

Lectures on Algebraic Geometry

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(Notes taken by Hang Yin)

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Preface

These lecture notes are for my graduate course Algebra Geometry in Fall 2020 at the University of the Chinese Academy of Sciences. The lectures were given in the Morningside Center of Mathematics. Apart from the original sources and classical textbooks, I have been much influenced by a course taught by Illusie in Spring 2004 and the Stacks Project [SP].

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I have not yet checked Chapters 3 through 5 and the end of Section 2.5.

Chapter 1

Schemes

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References:

- (1) Atiyah-MacDonald [AM]
- (2) Matsumura [M2, M1]
- (3) Hartshorne [H], Ch. 2–4.
- (4) Liu Qing [L], Ch. 1–7.
- (5) Fu Lei [F]
- (6) EGA [G, GD]
- (7) Stacks Project [SP]

1.1 Algebraic subsets

Let k be an algebraically closed field, $\mathbb{A}^n(k) = \{(a_1, \dots, a_n) \in k^n\}$ the affine n -space. Let $R = k[x_1, \dots, x_n]$ be the polynomial ring.

Notation 1.1.1. For $f \in R$, $Z(f) = \{P \in \mathbb{A}^n(k), f(P) = 0\}$. Similarly, for $T \subseteq R$, $Z(T) = \{P \in \mathbb{A}^n(k) : f(P) = 0, \forall f \in T\}$.

Remark 1.1.2. $Z(T) = Z(I)$, where I is the ideal generated by T .

Theorem 1.1.3 (Hilbert Basis). R is a Noetherian ring.

Remark 1.1.4. Every ideal $I \subseteq R$ is finitely generated. For $I = (f_1, \dots, f_m)$, we have $Z(I) = \bigcap_i Z(f_i)$.

Proposition 1.1.5. Some properties:

- (1) If $I_1 \subseteq I_2$, then $Z(I_1) \supseteq Z(I_2)$.
- (2) $Z(\sum_i I_i) = \bigcap Z(I_i)$.

$$(3) \ Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2).$$

$$(4) \ Z(0) = R, \ Z(R) = \emptyset.$$

Proof. 3: it is easy to see $Z(I_1 \cap I_2) \supseteq Z(I_1) \cup Z(I_2)$. We see $I_1 I_2 \subseteq I_1 \cap I_2$. Thus we have $Z(I_1 \cap I_2) \subseteq Z(I_1 I_2)$. Let $P \notin Z(I_1), P \notin Z(I_2)$. By definition, we have $f \in I_1, f(P) \neq 0, g \in I_2, g(P) \neq 0$. But $fg \in I_1 I_2$ and $f(P)g(P) \neq 0$.

□

Definition 1.1.6 (Zariski Topology and Algebraic Subsets). The above properties ensure that the $Z(I)$ are the closed subsets of a topology on $\mathbb{A}^n(k)$, called the **Zariski topology**. Closed subsets in $\mathbb{A}^n(k)$ are called **algebraic subsets**.

Example 1.1.7. (1) $\mathbb{A}^0(k) = \text{pt.}$

- (2) Consider $\mathbb{A}^1(k)$. Since $k[x]$ is a PID, every ideal $I = (f)$. Then $Z(I)$ consists of the roots of f . Therefore algebraic subsets $\subsetneq \mathbb{A}^1(k)$ are precisely finite subsets. This topology is called the **cofinite topology**. Since k is infinite, this topology is not Hausdorff.
- (3) In $\mathbb{A}^n(k)$, every point is closed. $P = (a_1, \dots, a_n)$ is defined by the ideal $\mathfrak{m}_P = (x_1 - a_1, \dots, x_n - a_n)$. Actually \mathfrak{m}_P is a maximal ideal. This shows $\mathbb{A}^n(k)$ is a T_1 space.
- (4) Consider $\mathbb{A}^2(k)$. As a set, it is in bijection with $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$. But its Zariski topology does not agree with the product topology. For example $Z(x - y)$ is closed in $\mathbb{A}^2(k)$ but not in the product topology.

Notation 1.1.8. For $Y \subseteq \mathbb{A}^n(k)$, $I(Y) = \{f \in R : f(P) = 0, \forall P \in Y\}$. It is the same thing as $\bigcap_{P \in Y} \mathfrak{m}_P$.

Proposition 1.1.9. Properties of $I(Y)$:

- (1) If $Y_1 \subseteq Y_2$, then $I(Y_1) \supseteq I(Y_2)$;
- (2) $I(\bigcup_i Y_i) = \bigcap_i I(Y_i)$;
- (3) $Z(I(Y)) = \overline{Y}$ (closure in $\mathbb{A}^n(k)$ for the Zariski topology).

Proof. 3: It is clear $Z(I(Y)) \supseteq Y$, hence contains its closure. Suppose $Y \subseteq Z(I)$, then $\forall f \in I$, f is zero on Y , thus $I \subseteq I(Y)$, hence $Z(I) \supseteq Z(I(Y))$. Hence $Z(I(Y))$ is the closure of Y .

□

Theorem 1.1.10 (Hilbert's Nullstellensatz). For each ideal I , we have $I(Z(I)) = \sqrt{I}$.

Corollary 1.1.11. There is an order-reversing one-to-one correspondence between algebraic subsets of $\mathbb{A}^n(k)$ and radical ideals of R , given by

$$\begin{aligned} Y &\mapsto I(Y) \\ Z(I) &\leftrightarrow I \end{aligned}$$

Corollary 1.1.12. *Every maximal ideal in R has the form \mathfrak{m}_P for some $P \in \mathbb{A}^n(k)$.*

Corollary 1.1.13. *For every ideal I in R , we have $\sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$.*

A ring satisfying the above property is called a **Jacobson ring**.

Lemma 1.1.14. *Let K be a field (not necessarily algebraic closed). Let E be a finitely generated K -algebra. Suppose E is also a field. Then E/K is a finite field extension.*

For a proof, see [AM, Corollary 5.24].

Proof of the Nullstellensatz. We have $I(Z(I)) \supseteq \sqrt{I}$. Suppose $f \notin \sqrt{I}$, then in R_f , IR_f is not unit ideal. Choose a maximal ideal $\mathfrak{m} \supseteq IR_f$, then R_f/\mathfrak{m} is a finitely generated k -algebra which is also a field, hence $R_f/\mathfrak{m} = k$. Now $R \rightarrow R_f \rightarrow R_f/\mathfrak{m} = k$ defines a ring homomorphism and a maximal ideal \mathfrak{n}_P such that $\mathfrak{n}_P \supseteq I$ and $f \notin \mathfrak{n}_P$, hence $P \in I(Z(I))$ but f is not zero on P . Thus $f \notin I(Z(I))$. \square

Notation 1.1.15. For a ring A , we let $\text{Max}(A)$ denote the set of all maximal ideals of A . This set is called the **maximal spectrum** of A .

From the Nullstellensatz, there are bijections

$$\begin{aligned}\mathbb{A}^n(k) &\simeq \text{Max}(R) \\ Z(I) &\simeq \text{Max}(R/I)\end{aligned}$$

We find that an algebraic subset is in bijection with the maximal spectrum of a finitely generated k -algebra.

1.2 Spectrum of a ring

For a ring homomorphism $f: A \rightarrow B$, there is no natural map $\text{Max}(B) \rightarrow \text{Max}(A)$ in general. The pull back of a maximal ideal is a prime ideal but not necessarily maximal. This shows the maximal spectrum behaves badly.

Notation 1.2.1. For a ring A , we let $\text{Spec}(A)$ denote the set of all prime ideals of A . We call $\text{Spec}(A)$ the **(prime) spectrum** of A .

Every ring homomorphism $\phi: A \rightarrow B$ induces a map

$$\begin{aligned}\text{Spec}(\phi): \text{Spec}(B) &\rightarrow \text{Spec}(A) \\ \mathfrak{p} &\mapsto \phi^{-1}(\mathfrak{p})\end{aligned}$$

Notation 1.2.2. For $T \subseteq A$, let $V(T) = \{\mathfrak{p} \in \text{Spec}(A) : T \subseteq \mathfrak{p}\}$. For $f \in A$, let $D(f) = \text{Spec}(A) \setminus V(f)$.

Remark 1.2.3. $V(T) = V(I)$, where I is the ideal generated by T .

Proposition 1.2.4. (1) For $I_1 \subseteq I_2$, we have $V(I_1) \supseteq V(I_2)$;

$$(2) \quad V(\sum_i I_i) = \bigcap_i V(I_i);$$

$$(3) \quad V(I_1 \cap I_2) = V(I_1) \cup V(I_2).$$

Proof. 3. It is clear $V(I_1 \cap I_2) \supseteq V(I_1) \cup V(I_2)$. Also, $V(I_1 I_2) \supseteq V(I_1 \cap I_2)$. Consider a prime ideal \mathfrak{p} not in $V(I_1) \cup V(I_2)$, that is $I_1 \subsetneq \mathfrak{p}, I_2 \subsetneq \mathfrak{p}$, then there exists $f \in I_1, g \in I_2$ such that $f, g \notin \mathfrak{p}$, hence $fg \in I_1 I_2$ but $fg \notin \mathfrak{p}$. This shows $\mathfrak{p} \notin V(I_1 I_2)$. \square

We equip $\text{Spec}(A)$ with the topology for which the closed subsets are exactly subsets of the form $V(I)$. We call it the **Zariski topology**.

Notation 1.2.5. For $Y \subseteq \text{Spec}(A)$, let $I(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. This is an ideal of A .

Proposition 1.2.6. *Properties of $I(Y)$:*

- (1) If $Y_1 \subseteq Y_2$, then $I(Y_1) \supseteq I(Y_2)$
- (2) $I(\bigcup_i Y_i) = \bigcap_i I(Y_i)$;
- (3) $V(I(Y)) = \overline{Y}$;
- (4) for an ideal I , $I(V(I)) = \sqrt{I}$.

Corollary 1.2.7. *There is an order-reversing one-to-one correspondence between closed subsets of $\text{Spec}(A)$ and radical ideals of A , given by*

$$\begin{aligned} Y &\mapsto I(Y) \\ V(I) &\leftrightarrow I \end{aligned}$$

Moreover, the closed points of $\text{Spec}(A)$ are the maximal ideals of A .

Example 1.2.8. (1) $A = 0 \Leftrightarrow \text{Spec}(A) = \emptyset$.

(2) Let k be a field. Then $\text{Spec}(k) = \text{pt}$.

- (3) $\mathbb{A}_k^1 = \text{Spec}(k[x])$. The closed points are of the form (f) , where f is an irreducible polynomials. The generic point is (0) . Closed subsets are either the whole space or a finite set of closed points. It is not even a T_1 space, but it is a T_0 space.
- (4) Consider $\text{Spec}(\mathbb{Z})$. The closed points are of the form (p) , where p is a prime number. The generic point is (0) . The topology is similar to that of \mathbb{A}_k^1 .

Corollary 1.2.9. $\text{Spec}(A)$ is quasi-compact (namely, every open cover has a finite subcover).

Proof. Suppose $\bigcap V(I_i) = \emptyset$. Then $\sqrt{\sum I_i} = A$, thus $1 \in \sum I_i$, hence there are some i_1, \dots, i_n such that $1 = a_1 + \dots + a_n$ where $a_j \in I_{i_j}$, hence I_{i_1}, \dots, I_{i_n} generate A , hence $V(I_{i_1}) \cap \dots \cap V(I_{i_n}) = \emptyset$. \square

Notation 1.2.10. Let $I^e = IB$ and $J^c = \phi^{-1}(J)$ denote the extension and contraction ideals with respect to certain ring homomorphism ϕ .

