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Fundamental Algebraic Geometry

Grothendieck's FGA Explained

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Preface

Without question, Alexander Grothendieck's work revolutionized Algebraic Geometry. He introduced many concepts — arbitrary schemes, representable functors, relative geometry, and so on — which have turned out to be astoundingly powerful and productive.

Grothendieck sketched his new theories in a series of talks at the Séminaire Bourbaki between 1957 and 1962, and collected his write-ups in a volume entitled "Fondements de la géométrie algébrique," commonly abbreviated FGA. In [**FGA**], he developed the following themes, which have become absolutely central:

- Descent theory,
- Hilbert schemes and Quot schemes,
- The formal existence theorem,
- The Picard scheme.

(FGA also includes a sketch of Grothendieck's extension of Serre duality for coherent sheaves; this theme is already elaborated in a fair number of works, and is not elaborated in the present book.)

Much of FGA is now common knowledge. Some of FGA is less well known, and few geometers are familiar with its full scope. Yet, its theories are fundamental ingredients in most of Algebraic Geometry.

Mudumbai S. Narasimhan conceived the idea of a summer school at the International Centre for Theoretical Physics (ICTP) in Trieste, Italy, to teach these theories. But this school was to be different from most ICTP summer schools. Most focus on current research: important new results are explained, but their proofs are sketched or skipped. This school was to teach the techniques: the proofs too had to be developed in sufficient detail.

Narasimhan's vision was realized July 7–18, 2003, as the "Advanced School in Basic Algebraic Geometry." Its scientific directors were Lother Göttsche of the ICTP, Conjeeveram S. Seshadri of the Chennai Mathematical Institute, India, and Angelo Vistoli of the Università di Bologna, Italy. The school offered the following courses:

(1) Angelo Vistoli: Grothendieck topologies and descent, 10 hours.
(2) Nitin Nitsure: Construction of Hilbert and Quot schemes, 6 hours.
(3) Lother Göttsche: Local properties of Hilbert schemes, and Hilbert schemes of points, 4 hours.
(4) Luc Illusie: Grothendieck's existence theorem in formal geometry, 5 hours.
(5) Steven L. Kleiman: The Picard scheme, 6 hours.

The school addressed advanced graduate students primarily and beginning researchers secondarily; both groups participated enthusiastically. The ICTP's administration was professional. Everyone had a memorable experience.

This book has five parts, which are expanded and corrected versions of notes handed out at the school. The book is not intended to replace [**FGA**]; indeed, nothing can ever replace a master's own words, and reading Grothendieck is always enlightening. Rather, this book fills in Grothendieck's outline. Furthermore, it introduces newer ideas whenever they promote understanding, and it draws connections to subsequent developments. For example, in the book, descent theory is written in the language of Grothendieck topologies, which Grothendieck introduced later. And the finiteness of the Hilbert scheme and of the Picard scheme, which are difficult basic results, are not proved using Chow coordinates, but using Castelnuovo–Mumford regularity, which is now a major tool in Algebraic Geometry and in Commutative Algebra.

This book is not meant to provide a quick and easy introduction. Rather, it contains demanding detailed treatments. Their reward is a far greater understanding of the material. The book's main prerequisite is a thorough acquaintance with basic scheme theory as developed in the textbook [**Har77**].

This book's contents are, in brief, as follows. Lengthier summaries are given in the introductions of the five parts.

Part 1 was written by Vistoli, and gives a fairly complete treatment of descent theory. Part 1 explains both the abstract aspects — fibered categories and stacks — and the most important concrete cases — descent of quasi-coherent sheaves and of schemes. Part 1 comprises Chapters 1–4.

Chapter 1 reviews some basic notions of category theory and of algebraic geometry. Chapter 2 introduces representable functors, Grothendieck topologies, and sheaves; these concepts are well known, and there are already several good treatments available, but the present treatment may be of greater appeal to a beginner, and can also serve as a warm-up to the more advanced theory that follows.

Chapter 3 is devoted to one basic notion, *fibered category*, which Grothendieck introduced in [**SGA1**]. The main example is the category of quasi-coherent sheaves over the category of schemes. Fibered categories provide the right abstract set-up for a discussion of descent theory. Although the general theory may be unnecessary for elementary applications, it is necessary for deeper comprehension and advanced applications.

Chapter 4 discusses *stacks*, fibered categories in which descent theory works. Chapter 4 treats, in full, the various ways of defining descent data, and it proves the main result of Part 1, which asserts that quasi-coherent sheaves form a stack.

Part 2 was written by Nitsure, and covers Grothendieck's construction of Hilbert schemes and Quot schemes, following his Bourbaki talk [**FGA**, 221], together with further developments by David Mumford and by Allen Altman and Kleiman. Part 2 comprises Chapter 5.

Specifically, given a scheme X, Grothendieck solved the basic problem of constructing another scheme Hilb_X, called the *Hilbert scheme of* X, which parameterizes, in a suitable universal manner, all possible closed subschemes of X. More generally, given a coherent sheaf E on X, he constructed a scheme $\mathrm{Quot}_{E/X}$, called the *Quot scheme of* E, which parameterizes, again in a suitable universal manner, all possible coherent quotients of E. These constructions are possible, in a relative set-up, where X is projective over a suitable base. The constructions make crucial use of several basic tools, including faithfully flat descent, flattening stratification, the semi-continuity complex, and Castelnuovo–Mumford regularity.

Part 3 was written jointly by Barbara Fantechi and Göttsche. It comprises Chapters 6 and 7.

Chapter 6 introduces the notion of an (infinitesimal) deformation functor, and gives several examples. Chapter 6 also defines a tangent-obstruction theory for such a functor, and explains how the theory yields an estimate on the dimension of the moduli space. The theory is worked out in some cases, and sketched in a few more, which are not needed in Chapter 7.

Chapter 7 studies the Hilbert scheme of points on a smooth quasi-projective variety, which parameterizes the finite subschemes of fixed length. The chapter constructs the Hilbert–Chow morphism, which maps this Hilbert scheme to the symmetric power by sending a subscheme to its support with multiplicities. For a surface, this morphism is a resolution of singularities. Finally, the chapter computes the Betti numbers of the Hilbert scheme, and sketches the action of the Heisenberg algebra on the cohomology.

Part 4 was written by Illusie, and revisits Grothendieck's Bourbaki talk [**FGA**, 182], where he presented a fundamental comparison theorem of "GAGA" type between algebraic geometry and formal geometry, and outlined some applications to the theory of the fundamental group and to that of infinitesimal deformations. A detailed account appeared shortly afterward in [**EGAIII1**], [**EGAIII2**] and [**SGA1**]. Part 4 comprises Chapter 8.

After recalling basic facts on locally Noetherian formal schemes, Chapter 8 explains the key points in the proof of the main comparison theorem, and sketches some corollaries, including Zariski's connectedness theorem and main theorem, and Grothendieck's criterion for algebraization of a formal scheme. Then Chapter 8 gives Grothendieck's applications to the fundamental group and to lifting vector bundles and smooth schemes, notably, curves and Abelian varieties. Chapter 8 ends with a discussion of Serre's celebrated examples of varieties in positive characteristic that do not lift to characteristic zero.

Part 5 was written by Kleiman, and develops in detail most of the theory of the Picard scheme that Grothendieck sketched in the two Bourbaki talks [**FGA**, 232, 236] and in his commentaries on them [**FGA**, pp. C-07–C-011]. In addition, Part 5 reviews in brief, in a series of scattered remarks, much of the rest of the theory developed by Grothendieck and by others. Part 5 comprises Chapter 9.

Chapter 9 begins with an extensive historical introduction, which serves to motivate Grothendieck's work on the Picard scheme by tracing the development of the ideas that led to it. The story is fascinating, and may be of independent interest.

Chapter 9 then discusses the four common relative Picard functors, which are successively more likely to be the functor of points of the Picard scheme. Next, Chapter 9 treats relative effective (Cartier) divisors and linear equivalence, two important preliminary notions. Then, Chapter 9 proves Grothendieck's main theorem about the Picard scheme: it exists for any projective and flat scheme whose geometric fibers are integral.

Chapter 9 next studies the union of the connected components of the identity elements of the fibers of the Picard scheme. Then, Chapter 9 proves two deeper finiteness theorems: the first concerns the set of points with a multiple in this union; the second concerns the set of points representing invertible sheaves with a given Hilbert polynomial. Chapter 9 closes with two appendices: one contains detailed answers to all the exercises; the other contains an elementary treatment of basic divisorial intersection theory — this theory is used freely in the proofs of the two finiteness theorems.

(Similarly, the isomorphism class of $\varphi^*\mathcal{O}(1)$ on $X_{\mathbb{C}}$ is independent of the choice of the isomorphism $\varphi\colon X_{\mathbb{C}} \xrightarrow{\sim} \mathbf{P}^1_{\mathbb{C}}$. So λ is independent too. But this fact too is not needed here.)

Finally, we must show λ is not in the image of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{zar})}(\mathbb{R})$. By way of contradiction, suppose λ is. Then λ arises from an invertible sheaf $\mathcal{L}$ on X. A priori, the pullback $\mathcal{L}|X_{\mathbb{C}}$ need not be isomorphic to $\varphi^*\mathcal{O}(1)$. Rather, these two invertible sheaves need only become isomorphic after they are pulled back to X_A where A is some étale $\mathbb{C}$-algebra.

However, cohomology commutes with flat base change. So

$$\dim_{\mathbb{R}}\mathrm{H}^0(\mathcal{L}) = \mathrm{rank}_A\,\mathrm{H}^0(\mathcal{L}|X_A) = \dim_{\mathbb{C}}\mathrm{H}^0(\varphi^*\mathcal{O}(1)) = 2.$$

Hence, $\mathcal{L}$ has a nonzero section. It defines an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_D \to 0.$$

Similarly $\mathrm{H}^1(\mathcal{O}_X) = 0$. Hence $\dim_{\mathbb{R}}\mathrm{H}^0(\mathcal{O}_D) = 1$. Therefore, D is an $\mathbb{R}$-point of X. But X has no $\mathbb{R}$-point. Thus λ is not in the image of $\mathrm{Pic}_{(X/\mathbb{R})\,(\mathrm{zar})}(\mathbb{R})$. □

ANSWER 9.2.6. First of all, we have $\mathrm{Pic}_{(X/S)\,(\mathrm{fppf})}(k) = \mathrm{Pic}_{(X_k/k)\,(\mathrm{fppf})}(k)$ essentially by definition, because a map $T' \to T$ of k-schemes is an fppf-covering if and only if it is an fppf-covering when viewed as a map of S-schemes. And a similar analysis applies to the other three functors. Now, $f_k\colon X_k \to k$ has a section; indeed, f_k is of finite type and k is algebraically closed, and so any closed point of X has residue field k by the Hilbert Nullstellensatz. Hence, by Part 2 of Theorem 9.2.5, the k-points of all four functors are the same. Finally, $\mathrm{Pic}_{X/S}(k) = \mathrm{Pic}(X_k)$ because $\mathrm{Pic}(T)$ is trivial whenever T has only one closed point.

Whether or not $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, all four functors have the same geometric points by Exercise 9.2.3; in fact, given an algebraically closed field k, the k-points of these functors are just the elements of $\mathrm{Pic}(X_k)$. □

ANSWER 9.3.2. By definition, a section in $\mathrm{H}^0(X,\mathcal{L})_{\mathrm{reg}}$ corresponds to an injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$. Its image is an ideal $\mathcal{I}$ such that $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{I}$. So $\mathcal{I}$ is the ideal of an effective divisor D. Then $\mathcal{O}_X(-D) = \mathcal{I}$. So $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{O}_X(-D)$. Taking inverses yields $\mathcal{O}_X(D) \simeq \mathcal{L}$. So $D \in |\mathcal{L}|$. Thus we have a map $\mathrm{H}^0(X,\mathcal{L})_{\mathrm{reg}} \to |\mathcal{L}|$.

If the section is multiplied by a unit in $\mathrm{H}^0(X,\mathcal{O}_X^*)$, then the injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$ is multiplied by the same unit, so has the same image $\mathcal{I}$; so then D is unaltered. Conversely, if D arises from a second section, corresponding to a second isomorphism $\mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{I}$, then these two isomorphism differ by an automorphism of $\mathcal{L}^{-1}$, which is given by multiplication by a unit in $\mathrm{H}^0(X,\mathcal{O}_X^*)$; so then the two sections differ by multiplication by this unit. Thus $\mathrm{H}^0(X,\mathcal{L})_{\mathrm{reg}}/\mathrm{H}^0(X,\mathcal{O}_X^*) \hookrightarrow |\mathcal{L}|$.

