', h') are all pairs (f, g) where $f : X \to X'$ and $g : Y \to Y'$ such that the following diagram commutes:*

$$\begin{tikzcd} F(X) \arrow[r, "Ff"] \arrow[d, "h"'] & F(X') \arrow[d, "h'"] \\ G(Y) \arrow[r, "Gg"'] & G(Y') \end{tikzcd}$$

Remark 1.2.11. *A special case is when $\mathcal{D} = \mathbf{1}$ (category with one element $\{*\}$ and the identity morphism) with $G(*) = Y \in \mathcal{E}$. Then an object of $(F \downarrow G)$, also written as $(F \downarrow Y)$, is a pair (X, h) where $X \in \mathrm{Ob}(\mathcal{C})$ and $h \in \mathrm{Hom}_{\mathcal{E}}(FX, Y)$; the morphism is defined in an obvious way.*

1.2.6 Limits, colimits

Let $\mathcal{I}$ be a category. Let $\mathcal{C}$ and $\mathcal{I}$ be categories and let $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}} := \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ be the diagonal functor sending X to the constant functor of value X. Hence, given $F \in \mathcal{C}^{\mathcal{I}}$, an object in $(\Delta \downarrow F)$ is just a pair (X, h), with $X \in \mathcal{C}$ and $h : \Delta X \to F$; since Δ is constant functor, $\Delta X(i) = X$ for any $i \in \mathcal{I}$, so h means a family of natural morphisms $X \to F(i)$. We sometimes refer to $(\Delta \downarrow F)$ as the category of objects of $\mathcal{C}$ over F and denote it by $\mathcal{C}_{/F}$. Dually we sometimes refer to $(F \downarrow \Delta)$ as the category of objects of $\mathcal{C}$ under F and denote it by $\mathcal{C}_{F/}$.

Definition 1.2.12. *Let $F : \mathcal{I} \to \mathcal{C}$ be a functor viewed as an element in $\mathcal{C}^{\mathcal{I}}$. A* limit *(also called projective limit) of F is a final object of $(\Delta \downarrow F)$ and a* colimit *(also called inductive limit) of F is an initial object of $(F \downarrow \Delta)$.*

Explicitly, a limit of F means an object (X, h) described above such that for any other pair (X', h'), there is a natural morphism $u : X' \to X$ fitting in a commutative diagram as in (1.2).

Notation. We let $\lim_{i \in \mathcal{I}} F(i)$ for the limit of F if it exists, and $\mathrm{colim}_{i \in \mathcal{I}} F(i)$ for the colimit of F if exists. When exist, limits and colimits are unique up to unique isomorphisms.

Remark 1.2.13. *When I is a directed set (or just a partially ordered set), we may view $\mathcal{I}$ as a category by letting*

$$\mathrm{Hom}_{\mathcal{I}}(j, i) = \begin{cases} \{*\} & i \leq j \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the above definition of limit (or colimit) generalizes an inverse limit (or direct limit).

Example 1.2.14. *For $\mathcal{I} = (\bullet \to \bullet \leftarrow \bullet)$, $F : \mathcal{I} \to \mathcal{C}$ is given by a pair of morphisms in $\mathcal{C}$:*

$$f : X \to Z, \qquad g : Y \to Z$$

A limit of F is then a pullback of (f, g) discussed above. Dually, for $\mathcal{I} = (\bullet \leftarrow \bullet \to \bullet)$, a colimit is a pushout.

Exercise 1: Prove: $(A \times_C B) \times_B D \cong A \times_C D$ and $(A \sqcup_C B) \sqcup_B D \cong A \sqcup_C D$.

Exercise 2: The *center* $Z(C)$ of a category $\mathcal{C}$ is the class of all natural transformations $\alpha : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$, where $1_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$. Let R be a ring with identity and put $\mathcal{C} = R$-**Mod**. Prove that there is a bijection $Z(R) \to Z(\mathcal{C})$, where $Z(R)$ is the center of R, that is, $Z(R) = \{z \in R | zr = rz \text{ for all } r \in R\}$.

Exercise 3: Let $\mathcal{C}$ be the following category:

$$\begin{tikzcd} x \arrow[loop left] \arrow[r, shift left] & y \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

Let $F : \mathcal{C} \to \mathcal{C}$ be the functor determined by the object map $x, y \to x$. Is F faithful? Is F full?

Exercise 4: Prove that, in the category R-**Mod**, the pullback diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is also a pushout with respect to (p_1, p_2) if and only if $(f, g) : A \oplus B \to C$ is surjective.

Exercise 5: For groups G_1 and G_2 let $\iota_i : G_i \to G_1 \times G_2$ be the injections defined by $\iota_1(g_1) = (g_1, e_2)$ and $\iota_2(g_2) = (e_1, g_2)$, where e_i is the identity of G_i. Prove that $(G_1 \times G_2, \{\iota_i\})$ is not, in general, a coproduct of the family $\{G_i\}$ in the category **Grp**.

1.3 Lecture 3 (2019-03-04)

Example 1.3.1. *For $\mathcal{I} = (\bullet \rightrightarrows \bullet)$, $F : \mathcal{I} \to \mathcal{C}$ is represented by a pair of morphisms $f, g : X \rightrightarrows Y$ with the same source and target. The limit of F is called the* equalizer *of the pair and is denoted by* $\mathrm{Eq}(f, g)$. *The colimit of F is called the* coequalizer *and denoted by* $\mathrm{Coeq}(f, g)$. *If $\mathcal{C}$ admits a zero object and g is the zero morphism, these are called the* kernel *and the* cokernel *of f:*

$$\mathrm{Ker}(f) = \mathrm{Eq}(f, 0), \qquad \mathrm{Coker}(f) = \mathrm{Coeq}(f, 0).$$

Remark 1.3.2. *An equalizer is always a monomorphism: if $E = \mathrm{Eq}(f, g) \xrightarrow{e} X \overset{(f,g)}{\rightrightarrows} Y$, and if $(a, b) : Z \rightrightarrows E$ is such that $ea = eb$, then $fea = geb$ and the uniqueness property forces $a = b$. By duality, an coequalizer is always an epimorphism.*

Definition 1.3.3. *We say $\mathcal{C}$ admits limits of shape $\mathcal{I}$ if every functor $F : \mathcal{I} \to \mathcal{C}$ has a limit in $\mathcal{C}$.*

We say $\mathcal{C}$ admits finite (resp. small) limits if $\mathcal{C}$ admits limits indexed by all finite (resp, small) category $\mathcal{I}$. In the latter case, we say $\mathcal{C}$ is a complete *category. Dually, one has the notion of a* cocomplete *category.*

Example 1.3.4. *The category* **Set** *admits small limits and small colimits. Let $F : \mathcal{I} \to$* **Set** *be a functor with $\mathcal{I}$ a small category.*

(1) $\lim F$ *is represented by the subset $L \subseteq \prod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)$ consisting of elements (x_i) such that $Ff(x_i) = x_j$ for every morphism $f : i \to j$.*

(2) $\mathrm{colim} F$ *is represented by the quotient*

$$Q = \coprod_{i \in \mathrm{Ob}(\mathcal{I})} F(i)/\sim$$

by the equivalence relation $\sim$ generated by $x_i \sim (Ff)(x_i)$ for $f : i \to j$ and $x_i \in F(i)$.

Proposition 1.3.5. *The following are equivalent for the category $\mathcal{C}$:*
(1) Equalizers and finite products exist in $\mathcal{C}$.
(2) Pullbacks and a terminal object exist in $\mathcal{C}$.

Proof. (1)⇒(2). Assume that equalizers and finite products exist in $\mathcal{C}$. A product of the empty family of objects is a final object of $\mathcal{C}$, so it remains to show that pullbacks exist.

Let $f_1 : X_1 \to Z$ and $f_2 : X_2 \to Z$ be morphisms in $\mathcal{C}$; then $X_1 \prod X_2$ exists in $\mathcal{C}$ with natural projections $p_i : X_1 \prod X_2 \to X_i$. Consider the composite

$$f_i p_i : X_1 \prod X_2 \to X_i \to Z$$

and let E be the equalizer of $(f_1 p_1, f_2 p_2)$. We claim that E is a pullback of (f_1, f_2).

Given an object Q of $\mathcal{C}$ with compatible morphisms $q_i : Q \to X_i$ such that $f_1 q_1 = f_2 q_2$. Then the universal property of products gives a morphism $u : Q \to X_1 \prod X_2$ such that $q_i = p_i u$. Since

$$(f_1 p_1)u = f_1(p_1 u) = f_1 q_1 = f_2 q_2 = f_2(p_2 u) = (f_2 p_2)u$$

and since E is the equalizer, the universal property implies a unique morphism $\delta : Q \to E$ which is compatible with everything.

(2)⇒(1). Exercise. □

Remark 1.3.6. *In fact, if the conditions of Proposition 1.3.5 hold, then $\mathcal{C}$ admits all finite limits. See Zheng's lecture, Remark 1.6.2.*

to check

Definition 1.3.7. *We say that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves all limits of shape $\mathcal{I}$ if for every functor $S : \mathcal{I} \to \mathcal{C}$ and every limit $\Delta X \to S$, $F(\Delta X) = \Delta(FX)$ is a limit of $FS : \mathcal{I} \to \mathcal{D}$.*

For example, one can say that F preserves products, equalizers, pullbacks, etc. A *continuous* functor is one that preserves all small limits and a *cocontinuous* functor is one that preserves all small colimits.

Proposition 1.3.8. *For every object Y of $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(Y, -) : \mathcal{C} \to \mathbf{Set}$ preserves limits.*

Proof. Let $F : \mathcal{I} \to \mathcal{C}$ be a functor. Then

$$\mathrm{Hom}_{\mathcal{C}}(Y, \lim F) \cong \mathrm{Hom}_{\mathcal{C}^{I}}(\Delta Y, F) \cong \lim \mathrm{Hom}_{\mathcal{C}}(Y, F-). \tag{1.3}$$

Here $\mathrm{Hom}_{\mathcal{C}}(Y, F-) : \mathcal{I} \to \mathbf{Set}$ is viewed as a functor from $\mathcal{I}$ to $\mathbf{Set}$. □

Remark 1.3.9. *None of the following forgetful functors*

$$\mathbf{Grp} \to \mathbf{Set}, \qquad \mathbf{Ab} \to \mathbf{Set}, \qquad \mathbf{Ab} \to \mathbf{Grp}$$

preserve finite coproducts.

Universal constructions

Before giving the precise definition, we look at some examples.

Example 1.3.10. *(1) If $\mathcal{C} = \mathbf{Vect}_k$ and $\mathcal{D} = \mathbf{Alg}_k$ and $U : \mathbf{Alg}_k \to \mathbf{Vect}_k$ be the forgetful functor. Given $V \in \mathbf{Vect}_k$, we can construct the tensor algebra $T(V)$, which satisfies the universal property: any linear map from V to an algebra A can be extended to an algebra homomorphism from $T(V)$ to A, i.e.*

$$\mathrm{Hom}_{\mathbf{Vect}}(V, A) \cong \mathrm{Hom}_{\mathbf{Alg}}(T(V), A).$$

In other words, $T(V)$ is an initial object in the comma category $(V \downarrow U)$.

(2) Let $\mathfrak{g}$ be a Lie algebra over k, $U(\mathfrak{g})$ be its universal enveloping algebra with a canonical morphism $h : \mathfrak{g} \to U(\mathfrak{g})$, which satisfies the following universal property: for any associate k-algebra A and any Lie algebra homomorphism $\mathfrak{g} \to A$ (where A is endowed with the bracket $[a, b] = ab - ba$), there is a unique k-algebra homomorphism $U(\mathfrak{g}) \to A$ such that

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "h"] \arrow[dr] & (U(\mathfrak{g}), [\ ,\ ]) \arrow[d, dotted] & U(\mathfrak{g}) \arrow[d, dotted] \\
& (A, [\ ,\ ]) & A
\end{tikzcd}$$

is commutative. As above, let $\mathrm{Lie} : \mathbf{Alg} \to \mathbf{LieAlg}$, then $(U(\mathfrak{g}), h)$ is an initial object in $(\mathfrak{g} \downarrow \mathrm{Lie})$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let Y be an object of $\mathcal{D}$. A *final morphism* is a final object in the comma category $(F \downarrow Y)$. In other words, it consists of a pair (X_0, ϵ) where $X_0 \in \mathcal{C}$ and $\epsilon : FX_0 \to Y$ satisfying the universal property of being a final object: for any pair (X, f), there is a unique morphism $g : X \to X_0$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, dotted, "g"'] & FX \arrow[d, dotted, "Fg"'] \arrow[dr, "f"] & \\ X_0 & FX_0 \arrow[r, "\epsilon"'] & Y. \end{tikzcd}$$

Dually, we can define an *initial morphism* to be an initial object in the category $(X \downarrow F)$. A *universal morphism* is a finial morphism or an initial morphism.

1.3.1 Adjoint functors

Definition 1.3.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction is a triple (F, G, ϕ) where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, and ϕ is a natural isomorphism*

$$\phi_{XY} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

We say that F is left adjoint *to G and G is* right adjoint *to F and write $\phi : F \dashv G$.*

We explain the naturality of ϕ. If $a : X' \to X$ is a morphism in $\mathcal{C}$, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(FX, Y) \arrow[r, "\phi_{XY}"] \arrow[d, "\circ Fa"'] & \mathrm{Hom}_{\mathcal{C}}(X, GY) \arrow[d, "\circ a"] \\ \mathrm{Hom}_{\mathcal{D}}(FX', Y) \arrow[r, "\phi_{X'Y}"] & \mathrm{Hom}_{\mathcal{C}}(X', GY), \end{tikzcd}$$

namely, if $f \in \mathrm{Hom}_{\mathcal{D}}(FX, Y)$ then

$$\phi(f \circ Fa) = \phi(f) \circ a.$$

If moreover, $b : Y \to Y'$ is a morphism in $\mathcal{D}$, then we obtain the formula:

$$\phi(b \circ f \circ Fa) = Gb \circ \phi(f) \circ a \tag{1.4}$$

Example 1.3.12. *(1) The free group functor F : **Set** $\to$ **Grp** (which sends X to the free groups generated by X) is left adjoint to the forgetful functor U : **Grp** $\to$ **Set**.*

(2) Let $\mathcal{I}$ be a category. If $\mathcal{C}$ admits limits indexed by $\mathcal{I}$, then $\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ is right adjoint to the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ by (1.3). If $\mathcal{C}$ admits colimits indexed by $\mathcal{I}$, then $\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ is left adjoint to Δ.

(3) Let R, S, T be rings and ${}_RM_S$, ${}_SN_T$, ${}_RP_T$ be bimodules. We have the adjunction formula:

$$\mathrm{Hom}_{(R,T)\text{-}\mathbf{Mod}}(M \otimes_S N, P) \cong \mathrm{Hom}_{(R,S)\text{-}\mathbf{Mod}}(M, \mathrm{Hom}_{\mathbf{Mod}\text{-}T}(N, P)),$$
$$\mathrm{Hom}_{(R,T)\text{-}\mathbf{Mod}}(M \otimes_S N, P) \cong \mathrm{Hom}_{(S,T)\text{-}\mathbf{Mod}}(N, \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, P)).$$

The first one implies that

$$- \otimes_S N : (R, S)\text{-}\mathbf{Mod} \to (R, T)\text{-}\mathbf{Mod}$$

is left adjoint to

$$\mathrm{Hom}_{\mathbf{Mod}\text{-}T}(N, -) : (R, T)\text{-}\mathbf{Mod} \to (R, S)\text{-}\mathbf{Mod},$$

and similarly the second one implies that $M \otimes_S -$ is left adjoint to $\mathrm{Hom}_{R-\mathbf{Mod}}(M, -)$.

(4) Let $G : \mathbf{Ab} \to \mathbf{Grp}$ be the inclusion functor, then it has a left adjoint functor Ab : $\mathbf{Grp} \to \mathbf{Ab}$*, sending G to $G/[G,G]$ and called* abelianization*, i.e.*

$$\mathrm{Hom}_{\mathbf{Ab}}(G^{\mathrm{ab}}, A) = \mathrm{Hom}_{\mathbf{Grp}}(G, A).$$

(5) The tensor algebra construction of a k-vector space V is functorial on V, so it gives rise to a functor $T : \mathbf{Vect}_k \to \mathbf{Alg}_k$. It is left adjoint to the forgetful functor $U : \mathbf{Vect}_k \to \mathbf{Alg}_k$.

Proposition 1.3.13. *Let $\phi : F \dashv G$. Then G is determined by F up to natural isomorphism.*

Proof. Assume $\phi' : F \dashv G'$. Consider the natural isomorphism

$$\phi'^{-1} \circ \phi : \mathrm{Hom}_{\mathcal{D}}(X, GY) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X, G'Y).$$

This is true for any $X \in \mathrm{Ob}(\mathcal{C})$, hence by Yoneda's lemma, this must be induced by an isomorphism $GY \to GY'$, which is natural by the naturality of ϕ and ϕ'. □

Proposition 1.3.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if for every object Y of $\mathcal{D}$, the functor $\mathrm{h}_{\mathcal{D}}(Y) \circ F = \mathrm{Hom}_{\mathcal{D}}(F-, Y)$ is representable.*

Proof. The necessity is clear because if $F \dashv G$, then $\mathrm{Hom}_{\mathcal{D}}(F-, Y) \cong \mathrm{Hom}_{\mathcal{C}}(-, GY)$ is represented by GY.

Assume $\mathrm{Hom}_{\mathcal{D}}(F-, Y)$ is representable and we construct an adjunction $\phi : F \dashv G$ as follows. For every $Y \in \mathrm{Ob}(\mathcal{D})$, choose an object GY of $\mathcal{C}$ with an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F-, Y) \cong \mathrm{h}_{GY}(-);$$

such an object GY is unique up to A natural isomorphism (by Yoneda's lemma). We need to check the association $Y \to GY$ defines a *functor* G from $\mathcal{D}$ to $\mathcal{C}$ (i.e. the naturality in Y). Moreover, by Yoneda's lemma, given a morphism $Y \to Y'$ in $\mathcal{D}$, we get a morphism $GY \to GY'$ in $\mathcal{C}$. □

Proposition 1.3.15. *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{D}' \to \mathcal{E}$, $G : \mathcal{D} \to \mathcal{C}$, $G' : \mathcal{E} \to \mathcal{D}$ be functors and let $\phi : F \dashv G$, $\phi' : F \dashv G'$ be adjunctions. Then $\phi\phi' : F'F \dashv GG'$.*

Unit, counit

Lemma 1.3.16. *Let $F \dashv G$. Then $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$, with $\eta_X = \phi(\mathrm{Id}_{FX}) : X \to GFX$ is a natural transformation. similarly $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, with $\epsilon_Y = \phi^{-1}(\mathrm{Id}_{GY}) : FGY \to Y$, is a natural transformation.*

Moreover, we have an equality of natural transformations:

$$\epsilon F \circ F\eta = \mathrm{Id}_F, \qquad G\epsilon \circ \eta G = \mathrm{Id}_G. \tag{1.5}$$

Proof. Exercise. □

Definition 1.3.17. *We call η the unit and ϵ the counit of the adjunction. The equations are called the first, and second, counit-unit equation respectively.*

Proposition 1.3.18. *Let $\phi : F \dashv G$. Then*
(1) F is fully faithful if and only if the unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ is a natural isomorphism.
(2) G is fully faithful if and only if the counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{C}}$ is a natural isomorphism.

Proof. For $f : X \to X'$, we have by (1.4)

$$\phi(Ff) = \phi(\mathrm{Id}_{FX'} \circ Ff) = \phi(\mathrm{Id}_{FX'}) \circ f = \eta_{X'} \circ f.$$

In other words, the composite

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(FX, FX') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, GFX')$$

is induced by $\eta_{X'}$. Then (1) follows from Yoneda's lemma. (2) is proved dually. □

Example 1.3.19. *In Example 1.3.12 (4), G is fully faithful and the counit is a natural isomorphism. In fact, the counit $\epsilon : \mathrm{Ab} \circ G \to \mathrm{Id}_{\mathbf{Ab}}$ is given by $\epsilon_A = \mathrm{Id}_A$. Note that the unit $\eta : \mathrm{Id}_{\mathbf{Grp}} \to G \circ \mathrm{Ab}$ is given by $\eta_G : G \to G^{\mathrm{ab}}$ (the natural projection), which is not an isomorphism.*

In Example 1.3.12 (5), the unit is given by the canonical morphism $V \to T(V)$ where $V \in \mathbf{Vect}_k$, and the counit is $T(A) \to A$ where $A \in \mathbf{Alg}_k$. Neither the unit or the counit is a natural isomorphism, so neither of the two functors is fully faithful.

Corollary 1.3.20. *Let $\phi : F \dashv G$. Then the following conditions are equivalent:*
(1) F is an equivalence of categories;
(2) G is an equivalence of categories;
(3) F and G are fully faithful;
(4) the unit η and the counit ϵ are natural transformations.
Under these conditions, F and G are quasi-inverse to each other.

Proof. Clear. □

Adjunction and (co)limits

Proposition 1.3.21. *Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then*
(1) $F^{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ is left adjoint to $G^{\mathcal{I}} : \mathcal{D}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$ for any category $\mathcal{I}$.
(2) G preserves limits and F preserves colimits whenever exist.

Proof. (2) Let $S : \mathcal{I} \to \mathcal{D}$ be a functor such that $\lim S$ exists. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "F"'] & \mathcal{C}^{\mathcal{I}} \arrow[d, "F^{\mathcal{I}}"] \\ \mathcal{D} \arrow[r, "\Delta"] & \mathcal{D}^{\mathcal{I}} \end{tikzcd}$$

we have

$$\mathrm{Hom}(X, \lim G^{\mathcal{I}} S) \cong \mathrm{Hom}(\Delta X, G^{\mathcal{I}} S) \cong \mathrm{Hom}(F^{\mathcal{I}} \Delta X, S) \cong \mathrm{Hom}(FX, \lim S) \cong \mathrm{Hom}(X, G \lim S).$$

so that $G \lim S$ is a limit of GS. □

1.4 Lecture 4 (2019-03-06)

1.4.1 Preadditive categories

Definition 1.4.1. *A* preadditive *category $\mathcal{C}$ is a category such that every* hom-set $\mathrm{Hom}(X,Y)$ *in $\mathcal{C}$ has the structure of an abelian group, and composition of morphisms is bilinear, in the sense that composition of morphisms distributes over the group operation. In formulas:*

$$f\circ(g+h)=(f\circ g)+(f\circ h),\qquad \text{resp. } (g+h)\circ f=(g\circ f)+(g\circ f),$$

where $+$ is the group operation and $f:X\to Y$, $g,h:W\to X$ (resp. $g,h:Y\to Z$).

Remark 1.4.2. *If $\mathcal{C}$ is a preadditive category, then for any $X\in\mathcal{C}$, $\mathrm{End}_{\mathcal{C}}(X):=\mathrm{Hom}_{\mathcal{C}}(X,X)$ becomes a ring. In fact, preadditive categories can be seen as a generalisation of rings. Given a ring R, we form the preadditive category with one single object $\{*\}$ and $\mathrm{Hom}(*,*)=R$ which is an abelian group and the composition of morphisms is bilinear.*

Example 1.4.3. *(i) The category R-**Mod** is preadditive. But the category **Grp** is not, because $\mathrm{Hom}(G_1,G_2)$ is in general not an abelian group. The category **CRing** is not preadditive neither.*

Proposition 1.4.4. *In a preadditive category $\mathcal{C}$, every finite product is a coproduct, i.e. the morphism $X_1\coprod X_2\to X_1\prod X_2$ described by $\left(\begin{smallmatrix}\mathrm{Id}_{X_1} & 0\\ 0 & \mathrm{Id}_{X_2}\end{smallmatrix}\right)$ is an isomorphism.*

Proof. We show that $X\prod Y$ (if exists) is a coproduct with respect to the morphisms

$$i_1=(\mathrm{Id}_{X_1},0):X_1\to X_1\prod X_2,\qquad i_2:(0,\mathrm{Id}_{X_2}):X_2\to X_1\prod X_2.$$

Namely, i_1 is induced, by the universal property, from morphisms $\mathrm{Id}:X_1\to X_1$ and $0:X_1\to X_2$. In particular, $p_2i_1=p_1i_2=0$ where p_i denotes the natural projection from $X_1\prod X_2$ to X_i. We claim that $i_1p_1+i_2p_2$ is equal to $\mathrm{Id}_{X_1\prod X_2}$. Indeed, there is a unique morphism u making the following diagram commute:

$$\begin{tikzcd}
X_1\prod X_2 \arrow[dr, dotted, "u"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_1\prod X_2 \arrow[r, "p_1"'] \arrow[d, "p_2"] & X_2 \\
& X_1 &
\end{tikzcd}$$

and both $i_1p_1+i_2p_2$ and $\mathrm{Id}_{X_1\prod X_2}$ enjoy this property.

Let Y be an object of $\mathcal{C}$ equipped with $h_1:X_1\to Y$ and $h_2:X_2\to Y$:

$$\begin{tikzcd}
X_1\prod X_1 \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & X_2 \arrow[dl, "i_2"'] \arrow[dd, "h_2"] \\
& X_1\prod X_2 \arrow[dr, dotted, "h"] & \\
X_1 \arrow[ur, "i_1"] \arrow[rr, "h_1"'] & & Y.
\end{tikzcd}$$

We want to construct a (unique) morphism $h : X_1 \prod X_2 \to Y$ such that $hi_1 = h_1$ and $hi_2 = h_2$. For this, we set

$$h = h_1p_2 + h_2p_2 : X_1 \prod X_2 \to Y.$$

Then $hi_1 = h_1p_1i_1 + h_2p_2i_1 = h_1 + 0 = h_1$ and similarly $hi_2 = h_2$. We also need to check the uniqueness of h. In fact, if $h' : X_1 \prod X_2 \to Y$ is another morphism such that $h'i_1 = h_1$ and $h'i_2 = h_2$, then

$$h' = h'(i_1p_1 + i_2p_2) = h_1p_1 + h_2p_2 = h.$$

□

Remark 1.4.5. *The proof gives the following. If there exists an object $Z \in \mathcal{C}$ together with morphisms $i_1 : X \to Z$, $i_2 : Y \to Z$, $p_1 : Z \to X$, $p_2 : Z \to Y$ satisfying*

$$\begin{aligned} &p_1i_1 = \mathrm{Id}_X, \quad p_1i_2 = 0 \\ &p_2i_1 = 0, \quad p_2i_2 = \mathrm{Id}_Y \\ &i_1p_1 + i_2p_2 = \mathrm{Id}_Z. \end{aligned} \tag{1.6}$$

Then the objects $X \prod Y$, $X \coprod Y$ and Z are naturally isomorphic.

1.4.2 Additive categories

Definition 1.4.6. *An* additive category *is a preadditive category $\mathcal{A}$ admitting all finite products (hence all finite coproducts).*

Proposition 1.4.7. *Let $\mathcal{A}$ be a category admitting a zero object, finite products, and finite coproducts, such that the natural map from the coproduct to the product*

$$\phi : Y \coprod Y \to Y \prod Y$$

is an isomorphism for every $Y \in \mathcal{O}(\mathcal{A})$. Then there exists a unique way to equip every $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ *with a unital magma such that composition is bilinear. Moreover,* $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ *is a commutative monoid.*

Proof. It is clear that the conditions are necessary. For the converse, we first show how to equip $\mathrm{Hom}(X, Y)$ a structure of group operation using (1). Let $f, f' : X \to Y$, we define $f + f'$ to be the composite

$$Y \overset{(f,f')}{\to} Y \prod Y \overset{\phi^{-1}}{\to} Y \coprod Y \overset{(\mathrm{Id}_Y,\mathrm{Id}_Y)}{\to} Y,$$

where the first map is induced by the universal property of product $Y \prod Y$ from $f, f' : X \to Y$, and similarly for the third map. We need to check that this operation enjoys the following properties:

(i) $f + 0 = f$, $0 + f = f$;

(ii) $(g + g')f = gf + g'f$, where $X \overset{f}{\to} Y \overset{(g,g')}{\rightrightarrows} Z$;

(iii) $f + f' = f' + f$;

(iv) $f + (f' + f'') = f + (f' + f'')$.

For (ii), we need to check the diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "{(f,f')}"] \arrow[dr, "{(gf,gf')}"'] & Y\prod Y \arrow[r, "\phi^{-1}"] \arrow[d, dotted, "g\prod g"] & Y\coprod Y \arrow[r, "{(\mathrm{Id}_Y,\mathrm{Id}_Y)}"] \arrow[d, dotted, "g\coprod g"] & Y \arrow[d] \\
& Z\prod Z \arrow[r, "\phi^{-1}"] & Z\coprod Z \arrow[r, "{(\mathrm{Id}_Z,\mathrm{Id}_Z)}"] & Z.
\end{tikzcd}$$

We may fill this diagram with morphisms $g\prod g$ and $g\coprod g$ which make each of the three small diagrams commutes. See exercise for the middle one.

For (iii), one checks with the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "{(f,f')}"] \arrow[dr, "{(f',f)}"'] & Y\prod Y \arrow[r, "\phi^{-1}"] \arrow[d, dotted, "{(p_2,p_2)}"] & Y\coprod Y \arrow[r, "{(\mathrm{Id}_Y,\mathrm{Id}_Y)}"] \arrow[d, dotted, "{(i_2,i_1)}"'] & Y \\
& Y\prod Y \arrow[r, "\phi^{-1}"'] & Y\coprod Y \arrow[ur, "{(\mathrm{Id}_Y,id_Y)}"'] &
\end{tikzcd}$$

where the morphism (p_2, p_1) is induced by the universal property of a product and similarly for (i_2, i_1).