Lemma 1.2.11. *Let $\phi: A \rightarrow B$ be a ring homomorphism and $f = \text{Spec}(\phi): \text{Spec}(B) \rightarrow \text{Spec}(A)$. Then*

- (1) $f^{-1}(V(I)) = V(I^e)$ for every ideal I of A ;
- (2) $\overline{f(V(J))} = V(J^c)$ for every ideal J of B .

Proof. (1) For $\mathfrak{q} \in \text{Spec}(B)$, $f(\mathfrak{q}) \in V(I)$ means $\mathfrak{q}^c \supseteq I$, which is equivalent to $\mathfrak{q} \supseteq I^e$. Hence $f^{-1}(V(I)) = V(I^e)$.

- (2) We have $I(f(V(J))) = \bigcap_{J \subseteq \mathfrak{q}} \phi^{-1}(\mathfrak{q}) = \phi^{-1}(\bigcap_{J \subseteq \mathfrak{q}} \mathfrak{q}) = \phi^{-1}(\sqrt{J}) = \sqrt{\phi^{-1}(J)}$. Applying V , we get $\overline{f(V(J))} = V(J^c)$.

□

Proposition 1.2.12. *Let $\phi: A \rightarrow B$ be a ring homomorphism. Then $\text{Spec}(\phi): \text{Spec}(B) \rightarrow \text{Spec}(A)$ is continuous.*

Proof. This follows immediately from Part 1 of the above lemma. □

Example 1.2.13. (1) For I an ideal in A , the quotient map $\pi: A \rightarrow A/I$ induces to $\text{Spec}(\pi): \text{Spec}(A/I) \rightarrow \text{Spec}(A)$, which is a closed embedding.

- (2) Suppose S is a multiplicative subsets in A . The localization map $\phi: A \rightarrow S^{-1}A$ induces $\text{Spec}(\phi): \text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$, which is also an embedding.

Lemma 1.2.14. *Let $\phi: A \rightarrow B$ be a ring homomorphism. Then $\text{Spec}(\phi)$ identifies $\text{Spec}(B)$ with a subspace of $\text{Spec}(A)$ if and only if every ideal $J \in B$ satisfies $\sqrt{J} = \sqrt{J^{ce}}$.*

Proof. It is easy to see that $f = \text{Spec}(\phi)$ is an embedding if and only if $f^{-1}(\overline{f(F)}) = F$ for every closed subset $F \subset \text{Spec}(B)$, which translates to the corresponding condition on ideals. □

It is easy to verify that $A \rightarrow A/I$ and $A \rightarrow S^{-1}A$ satisfy the condition in the lemma.

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Recall we define $\text{Spec}(A)$ as prime ideals of A with Zariski topology. A basis $D(f) = \{p \in \text{Spec}(A) | f \notin p\}$, $f \in A$.

For a ring homomorphism $\phi: A \rightarrow B$, we have

$$\begin{aligned}\phi^*: \text{Spec}(B) &\rightarrow \text{Spec}(A) \\ \mathfrak{q} &\mapsto \mathfrak{q}^c\end{aligned}$$

Some examples: $\pi: A \rightarrow A/I$ and $\phi: A \rightarrow S^{-1}A$, where S is a multiplicative system in A . The image of π^* is $V(I)$ and the image of ϕ^* is $\bigcap_{f \in S} D(f)$. If A is an integral domain, $S = A \setminus \{0\}$, then $S^{-1}A = \text{Frac}(A)$. $\text{Spec}(\text{Frac}(A))$ is a point, and its image in $\text{Spec}(A)$ is the generic point of $\text{Spec}(A)$, given by the zero ideal of A .

Example 1.2.15. Consider

$$\begin{array}{ccc} & k[x, y] & \\ \nearrow & & \swarrow \\ k[x] & & k[y] \end{array}$$

We have

$$\begin{array}{ccc} \text{Spec}(k[x, y]) = \mathbb{A}_k^2 & & \\ \swarrow & & \searrow \\ \text{Spec}(k[x]) = \mathbb{A}_k^1 & & \text{Spec}(k[y]) = \mathbb{A}_k^1 \end{array}$$

This defines a continuous map $\mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1 \times^{\text{top}} \mathbb{A}_k^1$ (product space). This is surjective but not injective, since the points (0) and $(x - y)$ in $\text{Spec}(k[x, y])$ both map to $((0), (0))$ in $\mathbb{A}_k^1 \times^{\text{top}} \mathbb{A}_k^1$.

Example 1.2.16 (Tangent Space). Consider $k[x_1, \dots, x_n]/(f_1, \dots, f_m)$, $I = (f_1, \dots, f_m)$. Let $P = (a_1, \dots, a_n) \in Z(I)$, then $\forall i, f_i(P) = 0$. Consider $\mathfrak{m}_P = (x_1 - a_1, \dots, x_n - a_n)/I$. Define

$$T_P = \{(t_1, \dots, t_n) \in k^n \mid \sum_i \frac{\partial f}{\partial x_i}(P)t_i = 0\}$$

which is a linear subspace of k^n .

We can write it in algebraic form. Let $k[\epsilon]/(\epsilon^2) = \{a + b\epsilon, a, b \in k\}$. Consider the diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A/\mathfrak{m}_P = k \\ & \searrow \phi & \uparrow \rho \\ & k[\epsilon]/\epsilon^2 & \end{array}$$

where ρ sends ϵ to 0 and $\psi(x_i) = a_i$. A homomorphism ϕ is defined by $\phi(x_i) = a_i + t_i\epsilon$. It factors through I if and only if

$$0 = f_j(a_1 + t_1\epsilon, \dots, a_n + t_n\epsilon) = f_j(a_1, \dots, a_n) + \sum_i \frac{\partial f}{\partial x_i}(P)t_i\epsilon = \sum_i \frac{\partial f}{\partial x_i}(P)t_i\epsilon$$

in $k[\epsilon]/(\epsilon^2)$. This is the same thing as a tangent vector defined above. Thus we have a bijection

$$T_P \simeq \{\phi : A \rightarrow k[\epsilon]/(\epsilon^2) \mid \rho\phi = \psi\}.$$

However, T_P cannot be read off from the induced maps of topological spaces:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xleftarrow{\psi^*} & \mathrm{Spec}(k) \\ \dashleftarrow_{\phi^*} & & \downarrow \rho^* \\ & & \mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \end{array}$$

Indeed, ρ^* is a homeomorphism.

1.3 Sheaves

Let X be a topological space, \mathcal{C} a category.

Definition 1.3.1. Let $\mathrm{Open}(X) = (\{\text{open subsets of } X\}, \subseteq)$. It is a poset and can be viewed as a category: there is a unique morphism $U \rightarrow V$ if $U \subseteq V$ and no such morphism otherwise.

- (1) A **presheaf** on X with values in \mathcal{C} is a contravariant functor $\mathrm{Open}(X)^{\mathrm{op}} \rightarrow \mathcal{C}$. Denote $\mathrm{PShv}(X, \mathcal{C}) = \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C})$.
- (2) A morphism between two sheaf \mathcal{F}, \mathcal{G} is a natural transformation $\phi: \mathcal{F} \rightarrow \mathcal{G}$.

In details, a presheaf $\mathcal{F}: \mathrm{Open}(X)^{\mathrm{op}} \rightarrow \mathcal{C}$ consists of $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$ for each open sets U , and $\rho_{UV}: \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ a morphism (called **restriction**) in \mathcal{C} for $U \subseteq V$. We require them to satisfy:

- (1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$,
- (2) For $U \subseteq V \subseteq W$, we have $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$

A morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ consists of $\phi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each open set U that satisfies for $U \subseteq V$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \rho_{UV}^{\mathcal{F}} \uparrow & & \rho_{UV}^{\mathcal{G}} \uparrow \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

In the rest of this section we assume that \mathcal{C} is Set, Ab, or Ring. Elements of $\mathcal{F}(U)$ are called **sections**. For the morphism $\rho_{UV}: \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, $s \in \mathcal{F}(V)$, we sometimes write $s|_U$ for $\rho_{UV}(s) \in \mathcal{F}(U)$.

Definition 1.3.2. A **sheaf** is a presheaf \mathcal{F} satisfying the following gluing property: $\forall U \subseteq X$ open, $\{U_i\}$ an open cover of U ,

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram. The latter two maps are induced respectively by two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$.

In other words, $\forall s_i \in \mathcal{F}(U_i)$, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then $\exists! s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Note that uniqueness is equivalent to the injectivity of $\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i)$. A presheaf satisfying the uniqueness is called **separated**.

Remark 1.3.3. Consider the empty set \emptyset . The empty cover is a cover of \emptyset . Now by definition, empty product is the terminal object and the equalizer of a pair of endomorphism of the terminal object is terminal. This shows that for any sheaf \mathcal{F} , $\mathcal{F}(\emptyset)$ is a terminal object of \mathcal{C} .

Example 1.3.4. Let X, Y be topological spaces. Then $\mathcal{F}_Y(U) = \text{Map}_{\text{cont}}(U, Y)$ defines a presheaf \mathcal{F}_Y on X . It is easy to see this is also a sheaf.

- (1) If Y is discrete, then $Y_X = \mathcal{F}_Y$ is called the **constant sheaf**: $Y_X(U) = \{f: U \rightarrow Y \mid f \text{ locally constant}\}$

Example 1.3.5. Let $f: Z \rightarrow X$ be a continuous map. For $U \subset X$ open, define $h_Z(U)$ as the set of continuous sections s of $f|_U: f^{-1}(U) \rightarrow U$ (namely, continuous maps $s: U \rightarrow f^{-1}(U)$ satisfying $f|_U \circ s = \text{id}$):

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & Z \\ s \uparrow \downarrow f|_U & & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

Such sections correspond bijectively to continuous maps $s: U \rightarrow Z$ such that $f \circ s = j$. This defines a sheaf on X .

Take $Z = X \times Y$ and $p: Z \rightarrow X$ the projection, then $p^{-1}(U) = U \times Y$:

$$\begin{array}{ccc} U \times Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

and a section $s: U \rightarrow U \times Y$ is determined by $U \rightarrow Y$. Thus $h_{X \times Y} = \mathcal{F}_Y$.

Example 1.3.6. Let X be a complex manifold. We have sheaves $\mathbb{C}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{F}_{\mathbb{C}}$ defined by

$$\begin{array}{ccccc} \mathbb{C}_X(U) & \xleftarrow{\quad} & \mathcal{O}_X(U) & \xleftarrow{\quad} & \mathcal{F}_{\mathbb{C}}(U) \\ \parallel & & \parallel & & \parallel \\ \{U \rightarrow \mathbb{C} \text{ locally constant}\} & & \{U \rightarrow \mathbb{C} \text{ holomorphic}\} & & \{U \rightarrow \mathbb{C} \text{ continuous}\} \end{array}$$

Example 1.3.7. Let $X = \text{pt}$, then

$$\begin{aligned} \text{Set} &\cong \text{Shv}(\text{pt}, \text{Set}) \\ \mathcal{F}(\text{pt}) &\leftrightarrow \mathcal{F} \\ S &\mapsto \begin{cases} \emptyset \mapsto \{\ast\} \\ \text{pt} \mapsto S. \end{cases} \end{aligned}$$

Proposition 1.3.8. Let \mathcal{F} be a presheaf. Then \exists a sheaf \mathcal{F}^+ and $\nu: \mathcal{F} \rightarrow \mathcal{F}^+$ such that \forall sheaf \mathcal{G} and a morphism of presheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$, there exists a unique $\phi^+: \mathcal{F}^+ \rightarrow \mathcal{G}$ such that

$$\begin{array}{ccc} & \mathcal{G} & \\ & \nearrow \phi & \uparrow \phi^+ \\ \mathcal{F} & \xrightarrow{\nu} & \mathcal{F}^+ \end{array}$$

Definition 1.3.9. We call \mathcal{F}^+ the **sheafification** of the presheaf \mathcal{F} . It is also called the **sheaf associated to \mathcal{F}** and sometimes denoted $a\mathcal{F}$.

