complex geometry

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1 Introduction to complex manifold.

lecture 1

1.1 complex manifolds

Lemma 1.1.1.
- *maximal principle;*
- *extend holomorphic function on $U \subset X$ with $\operatorname{codim} X \setminus U \geqslant 2$ to X.*

examples of cplx manifolds:

- $U \subset \mathbb{C}^n$;
- $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ and $X := \mathbb{C}^n/\mathbb{Z}^{2n}$: For any $z \in X$, take V_z be image of

  $$U_z = \{z : |z_i - z_i'| < \frac{1}{2}\} \subset \mathbb{C}^n$$

  This is not a "algebraic" tori.
- $\mathbb{P}^n$ ex: Show that $\mathbb{P}^n$ is compact.

1.2 complex submanifolds

Definition 1.2.1. *$Y \subset X$ is called submanifold of codimension k, if there is a atlas $\{u_i, \phi_i : U_i \to V_i \subset \mathbb{C}^n\}$ of X, such that $\phi_i|_{Y\cap U_i} \to V_i \cap \mathbb{C}^{n-k}$.*

Remark 1.2.2.
- *Jacobian martix: of $f : U \to \mathbb{C}^m$ at p is*

  $$J_f(p) := \left(\frac{\partial f_i}{\partial z_j}(p)\right)_{i,j}$$

- *holomorphic implicit function theorem: $f : U \to \mathbb{C}^m$, $f(p) = q$, $J_f(p)$ is rank m, then $f^{-1}(q)$ is a complex submanifold of U of codimension m.*

  example:
  - $f : \mathbb{C}^2 \to \mathbb{C}, (x, y) \mapsto x^2 + y^2$. *Then $f^{-1}(q \neq 0)$ is a complex submanifold of codimension* 1.
  - *affine hypersurface.*
  - *projective hypersurface. $V(f)$*

1.3 Projective manifolds

$X \subset \mathbb{P}^N$, such that $X = V(F_i)$.

Remark 1.3.1. *Let $F_1, \dots, F_{N-n}$ homogenuous polynomails, and $X = V(F_i)$. Then $J_{F^i}(x)$ has rank $N - n$, where $x \in U_i$ and F^i polynomial morphism from $U_i \cong \mathbb{C}^N$ induced by F, if and only if $J_F(x)$ has rank $N - n$.*

Theorem 1.3.2 (Chow's theorem)**.** *A compact complex submanifold $X \subset \mathbb{P}^n$ is projective.*

1.4 Vector bundles

Definition 1.4.1. *A vector bundle of rank r is $p : \mathscr{E} \to X$ such that*

1. $E_x = p^{-1}(x) \cong \mathbb{C}^r$;
2. *any x, there is a open submanifold $x \in U$ such that $h : p^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^r$, and $h_x : E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^r$ is an isomorphism as a complex vector space.*

The $\{U, h\}$ is called a local trivialization of E at x. Suppose $\{U_i, h_i\}, \{U_j, h_j\}$ are two trivialization, and any $x \in U_i \cap U_j$, there is an isomorphism $h_i \circ h_j^{-1} : \mathbb{C}^r \to \mathbb{C}^r$. Moreover, there is

$$g_{ij} := h_i \circ h_j^{-1} : U_i \cap U_j \to GL(r, \mathbb{C})$$

where $g_{ij} \in C^\infty$.

Vice versa, given $\{U_i, g_{ij}\}$, such that

- $g_{ij} \circ g_{jk} \circ g_{ki} = Id$ and $g_{ii} = Id$.

then we can construct a vector bundle.

Definition 1.4.2 (holomorphic vector bundle)**.** *A complex vector bundle is called holomorphic if g_{ij} are holomorphic.*

Construction of vector bundle: slogan: construction of vector space implies construction of vector bundles.