See Zheng's lecture for (i) and (iv). □

Corollary 1.4.8. *A category $\mathcal{A}$ is additive if and only if it admits a zero object, finite products, finite coproducts and such that*

(1) the natural map $\phi : Y\coprod Y \to Y\prod Y$ is an isomorphism for every $Y \in \mathcal{O}(\mathcal{A})$;

(2) $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group.

Proof. Clear from the definition and the above proposition 1.4.7. □

Example 1.4.9. *Let R be a ring. The category R-**Mod** is an additive category. The full subcategory of all finitely generated R-modules is also an additive category.*

Example 1.4.10. *Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ be a category. Then the functor category $\mathcal{A}^{\mathcal{I}}$ is also an additive category (proof omitted).*

Lemma 1.4.11. *(1) In a category with a zero object, every monomorphism $f : X \to Y$ has zero kernel, and every epimorphism has zero cokernel.*

(2) In an additive category $\mathcal{A}$, every morphism of zero kernel $f : X \to Y$ is a monomorphism and every morphism of zero cokernel is an epimorphism.

Proof. (1) Show that $0 \xrightarrow{0} X$ satisfies the universal property of $\mathrm{Ker}(f : X \to Y)$. Let $g : Z \to X$ be such that $fg = 0 = f0$, then since f is monic, $g = 0_{Z\to X}$, thus g factors as $Z \to 0 \to X$.

(2) We only check the statement in the first case. We need to show if $W \overset{(g_1,g_2)}{\rightrightarrows} X \xrightarrow{f} Y$ is such that $fg_1 = fg_2$, then $g_1 = g_2$. Since $\mathcal{A}$ is additive, we have $f(g_1 - g_2) = 0$. Thus there exists a morphism $W \to 0 = \mathrm{Ker}(f)$ such that

$$(g_1 - g_2 : W \to X) = (W \to 0 \to X) = 0_{W\to X}.$$

This implies $g_1 - g_2 = 0$. □

Additive functors

Proposition 1.4.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then the following conditions are equivalent:*

(1) F preserves products of pairs of objects;

(2) F preserves coproducts of pairs of objects;

(3) for every pair A, A' of $\mathcal{A}$, the map

$$\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(FA, FA')$$

induced by F is a group homomorphism.

Proof. (1)⇒(3). Let $f, g : A \to B$. Then $f+g$ is given by the composition

$$A \overset{(f,g)}{\to} B \prod B \overset{\phi^{-1}}{\to} B \coprod B \overset{(\mathrm{Id},\mathrm{Id})}{\to} B.$$

One checks that applying F to it, we obtain

$$FA \overset{(Ff,Fg)}{\to} FB \prod FB \overset{\phi^{-1}}{\to} FB \coprod FB \overset{(\mathrm{Id},\mathrm{Id})}{\to} FB,$$

so that that $F(f+g) = Ff + Fg$.

(3)⇒(1). We must show that $(Fp_1, Fp_2) : F(A_1 \prod A_2) \to FA_1 \prod FA_2$ is an isomorphism, where $p_j : A_1 \times A_2 \to A_j$ is the natural projection. We have morphisms

$$Fp_j : F(A_1 \prod A_2) \to A_j, \qquad Fi_j : FA_j \to F(A_1 \prod A_2),$$

and by assumption they also satisfy the relations listed in (1.6). □

Example 1.4.13. *(1) Let $\mathcal{A}$ be an additive category. The functor*

$$\mathcal{A} \to \mathcal{A}, \qquad A \mapsto A \times A$$

is additive. However, the functor $\mathcal{A} \to \mathcal{A}$, sending A to $A \times B$ is not additive unless $B = 0$.

(2) Be attention that, the functor

$$(R,R)\text{-}\mathbf{Mod} \to (R,R)\text{-}\mathbf{Mod}, \qquad A \mapsto A \otimes_S A$$

is not additive unless $S = 0$, because it does not preserve products.

1.4.3 Abelian categories

Definition 1.4.14. *Let $\mathcal{A}$ be an additive category admitting kernels and cokernels. Let $f : A \to B$ be a morphism. We define the* coimage *and* image *of f to be*

$$\mathrm{Coim}(f) = \mathrm{Coker}(g), \qquad \mathrm{Im}(f) = \mathrm{Ker}(h)$$

where $g : \mathrm{Ker}(f) \to A$ and $h : B \to \mathrm{Coker}(f)$ are the canonical morphisms.

Thus, every morphism $f : A \to B$ factors uniquely into

$$A \twoheadrightarrow \mathrm{Coim}(f) \to \mathrm{Im}(f) \hookrightarrow B. \tag{1.7}$$

Definition 1.4.15. *An* abelian category *is an additive category $\mathcal{A}$ satisfying the following axioms:*

(AB1) $\mathcal{A}$ admits kernels and cokernels.

(AB2) For each morphism $f : A \to B$, the morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Remark 1.4.16. *Since an additive category $\mathcal{A}$ admits all finite products and finite coproducts, hence combined with (AB1) which implies that $\mathcal{A}$ admits equalizers and coequalizers, we deduce that $\mathcal{A}$ admits all finite limits and finite colimits.*

Example 1.4.17. *(1) R-**Mod** is an abelian category, and as we shall see, it is the most important class of abelian category. But R-**mod** is not an abelian category in general, because if $I \subset R$ is an ideal which is not finitely generated then the quotient morphism $R \to R/I$ has no kernel in R-**mod**; it is abelian if and only if R is noetherian. On the other hand, the full subcategory of R-**Mod** consisting of all noetherian (or artinian) modules is an abelian category.*

(2) If $\mathcal{A}$ is an abelian category and $\mathcal{I}$ is a category, then the functor category $\mathcal{A}^{\mathcal{I}}$ is an abelian category.

(3) The category of topological abelian groups is an additive category admitting kernels and cokernels, but does not satisfies (AB2). For example, if $f : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}$ is the identity map (which is both monic and epi), then $\mathrm{Coim}(f) = \mathbb{R}_{\mathrm{disc}}$ and $\mathrm{Im}(f) = \mathbb{R}$.

(4) Another important non-abelian category is the category FilMod_k of filtered modules over k. Precisely, an object is a k-vector space M with a decreasing filtration $\{F_nM\}_{n\in\mathbb{Z}}$. A morphism between two objects is a morphism $f : M \to M'$ such that $f(F_nM) \subset F_nM'$. Take $M = k[X]$ with the filtration given by

$$F_nM = \begin{cases} k[X] & n \geq 0 \\ (X^n) & n \geq 0. \end{cases}$$

We can define another filtration on $k[X]$ by putting $F_nM' := (X^{2n})$. Clearly the identity morphism $M' \to M$ is not an isomorphism. Hence FilMod_k is not abelian.

Some properties in an abelian category $\mathcal{A}$:

(1) If a morphism is both a monomorphism and an epimorphism, then it is an isomorphism.

Proof. By (1.7), it suffices to prove $A \to \mathrm{Coim}(f)$ and $\mathrm{Im}(f) \to B$ are both isomorphisms. We prove only the first. By definition $\mathrm{Coim}(f) \cong \mathrm{Coker}(g)$ is an isomorphism, where $g : \mathrm{Ker}(f) \to A$. Since f is monic, we know $\mathrm{Ker}(f) = 0$ and $g = 0_{0\to A}$. Then it is direct to check that $\mathrm{Id}_A : A \to A$ is the cokernel of the zero morphism.

(2) Every monomorphism is the kernel of its cokernel.

Proof. Consider a monomorphism $0 \to X \xrightarrow{f} Y$. By definition, $\mathrm{Im}(f)$ is the kernel of $Y \to \mathrm{Coker}(f)$. But the morphism $X \to \mathrm{Im}(f)$ is monic and epi, hence is an isomorphism by (1).

(3) Every epimorphism is the cokernel of its kernel.

(4) Every morphism $f : A \to B$ can be decomposed into

$$A \xrightarrow{g} \mathrm{Im}(f) \xrightarrow{f} B$$

where g is an epimorphism and h is a monomorphism.

Definition 1.4.18. *We sat that a sequence $A \to B \to C$ with $gf = 0$ in $\mathcal{A}$ is* exact at B *is the morphism* $\mathrm{Coim}(f) \to \mathrm{Ker}(g)$ *is an isomorphism. We say that a sequence*

$$A^0 \to A^1 \to \cdots \to A^n$$

is exact if it is exact at each A^i, for $1 \leq i \leq n-1$.

Exercise 1. Prove the direction (2)⇒(1) in Proposition 1.3.5.

Exercise 2. Prove Lemma 1.3.16.

Exercise 3. Let $\mathcal{A}$ be as in Proposition 1.4.7 and $g : Y \to Z$ be a morphism in $\mathcal{A}$. Prove the commutativity of the diagram:

$$\begin{tikzcd} Y \coprod Y \arrow[r, "\phi_Y"] \arrow[d, "g \coprod g"'] & Y \prod Y \arrow[d, "g \prod g"] \\ Z \coprod Z \arrow[r, "\phi_Z"] & Z \prod Z, \end{tikzcd}$$

where $g \prod g$ and $g \coprod g$ are natural morphisms. For example, if $p_i : Y \prod Y \to Y$ denotes the two projections, then $g \prod g$ is the universal morphism induced from (gp_1, gp_2), i.e.

$$\begin{tikzcd} Y \prod Y \arrow[dr, dotted] \arrow[drr, bend left, "gp_2"] \arrow[ddr, bend right, "gp_1"'] & & \\ & Z \prod Z \arrow[r] \arrow[d] & Z \\ & Z. & \end{tikzcd}$$

Exercise 4. Let $\mathcal{C}$ be a category which admits inductive limits. One says that an object X of $\mathcal{C}$ is of finite type if for any functor $\alpha : I \to \mathcal{C}$ with I a directed set, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ injective. Show that this definition coincides with the classical one when $\mathcal{C} = \mathbf{Mod}(R)$, for a ring R.

Solution: We need to show in the category R-**Mod**, a module M is of finite type if and only if for any direct limit $\varinjlim \alpha(i)$, the map

$$\eta_M : \varinjlim \mathrm{Hom}_R(M, \alpha_i) \to \mathrm{Hom}_R(M, \varinjlim \alpha_i) \tag{1.8}$$

is injective.

⇒. is clear. In fact, if M is of finite type, it is a quotient of R^n for some n, i.e. $R^n \to M \to 0$, hence

$$\begin{tikzcd} & & 0 \arrow[d] \\ 0 \arrow[r] & \varinjlim \mathrm{Hom}_R(M, \alpha_i) \arrow[r] \arrow[d, "\eta_M"] & \varinjlim \mathrm{Hom}_R(R^n, \alpha_i) \arrow[d, "\eta_{R^n}"] \\ 0 \arrow[r] & \mathrm{Hom}_R(M, \varinjlim \alpha_i) \arrow[r] & \mathrm{Hom}_R(R^n, \varinjlim \alpha_i) \end{tikzcd}$$

Here the top horizontal arrow is injective because filtered direct colimit is exact and η_{R^n} is injective because η_R is.

$\Leftarrow$. Assume M is not of finite type. Since every element in M is contained in a submodule of finite type (indeed contained in the submodule generated by itself), we may write $M = \varinjlim_{i\in I} M_i$, where M_i runs over all submodules of finite type. We see that $\alpha := \{M/M_i\}_{i\in I}$ also form a direct system. We claim that for this system α, η_M in (1.8) is not injective. Indeed, for each i, let $p_i : M \to M/M_i$ be the natural projection, then $(p_i)_{i\in I}$ defines an element in $\varinjlim \mathrm{Hom}_R(M, M/M_i)$. However, $\eta_M\big((p_i)_{i\in I}\big)$ is identically zero because $\varinjlim M/M_i = 0$ (any $x \in M$ is contained in some M_i).

1.5 Lecture 5 (2019-03-11)

1.5.1 Exact functors

Definition 1.5.1. *Let $\mathcal{C}$ be a category admitting finite limits (resp. finite colimits). We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is left exact (resp.* right exact*) if it preserves finite limits (resp. finite colimits). For $\mathcal{C}$ admitting finite limits and finite colimits, we say that F is exact if it is both left exact and right exact.*

Proposition 1.5.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:*

(1) F is left exact.

(2) F preserves kernels; namely, for every exact sequence $0 \to X \to Y \to Z$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is an exact sequence in $\mathcal{B}$;

(3) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ$ is exact in $\mathcal{B}$.

Proof. (1)$\Rightarrow$ (2) $\Rightarrow$ (3), obvious.

(2)$\Rightarrow$ (3) This follows from ...and the assumption that F preserves finite products.

(3)$\Rightarrow$(2) Also easy. □

Corollary 1.5.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then the following conditions are equivalent:*

(1) F is exact.

(2) For every exact sequence $X \to Y \to Z$ in $\mathcal{A}$, $FX \to FY \to FZ$ is exact in $\mathcal{B}$.

(3) For every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, $0 \to FX \to FY \to FZ \to 0$ is a short exact sequence in $\mathcal{B}$.

(4) F is left exact and preserves epimorphisms.

(5) F is right exact and preserves monomorphisms.

Proof. Obvious. □

Proposition 1.5.4. *If a functor $F : \mathcal{A} \to \mathcal{B}$ admits a left (resp. right) adjoint, then F is left (resp. right) exact.*

Proof. Since F is left exact (resp. right exact) if and only if it preserves kernels (resp. cokernels), this follows from Proposition 1.3.21. □

Example 1.5.5. *We consider the category* **Ab** *in the following examples.*

(1) The functor $\mathrm{Hom}_{\mathbf{Ab}}(-, \mathbb{Z})$ is not exact. For example, consider the exact sequence $0 \to \mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, we obtain

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \overset{\times 2}{\to} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}).$$

It is easy to see that $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and the second morphism is not surjective: the identity map is not in the image.

(2) The functor $\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ is not exact. Still consider the above short exact sequence which implies

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0;$$

here the first morphism is clearly zero morphism, hence not injective.

(3) If $\mathcal{A}$ admits limits (resp. colimits) indexed by a category $\mathcal{I}$, then $\lim : \mathcal{A}^I \to \mathcal{A}$ (resp. $\mathcal{A}^{\mathcal{I}} \to \mathcal{A}$) is left (resp. right) exact.

The Mittag-Leffler condition

Example 1.5.6. *$\varprojlim : R\text{-}\mathbf{Mod}^{\mathcal{I}} \to R\text{-}\mathbf{Mod}$ is left exact, but not right exact in general.*

Consider $R = k[X]$ with k a field. Denote by $J = (x)$ be the principal ideal generated by x. One has

$$\varprojlim_n R/J^n \cong k[[X]],$$

the ring of formal series with coefficients in k. On the other hand, if $m \leq n$, the monomorphism $J^n \hookrightarrow J^m$ defines a projective system, i.e. $\cdots \hookrightarrow J^n \hookrightarrow \cdots \hookrightarrow J^1 \hookrightarrow R$, whose projective limit is 0 (the intersection of all J^n). Thus, if we consider the exact sequence of R-modules

$$0 \to J^n \to R \to R/J^n \to 0$$

and taking the projective limit, we get the sequence

$$0 \to 0 \to k[X] \to k[[X]] \to 0$$

*which is no more exact, neither in the category R-**Mod** nor in the category k-**Mod**.*

Let I be a directed set.

Definition 1.5.7. *Let $(A_i, \varphi_{ij} : A_j \to A_i)$ be a directed inverse system of sets over I. Then we say (A_i, φ_{ij}) is* Mittag-Leffler *if for each $i \in I$, the family $\varphi_{ij}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $f_{ik}(A_k) = \varphi_{ij}(A_j)$. If (A_i, φ_{ij}) is a directed inverse system of modules over a ring R, we say that it is* Mittag-Leffler *if the underlying inverse system of sets is Mittag-Leffler.*

Example 1.5.8. *If (A_i, φ_{ij}) is a directed inverse system of sets or of modules and the maps φ_{ij} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ij}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ij}(A_j)$ for $j \geq i$. Then $\varphi_{ij}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $\varprojlim A_i = \varprojlim A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ij}) can also be written as the limit of a directed inverse system over I with surjective maps.*

Lemma 1.5.9. *Let (A_i, φ_{ij}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ij}) is Mittag-Leffler and the A_i are nonempty, then $\varprojlim A_i$ is nonempty.*

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the subsequence $\{j_n\}_{n \geq 1}$ is increasing and forms a cofinal subset of I. Hence we may assume $I = 1, 2, 3, \ldots$ (with the natural total order). So by Example 1.5.8 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This is obvious. □

Theorem 1.5.10. *Let*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\varprojlim B_i \to \varprojlim C_i$. So let $(c_i) \in \varprojlim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ij} : B_j \to B_i$ restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 1.5.9 would show $\varprojlim E_i$ is nonempty, and taking any element of $\varprojlim E_i$ would give an element of $\varprojlim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ik}(A_k) = \varphi_{ij}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ik}(E_k) = \varphi_{ij}(E_j)$ for $k \geq j$. Always $\varphi_{ik}(E_k) \subset \varphi_{ij}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $e_k \in E_k$ such that $\varphi_{ik}(e_k) = \varphi_{ij}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{jk}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ik}(A_k) = \varphi_{ij}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ik}(a_k) = \varphi_{ij}(a_j)$. Hence

$$\varphi_{ik}(e'_k + a_k) = \varphi_{ij}(e'_j) + \varphi_{ij}(a_j) = \varphi_{ij}(e_j),$$

so we can take $e_k = e'_k + a_k$. □

Exactness of filtered colimits

Example 1.5.11. lim : *is not right exact in general. We give an example for* cokernels. *Consider the following diagram:*

$$\begin{tikzcd} 0 \arrow[r] & 4\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "="] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 2\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

the snake lemma gives

$$0 \to 0 \to 0 \to 2\mathbb{Z}/4\mathbb{Z}, \qquad 2\mathbb{Z}/4\mathbb{Z} \to 0 \to 0 \to 0$$

which shows that kernel is not right exact and cokernel is not left exact.

Definition 1.5.12. *A non-empty category $\mathcal{I}$ is said to be* filtered *if*

(1) For $i, j \in \mathrm{Ob}(\mathcal{I})$, there exists morphisms $i \to k$, $j \to k$ in $\mathcal{I}$.

(2) for morphisms $u, v : i \rightrightarrows j$, there exists a morphism $w : j \to k$ such that $wu = wv$.

Proposition 1.5.13. *In the category* **Set** *or R-***Mod** *for a ring R, small filtered colimits are exact.*

Proof. It remains to check that colim is left exact, i.e. preserves monomorphisms. Let $f : F \to G$ be a monomorphism in $\mathbf{Set}^{\mathcal{I}}$ and $\mathrm{colim} f : \mathrm{colim} F \to \mathrm{colim} G$ be the morphism in **Set**. Recall that $\mathrm{colim} F$ is represented by $\sqcup_{i \in \mathcal{I}} F(i)/\sim$, where if $x \in F(i)$ and $y \in F(j)$, then $x \sim y$ if and only if there exists $u : i \to k$, $v : j \to k$ such that $F(u)(x) = F(v)(y)$. Let $[x] \in \mathrm{Ker}(\mathrm{colim} f)$ and let $x \in F(i)$ be a representative of $[x]$. Since $\mathrm{colim} f([x]) = [f_i(x)] = 0$, there exists $u : i \to k$ such that $(Gu)(f_i(x)) = 0$. The commutative diagram below

$$\begin{tikzcd} F(i) \arrow[r, "f_i"] \arrow[d, "Fu"'] & G(i) \arrow[d, "Gu"] \\ F(k) \arrow[r, "f_k"] & G(k) \end{tikzcd}$$

shows that $f_k((Fu)x) = 0$. But f_k is injective, so $(Fu)(x) = 0$, hence $[x] = 0$. □

Remark 1.5.14. *In his Tōhoku article, Grothendieck listed four additional axioms (and their duals) that an abelian category $\mathcal{A}$ might satisfy. Some of them are:*

(AB3) For every family (A_i) of objects of $\mathcal{A}$, the coproduct of A_i exists in $\mathcal{A}$ (i.e. $\mathcal{A}$ is cocomplete). (Since $\mathcal{A}$ admits coequalizers, $\mathcal{A}$ admits all colimits.)

(AB5) $\mathcal{A}$ satisfies AB3), and filtered colimits of exact sequences are exact.

*Hence the categories R-**Mod** and **Set** satisfy (AB3), (AB5).*

The proof of the following result is omitted.

Proposition 1.5.15. *For any small filtered category $\mathcal{I}$, any finite category $\mathcal{J}$ and any functor $F : \mathcal{I} \times \mathcal{J} \to \mathbf{Set}$, the map*

$$\operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} F(i,j) \to \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} F(i,j)$$

is a bijection.

Proof. See Zheng, Prop. 1.6.25. □

1.6 Lecture 6 (2019-03-13)

Freyd-Mitchell Embedding Theorem

Theorem 1.6.1 (Freyd-Mitchell)**.** *Let $\mathcal{A}$ be a small abelian category. Then there exists a small ring and a fully faithful exact functor $F : \mathcal{A} \to R$-**Mod**.*

Remark 1.6.2. *For a fully faithful exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we have the following facts:*

(1) if $A \in \mathcal{A}$ is such that $FA = 0$, then $A = 0$;

(2) if $f : A \to A'$ is a morphism in $\mathcal{A}$ such that $Ff : FA \to FA'$ is the zero morphism, then f is the zero morphism.

Remark 1.6.3. *The functor F yields an equivalence between $\mathcal{A}$ and a full subcategory of R-**Mod** in such a way that kernels and cokernels computed in $\mathcal{A}$ correspond to the ordinary kernels and cokernels computed in R-**Mod**. The theorem thus essentially says that the objects of $\mathcal{A}$ can be thought of as R-modules, and the morphisms as R-linear maps, with kernels, cokernels, exact sequences and sums of morphisms being determined as in the case of modules.*

However, projective and injective objects in $\mathcal{A}$ do not necessarily correspond to projective and injective R-modules. There are more morphisms to test in $\mathcal{A}$.

For practical use, we also need the following lemma.

Lemma 1.6.4. *Let $\mathcal{A}$ be an abelian category and $S \subseteq \mathrm{Ob}(\mathcal{A})$ be a non-empty small set of objects. There is a full small abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ containing S.*

Proof. Define inductively a sequence $\{\mathcal{A}_n\}_{n\geq 0}$ of subcategories of $\mathcal{A}$ as follows. First, $\mathcal{A}_0$ is the category whose objects are the objects in S and whose morphisms are as in $\mathcal{A}$. Given $\mathcal{A}_n$, $\mathcal{A}_{n+1}$ is the full subcategory of $\mathcal{A}$ consisting of the objects in $\mathcal{A}_n$ together with

- single representatives for kernels and cokernels in $\mathcal{A}$ of every morphism in $\mathcal{A}_n$;
- single representatives for all finite products in $\mathcal{A}$ of objects in $\mathcal{A}_n$.

If $\mathcal{A}_n$ is small, then so is $\mathcal{A}_{n+1}$. Since $\mathcal{A}_0$ is small, so is $\mathcal{A}_n$ for all $n \geq 0$. Consequently, $\mathcal{B} = \cup_{n=0}^{\infty}\mathcal{A}_n$ is small, and $\mathcal{B}$ is a full abelian subcategory of $\mathcal{A}$. $\square$

With Theorem 1.6.4 and Lemma 1.6.4, we may reduce many proofs in general abelian category to the category R-**Mod** for some ring R.

1.6.1 Diagrams

Let $\mathcal{A}$ be an abelian category.

Lemma 1.6.5. *Consider a commutative diagram in $\mathcal{A}$ with exact rows:*

$$\begin{tikzcd} & X' \arrow[r, "f"] \arrow[d, "u'"] & X \arrow[r, "g"] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y''. & \end{tikzcd}$$

There exists an exact sequence

$$\ker(u') \to \ker(u) \to \ker(u'') \xrightarrow{\delta} \mathrm{Coker}(u') \to \mathrm{Coker}(u) \to \mathrm{Coker}(u'').$$

If, moreover, f is monic, then so is $\ker(u') \to \ker(u)$; if g' is epi, then so is $\mathrm{Coker}(u) \to \mathrm{Coker}(u'')$.

Proof. By diagram chasing, we check that the statement is true if $\mathcal{A}$ is replaced by R-**Mod**. Then let $\mathcal{B}$ be a small, full, abelian subcategory of $\mathcal{A}$ containing the X, Y, etc. Then the diagram is commutative and the rows are exact in $\mathcal{B}$. Let $F : \mathcal{B} \to R$-**Mod** be an exact fully faithful embedding. In $\mathcal{B}$ we have the desired long exact sequence in R-**Mod**:

$$F(\mathrm{Ker}(u')) \to F(\mathrm{Ker}(u)) \to F(\mathrm{Ker}(u'')) \stackrel{F(\delta)}{\to} F(\mathrm{Coker}(u')) \to F(\mathrm{Coker}(u)) \to F(\mathrm{Coker}(u''))$$

Since $F : \mathcal{B} \to R$-**Mod** is fully faithful, this gives morphisms in $\mathcal{B}$, in particular the existence of δ in $\mathcal{B}$, and the corresponding long sequence is exact. $\square$

Corollary 1.6.6. *Let $0 \to X' \stackrel{f}{\to} X \stackrel{g}{\to} X'' \to 0$ be a short exact sequence in $\mathcal{A}$. Then the following conditions are equivalent:*

(1) f admits a retraction, i.e. there exists $r : X \to X'$ such that $rf = \mathrm{Id}_{X'}$.

(2) g admits a section, i.e. there exists $s : X'' \to X$ such that $gs = \mathrm{Id}_{X''}$.

(3) The sequence is isomorphic to the short exact sequence $0 \to X' \to X' \times X'' \to X'' \to 0$ where i and p are the canonical morphisms, i.e. there is an isomorphism $\alpha : X \to X' \times X''$ and a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d] & X'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r, "i"] & X' \times X'' \arrow[r, "p"] & X'' \arrow[r] & 0
\end{tikzcd}$$

Definition 1.6.7. *A short exact sequence satisfying the above conditions is said to be* split.

Example 1.6.8. *The sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is not split.*

Corollary 1.6.9. *Consider a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd}
X^0 \arrow[r, "f^0"] \arrow[d, "u^0"] & X^1 \arrow[r, "f^2"] \arrow[d, "u^1"] & X^2 \arrow[r, "f^2"] \arrow[d, "u^2"] & X^3 \arrow[d, "u^3"] \\
Y^0 \arrow[r, "g^0"] & Y^1 \arrow[r, "g^1"] & Y^2 \arrow[r, "g^2"] & Y^3
\end{tikzcd}$$

with exact rows.

(1) If u^0 is an epimorphism, u^1 and u^3 are monomorphisms, then u^2 is a monomorphism.

(2) If u^3 is a monomorphism, u^0 and u^2 are epimorphisms, then u^1 is an epimorphism.

Proof. (1) We consider the following induced diagram with short exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f^2) \arrow[r] \arrow[d, "\alpha"] & X^2 \arrow[r] \arrow[d, "u^2"] & \mathrm{Im}(f^2) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \ker(g^2) \arrow[r] & Y^2 \arrow[r] & \mathrm{Im}(g^2) \arrow[r] & 0.
\end{tikzcd}$$

To show u^2 is monic, it suffices to show α, β are monic by Snake lemma. Since u^3 is monic, so is β. On the other hand, one has

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(f^0) \arrow[r] \arrow[d, "\overline{u}^0"] & X^1 \arrow[r] \arrow[d, "u^1"] & \ker(f^2) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Im}(g^0) \arrow[r] & Y^1 \arrow[r] & \ker(g^2) \arrow[r] & 0;
\end{tikzcd}$$

since u^0 is epi, so is $\overline{u}^0$, hence by Snake lemma again, α is monic (because u^1 is monic). $\square$

Corollary 1.6.10. *(Five lemma) Consider a commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d, "u^0"] & X^1 \arrow[r] \arrow[d, "u^1"] & X^2 \arrow[r] \arrow[d, "u^2"] & X^3 \arrow[r] \arrow[d, "u^3"] & X^4 \arrow[d, "u^4"] \\ Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3 \arrow[r] & Y^4 \end{tikzcd}$$

with exact rows. If u^0 is an epimorphism, u^4 is a monomorphism, and u^1, u^3 are isomorphisms, then u^2 is an isomorphism.

Proof. Obvious from Corollary 1.6.9. □

1.6.2 Complexes

Definition 1.6.11. *A (cochain) complex in $\mathcal{A}$ consists of $X = (X^n, d^n)_{n\in\mathbb{Z}}$ (or $X^\bullet$), where*

- *X^n is an object of $\mathcal{A}$,*
- *$d^n : X^n \to X^{n+1}$ is a morphism in $\mathcal{A}$, called* differential *such that*

$$d^{n+1}d^n = 0.$$

A cochain morphism *of complex $X \to Y$ is a collection of morphisms $(f^n)_{n\in\mathbb{Z}}$, where $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that $d^n_Y f^n = f^{n+1}d^n_X$. Namely*

$$\begin{tikzcd} \cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

Similarly, we may define a chain complex in $\mathcal{A}$ and morphisms between them.

We let $C(\mathcal{A})$ denote the category of complexes in $\mathcal{A}$. This is also an abelian category.

Proposition 1.6.12. *. For $f : X^\bullet \to Y^\bullet$ a chain map of complexes,*

(1) the complex $\ker(f)$ *of degreewise kernels in $\mathcal{A}$ is the kernel of f in the category $C(\mathcal{A})$.*

(2) the complex $\operatorname{Coker}(f)$ *of degreewise cokernels in $\mathcal{A}$ is the cokernel of f in the category $C(\mathcal{A})$.*

Remark 1.6.13. *A sequence of cochain complexes*

$$0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$$

is a short exact sequence in precisely if each component

$$0 \to X^n \to Y^n \to Z^n \to 0$$

is a short exact sequence in $\mathcal{A}$.