Finally, given $D \in |\mathcal{L}|$, by definition there exist an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{L}$. Since $O_X(-D)$ is the ideal $\mathcal{I}$ of D, the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ yields an injection $\mathcal{L}^{-1} \hookrightarrow \mathcal{O}_X$. The latter corresponds to a section in $\mathrm{H}^0(X,\mathcal{L})_{\mathrm{reg}}$, which yields D via the procedure of the first paragraph. Thus $\mathrm{H}^0(X,\mathcal{L})_{\mathrm{reg}}/\mathrm{H}^0(X,\mathcal{O}_X^*) \xrightarrow{\sim} |\mathcal{L}|$. □

ANSWER 9.3.5. Let $x \in D+E$. If $x \notin D\cap E$, then $D+E$ is a relative effective divisor at x, as $D+E$ is equal to D or to E on a neighborhood of x. So suppose $x \in D\cap E$. Then Lemma 9.3.4 says X is S-flat at x, and each of D and E is cut out at x by one element that is regular on the fiber X_s through x. Form the product of the two elements. Plainly, it cuts out $D+E$ at x, and it too is regular on X_s. Hence $D+E$ is a relative effective divisor at x by Lemma 9.3.4 again. □

ANSWER 9.3.8. Consider a relative effective divisor D on X_T/T. Each fiber D_t is of dimension 0. So its Hilbert polynomial $\chi(\mathcal{O}_{D_t}(n))$ is constant. Its value is $\dim \mathrm{H}^0(\mathcal{O}_{D_t})$, which is just the degree of D_t.

The assertions are local on S; so we may assume X/S is projective. Then $\mathrm{Div}_{X/S}$ is representable by an open subscheme $\mathbf{Div}_{X/S} \subset \mathbf{Hilb}_{X/S}$ by Theorem 9.3.7. And $\mathbf{Hilb}_{X/S}$ is the disjoint union of open and closed subschemes of finite type $\mathbf{Hilb}^{\varphi}_{X/S}$ that parameterize the subschemes with Hilbert polynomial φ. Set

$$\mathbf{Div}^m_{X/S} := \mathbf{Div}_{X/S} \cap \mathbf{Hilb}^m_{X/S}\,.$$

Then the $\mathbf{Div}^m_{X/S}$ have all the desired properties.

In general, whenever X/S is separated, X represents $\mathrm{Hilb}^1_{X/S}$, and the diagonal subscheme $\Delta \subset X \times X$ is the universal subscheme. Indeed, the projection $\Delta \to X$ is an isomorphism, so $\Delta \in \mathrm{Hilb}^1_{X/S}(X)$. Now, given any S-map $g : T \to X$, note $(1 \times g)^{-1}\Delta = \Gamma_g$ where $\Gamma_g \subset X \times T$ is the graph subscheme of g, because the T'-points of both $(1 \times g)^{-1}\Delta$ and Γ_g are just the pairs (gp, p) where $p \colon T' \to T$. So $\Gamma_g \in \mathrm{Hilb}^1_{X/S}(T)$.

Conversely, let $\Gamma \in \mathrm{Hilb}^1_{X/S}(T)$. So Γ is a closed subscheme of $X \times T$. The projection $\pi \colon \Gamma \to T$ is proper, and its fibers are finite; hence, it is finite by Chevalley's Theorem [**EGAIII1**, 4.4.2]. So $\Gamma = \mathrm{Spec}(\pi_*\mathcal{O}_\Gamma)$. Moreover, $\pi_*\mathcal{O}_\Gamma$ is locally free, being flat and finitely generated over $\mathcal{O}_T$. And forming $\pi_*\mathcal{O}_\Gamma$ commutes with passing to the fibers, so its rank is 1. Hence $\mathcal{O}_T \xrightarrow{\sim} \pi_*\mathcal{O}_\Gamma$. Therefore, π is an isomorphism. Hence Γ is the graph of a map $g \colon T \to X$. So, $(1 \times g)^{-1}\Delta = \Gamma$ by the above; also, g is the only map with this property, since a map is determined by its graph. Thus X represents $\mathrm{Hilb}^1_{X/S}$, and $\Delta \subset X \times X$ is the universal subscheme.

In the case at hand, $\mathrm{Div}^1_{X/S}$ is therefore representable by an open subscheme $U \subset X$ by Theorem 9.3.7. In fact, its proof shows U is formed by the points $x \in X$ where the fiber Δ_x is a divisor on X_x. Now, Δ_x is a k_x-rational point for any $x \in X$; so Δ_x is a divisor if and only if X_x is regular at Δ_x. Since X/S is flat, X_x is regular at Δ_x if and only if $x \in X_0$. Thus $X_0 = \mathbf{Div}^1_{X/S}$.

Finally, set $T := X_0^m$ and let $\Gamma_i \subset X \times T$ be the graph subscheme of the ith projection. By the above analysis, $\Gamma_i \in \mathrm{Div}^1_{X/S}(T)$. Set $\Gamma := \sum \Gamma_i$. Then $\Gamma \in \mathrm{Div}^m_{X/S}(T)$ owing to Exercise 9.3.5 and to the additivity of degree. Plainly Γ represents the desired T-point of $\mathbf{Div}^m_{X/S}$. □

ANSWER 9.3.11. Let $s \in S$. Let K be the algebraically closure of k_s, and set $A := \mathrm{H}^0(X_K, \mathcal{O}_{X_K})$. Since f is proper, A is finite dimensional as a K-vector space; so A is an Artin ring. Since X_K is connected, A is not a product of two nonzero rings by [**EGAIII2**, 7.8.6.1]; so A is an Artin local ring. Since X_K is reduced, A is reduced; so A is a field, which is a finite extension of K. Since K is algebraically closed, therefore $A = K$. Since cohomology commutes with flat base change, consequently $k_s \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{O}_{X_s})$.

The isomorphism $k_s \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{O}_{X_s})$ factors through $f_*(\mathcal{O}_X) \otimes k_s$:

$$k_s \to f_*(\mathcal{O}_X) \otimes k_s \to \mathrm{H}^0(X_s, \mathcal{O}_{X_s}).$$

So the second map is a surjection. Hence this map is an isomorphism by the implication (iv)⇒(iii) of Subsection 9.3.10 with $\mathcal{F} := \mathcal{O}_X$ and $\mathcal{N} := k_s$. Therefore, the first map is an isomorphism too.

It follows that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is surjective at s. Indeed, denote its cokernel by $\mathcal{G}$. Since tensor product is right exact and since $k_s \to f_*(\mathcal{O}_X) \otimes k_s$ is an isomorphism, $\mathcal{G} \otimes k_s = 0$. So by Nakayama's lemma, the stalk $\mathcal{G}_s$ vanishes, as claimed

Let $\mathcal{Q}$ be the $\mathcal{O}_S$-module associated to $\mathcal{F} := \mathcal{O}_X$ as in Subsection 9.3.10. Then $\mathcal{Q}$ is free at s by the implication (iv)⇒(i) of Subsection 9.3.10. And $\operatorname{rank} \mathcal{Q}_s = 1$ owing to the isomorphism in (9.3.10.1) with $\mathcal{N} := k_s$. But, with $\mathcal{N} := \mathcal{O}_S$, the isomorphism becomes $Hom(\mathcal{Q}, \mathcal{O}_X) \xrightarrow{\sim} f_*\mathcal{O}_X$. Hence $f_*\mathcal{O}_X$ too is free of rank 1 at s. Therefore, the surjection $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism at s. Since s is arbitrary, $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ everywhere.

Finally, let T be an arbitrary S-scheme. Then $f_T\colon X_T \to T$ too is proper and flat, and its geometric fibers are reduced and connected. Hence, by what we just proved, $\mathcal{O}_T \xrightarrow{\sim} f_T * \mathcal{O}_{X_T}$. □

ANSWER 9.3.14. By Theorem 9.3.13, L represents $\mathrm{LinSys}_{\mathcal{L}/X/S}$. So by Yoneda's Lemma [**EGAG**, (0,1.1.4), p. 20], there exists a $W \in \mathrm{LinSys}_{\mathcal{L}/X/S}(L)$ with the required universal property. And W corresponds to the identity map $p\colon L \to L$. The proof of Theorem 9.3.13 now shows $\mathcal{O}_{X_L}(W) = (\mathcal{L}|X_L) \otimes f_L^*\mathcal{O}_L(1)$. □

ANSWER 9.4.2. The structure sheaf $\mathcal{O}_X$ defines a section $\sigma\colon S \to \mathbf{Pic}_{X/S}$. Its image is a subscheme, which is closed if $\mathbf{Pic}_{X/S}$ is separated, by [**EGAG**, Cors. (5.1.4), p. 275, and (5.2.4), p. 278]. Let $N \subset T$ be the pullback of this subscheme under the map $\lambda\colon T \to \mathbf{Pic}_{X/S}$ defined by $\mathcal{L}$. Then the third property holds.

Both $\mathcal{L}_N$ and $\mathcal{O}_X$ define the same map $N \to \mathbf{Pic}_{X/S}$. So, since $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally, the Comparison Theorem, Theorem, Theorem 9.2.5, implies that there exists an invertible sheaf $\mathcal{N}$ on N such that the first property holds.

Consider the second property. Then $\mathcal{L}_{T'} \simeq f_{T'}^*\mathcal{N}'$. So $\lambda t\colon T' \to \mathbf{Pic}_{X/S}$ is also defined by $\mathcal{O}_{X_{T'}}$; hence, λt factors through $\sigma\colon S \to \mathbf{Pic}_{X/S}$. Therefore, $t\colon T' \to T$ factors through N. So, since the first property holds, $\mathcal{L}_{T'} \simeq f_{T'}^* t^*\mathcal{N}$. Hence $\mathcal{N}' \simeq t^*\mathcal{N}$ by Lemma (9.2.7). Thus the second property holds.

Finally, suppose the pair $(N_1, \mathcal{N}_1)$ also possesses the first property. Taking t to be the inclusion of N_1 into T, we conclude that $N_1 \subset N$ and $\mathcal{N}_1 \simeq \mathcal{N}|N$. Suppose $(N_1, \mathcal{N}_1)$ possess the second property too. Then, similarly, $N \subset N_1$. Thus $N = N_1$ and $\mathcal{N}_1 \simeq \mathcal{N}$, as desired. □

ANSWER 9.4.3. By Yoneda's Lemma [**EGAG**, (0,1.1.4), p. 20], a universal sheaf $\mathcal{P}$ exists if and only if $\mathbf{Pic}_{X/S}$ represents $\mathrm{Pic}_{X/S}$. Set $P := \mathbf{Pic}_{X/S}$.

Assume $\mathcal{P}$ exists. Then, for any invertible sheaf $\mathcal{N}$ on P, plainly $\mathcal{P} \otimes f_P^*\mathcal{N}$ is also a universal sheaf. Moreover, if $\mathcal{P}'$ is also a universal sheaf, then $\mathcal{P}' \simeq \mathcal{P} \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on P by the definition with $h := 1_P$.

Assume $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds universally. If $\mathcal{P} \otimes f_P^*\mathcal{N} \simeq \mathcal{P} \otimes f_P^*\mathcal{N}'$ for some invertible sheaves $\mathcal{N}$ and $\mathcal{N}'$ on P, then $\mathcal{N} \simeq \mathcal{N}'$ by Lemma 9.2.7.

By Part 2 of Theorem 9.2.5, if also f has a section, then $\mathbf{Pic}_{X/S}$ does represent $\mathrm{Pic}_{X/S}$; so then $\mathcal{P}$ exists. Furthermore, the curve $X/\mathbb{R}$ of Exercise 9.2.4 provides an example where no $\mathcal{P}$ exists, because $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable by Theorem 9.4.8, but $\mathrm{Pic}_{X/\mathbb{R}}$ is not since the two functors differ. □

ANSWER 9.4.4. Say $\mathbf{Pic}_{X/S}$ represents $\mathrm{Pic}_{(X/S)\,(\text{ét})}$. For any S'-scheme T,

$$\mathrm{Pic}_{(X_{S'}/S')\,(\text{ét})}(T) = \mathrm{Pic}_{(X/S)\,(\text{ét})}(T),$$

which holds essentially by definition, since a map of S'-schemes is an étale-covering if and only if it is an étale-covering when viewed as a map of S-schemes. However,

$$(\mathbf{Pic}_{X/S} \times_S S')(T) = \mathbf{Pic}_{X/S}(T)$$

because the structure map $T \to S'$ is fixed. Since the right-hand sides of the two displayed equations are equal, so are their left-hand sides. Thus $\mathbf{Pic}_{X/S} \times_S S'$ represents $\mathrm{Pic}_{(X_{S'}/S')\,(\text{ét})}$. Of course, a similar analysis applies when $\mathbf{Pic}_{X/S}$ represents one of the other relative Picard functors.

