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Yanze Wang

Basics of *Quot* and construction of *Gr*.

1 Introduction

Definition 1.1. *Let* S *be a* noetherian *scheme, then we define a functor* $Quot_{\mathcal{E}/X/S}$ *from category of* locally noetherian schemes *over* S *to category of sets, where* $\pi : X \to S$ *is a finite type scheme, and* $\mathcal{E}$ *is a coherent sheaf over* X*:*

$$T \mapsto \{\mathcal{E}_T \twoheadrightarrow \mathcal{F} | \mathcal{F} \in \mathfrak{Coh}(X_T) \text{ is parametrised by } T\} / \sim$$

"$\sim$" means commutative diagram

$$\begin{tikzcd} \mathcal{E}_T \arrow[r, two heads] \arrow[d, equal] & \mathcal{F} \arrow[d, "\sim"] \\ \mathcal{E}_T \arrow[r, two heads] & \mathcal{F}' \end{tikzcd}$$

parametrised by T *means schematic support of* $\mathcal{F}$ *is proper over* T*, and* $\mathcal{F}$ *is flat over* T*.*

In particular,

1. $Hilb_{X/S} := Qout_{\mathcal{O}_X/X/S}$
   $\mathcal{O}_X \twoheadrightarrow \mathcal{F}$ *gives a closed subscheme* $i : Y \hookrightarrow X$ *defined by ideal sheaf* $\ker(\mathcal{O}_X \twoheadrightarrow \mathcal{F})$*. In fact,* $i_*\mathcal{O}_Y = \mathcal{F}$*. Therefore* $Hilb_{X/S}(T)$ *parametrised all closed subscheme of* X_T *that are proper and flat over* T*.*

2. $Qout_{\mathcal{O}^r_{\mathbb{P}^n}} := Qout_{\mathcal{O}^r_{\mathbb{P}^n}/\mathbb{P}^n/\operatorname{Spec}\mathbb{Z}}$.

3. $Hilb_{\mathbb{P}^n} := Hilb_{\mathbb{P}^n/\operatorname{Spec}\mathbb{Z}}$.

Functor $Quot_{\mathcal{E}/X/S}$ is a zariski and fppf sheaf, and is proper over S.

Definition 1.2. *Notations as above, let* $P \in \mathbb{Q}[t]$ *be a polynomial and* $\mathcal{L}$ *be a invertible sheaf over* X*, then we have functor* $Quot^{P,\mathcal{L}}_{\mathcal{E}/X/S}$ *s.t.* $\mathcal{F} \in Quot^{P,\mathcal{L}}_{\mathcal{E}/X/S}(T)$ *satisfying* $\forall t \in T, \mathcal{F}_t$ *has hilbert polynomial* P *w.r.t.* $\mathcal{L}_t$*.*

In particular, we have a functor $Hilb^P_r = Hilb^{P,\mathcal{O}(1)}_{\mathbb{P}^r/\mathbb{Z}}$*.*

Clearly we have coproduct

$$Quot_{\mathcal{E}/X/S} = \coprod_{P \in \mathbb{Q}[t]} Quot^{P,\mathcal{L}}_{\mathcal{E}/X/S}$$

Functor $Quot_{\mathcal{E}/X/S}$ is a zariski and fpqc sheaf, and is proper over S.

2 Grassmannian

We can define Grassmannian scheme as a special Quot:

Definition 2.1.
$$Gr_S(d,r) := Qout^{d,\mathcal{O}}_{\mathcal{O}^r/S/S}$$
In particular, $S = \operatorname{Spec}\mathbb{Z}$. $Gr(r,d) := Gr_{\mathbb{Z}}(r,d) := Quot^{d,\mathcal{O}_{\mathbb{Z}}}_{\mathcal{O}^r_{\mathbb{Z}}/\mathbb{Z}/\mathbb{Z}}$

Notice that if $\mathcal{O}^r \twoheadrightarrow \mathcal{F}$ and $\mathcal{F}$ has constant hilbert polynomial d w.r.t $\mathcal{O}$, then $\mathcal{F}$ is alocally free sheaf of rank d.

To construct Quot scheme, we need to show $Gr(r,d)$ is representable.