Definition 1.6.14. *Let X be a complex in $\mathcal{A}$. We define*

$$Z^nX = \operatorname{Ker}(d^n : X^n \to X^{n+1})$$

$$B^nX = \operatorname{Im}(d^{n-1} : X^{n-1} \to X^n)$$

$$H^nX = \operatorname{Coker}(B^nX \hookrightarrow Z^nX),$$

and call them the cocycle, coboundary, cohomology *objects, of degree n.*

Definition 1.6.15. *A complex X is said to be* acyclic *if $H^n X = 0$ for all n. A morphism of complexes $X \to Y$ is called a quasi-isomorphism if $H^n f : H^n X \to H^n Y$ is an isomorphism for all n.*

Example 1.6.16. *(1) De Rham complex. The de Rham complex is the cochain complex of differential forms on a smooth manifold M of dimension n, with the exterior derivative as the differential:*

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,$$

where $\Omega^0(M)$ is the space of smooth functions on M (d^0 is the differential), $\Omega^k(M)$ is the space of k-forms. The cohomology of de Rham complex is called de Rham cohomology of M. For example, if $M = S^n$, the n-sphere, then

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n. \end{cases}$$

(2) Koszul chain complex. Let R be a commutative ring and $x_1, ..., x_n \in R$ (we abbreviate $\underline{x} = (x_1, ..., x_n)$). The Koszul chain complex $K_\bullet(\underline{x}, R) = (0 \to K_n \to K_{n-1} \to \cdots \to K_0 \to 0)$ is defined as follows.

- *$K_l = 0$ if $l \notin \{0, ..., n\}$.*
- *$K_0 = R$, and for $1 \leq l \leq n$, $K_l = \bigoplus Re_{i_1 \dots i_l}$ is the free R-module of rank $\binom{n}{l}$ with basis:*

$$\{e_{i_1 \dots i_l} \mid 1 \leq i_1 < \cdots < i_l \leq n\}.$$

- *The difference map $d : K_l \to K_{l-1}$ is defined as follows: if $l = 1$, $d(e_i) = x_i$; if $l \geq 2$, then*

$$d(e_{i_1 \dots i_l}) = \sum_{r=1}^{l} (-1)^{r-1} x_{i_r} e_{i_1 \dots \hat{i}_r \dots i_l};$$

For example, when $n = 2$, we get

$$0 \longrightarrow R \xrightarrow{\binom{-x_2}{x_1}} R \oplus R \xrightarrow{(x_1, x_2)} R \longrightarrow 0\,.$$

Its homology $H_0(K_\bullet)$ is just $R/(x_1, x_2)$, $H_2 = R[x_1, x_2]$ (annihilated by (x_1, x_2)), and

$$H_1(K_\bullet) = \frac{\{(r_1, r_2) \in R^2 | x_1 r_1 + x_2 r_2 = 0\}}{\{(-r x_2, r x_1) | r \in R\}}.$$

If $R = k[x_1, x_2]$ is a polynomial ring, then the Koszul complex is exact at degrees $n = 1, 2$; however, if $R = k[x_1, x_2]/(x_1^2, x_2^2)$, it is not exact.

Operations on complexes

1. **Shifting**: for any complex X and any $m \in \mathbb{Z}$ we denote by $X[m]$ the complex whose nth component is $X[k]^n = X^{k+n}$, whose nth morphism $X[k]^n \to X[k]^{n+1}$ is the map $(-1)^k d^{n+k}$. We have $H^n(X[k]) \cong H^{n+k}(X)$.

2. **Truncation functors**: for each n we define

$$\begin{array}{ll} \tau^{\leq n}X := & (\cdots \to X^{n-1} \to \ker(d_X^n) \to 0 \to \cdots) \\ \widetilde{\tau}^{\leq n}X := & (\cdots \to X^{n-1} \to X^n \to \mathrm{Im}(d_X^n) \to 0 \to \cdots) \\ \tau^{\geq n}X := & (\cdots \to 0 \to \mathrm{Coker}(d_X^{n-1}) \to X^{n+1} \to X^{n+2} \to \cdots) \\ \widetilde{\tau}^{\geq n} := & (\cdots \to 0 \to \mathrm{Im}(d_X^{n-1}) \to X^n \to X^{n+1} \to \cdots). \end{array}$$

There is a chain of morphisms in $C(\mathcal{A})$:

$$\tau^{\leq n}X \to \widetilde{\tau}^{\leq n}X \to X \to \widetilde{\tau}^{\geq n}X \to \tau^{\geq n}X$$

and there are exact sequences in $C(\mathcal{A})$:

$$\begin{cases} 0 \to \widetilde{\tau}^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \to 0 \\ 0 \to H^n(X)[-n] \to \tau^{\geq n}X \to \widetilde{\tau}^{\geq n+1}X \to 0 \\ 0 \to \tau^{\leq n}X \to X \to \widetilde{\tau}^{\geq n+1}X \to 0 \\ 0 \to \widetilde{\tau}^{\leq n-1}X \to X \to \tau^{\geq n}X \to 0. \end{cases}$$

3. **Direct sum**: given $X, Y \in C(\mathcal{A})$, we let $X \oplus Y = (X^n \oplus Y^n, d_X^n \oplus d_Y^n)$.

4. **Tensor products**. Let R be a commutative ring and consider the category R-**Mod**. The tensor product of two complexes $X, Y \in C(R\text{-}\mathbf{Mod})$ is defined as:

$$(X \otimes Y)^n = \bigoplus_{i+j=n} X^i \otimes Y^j, \qquad d^n(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db.$$

where $|a|$ denotes the degree of a. Precisely, the restriction of d to $X^i \otimes Y^j$ is given by:

$$a \otimes b \mapsto d_X^i a \otimes b + (-1)^i a \otimes d_Y^{n-i} b.$$

One checks that this is indeed a complex.

1.6.3 Long exact sequences

Theorem 1.6.17. *If $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ is a short exact sequence of cochain complexes, then there are natural maps $\partial : H^n(Z) \to H^{n+1}(X)$ and a long exact sequence*

$$\cdots \to H^{n-1}(Z) \xrightarrow{\partial} H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \xrightarrow{\partial} \cdots.$$

Similarly, if $0 \to X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet \to 0$ is a short exact sequence of chain complexes, then there are natural maps $\partial : H_n(C) \to H_{n-1}(A)$, called connecting homomorphisms, such that

$$\cdots \to H_{n+1}(Z) \xrightarrow{\partial} H_n(X) \to H_n(Y) \to H_n(Z) \xrightarrow{\partial} H_{n-1}(X) \to \cdots$$

is an exact sequence.

Proof. From the Snake lemma and the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^{n-1}X \arrow[r] \arrow[d] & Z^{n-1}Y \arrow[r] \arrow[d] & Z^{n-1}(Z) \arrow[d] & \\
0 \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "d"] & Y^{n-1} \arrow[r] \arrow[d, "d"] & Z^{n-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & X^n \arrow[r] \arrow[d] & Y^n \arrow[r] \arrow[d] & Z^n \arrow[r] \arrow[d] & 0 \\
 & \frac{X^n}{dX^{n-1}} \arrow[r] \arrow[d] & \frac{Y^n}{dY^{n-1}} \arrow[r] \arrow[d] & \frac{Z^n}{dZ^{n-1}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we see that the top and bottom rows are exact. Consider the following commutative diagram

$$\begin{tikzcd}
 & \frac{X^n}{dX^{n-1}} \arrow[r] \arrow[d] & \frac{Y^n}{dY^{n-1}} \arrow[r] \arrow[d] & \frac{Z^n}{dZ^{n-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}(X) \arrow[r] & Z^{n+1}(Y) \arrow[r] & Z^{n+1}(Z) &
\end{tikzcd}$$

whose kernels are H_n and cokernels are H^{n-1}; apply again the Snake lemma, we obtain

$$H^n(X) \to H^n(Y) \to H^n(Z) \xrightarrow{\partial} H^{n+1}(X) \to H^{n+1}(Y) \to H^{n+1}(Z).$$

Pasting these sequences together we obtain the result. $\square$

The way to obtain a long exact sequence from the short one $0 \to X \to Y \to Z \to 0$ is *natural*. Precisely, if there is a commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}$$

then there is a commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X) \arrow[r] \arrow[d] & H_n(Y) \arrow[r] \arrow[d] & H_n(Z) \arrow[r] \arrow[d] & H^{n+1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^n(X') \arrow[r] & H^n(Y') \arrow[r] & H^n(Z') \arrow[r] & H^{n+1}(X') \arrow[r] & \cdots
\end{tikzcd}$$

Corollary 1.6.18. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of complexes. If two of them have zero homology, then so is the third.*

Exercise 1. Let $\mathcal{A}$ be an abelian category. Prove that a complex $0 \to X \to Y \to Z$ is exact if and only if for any object $W \in \mathrm{Ob}(\mathcal{A})$ the complex of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{A}}(W, X) \to \mathrm{Hom}_{\mathcal{A}}(W, Y) \to \mathrm{Hom}_{\mathcal{A}}(W, Z)$$

is exact.

Exercise 2. Let $\mathcal{A}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \arrow[d] \\
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Y'' \arrow[d] \\
0 \arrow[r] & Z' \arrow[r] & Z \arrow[r] & Z''
\end{tikzcd}$$

Assume that all columns are exact. Prove: if the second and third rows are exact, then all rows are exact.

Exercise 3. We use the notation in Example . For each $x_i \in R$, we have a complex

$$K_\bullet(x_i, R): \qquad 0 \to R \xrightarrow{x_i} R \to 0$$

which can be viewed as the Koszul complex defined by x_i only. Prove that the Koszul complex $K_\bullet(\underline{x}, R)$ is isomorphic to the tensor product of $K_\bullet(x_i, R)$ for all $1 \le i \le n$.

Exercise 4. Calculate the homology of the Koszul complex $K_\bullet(x_1, x_2; R)$ in the following situation:

(a) $R = k[x_1, x_2]$;
(b) $R = k[x_1, x_2]/(x_1^2, x_2^2)$;
(c) $R = k[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$.

1.7 Lecture 7 (2019-03-18)

1.7.1 Mapping cones

Definition 1.7.1. *Let $f : X \to Y$ be a morphism in $C(\mathcal{A})$. We define the* mapping cone *of f to be the complex* $\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n$ *with differential*

$$d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

Namely, for $\begin{pmatrix} x \\ y \end{pmatrix} \in X^{n+1} \oplus Y^n$, $d^n_{\mathrm{Cone}(f)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d_X^{n+1} x \\ f^{n+1} x + d_Y^n y \end{pmatrix}$.

It is easy to check that $d^n_{\mathrm{Cone}(f)} d^{n-1}_{\mathrm{Cone}(f)} = 0$:

$$d^n_{\mathrm{Cone}(f)} d^{n-1}_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \begin{pmatrix} -d_X^{n} & 0 \\ f^{n} & d_Y^{n-1} \end{pmatrix} = \begin{pmatrix} d_X^{n+1} d_X^n & 0 \\ -f^{n+1} d_X^n + d_Y^n f^n & d_Y^n d_Y^{n-1} \end{pmatrix} = 0.$$

By construction, we have a short exact sequence in $C(\mathcal{A})$:

$$0 \to Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0$$

which induces a long exact sequence

$$\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta} H^n(Y) \to H^n(\mathrm{Cone}(f)) \to H^n(X[1]) \to \cdots$$

Proposition 1.7.2. *Via the isomorphism $H^{n-1}(X[1]) \cong H^n(X)$, the connecting morphism δ can be identified with $H^n f : H^n X \to H^n Y$. The long exact sequence thus has the form*

$$\cdots \to H^n X \xrightarrow{H^n f} H^n Y \xrightarrow{H^n i} H^n(\mathrm{Cone}(f)) \to H^{n+1} X$$

Proof. The connecting morphism is constructed using the snake lemma applied to the commutative diagram

$$\begin{tikzcd}
 & \frac{Y^{n-1}}{dY^{n-2}} \arrow[r] \arrow[d, "d_Y"] & \frac{C^{n-1}}{dC^{n-2}} \arrow[r] \arrow[d, "d_C"] & \frac{X^n}{dX^{n-1}} \arrow[r] \arrow[d, "d_X"] & 0 \\
0 \arrow[r] & Z^n(Y) \arrow[r] & Z^n(C) \arrow[r] & Z^{n+1}(X) &
\end{tikzcd}$$

where $C = \mathrm{Cone}(f)$. Recall the construction of the snake lemma (in the category R-**Mod**): for $x \in \ker(d_X)$, take $\binom{x}{0} + dC^{n-2}$ to be a lifting. Then $\delta(x)$ is by definition the image in $\mathrm{Coker}(d_Y)$ of the element $d_C\binom{x}{0} = \binom{0}{f^n(x)}$ (which belongs to $Z^n(Y)$). □

Proposition 1.7.3. *A morphism of complexes $f : X \to Y$ is a quasi-isomorphism if and only if its cone $\mathrm{Cone}(f)$ is acyclic.*

Proposition 1.7.4. *Consider a short exact sequence of complexes $0 \to X \to Y \to Z \to 0$. Then the map $\phi = (0, g) : \mathrm{Cone}(f) \to Z$ is a quasi-isomorphism.*

Proof. Easy. □

1.7.2 Chain homotopies

Definition 1.7.5. *(1) A morphism $f : X \to Y$ in $C(\mathcal{A})$ is* homotopic to zero *if for all n, there exists a morphism $s^n : X^n \to Y^{n-1}$, such that*

$$f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(2) An object $X \in C(\mathcal{A})$ is homotopic to zero if Id_X is homotopic to zero.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X^{n-1} \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n"] & X^{n+1} \arrow[dl, "s^{n+1}"] \\
Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r] & Y^{n+1}.
\end{tikzcd}$$

Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Proposition 1.7.6. *If $f : X \to Y$ is homotopic to zero, then every map $f^n : H^n(X) \to H^n(Y)$ is zero. If f and g are homotopic, then they induce the same maps $H^n(X) \to H^n(Y)$.*

Proof. It is enough to prove the first assertion, so suppose that $f = ds + sd$. Every element of $H^n(X)$ is represented by an n-cocycle x. But $f(x) = d(sx) + 0$. That is $f(x)$ is an n-coboundary in Y. Thus, $x = 0$ in $H^n(Y)$. $\square$

We say that $f : X \to Y$ is a cochain homotopy equivalence if there is a map $g : Y \to X$ such that gf and fg are chain homotopic to Id_X and Id_Y, respectively.

Homotopy category $K(\mathcal{A})$

Definition 1.7.7. *The homotopy category $K(\mathcal{A})$ is defined by:*

$$\begin{aligned}
\mathrm{Ob}(K(\mathcal{A})) &= \mathrm{Ob}(C(\mathcal{A})) \\
\mathrm{Hom}_{K(\mathcal{A})}(X, Y) &= \mathrm{Hom}_{C(\mathcal{A})}(X, Y)/\mathrm{Ht}(X, Y),
\end{aligned}$$

where $\mathrm{Ht}(X, Y) := \{f : X \to Y \mid f \text{ is homotopic to } 0\}$.

In other words, a morphism homotopic to zero in $C(\mathcal{A})$ becomes the zero morphism in $K(\mathcal{A})$ and a homotopy equivalence becomes an isomorphism.

1.7.3 Double complexes

A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathcal{A}$ consists of

- objects $X^{n,m}$ in $\mathcal{A}$;
- differentials $d_I^{n,m} : X^{n,m} \to X^{n+1,m}$ and $d_{II}^{n,m} : X^{n,m} \to X^{n,m+1}$ satisfy:

$$d_I^2 = d_{II}^2 = 0, \quad d_I \circ d_{II} = d_{II} \circ d_I.$$

One can represent a double complex by a commutative diagram:

$$\begin{tikzcd}
 & & & \\
\arrow[r] & X^{n,m+1} \arrow[r] \arrow[u] & X^{n+1,m+1} \arrow[r] \arrow[u] & {} \\
\arrow[r] & X^{n,m} \arrow[r] \arrow[u] & X^{n+1,m} \arrow[r] \arrow[u] & {} \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

One defines naturally the notion of a morphism of double complexes and obtains the abelian category $C^2(\mathcal{A})$. There are two functors $F_I, F_{II} : C^2(\mathcal{A}) \to C(C(\mathcal{A}))$ which associates to a double complex X the complex whose objects are the rows (resp. the columns) of X. They are both isomorphisms of categories.

Definition 1.7.8. *Let X be a double complex in $\mathcal{A}$. We define the total complex* $\mathrm{tot}(X)$ *to be (if the coproducts exist)*

$$\mathrm{tot}(X)^n = \oplus_{i+j=n} X^{i,j}, \quad d_{\mathrm{tot}} = d_I^{i,j} + (-1)^i d_{II}^{i,j}.$$

Remark 1.7.9. *One can similarly define* $\mathrm{tot}^{\Pi}(X)$ *be setting* $\mathrm{tot}^{\Pi}(X)^n = \prod_{i+j=n} X^{i,j}$ *(assuming the products exist).*

Remark 1.7.10. *We say that a double complex is* biregular *if for every n, $X^{i,j} = 0$ for all but finitely many (i,j) with $i+j=n$. We let $C^2_{\mathrm{reg}}(\mathcal{A}) \subset C^2(\mathcal{A})$ denote the full subcategory consisting of biregular double complexes. It is an additive (abelian) category. For $X^{\bullet,\bullet} \in C^2_{\mathrm{reg}}(\mathcal{A})$,* $\mathrm{tot}(X)$ *is well-defined and is an exact functor.*

Example 1.7.11. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$. Consider the double complex $Z^{\bullet,\bullet}$ such that*

$$Z^{-1,\bullet} = X^\bullet, \quad Z^{0,\bullet} = Y^\bullet, \quad Z^{i,\bullet} = 0, \ i \neq -1, 0,$$

with differentials $f^j : Z^{-1,j} \to Z^{0,j}$. Then we have an isomorphism in $C(\mathcal{A})$.

$$\mathrm{tot}(Z^{\bullet,\bullet}) \cong \mathrm{Cone}(f^\bullet).$$

Bifunctors

Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be additive categories. Let $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ be a functor that is additive in each variable. Then F extends to a functor

$$C^2(F) : C(\mathcal{A}) \times C(\mathcal{A}') \to C^2(\mathcal{A}'')$$

which is additive in each variable. For $X \in C(\mathcal{A})$, $Y \in C(\mathcal{A}')$, the double complex $C^2(F)(X,Y)$ is defined by $C^2(F)(X,Y)^{i,j} = F(X^i, Y^j)$, with

$$d_I^{i,j} = F(d_X^i, \mathrm{Id}_{Y^j}), \qquad d_{II}^{i,j} = F(\mathrm{Id}_{X^i}, d_Y^j).$$

Example 1.7.12. *Let R be a ring. The functor $- \otimes_R - :$ $\mathbf{Mod}$-$R \times R$-$\mathbf{Mod} \to \mathbf{Ab}$ is additive in each variable. thus it extends to*

$$- \otimes_R - : C(\mathbf{Mod}\text{-}R) \times C(R\text{-}\mathbf{Mod}) \to C^2(\mathbf{Ab}).$$

When R is commutative,one checks that the composite

$$C(R\text{-}\mathbf{Mod}) \times C(R\text{-}\mathbf{Mod}) \to C^2(R\text{-}\mathbf{Mod}) \xrightarrow{\mathrm{tot}} C(R\text{-}\mathbf{Mod})$$

is the tensor product we defined before.

1.7.4 Projective objects

Definition 1.7.13. *Let $\mathcal{C}$ be a category. An object P of $\mathcal{C}$ is said to be projective if given a morphism $f : P \to Y$ and an epimorphism $u : X \to Y$ in $\mathcal{C}$*

$$\begin{tikzcd} & P \arrow[dl, dotted] \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

there exists $g : P \to X$ rendering the diagram commutative.

Proposition 1.7.14. *Let $\mathcal{A}$ be an abelian category and $P \in \mathrm{Ob}(\mathcal{A})$. The following conditions are equivalent.*

(1) P is projective;

(2) the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is exact;

(3) Every short exact sequence $0 \to A \to B \to P \to 0$ splits.

Remark 1.7.15. *The coproduct of projective objects is projective.*

Projective modules

Proposition 1.7.16. *Let P be an R-module. The following conditions are equivalent:*

(1) P is projective;

(2) P is a direct summand of some free R-module.

Example 1.7.17. *Over $\mathbb{Z}$, projective = free: in fact, any submodule of a free $\mathbb{Z}$-module.*[3] *When M is finitely generated, this is also equivalent to being torsion free.*

1.7.5 Projective resolutions

Definition 1.7.18. *Let M be an object of $\mathcal{A}$. A* left resolution *of M is a chain complex $P_.$ with $P_i = 0$ for $i < 0$, together with a map $\epsilon : P_0 \to M$ so that the complex*

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. It is a projective resolution if each P_i is projective.

Remark 1.7.19. *Let $P_.$ be a complex of projectives with $P_i = 0$ for $i < 0$, then a map $\epsilon : P_\bullet \to M$ giving a resolution of M is the same thing as a chain map $\epsilon : P_\bullet \to M$, where M is considered as a complex concentrated in degree 0.*

Lemma 1.7.20. *Every R-module M has a projective resolution. More generally, if an abelian category $\mathcal{A}$ has enough projectives, then every object M in $\mathcal{A}$ has a projective resolution.*

Proof. Choose a projective P_0 and a surjection $\epsilon_0 : P_0 \to M$, and set $M_0 = \ker(\epsilon)$. Inductively, given M_{n-1}, we choose a projective P_n and a surjection $\epsilon_n : P_n \to M_{n-1}$. Set $M_n = \ker(\epsilon_n)$, and let d_n be the composite $P_n \to M_{n-1} \to P_{n-1}$. Since $d_n(P_n) = M_{n-1} = \ker(d_{n-1})$, the chain complex $P_\bullet$ is a resolution of M. $\square$

[3] The proof uses Zorn's lemma to find a basis for the submodule, see Lang, Algebra, p880.

Example 1.7.21. *(1) Let $R = k[x_1, x_2]$, then the Koszul complex $K_\bullet(x_1, x_2; R)$*

$$0 \to R \to R \oplus R \to R \to k \to 0$$

provides a resolution of $k = H_0(K_\bullet)$.

(2) Let $R = k[x]/(x^2)$. The following complex (of infinite length) gives a projective resolution of k:

$$\cdots \to R \xrightarrow{\times x} R \xrightarrow{\times x} R \to k \to 0.$$

We will see later that two projective resolutions are homotopy equivalent. This proves that a projective resolution need not be (left) bounded.

Theorem 1.7.22. *Let $P_\bullet \to M$ be a chain complex with the P_i projective and $f' : M \to N$ a map in $\mathcal{A}$. Then for every resolution $Q_\bullet \xrightarrow{\eta} N$ of N, there is a chain map $f : P_\bullet \to Q_\bullet$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map f is unique up to chain homotopy equivalence.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0.
\end{tikzcd}$$

Proof. For each n, the fact $f_{n-1}d = df_n$ means that f_n induces a map f_n' from $Z_n(P)$ to $Z_n(Q)$:

$$\begin{tikzcd}
0 \arrow[r] & Z_n(P) \arrow[r] \arrow[d, "f_n'"] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & Z_n(Q) \arrow[r] & Q_n \arrow[r] & Q_{n-1}.
\end{tikzcd}$$

Then by projectivitiy of P_{n+1}, we obtain a map f_{n+1} from P_{n+1} to Q_{n+1}, so that $df_{n+1} = f_n'd = f_nd$, illustrated as follows:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d"] \arrow[d, dashed, "f_{n+1}"] & Z_n(P) \arrow[d, "f_n'"] \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}$$

Note that the surjectivity in the lower row is crucial.

To see uniqueness of f up to chain homotopy, suppose that $g : P \to Q$ is another lift of f' and set $h = f - g$. We will construct a chain contraction $\{s_n : P_n \to Q_{n+1}\}$ of h by induction on n; namely $h_n = ds_n + s_{n-1}d$. If $n = 0$, note that since $\eta h_0 = (f' - f')\epsilon = 0$, the map h_0 sends P_0 to $Z_0(Q) = d(Q_1)$:

$$\begin{tikzcd}
 & P_0 \arrow[r] \arrow[d, "h_0"] \arrow[dl, dashed, "s_0"'] & M \arrow[d] \\
Q_1 \arrow[r, "d"] & Q_0 \arrow[r] & N
\end{tikzcd}$$

We use the lifting property of P_0 to get a map $s_0 : P_0 \to Q_1$ so that $h_0 = ds_0 = ds_0 + s_{-1}d$ (here $s_{-1} := 0$).

Inductively, suppose given maps s_i $(i < n)$ so that $h_{n-1} = ds_{n-1} + s_{n-2}d$ and consider the map $h_n - s_{n-1}d : P_n \to Q_n$. We compute that

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d = (dh_n - h_{n-1}d) + s_{n-2}dd = 0.$$

Therefore $h_n - s_{n-1}d$ lands in $Z_n(Q)$, a quotient of Q_{n+1}.

$$
\begin{tikzcd}
 & P_n \arrow[dl, dotted, "s_n"'] \arrow[d, "h_n - s_{n-1}d"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
$$

The lifting property of P_n yields the desired map $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$. ☐

Corollary 1.7.23. *Projective resolutions are unique up to chain homotopy. That is, any two projective resolutions $P^\bullet \to M$ and $Q^\bullet \to M$ are cochain homotopy equivalent.*

Lemma 1.7.24. *(Horseshoe lemma) Suppose given a commutative diagram*

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}
$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n, with suitable differentials, form a projective resolution P of A, and the right hand column lifts to an exact sequence of complexes

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$$

where $i_n : P_n' \to P_n$ and $\pi_n : P_n \to P_n''$ are the natural inclusion and projection, respectively.

Proof. Since π_A is epi, we may lift ϵ'' to a map $P_0'' \to A$. The direct sum of this with the map $i_A\epsilon' : P_0' \to A$ gives a map $\epsilon : P_0 \to A$. The diagram below commutes.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\epsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\epsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\epsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker(\epsilon'') \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The snake lemma shows that the left column is exact and $\mathrm{Coker}(\epsilon) = 0$, so that $P_0 \to A$ is epi. This finishes the initial step and reduces to the situation:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r, "d'"] & \ker(\epsilon') \arrow[r] \arrow[d] & 0 \\
 & & \ker(\epsilon) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_1'' \arrow[r, "d''"] & \ker(\epsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0. &
\end{tikzcd}$$

We then conclude by induction. $\square$

1.8 Lecture 8 (2019-03-20)

1.8.1 Injective objects

Definition 1.8.1. *Let $\mathcal{C}$ be a category. An object I of $\mathcal{C}$ is said to be injective if for any morphism $f : A \to I$ and any monomorphism $u : A \to B$ in $\mathcal{C}$, there exists g rendering the following diagram commutative:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl, dotted] \\ I & \end{tikzcd}$$

Proposition 1.8.2. *Let I be an object of an abelian category $\mathcal{A}$. The following conditions are equivalent:*

(1) I is injective;

(2) the (contravariant) functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact,

(3) every short exact sequence $0 \to I \to A \to B \to 0$ is split.

Remark 1.8.3. *The product of injective objects are injective.*

Remark 1.8.4. *I is injective in $\mathcal{A}$ if and only if I is projective in $\mathcal{A}^{\mathrm{op}}$.*

Injective modules

Let R be a ring.

Theorem 1.8.5. *(Baer's criterion) Let I be a left R-module. Then I is injective if and only if every morphism of R-modules $\mathfrak{a} \to I$, where $\mathfrak{a} \subset R$ is a left ideal, can be extended to $R \to I$. That is, in the above situation, we only need consider the case $A = \mathfrak{a}$ and $B = R$.*

Proof. $\Leftarrow$: Let $0 \to A \to B$, and $f : A \to I$ be given. Consider the set ordered by inclusion

$$\mathcal{S} = \{f' : A' \to I,\ A \subset A' \subset B,\ f'|_A = f\}.$$

First, $\mathcal{S}$ is non-empty since $f \in \mathcal{S}$. Next, it is inductive as is easily seen. Hence, by Zorn's lemma, there exists a maximal element of $\mathcal{S}$, that is, a module $A' \subset B$ (and containing A) together with a morphism $f' : A' \to I$ extending f. We intend to prove $A' = B$. If not, take $b \in B \backslash A'$ and consider $A' + Rb$. The set

$$\mathfrak{a} := \{r \in R : rb \in A'\}$$

is a left ideal of R and $f_0 : \mathfrak{a} \to I$ given by $f_0(r) := f'(rb)$ is an R-module homomorphism. By assumption it extends to $\tilde{f}_0 : R \to I$; since this is R-linear, if we let $u := \tilde{f}_0(1)$, then

$$\tilde{f}_0(r) = r\tilde{f}_0(1) = ru, \quad \forall r \in R.$$

Now define

$$A' + Rb \to I, \quad a + rb \mapsto f'(a) + ru$$

This is well defined because: if $rb \in A' \cap Rb$, then $r \in \mathfrak{a}$, thus

$$f'(rb) = f_0(r) = \tilde{f}_0(r) = ru.$$

This gives a contradiction to the choice of A'. $\square$

An R-module M is called *divisible* if every equation $m = rx$, for $r \in R\backslash\{0\}$ and $m \in M$, always has a solution $x \in M$. That is, $rM = M$ for any $r \in R\backslash\{0\}$. It is clear that a quotient of a divisible module is divisible, and a direct sum or direct product of divisible modules is divisible.

Proposition 1.8.6. *Every injective module M over an integral domain R (possibly non-commutative) is divisible. Over a PLID (i.e. principal left ideal domain), the converse also holds.*

Proof. $\Rightarrow$ Let $r \in R\backslash\{0\}$ and $m \in M$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\times r"] \arrow[d, "f"] & R \\ & M & \end{tikzcd}$$

where f sends 1 to m and the horizontal map is $r' \mapsto r'r$ (right multiplication by r), which is injective since R is a domain and $r \neq 0$. By injectivity of M, we can extend f to a map $\tilde{f} : R \to M$. Then

$$m = f(1) = \tilde{f}(r) = r \cdot \tilde{f}(1),$$

and $\tilde{f}(1) \in M$ solves the equation $m = rx$.