An example is provided by the curve $X \subset \mathbf{P}^2_{\mathbb{R}}$ of Exercise 9.2.4. Indeed, since the functors $\mathrm{Pic}_{X/\mathbb{R}}$ and $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ differ, $\mathrm{Pic}_{X/\mathbb{R}}$ is not representable. But $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable by the Main Theorem, 9.4.8. Finally, since $X_{\mathbb{C}}$ has a $\mathbb{C}$-point, all its relative Picard functors are equal by the Comparison Theorem, 9.2.5. □

ANSWER 9.4.5. An $\mathcal{L}$ on an X_k defines a map $\mathrm{Spec}(k) \to \mathbf{Pic}_{X/S}$; assign its image to $\mathcal{L}$. Then, given any field k'' containing k, the pullback $\mathcal{L}|X_{k''}$ is assigned the same scheme point of $\mathbf{Pic}_{X/S}$.

Consider an $\mathcal{L}'$ on an $X_{k'}$. If $\mathcal{L}$ and $\mathcal{L}'$ represent the same class, then there is a k'' containing both k and k' such that $\mathcal{L}|X_{k''} \simeq \mathcal{L}'|X_{k''}$; hence, then both $\mathcal{L}$ and $\mathcal{L}'$ are assigned the same scheme point of $\mathbf{Pic}_{X/S}$. Conversely, if $\mathcal{L}$ and $\mathcal{L}'$ are assigned the same point, take k'' to be any algebraically closed field containing both k and k'. Then $\mathcal{L}|X_{k''}$ and $\mathcal{L}'|X_{k''}$ define the same map $\mathrm{Spec}(k'') \to \mathbf{Pic}_{X/S}$. Hence $\mathcal{L}|X_{k''} \simeq \mathcal{L}'|X_{k''}$ by Exercise 9.2.3 or 9.2.6.

Finally, given any scheme point of $\mathbf{Pic}_{X/S}$, let k be the algebraic closure of its residue field. Then $\mathrm{Spec}(k) \to \mathbf{Pic}_{X/S}$ is defined by an $\mathcal{L}$ on X_k by Exercise 9.2.3 or 9.2.6. So the given point is assigned to $\mathcal{L}$. Thus the classes of invertible sheaves on the fibers of X/S correspond bijectively to the scheme points of $\mathbf{Pic}_{X/S}$. □

ANSWER 9.4.7. An S-map $h\colon T \to \mathbf{Div}_{X/S}$ corresponds to a relative effective divisor D on X_T. So the composition $\mathbf{A}_{X/S}h\colon T \to P$ corresponds to the invertible sheaf $\mathcal{O}_{X_T}(D)$. Hence $O_{X_T}(D) \simeq (1 \times \mathbf{A}_{X/S}h)^*\mathcal{P} \otimes f_P^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Therefore, if T is viewed as a P-scheme via $\mathbf{A}_{X/S}h$, then D defines a T-point η of $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$. Plainly, the assignment $h \mapsto \eta$ is functorial in T. Thus if $\mathbf{Div}_{X/S}$ is viewed as a P-scheme via $\mathbf{A}_{X/S}$, then there is a natural map Λ from its functor of points to $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$.

Furthermore, Λ is an isomorphism. Indeed, let T be a P-scheme. A T-point η of $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$ is given by a relative effective divisor D on X_T such that $\mathcal{O}_{X_T}(D) \simeq \mathcal{P}_T \otimes f_T^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on T. Then $\mathcal{O}_{X_T}(D)$ and $\mathcal{P}_T$ define the same S-map $T \to P$. But $\mathcal{P}_T$ defines the structure map. And $\mathcal{O}_{X_T}(D)$ defines the composition $\mathbf{A}_{X/S}h$ where $h\colon T \to \mathbf{Div}_{X/S}$ is the map defined by D. Thus $\eta = \Lambda(h)$, and h is determined by η; hence, Λ is an isomorphism.

In other words, $\mathbf{Div}_{X/S}$ represents $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$. But $\mathbf{P}(\mathcal{Q})$ too represents $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$ by Theorem 9.3.13. Therefore, $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$ as P-schemes. □

ANSWER 9.4.10. First, suppose $F \to G$ is a surjection. Given a map of étale sheaves $\varphi\colon F \to H$ such that the two maps $F \times_G F \to H$ are equal, we must show there is one and only one map $G \to H$ such that $F \to G \to H$ is equal to φ.

Let $\eta \in G(T)$. By hypothesis, there exist an étale covering $T' \to T$ and an element $\zeta' \in F(T')$ such that ζ' and η have the same image in $G(T')$. Set $T'' := T' \times_T T'$. Then the two images of ζ' in $F(T'')$ define an element ζ'' of $(F \times_G F)(T'')$. Since the two maps $F \times_G F \to H$ are equal, the two images of ζ'' in $H(T'')$ are equal. But these two images are equal to those of $\varphi(\zeta') \in H(T')$. Since H is a sheaf, therefore $\varphi(\zeta')$ is the image of a unique element $\theta \in H(T)$.

Note $\theta \in H(T)$ is independent of the choice of T' and $\zeta' \in F(T')$. Indeed, let $\zeta_1' \in F(T_1')$ be a second choice. Arguing as above, we find $\varphi(\zeta_1') \in H(T_1')$ and $\varphi(\zeta') \in H(T')$ have the same image in $H(T_1' \times_T T')$. So ζ_1' also leads to θ.

Define a map $G(T) \to H(T)$ by $\eta \mapsto \theta$. Plainly this map behaves functorially in T. Thus there is a map of sheaves $G \to H$. Plainly, $F \to G \to H$ is equal to $\varphi\colon F \to H$. Finally, $G \to H$ is the only such map, since the image of η in $H(T)$ is determined by the image of η in $G(T')$, and the latter must map to $\varphi(\zeta') \in H(T')$. Thus G is the coequalizer of $F \times_G F \rightrightarrows F$.

Conversely, suppose G is the coequalizer of $F \times_G F \rightrightarrows F$. Form the étale subsheaf $H \subset G$ associated to the presheaf whose T-points are the images in $G(T)$ of the elements of $F(T)$. Then the map $F \to G$ factors through H. So the two maps $F \times_G F \to H$ are equal. Since G is the coequalizer, there is a map $G \to H$ so that $F \to G \to H$ is equal to $F \to H$. Hence $F \to G \to H \hookrightarrow G$ is equal to $F \to G$. So $G \to H \hookrightarrow G$ is equal to 1_G by uniqueness. Therefore, $H = G$. Thus $F \to G$ is a surjection. □

ANSWER 9.4.11. Theorem 9.4.8 implies each connected component Z' of Z lies in an increasing union of open quasi-projective subschemes of $\mathbf{Pic}_{X/S}$. So Z' lies in one of them since Z' is quasi-compact. So Z' is quasi-projective. But Z has only finitely many components Z'. Therefore, Z is quasi-projective. □

ANSWER 9.4.12. Set $P := \mathbf{Pic}_{X/S}$, which exists by Theorem 9.4.8. If $\mathcal{P}$ exists, then $\mathbf{A}_{X/S}$ is, by Exercise 9.4.7, the structure map of the bundle $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ denotes the coherent sheaf on $\mathbf{Pic}_{X/S}$ associated to $\mathcal{P}$ as in Subsection 9.3.10. In particular, $\mathbf{A}_{X/S}$ is projective Zariski locally over S.

In general, forming P commutes with extending S by Exercise 9.4.4. Similarly, forming $\mathbf{A}_{X/S}$ does too. But a map is proper if it is after an fppf base extension by [**EGAIV2**, 2.7.1(vii)].

However, $f\colon X \to S$ is fppf. Moreover, $f_X\colon X \times X \to X$ has a section, namely, the diagonal. So use f as a base extension. Then, by Exercises 9.3.11 and 9.4.3, a universal sheaf $\mathcal{P}$ exists. Therefore, $\mathbf{A}_{X/S}$ is proper by the first case. □

ANSWER 9.4.13. Let's use the ideas and notation of Answer 9.4.12. Now, X_0 represents $\mathrm{Div}^1_{X/S}$ by Exercise 9.3.8. Hence the Abel map $\mathbf{A}_{X/S}$ induces a natural map $A\colon X_0 \to P$, and forming A commutes with extending S. But a map is a closed embedding if it is after an fppf base extension by [**EGAIV2**, 2.7.1(xii)]. So we may assume $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$.

The function $\lambda \mapsto \deg \mathcal{P}_\lambda$ is locally constant. Let $W \subset P$ be the open and closed subset where the function's value is 1. Plainly $\mathbf{P}(\mathcal{Q}_W) = \mathbf{Div}^1_{X/S}$ owing to the above. Therefore, $X_0 = \mathbf{P}(\mathcal{Q}_W)$, and $A\colon X_0 \to P$ is equal to the structure map of $\mathbf{P}(\mathcal{Q}_W)$. So it remains to show that this structure map is a closed embedding.

Fix $\lambda \in W$. Then $\dim_{k_\lambda}(\mathcal{Q} \otimes k_\lambda) = \dim_{k_\lambda} \mathrm{H}^0(X_\lambda, \mathcal{P}_\lambda)$. Suppose P_λ has two independent global sections. Each defines an effective divisor of degree 1, which is a k_λ-rational point x_i. Since neither section is a multiple of the other, the x_i are distinct. Hence the sections generate P_λ. So they define a map $h\colon X_\lambda \to \mathbf{P}^1_{k_\lambda}$ by [**EGAII**, 4.2.3] or [**Har77**, Thm. II, 7.1, p. 150]. Then h is birational since each x_i is the scheme-theoretic inverse image of a k_λ-rational point of $\mathbf{P}^1_{k_\lambda}$. Hence h is an isomorphism. But, by hypothesis, X_λ is of arithmetic genus at least 1. So there is a contradiction. Therefore, $\dim_{k_\lambda}(\mathcal{Q} \otimes k_\lambda) \le 1$.

By Nakayama's lemma, $\mathcal{Q}$ can be generated by a single element on a neighborhood $V \subset W$ of λ. So there is a surjection $\mathcal{O}_V \twoheadrightarrow \mathcal{Q}_V$. It defines a closed embedding $\mathbf{P}(\mathcal{Q}_V) \hookrightarrow \mathbf{P}(\mathcal{O}_V)$. But the structure map $\mathbf{P}(\mathcal{O}_V) \to V$ is an isomorphism. Hence $\mathbf{P}(\mathcal{Q}_V) \to V$ is a closed embedding. But $\lambda \in W$ is arbitrary. So $\mathbf{P}(\mathcal{Q}_W) \to W$ is indeed a closed embedding. □

ANSWER 9.4.15. Representing $\mathrm{Pic}_{X/S}$ is similar to representing $\mathrm{Pic}_{X'/S'}$ in Example 9.4.14, but simpler. Indeed, On $X \times_S \mathbb{Z}_S$, form an invertible sheaf $\mathcal{P}$ by placing $\mathcal{O}_X(n)$ on the nth copy of X. Then it suffices to show this: given any S-scheme T and any invertible sheaf $\mathcal{L}$ on X_T, there exist a unique S-map $q\colon T \to \mathbb{Z}_S$ and some invertible sheaf $\mathcal{N}$ on T such that $(1 \times q)^*\mathcal{P} = \mathcal{L} \otimes f_T^*\mathcal{N}$.

Plainly, we may assume T is connected. Then the function $s \mapsto \chi(X_t, \mathcal{L}_t)$ is constant on T by [**EGAIII2**, 7.9.11]. Now, X_t is a projective space of dimension at least 1 over the residue field k_t; so $\mathcal{L}_t \simeq \mathcal{O}_{X_t}(n)$ for some n by [**Har77**, Prp. 6.4, p. 132, and Cors. 6.16 and 6.17, p. 145]. Hence n is independent of t.

Set $\mathcal{M} := \mathcal{L}^{-1}(n)$. Then $\mathcal{M}_t \simeq \mathcal{O}_{X_t}$ for all $t \in T$. Hence $\mathrm{H}^1(X_t, \mathcal{M}_t) = 0$ and $\mathrm{H}^0(X_t, \mathcal{M}_t) = k_t$ by Serre's explicit computation [**EGAIII1**, 2.1.12]. Hence $f_{T*}\mathcal{M}$ is invertible, and forming it commutes with changing the base T, owing to the theory in Subsection 9.3.10.

Set $\mathcal{N} := f_{T*}\mathcal{M}$. Consider the natural map $u\colon f_T^*\mathcal{N} \to \mathcal{M}$. Forming u commutes with changing T, since forming $\mathcal{N}$ does. But u is an isomorphism on the fiber over each $t \in T$. So $u \otimes k_t$ is an isomorphism. Hence u is surjective by Nakayama's lemma. But both source and target of u are invertible; so u is an isomorphism. Hence $\mathcal{L} \otimes f_T^*\mathcal{N} = \mathcal{O}_{X_T}(n)$.

