MMP

August 20, 2025

1 Usual pairs

lc mmp:

some recipes:

1. flip type GMM

2. existence of flips

3. closure of GMM

4. mmp with scaling of **an** ample terminates for ample polarized lc pairs

5. mmp with scaling of **an** ample terminates for lc pairs admitting a MM

with conditions:

1. $(X, A + B)$ lc;

2. (X, B) dlt;

3. NQC;

4. $\mathbb{Q}$-factorial

2 gpair

2.1 Talk note

	Varieties/Pairs	NQC gpairs	Non-NOC gpairs
Non-Vanishing	Expect	False	False
Numerical Non-Vanishing	Expect	Expect	False
Finite generation	True	False	False

Example 2.1. *1. X is an elliptic curve, $B = 0$, and M is non-torsion numerical trivial divisor. Then $K_X + M$ is non-torsion numerical trivial.*

2. Let E be a general elliptic curve, and $X = E \times E$. Consider the curves

$$F_1 = E \times \{p\}, F_2 = \{q\} \times E, \Delta \text{ diagonal}$$

Then $M = F_1 + \sqrt{2}F_2 + (\sqrt{2} - 2)\Delta$ is nef and $K_X + M$ is never numerically effective.

Some natural ideas to proof the Cone theorem:

1. Classical approach: Kollar-Mori deal with klt, and one needs base-point-free theorem.

2. Hacon-Liu's proof for NQC case: need sub-adjunction formula which is still not known for non-NOC case.

Idea: use foliation theory.

Sketeched proof: Run MMP

- Cone theorem follows from ideas of ACSS.

- Contraction theorem and existence of flips for F-dlt gfqs:
  - $(X', \mathcal{F}', B', \mathbf{M}') \to (Z, \Sigma)$ a foliated log resolution of $(X, \mathcal{F}, B, \mathbf{M})$ and $\pi : X' to X$

    $$E + \pi^*(K_\mathcal{F} + B + \mathbf{M}_X) = K_{\mathcal{F}'} + B' + \mathbf{M}_{X'} \sim_Z K_{X'} + \Delta' + \mathbf{M}_{X'}$$

    where
    * $E \geqslant 0$ exceptional over X.
    * $(X', \Delta', \mathbf{M})$ is lc gpair.
  - run $K_{\mathcal{F}'} + B' + \mathbf{M}_{X'}$ MMP over X which is also $K_{X'} + \Delta' + \mathbf{M}_{X'}$ MMP over Z.
  - Since X is $\mathbb{Q}$-factorial, this MMP end with X. Therefore
    * $\mathcal{F}$ induced by $f : X \to Z$
    * X is $\mathbb{Q}$-factorial klt and $K_\mathcal{F} + B + \mathbf{M}_X \sim_Z K_X + \Delta + \mathbf{M}_X$
  - Then contraction and existence of flips given by theory of gpair.

Suffices to base-point-free: $(X, \mathcal{F}, B, \mathbf{M})$ is $\mathbb{Q}$-factorial F-dlt, A ample and $K_\mathcal{F} + B + A + \mathbf{M}_X$ is nef. Then $K_\mathcal{F} + B + A + \mathbf{M}_X$ is semi-ample.

Theorem 2.2 (CHLX). *Canonical bundle formula holds for lc-trivial gfqs.*

BPFness. BPF for gfqs:

- $K_X + \Delta + A + \mathbf{M}_X \sim_Z K_\mathcal{F} + B + A + \mathbf{M}_X$ is nef over Z and globally nef (replace Δ with $\Delta + f^*A_Z$, cone theorem for gfqs, length of extremal rays).

- $K_X + \Delta + A + \mathbf{M}_X$ is semi-ample and there is $\phi : X \to T$ (length of extremal ray)

- Fix ample H_T on T and $H = \phi^*H_T$. by canonical boundle formula:
  - $K_X + \Delta + \overline{(A - H)}_X + \mathbf{M}_X \sim \phi^*(K_T + \Delta_T + \mathbf{M}_T^T)$
  - $K_\mathcal{F} + B + \overline{(A - H)}_X + \mathbf{M}_X \sim \phi^*(K_{\mathcal{F}_T} + B_T + \mathbf{M}_T^T)$

- $K_T + \Delta_T + \mathbf{M}_T^T + H_T$ is ample and thus $K_T + \Delta_T + \mathbf{M}_T^T + (1 - \delta)H_T$ is ample.

- $K_T + \Delta_T + \mathbf{M}_T^T + (1 - \delta)H_T \sim_Z K_T + \Delta_T + \mathbf{M}_T^T + (1 - \delta)H_T$ nef over Z, thus nef (Cone thm)

- $K_T + \Delta_T + \mathbf{M}_T^T$ ample, implies $K_\mathcal{F} + B + A + \mathbf{M}_X$ semi-ample.

$\square$

Structure

- Cone theorem for gfqs $\implies$ MMP for ACSS gfqs.

- MMP for ACSS gfqs $\implies$ cbf for gfqs $\implies$ bpf and contraction for gpairs.

- MMP for ACSS gfqs $\implies$ (F-dlt $\implies$ ACSS)

- (F-dlt $\implies$ ACSS) + MMP for ACSS gfqs + bpf and contracion of gpairs $\implies$ MMP for a.i. F-dlt foliation

- Cone for gfqs + bpf and contraction for gpair + existence of flips $\implies$ MMP for gpairs.