On the Du Bois property of secant varieties

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1 Introduction to D.B

X smooth projective variety over $\mathbb{C}$, and $\dim X = n$ complex:

$$0 \to \mathcal{O}_X \xrightarrow{d=\partial} \Omega^1 \to \cdots \to Omega^n \to 0$$

there is a filtration of the cotangent sheaf:

$$F^0 : 0 \to \mathcal{O}_X \xrightarrow{d=\partial} \Omega^1 \to \cdots \to Omega^n \to 0$$
$$F^1 : 0 \to \Omega^1 \to \cdots \to \Omega^n \to 0$$

let $Gr_F^p \underline{\Omega}_X^{\cdot} = \Omega_X^p$
$E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \implies H^{p+q}(X, \mathbb{C})$

Definition 1.1. *If $\mathcal{O}_X \to \underline{\Omega}_{X^0}$ is quasi-isomorphism, then X is Du Bois.*

Proposition 1.2 (property of Du Bois)**.** *1. klt $\implies$ rational singularity $\implies$ Du Bois*

klt $\implies$ log canonical $\implies$ Du Bois

2. *A fibration $\mathcal{X} \to B$ such that $\mathcal{X}_b$ is D.B for all b. If $\mathcal{X}_{B_0}$ has S_k condition, then so is any $\mathcal{X}_b$*
3. *$R^q f_* \mathcal{O}_X$ is locally free and compatible with case change.*

Higher Du Bois

Definition 1.3 (Shen-Veriketesh-Vo)**.** *Fix $p \in \mathbb{N}$, X is p-Du bois if X is semi-normal, and*

1. *$h^0(\underline{\Omega}_X^k) \to \underline{\Omega}_X^k$ is quasi-isomorphism for $0 \le k \le p$ (pre-p-D.B.)*
2. *$Codim(X_{sing}, X) \ge 2p + 1$*
3. *$h^0(\underline{\Omega}_X^k) \xrightarrow{\sim} \underline{\Omega}_X^{[k]}$*

Definition 1.4. *X is p-rational singularity if X is normal and*

1. *$Codim(X_{sing}, X) > 2p + 1$*
2. *For log resolution $f : \hat{X} \to X$, and exceptional divisor E, there is $R^i f_*(\Omega_X^K(\log E)) = 0$ for $\forall i \ge -, 0 \le k \le p$ (pre-p-rational)*

Remark 1.5 (some results)**.** *1. And p-rational $\implies$ p=Du bois.*

2. *In lc I $p - DB \implies (p - q)$-rational*
3. *In Non lc I $p - DB \not\Longrightarrow (p - q)$-rational*

2 Singularity of secant varieties

Secant varieties Σ_k of a variety $X \hookrightarrow \mathbb{P}^N$ is defined as union of k-planes generated by points in X.

Remark 2.1. *For $X \hookrightarrow \mathbb{P}^N$ by $|L|$, if L hhas insufficient positive, then degenerate: $\dim \Sigma_k <$ expected. (expecte $\dim \Sigma^k = k + (k + 1)n$)*

Example: consider $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, then Σ is a hypersurface.

Remark 2.2. *Assume that*

Theorem 2.3 (Wlley,Vermeire)**.** *Σ is normal.*

Theorem 2.4 (Chou-S)**.** • *Σ has worst D.B.*

- *Σ is C.M. if and only if $h^1(X, \mathcal{O}) = \cdots = h^{n-1}(X, \mathcal{O}) = 0$.*
- *$\omega_X^{GR} \cong \omega$ if and only if $h^n(X, \mathcal{O}) = 0$*

3 Geometry of secant varieties

$X^{[2]}$ Hibert scheme of 2-points on X. Then

$$\begin{tikzcd} \Phi \arrow[r] & X^{[2]} \times X \arrow[dl] \arrow[dr] & \\ X^{[2]} \arrow[rr] & & X \end{tikzcd}$$

4 ref

ref: https://doi.org/10.1112/plms.12635