- $E_1 \oplus E_2, E_1 \otimes E_2, E^\vee, \wedge^{rkE} E = \det E, \mathrm{Sym}\, E$

Definition 1.4.3 (tautological bundle)**.** *$\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is called tautological bundle.*

$$([x], v) \in \mathcal{O}(-1) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$$

Where $v = \lambda x, \lambda \in \mathbb{C}$. trivialization is

$$h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}, ([x], v) \mapsto (x_i, v_i)$$

where $v = (v_0, \ldots, v_n) \in \mathbb{C}^{n+1}$.

lecture 2 omitted. All recalls in lecture 3:

1. Let X be a differential or complex manifold, and $\mathcal{A}_X$ ($\mathcal{O}_X$) be differential functions (holomorphic functions) . Let $E \to X$ be a vector bundle, and $\mathcal{E}$ be the sheaf of sections. then $\mathcal{E}$ is a $\mathcal{F}$ bundle.
2. Almost complex structure. Let X bea a differential manifold, and

   $$J : T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$$

   with $J^2 = id$.

   example: $X = \mathbb{R}^{2n} \cong \mathbb{C}^n$ with coordinates $(x_i, y_i) = (z_i = x_i + \sqrt{-1} y_i)$. Then

   $$\partial/\partial x_i \to \partial/\partial y_i, \partial/\partial y_i \to -\partial/\partial x_i$$

lecture 3

1.5 complex tangent bundle

Let X be a complex manifold, and consider real tangent bundle $T_{X,\mathbb{R}}$ of dimension $2n$. Let $T_{X,\mathbb{R}} := T_{X,\mathbb{R}} \otimes \mathbb{C}$ be a complexified tangent bundle.

There is a $J_{\mathbb{C}} : T_{X,\mathbb{C}} \to J_{X,\mathbb{C}}$ and $T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$, as eigenspace with respect to i and $-i$.

Locally, $X = \mathbb{C}^n$ and $T_{X,\mathbb{R}} = X \times \mathbb{R}^{2n}$, and $T_{X,\mathbb{C}} = X \times \mathbb{C}^{2n}$ and

$$J_{\mathbb{C}} : T_{X,\mathbb{C}} \to T_{X,\mathbb{C}}$$

such that $\partial/\partial x_j - i\partial/\partial y_j \mapsto \partial/\partial y_j - i(-\partial/\partial x_j)$ where $\partial/\partial x_j - i\partial/\partial y_j \in T_X^{1,0}$ basis of $T^{1,0}$

complex coordinates: $\partial/\partial z_j := 1/2(\partial/\partial x_j - i\partial/\partial y_j)$ and $\partial/\partial z_j(dz_j) = 1$ where $dz_j = dx_j + idy_j$

1.6 complex contangent

- $\Omega_{X,\mathbb{C}} = (T_{X,\mathbb{C}})^{\vee}$
- $\Omega^{1,0}_{X,\mathbb{C}} = (T^{1,0}_{X,\mathbb{C}})^{\vee}$

Then $dz_j = x_j + iy_j$ basis of $\Omega^{1,0}$ and $d\bar{z}_j = dx_j - idy_j$ basis of $\Omega^{0,1}$

$$\Omega^{p,q} := \wedge^p\Omega^{1,0} \otimes \wedge^q\Omega^{0,1}$$

Then a (p,q)-form is a section of $\Omega^{p,q} \subset \Omega^{k=p+q}$

Exercise: Check that $T_X^{1,0}$ and $\Omega_X^{1,0}$ are holomorphic vector bundles.

Remark 1.6.1. *In general, $T^{1,0}$ and $\Omega^{1,0}, \Omega^{p,0}$ are complex vector bundles. But consider $T_X := T^{1,0}, \Omega^p$ as holopmorphic vector bundles.*

1.7 Differential calculus on complex manifolds

recall differential in real manifolds:

$$d : \mathcal{A}^k \to \mathcal{A}^{k+1}$$

defined by

$$\alpha \wedge \beta \mapsto d\alpha \wedge \beta + (-1)^l \alpha \wedge d\beta$$

where $\alpha \in \mathcal{A}^l$

For complex manifolds, we have $\mathcal{A}^k_{\mathbb{C}}$:= sheaf of differential sections of $\Omega^k_{X,\mathbb{C}} = \oplus_{p+q=k}\Omega^{p,q}_X$, then

$$\partial : \mathcal{A}^{p,q}_{\mathbb{C}} \to \mathcal{A}^{p+1,q}_{\mathbb{C}}$$