Let $q\colon T \to \mathbb{Z}_S$ be the composition of the structure map $T \to S$ and the nth inclusion $S \hookrightarrow \mathbb{Z}_S$. Plainly $(1 \times q)^*\mathcal{P} = \mathcal{O}_{X_T}(n)$, and q is the only such S-map. Thus $\mathbb{Z}_S$ represents $\mathrm{Pic}_{X/S}$, and $\mathcal{P}$ is a universal sheaf. □

ANSWER 9.4.16. First of all, $\mathbf{Pic}_{X/\mathbb{R}}$ exists by Theorem 9.4.8. Now, $X_\mathbb{C} \simeq \mathbf{P}^1_\mathbb{C}$. Hence $\mathbf{Pic}_{X/\mathbb{R}} \times_\mathbb{R} \mathbb{C} \simeq \mathbb{Z}_\mathbb{C}$ by Exercises 9.4.4 and 9.4.15. The induced automorphism of $\mathbb{Z}_{\mathbb{C}\otimes_\mathbb{R}\mathbb{C}}$ is the identity; indeed, a point of this scheme corresponds to an invertible sheaf on $\mathbf{P}^1_\mathbb{C}$, and every such sheaf is isomorphic to its pullback under any $\mathbb{R}$-automorphism of $\mathbf{P}^1_\mathbb{C}$. Hence, by descent theory, $\mathbf{Pic}_{X/\mathbb{R}} = \mathbb{Z}_\mathbb{R}$.

The above reasoning leads to a second proof that $\mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$ is representable. Indeed, set $P := \mathrm{Pic}_{(X/\mathbb{R})\,(\text{ét})}$. By the above reasoning, the pair

$$(P \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{C} (\mathbb{C} \otimes_\mathbb{R} \mathbb{C}) \rightrightarrows P \otimes_\mathbb{R} \mathbb{C}$$

is representable by the pair $\mathbb{Z}_{\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}} \rightrightarrows \mathbb{Z}_{\mathbb{C}}$, whose coequalizer is $\mathbb{Z}_{\mathbb{R}}$. On the other hand, in the category of étale sheaves, the coequalizer is P owing to Exercise 9.4.10.

Notice in passing that $\mathbf{Pic}_{X/\mathbb{R}} = \mathbf{Pic}_{\mathbf{P}^1_{\mathbb{R}}/\mathbb{R}}$. But $\mathrm{Pic}_{X/\mathbb{R}}$ is not representable due to Exercise 9.2.4, where as $\mathrm{Pic}_{\mathbf{P}^1_{\mathbb{R}}/\mathbb{R}}$ is representable owing to Exercise 9.4.15. □

ANSWER 9.5.7. Exercise 9.4.11 implies Z is quasi-projective. Hence Z is projective if Z is proper. By [**EGAIV2**, 2.7.1], an S-scheme is proper if it is so after an fppf base change, such as $f\colon X \to S$. But $f_X\colon X \times X \to X$ has a section, namely, the diagonal. Thus we may assume f has a section.

Using the Valuative Criterion for Properness [**Har77**, Thm. 4.7, p. 101], we need only check this statement: given an S-scheme T of the form $T = \mathrm{Spec}(A)$ where A is a valuation ring, say with fraction field K, every S-map $u\colon \mathrm{Spec}(K) \to Z$ extends to an S-map $T \to Z$. We do not need to check the extension is unique if it exists; indeed, this uniqueness holds by the Valuative Criterion for Separatedness [**Har77**, Thm. 4.3, p. 97] since Z is quasi-projective, so separated.

Since f has a section, u arises from an invertible sheaf $\mathcal{L}$ on X_K by Theorem 9.2.5. We have to extend $\mathcal{L}$ over X_T. Indeed, this extension defines a map $t\colon T \to \mathbf{Pic}_{X/S}$ extending u, and t factors through Z because Z is closed and T is integral.

Plainly it suffices to extend $\mathcal{L}(n)$ for any $n \gg 0$. So replacing $\mathcal{L}$ if need be, we may assume $\mathcal{L}$ has a nonzero section. It is regular since X_K is integral. So X_K has a divisor D such that $\mathcal{O}(D) = \mathcal{L}$.

Let $D' \subset X_T$ be the closure of D. Now, X/S is smooth and T is regular, so X_T is regular by [**EGAIV2**, 6.5.2], so factorial by [**EGAIV2**, 21.11.1]. Hence D' is a divisor. And $\mathcal{O}(D')$ extends $\mathcal{L}$. □

ANSWER 9.5.16. Owing to Serre's Theorem [**Har77**, Thm. 5.2, p. 228], we have $\mathrm{H}^i(\Omega^2_X(n)) = 0$ for $i > 0$ and $n \gg 0$. So $\varphi(n) = \chi(\Omega^2_X(n))$. Hence

$$q = \mathrm{H}^1(\Omega^2_X) - \mathrm{H}^2(\Omega^2_X) + 1.$$

Serre duality [**Har77**, Cor. 7.13, p. 247] yields $\dim \mathrm{H}^i(\Omega^i_X) = \dim \mathrm{H}^{2-i}(\mathcal{O}_X)$ for all i. And $\dim \mathrm{H}^0(\mathcal{O}_X) = 1$ since X is projective and geometrically integral. So

$$q = \dim \mathrm{H}^1(\mathcal{O}_X).$$

Hence Corollary 9.5.14 yields dim $\mathbf{Pic}_{X/S} \le q$, with equality in characteristic 0. □

ANSWER 9.5.17. Set $P := \mathbf{Pic}_{X/S}$, which exists by Theorem 9.4.8. By Exercises 9.3.11 and 9.4.3, there exists a universal sheaf $\mathcal{P}$ on $X \times P$

Suppose $q = 0$. Then P is smooth of dimension 0 everywhere by Corollary 9.5.13. Let D be a relative effective divisor on X_T/T where T is a connected S-scheme. Then $\mathcal{O}_{X_T}(D)$ defines a map $\tau : T \to P$, and

$$\mathcal{O}_{X_T}(D) \simeq (1 \times \tau)^*\mathcal{P} \otimes f_T^*\mathcal{N}$$

for some invertible sheaf $\mathcal{N}$ on T. Now, T is connected and P is discrete and reduced; so τ is constant. Set $\lambda := \tau T$, and view $\mathcal{P}_\lambda$ as an invertible sheaf $\mathcal{L}$ on X. Then $\mathcal{L}_T = (1 \times \tau)^*\mathcal{P}$. So $\mathcal{O}_{X_T}(D) \simeq \mathcal{L}_T \otimes f_T^*\mathcal{N}$, as required.

Consider the converse. Again by Exercise 9.4.3, there is a coherent sheaf $\mathcal{Q}$ on P such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/S}$. Furthermore, $\mathcal{Q}$ is nonzero and locally free at any closed point λ representing an invertible sheaf $\mathcal{L}$ on X such that $\mathrm{H}^1(\mathcal{L}) = 0$ by Subsection 9.3.10; for example, take $\mathcal{L} := \mathcal{O}_X(n)$ for $n \gg 0$.

Let $U \subset P$ be a connected open neighborhood of λ on which Q is free. Let $T \subset \mathbf{P}(\mathcal{Q})$ be the preimage of U, and let D be the universal relative effective divisor on X_T/T. Then the natural map $A\colon T \to U$ is smooth with irreducible fibers. So T is connected. Moreover, A is the map defined by $\mathcal{O}_{X_T}(D)$.

Suppose $\mathcal{O}_{X_T}(D) \simeq \mathcal{M}_T \otimes f_T^*\mathcal{N}$ for some invertible sheaves $\mathcal{M}$ on X and $\mathcal{N}$ on T. Then $A\colon T \to U$ is also defined by $\mathcal{M}_T$. Say $\mu \in P$ represents $\mathcal{M}$. Then A factors through the inclusion of the closed point μ. Hence $\mu = \lambda$; moreover, since A is smooth and surjective, its image, the open set U, is just the reduced closed point λ. Now, there is an automorphism of P that carries 0 to λ, namely, "multiplication" by λ. So P is smooth of dimension 0 at 0. Therefore, $q = 0$ by Corollary 9.5.13.

In characteristic 0, a priori P is smooth by Corollary 9.5.14. Now, $A\colon T \to U$ is smooth. Hence, T is smooth too. But the preceding argument shows that, if the condition holds for this T, then $q = 0$, as required. □

ANSWER 9.5.23. By hypothesis, $\dim X_s = 1$ for $s \in S$; so $\mathrm{H}^2(\mathcal{O}_{X_s}) = 0$. Hence the $\mathbf{Pic}^0_{X_s/k_s}$ are smooth by Proposition 9.5.19, so of dimension p_a by Proposition 9.5.13. Hence, by Proposition 9.5.20, the $\mathbf{Pic}^0_{X_s/k_s}$ form a family of finite type, whose total space is the open subscheme $\mathbf{Pic}^0_{X/S}$ of $\mathbf{Pic}_{X/S}$. And $\mathbf{Pic}_{X/S}$ is smooth over S again by Proposition 9.5.19.

Hence $\mathbf{Pic}^0_{X/S}$ is quasi-projective by Exercise 9.4.11.

If X/S is smooth, then $\mathbf{Pic}^0_{X/S}$ is projective over S by Exercise 9.5.7. Alternatively, use Theorem 9.5.4 and Proposition 9.5.20 again to conclude $\mathbf{Pic}^0_{X/S}$ is proper, so projective since it is quasi-projective.

Conversely, assume $\mathbf{Pic}^0_{X/S}$ is proper, and let us prove X/S is smooth, Since X/S is flat, we need only prove each X_s is smooth. So we may replace S by the spectrum of the algebraic closure of k_s. If $p_a = 0$, then X is smooth, indeed $X = \mathbf{P}^1$, by [**Har77**, Ex. 1.8(b), p. 298].

Suppose $p_a > 0$. Let X_0 be the open subscheme where X is smooth. Then there is a closed embedding $A\colon X_0 \hookrightarrow \mathbf{Pic}_{X/S}$ by Exercise 9.4.13. Its image consists of points λ representing invertible sheaves of degree 1. Fix a rational point λ, and define an automorphism β of $\mathbf{Pic}_{X/S}$ by $\beta(\kappa) := \kappa\lambda^{-1}$. Then βA is a closed embedding of X_0 in $\mathbf{Pic}^0_{X/S}$.

By assumption, $\mathbf{Pic}^0_{X/S}$ is proper. So X_0 is proper. Hence $X_0 \hookrightarrow X$ is proper since X is separated. Hence X_0 is closed in X. But X_0 is dense in X since X is integral and the ground field is algebraically closed. Hence $X_0 = X$; in other words, X is smooth. □

ANSWER 9.6.4. As before, by Lemma 9.6.6, there is an m such that every $\mathcal{N}(m)$ is generated by its global sections. So there is a section that does not vanish at any given associated point of X; since these points are finite in number, if σ is a general linear combination of the corresponding sections, then σ vanishes at no associated point. So σ is regular, whence defines an effective divisor D such that $\mathcal{O}_X(-D) = \mathcal{N}^{-1}(-m)$.

Plainly $\mathcal{N}^{-1}$ is numerically equivalent to $\mathcal{O}_X$ too. So $\chi(\mathcal{N}^{-1}(n)) = \chi(\mathcal{O}_X(n))$ by Lemma 9.6.6. Hence the sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ yields

$$\chi(\mathcal{O}_D(n)) = \psi(n) \text{ where } \psi(n) := \chi(\mathcal{O}_X(n)) - \chi(\mathcal{O}_X(n-m)).$$

Let $T \subset \mathbf{Div}_{X/k}$ be the open and closed subscheme parameterizing the effective divisors with Hilbert polynomial $\psi(n)$. Then T is a k-scheme of finite type. Let $\mathcal{M}'$ be the invertible sheaf on X_T associated to the universal divisor; set $\mathcal{M} := \mathcal{M}'(-n)$. Then there exists a rational point $t \in T$ such that $\mathcal{N} = \mathcal{M}_t$. Thus the $\mathcal{N}$ numerically equivalent to $\mathcal{O}_X$ form a bounded family. $\square$

ANSWER 9.6.7. Suppose $a < a_r$. Suppose $\mathcal{L}(-1)$ has a nonzero section. It defines an effective divisor D, possibly 0. Hence

$$0 \le \textstyle\int h^{r-1}[D] = \int h^{r-1}\ell - \int h^r = a - a_r < 0,$$

which is absurd. Thus $\mathrm{H}^0(\mathcal{L}(-1)) = 0$.

Let H be a hyperplane section of X. Then there is an exact sequence

$$0 \to \mathcal{L}(n-1) \to \mathcal{L}(n) \to \mathcal{L}_H(n) \to 0.$$

It yields the following bound:

$$\dim \mathrm{H}^0(\mathcal{L}(n)) - \dim \mathrm{H}^0(\mathcal{L}(n-1)) \le \dim \mathrm{H}^0(\mathcal{L}_H(n)). \tag{A.9.6.7.1}$$

Since $\binom{n+i}{i} - \binom{n-1+i}{i} = \binom{n+i-1}{i-1}$, the sequence also yields the following formula:

$$\chi(\mathcal{L}_H(n)) = \textstyle\sum_{0\le i\le r-1} a_{i+1}\binom{n+i}{i}.$$

Suppose $r = 1$. Then $\dim \mathrm{H}^0(\mathcal{L}_H(n)) = \chi(\mathcal{L}_H(n)) = a_1$. Therefore, owing to Equation (A.9.6.7.1), induction on n yields $\dim \mathrm{H}^0(\mathcal{L}(n)) \le a_1(n+1)$, as desired.