$\Leftarrow$ Assume R is a PLID. Let $\mathfrak{a} = Rr$ be a left principal ideal and $f : \mathfrak{a} \to M$ be a morphism. To give a morphism $\tilde{f} : R \to M$ extending f is the same to give an element $x \in M$, serving as $\tilde{f}(1)$, which satisfies

$$f(r) = \tilde{f}(r) = r\tilde{f}(1) = rx,$$

but this equation has a solution by assumption. $\square$

Corollary 1.8.7. *$\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Also $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.*

Remark 1.8.8. *Let $R = \mathbb{Z}[X]$ (which is not PID) and $K = \mathbb{Q}(X)$ be its fraction field. Then the R-module K/R is divisible but not injective. Indeed, the homomorphism $\mathfrak{a} = (2, X) \to K/R$, with $2 \mapsto 0$, and $x \mapsto 1/2$, does not extend to a homomorphism $R \to K/R$. (Exercise)*

Enough injectives in R-Mod

Theorem 1.8.9. *Every R-module is a submodule of an injective module. That is, R-Mod has enough injective objects.*

Proof. Step 1. We first prove the theorem for $\mathbb{Z}$-module.Let M be a $\mathbb{Z}$-module. Choose F a free $\mathbb{Z}$-module such that $F \twoheadrightarrow M$ and let K be the kernel. Write $F = \mathbb{Z}^{(I)}$ and let $D = \mathbb{Q}^{(I)}$ (here D is injective because it is divisible), then $F/K \subset D/K$. Since D/K is a divisible group (being quotient of a divisible group), we get the result.

Step 2. The functor

$$\mathrm{Hom}_{\mathbb{Z}}(R, -) : \mathbb{Z}\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$$

admits a left adjoint, namely the restriction of scalars functor R-**Mod** $\to \mathbb{Z}$-**Mod**, which is exact. Thus $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ carries injective $\mathbb{Z}$-modules to injective R-modules by Proposition 1.8.10.

Let M be a left R-module. We embed the underlying $\mathbb{Z}$-module of M into an injective $\mathbb{Z}$-module I. Then $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module and we have injective homomorphisms

$$M \cong \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I).$$

This finishes the proof. $\square$

Proposition 1.8.10. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between two abelian categories.*

(1) If F admits a right adjoint G which is right exact, then F carries projective objects to projective objects.

(2) If F admits a left adjoint G which is left exact, then F carries injective objects to injective objects.

Proof. (2) We must show that $\mathrm{Hom}_{\mathcal{B}}(-, F(I))$ is exact. Given a monomorphism $f : B \to B'$ in $\mathcal{B}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{B}}(B', FI) \arrow[r, "f^*"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathcal{B}}(B, FI) \arrow[d, "\sim"] \\ \mathrm{Hom}_{\mathcal{A}}(GB', I) \arrow[r, "(Gf)^*"] & \mathrm{Hom}_{\mathcal{A}}(GA, I) \end{tikzcd}$$

commutes by naturality of $F \vdash G$. Since G is left exact and I is injective, $(Gf)^*$ is epi. Hence the map f^* is epi, proving that FI is an injective object in $\mathcal{B}$. □

1.8.2 Injective resolutions

Definition 1.8.11. *Let A be an object of $\mathcal{A}$. A right resolution of A is a cochain complex $I^\bullet$ with $I^i = 0$ for $i < 0$ and a map $A \to I^0$ such that the augmented complex*

$$0 \to A \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$$

is exact. It is called an injective resolution if each I^i is injective.

Lemma 1.8.12. *If the abelian category $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution. In particular, every M in R-**Mod** has an injective resolution.*

Theorem 1.8.13. *Let $A \to I^\bullet$ be a complex with all I^i injective and $f' : A' \to A$ a map in $\mathcal{A}$. Then for every resolution $A' \to J^\bullet$ there is a cochain map $f : J^\bullet \to I^\bullet$ lifting f'. The map f is unique up to cochain homotopy equivalence.*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

Proof. The proof is parallel to Theorem 1.7.22. □

Sheaves

Let X be a topological space. Denote by $\mathrm{Open}(X)$ the category open sets on X, i.e. whose objects are the open sets of X and whose morphisms are inclusions. A functor $\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Ab}$ is called a *presheaf* of abelian groups on X.

Definition 1.8.14. *A sheaf of abelian groups on X is a presheaf satisfying the following supplementary conditions:*

- *(Locality) If (U_i) is an open covering of an open set U, and if $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each set U_i of the covering, then $s = t$; and*

- *(Gluing) If (U_i) is an open covering of an open set U, and if for each i a section $s_i \in \mathcal{F}(U_i)$ is given such that for each pair U_i, U_j of the covering sets the restrictions of s_i and s_j agree on the overlaps: $s_i|U_i \cap U_j = s_j|U_i \cap U_j$, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i.*

We can define morphisms between sheaves on X, and the resulting category is denoted by $\mathbf{Shv}(X)$.

Lemma 1.8.15. $\mathbf{Shv}(X)$ *is an abelian category.*

Proof. See Hartshorne, §II.1. □

The stalk of a sheaf $\mathcal{F}$ at a point $x \in X$ is the abelian group

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

Lemma 1.8.16. *Taking stalk at x is an exact functor from* $\mathbf{Shv}(X)$ *to* $\mathbf{Ab}$. *Moreover, a sheaf $\mathcal{F}$ is 0 if and only if $\mathcal{F}_x$ is zero for each $x \in X$.*

Proof. The subcategory $\{U : x \in U\}$ of $\mathrm{Open}(X)^{\mathrm{op}}$ filtered in the sense of Definition 1.5.12, because given U, U', there are always morphisms $U, U' \to U \cap U'$. The result then follows from Proposition 1.5.13: filtered colimits are exact.

The second statement is Hartshorne, Chap. 2, Prop. 1.1. □

If A is any abelian group, the skyscraper sheaf x_*A at the point $x \in X$ is defined to be the presheaf

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

where the map $(x_*A)(U') \to (x_*A)(U)$ (for $U \subseteq U'$) is identity map if $x \in U$, and is zero if $x \notin U$.

Lemma 1.8.17. *x_*A is a sheaf and $(\cdot)_x \dashv x_*$, i.e.*

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\mathbf{Shv}(X)}(\mathcal{F}, x_*A) \tag{1.9}$$

for every sheaf $\mathcal{F}$. Thus, x_* carries injective abelian groups to injective objects in $\mathbf{Shv}(X)$.

Proof. Exercise. □

Theorem 1.8.18. *The category of sheaves* $\mathbf{Shv}(X)$ *has enough injectives.*

Proof. Given a fixed sheaf $\mathcal{F}$, choose an injection $\mathcal{F}_x \to I_x$ with I_x injective in $\mathbf{Ab}$ for each $x \in X$. By (1.9), there is a natural map $\mathcal{F} \to x_*\mathcal{F}_x$; combine it with $x_*\mathcal{F}_x \to x_*I_x$ yields a map

$$\mathcal{F} \to \mathcal{I} := \prod_{x \in X} x_*(I_x).$$

Being a product of injective objects, $\mathcal{I}$ is injective. The map $\mathcal{F} \to \mathcal{I}$ is an injection, because it is at each stalk (using Lemma 1.8.16). This finishes the proof. □

Exercise 1. Let $X^\bullet$ be a complex in $\mathcal{A}$. It is called split if there exists $s^n : X^{n+1} \to X^n$ such that $d^n s^n d^n = d^n$ for all n. Prove that X is split exact if and only if Id_X is homotopic to zero.

Exercise 2. Let $\mathcal{A}$ be an abelian category.

(a) Show that a cochain complex $P^\bullet$ is a projective object in $C(\mathcal{A})$ if and only if it is a split exact complex of projectives.

(b) Show that if an abelian category $\mathcal{A}$ has enough projectives, then so does the category $C(\mathcal{A})$.

Exercise 3. Prove Remark 1.8.8.

Exercise 4. Prove Lemma 1.8.17.

Chapter 2

Derived functors

2.1 Lecture 9 (2019-03-25)

2.1.1 δ-functors

Let $\mathcal{A}, \mathcal{B}$ be two abelian categories.

Definition 2.1.1. *A covariant cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ is a collection of additive functors $T^n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$, together with morphisms*

$$\delta^n : T^n(C) \to T^{n+1}(A)$$

defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. These two conditions are imposed:

1. *For each short exact sequence as above, there is a long exact sequence*

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta^0} T^1(A) \to \cdots \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \cdots$$

2. *For every morphism of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

the δ's give a commutative diagram

$$\begin{tikzcd}
T^n(C') \arrow[r, "\delta"] \arrow[d] & T^{n+1}(A') \arrow[d] \\
T^n(C) \arrow[r, "\delta"] & T^{n+1}(A).
\end{tikzcd}$$

A morphism $S \to T$ of δ-functors is a system of natural transformations $S^n \to T^n$ that commute with δ.

Similarly we may define a covariant homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$, i.e a collection of additive functors $T_n : \mathcal{A} \to \mathcal{B}$ for $n \geq 0$ together with morphisms

$$\delta_n : T_n(C) \to T_{n-1}(A)$$

defined for any short exact sequence $0 \to A \to B \to C \to 0$, with similar imposed conditions as in cohomological situation.

Example 2.1.2. *Cohomology gives a cohomological δ-functor H^* from $C^{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.*

Example 2.1.3. *If $n \in \mathbb{N}$ and $A \in \mathbf{Ab}$, the functor $T_0(A) = A/nA$ and*

$$T_1(A) = A[n] = \{a \in A : na = 0\}$$

form a homological δ-functor from $\mathbf{Ab}$ to $\mathbf{Ab}$. To see this, apply the Snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\times n"] & B \arrow[r] \arrow[d, "\times n"] & C \arrow[r] \arrow[d, "\times n"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

to get the exact sequence

$$0 \to A[n] \to B[n] \to C[n] \to A/nA \to B/nA \to C/nC \to 0.$$

Definition 2.1.4. *A cohomological δ-functor T is universal if, given any other δ-functor S and $\alpha^0 : T^0 \to S^0$ there exists a unique morphism $\{\alpha^n : T^n \to S^n\}$ of δ-functors extending α^0.*

A homological δ-functor T is universal if, given any other δ-functor S and a natural transformation $\alpha_0 : S_0 \to T_0$, there exists a unique morphism $\{\alpha_n : S_n \to T_n\}$ of δ-functors extending α_0.

A universal δ-functor T with given $T^0 = F$ (resp. $T_0 = F$), if exists, is unique (up to isomorphism).

Remark 2.1.5. *Given an additive functor $F : \mathcal{A} \to \mathcal{B}$, one may ask if there is any δ-functor T (not necessarily universal) such that $T^0 = F$ (resp. $T_0 = F$). If this is the case, then T^0 must be left exact (resp. T_0 must be right exact).*

Definition 2.1.6. *An additive functor $F : \mathcal{A} \to \mathcal{B}$ is* effaceable *if for each object A of $\mathcal{A}$, there is a monomorphism $u : A \to I$, for some I, such that $F(u) = 0$. It is* coeffaceable *if for each A there is an epimorphism $u : P \to A$ such that $F(u) = 0$.*

Theorem 2.1.7. *Let $T = (T^i)$ be a cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$. If T^i is effaceable for each $i > 0$, then T is universal.*

If $T = (T_i)$ is a homological δ-functor such that T_i is coeffaceable, then T is universal.

Proof. Let S be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and let $\alpha^0 : T^0 \to S^0$ be a morphism of functors. We have to show that there exists a unique morphism of δ-functors $\{\alpha^n\}_{n\geq 0} : T \to S$. We construct α^n by induction on n. Suppose we have already constructed a unique sequence of transformation of functors α^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in \mathrm{Ob}(\mathcal{A})$. By assumption we may choose an embedding $u : A \to I$ such that $T^{n+1}(u) = 0$. Let $M = I/u(A)$ so that we have a short exact sequence $0 \to A \to I \to M \to 0$. The long exact sequence attached to the δ-functor T gives

$$T^n(I) \to T^n(M) \to T^{n+1}(A) \to 0$$

By induction, we already have morphisms α^n,

$$\begin{tikzcd} T^n(I) \arrow[r] \arrow[d, "\alpha^n_I"] & T^n(M) \arrow[r] \arrow[d, "\alpha^n_M"] & T^{n+1}(A) \arrow[r] \arrow[d, dotted] & 0 \\ S^n(I) \arrow[r] & S^n(M) \arrow[r] & S^{n+1}(A) & \end{tikzcd}$$

and we define $\alpha_A^{n+1} : T^{n+1}(A) \to S^{n+1}(A)$ to be the unique morphism making the above diagram commute.

First, we check that α_A^{n+1} does not depend on the choice of effacement. Indeed, let $u' : A \to I'$ be another effacement and let $M' = I'/u'(A)$. It induces a morphism

$$\alpha_{A,u'}^{n+1} : T^{n+1}(A) \to S^{n+1}(A)$$

and we need to show $\alpha_{A,u}^{n+1} = \alpha_{A,u'}^{n+1}$. In general, we can not lift Id_A to a morphism $I \to I'$. But the universal property of a pushout provides a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & I \arrow[d] \\ I' \arrow[r] & I'', \end{tikzcd}$$

where $I'' := I \sqcup_A I'$. It is clear that the map $u'' : A \to I$ is also a monomorphism, and an effacement. So the exact sequence $0 \to A \to I'' \to M'' \to 0$ also allows to define a morphism

$$\alpha_{A,u''}^{n+1} : T^{n+1} \to S^{n+1}(A).$$

It suffices to check

$$\alpha_{A,u}^{n+1} = \alpha_{A,u''}^{n+1} = \alpha_{A,u'}^{n+1}.$$

Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & I'' \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

which induces

$$\begin{tikzcd} T^{n+1}(A) \arrow[r, "\alpha_{A,u}^{n+1}"] \arrow[d, equal] & S^{n+1}(A) \arrow[d, equal] \\ T^{n+1}(A) \arrow[r, "\alpha_{A,u''}^{n+1}"] & S^{n+1}(A) \end{tikzcd}$$

showing that $\alpha_{A,u}^{n+1} = \alpha_{A,u''}^{n+1}$. Similarly, $\alpha_{A,u'}^{n+1} = \alpha_{A,u''}^{n+1}$ which shows that α_A^{n+1} is well-defined, independent of the choice of effacement.

Next, we also need to check α^{n+1} is a natural transformation from $T^{n+1} \to S^{n+1}$. Namely, if $A \to A'$ is a morphism in $\mathcal{A}$, then there is a commutative diagram

$$\begin{tikzcd} T^{n+1}(A) \arrow[r] \arrow[d] & S^{n+1}(A) \arrow[d] \\ T^{n+1}(A') \arrow[r] & S^{n+1}(A'). \end{tikzcd}$$

We left this as an exercise. $\square$

2.1.2 Left derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant right exact functor between two abelian categories. Assume that $\mathcal{A}$ has enough projectives. We construct the left derived functors L_iF $(i \geq 0)$ as follows. If $A \in \mathcal{A}$, choose a projective resolution $P \to A$ and define the left derived functors of F

$$L_iF(A) := H_i(F(P)).$$

Note that since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, we have

$$L_0F(A) \cong F(A).$$

Lemma 2.1.8. *The objects $L_iF(A)$ of $\mathcal{B}$ are well-defined up to natural isomorphism, i.e. if $Q \to A$ is another projective resolution, then there is a canonical isomorphism*

$$L_iF(A) = H_i(F(P)) \cong H_i(F(Q)).$$

Proof. There is a morphism of complexes $f : P \to Q$ lifting Id_A, and also $g : Q \to P$. We know that $gf \sim \mathrm{Id}_P$ (homotopy equivalent) and $fg \sim \mathrm{Id}_Q$. Thus, $(Fg)(Ff) \sim \mathrm{Id}_{FP}$ and $(Ff)(Fg) \sim \mathrm{Id}_{FQ}$. taking homology, we see that Ff induces isomorphisms from $H_i(F(P)) \xrightarrow{\sim} H_i(F(Q))$. □

Corollary 2.1.9. *If A is projective, then $L_iF(A) = 0$, for any $i \neq 0$.*

Proof. Because then $(\cdots \to 0 \to A)$ is a projective resolution of A. □

Remark 2.1.10. *An object Q is called F-acyclic if $L_iF(Q) = 0$ for all $i \neq 0$.*

Projective objects are F-acyclic for every right exact functor F, but there are F-acyclic objects which are not projective. For example, flat modules are acyclic for tensor products.

Lemma 2.1.11. *If $f : A \to A'$ is a morphism in $\mathcal{A}$, then there exists natural maps*

$$L_iF(f) : L_iF(A) \to L_iF(A').$$

Proof. Let $P \to A$, $P' \to A'$ be the chosen projective resolutions. The comparison theorem yields a lift of f to a chain map $\tilde{f}_\bullet : P \to P'$, hence a map

$$\tilde{f}_* : H_iF(P) \to H_iF(P').$$

Any other lift is chain homotopic to $\tilde{f}$, so the map $\tilde{f}_*$ is independent of the choice of $\tilde{f}$. This is the desired map $L_iF(f)$. □

Proposition 2.1.12. *Each L_iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Exercise. □

Theorem 2.1.13. *The left derived functors L_iF form a homological δ-functor.*

Proof. Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

choose projective resolutions $P' \to A'$ and $P'' \to A''$. By the Horseshoed Lemma, there is a projective resolution $P \to A$ fitting into a short exact sequence

$$0 \to P' \to P \to P'' \to 0$$

of projective complexes in $\mathcal{A}$. Since the P''_n are projective, each sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. As F is additive, the induced sequence

$$0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$$

is also split exact in $\mathcal{B}$. Therefore,

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is a short exact sequence of chain complexes. Writing out the corresponding long exact homology sequence, we get

$$\cdots \to L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial_i} L_{i-1}F(A') \to L_{i-1}F(A) \to L_{i-1}F(A'') \to \cdots$$

this verifies the first condition of a δ-functor.

To see the naturality of the ∂_i, assume we are given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i_A"] \arrow[d, "f'"] & A \arrow[r, "\pi_A"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "i_B"] & B \arrow[r, "\pi_B"] & B'' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$, and projective resolutions of the corners:

$$\epsilon' : P' \to A', \quad \epsilon'' : P'' \to A'', \quad \eta' : Q' \to B', \quad \eta'' : Q'' \to B''.$$

Use the Horseshoe lemma to get projective resolutions

$$\epsilon : P \to A, \qquad \eta : Q \to B,$$

where $P_n \cong P'_n \oplus P''_n$ and $Q_n \cong Q'_n \oplus Q''_n$. Use the comparison theorem to obtain chain maps

$$\tilde{f}'_\bullet : P' \to Q' \qquad \tilde{f}''_\bullet : P'' \to Q''$$

lifting the maps f' and f'', respectively. We shall show that there is also a chain map $\tilde{f}_\bullet : P \to Q$ lifting f, and giving a commutative diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "\tilde{f}'_\bullet"] & P \arrow[r] \arrow[d, dotted, "\tilde{f}'_\bullet"] & P'' \arrow[r] \arrow[d, "\tilde{f}''_\bullet"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

The naturality of the connecting morphism in the long exact sequence then translates into the naturality of ∂. We will construct inductively such an $\tilde{f}_n : P_n \to Q_n$ of the form

$$\tilde{f}_n = \begin{pmatrix} \tilde{f}'_n & \gamma_n \\ 0 & \tilde{f}''_n \end{pmatrix} : \begin{matrix} P'_n \\ \oplus \\ P''_n \end{matrix} \longrightarrow \begin{matrix} Q'_n \\ \oplus \\ Q''_n \end{matrix}$$

for suitable maps $\gamma_n : P''_n \to Q'_n$.

Let us look at $\tilde{f}_0 : P'_0 \oplus P''_0 \to Q'_0 \oplus Q''_0$, as follows

$$\begin{tikzcd}
P'_0 \oplus P''_0 \arrow[r, "\epsilon"] \arrow[d, "\left(\begin{smallmatrix} \tilde{f}'_0 & \gamma_0 \\ 0 & \tilde{f}''_0 \end{smallmatrix}\right) = \tilde{f}_0"'] & A \arrow[d, "f"] \\
Q'_0 \oplus Q''_0 \arrow[r, "\eta"] & B,
\end{tikzcd}$$

and we need $f\epsilon = \eta\tilde{f}_0$. On P_0', this is clear because it becomes $f'\epsilon' = \eta'\tilde{f}_0'$ and by construction $\tilde{f}'$ lifts f'. On P_0'' this requires

$$f \circ (\epsilon|_{P_0''}) = \eta \circ (\gamma_0 + \tilde{f}_0'') = \eta'\gamma_0 + \eta|_{Q_0''}\tilde{f}_0'' \tag{2.1}$$

Let $\beta := f \circ (\epsilon|_{P_0''}) - \eta|_{Q_0''}\tilde{f}_0'' : P_0'' \to B$; if β takes values in B', then the projectivity of P_0'' will provide us a map γ_0, satisfying (2.1)

$$\begin{tikzcd}
 & P_0'' \arrow[dl, dotted, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q_0' \arrow[r, "\eta'"] & B' \arrow[r] & 0
\end{tikzcd}$$

So we only need to check $\pi_B \circ \beta = 0$, which is true:

$$\pi_B f\epsilon|_{P_0''} - \pi_B\eta|_{Q_0''}\tilde{f}_0'' = f''\pi_A\epsilon|_{P_0''} - \pi_B\eta|_{Q_0''}\tilde{f}_0'' = f''\epsilon'' - \eta''\tilde{f}_0''.$$

The general case can be done in a similar way. □

Remark 2.1.14. *Dimension shifting. If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic. Then the long exact sequence shows that*

$$L_iF(A) \cong L_{i-1}F(M)$$

for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \to F(P)$. More generally, if

$$0 \to M_m \to P_m \to \cdots \to P_0 \to A \to 0$$

is exact with the P_i projective or F-acyclic, then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$, and L_{m+1} is the kernel of $F(M_m) \to F(P_m)$.

Theorem 2.1.15. *Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F : \mathcal{A} \to \mathcal{B}$, the derived functors L_nF form a universal δ-functor.*

Proof. This follows from Theorem 2.1.7 and Corollary 2.1.9. □

2.2 Lecture 10 (2019-03-27)

2.2.1 Tor^n

Given a right R-module A and a left R-module B, we have two ways to construct the left derived functors $\mathrm{Tor}_i(A, B)$:

- as $L_i(- \otimes_R B)(A)$, i.e. take a projective resolution $P_\bullet \xrightarrow{\epsilon} A$, and
$$\mathrm{Tor}_i^R(A, B) = H_i(P_\bullet \otimes_R B);$$
- as $L_i(A \otimes_R -)(B)$, i.e. take a projective resolution $Q_\bullet \xrightarrow{\eta} B$, and
$$\overline{\mathrm{Tor}}_i^R(A, B) = H_i(A \otimes_R Q_\bullet).$$

Theorem 2.2.1. *We have natural isomorphisms* $\mathrm{Tor}_i^R(A, B) \cong \overline{\mathrm{Tor}}_i^R(A, B)$.

The case $i = 0$ is clear. Let's looked at the case $i = 1$. Let A_1, B_1 be defined so that $0 \to A_1 \to P_0 \to A \to 0$, and $0 \to B_1 \to Q_0 \to B \to 0$. We tensor these two complexes into a commutative diagram as below:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \overline{\mathrm{Tor}}_1^R(A,B) \arrow[d, hook] & \\
 & A_1 \otimes B_1 \arrow[r] \arrow[d] & P_0 \otimes B_1 \arrow[r] \arrow[d] & A \otimes B_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \otimes Q_0 \arrow[r] \arrow[d] & P_0 \otimes Q_0 \arrow[r] \arrow[d] & A \otimes Q_0 \arrow[r] \arrow[d] & 0 \\
 & A_1 \otimes B \arrow[r] \arrow[d] & P_0 \otimes B \arrow[r] \arrow[d] & A \otimes B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Snake lemma then implies that

$$0 \to \overline{\mathrm{Tor}}_1^R(A, B) \to A_1 \otimes B \to P_0 \otimes B \to A \otimes B \to 0.$$

On the other hand, $\{\mathrm{Tor}_i^R(-, B)\}$ form a δ-functor and, when applied to $0 \to A_1 \to P_0 \to A \to 0$ induces a long exact sequence

$$0 = \mathrm{Tor}_1^R(P_0, B) \to \mathrm{Tor}_1^R(A, B) \to A_1 \otimes B \to P_0 \otimes B \to A \otimes B \to 0.$$

All the maps being natural ones, we deduce the isomorphism

$$\mathrm{Tor}_1^R(A, B) \cong \overline{\mathrm{Tor}}_1^R(A, B).$$

One can treat the general case using dimension shifting trick to reduce to $i = 1$ case. We will rather give a more general proof.

Proof of Theorem 2.2.1. Tensoring $P_\bullet$ and $Q_\bullet \to B \to 0$ gives a map of double complexes

$$P_\bullet \otimes Q_\bullet \to P_\bullet \otimes B.$$

Taking tot, we obtain

$$f : \mathrm{tot}(P_\bullet \otimes Q_\bullet) \to \mathrm{tot}(P_\bullet \otimes B) = P_\bullet \otimes B.$$

We claim that it is a quasi-isomorphism, hence induces isomorphism

$$H_*(\mathrm{tot}(P_\bullet \otimes Q_\bullet)) \xrightarrow{\sim} \mathrm{Tor}_*^R(A, B). \tag{2.2}$$

Recall that a morphism f of complexes is a quasi-isomorphism if and only if the mapping cone $\mathrm{Cone}(f)$ is acyclic, see Proposition 1.7.3. It is easy to check that

$$\mathrm{Cone}(f) \cong \mathrm{tot}(P_\bullet \otimes Q'_\bullet)$$

where $Q'_\bullet$ is the augmented complex $Q_\bullet \to B \to 0$, with $B = Q'_{-1}$. The double complex $P_\bullet \otimes Q'_\bullet$ visually looks as:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_0 \arrow[d] & P_1 \otimes Q_0 \arrow[l] \arrow[d] & P_2 \otimes Q_0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes B & P_1 \otimes B \arrow[l] & P_2 \otimes B \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Since P_i are projective, hence flat, we see that each column is exact. By Lemma 2.2.2 below, we deduce that $P_\bullet \otimes Q'_\bullet$ is acyclic, proving (2.2).

Similarly, one checks that the morphism $\mathrm{tot}(P_\bullet \otimes Q_\bullet) \to A \otimes Q_\bullet$ induces isomorphisms

$$H_*(\mathrm{tot}(P_\bullet \otimes Q_\bullet)) \xrightarrow{\sim} H_*(A \otimes Q_\bullet) =: \overline{\mathrm{Tor}}_i^R(A, B).$$

The result follows. □

Lemma 2.2.2 (Acyclic Assembly Lemma). *Let C be a double complex, then*

- $\mathrm{tot}^\Pi(C)$ *is an acyclic chain complex, assuming either of the following conditions:*

  (1) C is an upper half-plane complex with exact columns.

  (2) C is a right half-plane complex with exact rows.

- $\mathrm{tot}(C)$ *is an acyclic chain complex, assuming either of the following:*

  (3) C is an upper half-plane complex with exact rows.

  (4) C is a right half plane complex with exact columns.

Here the convention for a double chain complex is as follows:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\leftarrow & C^{-1,1} \arrow[l] \arrow[d] & C^{0,1} \arrow[l] \arrow[d] & \arrow[l] \\
\leftarrow & C^{-1,0} \arrow[l] \arrow[d] & C^{0,0} \arrow[l] \arrow[d] & \arrow[l] \\
 & {} & {} &
\end{tikzcd}$$

Proof. Let's prove (1). Translating C left and right, it suffices to prove that $H_0(\mathrm{tot}(C)) = 0$. Let

$$c = (\cdots, c_{-p,p}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod C_{-p,p} = \mathrm{tot}(C)_0$$

be a 0-cycle; we will find elements $b_{-p,p+1}$ by induction on p so that (see Definition 1.7.8 for our convention)

$$d^I(b_{-p+1,p}) + (-1)^p d^{II}(b_{-p,p+1}) = c_{-p,p}.$$

Assembling the b's will yield an element b of $\mathrm{tot}(C)_1$ such that $d(b) = c$, proving that $H_0(\mathrm{tot}(C)) = 0$.

We begin the induction by choosing $b_{1,0} = 0$ for $p = -1$. Since $C_{0,-1} = 0$, $d^{II}(c_{00}) = 0$; since the 0^{th} column is exact, there is a $b_{01} \in C_{01}$ so that $d^{II}(b_{01}) = c_{00}$. We need to find an element $b_{-1,2}$ such that

$$c_{-1,1} = d^I(b_{0,1}) + (-1)d^{II}(b_{-1,2}).$$

Since the column is exact, such a $b_{-1,2}$ exists if and only if $c_{-1,1} - d^I(b_{0,1})$ lies in kernel of $d^{II} : C_{-1,1} \to C_{-1,0}$, i.e.

$$d^{II}(c_{-1,1} - d^I(b_{0,1})) = 0.$$

This is true, because

$$d^{II}(c_{-1,1}) - d^{II}d^I(b_{0,1}) = d^{II}(c_{-1,1}) - d^I d^{II}(b_{0,1}) = d^{II}(c_{-1,1}) - d^I(c_{0,0} = 0$$

where the last equality holds because c is a cycle. The following diagram chasing picture helps to understand the process:

$$\begin{tikzcd}
b_{-1,2} \arrow[d, "d^{II}"] & & \\
c_{-1,1} & b_{0,1} \arrow[l, "d^I"'] \arrow[d, "d^{II}"] & \\
 & c_{0,0} \arrow[d] & 0 \arrow[l] \\
 & 0 &
\end{tikzcd}$$

Inductively, we may construct b whose image is c.