Construction. For any open cover $\{U_i\}$ of an open subset U , consider

$$\text{Eq}\left(\Pi_i \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j} \mathcal{F}(U_i \cap U_j)\right).$$

We define a presheaf \mathcal{F}' on X by

$$\mathcal{F}'(U) = \underset{\text{Cov}(U)^{\text{op}}}{\text{colim}} \text{Eq}\left(\Pi_i \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j} \mathcal{F}(U_i \cap U_j)\right)$$

The category $\text{Cov}(U)$ of open covers of U is defined as follows. An object is an open cover $\{U_i\}_{i \in I}$. A morphism between two covers $\{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$ is a map $f: I \rightarrow J$ such that $U_i \subseteq V_{f(i)}$.

It is easy to see that \mathcal{F}' is a separated presheaf. Moreover, \mathcal{F}' is a sheaf if \mathcal{F} is separated. We take $\mathcal{F}^+ = (\mathcal{F}')'$. \square

Categorical point of view: We have $\text{Hom}(\mathcal{F}, \iota\mathcal{G}) \cong \text{Hom}(a\mathcal{F}, \mathcal{G})$.

$$\text{PShv} \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{\iota} \end{array} \text{Shv} \quad a \dashv \iota$$

Example 1.3.10. Let X be a topological space, A a set. Define the constant presheaf A^{psh} by $A_X^{\text{psh}}(U) = A$. Then $(A_X^{\text{psh}})^+ = A_X$ is the constant sheaf.

Definition 1.3.11 (Functionality). Let $f: X \rightarrow Y$ be a continuous map.

(1) For $\mathcal{F} \in \text{PShv}(X)$, define $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, $f_*\mathcal{F} \in \text{PShv}(Y)$. If \mathcal{F} is a sheaf, then $f_*\mathcal{F}$ is also a sheaf. This is called **pushforward** or **direct image**.

(2) for $\mathcal{G} \in \text{PShv}(Y)$, define

$$(f_{\text{psh}}^{-1}\mathcal{G})(U) = \underset{f(U) \subset V}{\text{colim}} \mathcal{G}(V)$$

It is clear $f_{\text{psh}}^{-1}\mathcal{G} \in \text{PShv}(X)$. This is called **pullback** or **inverse image**. We have $\text{Hom}(f_{\text{psh}}^{-1}\mathcal{G}, \mathcal{F}) \cong \text{Hom}(\mathcal{G}, f_*\mathcal{F})$.

(3) Unfortunately, even if \mathcal{G} is a sheaf on Y , $f_{\text{psh}}^{-1}\mathcal{G}$ may not be a sheaf. So we define $f^{-1}\mathcal{G} = (f_{\text{psh}}^{-1}\mathcal{G})^+$. We have $f^{-1} \dashv f_*$. Form the commutative diagram

$$\begin{array}{ccc} \text{PShv}(X) & \xleftarrow{\iota} & \text{Shv}(X) \\ \downarrow f_* & & \downarrow f_* \\ \text{PShv}(Y) & \xleftarrow{\iota} & \text{Shv}(Y) \end{array}$$

Taking left adjoints, we obtain the following diagram, which commutes up to natural isomorphism:

$$(1.3.1) \quad \begin{array}{ccc} \mathrm{PShv}(X) & \xrightarrow{a} & \mathrm{Shv}(X) \\ f_{\mathrm{psh}}^{-1} \uparrow & & f^{-1} \uparrow \\ \mathrm{PShv}(Y) & \xrightarrow{a} & \mathrm{Shv}(Y) \end{array}$$

Example 1.3.12. Consider $j: U \rightarrow X$ open. Then $j_{\mathrm{psh}}^{-1}\mathcal{F}(V) = \mathcal{F}(V \cap U)$. We usually denote it by $\mathcal{F}|_U$. We have $j^{-1}\mathcal{F} = j_{\mathrm{psh}}^{-1}\mathcal{F}$ if \mathcal{F} is a sheaf.

Example 1.3.13. We have $f^{-1}A_Y \simeq A_X$ by (1.3.1) applied to A_Y^{psh} .

Example 1.3.14. Consider $i_x: \mathrm{pt} \rightarrow X$, $\mathrm{pt} \mapsto x \in X$.

$$\begin{aligned} (i_x)^{-1}_{\mathrm{psh}}(\mathcal{F})(x) &= \operatorname{colim}_{x \in U} \mathcal{F}(U) \\ &= \{(U, s) | x \in U, s \in \mathcal{F}(U)\} / \sim \end{aligned}$$

where the equivalence relation is defined as follows: $(U, s) \sim (V, t)$ if and only if $\exists x \in W \subset U \cap V$ such that $s|_W = t|_W$. The same formula holds for i_x^{-1} .

This is also called the **stalk** of \mathcal{F} at x and is denoted by \mathcal{F}_x . For each $x \in U$, we have

$$\begin{aligned} \mathcal{F}(U) &\rightarrow \mathcal{F}_x \\ s &\mapsto [(U, s)] \end{aligned}$$

The image of s is called the **germ** of s at x and denoted s_x . We have $\mathcal{F}_x^+ \simeq \mathcal{F}_x$ by (1.3.1).

Lemma 1.3.15. Suppose \mathcal{F} a sheaf, $s, t \in \mathcal{F}(U)$ such that $s_x = t_x \in \mathcal{F}_x, \forall x \in U$. Then $s = t$.

Proof. By definition, for each $x \in U$, s, t agree on some neighborhood W_x of x . These W_x cover U when x varies in U , hence s, t agree on an open cover of U , hence they agree on U . \square

Proposition 1.3.16. Let \mathcal{F}, \mathcal{G} be sheaves.

(1) Suppose $\phi, \psi: \mathcal{F} \rightarrow \mathcal{G}$ morphisms of sheaves such that $\phi_x = \psi_x, \forall x \in X$. Then $\phi = \psi$.

(2) Suppose $\phi: \mathcal{F} \rightarrow \mathcal{G}$, ϕ_x is bijective for all $x \in X$. Then ϕ is an isomorphism.

Proof. (1) For U open, $s \in \mathcal{F}(U)$, then $\phi(U)(s)_x = \psi(U)(s)_x, \forall x \in U$, by above Lemma, we have $\phi(U)(s) = \psi(U)(s)$, hence $\phi = \psi$.

(2) We construct ψ to be the inverse of ϕ . For $t \in \mathcal{G}(U)$, and $x \in U$, since $\phi(U)_x$ is bijective, there exists open set $x \in V_x \subset U$ and $s_x \in \mathcal{F}(V_x)$ such that $\phi(V_x)(s_x) = t|_{V_x}$. Consider $x, y \in U, \forall z \in V_x \cap V_y$, we have $s_x|_z = t|_z = s_y|_z$, hence $s_x|_{V_x \cap V_y} = s_y|_{V_x \cap V_y}$, hence $s_x \in \mathcal{F}(V_x)$ glue to $s \in \mathcal{F}(U)$. \square

Consider continuous maps $f: X \rightarrow Y$ and $g: W \rightarrow X$. Then $g^{-1}f^{-1} = (fg)^{-1}$. In the case where $W = \mathrm{pt}$ and $g = i_x$, we get

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Limits and Colimits

Recall that $\mathcal{C} = \text{Set}$, Ab , or Ring .

The category $\text{PShv}(X, \mathcal{C})$ admits arbitrary small limits and colimits.

$$\begin{aligned} (\lim_i \mathcal{F}_i)(U) &= \lim_i \mathcal{F}_i(U) \\ (\operatorname{colim}_i^{\text{psh}} \mathcal{F}_i)(U) &= \operatorname{colim}_i^{\text{psh}} \mathcal{F}_i(U) \end{aligned}$$

It is easy to see the limit defined above takes sheaves to sheaves. But for colimits of sheaves, we need to sheafify: $\operatorname{colim}_i \mathcal{F}_i = (\operatorname{colim}_i^{\text{psh}} \mathcal{F}_i)^+$. The category $\text{Shv}(X, \mathcal{C})$ also admits small limits and colimits. The sheafification functor commutes with colimits and the functor ι commutes with limits.

Recall that filtered colimits in \mathcal{C} commute with finite limits, hence finite limits of sheaves do not need to be sheafified. The same remark shows that the sheafification functor is left exact, and hence exact. (Recall that a functor is called left (resp. right) exact if it commutes with finite limits (resp. finite colimits). A functor is called **exact** if it is left and right exact.)

The following special case will be used very often.

Definition 1.3.17. Let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of Abelian sheaves.

- (1) $\ker(\varphi)$ is defined to be $(\ker(\varphi))(U) = \ker(\varphi(U))$. It is already a sheaf.
- (2) $\operatorname{coker}(\varphi)$ is the sheafification of the presheaf $U \mapsto \operatorname{coker}(\varphi(U))$.

Proposition 1.3.18. $\text{Shv}(X, \text{Ab})$ is an Abelian category.

Proof. We first check that $\text{Shv}(X, \text{Ab})$ is an additive category:

- (1) It has a zero object (namely, an object that is initial and final): the constant sheaf 0 ;
- (2) Finite coproducts and finite products exist and coincide: we have $\mathcal{F} \times \mathcal{G} \simeq \mathcal{F} \oplus^{\text{psh}} \mathcal{G} \simeq \mathcal{F} \oplus \mathcal{G}$.
- (3) The commutative monoid $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ admits inverses: $(-\phi)_U(s) = -\phi_U(s)$.

Recall that an abelian category is an additive category admitting kernels, cokernels, and such that coimages coincides with images. The last property means that for every morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$, the canonical morphism $\psi: \operatorname{coker}(\phi) \rightarrow \ker(\phi)$ is an isomorphism, where

$$\ker(\phi) \xrightarrow{i} \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{p} \operatorname{coker}(\phi).$$

Since sheafification commutes with taking kernels, ψ is the sheafification of $\psi^{\text{psh}}: \operatorname{coker}^{\text{psh}}(\phi) \rightarrow \ker(p^{\text{psh}})$, where $p^{\text{psh}}: \mathcal{G} \rightarrow \operatorname{coker}^{\text{psh}}(\phi)$. Since ψ_U^{psh} is an isomorphism for every U , ψ is an isomorphism. \square

Let $f: X \rightarrow Y$ be a continuous map. The functor f^{-1} commutes with colimits and f_* commutes with limits.

Proposition 1.3.19. *Let $f: X \rightarrow Y$ be a continuous map between topological spaces. Then $f^{-1}: \text{Shv}(Y, \mathcal{C}) \rightarrow \text{Shv}(X, \mathcal{C})$ is an exact functor.*

In particular, taking stalks at a point is an exact functor.

