Cremona Group 1

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1 Conventions

All surface S means smooth projective 2-dimensional integral variety over $\mathbb{C}$. Let $f : X \dashrightarrow Y$ be a birational map, then there are maximal pairs $U \subset X$ and $V \subset Y$ such that $f : u \to V$ is an isomorphism. And we call

$$\operatorname{Exc} f = X \setminus U.$$

the exceptional set.

2 Blow up on Surface

Let $\pi : Y \to X$ be the blowing up of a point p on surface X, and let x, y be the local coordinates near p, then

1. If C is a curve on X defined by $f_m(x,y) + r(x,y)$ near p, where f_m is a homogeneous polynomial of degree m, then
$$m_p C = \operatorname{mult}_p C = m.$$
;
2. Let C' be the strict transform of C on Y, then
$$\pi^* C = C' + mE.$$
3. we have intersections
$$\begin{aligned} E^2 &= -1 \\ C_1'.C_2' &= C_1.C_2 - m_1 m_2 \\ K_{X'} &= \pi^* K_X + E \\ K_{X'}^2 &= K_X^2 - 1 \end{aligned}.$$

Let $Y \to X_n \to X_{n-1} \to \cdots \to X_0 = X$ be a sequence of blowing up points, then

1. If p is a point on $X_k, k \neq 0$, then define
$$\operatorname{mult}_p C = \operatorname{mult}_p C'.$$
where C' is strict transform of C on X_i.
2. If C_Y is strict transform of C on X which is smooth, then
$$p_a(C) = g(C_Y) + \sum_{i=0}^{n} \frac{m_i(m_{i-1})}{2}.$$
and $\operatorname{mult}_{p_i} \geqslant \operatorname{mult}_{p_j}$ for p_j over p_i.

Let Γ be a linear system on X, then define

$$\operatorname{mult}_p \Gamma = \min_{D \in \Gamma} \operatorname{mult}_p D.$$

Then we have

1.
$$\pi^* \Gamma = \Gamma' + mE.$$

3 Resolve

Lemma 3.1. *Let $f : X \dashrightarrow \mathbb{P}^n$ be a rational map, then there is a surface W and morphisms $p : W \to X$ $q : W \to \mathbb{P}^n$ resolving the raional map, and p is sequence of blowing ups.*

Corollary 3.2. *Let $f : X \dashrightarrow Y$ be a birational map, and $p \in \mathrm{Ind}(f^{-1})$, then there is a curve $C \in Y$ such that $(f^{-1})(C) = p$; If there is a point $q \notin \mathrm{Ind}(f^{-1})$, and $(f^{-1})(q) = p$, then there is a curve $C \subset Y$ such that $q \in C$.*

Lemma 3.3. *Let $f : X \to Y$ be a birational morphism and $p \in \mathrm{Ind}(f^{-1})$, then*

$$X \to \mathrm{Bl}_p Y.$$

Definition 3.4 (Minimal resolution)**.** *W is called the **minimal resolution** of birational map $f : X \dashrightarrow Y$ if any other resolution W' factors through W. Furthermore, the points blown up by $p : W \to X$ are called **base points** of f^{-1}.*

4 Hizebruch Surface

Define Hizebruch Surface as follows:

Definition 4.1.

$$\mathbb{F}_n \subset \mathbb{P}^2 \times \mathbb{P}^1 \quad [x : y : z] \times [u : v].$$

$\mathbb{F}_n$ is defined by $yv^n - zu^n$.

5 Groups

5.1 Birational groups

First of all, the Cremona group:

Definition 5.1. $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$.

Jonquière map:

Definition 5.2. *Let $\eta : \mathbb{P}^2 \dashrightarrow C$ be a rational map, then $\mathrm{Bir}(\mathbb{P}^2, \eta)$ is the group of birational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that*

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed] \arrow[d, "\eta"'] & \mathbb{P}^2 \arrow[d, "\eta"] \\ C \arrow[r, "\bar{f}"] & C \end{tikzcd}.$$

where $\bar{f}$ is an isomorphism. Furthermore, if η is

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ [x : y : z] &\mapsto [y : z]. \end{aligned}$$

then the corresponding group $\mathrm{Bir}(\mathbb{P}^2, \eta)$ is called **Jonquière group.**