Let's explain how to deduce (2),(3),(4) from (1). Rotating the complex interchanges rows and columns, so (1) is equivalent to (2), and (3) is equivalent to (4). We are left to show (4), using (1). Suppose we are in case (4), and let $\tau_n C$ be the double subcomplex of C obtained by truncating each column at level n. Then each $\tau_n C$ is, a first quadrant double complex with exact columns, so (1) implies that $\mathrm{tot}(\tau_C) = \mathrm{tot}^{\Pi}(\tau_n C)$ is acyclic. This implies that $\mathrm{tot}(C)$ is exact, because every cycle of $\mathrm{tot}(C)$ is a cycle in some subcomplex $\mathrm{tot}(\tau_n C)$, hence is a boundary. □

Corollary 2.2.3. *If R is commutative, then there is a canonical isomorphism $\mathrm{Tor}_i^R(A, B) \cong \mathrm{Tor}_i^R(B, A)$.*

Proof. Choose a projective resolution $P_\bullet \to A$, then

$$\mathrm{Tor}_i^R(A, B) = H_i(P_\bullet \otimes B).$$

On the other hand, since R is commutative, there is a natural isomorphism of complexes:

$$P_\bullet \otimes_R B \cong B \otimes_R P_\bullet$$

so that $H_i(P_\bullet \otimes B) \cong H_i(B \otimes P_\bullet)$, the latter is isomorphic to $\overline{\mathrm{Tor}}_i^R(B, A)$ by definition. Then we conclude by Theorem 2.2.1. □

2.2.2 Tor and colimits

First recall that, if $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint, then

(a) F preserves colimits whenever exist, see Proposition 1.3.21.

(b) F is right exact, see Proposition 1.5.4.

Proposition 2.2.4. *Assume $\mathcal{A}$ has enough projectives and colimits exist in $\mathcal{A}$. Let $F : \mathcal{A} \to \mathcal{B}$ be a left adjoint. Then for any set $\{A_i, i \in I\}$ of objects in $\mathcal{A}$:*

$$L_*F\Big(\bigoplus_{i\in I} A_i\Big) = \bigoplus_{i\in I} L_*F(A_i).$$

Proof. If $P_i \to A_i$ are projective resolutions, then so is $\oplus_i P_i \to \oplus_i A_i$. Hence

$$L_*F(\oplus A_i) = H_*(F(\oplus P_i)) = H_*(\oplus F(P_i)) \overset{(*)}{\cong} \oplus H_*(F(P_i)) = \oplus L_*F(A_i).$$

Here the isomorphism (*) holds because homology commutes with arbitrary direct sums of chain complexes. □

Corollary 2.2.5. *In the category R-**Mod**, we have $\mathrm{Tor}_*^R(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}_*^R(A_i, B)$.*

We also have the following.

Proposition 2.2.6. *In R-**Mod**, Tor_* commutes with filtered colimits (in particular, direct limits).*

Proof. Let $\{A_i,\ i \in I\}$ be a direct system of R-modules and let $A = \varinjlim_i A_i$. By Lemma 2.2.7 below, we can find a projective resolution P_i of A_i, which form a direct system and such that the direct limit $P := \varinjlim_i P_i$ is a projective resolution of A. Thus

$$\mathrm{Tor}_*(\varinjlim A_i, B) = H_*((\varinjlim P_i) \otimes B) \overset{(1)}{\cong} H_*(\varinjlim (P_i \otimes B)) \overset{(2)}{\cong} \varinjlim H_*(P_i \otimes B) = \varinjlim \mathrm{Tor}_*(A_i, B).$$

Here the isomorphisms (1), (2) follow from that filtered colimits are exact (see Prop. 1.5.13). □

Lemma 2.2.7. *Let $\{A_i,\ i \in I\}$ be a direct system of left R-modules, and $A = \varinjlim_i A_i$. Then there exist projective resolutions P_i of A_i forming a direct system such that $P := \varinjlim_i P_i$ is a projective resolution of A.*

Proof. This is taken from Cartan-Ellenberg, Lemma 9.5* (p.100).

For each $i \in I$, let $P_{0,i}$ be the free module $R^{(A_i)}$ (i.e. the free module with a basis indexed by elements in A_i) and let $P_0 = R^{(A)}$. The maps $A_i \to A_j$ induce maps $P_{0,i} \to P_{0,j}$ and P_0 may be identified with the limit $\varinjlim_i P_{0,i}$. Let K_i be the kernel of the natural map $P_{0,i} \to A_i$. The K_i forms a direct system of modules with $R := \ker(P_0 \to A)$ as limit. We now repeat the argument with A_i replaced by K_i. The complexes P_i are thus constructed by iteration. $\square$

Remark 2.2.8. *Be attention that we have no analogue of Lemma 2.2.7 for injective resolutions.*

2.2.3 Tor and flatness

Definition 2.2.9. *A left R-module B is flat if the functor $- \otimes_R B$ is exact. Similarly, a right R-module A is* flat *if the functor $A \otimes_R -$ is exact.*

Projective modules are flat. However, flat modules need not be projective: Q is flat $\mathbb{Z}$-module but not projective.

Proposition 2.2.10. *The following are equivalent for every left R-module B.*

(1) B is flat.

(2) $\mathrm{Tor}_n^R(A,B) = 0$ for all $n \neq 0$ and all right R-module A.

(3) $\mathrm{Tor}_1^R(A,B) = 0$ for all A.

Proof. (1)$\Rightarrow$(2): Let $P_\bullet \to A$ be a projective resolution of A, then $\mathrm{Tor}_*(A,B)$ is computed via $H_*(P_\bullet \otimes_R B)$. Since B is flat, $P_\bullet \otimes_R B$ is also acyclic, hence $\mathrm{Tor}_n^R(A,B) = 0$ for all $n \neq 0$.

The other implications are obvious. $\square$

Example 2.2.11. *Let $R = k[x,y]$. The Koszul complex $K_\bullet((x,y),R)$ gives a projective resolution of k:*

$$0 \to R \xrightarrow{\binom{-y}{x}} R \oplus R \xrightarrow{(x\ y)} R \to k \to 0.$$

Thus, for any R-module M, $\mathrm{Tor}_i^R(k,M)$ may be computed by taking the homology of

$$K_\bullet((x,y),M): \quad 0 \to M \xrightarrow{\binom{-y}{x}} M \oplus M \xrightarrow{(x\ y)} M \to 0 \tag{2.3}$$

(1) Take $M = k$, viewed as an R-module via $R \twoheadrightarrow R/(x,y) \cong k$; then the complex (2.3) becomes

$$0 \to k \xrightarrow{0} k \oplus k \xrightarrow{0} k \to 0$$

and $\mathrm{Tor}_0^R(k,k) = k$, $\mathrm{Tor}_1^R(k,k) = k \oplus k$, $\mathrm{Tor}_2^R(k,k) \cong k$.

(2) Take $M = \mathfrak{a} = (x,y)$, which is an ideal of R; we could compute $\mathrm{Tor}_i^R(k,\mathfrak{a})$ using (2.3). Alternatively, we may use the short exact sequence $0 \to \mathfrak{a} \to R \to k \to 0$ which induces

$$0 \to \mathrm{Tor}_1(k,k) \to \mathrm{Tor}_0(k,\mathfrak{a}) \to \mathrm{Tor}_0(k,R) \to \mathrm{Tor}_0(k,k) \to 0$$

$$\mathrm{Tor}_i(k,k) \cong \mathrm{Tor}_{i-1}(k,\mathfrak{a}), \quad \forall i \geq 2.$$

Note that $\mathrm{Tor}_0(k,R) \to \mathrm{Tor}_0(k,k)$ is an isomorphism because both of them are isomorphic to k and the morphism is surjective. Thus we deduce from (1) that

$$\mathrm{Tor}_0(k,\mathfrak{a}) \cong k \oplus k, \quad \mathrm{Tor}_1(k,\mathfrak{a}) \cong k, \quad \mathrm{Tor}_i(k,\mathfrak{a}) = 0,\ i \geq 2.$$

Definition 2.2.12. *The Pontrjagin dual B^* of a left R-module B is the right R-module $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$; an element r of R acts via $(fr)(b) = f(rb)$. It is also called the character module of B.*

Proposition 2.2.13. *A left R-module B is flat if and only if B^* is an injective right R-module.*

Proof. Use the tensor-hom adjunction. □

We also recall the following fact proved in the course *Algebra I*.

Theorem 2.2.14. *Every finitely presented flat R-module is projective.*

Flat resolution lemma

Lemma 2.2.15. *The groups $\mathrm{Tor}_*^R(A, B)$ may be computed using resolutions by flat modules. That is, if $F_\bullet \to A$ is a resolution of A with the F_n being flat modules, then*

$$\mathrm{Tor}_*^R(A, B) \cong H_*(F_\bullet \otimes_R B).$$

Similarly, if $F'_\bullet \to B$ is a resolution of B by flat modules, then $\mathrm{Tor}_*^R(A, B) \cong H_*(A \otimes_R F'_\bullet)$.

Proof. We use induction and dimension shifting to prove $\mathrm{Tor}_n(A, B) \cong H_n(F_\bullet \otimes B)$. The assertion is true for $n = 0$ because $- \otimes_R B$ is right exact. Let K be such that $0 \to K \to F_0 \to A \to 0$ is exact, so that $\cdots \to F_2 \to F_1 \to K$ is a resolution of K. Then

$$\begin{aligned} \mathrm{Tor}_1(A, B) &= \ker(K \otimes B \to F_0 \otimes B) \\ &= \ker\left\{\tfrac{F_1 \otimes B}{\mathrm{Im}(F_2 \otimes B)} \to F_0 \otimes B\right\} \\ &= \ker(F_1 \otimes B \to F_0 \otimes B)/\mathrm{Im}(F_2 \otimes B) \\ &= H_1(F \otimes B). \end{aligned}$$

For $n \geq 2$, we do induction using dimension shifting trick. □

Proposition 2.2.16 (Flat base change for Tor)**.** *Suppose $R \to S$ is a ring homomorphism, such that S is flat as an R-module. Then for all R-modules A, all S-modules C and all n,*

$$\mathrm{Tor}_n^R(A, C) \cong \mathrm{Tor}_n^S(A \otimes_R S, C).$$

Proof. First note that for any projective right R-module P, $P \otimes_R S$ is a projective S-module (this does not use the assumption on R-flatness of S). Indeed, we have isomorphisms

$$\mathrm{Hom}_S(P \otimes_R S, -) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_S(S, -)) \cong \mathrm{Hom}_R(P, -).$$

If $P_\bullet \to A$ is a projective resolution of A, then since S is R-flat, $P_\bullet \otimes_R S$ is a projective resolution of $A \otimes_R S$. Thus $\mathrm{Tor}_n^S(P_\bullet \otimes_R S, C)$ is the homology of the complex $P_\bullet \otimes_R S \otimes_S C \cong P_\bullet \otimes_R C$ as well. □

In the following we assume R is commutative, so that the $\mathrm{Tor}_*^R(A, B)$ are actually R-modules.

Lemma 2.2.17. *If $\mu : A \to A$ is multiplication by an element $r \in R$, then so are the induced maps*

$$\mu_* : \mathrm{Tor}_n^R(A, B) \to \mathrm{Tor}_n^R(A, B)$$

for all n and B.

Proof. Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map $\tilde{\mu} : P \to P$ lifting μ, and $\tilde{\mu} \otimes B$ is multiplication by r on $P \otimes B$. The induced map μ_* on the subquotient $\mathrm{Tor}_n(A, B)$ of $P_n \otimes B$ is therefore also multiplication by r. □

Corollary 2.2.18. *If A is an R/r-module, then for every R-module B, the modules $\mathrm{Tor}_*^R(A, B)$ are R/r-modules, i.e. annihilated by the ideal rR.*

Lemma 2.2.19. *Assume R is noetherian. Then for any finitely generated R-modules A, B, $\mathrm{Tor}_*^R(A, B)$ is finitely generated R-module.*

Proof. Since R is noetherian and A is finitely generated R-module, we may choose a projective resolution $P \to A$ with each P_n finitely generated. Since B is also finitely generated, so is each $P_n \otimes_R B$. Being a subquotient of $P_n \otimes_R B$, $\mathrm{Tor}_n^R(A, B)$ is also finitely generated over R, because R is noetherian. □

Exercise 1. Complete the proof of Theorem 2.1.7.

Exercise 2. Prove Proposition 2.1.12.

Exercise 3. Show that if $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules and if both B and C are flat, then A is also flat.

Exercise 4. In Example 2.2.11, compute $\mathrm{Tor}_i^R(k, \mathfrak{a}^n)$ for all $i \geq 0$ and $n \geq 1$. Here $\mathfrak{a}^n$ denotes the ideal.

2.3 Lecture 11 (2019-04-01)

External products for Tor

Let A, A', B, B' be R-modules (with R commutative). We may construct a map

$$\operatorname{Tor}_i(A,B) \otimes_R \operatorname{Tor}_j(A',B') \to \operatorname{Tor}_{i+j}(A \otimes_R A', B \otimes_R B')$$

which is called *external product*, as follows.

- There are natural maps $H_i(C) \otimes H_j(C') \to H_{i+j}(\operatorname{tot}(C \otimes C'))$ for every pair of complexes C, C': one maps the tensor product $c \otimes c'$ of two cycles $c \in C_i$ and $c' \in C'_j$ to $c \otimes c' \in C_i \otimes C'_j$. Choose projective resolutions $P_\bullet \to A$, $P'_\bullet \to A'$, so that

$$\operatorname{Tor}_i(A,B) = H_i(P_\bullet \otimes B), \qquad \operatorname{Tor}_j(A',B') = H_j(P'_\bullet \otimes B').$$

  Then we obtain a map

$$\operatorname{Tor}_i(A,B) \otimes \operatorname{Tor}_j(A',B') \to H_{i+j}(\operatorname{tot}(P_\bullet \otimes B \otimes P'_\bullet \otimes B')).$$

- Choose a projective resolution $P''_\bullet \to A \otimes A'$. The comparison theorem gives a chain map $\operatorname{tot}(P_\bullet \otimes P'_\bullet) \to P''_\bullet$ which is unique up to chain homotopy equivalence. This yields a natural map

$$H_n(\operatorname{tot}(P_\bullet \otimes B \otimes P'_\bullet \otimes B')) \cong H_n(\operatorname{tot}(P_\bullet \otimes P'_\bullet \otimes B \otimes B')) \to H_n(P''_\bullet \otimes B \otimes B').$$

Example 2.3.1. *Let $R = k[x,y]$ and $A = B = A' = B' = k$ viewed as R-module via $R \twoheadrightarrow k$. Then describe the map*

$$\operatorname{Tor}_1(k,k) \otimes \operatorname{Tor}_1(k,k) \to \operatorname{Tor}_2(k,k).$$

Describe this morphism.

Tor for $\mathbb{Z}$-modules

Lemma 2.3.2. *For any $\mathbb{Z}$-module B, we have*

$$\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = B/nB, \qquad \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = B[n] = \{b \in B : nB = 0\}.$$

Proof. We have a projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0,$$

which implies that $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B)$ is computed via the homology of the complex

$$0 \to B \xrightarrow{\times n} B \to 0.$$

The result then follows. $\square$

Example 2.3.3. *We have $\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m,n)$; and*

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong m_1\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

where $m = dm_1$.

Proposition 2.3.4. *For all $\mathbb{Z}$-modules A and B:*
(1) $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ *is a torsion* $\mathbb{Z}$*-module;*
(2) $\mathrm{Tor}_n^{\mathbb{Z}}(A,B)=0$ *for* $n\geq 2$*.*

Remark 2.3.5. *The above proposition says that $\mathbb{Z}$ has flat dimension* 1*.*

Proof. A is the direct limit of its finitely generated subgroups A_α, so $\mathrm{Tor}_i^{\mathbb{Z}}(A,B)$ is the direct limit of the $\mathrm{Tor}_i^{\mathbb{Z}}(A_\alpha,B)$, see Proposition 2.2.6. As the direct limit of torsion groups is a torsion group, we may assume that A is finitely generated, that is

$$A\cong\mathbb{Z}^m\oplus\mathbb{Z}/n_1\oplus\mathbb{Z}/n_2\oplus\cdots\oplus\mathbb{Z}/n_r$$

for some integers $m,n_1,...,n_r$. As $\mathbb{Z}^m$ is free, $\mathrm{Tor}_i(\mathbb{Z}^m,-)$ vanishes for $i\geq 1$, and so we have

$$\mathrm{Tor}_i^{\mathbb{Z}}(A,B)\cong\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n_1,B)\oplus\cdots\oplus\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n_r,B).$$

The proposition holds in by Lemma 2.3.2. $\square$

Proposition 2.3.6. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B)$ *is the torsion subgroup of B for every abelian group B.*

Proof. Since $\mathbb{Q}/\mathbb{Z}$ is the direct limit of its finite subgroups, each of which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some integer n, and Since Tor commutes with direct limits (by Proposition 2.2.6), we obtain

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B)\cong\varinjlim\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},B)\cong\varinjlim B[n]$$

which is the torsion subgroup of B. $\square$

Lemma 2.3.7. *Let $R=\mathbb{Z}/m\mathbb{Z}$ and $A=\mathbb{Z}/d\mathbb{Z}$ with $d|m$. Writing $m=dd'$, then we have a periodic free resolution of $\mathbb{Z}/d\mathbb{Z}$ as follows*

$$\cdots\xrightarrow{d}\mathbb{Z}/m\xrightarrow{d'}\mathbb{Z}/m\xrightarrow{d}\mathbb{Z}/m\to\mathbb{Z}/d\to 0,$$

so that for any $\mathbb{Z}/m\mathbb{Z}$-module B, we have

$$\mathrm{Tor}_i^{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z},B)=\begin{cases}B/dB & i=0,\\ B[d]/d'B & i>0\text{ odd},\\ B[d']/dB & i>0\text{ even}.\end{cases}$$

For example, if $m=8$, $d=4$, then $d'=2$ and

$$\mathrm{Tor}_i^{\mathbb{Z}/8\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z},\mathbb{Z}/4\mathbb{Z})=\begin{cases}\mathbb{Z}/4\mathbb{Z} & i=0,\\ (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z} & i>0\text{ odd},\\ (2\mathbb{Z}/4\mathbb{Z})/0\cong\mathbb{Z}/2\mathbb{Z} & i>0\text{ even}.\end{cases}$$

2.3.1 Right derived functors

Let $F:\mathcal{A}\to\mathcal{B}$ be a left exact functor between two abelian categories. Assume that $\mathcal{A}$ has enough injectives. We construct the right derived functors R^iF $(i\geq 0)$ of F as follows. If A is an object in $\mathcal{A}$, choose an injective resolution $A\to I^\bullet$ and define

$$R^iF(A)=H^iF(I).$$

Note that since $0\to F(A)\to F(I^0)\to F(I^1)$ is exact, we have $R^0F(A)\cong F(A)$.

Theorem 2.3.8. *The objects $R^iF(A)$ are independent of the choice of injective resolutions, $\{R^iF\}$ is a universal cohomological δ-functor, and $R^iF(I) = 0$ for $i \geq 1$ and I injective.*

Proof. We may view F as a (covariant) right exact functor

$$F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}},$$

and $\mathcal{A}^{\mathrm{op}}$ has enough projectives. Since $I^\bullet$ becomes a projective resolution of A in $\mathcal{A}^{\mathrm{op}}$, we see that

$$R^iF(A) = (L_iF^{\mathrm{op}})^{\mathrm{op}}(A).$$

Therefore all the results about right exact functors apply to left exact functors. □

Contravariant functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant left exact functor. This is the same as a covariant left exact functor from $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. So if $\mathcal{A}$ has enough projectives (i.e. $\mathcal{A}^{\mathrm{op}}$ has enough injectives), we can define right derived functors $R^*F(A)$ to be

$$H^*F(P_\bullet),$$

where $P_\bullet \to A$ is a projective resolution of A in $\mathcal{A}$. *Convention*: we numerate $P_\bullet$ as $\cdots \to P_n \to \cdots \to P_1 \to P_0$, and since F is contravariant, we get

$$F(P_0) \to F(P_1) \to \cdots \to F(P_n) \to \cdots .$$

As usual, this is a universal δ-functor.

Example 2.3.9. *Given a left R-module A, the functor $\mathrm{Hom}_R(A, -)$ is covariant and left exact, so we can define right derived functors $R^*\mathrm{Hom}(A, -)(B)$ for any R-module B. Correspondingly, given a left R-module B, the functor $\mathrm{Hom}_R(-, B)$ is contravariant and left exact, so we can define $R^*\mathrm{Hom}(-, B)(A)$ for any A. We will see that these two construction give the same object in* **Ab***, which we will denote by*

$$\mathrm{Ext}^*_R(A, B).$$

Example 2.3.10. *Let X be a topological space. The* global sections *functor $\Gamma : \mathbf{Shv}(X) \to$* **Ab** *is the functor*

$$\Gamma(\mathcal{F}) := \mathcal{F}(X).$$

We can show that Γ is left exact (it is right adjoint to the constant sheaves functor). The right derived functors of Γ are the so-called cohomology functors *on X:*

$$H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F}).$$

Example 2.3.11. *Let G be a group. A left G-module is an abelian group A on which G acts. With obvious notion of morphisms between G-modules, we obtain the category G-**Mod** of left G-modules. It can be identified with the category $\mathbb{Z}G$-**Mod** of left modules over the integral group ring $\mathbb{Z}G$. Consider the following two functors from G-**Mod** to **Ab**:*

- *The invariant subgroup A^G of a G-module A,*

$$A^G = \{a \in A : ga = a \ \forall g \in G\}.$$

- *The coinvariants A_G of A,*

$$A_G = A/\textit{submodule generated by } \{ga - a : g \in G, a \in A\}.$$

It is easy to see that $-^G$ is a covariant left exact functor and $-_G$ is covariant right exact. We let

$$H^*(G, A) \quad (\text{resp. } H_*(G, A))$$

denote the right derived functors of $-^G$ (resp. left derived functors of $-_G$).

2.3.2 Ext^n

Given R-modules A and B, we may calculate $\mathrm{Ext}^n_R(A, B)$ in two ways:

- take a projective resolution $P_\bullet$ of A, then

$$\mathrm{Ext}^n_R(A, B) = H_n(\mathrm{Hom}_R(P_\bullet, B).$$

- take an injective resolution $I^\bullet$ of B, then

$$\overline{\mathrm{Ext}}^n_R(A, B) = H^n(\mathrm{Hom}_R(A, I^\bullet)).$$

Theorem 2.3.12. *We have isomorphisms $\mathrm{Ext}^n_R(A, B) \cong \overline{\mathrm{Ext}}^n_R(A, B)$.*

Proof. The proof is analogous to Theorem 2.2.1. Choose a projective resolution $P_\bullet$ of A and an injective resolution $I^\bullet$ of B. Form the first quadrant double cochain complex $\mathrm{Hom}(P_\bullet, I^\bullet) = \{\mathrm{Hom}(P_p, I^q)\}$. We have induced maps

$$\mathrm{Hom}(P_\bullet, B) \to \mathrm{Hom}(P_\bullet, I^\bullet), \qquad \mathrm{Hom}(A, I^\bullet) \to \mathrm{Hom}(P_\bullet, I^\bullet).$$

We then show that the induced maps of complexes

$$\mathrm{Hom}(A, I^\bullet) \to \mathrm{tot}(\mathrm{Hom}(P_\bullet, I^\bullet)) \leftarrow \mathrm{Hom}(P_\bullet, B)$$

are quasi-isomorphisms by showing the mapping cones are acyclic using the Acyclic Assembly Lemma. □

Proposition 2.3.13. *The following are equivalent:*
(1) B is an injective R-module.
(2) $\mathrm{Hom}_R(-, B)$ is an exact functor.
(3) $\mathrm{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all A.
(4) $\mathrm{Ext}^1_R(A, B)$ vanishes for all A.

Proposition 2.3.14. *The following are equivalent:*
(1) A is a projective R-module.
(2) $\mathrm{Hom}_R(A, -)$ is an exact functor.
(3) $\mathrm{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all B.
(4) $\mathrm{Ext}^1_R(A, B)$ vanishes for all B.

2.3.3 Ext^n and direct sum/product

Proposition 2.3.15. *For all n and all rings R,*
(1) $\mathrm{Ext}^n_R(\oplus_i A_i, B) \cong \prod_i \mathrm{Ext}^n_R(A_i, B)$.
(2) $\mathrm{Ext}^n_R(A, \prod_1 B_i) \cong \prod_i \mathrm{Ext}^n_R(A, B_i)$.

Proof. If $P_i \to A_i$ are projective resolutions, so is $\oplus P_i \to \oplus A_i$. Since $\mathrm{Hom}(\oplus P_i, B) \cong \prod \mathrm{Hom}(P_i, B)$, we obtain

$$\mathrm{Ext}^*_R(\oplus A_i, B) \cong H^*(\mathrm{Hom}(\oplus P_i, B)) \cong H^*(\prod \mathrm{Hom}(P_i, B)) \overset{(*)}{\cong} \prod H^*(\mathrm{Hom}(P_i, B)) \cong \prod \mathrm{Ext}^*(A_i, B)$$

where (*) holds because for any family of cochain complexes C_i,

$$H^*(\prod C_i) \cong \prod H^*(C_i).$$

Similarly, if $B_i \to I_i$ are injective resolutions, so is $\prod B_i \to \prod I_i$, and we conclude as before. □

2.3.4 Ext for commutative rings

We assume R is commutative in this subsection. If A, B are R-modules, then so are $\mathrm{Hom}_R(A, B)$ and $\mathrm{Ext}^*_R(A, B)$.

Proposition 2.3.16. *If $\mu : A \to A$ and $\nu : B \to B$ are multiplication by $r \in R$, so are the induced endomorphisms μ^* and v_* of $\mathrm{Ext}^n_R(A, B)$ for all n.*

Proof. Pick a projective resolution $P_\bullet \to A$. Multiplication by r gives an R-module chain map

$$\tilde{\mu} : P_\bullet \to P_\bullet.$$

The induced map $\mathrm{Hom}(P_n, B) \to \mathrm{Hom}(P_n, B)$ is also multiplication by r, because it sends $f \in \mathrm{Hom}(P_n, B)$ to $f\tilde{\mu}$, which takes $x \in P_n$ to $f(rx) = rf(x)$. Hence the map μ^* on the subquotient $\mathrm{Ext}^n(A, B)$ of $\mathrm{Hom}(P_n, B)$ is also multiplication by r. The argument for v_* is similar, using an injective resolution $B \to I^\bullet$. □

Corollary 2.3.17. *If R is commutative, and A is actually an R/r-module (i.e. annihilated by r), then for any R-module B, $\mathrm{Ext}^*_R(A, B)$ are also R/r-modules.*

Proposition 2.3.18. *Assume R is noetherian. Then for any finitely generated R-modules A, B, the R-modules $\mathrm{Ext}^*_R(A, B)$ are also finitely generated.*

Proof. Since A is finitely generated and R is noetherian, we may find a projective resolution $P_\bullet$ of A, such that each P_n is projective and finitely generated. This implies that $\mathrm{Hom}_R(P_n, B)$ is also finitely generated. Since R is noetherian, $\mathrm{Ext}^n_R(A, B)$ is also finitely generated (because it is a subquotient of $\mathrm{Hom}_R(P_n, B)$). □

2.3.5 Computations of Ext^n

Lemma 2.3.19. $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ *for $n \geq 2$ and all abelian groups A, B.*

Proof. Embed B in an injective abelian group I^0. Since I^0 is injective, the quotient I^1 is also divisible, hence injective. Therefore $B \to I^0 \to I^1 \to 0$ is an injective resolution of length 2, and the result follows. □

Lemma 2.3.20. *We have* $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) \cong B[n]$ *and* $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) \cong B/nB$*.*

Proof. Use the projective resolution

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

and the fact that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \cong B$. □

Remark 2.3.21. *One may realize that the result is dual to* $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B)$*. We will see the dual relation between* Tor *and* Ext*.*

Example 2.3.22. *Write* $R = \mathbb{Z}/m\mathbb{Z}$ *and* $A = \mathbb{Z}/d\mathbb{Z}$ *with* $d|m$*. Writing* $m = dd'$*, then*

$$\cdots \xrightarrow{d} \mathbb{Z}/m \xrightarrow{d'} \mathbb{Z}/m \xrightarrow{d} \mathbb{Z}/m \to \mathbb{Z}/d \to 0$$

is an infinite periodic injective resulution of A*. Thus for any* $\mathbb{Z}/m\mathbb{Z}$*-module* B*, we have*

$$\mathrm{Ext}^i_{\mathbb{Z}/m\mathbb{Z}}(A, B) \cong \begin{cases} B[d] & i = 0 \\ B[d']/dB & i > 0 \text{ odd} \\ B[d]/d'B & i > 0 \text{ even.} \end{cases}$$

2.3.6 $\mathrm{Ext}^n_{\mathbb{Z}}(A, \mathbb{Z})$

Let A be a torsion abelian group, and recall that A^* denotes its Pontryagin dual $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. If A is finite abelian group, then

$$\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong A.$$

Using the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain that

$$\mathrm{Ext}^0(A, \mathbb{Z}) = 0, \qquad \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = A^*.$$

For example, let $\mathbb{Z}_{p^\infty} := \varinjlim p^{-n}\mathbb{Z}/\mathbb{Z}$, then

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = (\mathbb{Z}_{p^\infty})^* \cong \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}) =: \hat{\mathbb{Z}}_p,$$

where $\hat{\mathbb{Z}}_p$ denotes the p-adic integer numbers.