Proof. It suffices to show that f^{-1} is left exact. By definition,

$$(f_{\text{psh}}^{-1}\mathcal{G})(U) = \underset{f(U) \subset V}{\text{colim}} \mathcal{G}(V)$$

which is a filtered colimit, hence commutes with finite limits. The sheafification functor also left exact, hence the result. \square

Proposition 1.3.20. *A sequence $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ in $\text{Shv}(X, \text{Ab})$ is exact if and only if it is exact on stalks: $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for every $x \in X$.*

Proof. This follows from the exactness of taking stalks. For the “if” part, we also need Proposition 1.3.16. \square

Example 1.3.21. Consider $i: Y \subset X$ a closed embedding, \mathcal{G} a sheaf on Y . Then $i_*\mathcal{G}(U) = \mathcal{G}(U \cap Y)$. For $x \in X$,

$$\begin{aligned} (i_*\mathcal{G})_x &= \underset{x \in U}{\text{colim}} \mathcal{G}(U \cap Y) \\ &= \begin{cases} *, & x \notin Y \\ \mathcal{G}_x, & x \in Y, \end{cases} \end{aligned}$$

where $*$ denotes a final object of \mathcal{C} . It follows that the functor $i_*: \text{Shv}(Y, \text{Ab}) \rightarrow \text{Shv}(X, \text{Ab})$ is exact. (The functor $i_*: \text{Shv}(Y, \text{Set}) \rightarrow \text{Shv}(X, \text{Set})$ does not preserve initial objects unless i is a homeomorphism.)

Let $\phi: i^{-1}i_*\mathcal{G} \rightarrow \mathcal{G}$ be the canonical morphism. Then ϕ_y can be identified with $\text{id}_{\mathcal{G}_y}$. Hence ϕ is an isomorphism. In the other direction, for every an abelian sheaf \mathcal{F} on X , the canonical morphism $\psi: \mathcal{F} \rightarrow i_*i^{-1}\mathcal{F}$ is an epimorphism. This is easy to check on stalks.

Warning 1.3.22. An epimorphism of sheaves is **not** surjective on sections in general. Let X be a connected topological space, $Y = \{x, y\}$ two distinct closed points in X , $\iota: Y \rightarrow X$. Consider the constant sheaf \mathbb{Z}_X on X . Then $\mathbb{Z}_X(X) = \mathbb{Z}$ since X is connected. But $(\iota_*\iota^{-1}\mathbb{Z}_X)(X) \simeq \mathbb{Z}_Y(Y) \simeq \mathbb{Z} \times \mathbb{Z}$. The map $\psi_X: \mathbb{Z}_X(X) \rightarrow \iota_*\iota^{-1}\mathbb{Z}_X(X)$ is not surjective.

Let us describe epimorphisms of sheaves of sets or abelian groups. A morphism of sheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is an epimorphism if and only if $\forall U$ open in X , $s \in \mathcal{G}(U)$, $\exists \{U_i\}$ an open cover of U and $t_i \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(t_i) = s|_{U_i}$.

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Remark 1.3.23. We have defined for every continuous map $f: Z \rightarrow X$ between topological spaces, a sheaf of sections $\mathcal{F} = h_Z$ such that $h_Z(U)$ is the set of continuous sections $U \rightarrow Z$ of f over U . Conversely, every sheaf of sets has the form $\mathcal{F} \simeq h_Z$. Here $Z = \coprod_{x \in X} \mathcal{F}_x$. An element in Z has the form (x, s) , where $x \in X$, $s \in \mathcal{F}_x$. We equip Z with the strongest topology such that for all $U \subset X$ open and $s \in \mathcal{F}(U)$, the map

$$\begin{aligned}\varphi_s: U &\rightarrow Z \\ x &\mapsto (x, s_x)\end{aligned}$$

is continuous. A basis for the topology is given by the subsets $\varphi_s(U)$. The space Z is called the **espace étalé** of \mathcal{F} .

1.4 Schemes

Let A be a ring and let $X = \text{Spec}(A)$. We now proceed to equip X with a sheaf of rings \mathcal{O}_X such that $\mathcal{O}_X(X) = A$ and for $f \in A$, $\mathcal{O}_X(D(f)) = A_f$. Recall $D(f) = \{\mathfrak{p} \in A \mid f \notin \mathfrak{p}\}$.

Consider the poset $\mathcal{B} = (\{D(f) \mid f \in A\}, \subseteq)$. Define a functor

$$\begin{aligned}\mathcal{B}^{\text{op}} &\rightarrow \text{Ring} \\ D(f) &\mapsto A_f\end{aligned}$$

If $D(f) \subseteq D(g)$, we have $V(f) \supseteq V(g)$ hence $\sqrt{f} \subseteq \sqrt{g}$, which means that $f^n = ga$ for some $n \geq 1$ and $a \in A$. This implies that g is invertible in A_f and there is a natural ring morphism $A_g \rightarrow A_f$. This finishes the definition of functor.

Lemma 1.4.1. *Let X be a topological space, \mathcal{B} an open basis such that $U, V \in \mathcal{B} \Rightarrow U \cap V \in \mathcal{B}$ and $\emptyset \in \mathcal{B}$. We let $\text{Shv}(\mathcal{B}, \mathcal{C})$ denote the category of \mathcal{B} -sheaves, namely the full subcategory of $\text{Fun}(\mathcal{B}^{\text{op}}, \mathcal{C})$ spanned by functors \mathcal{F} satisfying the following gluing condition: for every open cover $\{U_i\}$ of $U \in \mathcal{B}$ with $U_i \in \mathcal{B}$,*

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram. Then the restriction functor

$$\Phi: \text{Shv}(X, \mathcal{C}) \rightarrow \text{Shv}(\mathcal{B}, \mathcal{C})$$

is an equivalence of categories, where $(*)$ denotes

Proof. We first prove that Φ is fully faithful, which means that for \mathcal{F}, \mathcal{G} sheaves on X , we have $\text{Hom}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}|_{\mathcal{B}}, \mathcal{G}|_{\mathcal{B}})$. This is clear by the gluing condition for sheaves on X , since \mathcal{B} is a basis.

We next prove essential surjectivity. Let \mathcal{G} be a \mathcal{B} -sheaf. We define a sheaf \mathcal{F} on X by

$$\mathcal{F}(U) = \underset{\{U_i\} \in \text{Cov}(U)^{\text{op}}}{\text{colim}} \text{Eq} \left(\prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j). \right)$$

Here $\text{Cov}(U)$ is the category of open covers of U in \mathcal{B} . In more detailed words, an element $s \in \mathcal{F}(U)$ is an equivalence class of pairs $(\{U_i\}_{i \in I}, \{s_i\}_{i \in I})$, where $\{U_i\}$ is an open cover of U in \mathcal{B} and $s_i \in \mathcal{G}(U_i)$. We require $\{s_i\}_{i \in I}$ to be compatible, namely $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Two pairs $(\{U_i\}_{i \in I}, \{s_i\}_{i \in I})$ and $(\{V_j\}_{j \in J}, \{t_j\}_{j \in J})$ are equivalent if there exists a common refinement $\{W_k\}_{k \in K}$ in \mathcal{B} of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ such that $\{s_i\}$ and $\{t_j\}$ restrict to the same family on $\{W_k\}$.

□

Proposition 1.4.2. *Let $X = \text{Spec}(A)$, $\mathcal{B} = \{D(f) \mid f \in A\}$. Then the functor*

$$\begin{aligned} \mathcal{B}^{\text{op}} &\rightarrow \text{Ring} \\ D(f) &\mapsto A_f \end{aligned}$$

extends uniquely to a sheaf \mathcal{O}_X on X up to isomorphism. Moreover, $\forall \mathfrak{p} \in X, \mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

Proof. The second assertion is clear. For the first assertion, let $U = D(f)$ be open and $\{D(f_i)\}_{i \in I}$ an open cover of U . Since $D(f) = \text{Spec}(A_f)$, we may assume $U = X$. The gluing property in this case says that

$$A \xrightarrow{\lambda} \prod_i A_{f_i} \rightrightarrows \prod_{i,j} A_{f_i f_j}$$

is an equalizer diagram.

Let us first show that the general case follows from the case of a finite cover. Since X is compact, there exists a subset $J \subset I$ such that $\{D(f_j)\}_{j \in J}$ covers X . The injectivity of λ follows from the case of a finite cover. Let $(s_i) \in \prod_i A_{f_i}$ such that $s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)}$ for all $i, j \in I$. By the case of a finite cover, there exists $s \in A$ such that $s_j = s|_{D(f_j)}$ for all $j \in J$. Then, for all i , $s_i|_{D(f_i f_j)} = s|_{D(f_i f_j)}$ and $s_i = s|_{D(f_i)}$ by the injectivity of λ for the cover $\{D(f_i f_j)\}_{j \in J}$ of $\{D(f_i)\}$.

Thus we may assume that I is finite. In this case λ is fully faithful and the result follows from Proposition 1.4.3 below. We also give a more direct proof as follows. Let $a \in A$ such that $a|_{D(f_i)} = 0$ for all i . Then for each i , there exists m_i such that $f_i^{m_i} a = 0$. But $\{D(f_i)\} = \{D(f_i^{m_i})\}$ cover X , so that $f_i^{m_i}$ generates the unit ideal. Therefore, 1 annihilates a and $a = 0$. It remains to check that every $(s_i) \in \prod_i A_{f_i}$ satisfying $s_i|_{D(f_i f_j)} = s_j|_{D(f_i f_j)}$ for all $i, j \in I$ is in the image of λ . Write $s_i = \frac{b_i}{f_i^{m_i}}$. We may multiply b_i with powers of f_i to assume $\forall i, m_i = m$. We have $\frac{b_i}{f_i^m} = \frac{b_j}{f_j^m} \in A_{f_i f_j}$. Hence there exists r such that $(f_i f_j)^r (b_i f_j^m - b_j f_i^m) = 0$. Up to replacing b_i by $b_i f_i^r$ and m by $m + r$, we may assume $b_i f_j^m - b_j f_i^m = 0$. Since $D(f_i^m)$ cover X , we have $1 = \sum a_i f_i^m$. Let $s = \sum a_i b_i$. Then, on $D(f_i)$, $s f_i^m = \sum_j a_j b_j f_i^m = \sum_j a_j b_i f_j^m = b_i$, so that $s|_{D(f_i)} = s_i$. □

Proposition 1.4.3. *Let $\phi: A \rightarrow B$ be a faithfully flat ring homomorphism. Then*

$$A \xrightarrow{\phi} B \rightrightarrows \begin{matrix} i_1 \\ i_2 \end{matrix} B \otimes_A B$$

is an equalizer diagram in the category $A\text{-Mod}$. The morphism i_1, i_2 are defined by $i_1(b) = b \otimes 1$ and $i_2(b) = 1 \otimes b$.

Recall that $\phi: A \rightarrow B$ is called **faithfully flat** if for every sequence of A -modules

$$M \longrightarrow N \longrightarrow P,$$

it is exact if and only if it is exact after tensoring with B :

$$M \otimes_A B \longrightarrow N \otimes_A B \longrightarrow P \otimes_A B.$$

Note that ϕ is faithfully flat if and only if ϕ is flat and $\text{Spec}(\phi)$ is surjective ([AM, Exercise 3.16], [M2, Theorem 7.3]).