Furthermore, $\mathcal{L}_H$ is 0-regular. Set $m := \dim \mathrm{H}^1(\mathcal{L}(-1))$. Then $\mathcal{L}$ is m-regular by Mumford's conclusion at the bottom of [**Mum66**, p. 102]. But

$$m = \dim \mathrm{H}^0(\mathcal{L}(-1)) - \chi(\mathcal{L}(-1)) = 0 - a_1(-1+1) - a_0 = -a_0.$$

Thus we may take $\Phi_1(u_0) := -u_0$ where u_0 is an indeterminate.

Suppose $r \ge 2$. Then we may take H irreducible by Bertini's Theorem [**Sei50**, Thm. 12, p. 374] or [**Jou79**, Cor. 6.7, p. 80]. Set $h_1 := c_1 O_H(1)$ and $\ell_1 := c_1\mathcal{L}_H$. Then $\int \ell_1 h_1^{r-2} = \int \ell h^{r-2}[H] = a < a_r$. So by induction on r, we may assume

$$\dim \mathrm{H}^0(\mathcal{L}_H(n)) \le a_r\binom{n+r-1}{r-1}.$$

Therefore, owing to Equation (A.9.6.7.1), induction on n yields the desired bound.

Furthermore, we may assume $\mathcal{L}_H$ is m_1-regular where $m_1 := \Phi_{r-1}(a_1 \ldots, a_{r-1})$. Set $m := m_1 + \dim \mathrm{H}^1(\mathcal{L}(m_1-1))$. By Mumford's same work, $\mathcal{L}$ is m-regular. But

$$\begin{aligned} m &= m_1 + \dim \mathrm{H}^0(\mathcal{L}(m_1-1)) - \chi(\mathcal{L}(m_1-1)) \\ &\le m_1 + a_r\binom{m_1-1+r}{r} - \textstyle\sum_{0\le i\le r} a_i\binom{m_1-1+i}{i}. \end{aligned}$$

The latter expression is a polynomial in $a_0, \ldots, a_{r-1}$ and m_1. So it is a polynomial Φ_r in $a_0, \ldots, a_{r-1}$ alone, as desired.

In general, consider $\mathcal{N} := \mathcal{L}(-a)$. Then

$$\chi(\mathcal{N}(n)) = \textstyle\sum_{0\le i\le r} b_i\binom{n+i}{i} \text{ where } b_i := \sum_{j=0}^{r-i} a_{i+j}(-1)^j\binom{a-i-j}{j}.$$

Set $\nu := c_1\mathcal{N}$ and $b := \int \nu h^{r-1}$. Then

$$b = \textstyle\int \ell h^{r-1} - a\int h^r = a - aa_r \le 0 < a_r.$$

Hence $\mathcal{N}$ is m-regular where $m := \Phi_r(b_0, \ldots, b_{r-1})$. But the b_i are polynomials in $a_0, \ldots, a_r$ and a. Hence there is a polynomial Ψ_r depending only on r such that $m := \Psi_r(a_0, \ldots, a_r; a)$, as desired. □

ANSWER 9.6.10. Let k' be the algebraic closure of k. If $H \otimes k' \subset G^\tau \otimes k'$, then $H \subset G^\tau$. But $G^\tau \otimes k' = (G \otimes k')^\tau$ by Lemma 9.6.10. Thus we may assume $k = k'$.

Then $H \subset \bigcup_{h \in H(k)} hG^0$. But G^0 is open, so hG^0 is too. And H is quasi-compact. So H lies in finitely many hG^0. So $G^0(k)$ has finite index in $H(k)G^0(k)$, say n. Then $h^n \in G^0(k)$ for every $h \in H(k)$. So $\varphi_n(H) \subset G^0$. Thus $H \subset G^\tau$. □

ANSWER 9.6.11. Given n, plainly $\mathcal{L}^{\otimes n}$ corresponds to $\varphi_n\lambda$. And $\mathcal{L}^{\otimes n}$ is algebraically equivalent to $\mathcal{O}_X$ if and only if $\varphi_n\lambda \in \mathbf{Pic}^0_{X/k}$ by Proposition 9.5.10. So $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$ if and only if $\lambda \in \mathbf{Pic}^\tau_{X/k}$ by Definitions 9.6.1 and 9.6.8. □

ANSWER 9.6.13. Theorem 9.4.8 says $\mathbf{Pic}_{X/k}$ exists and represents $\mathrm{Pic}_{(X/k)\,(\text{ét})}$. So $\mathbf{Pic}^\tau_{X/k}$ is of finite type by Proposition 9.6.12. Hence $\mathbf{Pic}^\tau_{X/k}$ is quasi-projective by Exercise 9.4.11.

Suppose X is also geometrically normal. Since $\mathbf{Pic}^\tau_{X/k}$ is quasi-projective, to prove it is projective, it suffices to prove it is complete. By Proposition 9.6.12, forming $\mathbf{Pic}^0_{X/k}$ commutes with extending k. And by [**EGAIV2**, 2.7.1(vii)], a k-scheme is complete if (and only if) it is after extending k. So assume k is algebraically closed.

As λ ranges over the k-points of $\mathbf{Pic}^\tau_{X/k}$, the cosets $\lambda\,\mathbf{Pic}^0_{X/k}$ cover $\mathbf{Pic}^\tau_{X/k}$. So finitely many cosets cover, since $\mathbf{Pic}^0_{X/k}$ is an open by Proposition 9.5.3 and since $\mathbf{Pic}^\tau_{X/k}$ is quasi-compact, Now, $\mathbf{Pic}^0_{X/k}$ is projective by Theorem 9.5.4, so complete, And $\mathbf{Pic}^\tau_{X/k}$ is closed, again by Proposition 9.6.12. Hence $\mathbf{Pic}^\tau_{X/k}$ is complete. □

ANSWER 9.6.15. First, suppose that L is bounded. Then $\mathcal{M}$ defines a map $\theta\colon T \to \mathbf{Pic}_{X/S}$, and $\theta(T) \supset \Lambda$. Since T is Noetherian, plainly so is $\theta(T)$; whence, plainly so is any subspace of $\theta(T)$. Thus Λ is quasi-compact.

Conversely, suppose Λ is quasi-compact. Since $\mathbf{Pic}_{X/S}$ is locally of finite type by Proposition 9.4.17, there is an open subscheme of finite type containing any given point of Λ. So finitely many of the subschemes cover Λ. Denote their union by U.

The inclusion $U \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to U$. Replace T be an open subscheme so that $T \to U$ is of finite type and surjective. Since U is of finite type, so is T. Given $\lambda \in \Lambda$, let $t \in T$ map to λ. Then λ corresponds to the class of $\mathcal{M}_t$. □

ANSWER 9.6.18. Theorem 9.6.16 asserts $\mathbf{Pic}^\tau_{X/S}$ is of finite type. So it is projective by Exercise 9.5.7. □

ANSWER 9.6.21. Plainly, replacing S by an open subset, we may assume X/S is projective and S is connected. Given $s \in S$, set $\psi(n) := \chi(\mathcal{O}_{X_s}(n))$. Then $\psi(n)$ is independent of s. Given m, set $\varphi(n) := m + \psi(n)$.

Let $\lambda \in \mathbf{Pic}_{X/S}$. Then $\lambda \in \mathbf{Pic}^m_{X/S}$ if and only if λ represents an invertible sheaf $\mathcal{L}$ of degree m. And $\lambda \in \mathbf{Pic}^{\varphi}_{X/S}$ if and only if $\chi(\mathcal{L}(n)) = \varphi(n)$. But,

$$\chi(\mathcal{L}(n)) = \deg(\mathcal{L}(n)) + \psi(0) = \deg(\mathcal{L}) + \psi(n)$$

by Riemann's Theorem and the additivity of $\deg(\bullet)$. Hence $\mathbf{Pic}^m_{X/S} = \mathbf{Pic}^{\varphi}_{X/S}$. So Theorem 9.6.20 yields all the assertions, except for the two middle about $\mathbf{Pic}^0_{X/S}$.

To show $\mathbf{Pic}^0_{X/S} = \mathbf{Pic}^{\tau}_{X/S}$, similarly we need only show $\deg \mathcal{L} = 0$ if and only if $\mathcal{L}$ is τ-equivalent to $\mathcal{O}_X$, for, by Exercise 9.6.11, the latter holds if and only if $\lambda \in \mathbf{Pic}^{\tau}_{X/S}$. Plainly, we may assume $\mathcal{L}$ lives on a geometric fiber of X/S. Then the two conditions on $\mathcal{L}$ are equivalent by Theorem 9.6.3.

Since deg is additive, multiplication carries $\mathbf{Pic}^0_{X/S} \times \mathbf{Pic}^m_{X/S}$ set-theoretically into $\mathbf{Pic}^m_{X/S}$. So $\mathbf{Pic}^0_{X/S}$ acts on $\mathbf{Pic}^m_{X/S}$ since these two sets are open in $\mathbf{Pic}_{X/S}$.

Since X/S is flat with integral geometric fibers, its smooth locus X_0 provides an fppf covering of S. Temporarily, make the base change $X_0 \to S$. After it, the new map $X_0 \to S$ has a section. Its image is a relative effective divisor D, and tensoring with $\mathcal{O}_X(mD)$ defines the desired isomorphism from $\mathbf{Pic}^0_{X/S}$ to $\mathbf{Pic}^m_{X/S}$.

Finally, to show there is no abuse of notation, we must show the fiber $(\mathbf{Pic}^0_{X/S})_s$ is connected. To do so, we instead make the base change to the spectrum of an algebraically closed field $k \supset k_s$. Then X_0 has a k-rational point D, and again tensoring with $\mathcal{O}_X(mD)$ defines an isomorphism from $\mathbf{Pic}^0_{X/k}$ to $\mathbf{Pic}^m_{X/k}$. So it suffices to show $\mathbf{Pic}^m_{X/k}$ is connected for some $m \geq 1$.

Let $\beta\colon X_0^m \to \mathbf{Div}^m_{X/k} \to \mathbf{Pic}^m_{X/S}$ be composition of the map α of Exercise 9.3.8 and the Abel map. Since X is integral, so is the m-fold product X_0^m. Hence it suffices to show β is surjective for some $m \geq 1$.

By Exercise 9.6.7, there is an $m_0 \geq 1$ such that every invertible sheaf on X of degree 0 is m_0-regular. Set $m := \deg(\mathcal{O}_X(m_0))$. Then every invertible sheaf $\mathcal{L}$ on X of degree m is 0-regular, so generated by its global sections.

In particular, for each singular point of X, there is a global section that does not vanish at it. So, since k is infinite, a general linear combination of these sections vanishes at no singular point. This combination defines an effective divisor E such that $\mathcal{O}_X(E) = \mathcal{L}$. It follows that β is surjective, as desired. □

ANSWER 9.6.29. Suppose Λ is quasi-compact. Then, owing to Exercise 9.6.15, there exist an S-scheme T of finite type and an invertible sheaf $\mathcal{M}$ on X_T such that every polynomial $\varphi \in \Pi$ is of the form $\varphi(n) = \chi(\mathcal{M}_t(n))$ for some $t \in T$. Hence, by [**EGAIII2**, 7.9.4], the number of φ is at most the number of connected components of T. Thus Π is finite.

Suppose Λ is connected. Then its closure is too. So we may assume Λ is closed. Give Λ its reduced subscheme structure. Then the inclusion $\Lambda \hookrightarrow \mathbf{Pic}_{X/S}$ is defined by an invertible sheaf $\mathcal{M}$ on X_T for some fppf covering $T \to \Lambda$. Fix $t \in T$ and set $\varphi(n) := \chi(\mathcal{M}_t(n))$. Fix n, and form the set T' of points t' of T such that $\chi(\mathcal{M}_{t'}(n)) = \varphi(n)$. By [**EGAIII2**, 7.9.4], the set T' is open, and so is its complement. Hence their images are open in Λ, and plainly these images are disjoint. But Λ is connected. Hence $T' = T$. Thus $\Pi = \{\varphi\}$. □

Appendix B. Basic intersection theory

This appendix contains an elementary treatment of basic intersection theory, which is more than sufficient for many purposes, including the needs of Section 9.6. The approach was originated in 1959–60 by Snapper. His results were generalized and his proofs were simplified immediately afterward by Cartier [**Car60**]. Their work was developed further in fits and starts by the author.