2.4 Lecture 12 (2019-04-03)

2.4.1 Yoneda extension

Although the material discussed below work for any abelian category $\mathcal{A}$, we will assume $\mathcal{A} = R$-**Mod** for a ring R.

An *extension* ξ of A by B is an exact sequence $0 \to B \to X \to A \to 0$. Two extensions ξ and ξ' are equivalent if there is a commutative diagram

(2.4)
$$\begin{tikzcd} \xi: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\ \xi': & 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

An extension is *split* if it is equivalent to $0 \to B \xrightarrow{(0,\mathrm{Id}_B)} A \oplus B \to A \to 0$. We denote by $\mathrm{YExt}^1(A, B)$ the (pointed) set of equivalence classes of extensions of A by B.

Lemma 2.4.1. *If* $\mathrm{Ext}^1(A, B) = 0$, *then every extension of A by B is split.*

Proof. Given $0 \to B \to X \to A \to 0$, applying $\mathrm{Hom}(-, B)$ to it gives

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(B, B) \to 0 = \mathrm{Ext}^1(A, B).$$

Thus there exists a retraction $X \to B$, showing that X is split. □

Definition 2.4.2. *Given an extension ξ of A by B, apply* $\mathrm{Hom}(-, B)$ *to get*

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(B, B) \xrightarrow{\partial} \mathrm{Ext}^1(A, B).$$

Define $\Theta(\xi)$ *to be* $\partial(\mathrm{Id}_B)$.

Remark 2.4.3. *Given ξ as above, we could alternatively apply* $\mathrm{Hom}(A, -)$ *to get*

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \to \mathrm{Ext}^1(A, B)$$

and define $\Theta'(\xi)$ *to be* $\partial(\mathrm{Id}_A)$. *Check that we get the same thing. Exercise.*

The map Θ is well-defined. Indeed, consider the diagram (2.4) which is an equivalence; it induces a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[r, "\partial"] \arrow[d, equal] & \mathrm{Ext}^1(A, B) \arrow[d, equal] \\ \mathrm{Hom}(X', B) \arrow[r] \arrow[u] & \mathrm{Hom}(B, B) \arrow[r, "\partial"] & \mathrm{Ext}^1(A, B) \end{tikzcd}$$

thus $\Theta(\xi) = \Theta(\xi')$.

Lemma 2.4.4. *Consider a pushout diagram*

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "j'"'] & X \arrow[d] \\ X' \arrow[r, "i"] & X \sqcup_Y X' \end{tikzcd}$$

If j is injective, then so is i and

$$\mathrm{Coker}(Y \to X) \cong \mathrm{Coker}(X' \to X \sqcup_Y X').$$

Proof. Recall that $X \sqcup_Y X' = X \oplus X'/\{(j(y), -j'(y)) : y \in Y\}$. So if $i(x') = 0$, i.e. $(0, x') = (j(y), j'(y))$ for some $y \in Y$, then $y = 0$ (as j is injective), hence $x' = 0$. Left as an exercise to show $\mathrm{Coker}(j) = \mathrm{Coker}(i)$. □

Theorem 2.4.5. *Given two R-modules A and B, the mapping $\Theta : \xi \to \partial(\mathrm{Id}_A)$ establishes a $1:1$ correspondence*

$$\mathrm{YExt}^1(A, B) \overset{1:1}{\longleftrightarrow} \mathrm{Ext}^1(A, B)$$

in which the split extension corresponds to the element $0 \in \mathrm{Ext}^1(A, B)$.

Proof. Fix an exact sequence $0 \to M \xrightarrow{j} P \to A \to 0$ with P projective. Applying $\mathrm{Hom}(-, B)$ yields an exact sequence

$$\mathrm{Hom}(P, B) \to \mathrm{Hom}(M, B) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \to 0.$$

Given $x \in \mathrm{Ext}^1(A, B)$, choose $\beta \in \mathrm{Hom}(M, B)$ with $\delta(\beta) = x$. Let X be the pushout of j and β, i.e. the cokernel of $M \xrightarrow{(j,-\beta)} P \oplus B$, so that we obtain a commutative diagram with exact rows (by Lemma 2.4.4):

$$\begin{tikzcd} & 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ \xi : & 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

This induces the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(P, B) \arrow[r] & \mathrm{Hom}(M, B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A, B) \\ \mathrm{Hom}(X, B) \arrow[r] \arrow[u] & \mathrm{Hom}(B, B) \arrow[r, "\partial"] \arrow[u, "\circ\beta"'] & \mathrm{Ext}^1(A, B). \arrow[u, equal] \end{tikzcd}$$

By the naturality of the connecting map ∂, we see that $\Theta(\xi) = \partial(\mathrm{Id}_B) = \delta(\beta)$, that is, Θ is surjective.

It remains to prove that Θ is injective. Suppose that Θ takes both ξ and ξ' to the same element in $\mathrm{Ext}^1(A, B)$. Let $0 \to M \to P \to A \to 0$ be constructed as above with P projective. Then there exists $\alpha : P \to X$, with an induced morphism β, fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, dotted, "\beta"] & P \arrow[r] \arrow[d, dotted, "\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

from which we get the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(P, B) \arrow[r, "\circ j"] & \mathrm{Hom}(M, B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A, B) \arrow[r] & 0 \\ \mathrm{Hom}(X, B) \arrow[r] \arrow[u, "\circ\alpha"'] & \mathrm{Hom}(B, B) \arrow[r, "\partial"] \arrow[u, "\circ\beta"'] & \mathrm{Ext}^1(A, B). \arrow[u, equal] & \end{tikzcd}$$

We obtain that $\Theta(\xi) := \partial(\mathrm{Id}_B) = \delta(\beta)$. Similarly, we have a morphism $\alpha' : P \to X'$ (with induced β') and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta'"] & P \arrow[r] \arrow[d, "\alpha'"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

so that $\Theta(\xi') := \partial'(\mathrm{Id}_B) = \delta(\beta')$. Since $\Theta(\xi) = \Theta(\xi')$ by assumption, we obtain $\delta(\beta) = \delta(\beta')$. Hence there exists $\gamma \in \mathrm{Hom}(P, B)$ such that

$$\gamma \circ j = \beta' - \beta.$$

Furthermore, X is the pushout of (j, β) and X' is the pushout of (j, β') respectively, namely

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[d, "\sigma"] \\
B \arrow[r, "i"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta'"'] & P \arrow[d, "\sigma"] \\
B \arrow[r, "i'"] & X'
\end{tikzcd}$$

This implies that X is isomorphic to X'. Indeed, if we set $\alpha = \sigma + i\gamma$, then

$$i\beta' = i(\beta + \gamma j) = \sigma j + i\gamma j = \alpha j$$

as illustrated as follows:

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta'"'] & P \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
B \arrow[r, "i'"] \arrow[drr, bend right, "i"'] & X' \arrow[dr, dotted] & \\
& & X
\end{tikzcd}$$

Hence there is an induced morphism $X' \to X$; one checks that this is an isomorphism. $\square$

Baer sum

By Theorem 2.4.5, the abelian group structure on $\mathrm{Ext}^1(A, B)$ naturally translates to an abelian group structure on $\mathrm{YExt}^1(A, B)$. The precise definition is as follows.

Let $\xi : 0 \to B \xrightarrow{f} X \xrightarrow{g} A \to 0$ and $\xi' : 0 \to B \xrightarrow{f'} X' \xrightarrow{g'} A \to 0$ be two extensions of A by B. Let X'' be the pullback

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[d, "g"] \\
X' \arrow[r, "g'"] & A
\end{tikzcd}$$

i.e. $X'' = \{(x, x') \in X \times X' : g(x) = g(x') \text{ in } A\}$. Then X'' contains the skew diagonal $\{(f(b), -f'(b)) : b \in B\}$; set Y to be the quotient. Then we obtain an exact sequence

$$\phi : \quad 0 \to B \to Y \to A \to 0,$$

where the first map is $b \mapsto [(f(b), 0)] = [(0, f'(b))]$ and the second is $(x, x') \mapsto g(x) = g'(x')$.

Set $\phi = \xi + \xi'$, and call it the *Baer sum* of ξ and ξ' (first introduced by Baer in 1934).

Corollary 2.4.6. *The set* $\mathrm{YExt}^1(A, B)$ *under Baer sum, with zero being the class of the split extension. The map* Θ *is an isomorphism of abelian groups.*

Proof. Exercise. $\square$

Yoneda Ext **groups**

Definition 2.4.7. *An element of the Yoneda* $\mathrm{YExt}^n(A, B)$ *is an equivalence class of exact sequences of the form*

$$\xi: \quad 0 \to B \to X_n \to \cdots \to X_1 \to A \to 0.$$

It is called similar *to an extension* ξ' *if there is a commutative diagram*

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d] & \cdots \arrow[r] & X_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi': & 0 \arrow[r] & B \arrow[r] & X_n' \arrow[r] & \cdots \arrow[r] & X_1' \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

Two extensions ξ *and* ξ' *are called* equivalent *if there exist* $\xi = \xi^0, \xi^1, ..., \xi^m = \xi'$ *for some* m*, such that* ξ^i *and* ξ^{i+1} *are similar (hence* $\xi^i \to \xi^{i+1}$ *or* $\xi^i \leftarrow \xi^{i+1}$*).*

To define $\xi + \xi'$ *when* $n \geq 2$*, let* X_1'' *be the pullback of* X_1 *and* X_1'' *over* A*, let* X_n'' *be the pushout of* X_n *and* X_n' *under* B*. Then* $\xi + \xi'$ *is the class of the extension*

$$0 \to B \to X_n'' \to X_{n-1} \oplus X_{n-1}' \to \cdots \to X_2 \oplus X_2'' \to X_1'' \to A \to 0.$$

Theorem 2.4.8. *There is an ismorphism of abelian groups*

$$\mathrm{YExt}^n(A, B) \xrightarrow{\sim} \mathrm{Ext}^n(A, B).$$

Proof. Choose $P \to A$ to be a projective resolution, then the Comparison Theorem yields a map from P to ξ, hence a diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r] \arrow[d, "\beta"] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi: & 0 \arrow[r] & B \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_a \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

By dimension shifting, there is an exact sequence

$$\mathrm{Hom}(P_{n-1}, B) \to \mathrm{Hom}(M, B) \xrightarrow{\partial} \mathrm{Ext}^n(A, B) \to 0$$

and we define a map $\Theta : \mathrm{YExt}^n(A, B) \to \mathrm{Ext}^n(A, B)$ by $\Theta(\xi) := \partial(\beta)$. Then we check that this is well-defined and is an isomorphism. See GTM 4, Chap. IV, §9 for details. ☐

Remark 2.4.9. *We can define the group* $\mathrm{YExt}^n(A, B)$ *in any abelian category, even without enough projectives or injectives.*

Yoneda products

Given $A, B, C \in R$-**Mod**, we can define the *Yoneda product*

$$\mathrm{Ext}^m_R(B, C) \otimes \mathrm{Ext}^n_R(A, B) \to \mathrm{Ext}^{m+n}_R(A, C)$$

as follows. Given extensions

$$\xi: \quad 0 \to B \xrightarrow{\alpha} X_n \to \cdots \to X_1 \to A \to 0$$

and

$$\eta: \quad 0 \to C \to Y_m \to \cdots \to Y_1 \xrightarrow{\beta} B \to 0$$

we define the product of $\eta \otimes \xi$ to be the equivalence class of the extension

$$0 \to C \to Y_m \to \cdots \to Y_1 \xrightarrow{\alpha\circ\beta} X_n \to \cdots \to X_1 \to A \to 0.$$

Yoneda product is associative. As a result, $\mathrm{Ext}_R^*(A, A)$ is a graded ring. Moreover, for any R-module B, $\mathrm{Ext}_R^*(A, B)$ is a graded module over $\mathrm{Ext}_R^*(A, A)$.

Remark 2.4.10. *One may also define a product using projective resolutions and injective resolutions, and can show that this coincides with the Yoneda products.*

2.4.2 Kunneth formula

Theorem 2.4.11. *Let P be a chain complex of flat right R-modules such that each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is an exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

Proof. Since $\mathrm{Tor}_1^R(d(P_n), M) = 0$, we get a short exact sequence

$$0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0$$

for every n. These give a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes M \to P_\bullet \otimes M \to d(P_\bullet) \otimes M \to 0$$

with the differentials in $Z_\bullet$ and $d(P_\bullet)$ are zero. The homology sequence is

$$H_{n+1}(dP_\bullet \otimes M) \xrightarrow{\partial_{n+1}} H_n(Z_\bullet \otimes M) \to H_n(P_\bullet \otimes M) \to H_n(dP_\bullet \otimes M) \xrightarrow{\partial_n} H_{n-1}(Z_\bullet \otimes M), \tag{2.5}$$

with

$$H_i(Z_\bullet \otimes M) = Z_i \otimes M, \qquad H_i(dP_\bullet \otimes M) = d(P_n) \otimes M.$$

Moreover, the morphism ∂ is just induced from the inclusion map $d(P_n) \subset Z_{n-1}$ (tensored with M). We deduce from (2.5) a short exact sequence for each n

$$0 \to \mathrm{Coker}(\partial_{n+1}) \to H_n(P_\bullet \otimes M) \to \ker(\partial_n) \to 0.$$

On the other hand, we have for any n two short exact sequences

$$0 \to Z_n \to P_n \to d(P_n) \to 0, \qquad 0 \to d(P_{n+1}) \to Z_n \to H_n(P) \to 0.$$

The first one shows that Z_n is also flat, since P_n and $d(P_n)$ are both flat. Thus the second one is a flat resolution of $H_n(P)$, and $\mathrm{Tor}_*(H_n(P), M)$ is the homology of the complex

$$0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial} Z_n \otimes M \to 0$$

which are respectively the kernel and cokernel of ∂. □

Theorem 2.4.12 (Universal Coefficient Theorem)**.** *Let P be a chain complex of free abelian groups. Then for every n and every abelian group M the Kunneth formula splits noncanonically, yielding a direct sum decomposition*

$$H_n(P \otimes M) \cong H_n(P) \otimes M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M).$$

Remark 2.4.13. *This uses the key fact that $\mathbb{Z}$ has flat dimension* 1.

Proof. We use the fact that every subgroup of a free abelian group is still free. Thus, $d(P_n)$ is free and the exact sequence $0 \to Z_n \to P_n \to d(P_n) \to 0$ splits, giving a non-canonical isomorphism $P_n \cong Z_n \oplus d(P_n)$. This allows to conclude by the proof of Theorem 2.4.11. $\square$

Similarly, we have the Universal Coefficient Theorem for Cohomology.

Theorem 2.4.14. *Let P be a chain complex of projective R-modules such that each $d(P_n)$ is also projective. Then for every n and every R-module M, there is a (noncanonically) split exxact sequence*

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(P), M) \to H^n(\operatorname{Hom}_R(P, M)) \to \operatorname{Hom}_R(H_n(P), M) \to 0.$$

Exercise 1. Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \cong \hat{\mathbb{Z}}_p/\mathbb{Z}$.

Exercise 2. Prove Remark 2.4.3.

Exercise 3. Prove Corollary 2.4.6.

Exercise 4. Let $M^* := \operatorname{Hom}_R(M, R)$ for any R-module M. Let $P_1 \to P_0 \to M \to 0$ be an exact sequence of R-modules with P_0, P_1 finitely generated and projective. Let

$$D = \operatorname{Coker}(P_0^* \to P_1^*).$$

Prove that the sequence

$$0 \to \operatorname{Ext}^1_R(D, R) \to M \to M^{**} \to \operatorname{Ext}^2_R(D, R) \to 0$$

is exact.

2.5 Lecture 13 (2019-04-08)

2.5.1 Derived functors of the inverse limit

Let $\mathcal{A}$ be an abelian category and I be a directed poset (or more generally a small category).

Proposition 2.5.1. *If $\mathcal{A}$ is complete (i.e. any product exists, hence any limit exists) and has enough injectives, then $\mathcal{A}^I$ has enough injectives.*

Proof. The construction is similar to Theorem 1.8.18. Given $\{A_i\} \in \mathcal{A}^I$, we may associate the k-th coordinate $A_i \in \mathcal{A}$, which gives an exact functor from $\mathcal{A}^I \to \mathcal{A}$. On the other hand, given $B \in \mathcal{A}$, we may define an object $k_*(B) \in \mathcal{A}^I$ by:

$$(k_*B)_i = \begin{cases} B & i \geq k \\ 0 & \text{otherwise} \end{cases}$$

where for $i \geq j \geq k$, the transition morphism $(k_*B)_i \to (k_*B)_j$ is just the identity map on B. It is easy to check that

$$\operatorname{Hom}_{\mathcal{A}^I}(\{A_i\}, k_*B) \cong \operatorname{Hom}_{\mathcal{A}}(A_k, B),$$

i.e. $k_* : \mathcal{A} \to \mathcal{A}^I$ is a right adjoint of an exact functor, hence sends injectives to injectives by Proposition 1.8.10. Thus for any injective object E in $\mathcal{A}$, k_*E is injective in $\mathcal{A}^I$.

Now given $\{A_i\} \in \mathcal{A}^I$, choose a monomorphism $A_k \hookrightarrow E_k$ for each $k \in I$, with E_k injective. Then k_*E_k is injective, and so is $\prod_{k\in I} k_*E_k$. Moreover, by the adjointness there is a morphism $\{A_i\} \to k_*E_k$ for each k, hence a morphism (note that $\mathcal{A}^I$ is also complete)

$$\{A_i\} \to \prod_{k\in I} k_*(E_k). \tag{2.6}$$

Finally one checks that this is a monomorphism, which finishes the proof. $\square$

Also recall that the functor $\lim : \mathcal{A}^I \to \mathcal{A}$ is left exact. Thus, assuming $\mathcal{A}$ has enough injectives, we can define the right derived functors $R^n \lim : \mathcal{A}^I \to \mathcal{A}$. In the following, we will consider the case $\mathcal{A} = \mathbf{Ab}$ or more generally $\mathcal{A} = R\text{-}\mathbf{Mod}$, and I is the poset $\cdots \to 2 \to 1 \to 0$ of whole natural numbers $\mathbb{N}$. Recall the following Theorem 1.5.10.

Theorem 2.5.2. *Let*

$$0 \to \{A_i\} \xrightarrow{f_i} \{B_i\} \xrightarrow{g_i} \{C_i\} \to 0$$

be an exact sequence of inverse systems of abelian groups over $\mathbb{N}$. If $\{A_i\}$ is Mittag-Leffler, then the induced sequence

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 0$$

is exact.

This may be considered as that $R^1 \varprojlim A_i = 0$ for an inverse system satisfying the Mittag-Leffler condition. In fact, taking a monomorphism $\{A_i\} \hookrightarrow \{B_i\}$ to an injective object, we obtain

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim B_i/A_i \to R^1 \varprojlim A_i \to 0,$$

hence $R^1 \varprojlim A_i = 0$ by Theorem 2.5.2.

Definition 2.5.3. *Given an inverse system $\{A_i\}$ in $\mathbf{Ab}^{\mathbb{N}}$, define the map*

$$\Delta : \prod_{i=0}^{\infty} A_i \to \prod_{i=0}^{\infty} A_i$$

by the formula

$$\Delta(\cdots, a_i, \cdots, a_0) = (\cdots, a_i - \bar{a}_{i+1}, \cdots, a_1 - \bar{a}_2, a_0 - \bar{a}_1)$$

where $\bar{a}_{i+1}$ denote the image of $a_{i+1} \in A_{i+1}$ in A_i. The kernel of Δ is $\varprojlim A_i$ as is easily checked. We define $\varprojlim^1 A_i$ to be the cokernel of Δ, making $\varprojlim^1$ to be a functor $\mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}$.

We also set

$$\varprojlim{}^0 A_i = \varprojlim A_i, \qquad \varprojlim{}^n A_i = 0, \quad \forall n \neq 0, 1.$$

Lemma 2.5.4. *The functors $\{\varprojlim^n\}$ form a universal cohomological δ-functor. Hence we have $\varprojlim^n A_i = R^n \varprojlim$ for any n.*

Proof. If $0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$ is a short exact sequence of inverse systems, apply the Snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & \prod A_i \arrow[r] \arrow[d, "\Delta"] & \prod B_i \arrow[r] \arrow[d, "\Delta"] & \prod C_i \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \prod A_i \arrow[r] & \prod B_i \arrow[r] & \prod C_i \arrow[r] & 0
\end{tikzcd}$$

to get the natural long exact sequence.

In order to show that the $\{\varprojlim^n\}$ form a universal δ-functor, we only need to check they are effaceable for $n \geq 1$, for which it suffices to check $\varprojlim^1$ vanishes on enough injectives, namely the injectives in (2.6). By construction, it suffices to show $\varprojlim^1(k_* E)$ vanishes, where E is an injective object in $\mathcal{A}$. The inverse system $k_* E$ is simply

$$k_* E : \quad \cdots \xrightarrow{\mathrm{Id}} E \xrightarrow{\mathrm{Id}} E \to 0 \to 0 \to \cdots \to 0;$$

In particular, all the transition maps are surjective, and it is direct to check that Δ is surjective. $\square$

Corollary 2.5.5. *For $n \geq 2$, $R^n \varprojlim(-)$ vanishes.*

Exercise. If $\{A_i\}$ satisfies the Mittag-Leffler condition, show $\varprojlim^1 A_i = 0$ using the definition.

Theorem 2.5.6. *Let $\cdots \to C_1^\bullet \to C_0^\bullet$ be a system of cochain complexes of abelian groups satisfying the Mittag-Leffler condition, and set $C^\bullet = \varprojlim C_i^\bullet$. Then there is an exact sequence for each q:*

$$0 \to \varprojlim{}^1 H^{q-1}(C_i^\bullet) \to H^q(C^\bullet) \to \varprojlim H^q(C_i^\bullet) \to 0.$$

Proof. We first note the following facts:

(a) Since $\varprojlim$ is left exact, if we have a morphism $\{A_i\} \xrightarrow{f_i} \{A_i'\}$, with limit $A \xrightarrow{f} A'$, then $\ker(f) = \varprojlim \ker(f_i)$.

(b) However, it is subtler for image: the exact sequence $0 \to \ker(f_i) \to A_i \to \mathrm{Im}(f_i) \to 0$ implies

$$0 \to \ker(f) \to A \to \varprojlim \mathrm{Im}(f_i) \to \varprojlim{}^1 \ker(f_i) \to \varprojlim{}^1 A_i \to \varprojlim{}^1 \mathrm{Im}(f_i) \to 0$$

In the case $\{A_i\}$ satisfies the Mittag-Leffler condition, we have $\varprojlim^1 A_i = 0$, hence by (a) we obtain

$$0 \to \mathrm{Im}(f) \to \varprojlim \mathrm{Im}(f_i) \to \varprojlim{}^1 \ker(f_i) \to 0, \qquad \varprojlim{}^1 \mathrm{Im}(f_i) = 0.$$

Now we prove the theorem. Define $B_i^\bullet \subset Z_i^\bullet \subset C_i^\bullet$ to be the cocycles and coboundaries, and similarly $B^\bullet \subset Z^\bullet \subset C$. Then we have a short exact sequence for each q

$$0 \to Z_i^{q-1} \to C_i^{q-1} \to B_i^q \to 0,$$

which implies (as seen above): $Z^q = \varprojlim Z_i^q$, $\varprojlim^1 B_i^q = 0$, and

$$0 \to B^q \to \varprojlim B_i^q \to \varprojlim{}^1 Z_i^{q-1} \to 0. \tag{2.7}$$

On the other hand, the exact sequence $0 \to B_i^q \to Z_i^q \to H^q(C_i^\bullet) \to 0$ induces an isomorphism $\varprojlim^1 Z_i^q \cong \varprojlim^1 H^q(C_i^\bullet)$ and

$$0 \to \varprojlim B_i^q \to \varprojlim Z_i^q = Z^q \to \varprojlim H^q(C_i^\bullet) \to 0 \tag{2.8}$$

The result then follows: indeed, C^q has a filtration

$$0 \subseteq B^q \subseteq \varprojlim_i B_i^q \subseteq Z^q \subseteq C^q,$$

hence a short exact sequence by (2.7) and (2.8)

$$0 \to \varprojlim{}^1 H^{q-1}(C_i^\bullet) \to H^q(C^\bullet) = Z^q/B^q \to \varprojlim H^q(C_i^\bullet) \to 0,$$

as desired. □

Application

Corollary 2.5.7. *Let A be a left R-module that is the union of submodules: $A_0 \subseteq \cdots \subset A_i \subseteq A_{i+1} \subseteq \cdots$. Then, for every R-module B and every q, the sequence*

$$0 \to \varprojlim{}^1 \mathrm{Ext}_R^{q-1}(A_i, B) \to \mathrm{Ext}_R^1(A, B) \to \varprojlim \mathrm{Ext}_R^q(A_i, B) \to 0$$

is exact.

For example, take $\mathbb{Z}_{p^\infty} = \cup_i \mathbb{Z}/p^i$, this gives a short exact sequence for every B:

$$0 \to \varprojlim{}^1 \mathrm{Hom}(\mathbb{Z}/p^i, B) \to \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, B) \to \varprojlim B/p^i B \to 0.$$

If B is torsion-free, then the first term vanishes and we obtain $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, B) \cong \varprojlim B/p^i B$. Furthermore, if B is a finitely generated $\mathbb{Z}$-module, then each $\mathrm{Hom}(\mathbb{Z}/p^i, B)$ is also finitely generated $\mathbb{Z}/p^i$-module (see Proposition 2.3.18 and Corollary 2.3.17), hence a *finite* abelian group. As a consequence, the inverse system $\mathrm{Hom}(\mathbb{Z}/p^i, B)$ satisfies the Mittag-Leffler condition, so that $\varprojlim^1$ vanishes and we still have an isomorphism $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, B) \cong \varprojlim B/p^i B$.

Proof. Choose an injective resolution $B \hookrightarrow E^\bullet$; then we obtain a system of cochain complexes

$$\cdots \to \mathrm{Hom}_R(A_{i+1}, E^\bullet) \to \mathrm{Hom}_R(A_i, E^\bullet) \to \cdots \to \mathrm{Hom}_R(A_0, E^\bullet).$$

The Mittag-Leffler condition is satisfied because each E^q is injective, so that $\mathrm{Hom}(A_{i+1}, E^q) \to \mathrm{Hom}(A_i, E^q)$ is surjective. Since the cohomology of $\mathrm{Hom}(A, E^\bullet)$ (resp. $\mathrm{Hom}(A_i, E^\bullet)$) computes $\mathrm{Ext}^q(A, B)$ (resp. $\mathrm{Ext}^q(A_i, B)$), the result follows from Theorem 2.5.6. □

2.5.2 Homological dimensions

Dimensions

Definition 2.5.8. *Let A be a left R-module.*

(1) The projective dimension *$pd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by projective modules*

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$

(2) The injective dimension *$id(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by injective rmodules*

$$0 \to A \to E^0 \to E^1 \to \cdots \to E^n \to 0.$$

(3) The flat dimension *$fd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by flat modules*

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0.$$

If no finite resolution exists, we set $pd(A), id(A)$ or $fd(A)$ equal to ∞.

Remark 2.5.9. *As projective modules are flat, $fd(A) \leq pd(A)$ for every R-modules A. We need not have equality: over $\mathbb{Z}$, $fd(\mathbb{Q}) = 0$ but $pd(\mathbb{Q}) = 1$.*

Lemma 2.5.10. *(1) The following are equivalent for a left R-module A:*

(a) $pd(A) \leq d$;

(b) $\mathrm{Ext}^n_R(A, B) = 0$ for all $n > d$ and all R-module B;

(c) $\mathrm{Ext}^{d+1}_R(A, B) = 0$ for all R-modules B;

(d) If $0 \to M_d \to P_{d-1} \to P_{d-2} \to \cdots \to P_0 \to A \to 0$ is any resolution with the P_i projective, then the syzygy M_d is also projective.

(2) The following are equivalent for a left R-module B:

(a) $id(B) \leq d$;

(b) $\mathrm{Ext}^n_R(A, B) = 0$ for all $n > d$ and all R-module A;

(c) $\mathrm{Ext}^{d+1}_R(A, B) = 0$ for all R-modules A;

(d) If $0 \to B \to E^0 \to E^1 \to \cdots \to E^{d-1} \to M^d \to 0$ is any resolution with the E^i injective, then the syzygy M^d is also injective.

(3) The following are equivalent for a left R-module A:

(a) $fd(A) \leq d$;

(b) $\mathrm{Tor}_n^R(A, B) = 0$ for all $n > d$ and all R-modules B;

(c) $\mathrm{Tor}_{d+1}^R(A, B) = 0$ for all R-modules B;

(d) If $0 \to M_d \to F_{d-1} \to \cdots \to F_0 \to A \to 0$ is any resolution with the F_i flat, then the syzygy M_d is also flat.

Proof. (1) Easy to see (d)⇒(a)⇒(b)⇒(c). For (c)⇒(d): given the resolution as in the statement, by (c) we know $\mathrm{Ext}^1_R(M_d, B) \cong \mathrm{Ext}^{d+1}_R(A, B)$ vanishes for all B, hence M_d is projective. □

Lemma 2.5.11. *A left R-module B is injective iff $\mathrm{Ext}^1_R(R/I, B) = 0$ for all left ideals I.*

Proof. Applying $\mathrm{Hom}(-, B)$ to $0 \to I \to R \to R/I \to 0$, we see that

$$\mathrm{Hom}(R, B) \to \mathrm{Hom}(I, B) \to \mathrm{Ext}^1(R/I, B) \to 0$$

is exact. By Baer's criterion, B is injective iff the first map is surjective, that is, iff $\mathrm{Ext}^1(R/I, B) = 0$. □

Example 2.5.12. *(1) As a module over $\mathbb{Z}$,*

(a) $A = \mathbb{Z}$: $pd = 0$, $id = 1$, $fd = 0$;

(b) $A = \mathbb{Z}/n\mathbb{Z}$: $pd = 1$, $id = 1$, $fd = 1$;

(c) $A = \mathbb{Q}$: $pd = 1$, $id = 0$, $fd = 0$

(2) As a module over $\mathbb{Z}/n\mathbb{Z}$,

(a) $A = \mathbb{Z}/n\mathbb{Z}$: $pd = 0$, $id = 0$, $fd = 0$. Here $id = 0$ follows from Lemma 2.5.11 and the calculation in Example 2.3.22.