Proof. Since ϕ is faithfully flat, we only need to prove that the diagram is an equalizer after tensoring with B on the right:

$$B \xrightarrow{\phi \otimes B} B \otimes_A B \xrightarrow{i_1 \otimes B} B \otimes_A B \otimes_A B \xrightarrow{i_2 \otimes B}$$

Define

$$\begin{aligned} f: B \otimes_A B &\rightarrow B \\ b_1 \otimes b_2 &\mapsto b_1 b_2 \\ g: B \otimes_A B \otimes_A B &\rightarrow B \otimes_A B \\ b_1 \otimes b_2 \otimes b_3 &\mapsto b_1 \otimes b_2 b_3 \end{aligned}$$

One readily checks that

$$\begin{aligned} f \circ \phi &= \text{id} \\ g \circ (i_1 \otimes B) &= \text{id} \\ \phi \circ f &= (i_2 \otimes B) \circ g \end{aligned}$$

This is called a **split equalizer** and one can show directly that a split equalizer is an equalizer. \square

Next we consider the functoriality of the sheaf of rings defined above with respect to ring homomorphisms. Let $\phi: A \rightarrow B$ a ring homomorphism. We have the corresponding continuous map

$$\begin{aligned} \phi^*: \text{Spec}(B) &\rightarrow \text{Spec}(A) \\ \mathfrak{q} &\mapsto \mathfrak{q}^c \end{aligned}$$

Let $X = \text{Spec}(B)$, $Y = \text{Spec}(A)$. For $g \in A$, we have

$$\begin{aligned} (\phi^*)^{-1}(D(g)) &= D(\phi(g)) \\ \mathcal{O}_Y(D(g)) &= A_g \\ \phi_* \mathcal{O}_X(D(g)) &= \mathcal{O}_X(D(\phi(g))) = B_{\phi(g)} \end{aligned}$$

The homomorphism ϕ naturally induces a homomorphism $A_g \rightarrow B_{\phi(g)}$. This defines a morphism of sheaves $f^\flat: \mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_X$, which corresponds by adjunction to $f^\sharp: \phi^{-1} \mathcal{O}_Y \rightarrow \mathcal{O}_X$. For $\mathfrak{p} \in \text{Spec}(B)$, f^\sharp induces $\mathcal{O}_{Y, \phi(\mathfrak{p})} \simeq (\phi^{-1} \mathcal{O}_Y)_{\mathfrak{p}} \rightarrow \mathcal{O}_{X, \mathfrak{p}}$.

Definition 1.4.4. A **ringed space** consists of a pair (X, \mathcal{O}_X) , where X is a topological space and \mathcal{O}_X is a sheaf of rings on X . A **locally ringed space** is a ringed space such that $\forall x \in X$, $\mathcal{O}_{X,x}$ is a local ring.

A morphism of ringed spaces is a pair $(f, f^\sharp): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, where $f: X \rightarrow Y$ is a continuous map and $f^\sharp: f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ is a morphism of sheaves of rings. A morphism of locally ringed spaces is a morphism of ringed spaces such that $\forall x \in X$, $f_x^\sharp: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local ring homomorphism. (Recall that a homomorphism between local rings $\phi: B \rightarrow A$ is called **local** if $\phi(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$, or, equivalently, $\phi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_B$.) For two morphisms of locally ringed spaces

$$(X, \mathcal{O}_X) \xrightarrow{(f, f^\sharp)} (Y, \mathcal{O}_Y) \xrightarrow{(g, g^\sharp)} (Z, \mathcal{O}_Z)$$

the composition is $(gf, (gf)^\sharp)$, where $(gf)^\sharp$ is defined by

$$(g \circ f)^{-1}\mathcal{O}_Z \cong f^{-1}g^{-1}(\mathcal{O}_Z) \xrightarrow{f^{-1}(g^\sharp)} f^{-1}\mathcal{O}_Y \xrightarrow{g^\sharp} \mathcal{O}_X.$$

Definition 1.4.5. An **affine scheme** is a locally ringed space that is isomorphic to $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ for some ring A . A **scheme** X is a locally ringed space (X, \mathcal{O}_X) such that there exists an open cover $\{U_i\}$ of X such that the restriction $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for all i . For schemes X and Y , a morphism of schemes $X \rightarrow Y$ is a morphism of locally ringed spaces.

We denote the category of schemes by Sch , which is a full subcategory of the category of locally ringed spaces.

Proposition 1.4.6. *The functor*

$$\begin{aligned} \text{Spec}: \text{Ring}^{\text{op}} &\rightarrow \text{Sch} \\ A &\mapsto (\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)}) \end{aligned}$$

is fully faithful.

Proof. For A, B rings, we need to check that the map

$$\Psi: \text{Hom}_{\text{Ring}}(A, B) \rightarrow \text{Hom}_{\text{Sch}}(\text{Spec}(B), \text{Spec}(A))$$

is a bijection. Let $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. We define

$$\Phi: \text{Hom}_{\text{Sch}}(Y, X) \rightarrow \text{Hom}_{\text{Ring}}(A, B)$$

by $(f, f^\sharp) \mapsto f_Y^\sharp: A = \mathcal{O}_X(X) \rightarrow \mathcal{O}_Y(Y) = B$. It is easy to see $\Phi \circ \Psi = \text{id}$. It remains to show $\Psi \circ \Phi = \text{id}$. Let $(f, f^\sharp): \text{Spec}(B) \rightarrow \text{Spec}(A)$ be a morphism and let $\phi = \Phi(f, f^\sharp): A \rightarrow B$. For $\mathfrak{q} \in \text{Spec}(B)$, we have a natural commutative diagram defined by restricting to stalks:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{q})} & \xrightarrow{f_q^\sharp} & B_\mathfrak{q} \end{array}$$

Since f^\sharp is a local ring morphism we have $f(\mathfrak{q}) = \phi^{-1}(\mathfrak{q})$. Moreover, by the universal property of localization, f_q^\sharp must be the morphism induced by ϕ . This concludes that $\Psi \circ \Phi = \text{id}$. \square

If we did not require f^\sharp to induce local homomorphisms, then the above proposition would fail to hold. For example, $\text{Hom}_{\text{Ring}}(\mathbb{Z}, \mathbb{Q})$ has only one element, but for every $\mathfrak{p} \in \text{Spec}(\mathbb{Z})$, we can define a morphism of ringed spaces $(f, f^\sharp): \text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z})$ of image $\{\mathfrak{p}\}$ with $f_{\text{Spec}(\mathbb{Q})}^\sharp$ given by $\mathbb{Z}_\mathfrak{p} \rightarrow \mathbb{Q}$.

Example 1.4.7. (1) $\text{Spec}(0) = \emptyset$ is the initial object of Sch .

- (2) For a field k , $\text{Spec}(k)$ is a point equipped with the constant sheaf of value k .
- (3) For $A = k[\epsilon]/(\epsilon^2)$, $\text{Spec}(A)$ is a point equipped with the constant sheaf of value A .
- (4) For a discrete valuation ring (DVR) A , $X = \text{Spec}(A) = \{\eta, s\}$ where $\eta = (0)$ and $s = \mathfrak{m}$ is the unique maximal ideal of A . We have $\mathcal{O}_X(\eta) = \text{Frac}(A)$, $\mathcal{O}_X(X) = A$.
- (5) $\text{Spec}(\mathbb{Z})$.
- (6) For a ring A , $\mathbb{A}_A^n := \text{Spec}(A[x_1, \dots, x_n])$ is called the **affine n -space** over A . For $n \geq 2$, not all opens are principal (see below).

Definition 1.4.8. Let X be a scheme, U an open subset of X . It is easy to see that $(U, \mathcal{O}_X|_U)$ is a scheme. This is called an **open subscheme** of X .

A morphism of scheme $f: Y \rightarrow X$ is called an **open immersion** if f identifies Y with an open subscheme of X , i.e. f is a composition $Y \xrightarrow{g} U \xrightarrow{j} X$, where g is an isomorphism and j is the inclusion of an open subscheme.

Not all schemes are affine.

Example 1.4.9. Let $X = \mathbb{A}_k^2$, $U = X \setminus V(x, y)$. Namely U is the open subset formed by removing the origin. We observe that $U = D(x) \cup D(y)$, so that $\mathcal{O}(U)$ is

$$\text{Eq}(\mathcal{O}(D(x)) \times \mathcal{O}(D(y))) \rightrightarrows \mathcal{O}(D(x) \cap D(y)))$$

$$k[x, y, x^{-1}] \times k[x, y, y^{-1}] \quad k[x, y, x^{-1}, y^{-1}]$$

The equalizer is $k[x, y, x^{-1}] \cap k[x, y, y^{-1}] = k[x, y]$. Thus the map

$$\Phi: \text{Hom}_{\text{Sch}}(X, U) \rightarrow \text{Hom}_{\text{Ring}}(\mathcal{O}_U(U), \mathcal{O}_X(X))$$

defined by $(f, f^\sharp) \mapsto f_U^\flat$ is not surjective. In particular, U is not affine.

Example 1.4.10. For a family of schemes $\{X_i\}_{i \in I}$, the coproduct is $X = \coprod_i X_i$, equipped with \mathcal{O}_X defined by $\mathcal{O}_X(\coprod_i U_i) = \prod_i \mathcal{O}_{X_i}(U_i)$. If I is infinite and X_i is non-empty for all i , then X is not quasi-compact, and hence not an affine scheme. On the other hand, if I is finite with $X_i = \text{Spec}(A_i)$, then $X \cong \text{Spec}(\prod_i A_i)$.

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Definition 1.4.11. Let X be a topological space, $\{U_i\}$ an open cover. A **Gluing Datum** consists of a family of sheaves \mathcal{F}_i over U_i and a family of morphisms $\gamma_{ij}: \mathcal{F}_i|_{U_i \cap U_j} \rightarrow \mathcal{F}_j|_{U_i \cap U_j}$, such that

- (1) $\gamma_{ii} = \text{id}$ and
- (2) $\gamma_{ik} = \gamma_{jk} \circ \gamma_{ij}$ on $U_i \cap U_j \cap U_k$.

A morphism of gluing data $(\mathcal{F}_i, \gamma_{ij}) \rightarrow (\mathcal{G}_i, \delta_{ij})$ is a family of morphisms of sheaves $\phi_i: \mathcal{F}_i \rightarrow \mathcal{G}_i$ such that

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{\phi_i} & \mathcal{G}_i \\ \downarrow \gamma_{ij} & & \downarrow \delta_{ij} \\ \mathcal{F}_j & \xrightarrow{\phi_j} & \mathcal{G}_j \end{array}$$

is commutative.

Lemma 1.4.12 (Gluing sheaves). *We have an equivalence of categories $\text{Shv}(X, \mathcal{C}) \cong \{\text{gluing data}\}$.*

Proof. Let $(\mathcal{F}_i, \gamma_{ij})$ be a gluing datum. Define

$$\mathcal{F}(U) = \text{Eq} \left(\prod_i \mathcal{F}_i(U \cap U_i) \rightrightarrows_{\pi_2}^{\pi_1} \prod_{ij} \mathcal{F}_i(U \cap U_i \cap U_j) \right)$$

where π_1 is induced by the restriction $\mathcal{F}_i(U \cap U_i) \longrightarrow \mathcal{F}_i(U \cap U_i \cap U_j)$ and π_2 is induced by $\mathcal{F}_i(U \cap U_i) \longrightarrow \mathcal{F}_i(U \cap U_i \cap U_j) \xrightarrow{\gamma_{ij}} \mathcal{F}_j(U \cap U_j \cap U_i)$.