defined by

$$\partial : \mathcal{A}^{p,q}_{\mathbb{C}} \subset \mathcal{A}^k_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^{k+1}_{\mathbb{C}} \xrightarrow{projection} \mathcal{A}^{p+1,q}_{\mathbb{C}}$$

and

$$\overline{\partial} : \mathcal{A}^{p,q}_{\mathbb{C}} \to \mathcal{A}^{p,q+1}_{\mathbb{C}}$$

defined by

$$\overline{\partial} : \mathcal{A}^{p,q}_{\mathbb{C}} \subset \mathcal{A}^k_{\mathbb{C}} \xrightarrow{d} \mathcal{A}^{k+1}_{\mathbb{C}} \xrightarrow{projection} \mathcal{A}^{p,q+1}_{\mathbb{C}}$$

Proposition 1.7.1. *1. $d = \partial + \overline{\partial}, \partial^2 = \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0$*

2. $\partial, \overline{\partial}$ satisfy Leibniz rule.

1.8 E-valued differential operators

Let X be a complex manifold, and $E \to X$ be a holomorphic vector bundle. Let $\mathcal{E}$ be the sheaf of sections of E. Then we can define

Definition 1.8.1. *Sheaf $\mathcal{A}^{p,q}(E)$ such that*

$$\mathcal{A}^{p,q}(U,E) := \{s : U \to \Omega^{p,q}_X \otimes E|_U : s \textit{ differential section}\}$$

Define $\overline{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ locally:

Take a trivialization $E|_U \cong U \times \mathbb{C}^r$ and $\alpha = \sum_{J,K} s_{I,K} dz_I \wedge d\bar{z}_J$ where $s = (s_1, \ldots, s_r)_{J,K}$, then

$$\overline{\partial}_E(\alpha) = \sum_{J,K} \overline{\partial} s_{I,K} dz_I \wedge d\bar{z}_J = \sum_{J,K}\sum_{l=1}^{r} (\partial/\partial\bar{z}_l s_{I,K}) dz_I \wedge d\bar{z}_J \wedge d\bar{z}_l.$$

Since E is holomorphic, this is well-defined.

1.9 Cohomologies of sheaves

- Cech cohomology: Let X be a topology space, and $\mathcal{F}$ be a sheaf of abelian group. Let $\{U_i\}$ be an open cover. Denote $U_{i_0 i_1 \ldots i_p} = \cap_{k=0}^p U_{i_k}$
- sheaf cohomology: a sheaf $\mathcal{I}$ is injective if $Hom(-, \mathcal{I})$ is an exact functor. Let $\mathcal{I}^* = \mathcal{I}_0, \ldots, \mathcal{I}_i, \ldots$ be injective resolution, then

$$H^i(X, \mathcal{F}) = H^i(\mathcal{I}(X)^*)$$

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1.10 Fine and a cyclic sheaves

Definition 1.10.1. • *sheaf $\mathcal{F}$ is acyclic if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

- *sheaf $\mathcal{F}$ is fine if for any locally finite open cover $\{U_i\}$, there is a family of homomorphism $\{h_i : \mathcal{F} \to \mathcal{F}\}$ such that*

  1. $\operatorname{Supp} h_i \subset U_i$
  2. $\sum_i h_i = Id.$

  example: Let $\mathcal{F}$ be a sheaf of module, and h_i is partion of unit, then $\mathcal{F} \xrightarrow{\times h_i} \mathcal{F}$ is fine.

Proposition 1.10.2. *Sheaf cohomology can be computed by acyclic resolution. And fine sheaf is acyclic.*

1.11 Comparison theorem

Theorem 1.11.1 (Leray)**.** *Let X be a topologic space and $\mathcal{F}$ be a sheaf of abelian group. Let $\{U_i\}$ be an open cover, then*

1. *There is a canonical map from Cech cohomology to sheaf cohomology*

$$l_i : \check{H}^i(X, \mathcal{F}) \to H^i(X, \mathcal{F})$$

2. *if $\mathcal{F}|_{U_{i_0 \cdots i_r}}$ is acyclic for all r, then l_i is an isomorphism.*

References:

- Cech cohomology and fine sheaves: Kodaira: complex manifolds and deformation of complex structure. chapter 3.
- Cohomologies: Voisin chapter 4. Demeilly chapter 4.

