Some properties concerning derviations of affine domains

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1 Derivations theory and affine algebraic geometry

Affine algebraic geometry mainly studies $\mathbb{A}^n$. (Mathematics Subject Classification: 14R)

relations:

- lnd and $\mathbb{G}_a$ group actions on varieties
- kernel of lnd and Hilbert 14'th problem
- exponential of lnd and automorphism group
- images of derivations and Jacobian conjecture
- ML-inv and concellation problem for varieties

A derivation D of B is locally nilpotent if for any $a \in b$ there is a integer n_a such that $D^{n_a}(a) = 0$. Then for $char(k) = 0$, lnd D, then

$$\exp D = \sum_{n=0}^{\infty} \frac{D^n}{n!}$$

Let X be an affine variety, then there are exponential automorphisms of X.

Conjecture: $\operatorname{Aut} \mathbb{A}^n$ is generated by affine automorphism and exponential automorphism.

Hilbert 14:

a lnd D of $k[x_1, \ldots, x_n]$

$$Frac(\ker D) \cap k[x_1, \ldots, x_n] = \ker D$$

Then there are conterexamples for hilbert 14 with $n = 5, 6, 7$

2 image of LND

Jacobson conjecture: $\det JF$ non-zero constant, then F automorphism.

Image conjecture (Zhao,2010) implies Jacobian conjecture.

MWZ problem (Essen-Wright-Zhao, 2011): D derivation of $k[x, y]$, with $\operatorname{div} D = 0$, Is $\operatorname{Im} D$ a MZ subspace.

Theorem 2.1. *JC of dimension 2* $\implies$ *MWZ holds in the case* $1 \in \operatorname{Im} D$

Definition 2.2. *A subspace of* M *of* S *is MZ subspace if*

$$f^m \in M, \forall n \implies \forall g \in B, gf^m \in M, m \gg 0$$

lnd conjecture: Let D be a lnd of $B = k[x_1, \ldots, x_n]$, any idea I of B, then DI is a MZ subspace. Open for $n \geqslant 2$.

Crucial problem: Describe polynomials f with $f^m \in \operatorname{Im} D \forall m \geqslant 1$

- polytopes of Laurent polynomial
- Integrals of polynomial functions
- local slices (local slice construction) (Daigle algebraic explains)
- Grobner basis

3 ML-invariant

Cancellation problem: for affine variety X, Y, does $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ imply $X \cong Y$? Fail for surfaces.

ML-invariant:

$$ML(B) = \cap_{D \in LND(B)} \ker D$$