2.1 construction

Idea: let V be a vector space of dimension r, then $Gr(d,V) = Gr(d,r)$ is set of all d-dimensional subvector spaces of V. One can verify a subspace by a $d \times r$ metrix of rank d, of course up to equivalence. Then $Gr(d,r) = \bigcup_I \{M_{d\times r}$: I-th minor is of rank d $\}/\sim = \bigcup U^I$, where $I \subset \{1,2,\ldots,r\}, \#I = d$. For each equivalent class in of U^I, one can choose a metrix M representing it, whose I-th minor is E, then $U^I \cong k^{d\times(r-d)}$, i.e. coordinates outside the I-th minor are free. However $\bigcup U^I$ is not a disjoint union, we have to glue these sets.

Now construct $\mathcal{G}r_{\mathbb{Z}}(r,d)$ as follows: Let X^I be a matrix whose I-th minor is identity matrix E, and other coordinates x^I_{pq} are independent variables, and X^I_J means its J-th minor. Let $\mathbb{Z}[x^I]$ be the polynomial ring in variables x^I_{pq} and let $U^I := \operatorname{Spec}\mathbb{Z}[x^I] \cong \mathbb{A}^{d\times(r-d)}$, an A-point of which is a matrix $M^I(A)$ with coordinates in ring A and I-th minor being identity martix. Let $P^I_J = \det X^I_J \in \mathbb{Z}[x^I]$, then $U^I_J = \operatorname{Spec}\mathbb{Z}[x^I, 1/P^I_J]$ is an open subscheme of U^I, whose A-point is a matrix $M^I(A)$ with J-th minor invertible. Then we will glue U^I through isomorphisms $U^I_J \to U^J_I$. For any ring A, there is a functorial bijection $U^I_J(A) \to U^J_I(A), M^I(A) \mapsto (M^I_J(A))^{-1}M^I(A)$, hence a natural isomorphism $\operatorname{Hom}(\operatorname{Spec} -, U^I_J) \to \operatorname{Hom}(\operatorname{Spec} -, U^J_I)$, and therefore an isomorphism $U^I_J \to U^J_I$ (acturally this is the map $X^I \mapsto (X^I_J)^{-1}\cdot X^I$). Furthermore, over U^I we have a quotient of locally free sheaves

$$\bigoplus_{i=1}^r \mathcal{O}e_i \to \bigoplus_{j=1}^d \mathcal{O}u_j, e_i \mapsto \sum_{j=1}^d x_{ji}u_j$$

Gluing U^I and the quotient we have a scheme $\mathcal{G}r(r,d)$ and a quotient $\bigoplus_{i=1}^r \mathcal{O} \to \mathcal{U}$, where $\mathcal{U}$ is locally free of rank d (trival over open subschemes U^I).

2.2 representibility

To show this scheme indeed represents the functor, we have to show the quotient is universal, which is difficult. However, one can prove these by a criterion for representability.

If a functor $F : \mathfrak{Sch}_S \to \mathfrak{Set}$ is representable, then it is a sheaf w.r.t. zariski and fpqc topology on $\mathfrak{Sch}$ (FGA section 2.3.6). Conversely, we have a criterion for a functor:

Definition 2.2. *Let $F \to G$ be a natural transform of functors in $[\mathfrak{Sch}^{op}, \mathfrak{Set}]$. It is a* <u>*open/closed/locally closed subfunctor*</u>*, if for any scheme T, $F(T) \to G(T)$ is injective, and $F \times_G h_T$ is represented by an open/closed/locally closed subscheme T' of T.*

Open subfunctors $\{F_i\}$ of F forms an <u>*open covering*</u>*, if for any scheme T, corresponding open subschemes T_i forms an open covering of T.*

Theorem 2.3. *A functor $F : \mathfrak{Sch}_S \to \mathfrak{Set}$ is representable, if*

1. *F is a sheaf w.r.t. zatiski topology over $\mathfrak{Sch}_S$, and*

2. *F has an open covering $\{F_i\}$ s.t. each F_i is representable.*

Proof. <u>Hint</u>: Glue schemes X_i in the way functors F_i.