1.12 de Rham Theorem

Let X be a topologic space, and G_X be the constant sheaf assosiated to an abelian group G. Then

$$0 \to G_X \to \mathcal{A}_G^0 \xrightarrow{d^0} \mathcal{A}_G^1 \to \cdots$$

Differential of G-valued k-form. By Poincare lemma, it is exact.

Theorem 1.12.1 (de Rham isomorphism)**.** *Let X be a differential manifold, then there is a canonical isomorphism*

$$H^i(X, G_X) \cong H^i(\mathcal{A}_G^*(X))$$

The formar is cohomology about topology, and the latter is cohomology about differential.

Proof. Ony need to show $\mathcal{A}_G^*$ is fine, thus acyclic. □

Exercise: Prove that $H^i(X, \mathbb{Z}_X) \cong H^i_{sing}(X, \mathbb{Z})$, the latter is singular cohomology. Hint: consider

$$U \mapsto C^i_{sing}(U, \mathbb{Z})$$

and inducing sheaves $\mathcal{C}^i_{sing}$, which is an acyclic resolution.

1.13 Dolbeault cohomology

Definition 1.13.1. *Let X be a complex manifold of dimension n, for any $0 \leqslant q \leqslant n$, there is a complex*

$$0 \to \mathcal{A}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(X) \xrightarrow{\bar{\partial}} \cdots$$

And

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,*}(X))$$

Similarly, we can define $H^{p,q}(X, E)$ for holomorphic vector bundle, by $\bar{\partial}_E$.

$0 \leqslant h^{p,q} = \dim H^{p,q} \leqslant +\infty$ *called (p,q)-hodge number.*

1.14 comparison theorem

$$H^{p,0}(X) = H^0(X, \Omega_X^p)$$

Here Ω_X^p is the sheaf of holomorphic p-form. Holomorphic section of holomorphic vector bundle.

Theorem 1.14.1 (Dolbeault's isomorphism)**.**

$$H^{p,q}(X) = H^q(X, \Omega_X^p)$$

Formar is Dolbeault cohomology, and the latter is sheaf cohomology of Ω^p

remark: Hormander 4.2.6, for pseudoconvex or polydisc U, then $H^{p,q}(U) = 0, p \geqslant 1, p \geqslant 0$

Theorem 1.14.2 (A)**.**

$$0 \to \Omega_X^p \to \mathcal{A}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(X) \xrightarrow{\bar{\partial}} \cdots$$

is exact. As result,

$$H^q(X, \Omega_X^p) \cong H^q(\mathcal{A}^{p,*}(X), \bar{\partial}) = H^{p,q}(X)$$

Theorem 1.14.3 (B)**.** *Let $\mathcal{U} = \{U_i\}$ be an open cover with pseudoconvex U_i of X complex manifold, then*

$$\check{H}(\mathcal{U}, \Omega_X^p) \cong H^q(X, \Omega_X^p)$$

Remark 1.14.4. *Same for (p,q)-Dolbeault cohomology of a holomorphic vector bundle E. That is*

$$H^{p,q}(X, E) \cong H^q(X, \Omega_X^p \otimes E)$$

2 Connection and curvature

2.1 connection

Let $E \to X$ be complex vector bundle, we want

$$d : \mathcal{A}_{\mathbb{C}}^k(E) \to \mathcal{A}_{\mathbb{C}}^{k+1}(E)$$

but can NOT do it locally, since not well-defined under different trivialization of E.

Definition 2.1.1. *X differential manifold and $E \to X$ complex vector bundle. A connection on E is a $\mathbb{C}$-linear homorphism*

$$\nabla : \mathcal{A}_{\mathbb{C}}^0(E) \to \mathcal{A}_{\mathbb{C}}^1$$

such that

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

where f is a locally differential function and s is a locally differential section of E.