□

Lemma 1.4.13 (Gluing morphisms of schemes). *Let X, Y be schemes, $\{U_i\}_{i \in I}$ an open cover of X . Then*

$$\text{Hom}_{\text{Sch}}(X, Y) \longrightarrow \prod_i \text{Hom}_{\text{Sch}}(U_i, Y) \rightrightarrows \prod_{ij} \text{Hom}_{\text{Sch}}(U_i \cap U_j, Y)$$

is an equalizer diagram. More generally, $U \mapsto \text{Hom}_{\text{Sch}}(-, Y)$ is a sheaf of sets on X .

Proof. Let $(f_i: U_i \rightarrow Y)$ be a compatible family of morphism. We first glue them in the category of topological spaces and get a continuous map $f: X \rightarrow Y$. Then $f_i^\sharp: (f^{-1}\mathcal{O}_Y)|_{U_i} \rightarrow \mathcal{O}_X|_{U_i}$ is a compatible family of morphisms of sheaves, namely a morphism of gluing data and the previous lemma tells us that there exists a unique $f^\sharp: f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ that restricts to f_i^\sharp . □

Remark 1.4.14. The above lemma implies that if X is a scheme and $\{U_i\}$ is an open cover, $U_{ij} = U_i \cap U_j$, then

$$\coprod_{ij} U_{ij} \rightrightarrows \coprod_i U_i \longrightarrow X$$

is a coequalizer diagram in the category Sch .

Lemma 1.4.15 (Gluing schemes). *Let $\{X_i\}_{i \in I}$ be a family of schemes. Let $X_{ij} \subseteq X_i$ be open sub-schemes and $f_{ij}: X_{ij} \rightarrow X_{ji}$ isomorphisms of schemes for all $i, j \in I$. We require*

- (1) $f_{ii} = \text{id}$
- (2) $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$
- (3) $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \cap X_{ik}$.

Then there exists a scheme X and open immersions $f_i: X_i \rightarrow X$ such that

$$\begin{array}{ccc} X_{ij} & \hookrightarrow & X_i \\ \downarrow f_{ij} & & \searrow f_i \\ X_{ji} & \hookrightarrow & X_j \xrightarrow{f_j} X \end{array}$$

and has the universal property: For every scheme Y and a family of morphisms of schemes $g_i: X_i \rightarrow Y$ satisfying

$$\begin{array}{ccc} X_{ij} & \hookrightarrow & X_i \\ \downarrow f_{ij} & & \searrow g_i \\ X_{ji} & \hookrightarrow & X_j \xrightarrow{g_j} Y \end{array}$$

then there exists a unique $g: X \rightarrow Y$ such that

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & X \\ & \searrow g_i & \downarrow g \\ & & Y \end{array}$$

is commutative.

Proof. Let $X = \coprod_i X_i / \sim$, where $x \in X_i \sim y \in X_j \Leftrightarrow y = f_{ij}x$. This makes a topological space X with open subsets $X_i \subseteq X$. We have a sheaf $\mathcal{O}_{X,i}$ on each X_i , and we glue them to get \mathcal{O}_X . \square

Example 1.4.16. Consider $\mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n])$, $n \geq 1$ and the origin $O = V(x_1, \dots, x_n)$. Let $X_0 = X_1 = \mathbb{A}_k^n$, $X_{01} = X_{10} = \mathbb{A}_k^n \setminus \{O\}$. We then glue them by $X_{01} \xrightarrow{\text{id}} X_{10}$. The resulting scheme X is called the affine n -space with doubled origin. We have

$$\mathcal{O}_X(X) = \text{Eq} \left(\mathcal{O}(X_0) \times \mathcal{O}(X_1) \rightrightarrows \mathcal{O}(X_0 \cap X_1) \right) = k[x_1, \dots, x_n].$$

Since the two morphisms $f_i: \mathbb{A}_k^n = X_i \rightarrow X$, $i = 0, 1$ induce the same ring homomorphism on global sections, the map

$$\Phi: \text{Hom}_{\text{Sch}}(\mathbb{A}_k^n, X) \rightarrow \text{Hom}_{\text{Ring}}(\mathcal{O}_X(X), \mathcal{O}_{\mathbb{A}_k^n}(\mathbb{A}_k^n))$$

is not an injection. This shows that X is not affine.

Example 1.4.17. Let $X_0 = X_1 = \mathbb{A}_k^1$, $X_{01} = X_{10} = \mathbb{A}_k^1 \setminus \{O\}$. Write $X_{01} = \text{Spec}(k[x, x^{-1}])$, $X_{10} = \text{Spec}(k[y, y^{-1}])$. Gluing them by $x \mapsto y^{-1}$, we get the projective line \mathbb{P}_k^1 over k .

Example 1.4.18. More generally, let A be a ring, $X_i = \text{Spec}(A[T_i^{-1}T_0, \dots, T_i^{-1}T_n]) \simeq \mathbb{A}_A^n$. Let $X_{ij} = D(T_i^{-1}T_j) \subset X_i$. Then

$$X_{ij} = \text{Spec}(A[T_i^{-1}T_k, T_j^{-1}T_l]_{k=0}^n) = \text{Spec}(A[T_i^{-1}T_k, T_i^{-1}T_l]_{k=0}^n) = X_{ji}.$$

Gluing them by the identity morphisms, we get $X = \mathbb{P}_A^n$, the projective n -space over A . It can be shown from the construction that $\mathcal{O}_X(X) = \bigcap_i A[T_i^{-1}T_0, \dots, T_i^{-1}T_n] = A$. For $A \neq 0$ and $n \geq 1$, \mathbb{P}_A^n is not affine.

Proposition 1.4.19. *Let X be a scheme, $Y = \text{Spec}(A)$ an affine scheme. Then the map $\text{Hom}_{\text{Sch}}(X, Y) \rightarrow \text{Hom}_{\text{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ sending f to f_Y^\flat is a bijection.*

Proof. Let $X = \bigcup_i U_i$, U_i open affine. Then by gluing morphisms of schemes, we have

$$\text{Hom}_{\text{Sch}}(X, Y) = \text{Eq} \left(\prod_i \text{Hom}_{\text{Sch}}(U_i, Y) \rightrightarrows \prod_{ij} \text{Hom}_{\text{Sch}}(U_i \cap U_j, Y) \right)$$

Write $U_i \cap U_j = \bigcup_k U_{ijk}$ with U_{ijk} open affine, then

$$\text{Hom}_{\text{Sch}}(X, Y) = \text{Eq} \left(\prod_i \text{Hom}_{\text{Sch}}(U_i, Y) \rightrightarrows \prod_{ijk} \text{Hom}_{\text{Sch}}(U_{ijk}, Y) \right)$$

but for X affine, we have $\text{Hom}_{\text{Sch}}(X, Y) \cong \text{Hom}_{\text{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$. Therefore the above equalizer diagram is isomorphism to

$$\text{Eq} \left(\prod_i \text{Hom}_{\text{Ring}}(\mathcal{O}_Y(Y), \mathcal{O}_X(U_i)) \rightrightarrows \prod_{ijk} \text{Hom}_{\text{Sch}}(\mathcal{O}_Y(Y), \mathcal{O}_X(U_{ijk})) \right)$$

Since

$$\mathcal{O}_X(X) \longrightarrow \prod_i \mathcal{O}_X(U_i) \rightrightarrows \prod_{ijk} \mathcal{O}_X(U_{ijk})$$

is an equalizer diagram by the sheaf condition, we get the desired equalizer diagram by applying $\text{Hom}_{\text{Ring}}(\mathcal{O}_Y(Y), -)$. \square

Remark 1.4.20. We have

$$\text{Sch} \xrightleftharpoons[\text{Spec}]{\Gamma} \text{Ring}^{\text{op}} \quad \Gamma \dashv \text{Spec},$$

where Γ is the functor sending X to $\mathcal{O}_X(X)$. It follows that Spec transforms colimits in Ring to limits in Sch . Moreover, Spec is fully faithful and equivalently $\Gamma \circ \text{Spec} \cong \text{id}$.

Example 1.4.21. (1) Since \mathbb{Z} is initial in Ring , $\text{Spec}(\mathbb{Z})$ is the final object of Sch .

(2) Pushouts in Ring are given by tensor product. Hence $\text{Spec}(B \otimes_A C) \cong \text{Spec}(B) \times_{\text{Spec}(A)} \text{Spec}(C)$.

Example 1.4.22.

$$\mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(\mathbb{Z}[T])) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[T], \mathcal{O}_X(X)) \cong \mathcal{O}_X(X)$$

$$f \longmapsto \Gamma(f) \longmapsto \Gamma(f)(T)$$

Recall the **Yoneda embedding**. Let \mathcal{C} be a locally small category. For every object X , consider the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{Set}$. The Yoneda embedding is the functor

$$h: \mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \quad X \mapsto h_X,$$

which is fully faithful.

In the case of $\mathrm{Ring}^{\mathrm{op}}$ and Sch , we have functors

$$\begin{array}{ccc} \mathrm{Ring}^{\mathrm{op}} & \xrightarrow{\mathrm{Spec}} & \mathrm{Sch} \\ \downarrow h & & \downarrow h \\ \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) & \xleftarrow{\circ \mathrm{Spec}} & \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set}) \end{array}$$

The diagram commutes up to isomorphism by the full faithfulness of Spec . The functor $\circ \mathrm{Spec}$ is not fully faithful. However, by gluing morphisms of schemes one obtains the following.

Proposition 1.4.23. *The functor*

$$\begin{aligned} \mathrm{Sch} &\longrightarrow \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) \\ Y &\longmapsto (B \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), Y)) \end{aligned}$$

is fully faithful.

Proof. Let X and Y be schemes. Denote the functor

$$B \mapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(B), X)$$

by F_X . We construct an inverse of the map $f \mapsto F_f$ as follows. Let φ be a natural transformation from F_X to F_Y . Cover X by open affine subsets $\{U_i\}$ and cover $U_i \cap U_j$ by open affine subsets U_{ijk} . Then by Remark 1.4.14,

$$\coprod_{ijk} U_{ijk} \rightrightarrows \coprod_i U_i \longrightarrow X$$

is a coequalizer diagram. Apply φ we get a corresponding diagram involving Y and a unique morphism f making the diagram commutative:

$$\begin{array}{ccc} \coprod_{ijk} U_{ijk} & \rightrightarrows & \coprod_i U_i \longrightarrow X \\ & & \searrow f \\ & & Y \end{array}$$

One then checks that $\varphi \mapsto f$ is the desired inverse. \square

Remark 1.4.24. The proposition implies that a morphism of schemes $f: X \rightarrow Y$ is an isomorphism if and only if for every ring B , $\text{Hom}_{\text{Sch}}(\text{Spec}(B), X) \rightarrow \text{Hom}_{\text{Sch}}(\text{Spec}(B), Y)$ is an isomorphism.

We sometimes regard Sch via these fully faithful functors as subcategories of $\text{Fun}(\text{Ring}, \text{Set})$ or $\text{Fun}(\text{Sch}^{\text{op}}, \text{Set})$.

Definition 1.4.25. Let S be a scheme. The category Sch/S of S -schemes or schemes over S is defined as follows. An object of Sch/S is a scheme X equipped with a morphism of schemes $f: X \rightarrow S$. A morphism from $(X, f: X \rightarrow S)$ to $(Y, g: Y \rightarrow S)$ is a morphism of schemes $h: X \rightarrow Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

is commutative.