Assume X_i represents F_i and $x_i \in F_i(X_i)$ corresponds to 1_{X_i}. For any i, j, since F_j is an open subscheme, $h_{X_i} \times_F F_j$ is represented by an open subscheme U_{ij} of X_i, and is isomorphic to $F_{ij} := F_i \times_F F_j$.

$$\begin{tikzcd}
h_{U_{ij}} \arrow[r] \arrow[d, "\sim"] & h_{X_i} \arrow[d, "\sim"] \\
F_{ij} \arrow[r] \arrow[d] & F_i \arrow[d] \\
F_j \arrow[r] & F
\end{tikzcd}$$

However, we also have $F_{ij} \cong h_{U_{ji}}$ for an open subscheme U_{ji} of X_{ji}. Therefore we have an isomorphism $f_{ij} : U_{ij} \to U_{ji}$ and its inverse$f_{ji} : U_{ji} \to U_{ij}$, satisfying cocycle condition. Gluing X_i through f_{ij} we obtain a scheme X. To show X represents F, using Yoneda lemma: for any scheme T and $h_T \to F$ given by a element $y \in F(T)$, we have an open covering T_i ($h_{T_i} \cong h_T \times_F F_i$) of T and $(f_i : T_i \to X_i) \in F_i(T_i)$. Gluing f_i we have $f : T \to X$. This means $\mathrm{Hom}\,(h_T, F) = \mathrm{Hom}\,(h_T, h_X)$ for any T. □

Back to the Grassmannin, using the theorem above. Define subfuncors F_I of $Gr(d, r)$: for each $I = \{i_1, \dots, i_d\} \subset \{1, \dots, r\}$, we have an injection $s_I : \bigoplus_{k=1}^d \mathcal{O}e_{i_k} \to \bigoplus_{i=1}^r \mathcal{O}e_i$. Define subfuctor

$$F_I(T) = \{< \mathcal{F}, q > |\text{composition } \bigoplus_{k=1}^{d} \mathcal{O}e_{i_k} \xrightarrow{s_I} \bigoplus_{i=1}^{r} \mathcal{O}e_i \xrightarrow{q} \mathcal{F} \text{ is surjective}\}$$

Need to show F_I are representable and form an open covering.

1. This is represented by $\mathbb{A}^{d(r-d)}$(In fact this is U^I defined above): take an open affine covering $\{T_\alpha \cong \operatorname{Spec} A_\alpha\}$ of T such that $\mathcal{F}$ is trival over T_α, then we have a surjection $\bigoplus_{k=1}^d A_\alpha e_{i_k} \to \mathcal{F}(T_\alpha) \cong A_\alpha^d$. But both are free A_α-modules of rank d, this in fact is an isomorphism, and images of e_{ij} form a basis u_j of $\mathcal{F}(T_\alpha) \cong \bigoplus_{j=1}^d A_\alpha u_j$. Then $q|_{T_\alpha} : \bigoplus_{i=1}^r A_\alpha e_i \to \bigoplus_{j=1}^d A_\alpha u_j$ is give by $e_i \mapsto \sum_{j=1}^d a_{ji} u_j, a_{ji} \in A_\alpha$, cooresponds to an A_α-point of $\mathbb{A}^{d(r-d)}$, i.e. a morphism $T_\alpha \to \mathbb{A}^{d(r-d)}$. Gluing all these maps we have a morphism $T \to \mathbb{A}^{d(r-d)}$, which implies $\mathbb{A}^{d(r-d)}$ represents F_I.

2. Let T be a scheme and $h_T \to Gr$ be a natural transform given by a quotient of locally free sheaves $\bigoplus_{i=1}^r \mathcal{O}_T e_i \xrightarrow{q} \mathcal{F}$ over T. Let q_I be the composition of $(\bigoplus_{k=1}^d \mathcal{O}_T e_{i_k} \xrightarrow{s_I} \bigoplus_{i=1}^r \mathcal{O}_T e_i \xrightarrow{q} \mathcal{F})$ and $\mathcal{K} = \operatorname{coker}(q_I)$. Then q_I is surjective over $T_I := T - \operatorname{Supp}\mathcal{K} \underset{open}{\subset} T$. This open subscheme actually has a universal property that a map $U \to T$ factors through if and only if pull back of q_I is surjective, which implies $h_T \times_{Gr} F_I \cong h_{T_I}$, hence F_I is an open subfunctor.

3. To show F_I forms an open covering, need to show T_I covers T. Indeed, any point $t \in T$, pull back of q is a surjection $\bigoplus_{i=1}^r k(t) e_i \xrightarrow{q} k(t)^d$ of vector space over the reside field $k(t)$. There is a subspace $\bigoplus_{k=1}^d k(t) e_{i_k}$ for some $I = \{i_1, \ldots, i_d\}$ and the compotision is an isomorphism. This implies $t \in T_I$, hence T_I covers T.

By the criterion for representibility, one can glue these $\mathbb{A}^{d(r-d)}$s and represents $Gr(r,d)$. In fact, this is exact the same way we construct $\mathcal{G}r(r,d)$ above.