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Remark 2.1.2. *∇ is NOT a morphism of $\mathcal{A}_X$-modules, where $\mathcal{A}_X$ is the sheaf of $\mathbb{C}$-valeud differential functions on X. That is*

$$\nabla(f \cdot s) \neq f \cdot \nabla(s)$$

But it is $\mathbb{C}$-linear, therefore it extends to

$$\nabla : \mathcal{A}_{\mathbb{C}}^k(E) \to \mathcal{A}_{\mathbb{C}}^{k+1}(E)$$

such that

$$\nabla(\alpha \wedge \beta) = \nabla(\alpha) \wedge \beta + (-1)^l \alpha \wedge \nabla(\beta)$$

where $\alpha \in \mathcal{A}_{\mathbb{C}}^l$.

example:

1. k=1. Locally let $U \times \mathbb{C}^r$ be a trivialization, and e_i are sections as a basis. Then a section s has form $s = \sum s_i e_i \in \mathcal{A}_{\mathbb{C}}^0(E)$, where s_i are $\mathbb{C}$-valued functions. Then

$$\nabla(s) = \sum ds_i \otimes e_i + s_i \otimes \nabla(e_i)$$

Supppose $\nabla(e_i) = \sum_j a_{ij} e_j$, where a_{ij} are $\mathbb{C}$-valued differential 1-form, and set $A = (a_{ij})$. Then

$$\nabla(s) = ds + A \cdot s$$

2.2 Hermition structure

Recall: A Hermitian structure on E is a $\mathbb{C}$-valued $\mathbb{R}$-bilinear form

$$h : E \times E \to \mathbb{C}$$

such that

1. h is $\mathbb{C}$-linear in the first variable;
2. $h(s_1, s_2) = \overline{h(s_2, s_1)}$. Therefore h is called positive definite if $h(s, s) \geqslant 0$ for any s, and $h(s, s) = 0 \Leftrightarrow s = 0$.

Definition 2.2.1. *A Hermitian martix h on E is a positivi definite Hermitian form h_x on E_x for any $x \in X$.*

Every complex vector bundle has a Hermitian structure.

Definition 2.2.2. *A Hermitian vector bundle (E, h).*

Extends h to forms

$$\mathcal{A}^p_{\mathbb{C}}(E) \times \mathcal{A}^q_{\mathbb{C}}(E) \longrightarrow \mathcal{A}^{p+q}_{\mathbb{C}}(E)$$
$$(\alpha, \beta) \longmapsto h(\alpha, \beta) = \sum \alpha_i \wedge \bar{\beta}_j h(e_i, e_j).$$

Proposition 2.2.3 (Exercise)**.** *1. Every hermitian vector bundle has a connection.*

2. trivialiation $E|_U \cong U \times \mathbb{C}^r \xrightarrow{g=(g_{ij})} U \times \mathbb{C}^r, e_i = \sum g_{ij} e'_j$ Then $H = g^t H' \bar{g}$.

tired, omitted.

Definition 2.2.4. *A connection ∇ on a holomorphic vetor bundle E is called compatible with a Hermitian structure h if*

$$\nabla^{0,1} = \bar{\partial}_E$$

that is, $A^{0,1} = 0$.

lecture 6 recall:

- hermition metric:
- hermition connection:

$$dh(s_1, s_2) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2))$$

- a general fact. Let X be a differential manifold, and E, F two complex vector bundle. And $\phi : \mathcal{E} \to \mathcal{F}$ a morphism of sheaves of abelian groups. Then TFAE:
  - $\phi \in Hom(E, F)$
  - if and only if ϕ is $\mathcal{A}_{X,\mathbb{C}}$-linear
  - $\phi(f \cdot s) = f\phi(s)$
  - ϕ is a morphism of $\mathcal{A}_{X,\mathbb{C}}$-modules.

2.3 Holomorphic connection

Definition 2.3.1. *Let $E \to X$ be a holomorphic vector bundle. A holomorphic connection D is a $\mathbb{C}$-linear map of sheaves*

$$D : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$$

where $\mathcal{E}$ sheaf of holomorphic section of E, and Ω^1_X holomorphi cotangent bundle, and satisfies

$$D(f \cdot s) = \partial f \cdot s + f \cdot D(s)$$

Remark 2.3.2. *holomorphic connection is NOT a connection compatible with holomorphic structure.*

Locally, choose a trivialization, then we can write D as $\partial + A$ and $A = (a_{ij}), a_{ij} = \sum_k \alpha_k dz_k$. There is a connection induced by D:

$$\nabla = D + \bar{\partial}_E$$

locally $d + A$.

exercise: $\nabla = D + \bar{\partial}_E$ is a connection compatible with holomorphic structure.

