Fano Threefold

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1 singularities of pairs

Remark 1.1. *terminal* $\subset$ *canonical* $\subset$ *klt* $\subset$ *lc* $\subset$ *slc* $\subset$ *qlc* $\subset$ *DB*

A ref: Miles Reid 1975, 25 years of 3-folds an old person's view

Remark 1.2. $-K_X$ *is nef and big, then* X *is weak fano.*

Remark 1.3. *Let* Y *be canonical, then there is terminalization* $f : X \to Y$ *and* $f^*K_Y = K_X$*. If* Y *is Fano, then* X *weak fano.*

Remark 1.4. $\mathbb{Q}$*-factorialization always exists for klt varieties.*

X fano var with at most canonical singularities, called canonical Fano varieties.
Goal: classify canonical Fano 3-folds under some invariant:

- Picard number $\rho(X)$
- Fano index
- degree $c_1(X)^3 = (-K_X)^3$

About picard group $\mathrm{Pic}(X)$ Finitely generated torison-free. and $/\sim_{\mathbb{Q}}$ coincides with $/\equiv$, thus

$$\mathbb{Z}^{\rho X} \cong \mathrm{Cl}(X)/\sim_{\mathbb{Q}} \cong \mathrm{Cl}(X)/\equiv$$

Weil Fano index $q_W = \max\{-K_X \sim qA, A \in \mathrm{Cl}(X)\}$, and $\mathbb{Q}$-index $q_{\mathbb{Q}} = \max\{-K_X \sim_{\mathbb{Q}} qA, A \in \mathrm{Cl}(X)\}$ Then

- $q_W, q_{\mathbb{Q}} \in \mathbb{Z}$
- $q_W | q_{\mathbb{Q}}$
- X smooth, then $q_W = q_{\mathbb{Q}} = i(X) = \max\{-k_X \sim_{\mathbb{Q}} qH, H \in \mathrm{Pic}(X)\}$

Theorem 1.5. X *weak Fano 3-fano, then* $q_{\mathbb{Q}}(X) \leqslant 66$

Remark 1.6. *isolated can* $q_{\mathbb{Q}}(X) \leqslant 61$*, and* $q_{\mathbb{Q}}(X) = 66$ *by* $X = \mathbb{P}(5, 6, 22, 33)$

Remark 1.7. *New tech*

- *A new R-R*
- *kawamata-Miyaoka type inequality, and Foliation of rank 2.*

2 Kawamata-Miyaka type inequality