For two S -schemes $T \rightarrow S$ and $X \rightarrow S$, the set of T -points of X is defined by $X(T) = \text{Hom}_{\text{Sch}/S}(T, X)$. For $T = \text{Spec}(A)$, we write $X(A) := X(\text{Spec}(A))$ and we refer to $\text{Spec}(A)$ -points as A -points.

Example 1.4.26. Let $\mathbb{A}_A^n = \text{Spec}(A[x_1, \dots, x_n])$ and let $a: \mathbb{A}_A^n \rightarrow \text{Spec}(A)$ be the canonical morphism. In $\text{Sch}/A := \text{Sch}/\text{Spec}(A)$, an A -point of \mathbb{A}_A^n is a morphism $s: \text{Spec}(A) \rightarrow \mathbb{A}_A^n$ that makes

$$\begin{array}{ccc} \text{Spec}(A) & \xrightarrow{f} & \mathbb{A}_A^n \\ & \searrow \text{id} & \swarrow a \\ & \text{Spec}(A) & \end{array}$$

commutative, namely a section of a . This corresponds to an A -algebra homomorphism $\phi: A[x_1, \dots, x_n] \rightarrow A$, which is uniquely determined by $(\phi(x_1), \dots, \phi(x_n)) \in A^n$. Thus the set $\mathbb{A}_A^n(A)$ can be identified with A^n .

1.5 Topology of schemes

Lemma 1.5.1. *Let X be a scheme. Then $\mathcal{O}_X(X) = 0$ if and only if $X = \emptyset$.*

Proof. Let X be a scheme such that $\mathcal{O}_X(X) = 0$. Take U open affine subset, since $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(U)$ is a ring homomorphism, it sends $1 = 0$ to 1 , hence $\mathcal{O}_X(U) = 0$ and $U = \text{Spec}(0) = \emptyset$. \square

Definition 1.5.2. Let X be a scheme, $f \in \mathcal{O}_X(X)$. Define $X_f = \{x \in X \mid f_x \in \mathcal{O}_{X,x}^\times\}$.

Example 1.5.3. $(\text{Spec}(A))_f = D(f)$. If $U = \text{Spec}(A) \subset X$ is an open affine, then $X_f \cap U = D(f|_U)$. It follows that $X_f \subseteq X$ is open.

Remark 1.5.4. It is easy to see that $X_f \cap X_g = X_{fg}$, $X_{f+g} \subseteq X_f \cup X_g$, $X_0 = \emptyset$, and $X_f = X$ for $f \in \mathcal{O}(X)^\times$.

Proposition 1.5.5. *For any scheme X , we have a bijection*

$$\{\text{Open and closed subsets of } X\} \cong \{\text{idempotent elements in } \mathcal{O}_X(X)\}.$$

Proof. Let U be open and closed. Then $X = U \amalg U^c$, where U^c is the complement of U . Let $e_U \in \mathcal{O}_X(X)$ such that $e_U|_U = 1$ and $e_U|_{U^c} = 0$. Then e_U is an idempotent element.

Let $e \in \mathcal{O}_X(X)$ be an idempotent element and consider X_e and X_{1-e} . By the remark preceding the proposition, $X = X_e \amalg X_{1-e}$. Thus X_e is open and closed.

It is clear that $X_{e_U} = U$. Moreover, let $s = e_{X_e}$. The only idempotents in a local ring are 0 and 1. It follows that the germs of s and e agree at every point. This implies $s = e$. \square

We say that a scheme is connected if its underlying topological space is connected.

Corollary 1.5.6. *A scheme X is connected if and only if the only idempotents of $\mathcal{O}_X(X)$ are 0, 1.*

Definition 1.5.7. A topological space X is called **irreducible** if it is nonempty and if $X = F_1 \cup F_2$ with F_1, F_2 closed implies $X = F_1$ or $X = F_2$.

Remark 1.5.8. • Irreducible \Rightarrow connected.

- A Hausdorff space cannot be irreducible unless X is a point.

Lemma 1.5.9. *Let X be a topological space, $Y \subseteq X$.*

(1) *Y is irreducible if and only if Y is nonempty and whenever $Y \subseteq F_1 \cup F_2$, for closed subsets F_1, F_2 in X , we have $Y \subseteq F_1$ or $Y \subseteq F_2$.*

(2) *Y is irreducible if and only if \overline{Y} is irreducible.*

Lemma 1.5.10. *Let X be a nonempty topological space. Then X is irreducible if and only if every non-empty open subset U is dense in X . In that case, U is irreducible as well.*

Example 1.5.11. Let $X = \text{Spec}(k[x, y])$, $Y = V(xy)$. Then Y is not irreducible since $Y = V(x) \cup V(y)$.

Definition 1.5.12. Let X be a topological space. If $X = \overline{\{\eta\}}$, then we call η a **generic point** of X .

If X has a generic point, then X is irreducible.

Lemma 1.5.13. *Let A be a ring, $I \subseteq A$ an ideal. Then $V(I)$ is irreducible if and only if $\sqrt{I} = \mathfrak{p}$ is a prime ideal. In that case, \mathfrak{p} is the only generic point of $V(I)$.*

Proof. Up to replacing A by A/\sqrt{I} we may assume that $I = 0$ and is radical. Then $\text{Spec}(A)$ is irreducible if and only if whenever $D(f), D(g) \neq \emptyset$, we have $D(f) \cap D(g) = D(fg) \neq \emptyset$ if and only if whenever $f, g \neq 0$, we have $fg \neq 0$. Moreover, if $\mathfrak{p} = (0)$ is a prime, then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \text{Spec}(A)$, so that \mathfrak{p} is the generic point. \square

Definition 1.5.14. Let X be a topological space. We say that X is **sober** if every irreducible subset has a generic point. We say that X is a **T_0 space** (or Kolmogorov space) if for all $x \neq y \in X$, there exists either an open neighborhood U of x such that $y \notin U$ or an open neighborhood V of y such that $x \notin V$.

Consider the map

$$\begin{aligned} F: X &\rightarrow \{\text{irreducible closed subsets of } X\} \\ x &\mapsto \overline{\{x\}} \end{aligned}$$

Observe that X is T_0 if and only if F is injective, and X is sober if and only if F is bijective. Thus we have sober $\Rightarrow T_0$.

Proposition 1.5.15. *The underlying topological space of every scheme is sober.*

This follows from Lemma 1.5.13 and the following.

Lemma 1.5.16. *Any locally closed subspace of a sober space is sober. A topological space admitting an open cover by sober spaces is sober.*

Proof. Exercise. □

Definition 1.5.17. Let X be a scheme. We say that X is **irreducible** if its underlying topological space is irreducible. We say that X is reduced **reduced** if for every open subset U , $\mathcal{O}_X(U)$ is reduced. (Recall that a ring A is called **reduced** if $\sqrt{(0)} = (0)$). We say that X is **integral** if $X \neq \emptyset$ and for every nonempty open subset U , $\mathcal{O}_X(U)$ is an integral domain.

Proposition 1.5.18. *Let X be a scheme, then*

- (1) X is reduced if and only if $\forall x \in X$, $\mathcal{O}_{X,x}$ is reduced.
- (2) X is integral if and only if X is irreducible and reduced.

Proof. (1) \Rightarrow since localization preserves reduced.

\Leftarrow Let $s \in \mathcal{O}_X(U)$ and $s^n = 0$. For all $x \in U$, since $\mathcal{O}_{X,x}$ is reduced, we have $s_x = 0$. It follows that $s = 0$.

- (2) \Rightarrow X is easily seen to be reduced. Suppose $U_1, U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$. Then $\mathcal{O}_X(U_1 \cup U_2) \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not a domain.

\Leftarrow It suffices to show that $\mathcal{O}_X(X)$ is integral. Suppose $f, g \in \mathcal{O}_X(X)$, $fg = 0$. Then $X_f \cap X_g = \emptyset$, and hence $X_f = \emptyset$ or $X_g = \emptyset$. Say $X_f = \emptyset$. Then for each open affine subset $V = \text{Spec}(A)$, $V \cap X_f = D(f|_V) = \emptyset$. This implies that $f|_V$ is nilpotent. Since V is arbitrary, f must be 0.

□

Warning 1.5.19. It is **not** true in general without assuming X quasi-compact that a global section of \mathcal{O}_X is nilpotent if and only if every germ of it is nilpotent.

Example 1.5.20. (1) $\text{Spec}(A)$ is reduced if and only if A is reduced.

- (2) $\text{Spec}(A)$ is irreducible if and only if $\sqrt{0}$ is a prime ideal.
- (3) $\text{Spec}(A)$ is integral if and only if A is integral.

Definition 1.5.21. A **spectral space** is a sober, quasi-compact space such that

- (1) quasi-compact opens form a basis.
- (2) Finite intersections of quasi-compact opens are quasi-compact.

A continuous map between spectral spaces $f: X \rightarrow Y$ is called **spectral** if $\forall V$ quasi-compact open of Y , $f^{-1}(V)$ is quasi-compact.

We denote by Sp the category of spectral spaces and spectral maps.

Theorem 1.5.22 (Hochster). *The essential image of the functor $\text{Spec}: \text{Ring} \rightarrow \text{Top}$ is Sp .*

Date: 9.29

Lemma 1.5.23. *Let X be an integral scheme with a generic point η . Then*

- (1) $\mathcal{O}_{X,\eta}$ is a field called the **function field** of X .
- (2) For $U \subseteq X$ open, the natural map $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,\eta}$ is injective.

Proof. To see that $\mathcal{O}_{X,\eta}$ is a field, we may take an arbitrary nonempty open affine subset $U = \text{Spec}(A)$ and observe that $\mathcal{O}_{X,\eta} = A_{(0)}$ is the fraction field of A .

For the second statement, we may replace U by an nonempty open affine subset and reduce to the case where $U = \text{Spec}(A)$ is affine. In this case $\mathcal{O}_X(U) = A \rightarrow \text{Frac}(A) = \mathcal{O}_{X,\eta}$ is injective. \square

Corollary 1.5.24. *For X integral and open subsets $\emptyset \neq U \subseteq V \subseteq X$, the restriction map $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$ is injective.*

Recall that every topological space X is the disjoint union of connected components. Each connected component is closed but not necessarily open.

Definition 1.5.25. Let X be a topological space, an **irreducible component** of X is a maximal irreducible subset of X .

An irreducible component is necessarily closed. By Zorn's Lemma, every irreducible subset is contained in some irreducible component. Since every point is irreducible, every topological space X is the union of its irreducible components.

Lemma 1.5.26. *Let $X = \bigcup_{i=1}^n Y_i$ be a finite union of irreducible closed subsets. Then the irreducible components of X are the maximal elements of the family $\{Y_i\}_{i=1}^n$. In particular, if there are no inclusions among the Y_i 's, then the irreducible components of X are $\{Y_i\}_{i=1}^n$.*

Proof. Indeed, every irreducible subset of X is contained in some Y_i . \square

Example 1.5.27. For $A = k[x,y]/(xy)$, $\text{Spec}(A) = V(x) \cup V(y)$. The irreducible components of $\text{Spec}(A)$ are $V(x)$ and $V(y)$.