The Index Theorem was proved by Hodge in 1937. Immediately afterward, B. Segre [**Seg37**, § 1] gave an algebraic proof for surfaces, and this proof was rediscovered by Grothendieck in 1958. Their work was generalized a tad in [**Kle71**, p. 662], and a variation appears below in Theorem B.27. From the index theorem, Segre [**Seg37**, §6] derived a connectedness statement like Corollary B.29 for surfaces, and the proof below is basically his.

DEFINITION B.1. Let $\mathbf{F}(X/S)$ or $\mathbf{F}$ denote the Abelian category of coherent sheaves $\mathcal{F}$ on X whose support $\operatorname{Supp}\mathcal{F}$ is proper over an Artin subscheme of S, that is, a 0-dimensional Noetherian closed subscheme. For each $r \geq 0$, let $\mathbf{F}_r$ denote the full subcategory of those $\mathcal{F}$ such that $\dim \operatorname{Supp}\mathcal{F} \leq r$.

Let $\mathbf{K}(X/S)$ or $\mathbf{K}$ denote the "Grothendieck group" of $\mathbf{F}$, namely, the free Abelian group on the $\mathcal{F}$, modulo short exact sequences. Abusing notation, let $\mathcal{F}$ also denote its class. And if $\mathcal{F} = \mathcal{O}_Y$ where $Y \subset X$ is a subscheme, then let $[Y]$ also denote the class. Let $\mathbf{K}_r$ denote the subgroup generated by $\mathbf{F}_r$.

Let $\chi\colon \mathbf{K} \to \mathbb{Z}$ denote the homomorphism induced by the Euler characteristic, which is just the alternating sum of the lengths of the cohomology groups.

Given $\mathcal{L} \in \operatorname{Pic}(X)$, let $c_1(\mathcal{L})$ denote the endomorphism of $\mathbf{K}$ defined by the following formula:

$$c_1(\mathcal{L})\mathcal{F} := \mathcal{F} - \mathcal{L}^{-1} \otimes \mathcal{F}.$$

Note that $c_1(\mathcal{L})$ is well defined since tensoring with $\mathcal{L}^{-1}$ preserves exact sequences.

LEMMA B.2. *Let $\mathcal{L} \in \operatorname{Pic}(X)$. Let $Y \subset X$ be a closed subscheme with $\mathcal{O}_Y \in \mathbf{F}$. Let $D \subset Y$ be an effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_Y$. Then*

$$c_1(\mathcal{L}) \cdot [Y] = [D].$$

PROOF. The left side is defined since $\mathcal{O}_Y \in \mathbf{F}$. The equation results from the sequence $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$ since $\mathcal{O}_Y(-D) \simeq \mathcal{L}^{-1} \otimes \mathcal{O}_Y$. □

LEMMA B.3. *Let $\mathcal{L}$, $\mathcal{M} \in \operatorname{Pic}(X)$. Then the following relations hold:*

$$c_1(\mathcal{L})c_1(\mathcal{M}) = c_1(\mathcal{L}) + c_1(\mathcal{M}) - c_1(\mathcal{L} \otimes \mathcal{M});$$
$$c_1(\mathcal{L})c_1(\mathcal{L}^{-1}) = c_1(\mathcal{L}) + c_1(\mathcal{L}^{-1});$$
$$c_1(\mathcal{O}_X) = 0.$$

Furthermore, $c_1(\mathcal{L})$ and $c_1(\mathcal{M})$ commute.

PROOF. Let $\mathcal{F} \in \mathbf{K}$. By definition, $c_1(\mathcal{O}_X)\mathcal{F} = 0$; thus the third relation holds. Plainly, each side of the first relation carries $\mathcal{F}$ into

$$\mathcal{F} - \mathcal{L}^{-1} \otimes \mathcal{F} - \mathcal{M}^{-1} \otimes \mathcal{F} + \mathcal{L}^{-1} \otimes \mathcal{M}^{-1} \otimes \mathcal{F}.$$

Thus the first relation holds. It and the third relation imply the second. Furthermore, the first relation implies that $c_1(\mathcal{L})$ and $c_1(\mathcal{M})$ commute. □

LEMMA B.4. Given $\mathcal{F} \in \mathbf{F}_r$, let $Y_1, \ldots, Y_s$ be the r-dimensional irreducible components of $\operatorname{Supp}\mathcal{F}$ equipped with their induced reduced structure, and let l_i be the length of the stalk of $\mathcal{F}$ at the generic point of Y_i. Then, in $\mathbf{K}_r$,

$$\mathcal{F} \equiv \sum l_i \cdot [Y_i] \bmod \mathbf{K}_{r-1}.$$

PROOF. The assertion holds if it does after we replace S by a neighborhood of the image of $\operatorname{Supp}\mathcal{F}$. So we may assume S is Noetherian.

Let $\mathbf{F}' \subset \mathbf{F}_r$ denote the family of $\mathcal{F}$ for which the assertion holds. Since length($\bullet$) is an additive function, $\mathbf{F}'$ is "exact" in the following sense: for any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ such that two of the $\mathcal{F}$s belong to $\mathbf{F}'$, then the third does too. Trivially, $\mathcal{O}_Y \in \mathbf{F}'$ for any closed integral subscheme $Y \subset X$ such that $\mathcal{O}_Y \in \mathbf{F}_r$. Hence $\mathbf{F}' = \mathbf{F}_r$ by the "Lemma of Dévissage," [**EGAIII1**, Thm. 3.1.2]. □

LEMMA B.5. Let $\mathcal{L} \in \operatorname{Pic}(X)$. Then $c_1(\mathcal{L})\mathbf{K}_r \subset \mathbf{K}_{r-1}$ for all r.

PROOF. Let $\mathcal{F} \in \mathbf{F}_r$. Then $\mathcal{F}$ and $\mathcal{L}^{-1} \otimes \mathcal{F}$ are isomorphic at the generic point of each component of $\operatorname{Supp}\mathcal{F}$. So Lemma B.4 implies $c_1(\mathcal{L})\mathcal{F} \in \mathbf{K}_{r-1}$. □

LEMMA B.6. Let $\mathcal{L} \in \operatorname{Pic}(X)$, let $\mathcal{F} \in \mathbf{K}_r$, and let $m \in \mathbb{Z}$. Then

$$\mathcal{L}^{\otimes m} \otimes \mathcal{F} = \textstyle\sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{L})^i \mathcal{F}.$$

PROOF. Let x be an indeterminate, and consider the formal identity

$$(1-x)^n = \textstyle\sum_{i \geq 0} (-1)^i \binom{n}{i} x^i.$$

Replace x by $1 - y^{-1}$, set $n := -m$, and use the familiar identity

$$(-1)^i \tbinom{n}{i} = \tbinom{m+i-1}{i},$$

to obtain the formal identity

$$y^m = \textstyle\sum \binom{m+i-1}{i} (1 - y^{-1})^i.$$

It yields the assertion, because $c_1(L)^i \mathcal{F} = 0$ for $i > r$ owing to Lemma B.5. □

THEOREM B.7 (Snapper). Let $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \operatorname{Pic}(X)$, let $m_1, \ldots, m_n \in \mathbb{Z}$, and let $\mathcal{F} \in \mathbf{K}_r$. Then the Euler characteristic $\chi\big(\mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes m_n} \otimes \mathcal{F}\big)$ is given by a polynomial in the m_i of degree at most r. In fact,

$$\chi\big(\mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes m_n} \otimes \mathcal{F}\big) = \textstyle\sum a(i_1, \ldots, i_n) \binom{m_1+i_1-1}{i_1} \cdots \binom{m_n+i_n-1}{i_n}$$

where $i_j \geq 0$ and $\sum i_j \leq r$ and where $a(i_1, \ldots, i_n) := \chi\big(c_1(\mathcal{L}_1)^{i_1} \cdots c_1(\mathcal{L}_n)^{i_n} \mathcal{F}\big)$.

PROOF. The theorem follows from Lemmas B.6 and B.5. □

DEFINITION B.8. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \operatorname{Pic}(X)$, repetitions allowed. Let $\mathcal{F} \in \mathbf{K}_r$. Define the *intersection number* or *intersection symbol* by the formula

$$\textstyle\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} := \chi\big(c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F}\big) \in \mathbb{Z}.$$

If $\mathcal{F} = \mathcal{O}_X$, then also write $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)$ for the number. If $\mathcal{L}_j = \mathcal{O}_X(D_j)$ for a divisor D_j, then also write $(D_1 \cdots D_r \cdot \mathcal{F})$, or just $(D_1 \cdots D_r)$ if $\mathcal{F} = \mathcal{O}_X$.

THEOREM B.9. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{K}_r$.

(1) If $\mathcal{F} \in \mathbf{F}_{r-1}$, then $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = 0$.

(2) (symmetry and additivity) The symbol $\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F}$ is symmetric in the $\mathcal{L}_j$. Furthermore, it is a homomorphism separately in each $\mathcal{L}_j$ and in $\mathcal{F}$.

(3) Set $\mathcal{E} := \mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_r^{-1}$. Then

$$\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = \textstyle\sum_{i=0}^{r} (-1)^i \chi\big((\bigwedge^i \mathcal{E}) \otimes \mathcal{F}\big).$$

PROOF. Part (1) results from Lemma B.5. So the symbol is a homomorphism in each $\mathcal{L}_j$ owing to the relations asserted in Lemma B.3. Furthermore, the symbol is symmetric owing to the commutativity asserted in Lemma B.3. Part (3) results from the definitions. □

COROLLARY B.10. Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{K}_2$. Then

$$\int c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\mathcal{F} = \chi(\mathcal{F}) - \chi(\mathcal{L}_1^{-1} \otimes \mathcal{F}) - \chi(\mathcal{L}_2^{-1} \otimes \mathcal{F}) + \chi(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \otimes \mathcal{F}).$$

PROOF. The assertion is a special case of Part (3) of Proposition B.9. □

COROLLARY B.11. Let $D_1, \dots, D_r$ be effective divisors on X, and $\mathcal{F} \in \mathbf{F}_r$. Set $Z := D_1 \cap \cdots \cap D_r$. Suppose $Z \cap \operatorname{Supp} F$ is finite, and at each of its points, F is Cohen–Macaulay. Then

$$(D_1 \cdots D_r \cdot \mathcal{F}) = \operatorname{length} \mathrm{H}^0(\mathcal{F}_Z) \text{ where } \mathcal{F}_Z := \mathcal{F} \otimes \mathcal{O}_Z.$$

PROOF. For each j, set $\mathcal{L}_j := \mathcal{O}_X(D_j)$ and let $\sigma_j \in \mathrm{H}^0(\mathcal{L}_j)$ be the section defining D_j. Set $\mathcal{E} := \mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_r^{-1}$. Form the corresponding Koszul complex $(\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}$ and its cohomology sheaves $\mathcal{H}^i\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big)$. Then

$$\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = \textstyle\sum_{i=0}^{r} (-1)^i \chi\big(\mathcal{H}^i\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big)\big).$$

owing to Part (3) of Proposition B.9 and to the additivity of χ. Furthermore, essentially by definition, $\mathcal{H}^0\big((\bigwedge^\bullet \mathcal{E}) \otimes \mathcal{F}\big) = \mathcal{F}_Z$. And by standard local algebra, the higher $\mathcal{H}^i$ vanish. Thus the assertion holds. □

LEMMA B.12. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r$. Let $Y_1, \dots, Y_s$ be the r-dimensional irreducible components of $\operatorname{Supp} \mathcal{F}$ given their induced reduced structure, and let l_i be the length of the stalk of $\mathcal{F}$ at the generic point of Y_i. Then

$$\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} = \textstyle\sum_i l_i \int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)[Y_i].$$

PROOF. Apply Lemma B.4 and Parts (1) and (2) of Proposition B.9. □

LEMMA B.13. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $Y \subset X$ a closed subscheme with $\mathcal{O}_Y \in \mathbf{F}$. Let $D \subset Y$ be an effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_r|Y$. Then

$$\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)[Y] = \int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{r-1})[D].$$

PROOF. Apply Lemma B.2. □

PROPOSITION B.14. *Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r$. If all the $\mathcal{L}_j$ are relatively ample and if $\mathcal{F} \notin \mathbf{K}_{r-1}$, then*

$$\int c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_r)\mathcal{F} > 0.$$

PROOF. Proceed by induction on r. If $r = 0$, then $\int \mathcal{F} = \dim \mathrm{H}^0(\mathcal{F})$ essentially by definition, and $H^0(\mathcal{F}) \neq 0$ since $\mathcal{F} \notin \mathbf{K}_{r-1}$ by hypothesis.