2.4 Line bundle and divisor

2.4.1 Picard groups

Definition 2.4.1. *$Pic(X) = \{L \to X$ holomorphic line bundle$\}/\sim$ where $L \sim L'$ is isomorphic:*

$$\begin{tikzcd} L \arrow[rr] \arrow[dr] & & L' \arrow[dl] \\ & X & \end{tikzcd}$$

consider $s \in H^0(X, L)$ as

$$\begin{tikzcd} U_i \times \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \\ U_i \arrow[ur, "s"'] & \end{tikzcd}$$

or $\phi_{ij} s_j = s_i$. L given by $(\mathcal{U} = \{U_i\}), \phi_{ij}$, there is $\alpha_L \in C^1(\mathcal{U}, \mathcal{O}_X^*)$ where $\mathcal{O}^*$ sheaf of invertible holomorphic functions. $(\alpha_L)_{ij} := \phi_{ji}$ (not a typo, it is ϕ_{ji}).

Tired. All we what to show is that $Pic(X) = H^1(X, \mathcal{O}^*)$

2.4.2 Analytic subvarieties

Definition 2.4.2. 1. *An analytic subvariety of X is a closed subset $Y \subset X$ such that for any $x \in X$ there is a open neighbourhood $x \in U_x \subset X$ such that $Y \cap U_x = \{f_1 = \cdots = f_k = 0\} \subset U_x$ where f_i are holomorphic functions on U_x.*

2. *A point $yinY$ is regular or smooth if there are f_i such that Jacobian $J_f(y)$ has rank k.*

3. *Irreducible analytic subvariety.*

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Definition 2.4.3. *Ideal sheaf of Y is $\mathcal{I}_Y = \{f \in \mathcal{O}_X : f|_Y = 0\}$.*

2.4.3 Divisors

Definition 2.4.4. *An irreducible analytic hypersurface of X is an irreducible analytic subvariety of codimension 1.*

fact-exercises

- Let g be a meroimorphic function on $U \subset X$, for any $x \in X$, there is $g = \frac{g_1}{g_2}, g_i \in \mathcal{O}(U)$.
- principle divisor associated to g is $(g) = \sum_x ord_x(g)[x]$.

2.4.4 From divisors to line bundles

Let $D = \sum_i a_i Y_i$. To define holomorphic line bundle L_D, need $\mathcal{U} = \{U_i\}, \phi_{ij}$. Take $\mathcal{U}$ such that $Y|_{U_i} = (f_i)$ is a principle divisor associated to meroimorphic function f_i. And $\phi_{ij} = \frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$

Definition 2.4.5. *$D \sim D'$ if $D - D' = (f)$ principle divisor associated to f.*

THere is a homorphism

$$Div(X)/\sim \xrightarrow{injective} Pic(X)$$

which in general is NOT surjective.

2.5 Chern classes

2.5.1 First chern class

Proposition 2.5.1 (Fact)**.**

$$\{complex\ line\ bundles\ over\ X\}/\sim \cong H^1(X, \mathcal{O}^*)$$

Definition 2.5.2. *exponential sequence on X is*

$$0 \to \mathbb{Z}_X \to \mathcal{A}_{X,\mathbb{C}} \to \mathcal{A}^*_{X,\mathbb{C}} \to 0$$

where the seconde map is $s \mapsto \exp(2\pi\sqrt{-1}\cdot s)$

Then there is long exact sequence:

$$0 = H^1(X, \mathcal{A}_{X,\mathbb{C}}) \to H^1(X, \mathcal{A}^*_{X,\mathbb{C}}) \xrightarrow{\sim} H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{A}_{X,\mathbb{C}}) = 0$$