Construct *Quot*.

3 Construct

3.1 lemmas

We will show that functor $Quot_{\mathcal{E}/X/S}$ is represented for some special cases.

Theorem 3.1. *FGA,5.15 Let S be a noetherian scheme, and $\mathcal{V}, \mathcal{W}$ two locally free sheaves over it. Take $\pi : X = \mathbb{P}(V) \to S$ and $\mathcal{E} = \pi^*\mathcal{W}, \mathcal{L} = \mathcal{O}_X(1)$, then $Q = Quot_{\mathcal{E}/X/S}^{\Phi,\mathcal{L}}$ is represented.*

Proof. Hint: We will prove it in several steps.

1. Given a morphism $f : T \to S$ and

$$0 \to \mathcal{K} \hookrightarrow \mathcal{E}_T \twoheadrightarrow \mathcal{F} \to 0$$

over X_T s.t. for any point $t \in T$ we have $h_{\mathcal{F}_t} = \Phi$ (i.e an element in $Q(T)$). Then there is an integer m depended only on rank of $\mathcal{V}, \mathcal{W}$ and Φ, s.t. over T we have SES of locally free sheaves

$$0 \to \pi_{T*}\mathcal{K}(m)\pi_{T*}\mathcal{E}_T(m) \twoheadrightarrow \pi_{T*}\mathcal{F}(m) \to 0$$

In particular, $\pi_{T*}\mathcal{E}_T(m) \cong f^*(\mathrm{Sym}^m\mathcal{V} \otimes \mathcal{W})$ and $\mathrm{rk}\, \pi_{T*}\mathcal{F}(m) = \Phi(m)$. Furthermore, $\pi_T^*\pi_{T*}\mathcal{K}(m) \twoheadrightarrow \mathcal{K}(m)$ is surjective, and so is $\mathcal{E}_T(m), \mathcal{F}(m)$.

$$\begin{tikzcd}
0 \arrow[r] & \pi_T^*\pi_{T*}\mathcal{K}(m) \arrow[r] \arrow[d, two heads] \arrow[dr, dashed, "h"] & \pi_T^*\pi_{T*}\mathcal{E}_T(m) \arrow[r] \arrow[d, two heads] & \pi_T^*\pi_{T*}\mathcal{F}(m) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{K}(m) \arrow[r] & \mathcal{E}_T(m) \arrow[r] & \mathcal{F}(m) \arrow[r] & 0
\end{tikzcd}$$

Define a map from $Q(T)$ to $G(T) := Gr_S(\mathrm{Sym}^m\mathcal{V} \otimes \mathcal{W}, \Phi(m))(T)$. This is an injective map: Given any element $(0 \to \mathcal{G} \to \pi_{T*}\mathcal{E}(m) \to \mathcal{Q} \to 0)$ in the image of the map , one can recover $\mathcal{F}(m)$ from cokernel of

$$h : \pi_T^*\pi_{T*}\mathcal{K}(m) = \pi^*\mathcal{G} \to \pi_T^*\pi_{T*}\mathcal{E}(m) \to \mathcal{E}(m)$$

. Hence Q is a subfunctor of G.

2. Functor G is represented by a projective scheme $\mathcal{G}$ embedded into a projective space $\mathbb{P}(\wedge^{\Phi(m)}\mathcal{W} \otimes \mathrm{Sym}^m\mathcal{V})$.

3. Now we have a subfunctor $Q \to G$, need to show this is a locally closed subfunctor. A map $h_T \to G$ is determined by an element in $G(T)$, i.e. $\pi_{T*}\mathcal{E}(m) \to \mathcal{H}$.

$$\begin{tikzcd}
h_T \times_G Q \arrow[r] \arrow[d] & h_T \arrow[d] \\
Q \arrow[r] & G
\end{tikzcd}$$

We need to show there is a locally closed subscheme $T' \to T$ s.t. a morphism $U \to T$ factors through T' if and only if the corresponding $\mathcal{E}_U \twoheadrightarrow \mathcal{F}$ is in $Q(U)$. This is given by flattening stratification theorem.

4. Hence functor Q is represented by a locally closed subscheme $\mathcal{Q} \to \mathcal{G}$. Since Q is proper over S, $\mathcal{Q}$ is a closed subscheme of $\mathcal{G}$, hence is a projective scheme.

