WEEK 8: 2021-07-06, 2021-07-08

Exercise 1. Let $(A, \mathfrak{m})$ be a noetherian local ring and M be a finite A-module. Consider the following quantities attached to M:

- $\dim(M)$, the Krull dimension of M, defined to be $\dim(A/\mathrm{ann}(M))$;
- $d(M)$, degree of the Hilbert-Samuel polynomial of M;
- $\delta(M)$, called the Chevalley dimension, the smallest $n \geq 0$ such that there exist $a_1, \ldots, a_n \in \mathfrak{m}$ such that
$$l(M/(a_1, \ldots, a_n)M) < +\infty.$$

Prove that: $\dim(M) = d(M) = \delta(M)$.

Exercise 2. Let $(A, \mathfrak{m})$ be a noetherian local ring. Consider
$$0 \to M' \to M \to M'' \to 0$$
an exact sequence of finite A-modules. Prove the following statements:

- $\mathrm{depth}(M) \geq \min\{\mathrm{depth}(M'), \mathrm{depth}(M'')\}$;
- $\mathrm{depth}(M') \geq \min\{\mathrm{depth}(M), \mathrm{depth}(M'') + 1\}$;
- $\mathrm{depth}(M'') \geq \min\{\mathrm{depth}(M') - 1, \mathrm{depth}(M)\}$.

Exercise 3. Let $(A, \mathfrak{m})$ be a noetherian local ring and $I, J \subset \mathfrak{m}$ be ideals such that $I \cap J = 0$. Suppose that A/I, A/J are Cohen-Macaulay rings of the same dimension d and that $A/(I + J)$ is of dimension $d - 1$. Show that A is Cohen-Macaulay if and only if $A/(I + J)$ is.
Hint: first prove that, given ideals $I, J \subset A$, there exists an exact sequence
$$0 \to A/(I \cap J) \to A/I \oplus A/J \to A/(I + J) \to 0.$$

Exercise 4. Show that a one-dimensional reduced noetherian ring is Cohen-Macaulay. (Recall that A is reduced if $\mathrm{Nil}(A) = 0$.)

Exercise 5. Prove that the ring
$$A = k[x^4, x^3y, x^2y^2, xy^3, y^4] \subset k[x, y]$$
is a Cohen-Macaulay ring, but
$$A' = k[x^4, x^3y, xy^3, y^4] \subset k[x, y]$$
is not.
Hint: A (or A') is CM if and only if $\{x^4, y^4\}$ is a regular sequence.