Suppose $r \geq 1$. Owing to Proposition B.12, we may assume $\mathcal{F} = \mathcal{O}_Y$ where Y is integral. Owing to Part (2) of Theorem B.9, we may replace $\mathcal{L}_r$ by a multiple, and so assume it is very ample. Then, for the corresponding embedding of Y, a hyperplane section D is a nonempty effective divisor such that $\mathcal{O}_Y(D) \simeq \mathcal{L}_r|Y$. Hence the assertion results from Proposition B.13 and the induction hypothesis. □

LEMMA B.15. Let $g\colon X' \to X$ be an S-map. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and let $\mathcal{F} \in \mathbf{F}_r(X'/S)$. Then

$$\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r) g_*\mathcal{F}.$$

PROOF. Let $\mathcal{G} \in \mathbf{F}_r(X'/S)$. Then, by hypothesis, $\mathrm{Supp}\,\mathcal{G}$ is proper over an Artin subscheme of S, and $\dim \mathrm{Supp}\,\mathcal{G} \leq r$; furthermore, X/S is separated. Hence, the restriction $g|\mathrm{Supp}\,\mathcal{G}$ is proper; so $g(\mathrm{Supp}\,\mathcal{G})$ is closed. And by the dimension theory of schemes of finite type over Artin schemes, $\dim g(\mathrm{Supp}\,\mathcal{G}) \leq r$. Therefore, $\mathrm{R}^i g_*\mathcal{G} \in \mathbf{F}_r(X/S)$ for all i.

Define a map $\mathbf{F}_r(X'/S) \to \mathbf{K}_r(X/S)$ by $\mathcal{G} \mapsto \sum_{i=0}^{r}(-1)^i \mathrm{R}^i g_*\mathcal{G}$. It induces a homomorphism $\mathrm{R}g_*\colon \mathbf{K}_r(X'/S) \to \mathbf{K}_r(X/S)$. And $\chi(Rg_*(\mathcal{G})) = \chi(\mathcal{G})$ owing to the Leray Spectral Sequence [**EGAIII1**, **0**-12.2.4] and to the additivity of χ [**EGAIII1**, **0**-11.10.3]. Furthermore, $\mathcal{L} \otimes \mathrm{R}^i g_*(\mathcal{G}) \xrightarrow{\sim} \mathrm{R}^i g_*(g^*\mathcal{L} \otimes \mathcal{G})$ for any $\mathcal{L} \in \mathrm{Pic}(X)$ by [**EGAIII1**, **0**-12.2.3.1]. Hence $c_1(\mathcal{L})Rg_*(\mathcal{G}) = Rg_*(c_1(g^*\mathcal{L})\mathcal{G})$. Therefore,

$$\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathrm{R}g_*\mathcal{F}.$$

Finally, $\mathrm{R}g_*\mathcal{F} \equiv \mathrm{g}_*\mathcal{F} \bmod \mathbf{K}_{r-1}(X/S)$, because $\mathrm{R}^i g_*\mathcal{F} \in \mathbf{F}_{r-1}$ for $i \geq 1$ since, if $W \subset X'$ is the locus where $\mathrm{Supp}\,\mathcal{F} \to X$ has fibers of dimension at least 1, then $\dim g(W) \leq r-1$. So Part (1) of Theorem B.9 yields the asserted formula. □

PROPOSITION B.16 (Projection Formula). *Let $g\colon X' \to X$ be an S-map. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$. Let $Y' \subset X'$ be an integral subscheme with $\mathcal{O}_{Y'} \in \mathbf{F}_r(X'/S)$. Set $Y := gY' \subset X$, give Y its induced reduced structure, and let* $\deg(Y'/Y)$ *be the degree of the function field extension if finite and be* 0 *if not. Then*

$$\int c_1(g^*\mathcal{L}_1)\cdots c_1(g^*\mathcal{L}_r)[Y'] = \deg(Y'/Y)\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)[Y].$$

PROOF. Lemma B.4 yields $g_*\mathcal{O}_{Y'} \equiv \deg(Y'/Y)[Y] \bmod \mathbf{K}_{r-1}(X/S)$. So the assertion results from Lemma B.15 and from Part (1) of Theorem B.9. □

PROPOSITION B.17. *Assume S is the spectrum of a field, and let T be the spectrum of an extension field. Let $\mathcal{L}_1, \dots, \mathcal{L}_r \in \mathrm{Pic}(X)$ and $\mathcal{F} \in \mathbf{F}_r(X/S)$. Then*

$$\int c_1(\mathcal{L}_{1,T})\cdots c_1(\mathcal{L}_{r,T})\mathcal{F}_T = \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

PROOF. The base change $T \to S$ preserves short exact sequences. So it induces a homomorphism $\kappa\colon \mathbf{K}_r(X/S) \to \mathbf{K}_r(X_T/T)$. Plainly κ preserves the Euler characteristic. The assertion now follows. □

PROPOSITION B.18. *Let* $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \operatorname{Pic}(X)$. *Let* $\mathcal{F}$ *be a flat coherent sheaf on* X. *Assume* $\operatorname{Supp}\mathcal{F}$ *is proper and of relative dimension* r. *Then the function*

$$y \mapsto \int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F}_y$$

is locally constant.

PROOF. The assertion results from Definition B.8 and [**Mum70**, Cor., top p. 50]. □

DEFINITION B.19. Let $\mathcal{L}, \mathcal{N} \in \operatorname{Pic}(X)$. Call them *numerically equivalent* if $\int c_1(\mathcal{L})[Y] = \int c_1(\mathcal{N})[Y]$ for all closed integral curves $Y \subset X$ with $\mathcal{O}_Y \in \mathbf{F}_1$.

PROPOSITION B.20. *Let* $\mathcal{L}_1, \ldots, \mathcal{L}_r; \mathcal{N}_1, \ldots, \mathcal{N}_r \in \operatorname{Pic}(X)$ *and* $\mathcal{F} \in \mathbf{K}_r$. *If* $\mathcal{L}_j$ *and* $\mathcal{N}_j$ *are numerically equivalent for each* j, *then*

$$\int c_1(\mathcal{L}_1)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)\cdots c_1(\mathcal{N}_r)\mathcal{F}.$$

PROOF. If $r = 1$, then the assertion results from Lemma B.12. Suppose $r \geq 2$. Then $c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F} \in \mathbf{K}_1$ by Lemma B.5. Hence

$$\int c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)c_1(\mathcal{L}_2)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

Similarly, $c_1(\mathcal{N}_1)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F} \in \mathbf{K}_1$, and so

$$\int c_1(\mathcal{N}_1)c_1(\mathcal{L}_2)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F} = \int c_1(\mathcal{N}_1)c_1(\mathcal{N}_2)c_1(\mathcal{L}_3)\cdots c_1(\mathcal{L}_r)\mathcal{F}.$$

Continuing in this fashion yields the assertion. □

PROPOSITION B.21. *Let* $g\colon X' \to X$ *be an* S*-map. Let* $\mathcal{L}, \mathcal{N} \in \operatorname{Pic}(X)$.

(1) *If* $\mathcal{L}$ *and* $\mathcal{N}$ *are numerically equivalent, then so are* $g^*\mathcal{L}$ *and* $g^*\mathcal{N}$.

(2) *Conversely, when* g *is proper and surjective, if* $g^*\mathcal{L}$ *and* $g^*\mathcal{N}$ *are numerically equivalent, then so are* $\mathcal{L}$ *and* $\mathcal{N}$.

PROOF. Let $Y' \subset X'$ be a closed integral curve with $\mathcal{O}_{Y'} \in \mathbf{F}_1(X'/S)$. Set $Y := g(Y')$ and give Y its induced reduced structure. Then Proposition B.16 yields

$$\int c_1(g^*\mathcal{L})[Y'] = \deg(Y'/Y)\int c_1(\mathcal{L})[Y] \text{ and}$$
$$\int c_1(g^*\mathcal{N})[Y'] = \deg(Y'/Y)\int c_1(\mathcal{N})[Y].$$

Part (1) follows.

Conversely, suppose g is proper and surjective. Let $Y \subset X$ be a closed integral curve with $\mathcal{O}_Y \in \mathbf{F}_1(X/S)$. Then Y is a complete curve in the fiber X_s over a closed point $s \in S$. Hence, since g is proper, there exists a complete curve Y' in X'_s mapping onto Y. Indeed, let $y \in Y$ be the generic point, and $y' \in g^{-1}Y$ a closed point; let Y' be the closure of y' given Y' its induced reduced structure. Plainly $\mathcal{O}_{Y'} \in \mathbf{F}_1(X'/S)$ and $\deg(Y'/Y) \neq 0$. The two equations displayed above now yield Part (2). □

DEFINITION B.22. Assume S is Artin, and X a proper curve. Let $\mathcal{L} \in \operatorname{Pic}(X)$. Define its *degree* $\deg(\mathcal{L})$ by the formula

$$\deg(\mathcal{L}) := \int c_1(\mathcal{L}).$$

Let D be a divisor on X. Define its *degree* $\deg(D)$ by $\deg(D) := \deg(\mathcal{O}_X(D))$.

PROPOSITION B.23. *Assume S is Artin, and X a proper curve.*
(1) *The map* $\deg\colon \operatorname{Pic}(X) \to \mathbb{Z}$ *is a homomorphism.*
(2) *Let $D \subset X$ be an effective divisor. Then*

$$\deg(D) = \dim \mathrm{H}^0(\mathcal{O}_D).$$

(3) (Riemann's Theorem) *Let $\mathcal{L} \in \operatorname{Pic}(X)$. Then*

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X).$$

(4) *Suppose X is integral, and let $g\colon X' \to X$ be the normalization map. Then*

$$\deg(\mathcal{L}) = \deg(g^*\mathcal{L}).$$

PROOF. Part (1) results from Theorem B.9 (2). And Part (2) results from Lemma B.13. As to Part (3), note $\deg(\mathcal{L}^{-1}) = -\deg(\mathcal{L})$ by Part (1). And the definitions yield $\deg(\mathcal{L}^{-1}) = \chi(\mathcal{O}_X) - \chi(\mathcal{L})$. Thus Part (3) holds. Finally, Part (3) results from the definition and the Projection Formula. □

DEFINITION B.24. Assume S is Artin, and X a proper surface. Given a divisor D on X, set

$$p_a(D) := 1 - \chi\big(c_1(\mathcal{O}_X(D))\,\mathcal{O}_X\big).$$

PROPOSITION B.25. *Assume S is Artin, and X a proper surface. Let D and E be divisors on X. Then*

$$p_a(D+E) = p_a(D) + p_a(E) + (D \cdot E) - 1.$$

Furthermore, if D is effective, then

$$p_a(D) = 1 - \chi(\mathcal{O}_D);$$

in other words, $p_a(D)$ is equal to the arithmetic genus *of D.*

PROOF. The assertions result from Lemmas B.3 and B.2. □

PROPOSITION B.26 (Riemann–Roch for surfaces). *Assume S is the spectrum of a field, and X is a reduced, projective, equidimensional, Cohen–Macaulay surface. Let ω be a dualizing sheaf, and set $\mathcal{K} := \omega - \mathcal{O}_X$. Let D be a divisor on X. Then $\mathcal{K} \in \mathbf{K}_1$; furthermore,*

$$p_a(D) = \frac{(D^2) + (D \cdot \mathcal{K})}{2} + 1 \quad \textit{and} \quad \chi(\mathcal{O}_X(D)) = \frac{(D^2) - (D \cdot \mathcal{K})}{2} + \chi(\mathcal{O}_X).$$

If X/S is Gorenstein, that is, $\omega = \mathcal{O}_X(K)$ for some "canonical" divisor K, then

$$p_a(D) = \frac{\big(D \cdot (D+K)\big)}{2} + 1 \quad \textit{and} \quad \chi(\mathcal{O}_X(D)) = \frac{\big(D \cdot (D-K)\big)}{2} + \chi(\mathcal{O}_X).$$

PROOF. Since X is reduced, ω is isomorphic to $\mathcal{O}_X$ on a dense open subset of X by [**AK70**, (2.8), p. 8]. Hence $\mathcal{K} \in \mathbf{K}_1$.

Set $\mathcal{L} := \mathcal{O}_X(D)$. Then $(D^2) := \int c_1(\mathcal{L})^2 = -\int c_1(\mathcal{L})c_1(\mathcal{L}^{-1})$ by Parts (1) and (2) of Theorem B.9. Now, the definitions yield

$$\begin{aligned} c_1(\mathcal{L})(-c_1(\mathcal{L}^{-1})\mathcal{O}_X + \mathcal{K}) &= c_1(\mathcal{L})(\mathcal{L} - 2\mathcal{O}_X + \omega) \\ &= \mathcal{L} + \omega - 3\mathcal{O}_X + 2\mathcal{L}^{-1} - \mathcal{L}^{-1} \otimes \omega \quad \text{and} \end{aligned}$$

$$c_1(\mathcal{L})(-c_1(\mathcal{L}^{-1})\mathcal{O}_X - \mathcal{K}) = c_1(\mathcal{L})(\mathcal{L} - \omega) = \mathcal{L} - \omega - \mathcal{O}_X + \mathcal{L}^{-1} \otimes \omega.$$

But, $\mathrm{H}^i(\mathcal{L})$ is dual to $\mathrm{H}^{2-i}(\mathcal{L}^{-1} \otimes \omega)$ by duality theory; see [**Har77**, Cor. 7.7, p. 244], where k needn't be taken algebraically closed. So $\chi(\mathcal{L}) = \chi(\mathcal{L}^{-1} \otimes \omega)$. Similarly, $\chi(\mathcal{O}_X) = \chi(\omega)$. Therefore,

$$(D^2) + (D \cdot \mathcal{K}) = 2(\chi(\mathcal{L}^{-1}) - \chi(\mathcal{O}_X)) \quad \text{and} \quad (D^2) - (D \cdot \mathcal{K}) = 2(\chi(\mathcal{L}) - \chi(\mathcal{O}_X)).$$

Now, $-c_1(\mathcal{O}_X(D))\,\mathcal{O}_X = \mathcal{L}^{-1} - \mathcal{O}_X$. The first assertion follows.