As $\mathcal{A}_{X,\mathbb{C}}$ is a fine sheaf, thus $H^q(X, \mathcal{A}_{X,\mathbb{C}}) = 0, q \geqslant 1$. Then there is isomorphism

$$c_1 : H^1(X, \mathcal{A}^*_{X,\mathbb{C}}) \to H^2(X,\mathbb{Z}), [\alpha_L] \mapsto c_1(L)$$

2.5.2 Axiomatic approach

Theorem 2.5.3 (def). *Let X be a differential manifold, there is a unique map c such that for any complex vector bundle $E \to X$, $c(E) \in H^2(X,\mathbb{Z})[t]$ and $c_i(E) \in H^{2i}(X,\mathbb{Z})$, that is*

$$c(E) = \sum c_i(E)t^i$$

satisfying

1. *if $rkE = 1$, then $c(E) = c_1(E)$;*
2. *for $f : X \to Y$, we have $c(f^*E) = f^*c(E)$;*
3. *$c(E \oplus F) = c(E)\cdot c(F)$*

2.6 Chern-Weil theory

Let ∇ be a connection on X, locally $\nabla = d + A$ where $A = (a_{ij})$, and a_{ij} are 1-forms.

$$\Theta_\nabla := dA + A \wedge A$$

called curvature of ∇. Consider chen polynomial

$$\det(1 + \frac{\sqrt{-1}}{2\pi}\Theta_\nabla t) = 1 + \sum_j^r P_j(E,\nabla)t^j$$

here $P_j(E,\nabla)$ are differential $2j$-forms.

example: Let $L \to X$ be a complex line bundle, then $\Theta_\nabla = dA + A\wedge A = dA$ and

$$\det(1 + \frac{\sqrt{-1}}{2\pi}\Theta_\nabla t) = 1 + dA \cdot t$$

Lemma 2.6.1.

1. *$P_j(E,\nabla)$ is d-closed,*
2. *$[P_j(E,\nabla)] \in H^{2j}(X, \mathcal{A}^{\cdot}_{X,\mathbb{C}}) \xrightarrow{derhomisomorphic} H^{2j}(X,\mathbb{C})$ independent of ∇.*

Definition 2.6.2. *j-th chen class $c_j(E)$ of a complex vector bundle E is defined as*

$$[P_j(E,\nabla)] \in H^{2j}(X,\mathbb{C})$$

Definition 2.6.3. *X complex manifold. j-th chern class is*

$$c_j(X) = c_j(T_X) = c_j(T_X^{1,0})$$

Recall:

1. (X,g) Kahler if and only if $d\omega_g = 0$
2. g Kahler, then locally $\omega_g = \frac{\sqrt{-1}}{2}\sum_{i,j} dz_i \wedge z_j + O(|z|^2)$
3. $\mathbb{P}^n$ is Kahler, so are projective algebraic varieties. And

$$\omega_i = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(1 + |z_i|^2)$$

2.7 Compare two defitions of the first chern class

2024-12-20 recall

- Definition of Hodge $*$ operator.
- Laplace(-Beltrami) operator: $\Delta = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$, where $d^* = (-1)^{nk+n+1} * d*$
- Hodge isomorphism.
- Poincare duality: $H^k(X, \mathbb{R}) \cong H_{n-k}(X, \mathbb{R})$.

2.8 About Kahler manifolds

12/27 recall:

- Kahler identity $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$
- Hodge decomposition
  1. $H^k(X, \mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X)$
  2. $H^{p,q}(X) = \overline{H^{q,p}(X)}$
- Bott-Chern cohomology
$$\frac{\ker(d)}{\operatorname{im}(\partial\bar{\partial})} = H^{p,q}_{BC}(X) \xrightarrow{\sim} H^{p,q}(X) \subset H^{p+q}(X, \mathbb{C})$$
  using $\partial\bar{\partial}$-lemma

Example:

1. Is S_6 a complex manifold?
2. simple computes

2.8.1 Hodge diamond

Corollary 2.8.1. *1. Serre duality*
$$H^{p,q}(X) \cong (H^{n-p,n-q}(X))^*$$
2. Hodge $*$-isomorphism:
$$* : H^{p,q}(X) \to H^{n-q,n-p}(X)$$