$\square$

3.2 m-regular

Lemma 3.2. *numerical polynomial, Hartshorne: $P(t) \in \mathbb{Q}[t]$ is a numerical polynomial if $P(n) \in \mathbb{Z}$ for $n \gg 0$.*

1. *If P is a numercial polnomial, then there are integers $c_0, \dots, c_r$ s.t.*

$$P(n) = c_0\binom{n}{r} + c_1\binom{n}{r-1} + \cdots + c_r$$

2. *If function $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $\Delta f(n) = f(n+1) - f(n) = Q(n)$ for $n \gg 0$ and Q numerical polynomial, then $f(n) = P(n), n \gg 0$ for some numerical polynomial P.*

Proof. Hint: Induction on degree of the polynomial. $\square$

Definition 3.3. *A coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^n_k$ over a field k is called m-regular if*

$$H^i(\mathbb{P}^n_k, \mathcal{F}(m-i)) = 0, i > 0$$

Remark 3.4. *By taking base change, WMA k is an infinite field (this change is flat and cohomology groups well behaved), then one can take a hyperplane H passing through none of asscoiated points of $\mathcal{F}$, thus we have a SES*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0$$

(injectivity follows from the choice of H) and hence

$$0 \to \mathcal{F}(r-1) \to \mathcal{F}(r) \to \mathcal{F}_H(r) \to 0$$

Taking LES, one can show $\mathcal{F}_H$ is m-regular for same m.

Lemma 3.5. *If $\mathcal{F}$ is m-regular, then*

1. *$H^0(\mathbb{P}^n, \mathcal{F}(r)) \times H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+1))$ is surjective for $r \geqslant m$;*
2. *$\mathcal{F}$ is m'-regular for all $m' > m$;*
3. *$\mathcal{F}(r)$ is globally generated, and higher cohomologies vanish for $r \geqslant m$.*

Remark 3.6. *SES: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, if $\mathcal{F}', \mathcal{F}''$ are m-regular, then so is $\mathcal{F}$; if $\mathcal{F}'$ is $(m+1)$-regular and $\mathcal{F}$ is m-regular, then $\mathcal{F}''$ is m-regular; if $\mathcal{F}$ is m-regular and $\mathcal{F}''$ is $(m-1)$-regular, then $\mathcal{F}'$ is m-regular.*

Theorem 3.7. *m-regular: For any integers n, p, there is a polynomial $F_{p,n}$ in $n+1$ varibles satisfying:*

For any coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^n_k$ isomorphic to a subsheaf of $\mathcal{O}_{\mathbb{P}^n}^{\oplus p}$, with hilbert polynomial

$$\chi(\mathcal{F}(r)) = \sum_{i=0}^{n} a_i \binom{r}{i}$$

where $a_i \in \mathbb{Z}$, $\mathcal{F}$ is m-regular for $m = F_{p,n}(a_0, \ldots, a_n)$.

4 flattening stratification

Lemma 4.1. *Let A be a noetherian domain, B a finitely generated A-algebra, and M a finitely generated B-module. Then there is a nonzero element $f \in A$ s.t. M_f is a free A_f-module.*

Theorem 4.2. *FGA,5.12 Let S be a noetherian integral scheme, and $X \to S$ of finite type. Let $\mathcal{F}$ be a coherent sheaf over X, then there is a nonempty open subset $U \subset S$ s.t. $\mathcal{F}_U$ flat over U.*

Theorem 4.3. *FGA,5.13 Let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}^n_S$, where S is a noetherian scheme, let $\mathcal{P}$ be the set of hilbert polynomials of restrictions $\mathcal{F}_s$ of $\mathcal{F}$ to the fibres of $\pi : \mathbb{P}^n_S \to S$. Then $\#\mathcal{P} < \infty$ and for each $P \in \mathcal{P}$ there is a locally closed subscheme $S_P \hookrightarrow S$, satisfying*

1. *$\forall s \in S_P$, hilbert polynomial $h_{\mathcal{F}_s} = P$, and $\cup_{P\in\mathcal{P}}\mathrm{sp}(S_p) = \mathrm{sp}(S)$ as sets;*
2. *$S' = \coprod_{P\in\mathcal{P}} S_P \to S$ has a universal property: any $T \to S$ factors through $S' \to S$ if and only if $\mathcal{F}_T$ flat over T.*
3. *Define $P \leqslant Q$ if $P(n) \leqslant Q(n)$ for $n \gg 0$, then $\overline{\mathrm{sp}(S_P)} \subset \cap_{P\leqslant Q}\mathrm{sp}(S_Q)$.*