Suppose ω is invertible. Then $-c_1(\omega^{-1})\mathcal{O}_X = \mathcal{K}$ owing to the definitions. And $-\int c_1(\mathcal{L})c_1(\omega^{-1}) = \int c_1(\mathcal{L})c_1(\omega)$ by Part (2) of Theorem B.9. Therefore, $(D \cdot \mathcal{K}) = (D \cdot K)$. Hence Part (2) of Theorem B.9 yields the second assertion. □

THEOREM B.27 (Hodge Index). Assume S is the spectrum of a field, and X is a geometrically irreducible complete surface. Assume there is an $\mathcal{H} \in \mathrm{Pic}(X)$ such that $\int c_1(\mathcal{H})^2 > 0$. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Assume $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$ and $\int c_1(\mathcal{L})^2 \geq 0$. Then $\mathcal{L}$ is numerically equivalent to $\mathcal{O}_X$.

PROOF. We may extend the ground field to its algebraic closure owing to Proposition B.17. Furthermore, we may replace X by its reduction; indeed, the hypotheses are preserved due to Lemma B.12, and the conclusion is preserved due to Definition B.19.

By Chow's Lemma, there is a surjective map $g\colon X' \to X$ where X' is an integral projective surface. Furthermore, we may replace X' by its normalization. Now, we may replace X by X' and $\mathcal{H}$ and $\mathcal{L}$ by $g^*\mathcal{H}$ and $g^*\mathcal{L}$. Indeed, the hypotheses are preserved due to the Projection Formula, Proposition B.16. And the conclusion is preserved due to Part (2) of Proposition B.21.

By way of contradiction, assume that there exists a closed integral subscheme $Y \subset X$ such that $\int c_1(\mathcal{L})\mathcal{O}_Y \neq 0$. Let $g\colon X' \to X$ be the blowing-up along Y, and $E := g^{-1}Y \subset X$ the exceptional divisor. Let $E_1, \ldots, E_s$ be the irreducible components of E, and give them their induced reduced structure.

Since X is normal, it has only finitely may singular points. Off them, Y is a divisor, and g is an isomorphism. Hence one of the E_i, say E_1 maps onto Y, and the remaining E_i map onto points. Therefore, $\int c_1(g^*\mathcal{L})[E_1] = \int c_1(g\mathcal{L})[Y]$ and $\int c_1(g^*\mathcal{L})[E_i] = 0$ for $i \geq 2$ by the Projection Formula. Hence Lemma B.12 yields $\int c_1(g^*\mathcal{L})[E] = \int c_1(\mathcal{L})[Y]$. The latter is nonzero by the new assumption, and the former is equal to $\int c_1(g^*\mathcal{L})c_1(\mathcal{O}_{X'}(E))$ by Lemma B.13.

Set $\mathcal{M} := \mathcal{O}_{X'}(E)$. Then $\int c_1(g^*\mathcal{L})c_1(\mathcal{M}) \neq 0$. Moreover, by the Projection Formula, $\int c_1(g^*\mathcal{H}) > 0$ and $\int c_1(g^*\mathcal{L})c_1(g^*\mathcal{H}) = 0$ and $\int c_1(g^*\mathcal{L})^2 \geq 0$. Let's prove this situation is absurd. First, replace X by X' and $\mathcal{H}$ and $\mathcal{L}$ by $g^*\mathcal{H}$ and $g^*\mathcal{L}$.

Let $\mathcal{G}$ be an ample invertible sheaf on X. Set $\mathcal{H}_1 := \mathcal{G}^{\otimes m} \otimes \mathcal{M}$. Then

$$\textstyle\int c_1(\mathcal{L})c_1(\mathcal{H}_1) = m \int c_1(\mathcal{L})c_1(\mathcal{G}) + \int c_1(\mathcal{L})c_1(\mathcal{M})$$

by additivity (see Part (2) of Theorem B.9). Now, $\int c_1(\mathcal{L})c_1(\mathcal{M}) \neq 0$. Hence there is an $m > 0$ so that $\int c_1(\mathcal{L})c_1(\mathcal{H}_1) \neq 0$ and so that $\mathcal{H}_1$ is ample.

Set $\mathcal{L}_1 := \mathcal{L}^{\otimes p} \otimes \mathcal{H}^{\otimes q}$. Since $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$, additivity yields

$$\textstyle\int c_1(\mathcal{L}_1)^2 = p^2 \int c_1(\mathcal{L})^2 + q^2 \int c_1(\mathcal{H})^2, \text{ and}$$

$$\textstyle\int c_1(\mathcal{L}_1)c_1(\mathcal{H}_1) = p \int c_1(\mathcal{L})c_1(\mathcal{H}_1) + q \int c_1(\mathcal{H})c_1(\mathcal{H}_1).$$

Since $\int c_1(\mathcal{L})c_1(\mathcal{H}_1) \neq 0$, there are p, q with $q \neq 0$ so that $\int c_1(\mathcal{L}_1)c_1(\mathcal{H}_1) = 0$. Then $\int c_1(\mathcal{L}_1)^2 > 0$ since $\int c_1(\mathcal{L})^2 \geq 0$ and $\int c_1(\mathcal{H})^2 > 0$. Replace $\mathcal{L}$ by $\mathcal{L}_1$ and $\mathcal{H}$ by $\mathcal{H}_1$. Then $\mathcal{H}$ is ample, $\int c_1(\mathcal{L})c_1(\mathcal{H}) = 0$ and $\int c_1(\mathcal{L})^2 > 0$.

Set $\mathcal{N} := \mathcal{L}^{\otimes n} \otimes \mathcal{H}^{-1}$ and $\mathcal{H}_1 := \mathcal{L} \otimes \mathcal{H}^a$. Take $a > 0$ so that $\mathcal{H}_1$ is ample. By additivity,

$$\textstyle\int c_1(\mathcal{N})c_1(\mathcal{H}_1) = n \int c_1(\mathcal{L})^2 - a \int c_1(\mathcal{H})^2.$$

Take $n > 0$ so that $\int c_1(\mathcal{N})c_1(\mathcal{H}_1) > 0$. Then additivity and Proposition B.14 yield

$$\textstyle\int c_1(\mathcal{N})c_1(\mathcal{H}) = -\displaystyle\int c_1(\mathcal{H})^2 < 0,$$

$$\int c_1(\mathcal{N})^2 = n^2 \int c_1(\mathcal{L})^2 + \int c_1(\mathcal{H})^2 > 0.$$

But this situation stands in contradiction to the next lemma. □

LEMMA B.28. Assume S is the spectrum of a field, and X is an integral surface. Let $\mathcal{N} \in \operatorname{Pic}(X)$, and assume $\int c_1(\mathcal{N})^2 > 0$. Then these conditions are equivalent:

(i) For every ample sheaf $\mathcal{H}$, we have $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$.
(i′) For some ample sheaf $\mathcal{H}$, we have $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$.
(ii) For some $n > 0$, we have $\mathrm{H}^0(\mathcal{N}^{\otimes n}) \neq 0$.

PROOF. Suppose (ii) holds. Then there exists an effective divisor D such that $\mathcal{N}^{\otimes n} \simeq \mathcal{O}_X(D)$. And $D \neq 0$ since $\int c_1(\mathcal{N})^2 > 0$. Hence (i) results as follows:

$$\textstyle\int c_1(\mathcal{N})c_1(\mathcal{H}) = \int c_1(\mathcal{H})c_1(\mathcal{N}) = \int c_1(\mathcal{H})[D] > 0$$

by symmetry, by Lemma B.13, and by Proposition B.14.

Trivially, (i) implies (i′). Finally, assume (i′), and let's prove (ii). Let ω be a dualizing sheaf for X; then ω is torsion free of rank 1, and $\mathrm{H}^2(\mathcal{L})$ is dual to $\operatorname{Hom}(\mathcal{L}, \omega)$ for any coherent sheaf $\mathcal{F}$ on X; see [**FGA**, p. 149-17], [**AK70**, (1.3), p. 5, and (2.8), p. 8], and [**Har77**, Prp, 7.2, p. 241]. Set $\mathcal{K} := \omega - \mathcal{O}_X \in \mathbf{K}_1$.

Suppose $\mathcal{L}$ is invertible and $\mathrm{H}^2(\mathcal{L})$ is nonzero. Then there is a nonzero map $\mathcal{L} \to \omega$, and it is injective since X is integral. Let $\mathcal{F}$ be its cokernel. Then

$$\mathcal{K} = \mathcal{F} - c_1(\mathcal{L}^{-1})\mathcal{O}_X \text{ in } \mathbf{K}_1.$$

Hence Proposition B.14, symmetry, and additivity yield

$$\textstyle\int c_1(\mathcal{H})\mathcal{K} \geq \int c_1(\mathcal{L})c_1(\mathcal{H}).$$

Take $\mathcal{L} := \mathcal{N}^{\otimes n}$. Then $\int c_1(\mathcal{H})c_1(\mathcal{L}) = n \int c_1(\mathcal{H})c_1(\mathcal{N})$ by additivity. But $\int c_1(\mathcal{N})c_1(\mathcal{H}) > 0$ by hypothesis. Hence $\mathrm{H}^2(\mathcal{N}^{\otimes n})$ vanishes for $n \gg 0$. Now,

$$\textstyle\chi(\mathcal{N}^{\otimes n}) = \int c_1(\mathcal{N})^2 \binom{n+1}{2} + a_1 n + a_0$$

for some a_1, a_0 by Snapper's Theorem, Theorem B.7. But $\int c_1(\mathcal{N})^2 > 0$ by hypothesis. Therefore, (ii) holds. □

COROLLARY B.29. Assume S is the spectrum of a field, and X a geometrically irreducible projective r-fold with $r \geq 2$. Let D, E be effective divisors, with E possibly trivial. Assume D is ample. Then $D + E$ is connected.

PROOF. Plainly we may assume the ground field is algebraically closed and X is reduced. Fix $n > 0$ so that $nD + E$ is ample; plainly we may replace D and E by $nD + E$ and 0. Proceeding by way of contradiction, assume D is the disjoint union of two closed subschemes D_1 and D_2. Plainly D_1 and D_2 are divisors; so $D = D_1 + D_2$.

Proceed by induction on r. Suppose $r = 2$. Then, since D_1 and D_2 are disjoint, $(D_1 \cdot D_2) = 0$ by Lemma B.13. Now, D is ample. Therefore, Proposition B.14 yields

$$(D_1^2) = (D.D_1) > 0 \text{ and } (D_2^2) = (D.D_2) > 0.$$

These conclusions contradict Theorem B.27 with $\mathcal{H} := \mathcal{O}_X(D_1)$ and $\mathcal{L} := \mathcal{O}_X(D_2)$.

Finally, suppose $r \geq 3$. Let H be a general hyperplane section of X. Then H is integral by Bertini's Theorem [**Sei50**, Thm. 12, p. 374]. And H is not a component of D. Set $D' := D \cap H$ and $D'_i := D_i \cap H$. Plainly D'_1 and D'_2 are disjoint, and $D' = D'_1 + D'_2$; also, D' is ample. So induction yields the desired contradiction. □

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Alexander Grothendieck introduced many concepts into algebraic geometry; they turned out to be astoundingly powerful and productive and truly revolutionized the subject. Grothendieck sketched his new theories in a series of talks at the Séminaire Bourbaki between 1957 and 1962 and collected his write-ups in a volume entitled "Fondements de la Géométrie Algébrique," known as FGA.

Much of FGA is now common knowledge; however, some of FGA is less well known, and its full scope is familiar to few. The present book resulted from the 2003 "Advanced School in Basic Algebraic Geometry" at the ICTP in Trieste, Italy. The book aims to fill in Grothendieck's brief sketches. There are four themes: descent theory, Hilbert and Quot schemes, the formal existence theorem, and the Picard scheme. Most results are proved in full detail; furthermore, newer ideas are introduced to promote understanding, and many connections are drawn to newer developments.

The main prerequisite is a thorough acquaintance with basic scheme theory. Thus this book is a valuable resource for anyone doing algebraic geometry.

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