(b) $A = \mathbb{Z}/d\mathbb{Z}$, with $n = dd'$ and $d' > 1$: $pd = \infty$, $id = 1$, $fd = \infty$.

Theorem 2.5.13. *The following numbers are the same for any ring R:*
(1) $\sup\{id(B) : B \in R\text{-}\mathbf{Mod}\}$
(2) $\sup\{pd(A) : A \in R\text{-}\mathbf{Mod}\}$
(3) $\sup\{pd(R/I) : I \text{ is a left ideal of } R\}$
(4) $\sup\{d : \operatorname{Ext}^d_R(A, B) \neq 0 \text{ for some right modules } A, B\}$.
This common number (possibly ∞) is called the (left) global dimension of R, denoted by $l.gl.dim(R)$.

Remark 2.5.14. *One may define the right global dimension similarly. If R is commutative, then clearly $l.gl.dim(R) = r.gl.dim(R)$; also the equality holds if R is left and right noetherian. However, they could be different in general.*

Proof. Lemma 2.5.10 shows that (2)=(4)=(1), also we know (2)≥(3). So we may assume $d = \sup\{pd(R/I)\}$ is finite and that $id(B) > d$ for some R-module B. For this B, choose a resolution $0 \to B \to E^0 \to \cdots \to E^{d-1} \to M^d \to 0$ with the E^i injective. But then for all left ideals I we have
$$0 = \operatorname{Ext}^{d+1}_R(R/I, B) \cong \operatorname{Ext}^1_R(R/I, M^d).$$
By Lemma 2.5.11, M^d is injective, giving a contradiction. □

Tor**-dimension**

Theorem 2.5.15. *the following numbers are the same for any ring R:*
(1) $\sup\{fd(A) : A \in \mathbf{Mod}\text{-}R\}$
(2) $\sup\{fd(R/J) : J \text{ right ideal}\}$
(3) $\sup\{fd(B) : B \in R\text{-}\mathbf{Mod}\}$
(4) $\sup\{fd(R/I) : I \text{ left ideal}\}$
(5) $\sup\{\operatorname{Tor}^R_d(A, B) \neq 0$ for some *R-modules* $A, B\}$.
This common number (possibly ∞) is caleld the Tor*-dimension of R.*

Proof. Proved in a similar way. □

Since $fd(A) \leq pd(A)$ for every R-module A, we have $\operatorname{Tor-dim}(R) \leq l.gl.\dim(R)$. Equality need not hold in general.

Proposition 2.5.16. *If R is left noetherian, then*
(1) $fd(A) = pd(A)$ for every finitely generated left R-module A.
(2) $\operatorname{Tor-dim}(R) = l.gl.\dim(R)$.

Proof. Since we can compute Tor-dim and $l.gl.\dim$ using the modules R/I, it suffices to prove (1). Since $fd(A) \leq pd(A)$, it suffices to suppose that $fd(A) = d < \infty$ and prove that $pd(A) \leq d$. As R is noetherian, there is a resolution

$$0 \to M \to P_{d-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

in which the P_i are finitely generated projective modules and M is finitely presented. Since $pd(A) \leq d$, M is a flat R-module by Lemma 2.5.10. But then M is also projective (being finitely presented), see Theorem 2.2.14. This proves that $pd(A) \leq d$. $\square$

2.6 Lecture 14 (2019-04-10)

2.6.1 Examen

Chapter 3

Spectral sequences

3.1 Lecture 15 (2019-04-15)

3.1.1 Motivating examples

Before giving the precise definition of a spectral sequence, we look at some basic examples showing that spectral sequences are ubiquitous.

(1) Non-acyclic resolutions

Let us work in the category R-**Mod**. When we define $\mathrm{Ext}^i_R(A, B)$, we choose first an injective resolution $B \to I^\bullet$, and calculate the cohomology of $\mathrm{Hom}_R(A, I^\bullet)$. One may ask the following question:

Question 1: What happens if we start with a resolution $B \to I^\bullet$ with I^i not necessarily injective?

Let us look at some special cases.

(a) Suppose given a resolution $0 \to B \to I^0 \xrightarrow{d^0} I^1 \to 0$, i.e. of length 2. Then the long exact sequence gives

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, I^0) \xrightarrow{d^{00}} \mathrm{Hom}(A, I^1) \to \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(A, I^0) \xrightarrow{d^{01}} \mathrm{Ext}^1(A, I^1) \to \cdots$$

In particular, if we denote $d^{0q} : \mathrm{Ext}^q(A, I^0) \to \mathrm{Ext}^q(A, I^1)$, then there is an exact sequence

$$0 \to \mathrm{Coker}(d^{0,q-1}) \to \mathrm{Ext}^q(A, B) \to \ker(d^{0q}) \to 0.$$

One way to graphically draw it is as follows: put at (p, q) coordinate the module $\mathrm{Ext}^q(A, I^p)$ (with 0 for $p \neq 0, 1$ or $q < 0$):

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
\mathrm{Ext}^1(A, I^0) \arrow[r, "d^{01}"] & \mathrm{Ext}^1(A, I^1) \arrow[r] & 0 \\
\mathrm{Hom}(A, I^0) \arrow[r, "d^{00}"] & \mathrm{Hom}(A, I^1) \arrow[r] & 0
\end{tikzcd}$$

Each row $\mathrm{Ext}^q(A, I^\bullet)$ is a complex induced from $I^\bullet$, and the cohomology are related to $\mathrm{Ext}^i(A, B)$ via

$$0 \to H^1(\mathrm{Ext}^{q-1}(A, I^\bullet)) \to \mathrm{Ext}^q(A, B) \to H^0(\mathrm{Ext}^q(A, I^\bullet)) \to 0.$$

(b) Suppose given a resolution $0 \to B \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to 0$. Then we may split it as

$$0 \to B \xrightarrow{\epsilon} I^0 \xrightarrow{\eta} M \to 0, \qquad 0 \to M \to I^1 \to I^2 \to 0$$

which induce respectively

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, I^0) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1(A, B) \xrightarrow{\epsilon^1} \mathrm{Ext}^1(A, I^0) \xrightarrow{\eta^1} \mathrm{Ext}^1(A, M) \to \cdots$$

$$0 \to \mathrm{Hom}(A, M) \to \mathrm{Hom}(A, I^1) \to \mathrm{Hom}(A, I^2) \to \mathrm{Ext}^1(A, M) \to \mathrm{Ext}^1(A, I^1) \to \mathrm{Ext}^1(A, I^2) \to \cdots$$

We draw the picture as follows:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & 0 \\
\mathrm{Ext}^1(A, I^0) \arrow[r, "d^{01}"] & \mathrm{Ext}^1(A, I^1) \arrow[r, "d^{11}"] & \mathrm{Ext}^1(A, I^2) \arrow[r] & 0 \\
\mathrm{Hom}(A, I^0) \arrow[r, "d^{00}"] & \mathrm{Hom}(A, I^1) \arrow[r, "d^{10}"] & \mathrm{Hom}(A, I^2) \arrow[r] & 0
\end{tikzcd}$$

We check that:

- $H^1(\mathrm{Hom}(A, I^\bullet)) := \ker(d^{10})/\mathrm{Im}(d^{00}) = \mathrm{Hom}(A, M)/(\mathrm{Im}(\mathrm{Hom}(A, I^0))) \cong \ker(\epsilon^1)$.
- there is an inclusion $\mathrm{Im}(\epsilon^1) = \ker(\eta^1) \subset \ker(d^{01})$, but in general not equality. Rather, there exists a map $d_2^{01} : \ker(d^{01}) \to \mathrm{Coker}(d^{10})$ fitting in the diagram below

$$\begin{tikzcd}
0 \arrow[r] & \ker(d^{01}) \arrow[r] \arrow[d, dashed, "d_2^{01}"'] & \mathrm{Ext}^1(A, I^0) \arrow[r, "d^{01}"] \arrow[d, "\eta^1"] & \mathrm{Ext}^1(A, I^1) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Coker}(d^{10}) \arrow[r] & \mathrm{Ext}^1(A, M) \arrow[r] & \mathrm{Ext}^1(A, I^1)
\end{tikzcd}$$

and crucially $\ker(d_2^{01}) = \mathrm{Im}(\epsilon^1)$ by Snake lemma.

Summary: setting $E_2^{pq} := H^p(\mathrm{Ext}^q(A, I^\bullet))$, we obtain again a family of objects indexed by (p, q):

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
E_2^{01} \arrow[drr, dashed, "d_2^{01}"] & E_2^{11} & E_2^{21} \\
E_2^{00} & E_2^{10} & E_2^{20}
\end{tikzcd}$$

together with an extra (but not obvious) morphism $d_2^{01} : E_2^{01} \to E_2^{20}$ such that there is a short exact sequence

$$0 \to E_2^{10} \to \mathrm{Ext}^1(A, B) \to \ker(d_2^{01}) \to 0.$$

Note that if $I^\bullet$ is injective resolution, then all $E_2^{pq} = 0$ for $q \geq 1$, hence the above sequence degenerates to $E_2^{10} \cong \mathrm{Ext}^1(A, B)$ as usual.

Remark 3.1.1. *The argument goes through for any left exact covariant functor in place of* $\mathrm{Hom}_R(A, -)$.

(2) Universal Coefficients Theorem

Question 2: Given a (not necessarily exact) complex $I^\bullet$ of injectives, can we compute $H^n(\mathrm{Hom}(A, I^\bullet))$ using $\mathrm{Ext}^i(A, H^i(I))$.

Remark 3.1.2. *If $I^\bullet$ is exact, then it is an injective resolution of $H^0(I)$ and we have $H^n(\mathrm{Hom}(A, I^\bullet)) \cong \mathrm{Ext}^n(A, H^0(I^\bullet))$.*

Recall the Universal Coefficient Theorem for Cohomology.

Theorem 3.1.3. *Let $I^\bullet$ be a cochain complex of injective R-modules such that each $d(I^n)$ is also injective. Then for every n and every R-module A, there is a (noncanonically) split exxact sequence*

$$0 \to \mathrm{Ext}^1_R(A, H^{n-1}(I^\bullet)) \to H^n(\mathrm{Hom}_R(A, I^\bullet)) \to \mathrm{Hom}_R(A, H^n(I^\bullet)) \to 0.$$

Let's drop the assumption that "each $d(I^n)$ is injective", and assume simply that $I^\bullet = (0 \to I^0 \to I^1 \to 0)$. Then we obtain two short exact sequences:

$$0 \to H^0(I^\bullet) \to I^0 \to B^0 \to 0, \qquad 0 \to B^0 \to I^1 \to H^1(I^\bullet) \to 0.$$

Applying $\mathrm{Hom}(A, -)$ we obtain (using the injectivity of I^0, I^1):

$$0 \to \mathrm{Hom}(A, H^0(I^\bullet)) \to \mathrm{Hom}(A, I^0) \xrightarrow{\alpha} \mathrm{Hom}(A, B^0) \to \mathrm{Ext}^1(A, H^0(I^\bullet)) \to 0,$$

$$\mathrm{Ext}^1(A, B^0) \cong \mathrm{Ext}^2(A, H^0(I^\bullet)),$$

and

$$0 \to \mathrm{Hom}(A, B^0) \xrightarrow{\beta} \mathrm{Hom}(A, I^1) \to \mathrm{Hom}(A, H^1(I^\bullet)) \xrightarrow{\gamma} \mathrm{Ext}^1(A, B^0) \to 0.$$

Since β is injective, we have an isomorphism

$$\mathrm{Hom}(A, H^0(I^\bullet)) = \ker(\alpha) \cong \ker(\beta\alpha) = H^0(\mathrm{Hom}(A, I^\bullet)).$$

On the other hand, since $\ker(\beta) = 0$, we also have $0 \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta\alpha) \to \mathrm{Coker}(\beta) \to 0$ which translates to

$$0 \to \mathrm{Ext}^1(A, H^0(I^\bullet)) \to H^1(\mathrm{Hom}(A, I^\bullet)) \to \ker(\gamma) \to 0.$$

Note that we wish to view γ as $\mathrm{Hom}(A, H^1(I^\bullet)) \xrightarrow{\gamma} \mathrm{Ext}^1(A, B^0) \cong \mathrm{Ext}^2(A, H^0(I^\bullet))$, because we want an expression using only $H^i(I^\bullet)$ (not B^i!). Thus, we arrive at the following situation (again !)

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
E_2^{01} \arrow[ddrr, dashed, "d_2^{01}"] & E_2^{11} & E_2^{21} \\
 & & \\
E_2^{00} & E_2^{10} & E_2^{20}
\end{tikzcd}$$

where $E_2^{pq} := \mathrm{Ext}^p_R(A, H^q(I^\bullet))$. With this notation, we have a long exact sequence

$$0 \to E_2^{10} \to H^1(\mathrm{Hom}(A, I^\bullet)) \to E_2^{01} \xrightarrow{\gamma} E_2^{20} \to 0. \tag{3.1}$$

3.2 Lecture 16 (2019-04-17)

3.2.1 Spectral sequence: definition

Definition 3.2.1. *A cohomological spectral sequence in $\mathcal{A}$ starting on page $a \geq 0$ consisting of the following data:*

(a) an object E_r^{pq} of $\mathcal{A}$ for every $p, q \in \mathbb{Z}$ and $r \geq a$;

(b) a morphism $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ for $p, q \in \mathbb{Z}$ and $r \geq a$ such that $d_r^{p+r,q-r+1} d_r^{pq} = 0$;

(c) an isomorphism $\alpha_r^{pq} : E_{r+1}^{pq} \cong \ker(d_r^{pq})/\operatorname{Im}(d_r^{p-r,q+r-1})$ for $p, q \in \mathbb{Z}$ and $r \geq a$.

Note that E_{r+1}^{pq} is a subquotient of E_r^{pq}. The total degree of E_r^{pq} is $n = p + q$. Each differential d_r^{pq} increases the total degree by one.

A morphism $f : E \to E'$ between two spectral sequence is a family of maps $f_r^{pq} : E_r^{pq} \to E_r'^{pq}$ with $d_r' f_r = f_r d_r$. With this, spectral sequences form a category.

Example 3.2.2. *A first quadrant spectral sequence is one with $E_r^{pq} = 0$, unless $p \geq 0$ and $q \geq 0$. For fixed p, q, $E_r^{pq} = E_{r+1}^{pq}$ for large r, i.e. $r > \max\{p, q+1\}$.*

Example 3.2.3. *Let $A, B \in \mathcal{A}$ (with enough projectives and also enough injectives). Choose a projective resolution $P_\bullet \to A$ and an injective resolution $B \to I^\bullet$, we form the double complex $\operatorname{Hom}(P_\bullet, I^\bullet)$. Set $a = 0$ and $E_0^{pq} = \operatorname{Hom}(P_q, I^p)$, we obtain the E_0 page of the spectral sequence ($E_0^{pq} \to E_0^{p,q+1}$)*

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
E_0^{01} \arrow[u] & E_0^{11} \arrow[u] & E_0^{21} \arrow[u] \\
E_0^{00} \arrow[u] & E_0^{10} \arrow[u] & E_0^{20} \arrow[u]
\end{tikzcd}$$

Taking the cohomology of each column, we obtain $E_1^{pq} = \operatorname{Ext}^q(A, I^p)$ since $P_\bullet$ is a projective resolution of A, and the E_1 page is as follows:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
E_1^{01} \arrow[r] & E_1^{11} \arrow[r] & \cdots \\
E_1^{00} \arrow[r] & E_1^{10} \arrow[r] & \cdots .
\end{tikzcd}$$

Again taking the cohomology, we obtain the E_2 page, etc:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
E_2^{01} \arrow[drr, dashed, "d_2^{01}"] & E_2^{11} & E_2^{21} \\
E_2^{00} & E_2^{10} & E_2^{20}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots \\
E_3^{02} \arrow[rrrdd, dashed, "d_3^{02}"] & E_3^{12} & E_3^{22} & E_3^{32} \\
E_3^{01} & E_3^{11} & E_3^{21} & E_3^{31} \\
E_3^{00} & E_3^{10} & E_3^{20} & E_3^{30}
\end{tikzcd}$$

E_∞ terms

By definition, each E_{r+1}^{pq} is a subquotient of E_r^{pq}, hence inductively a subquotient of E_a^{pq}. Thus, there is a sequence of subobjects B_r^{pq}, Z_r^{pq} of E_a^{pq} for all $r \geq a$:

$$0 = B_a^{pq} \subseteq B_{a+1}^{pq} \subseteq \cdots B_r^{pq} \subseteq \cdots \subseteq Z_r^{pq} \subseteq \cdots \subseteq Z_{a+1}^{pq} \subseteq Z_a^{pq} = E_a^{pq}$$

such that $E_r^{pq} \cong Z_r^{pq}/B_r^{pq}$.

Definition 3.2.4. *We define*

$$B_\infty^{pq} = \bigcup_{r=a}^{\infty} B_r^{pq} = \operatorname*{colim}_r B_r^{pq}, \quad \text{infinite cocycles}$$

$$Z_\infty^{pq} := \bigcap_{r=a}^{\infty} Z_r^{pq} = \lim_r Z_r^{pq}, \quad \text{infinite coboundaries}$$

and E_∞-terms

$$E_\infty^{pq} := Z_\infty^{pq}/B_\infty^{pq}.$$

Remark 3.2.5. *One may also define a derived E_∞-term*

$$RE_\infty^{pq} = \varprojlim{}^1 Z_r^{pq}.$$

Lemma 3.2.6. *Let $f : E \to E'$ be a morphism of spectral sequences, such that for some r, $f_r^{pq} : E_r^{pq} \to E_r'^{pq}$ is an isomorphism for all $p, q \in \mathbb{Z}$. Then $f_\infty^{pq} : E_\infty^{pq} \to E_\infty'^{pq}$ is an isomorphism.*

Proof. It follows from five lemma. □

By construction, we have inclusions for any r:

$$B_r^{pq} \subseteq B_\infty^{pq} \subseteq Z_\infty^{pq} \subseteq Z_r^{pq}.$$

Definition 3.2.7. *A spectral sequence is* regular *if for each (p, q), the differentials d_r^{pq} are zero for all large r; this is equivalent to $Z_\infty^{pq} = Z_r^{pq}$ for large r. It is* coregular *if the differential going to E_r^{pq} are zero all large r; this is equivalent to $B_\infty^{pq} = B_r^{pq}$ for large r. It is* biregular *if both regular and coregular.*

Convergence

Definition 3.2.8. *Let $A \in \mathrm{Ob}(\mathcal{A})$. A filtration (or decreasing filtration) of A is a sequence of subobjects of A*

$$\cdots \supseteq F^0(A) \supseteq F^1(A) \supseteq \cdots \supseteq F^p(A) \supseteq \cdots$$

If they exist, we write $\cap_p F^p(A)$ for the intersection (or limit) and $\cup_p F^p(A)$ for the union (or colimit). We say that the filtration is separated if $\cap_p F^p(A) = 0$ and exhaustive if $\cup_p F^p(A) = A$. We say the filtration is bounded below *if there exists $p \in \mathbb{Z}$ with $F^p(A) = 0$ (trivially implies separated), and* bounded above *if there exists $p \in \mathbb{Z}$ with $F^p(A) = A$ (trivially implies exhaustive). It is called bounded if both bounded above and below.*

Definition 3.2.9. *(1) We say the spectral sequence* weakly converges *to given objects H^n of $\mathcal{A}$ if each H^n has a decreasing filtration*

$$H^n \supseteq \cdots \supseteq F^p H^n \supseteq F^{p+1} H^n \supseteq \cdots$$

together with isomorphisms

$$\beta^{pq} : E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$$

for all $p, q \in \mathbb{Z}$. Note that we don't require the filtration on H^n to be separated or exhaustive.

Notation: $E_a^{pq} \Rightarrow_p H^{p+q}$.

(2) We say that the spectral sequence $\{E_r^{pq}\}$ abuts *to H^n if it weakly converges to H^n and the filtration on H^n is separated and exhaustive.*

(3) We say that the spectral sequence converges *to H^n if it abuts to H^n, it is regular, and $H^n = \varprojlim_p (H^n / F^p H^n)$ for each n.*

Remark 3.2.10. *(i) Every weakly convergent spectral sequence abuts to $\cup F^p H^n / \cap F^p H^n$.*

(ii) In some literatures, abutment in our sense is called convergence, while convergence in our sense is called strongly convergence.

(iii) The same spectral sequence may converge to two different objects H^*.

Definition 3.2.11. *We say that the spectral sequences degenerates at page r if, for any $s \geq r$, the differential d_s is zero. This implies that $E_r \cong E_{r+1} \cong E_{r+2} \cong \cdots$. In particular, it implies that E_r is isomorphic to E_∞.*

Bounded spectral sequence

Definition 3.2.12. *A spectral sequence is said to be* bounded *if there are only finitely many non-zero terms of each total degree n in E_a^{pq} $(p + q = n)$. This implies the same statement for E_r^{pq} for all $r \geq a$.*

For a bounded spectral sequence:

- for each p, q there is an r_0 such that $E_r^{pq} = E_{r_0}^{pq}$ for all $r \geq r_0$, and consequently $E_\infty^{pq} = E_r^{pq}$ for large r.
- If E_r^{pq} weakly converges to H^n, then the filtration on H^n is *finite*, meaning that $F^p H^n$ stabilizes when p goes to ∞ or $-\infty$.

- it converges to H^* whenever it abuts to H^*.

Example 3.2.13. *A first quadrant spectral sequence is bounded. If it converges to H^n, then each H^n has a finite filtration of length $n+1$:*

$$H^n = F^0H^n \supseteq F^{-1}H^n \supseteq \cdots \supseteq F^nH^n \supseteq F^{n+1}H^n = 0.$$

Edge morphisms. Since the spectral sequence lives in the first quadrant, each E_{r+1}^{0n} is a subobject of E_r^{pq}, hence E_∞^{pq} is a subobject of E_a^{pq}. On the other hand, E_∞^{0n} is isomorphic to H^n/F^1H^n, hence is a quotient of H^n. The composite morphisms

$$H^n \twoheadrightarrow E_\infty^{pq} \hookrightarrow E_a^{pq}$$

is called edge morphisms*. Similarly we have edge morphisms*

$$E_a^{n0} \twoheadrightarrow E_\infty^{n0} \hookrightarrow H^n.$$

Example 3.2.14. *(2 columns) Suppose E is a spectral sequence with $E_2^{pq} = 0$ except $p = 0, 1$. If it abuts to H^*, then there are exact sequences:*

$$0 \to E_2^{1,n-1} \to H^n \to E_2^{0n} \to 0.$$

Indeed, the spectral sequence degenerates at page $r = 2$ and for each total degree n, only two terms $E_2^{1,n-1}$, $E_2^{0,n}$ are (possibly) non-zero.

Example 3.2.15. *Suppose E is a spectral sequence with $E_2^{pq} = 0$ except $q = 0, 1$. The spectral sequence degenerates at $r = 3$ and if it abuts to H^*, we have exact sequences for each n:*

$$0 \to E_3^{n0} \to H^n \to E_3^{n-1,1} \to 0.$$

But, by definition, E_3^{n0} is just the cokernel pf $d_2^{n-2,1}$ and $E_3^{n-1,1}$ is the kernel of $d_2^{n-1,1}$, hence there is a long exact sequences:

$$\cdots \to H^{n-1} \to E_2^{n-2,1} \xrightarrow{d_2^{n-2,1}} E_2^{n0} \to H^n \to E_2^{n-1,1} \xrightarrow{d_2^{n-1,1}} E_2^{n+1,0} \to H^{n+1} \to \cdots .$$

Exercise 1. In the example that $0 \to B \to I^0 \to I^1 \to I^2 \to I^3 \to 0$ is a resolution, with I^i not necessarily injective. Write down the definition of the morphism

$$d_3^{02} : E_3^{02} \to E_3^{30}.$$

Describe a filtration of $\mathrm{Ext}^2(A, B)$:
(a) use an elementary way as in the example;
(b) with the help of the convergent spectral sequence

$$E_1^{pq} := \mathrm{Ext}^q(A, I^p) \Rightarrow \mathrm{Ext}^{p+q}(A, B).$$

Exercise 2. Prove that In the example of "universal coefficient theorem", consider the situation $I^\bullet = (0 \to I^0 \to I^1 \to I^2 \to 0)$. Prove that there is a long exact sequence:

$$0 \to E_2^{10} \to H^1(\mathrm{Hom}(A, I^\bullet)) \to E_2^{01} \to E_2^{20} \to H^2(\mathrm{Hom}(A, I^\bullet))$$

where $E_2^{pq} := \mathrm{Ext}_R^p(A, H^q(I^\bullet))$.

3.3 Lecture 17 (2019-04-22)

3.3.1 The spectral sequence of a filtration

Let $C^\bullet$ be a complex in $\mathcal{A}$, with a decreasing filtration $\{F^pC^\bullet\}_{p\in\mathbb{Z}}$ of $C^\bullet$; the connecting morphism $\partial : C^n \to C^{n+1}$ satisfies

$$\partial(F^pC^n) \subset F^pC^{n+1}, \quad \forall p \in \mathbb{Z},$$

so that ∂ induces morphisms $F^pC^n/F^{p+1}C^{n+1} \to F^pC^{n+1}/F^{p+1}C^{n+1}$. The aim is to construct a spectral sequence whose E_0 page is $E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$:

$$\begin{tikzcd}
\vdots & \vdots \\
E_0^{p,q+1} \arrow[u] & E_0^{p+1,q+1} \arrow[u] \\
E_0^{pq} \arrow[u, "\partial"] & E_0^{p+1,q} \arrow[u, "\partial"]
\end{tikzcd}$$

We first define the subobjects Z_r^{pq} and B_r^{pq} of E_0^{pq}. The idea is that as r is made large, the approximate cocycles and coboundaries of degree r approach the real cocycles and coboundaries, and therefore E_r^{pq} approaches something related to the real cohomology $H^{p+q}(C^\bullet)$.

Definition 3.3.1. *Define A_r^{pq} and $\ddot{A}_r^{pq}$ to be:*

$$A_r^{pq} := F^pC^{p+q} \cap \partial^{-1}(F^{p+r}C^{p+q+1}) = \{x \in F^pC^{p+q} : \partial(x) \in F^{p+r}C^{p+q+1}\}$$

$$\ddot{A}_r^{pq} = F^pC^{p+q} \cap \partial(F^{p-r+1}C^{p+q-1}) = \{x \in F^pC^{p+q} : x = \partial(y) \text{ for } y \in F^{p-r+1}C^{p+q-1}\},$$

so that

$$\ddot{A}_0^{pq} \subseteq \cdots \subseteq \ddot{A}_r^{pq} \subseteq \cdots \subseteq F^pC^{p+q} \cap \ker(\partial) \subseteq \cdots \subseteq A_r^{pq} \subseteq \cdots \subseteq A_0^{pq} = F^pC^{p+q}.$$

Let Z_r^{pq} and B_r^{pq} be the image of A_r^{pq} and $\ddot{A}_r^{pq}$ in $E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$, namely:

$$Z_r^{pq} = \frac{F^pC^{p+q} \cap \partial^{-1}(F^{p+r}C^{p+q+1}) + F^{p+q}C^{p+q}}{F^{p+1}C^{p+q}}$$

$$B_r^{pq} = \frac{F^pC^{p+q} \cap \partial(F^{p-r+1}C^{p+q-1}) + F^{p+1}C^{p+q}}{F^{p+1}C^{p+q}},$$

so that[1]

$$0 = B_0^{pq} \subseteq \cdots \subseteq B_r^{pq} \subseteq \cdots \subseteq Z_r^{pq} \subseteq \cdots \subseteq Z_0^{pq} = E_0^{pq}.$$

Finally define $E_r^{pq} := Z_r^{pq}/B_r^{pq}$.

Lemma 3.3.2. *(1) $\ddot{A}_r^{pq} = \partial(A_{r-1}^{p-r+1,q+r-2})$, so that taking images gives $B_r^{pq} = \partial(Z_{r-1}^{p-r+1,q+r-2})$.*

(2) $A_{r-1}^{p+1,q-1} = A_r^{pq} \cap F^{p+1}C^{p+q}$ so that $Z_r^{pq} \cong A_r^{pq}/A_{r-1}^{p+1,q-1}$.

(3) $\ddot{A}_r^{p,q} \cap F^{p+1}C^{p+q} = \ddot{A}_{r+1}^{p+1,q-1}$, so that

$$B_r^{pq} = \ddot{A}_r^{pq}/\ddot{A}_{r+1}^{p+1,q-1} \cong \partial(A_{r-1}^{p-r+1,q+r-2})/\partial(A_r^{p-r+1,q+r-2}).$$

(3) The map $\partial : A_r^{pq} \to \ddot{A}_{r+1}^{p+r,q-r+1}$ induces an isomorphism

$$Z_r^{pq}/Z_{r+1}^{pq} \cong B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

[1] since $A_0^{pq} \subset F^{p+1}C^{p+q}$

Proof. (1) We have $\partial(A \cap A') \subset \partial(A) \cap \partial(A')$, hence

$$\partial(A_{r-1}^{p-r+1,q+r-2}) \subset \partial(F^{p-r+1}C^{p+q-1}) \cap \partial(\partial^{-1}(F^pC^{p+q})) \subset \partial(F^{p-r+1}C^{p+q-1}) \cap F^pC^{p+q}$$

The converse inclusion is also clear: if $\partial(y) \in F^pC^{p+q}$ for some $y \in F^{p-r+1}C^{p+q-1}$, then $y \in A_{r-1}^{p-r+1,q+r-2}$.