2.9 Lefschetz Theorems

2.9.1 Motivation

(X, ω) cpt kahler mfd, ω Kahler form. Let ω be a closed form, then $[\omega] \in H^{1,1}(X)$

2.9.2 Primitive decomposition

Definition 2.9.1. *primitive cohomology*

2.9.3 Lefschetz operator and $\mathcal{H}_\in(\mathbb{C})$-representation

3 Summary

3.1 Complex manifold

- Definition and examples $\mathbb{P}^n, \mathbb{C}^n/\mathbb{Z}^{2n}$
- differential: (p, q)-form, operators: $\partial, \overline{\partial}$
- de Rham cohomology and Dolbeault cohomology.
$$H^k(X, \mathbb{R}/\mathbb{C}) = \frac{\operatorname{Ker} d}{\operatorname{Im} d}, H^{p,q}(X) = \frac{\ker \overline{\partial}}{\operatorname{Im} \overline{\partial}}$$
- Sheaf and Cech cohomology

3.2 vector bundles

- Definition of complex and holomorphic vector bundles.
- examples. $\mathcal{O}_{\mathbb{P}^n}(k), \mathcal{T}_X, \mathcal{T}_{X,\mathbb{C}/\mathbb{R}}$, where $\mathcal{T}_X$ is holomorphic vector bundle note that $T_X^{1,0} = \mathcal{T}_X$. And

$$\bigwedge^{p,q} T_X, \bigwedge^{p,q} \Omega_X$$

- Sheaf of sections.
- Connections. Definitions, Hermitian metric. Chern connection and curvature. compute.
- For holomorphic vector bundle, there is an operator $\overline{\partial}_E$, and there is Dolbeault cohomology.
- Divisor and Picard group.
- Chern classes, definition of c_1. Show that $Pic(X) \cong H^1(X, \mathbb{C})$ by exponential sequence.

$$0 \to \mathbb{Z}_X \to \mathcal{O}_{X,\mathbb{C}} \xrightarrow{\exp(2\pi\cdot)} \mathcal{O}^*_{X,\mathbb{C}} \to 0$$

and curvature

$$[\frac{\sqrt{-1}}{2\pi}\Theta_\nabla] = H^{1,1}(X, \mathbb{R}) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$$

3.3 Kahler manifold

3.3.1 Hermitian metric/ Hermitian structure

Conside Riemannian metric g on X compatible with J. Hermition metric on $T_X^{1,0}$
ω_g fundemental form, Hermition metric on $(T_{X,\mathbb{R}}, J)$
These four things, one to another.

3.3.2 Hodge * operator

operator $*, d^*, \partial^*, \overline{\partial}^*, \Delta_d, \Delta_\partial, \Delta_{\overline{\partial}}$, Hodge decomposition, Serre duality, Hodge $*$-isomorphism. Definition adn computation in local structure with respect to standard

3.3.3 Harmonic forms

$$H^k(X, \mathbb{R}) \cong \mathcal{H}^{X,g}, H^{p,q}(X) = \mathcal{H}^{p,q}_{\overline{\partial}}(X)$$

3.3.4 Kahler manifolds

$d\omega_g = 0$ examples:

- Fubini-Study metric on $\mathbb{P}^n$, Kahler form on $\mathbb{P}^n$, Kahler form on projective algebraic variety.

Kahler identity:

- $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$
- Δ_d commutes with $\partial, \overline{\partial}, *, d$

3.3.5 Hodge decomposition

$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$ and $H^{p,q}(X) = \overline{H^{q,p}(X)}$

3.3.6 Serre duality

$H^{p,q}(X) \cong \overline{H^{n-p,n-q}(X)}^*$ by $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$

3.3.7 Hard Lefschetz

Primitive decomposition:

$$H^k(X, \mathbb{C}) = \oplus_i L^i(H^{k-2i}(X, \mathbb{C})_{primitive})$$
$$H^{n-k}(X, \mathbb{C})_{primitive} = \ker(L^{k+1})$$

Hard Lefschetz:

$$L^k : H^{n-k}(X, \mathbb{C}) \xrightarrow{\sim} H^{n+k}(X, \mathbb{C})$$