(2) (3) follow by definition.

(4) By Lemma , we have

$$Z_r^{pq}/Z_{r+1}^{pq} \cong \frac{A_r^{pq}}{A_{r-1}^{p+1,q-1}} \Big/ \frac{A_{r+1}^{pq}}{A_r^{p+1,q-1}} \cong \frac{A_r^{pq}}{A_{r+1}^{pq} + A_{r-1}^{p+1,q-1}}$$

On the other hand, by (3),

$$\frac{B_{r+1}^{p+r,q-r+1}}{B_r^{p+r,q-r+1}} = \frac{\partial(A_r^{pq})}{\partial(A_{r+1}^{pq})} \Big/ \frac{\partial(A_{r-1}^{p+1,q-1})}{\partial(A_r^{p+1,q-1})} \cong \frac{\partial(A_r^{pq})}{\partial(A_{r+1}^{pq}) + \partial(A_{r-1}^{p+1,q-1})}$$

We need[2] to check that $\ker(\partial|_{A_r^{pq}}) \subset A_{r+1}^{pq} + A_{r-1}^{p+1,q-1}$; this is clear. □

To show this well defines a spectral sequence, we need to check:

(b) there is a differential morphism $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$;

(c) there is an isomorphism $E_{r+1}^{pq} \cong \ker(d_r^{pq})/\mathrm{Im}(d^{p-r,q+r-1})$.

Let's first treat (b). By construction the morphism ∂ induces

$$\partial : A_r^{pq} \to A_r^{p+r,q-r+1}. \tag{3.2}$$

On the other hand, we have epimorphisms $A_r^{pq} \to Z_r^{pq} \to E_r^{pq}$. It induces a morphism

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}.$$

In fact, one checks that this morphism is equal to the composite

$$Z_r^{pq}/B_r^{pq} \twoheadrightarrow Z_r^{pq}/Z_{r+1}^{pq} \xrightarrow[\sim]{\partial} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \hookrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1} \tag{3.3}$$

It is clear that $d_r^{p+r,q-r+1}d_r^{pq} = 0$ (as $\partial \circ \partial = 0$), so we have defined (b) of a spectral sequence. Next, we prove (c): namely the cohomology of

$$E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{pq} \xrightarrow{d_r^{pq}} E_r^{p+r,q-r+1}$$

is isomorphic to E_{r+1}^{pq}.

Lemma 3.3.3.

$$\ker(d_r^{pq}) = \frac{Z_{r+1}^{pq}}{B_r^{pq}}$$

$$\mathrm{Im}(d_r^{p-r,q+r-1}) = \frac{B_{r+1}^{pq}}{B_r^{pq}}$$

As a result, we deduce the required canonical isomorphism

$$\alpha_r^{pq} : \ker(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1}) \cong E_{r+1}^{pq}.$$

Proof. The result follows from (3.3). □

[2] If $f : X \to Y$ is a morphism and $X_1 \subset X$ is a sub-object, then the induced morphism $X/X_1 \to f(X)/f(X_1)$ is an isomorphism if and only if $\ker(f) \subset X_1$.

E_∞-terms

By definition of E_∞-terms, we have $E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}$, where

$$Z_\infty^{pq} = \bigcap_r Z_r^{pq}, \qquad B_\infty^{pq} = \bigcup_r B_r^{pq}.$$

Remark 3.3.4. *Be attention that Z_∞^{pq} is in general* not *the image of*

$$A_\infty^{pq} := \bigcap_r A_r^{pq} = \{x \in F^pC^{p+q} : \partial(x) \in \cap_{p'} F^{p'}C^{p+q+1}\}$$

in E_0^{pq}; while B_∞^{pq} is the image of

$$\ddot{A}_\infty^{pq} := \bigcup_r \ddot{A}_r^{pq} = F^pC^{p+q} \cap \partial(\cup_{p'} F^{p'}C^{p+q-1}).$$

Indeed, there are exact sequences $0 \to A_r^{pq} \cap F^{p+1}C^{p+q} \to A_r^{pq} \to Z_r^{pq} \to 0$ by construction and taking limit (over r) is not exact in general (while taking filtered colimit is). Set

$$e_\infty^{pq} = \frac{\mathrm{Im}(A_\infty^{pq} \to Z_\infty^{pq})}{B_\infty^{pq}} \subset E_\infty^{pq}.$$

Observation: *(a) $e_\infty^{pq} = E_\infty^{pq}$ if $\varprojlim^1(A_r^{pq} \cap F^{p+1}C^{p+q}) = 0$. This holds when the filtration on $C^\bullet$ is bounded below.*

(b) If the filtration on C is Hausdorff and exhaustive, then $\ddot{A}_\infty^{pq} = F^pC^{p+q} \cap \mathrm{Im}(\partial)$ and

$$A_\infty^{pq} = \{x \in F^pC^{p+q} : \partial(x) = 0\} = \ker(\partial) \cap F^pC^{p+q}.$$

Convergence

We assume the filtration on C is Hausdorff and exhaustive, so that $\ker(\partial) \cap F^pC^{p+q} = A_\infty^{pq}$ and $\mathrm{Im}(\partial) \cap F^pC^{p+q} = \ddot{A}_\infty^{pq}$.

For $p \in \mathbb{Z}$, the inclusion $F^p(C^\bullet) \to C^\bullet$ of complexes induces a morphism

$$H^{p+q}(F^pC) \to H^{p+q}(C);$$

denote the image of this morphism by $F^pH^{p+q}(C)$. This defines a filtration on $H^{p+q}(C)$. Precisely, we have

$$F^pH^{p+q}(C) = \frac{\ker(\partial) \cap F^pC^{p+q} + \mathrm{Im}(\partial)}{\mathrm{Im}(\partial)} \cong \frac{A_\infty^{pq}}{A_\infty^{pq} \cap \mathrm{Im}(\partial)} \cong \frac{A_\infty^{pq}}{\ddot{A}_\infty^{pq}}$$

hence

$$\mathrm{gr}^pH^{p+q}(C) := F^p(H^{p+q}(C))/F^{p+1}(H^{p+q}(C)) \cong \frac{A_\infty^{pq}}{\ddot{A}_\infty^{pq}} \Big/ \frac{A_\infty^{p+1,q}}{\ddot{A}_\infty^{p+1,q}} \cong \frac{A_\infty^{pq}}{A_\infty^{p+1,q}} \Big/ \frac{\ddot{A}_\infty^{pq}}{\ddot{A}_\infty^{p+1,q}}.$$

Since $A_\infty^{pq}/A_\infty^{p+1,q}$ is just the image of $A_\infty^{pq} \to Z_\infty^{pq}$, and similarly for $\ddot{A}_\infty^{pq}/\ddot{A}_\infty^{p+1,q}$, we deduce

$$\mathrm{gr}^pH^{p+q}(C) \cong \frac{\mathrm{Im}(A_\infty^{pq} \to Z_\infty^{pq})}{\mathrm{Im}(\ddot{A}_\infty^{pq} \to B_\infty^{pq})} = e_\infty^{pq}.$$

Theorem 3.3.5. *Suppose that the filtration on $C^\bullet$ is bounded. Then the spectral sequence $(E_r^{pq})_{r\geq 0}$, with $E_0^{pq} = F^pC^{p+q}/F^{p+1}C^{p+q}$ is bounded and converges to $H^*(C^\bullet)$:*

$$E_1^{pq} := H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet) \Rightarrow H^{p+q}(C^\bullet).$$

Proof. By boundedness, the filtration on $C^\bullet$ is Hausdorff and exhaustive, so is the filtration induced on $H^*(C^\bullet)$. Hence we only need to show the weak convergence, i.e. an isomorphism

$$E_\infty^{pq} \cong F^pH^{p+q}(C)/F^{p+1}H^{p+q}(C).$$

But, we have $e_\infty^{pq} = E_\infty^{pq}$, hence the result. □

First quadrant filtration

Suppose that $C^i = 0$ for $i < 0$ and that the filtration on C^n for $n \geq 0$ has the following form

$$C^n = F^0C^n \supseteq \cdots F^nC^n \supseteq F^{n+1}C^n = 0.$$

That is, $F^jC^n = 0$ for $j > n$ and $F^jC^n = C^n$ for $j \leq 0$. Such a filtration is called *canonically bounded*. This means that when we place the filtered pieces of C is a diagram, everything essentially lives in the first quadrant:

$$\begin{tikzcd}
F^0C^3 & & & \\
F^0C^2 \arrow[u] & F^1C^3 \arrow[ul, dotted] & & \\
F^0C^1 \arrow[u] & F^1C^2 \arrow[u] \arrow[ul, dotted] & F^2C^3 \arrow[ul, dotted] & \\
F^0C^0 \arrow[u] & F^1C^1 \arrow[u] \arrow[ul, dotted] & F^2C^2 \arrow[u] \arrow[ul, dotted] & F^3C^3 \arrow[ul, dotted]
\end{tikzcd}$$

Then the associated spectral sequence converges.

Remark 3.3.6. *The reason to put the objects in such a way is that we may view it as the "page -1" of the spectral sequence (see the dotted arrows).*

3.4 Lecture 18 (2019-04-24)

3.4.1 The spectral sequence of a double complex

Let C be a double complex in $\mathcal{A}$ with total complex $\mathrm{tot}(C)$:

$$\begin{tikzcd}
 & \vdots \arrow[d, phantom] & \vdots & \vdots & \\
\cdots \arrow[r, dotted] & C^{i-1,j+1} \arrow[r] \arrow[u, dotted] & C^{i,j+1} \arrow[r] \arrow[u, dotted] & C^{i+1,j+1} \arrow[r, dotted] \arrow[u, dotted] & \cdots \\
\cdots \arrow[r, dotted] & C^{i-1,j} \arrow[r] \arrow[u] & C^{ij} \arrow[r, "d_I^{ij}"] \arrow[u, "d_{II}^{ij}"] & C^{i+1,j} \arrow[r, dotted] \arrow[u] & \cdots \\
\cdots \arrow[r, dotted] & C^{i-1,j-1} \arrow[r] \arrow[u] & C^{i,j-1} \arrow[r] \arrow[u] & C^{i+1,j-1} \arrow[r, dotted] \arrow[u] & \cdots \\
 & \vdots \arrow[u, dotted] & \vdots \arrow[u, dotted] & \vdots \arrow[u, dotted] &
\end{tikzcd}$$

At each position we can define the vertical and horizontal cohomology by

$$H_I^{ij}(C) = \frac{\ker(d_{II}^{ij})}{\mathrm{Im}(d_{II}^{i,j-1})}$$

$$H_{II}^{ij}(C) = \frac{\ker(d_I^{ij})}{\mathrm{Im}(d_I^{i-1,j})}$$

and these become complexes $H_I^{\bullet,j}(C)$ and $H_{II}^{i,\bullet}(C)$ in the obvious way. This yields at each position two cohomology objects $H^p(H_I^{\bullet,q}(C))$ and $H^1(H_{II}^{p,\bullet}(c))$. In general, these are not necessarily equal, but we shall see that they "converge" to the same limit in some sense.

Recall that $\mathrm{tot}(C)^n = \oplus_{i+j=n} C^{ij}$, with $d_{\mathrm{tot}} = d_I^{i,j} + (-1)^i d_{II}^{i,j}$. We define two decreasing filtrations on $\mathrm{tot}(C)$:

- $F_I^p(\mathrm{tot}(C))^n := \bigoplus_{r\geq p} C^{r,n-r}$;
- $F_{II}^p(\mathrm{tot}(C))^n := \bigoplus_{r\geq p} C^{n-r,r}$.

It is easy to check that these are complexes in $\mathcal{A}$ that admit canonical monomorphisms into $\mathrm{tot}(C)$, i.e. d_{tot} restricts to

$$F_I^p(\mathrm{tot}(C))^n = \oplus_{r\geq p} C^{r,n-r} \to \oplus_{n\geq p} C^{r,n+1-r} = F_I^p(\mathrm{tot}(C))^{n+1}.$$

Moreover, the two filtrations are always Hausdorff and exhaustive.

Remark 3.4.1. *We may also consider* $\mathrm{tot}^{\Pi}(C)$ *and define filtrations on it. However, the filtrations are not exhaustive in general.*

By construction in §3.3, we obtain two spectral sequences $'E$ and $''E$ corresponding to the filtrations $\{F_I^p(\mathrm{tot}(C))\}_{p\in\mathbb{Z}}$ and $\{F_{II}^p(\mathrm{tot}(C))\}_{p\in\mathbb{Z}}$ respectively. The following lemma computes the first pages of $'E$ and $''E$.

Lemma 3.4.2. *(1) For $p, q \in \mathbb{Z}$ there are canonical isomorphisms*

$$'E_0^{pq} \to C^{pq}, \qquad ''E_0^{pq} \to C^{q,p}$$

making the following diagrams commute

$$\begin{tikzcd} 'E_0^{p,q+1} \arrow[r] & C^{p,q+1} \\ 'E_0^{p,q} \arrow[r] \arrow[u, "'d_0^{pq}"] & C^{pq}, \arrow[u, "(-1)^p\partial_2^{pq}"'] \end{tikzcd} \qquad \begin{tikzcd} ''E_0^{p,q+1} \arrow[r] & C^{q+1,p} \\ ''E_0^{p,q} \arrow[r] \arrow[u, "''d_0^{pq}"] & C^{qp}. \arrow[u, "\partial_1^{qp}"'] \end{tikzcd}$$

(2) For $p, q \in \mathbb{Z}$ there are canonical isomorphisms

$$'E_1^{pq} \to H_I^{pq}(C), \qquad ''E_1^{pq} \to H_{II}^{qp}(C)$$

making the following diagrams commute

$$\begin{tikzcd} 'E_1^{p,q} \arrow[r, "'d_1^{pq}"] \arrow[d, "\sim"] & 'E_1^{p+1,q} \arrow[d, "\sim"] \\ H_I^{pq}(C) \arrow[r] & H_I^{p+1,q}(C), \end{tikzcd} \qquad \begin{tikzcd} ''E_1^{p,q} \arrow[r, "(-1)^q\, ''d_1^{pq}"] \arrow[d, "\sim"] & ''E_1^{p+1,q} \arrow[d, "\sim"] \\ H_{II}^{qp}(C) \arrow[r] & H_{II}^{q,p+1}(C). \end{tikzcd}$$

(3) For $p, q \in \mathbb{Z}$ there are canonical isomorphisms

$$'E_2^{pq} \cong H^p(H_I^{\bullet,q}(C)), \qquad ''E_2^{pq} \cong H^p(H_{II}^{q,\bullet}(C)).$$

Proof. Easy but long check. □

Convergence

Theorem 3.4.3. *If $C^{pq} = 0$ for (p, q) in the fourth quadrant, then the first filtration is bounded below and exhaustive, and we have a convergent spectral sequence*

$$'E_2^{pq} = H^p(H_I^{\bullet,q}(C)) \Rightarrow H^{p+q}(\mathrm{tot}(C)).$$

If $C^{pq} = 0$ for (p, q) in the second quadrant, then the second filtration is bounded below and exhaustive, and we have a convergent spectral sequence

$$''E_2^{pq} = H^p(H_{II}^{q,\bullet}(C)) \Rightarrow H^{p+q}(\mathrm{tot}(C)).$$

Remark 3.4.4. *In the case C is a right half-plan complex, $'E$ is right half-plan while $''E$ is upper half-plan.*

Proof. the two filtrations are always exhaustive and the below boundedness is also clear under the condition imposed. We conclude by Theorem 3.3.5. □

Corollary 3.4.5. *If C is a first quadrant double complex, then both the filtrations F_I^p and F_{II}^p on $\mathrm{tot}(C)$ are canonically bounded, and we have convergent spectral sequences*

$$'E_2^{pq} = H^p(H_I^{\bullet,q}(C)) \Rightarrow H^{p+q}(\mathrm{tot}(C)), \qquad ''E_2^{pq} = H^p(H_{II}^{q,\bullet}(C)) \Rightarrow H^{p+q}(\mathrm{tot}(C)).$$

Application: Künneth spectral sequence

We first introduce the following definition.

Definition 3.4.6. *A spectral sequence* collapses *at E_r $(r \geq 2)$ if there is exactly one non-zero row or column in $\{E_r^{pq}\}$. This implies that all connecting morphisms are zero, hence if E_r^{pq} converges to H^n, then $H^n \cong E_r^{pq}$, the unique term in the row (or column) with $p+q=n$.*

Theorem 3.4.7. *Let $P_\bullet$ be a bounded below chain complex of projective R-modules (i.e. $P_n = 0$ for $n \ll 0$). Then for any R-module M, there is a convergent spectral sequence*

$$E_2^{pq} = \operatorname{Ext}_R^p(H_q(P_\bullet), M) \Rightarrow H^{p+q}(\operatorname{Hom}_R(P_\bullet, M)).$$

Proof. Choose an injective resolution of M: $M \hookrightarrow I^\bullet$ and consider the double complex $C^{pq} := \operatorname{Hom}_R(P_q, I^p)$. Since C^{pq} is (essentially) in the first quadrant, both the spectral sequences $'E$ and $''E$ converge to $H^n(\operatorname{tot}(C))$ by Corollary 3.4.5.

About $'E$: since I^p is injective, we have $H_I^{\bullet,q}(C) \cong \operatorname{Hom}_R(H_q(P), I^\bullet)$, hence

$$'E_2^{pq} = H^p(\operatorname{Hom}_R(H_q(P), I^\bullet)) \cong \operatorname{Ext}^p(H_q(P), M)$$

because $M \to I^\bullet$ is an injective resolution.

About $''E$: it also converges to $H^n(\operatorname{tot}(C))$ because P is bounded below. Since $M \to I^\bullet$ is an injective resolution, we obtain

$$H_I^{\bullet,q}(C) = \begin{cases} \cong \operatorname{Hom}(P_\bullet, M) & q = 0 \\ 0 & q > 0 \end{cases}$$

so that the spectral sequence collapses yielding

$$H^n(\operatorname{Hom}(P_\bullet, M)) = H^n(\operatorname{tot}(C)).$$

The proof is complete. □

Remark 3.4.8. *Note that, in the Universal coefficient theorem, the chain complex $P_\bullet$ is not required to be bounded below, but with another condition that each $d(P_n)$ is projective. It can be proved via a spectral sequence argument as above, because the spectral sequences concentrate on a vertical (or horizontal) strip, hence are bounded (below and above).*

Similarly we have a Künneth homological spectral sequence for Tor.

Theorem 3.4.9. *Let $P_\bullet$ be a bounded below complex of flat R-modules and M an R-module. Then there is a boundedly converging spectral sequence*

$$E_{pq}^2 = \operatorname{Tor}_p^R(H_q(P), R) \Rightarrow H_{p+q}(P \otimes_R M).$$

3.4.2 Cartan-Eilenberg resolution

Definition 3.4.10. *Given a complex C in $\mathcal{A}$, an injective resolution of C is a commutative diagram*

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & I^{p,1} \arrow[r] \arrow[u] & I^{p+1,1} \arrow[r] \arrow[u] & I^{p+2,1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & I^{p,0} \arrow[r] \arrow[u] & I^{p+1,0} \arrow[r] \arrow[u] & I^{p+2,0} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p} \arrow[r] \arrow[u] & C^{p+1} \arrow[r] \arrow[u] & C^{p+2} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

in which the rows are complexes and each column is an injective resolution. That is, $C^p \to I^{p,\bullet}$ is an injective resolution in $\mathcal{A}$. For each $p \in \mathbb{Z}$, we have complexes

$$0 \to Z^p(C) \to Z^p(I^{\bullet,0}) \to Z^p(I^{\bullet}, 1) \to \cdots$$

$$0 \to B^p(C) \to B^p(I^{\bullet}, 0) \to B^p(I^{\bullet}, 1) \to \cdots$$

$$0 \to H^p(C) \to H^p(I^{\bullet,0}) \to H^p(I^{\bullet}, 1) \to \cdots$$

and we say the injective resolution is fully resolution *if for each $p \in \mathbb{Z}$ these complexes are all injective resolutions.*

Lemma 3.4.11. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then every complex C in $\mathcal{A}$ has a fully injective resolution.*

If the complex C is acyclic, then we may choose a fully injective resolution $I^{\bullet,\bullet}$ with each row acyclic.

Proof. For each $n \in \mathbb{Z}$, we have short exact sequences

$$0 \to Z^n(C) \to C^n \to B^{n+1}(C) \to 0$$

$$0 \to B^n(C) \to Z^n(C) \to H^n(C) \to 0.$$

Choose for each $n \in \mathbb{Z}$ injective resolutions of $H^n(C)$ and $B^n(C)$. By Horseshoe lemma, we may find an injective resolution of $Z^n(C)$ fitting a short exact sequence with the original two filtrations. Applying the lemma again to get an injective resolution of C^n.

In the case when C is acyclic, i.e. $H^p(C) = 0$, we set 0 to be the injective resolution of $H^p(C)$ in the above construction. Clearly the resulting resolution has acyclic rows. □

Definition 3.4.12. *A fully injective resolution constructed as in the proof of Lemma 3.4.11 is called a* Cartan-Eilenberg resolution.

3.4.3 The Grothendieck spectral sequence

Theorem 3.4.13. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive left exact functors between abelian categories where $\mathcal{A}, \mathcal{B}$ have enough injectives and $\mathcal{C}$ is cocomplete. Suppose that F sends injectives to G-acyclics (i.e. $R^iG(-) = 0$ for $i \geq 1$). Then for any object $A \in \mathcal{A}$ there is a converging first quadrant spectral sequence E starting on page zero, such that*

$$E_2^{pq} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(GF)(A).$$

The exact sequence of low degree terms is

$$0 \to R^1G(FA) \to R^1(GF)(A) \to G(R^1F(A)) \to (R^2G)(FA) \to R^2(GF)(A).$$

Proof. Let the complex C be an injective resolution of A, and let I be a Cartan-Eilenberg resolution of the complex $F(C)$ (with $I^{p,q} = 0$ unless $p, q \geq 0$). That is, we have a commutative diagram with exact columns

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I^{0,1} \arrow[r] \arrow[u] & I^{1,1} \arrow[r] \arrow[u] & I^{2,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & I^{0,0} \arrow[r] \arrow[u] & I^{1,0} \arrow[r] \arrow[u] & I^{2,0} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & F(C^0) \arrow[r] \arrow[u] & F(C^1) \arrow[r] \arrow[u] & F(C^2) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We can apply the results in §3.4.1 to the double complex $G(I^{\bullet,\bullet})$ to obtain two canonical filtrations

$$F_I^p(\mathrm{tot}(GI))^n = \bigoplus_{r \geq p} G(I^{r,n-r}), \qquad F_{II}^p(\mathrm{tot}(GI))^n = \bigoplus_{r \geq p} G(I^{n-r,r})$$

and spectral sequences $'E_r^{pq}$, $''E_r^{pq}$ converging to the cohomology of $\mathrm{tot}(C)$ (since these are both biregular first quadrant spectral sequences). We have a canonical isomorphism

$$'E_2^{pq} \cong H^p(H_I^{\bullet,q}(GI)) \cong H^p(R^qG(FC^\bullet)).$$

But C is a complex of injectives and F sends injectives to G-acyclics by assumption, so for $q > 0$ the complex $R^qG(FC^\bullet)$ is zero, and for $q = 0$ it is canonically isomorphic to $(GF)(C)$. In other words, we have

$$'E_2^{pq} = \begin{cases} 0 & q > 0 \\ R^p(GF)(A) & q = 0. \end{cases}$$

i.e. this spectral sequence collapses, hence

$$R^n(GF)(A) = H^n(\mathrm{tot}(GI)). \tag{3.4}$$

Now we turn to the second spectral sequence $''E$. By Lemma 3.4.2(3), we know $''E_2^{pq} = H^p(H_{II}^{q,\bullet}(GI))$.

Claim: there is a canonical isomorphism of complexes: $H_{II}^{q,\bullet}(GI) \cong G(H_{II}^{q,\bullet}(I))$.

Proof: since the resolution $I^{\bullet,\bullet}$ of $F(C)$ is fully injective, we have exact sequences

$$0 \to Z^{pq} \to I^{pq} \to B^{p+1,q} \to 0$$

$$0 \to B^{pq} \to Z^{pq} \to H_{II}^{pq} \to 0$$

which must be split exact because all the objects are injective. It follows that the images under G of these sequences remain exact, so that $Z^{pq}(GI) = G(Z^{pq})$, $B^{pq}(GI) = G(B^{pq})$, and finally $H^{pq}(GI) = G(H^{pq}_{II})$, proving the claim.

On the other hand, since the resolution I of $F(C)$ is fully injective, $H^{q,\bullet}_{II}(I)$ is an injective resolution of $H^q(FC) \cong R^qF(A)$:

$$0 \to H^p(FC) \to H^{p,0}_{II}(I) \to H^{p,1}_{II}(I) \to \cdots .$$

Thus, there is a canonical isomorphism

$$H^pG(H^{q,\bullet}_{II}(I)) \cong R^pG(R^qF(A)).$$

Putting everything together, we see that $''E_2$ is a spectral sequence whose second page has all its entries isomorphic to $R^pG(R^qF(A))$, and which converges to $H^{p+q}(\mathrm{tot}(GI))$, which we know is isomorphic to $R^{p+q}(GF)(A)$ by (3.4). The proof is complete. □

3.4.4 Examples of spectral sequences

Base change spectral sequences

Theorem 3.4.14. *Let $f : R \to S$ be a ring map. Then there is a first quadrant cohomological spectral sequence*

$$E_2^{pq} = \mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \Rightarrow \mathrm{Ext}^{p+q}_R(A, B)$$

for every S-module A and R-module B.

Proof. This is an example of Grothendieck's spectral sequences, by setting

- **categories**: $\mathcal{A} = R$-**Mod**, $\mathcal{B} = S$-**Mod**, $\mathcal{C} = \mathbf{Ab}$
- **functors**: $F = \mathrm{Hom}_R(S, -)$, $G = \mathrm{Hom}_S(M, -)$ so that

$$G \circ F = \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, -)) \cong \mathrm{Hom}_R(M \otimes_S S, -) = \mathrm{Hom}_R(M, -).$$

Here, by Proposition 1.8.10 and the proof of Theorem 1.8.9, F admits a left adjoint which is exact, hence it sends injectives to injectives. □

Theorem 3.4.15. *Let $f : R \to S$ be a ring map. Then there is a first quadrant homological spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B) \Rightarrow \mathrm{Tor}^R_{p+q}(A, B).$$

for every R-module A and S-module B.

Proof. This is again an example of Grothendieck spectral sequences, by taking

- **categories**: $\mathcal{A} = R$-**Mod**, $\mathcal{B} = S$-**Mod**, $\mathcal{C} = \mathbf{Ab}$
- **functors**: $F = - \otimes_R S$, $G = - \otimes_S M$. □

The Leray spectral sequence

Corollary 3.4.16. *Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf of abelian groups on X. Then there is a biregular first quadrant spectral sequence*

$$E_2^{pq} = H^p(Y, R^qf_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Proof. This is the Grothendieck spectral sequence, by taking

- **categories**: $\mathcal{A} = \mathbf{Shv}(X)$, $\mathcal{B} = \mathbf{Shv}(Y)$, $\mathcal{C} = \mathbf{Ab}$;
- **functors**: $F = f_*$, $G = \Gamma(X, -)$.

One need to check that F sends injectives to flasque sheaves. □

3.4.5

Finally we go back to the example discussed in §3.1.1.

Theorem 3.4.17. *Let B be an R-module and $B \to C^\bullet$ be a resolution with C^p not necessarily injective. Then for any R-module A, there is a convergent spectral sequence*

$$E_1^{pq} = \mathrm{Ext}_R^q(A, C^p) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B).$$

Proof. Choose a Cartan-Eilenberg resolution $C^\bullet \to I^{\bullet,\bullet}$, in such a way that for $p \geq 1$, $H_{II}^{pq}(I) = 0$, see Lemma 3.4.11. For $p = 0$, $H_{II}^{0,\bullet}(I)$ is an injective resolution of B ($\cong H^0(C^\bullet)$). Consider the double complex $F(I^{\bullet,\bullet})$ and the two associated spectral sequence $'E$ and $''E$. A computation shows that

- $'E_1^{pq} = \mathrm{Ext}^q(A, C^p)$ and $'E_1^{pq} \Rightarrow H^{p+q}(F(I))$;
- $''E_1^{pq} = 0$ for $q > 0$ and $''E_1^{p0} = H_{II}^{p0}(I)$, and hence

$$''E_2^{pq} = \begin{cases} \mathrm{Ext}_R^p(A, B) & q = 0 \\ 0 & q > 0. \end{cases}$$

Namely, the spectral sequence $''E$ *collapses* and $H^n(F(I)) \cong {}''E_2^{n0} = \mathrm{Ext}_R^n(A, B)$.

We conclude as usual. □

Remark 3.4.18. *The same argument goes through by replacing* $\mathrm{Hom}_R(A, -)$ *by any covariant left exact functor F, giving a convergent spectral sequence*

$$E_1^{pq} = R^q F(C^p) \Rightarrow R^{p+q}(B).$$

The classical Hodge-to-de Rham spectral sequence for a proper smooth variety has this form:

$$E_1^{pq} = H^q(X, \Omega_{X/\mathbb{C}}^p) \Rightarrow H_{\mathrm{dR}}^{p+q}(X/\mathbb{C}).$$

The proof builds on the Poincaré lemma.

3.5 Lecture 19 (2019-04-29)

3.5.1 *******

3.6 Lecture 20 (2019-05-06)

3.6.1 Examen