Example 1.5.28. Let S be a profinite set and k a field. Consider the constant sheaf k_S on S and $A = k_S(S) = \{f: S \rightarrow k \text{ locally constant}\}$. It is easy to see that $\text{Spec}(A) \cong (S, k_S)$. Thus, for S an infinite profinite set (e.g. the Cantor set), $\text{Spec}(A)$ has infinitely many irreducible components.

In the case $k = \mathbb{F}_2$, k_S can be identified with the Boolean algebra of open closed subsets of S .

Definition 1.5.29. Let X be a topological space, $x, y \in X$. We say that x **specializes** to y or y **generalizes** to x and we write $x \rightsquigarrow y$, if $y \in \overline{\{x\}}$.

Let X be a T_0 space. Generalization defines a partial order: $x \leqslant y \iff x \in \overline{\{y\}} \iff \overline{\{x\}} \subseteq \overline{\{y\}}$.

- The minimal points are the closed points.

- If X is sober, then the maximal points are the generic points of irreducible components.

Example 1.5.30. In $\text{Spec}(A)$, $\overline{\{x_{\mathfrak{p}}\}} = V(\mathfrak{p})$. Here $x_{\mathfrak{p}} \in \text{Spec}(A)$ denotes the point corresponding to the prime ideal \mathfrak{p} . We have

$$\begin{aligned} \mathfrak{p} \subseteq \mathfrak{q} &\Leftrightarrow \overline{\{x_{\mathfrak{p}}\}} \supseteq \overline{\{x_{\mathfrak{q}}\}} \\ &\Leftrightarrow x_p \rightsquigarrow x_q. \end{aligned}$$

Thus we have a bijection

$$\begin{aligned} \{\text{irreducible components of } \text{Spec}(A)\} &\longleftrightarrow \{\text{minimal primes of } A\} \\ V(\mathfrak{p}) &\leftarrow \mathfrak{p}. \end{aligned}$$

Warning 1.5.31. Schwede gave an example of a scheme without a closed point. The underlying topological space looks like $x_0 \rightsquigarrow x_1 \rightsquigarrow \dots$. Note that an affine scheme must have closed points which correspond to maximal ideals.

Noetherian Spaces

Definition 1.5.32. A topological space X is called **Noetherian** if its closed subsets satisfy the descending chain condition, i.e. if $Y_1 \supseteq Y_2 \supseteq \dots$ is a descending chain of closed subsets, there exists N such that $Y_N = Y_{N+1} = \dots$. Equivalently, any nonempty family of closed subsets admits a minimal element.

Example 1.5.33. If A is a Noetherian ring, then $\text{Spec}(A)$ is Noetherian space.

Warning 1.5.34. If $\text{Spec}(A)$ is a Noetherian space, A may not be a Noetherian ring. Let $A = \bigcup_n k[[x^{1/n}]]$ be a union of rings of formal power series. Then $\text{Spec}(A) = \{\eta, s\}$ is a Noetherian space. Here η corresponds to the 0 ideal and s corresponds to the ideal generated by $x^{1/n}, n \in \mathbb{N}$. The ring A is not Noetherian.

Lemma 1.5.35. *Let X be a topological space. The following are equivalent:*

- (1) *X is Noetherian.*
- (2) *Every open subset of X is quasi-compact.*
- (3) *Every subset of X is quasi-compact.*

Proof. (3) \Rightarrow (2) is obvious.

For (2) \Rightarrow (1), note that the union U of an ascending chain of open subsets $U_1 \subseteq U_2 \subseteq \dots$ is open, hence is quasi-compact by assumption (2).

For (1) \Rightarrow (3), let $Y \subseteq X$ be a subset and $Z_1 \supseteq Z_2 \supseteq \dots$ be a descending chain of closed subsets in Y . Then $\overline{Z_i} \cap Y = Z_i$ where $\overline{Z_i}$ is the closure in X , and $\overline{Z_i}$ forms a descending chain of closed subsets in X . \square

Corollary 1.5.36. *If X is Noetherian, $Y \subseteq X$ with subspace topology. Then Y is Noetherian.*

Corollary 1.5.37. *X is Noetherian and sober $\Rightarrow X$ is a spectral space.*

Lemma 1.5.38. *X is Noetherian $\Rightarrow X$ has only finitely many irreducible components.*

Proof. Consider $\mathcal{F} = \{Y \subseteq X \mid Y \text{ is not a finite union of irreducible closed subsets}\}$. If it is not empty, we can find a minimal element Y by Noetherian hypothesis. Y cannot be irreducible, hence $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper closed subset. But at least one of Y_1, Y_2 must be in \mathcal{F} , hence there exists a smaller one, say $Y_1 \in \mathcal{F}$, violating the minimal property of Y . \square

Definition 1.5.39. Let X be a scheme.

- X is **quasi-compact** if its underlying space $\text{sp}(X)$ is quasi-compact.
- X is **locally Noetherian** if X can be covered by open affine subsets $U_i = \text{Spec}(A_i)$ with A_i Noetherian rings.
- X is **Noetherian** if X is quasi-compact and locally Noetherian.

Proposition 1.5.40. *Let X be a locally Noetherian scheme and $U = \text{Spec}(A)$ is an open affine subset. Then A is a Noetherian ring. In particular, a ring A is Noetherian if and only if $\text{Spec}(A)$ is a Noetherian scheme.*

Definition 1.5.41. Let \mathcal{P} be a collection of rings. We say that \mathcal{P} is **local** if it satisfies the following properties:

- (1) $A \in \mathcal{P}$ implies for any $f \in A, A_f \in \mathcal{P}$.
- (2) If there are $f_i \in A, 1 \leq i \leq n$ such that $\text{Spec}(A) = \bigcup_{i=1}^n D(f_i)$ and $A_{f_i} \in \mathcal{P}$, then $A \in \mathcal{P}$.

Lemma 1.5.42. *Let \mathcal{P} be a local collection of rings, and X a scheme with an open affine cover $\{U_i\}_{i \in I}$ with $U_i = \text{Spec}(A_i)$, where each $A_i \in \mathcal{P}$. Then for every open affine subset $U = \text{Spec}(A)$, we have $A \in \mathcal{P}$.*

The proof relies on the following technical result.

Lemma 1.5.43. *Let X be a scheme and $U = \text{Spec}(A), V = \text{Spec}(B)$ open affine subsets. Then $U \cap V$ can be written as a union of open affine subsets which are principal open subsets of both U and V .*

Proof. For $x \in U \cap V$, choose a principal open subset W of U that covers x and is contained in V . Up to replacing U by W , we may assume $U \subseteq V$. Choose $f \in B$ such that $V_f = \text{Spec}(B_f) \subseteq U$. We observe that $V_f = U_{\bar{f}} = \text{Spec}(A_{\bar{f}})$ is also a principal open subset of U . Here $\bar{f} = f|_U$. \square

Proof of Lemma 1.5.42. Let $U = \text{Spec}(A)$ be an open affine subset. Then $U = \bigcup_i (U \cap U_i)$. By the previous lemma, $U \cap U_i$ can be covered by open affine subsets U_{ij} which are both principal in U and U_i . By hypothesis 1 of \mathcal{P} , each U_{ij} is the spectrum of a ring in \mathcal{P} . Since U is quasi-compact, we may choose finitely many of them and apply hypothesis 2 in the definition. \square

Proof of Proposition 1.5.40. It remains to show that $\mathcal{P} = \{\text{Noetherian rings}\}$ is a local collection. The first condition is easy to verify. For the second one, let $f_i \in A$, $1 \leq i \leq n$, satisfying $\text{Spec}(A) = \bigcup_i D(f_i)$ with each A_{f_i} Noetherian. We will show that every ideal $I \subseteq A$ is finitely generated. For each i , the ideal $IA_{f_i} \subseteq A_{f_i}$ is finitely generated. Let $\{a_{ij}\}_{j=1}^{m_i}$ be a family of generators of IA_{f_i} in I . Then $\{a_{ij}\}_{i,j}$ generates I . Indeed, if $\phi: A^{m_1+\dots+m_n} \rightarrow I$ denotes the homomorphism of A -modules given by $\{a_{ij}\}_{i,j}$, then ϕ_{f_i} is a surjection for every i , which implies that ϕ is a surjection. \square

Remark 1.5.44. If X is a Noetherian scheme, then its underlying space $\text{sp}(X)$ is Noetherian.

Warning 1.5.45. There exists a Noetherian space which is not the underlying space of any Noetherian scheme.

In fact, it follows from Krull's principal ideal theorem that a Noetherian scheme of dimension ≥ 2 (see below for the definition of dimension) must have infinitely many points. Thus spaces such as $X = \{x \rightsquigarrow y \rightsquigarrow z\}$ cannot be the underlying space of a Noetherian scheme.

Warning 1.5.46. For a Noetherian scheme X , $\mathcal{O}_X(X)$ is not a Noetherian ring in general. Consider the projective space \mathbb{P}_k^3 over a field k , with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$. Let $D = V(x_0)$ and $E = V(x_1)$ be distinct planes of \mathbb{P}_k^3 and let $l = V(x_0, x_2) \neq D \cap E$ be a projective line on D . Let $Y = D \cup E$ and $X = Y \setminus l$. Then X is a Noetherian scheme. We have $X = (D \setminus l) \cup (E \setminus O)$, where $O = E \cap l$. We have $\mathcal{O}_X(D \setminus l) = k[x, y]$, where $x = \frac{x_1}{x_2}$ and $y = \frac{x_3}{x_2}$, and $\mathcal{O}_X(E \setminus O) = k$. The restriction map $\mathcal{O}_X(D \setminus l) \rightarrow \mathcal{O}_X((D \setminus l) \cap E)$ is given by substituting $x = 0$. One can deduce that $\mathcal{O}_X(X) \cong k + xk[x, y] \subseteq k[x, y]$. This is not a Noetherian ring: there is an ascending chain of ideals $(x) \subseteq (x, xy) \subseteq (x, xy, xy^2) \subseteq \dots$.

Dimension

Definition 1.5.47. Let X be a topological space. The **dimension** of X , denoted by $\dim X$, is defined to be

$$\sup_n \{n \mid \exists Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \text{ such that } Y_i \text{ are irreducible closed}\}$$

Let $Y \subseteq X$ be an irreducible closed subset. The **codimension** $\text{codim}(Y, X)$ of Y is defined to be

$$\sup_n \{n \mid \exists Y = Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \text{ such that } Y_i \text{ are irreducible closed}\}$$

If Y is an arbitrary closed subset, we define its codimension as

$$\inf \{\text{codim}(Y', X) \mid Y' \subseteq Y \text{ irreducible and closed}\}$$

Example 1.5.48. $\dim \emptyset = -\infty$, $\text{codim}(\emptyset, X) = \infty$.

Lemma 1.5.49. Let X be a topological space.

- $\dim X = \sup\{\dim X_i \mid X_i \subseteq X \text{ are irreducible components}\}$
- If $Z \subseteq X$, $\dim Z \leq \dim X$.
- If $\{U_i\}$ is an open cover of X , then $\dim X = \sup_i(\dim U_i)$.

If X is a sober space,

$$\dim X = \sup\{n \mid \exists x_n \rightsquigarrow x_{n-1} \rightsquigarrow \cdots \rightsquigarrow x_0 \text{ with all } x_i \text{ distinct}\}$$

$$\operatorname{codim}(\overline{\{x\}}, X) = \sup\{n \mid \exists x_n \rightsquigarrow x_{n-1} \rightsquigarrow \cdots \rightsquigarrow x_0 = x \text{ with all } x_i \text{ distinct}\}$$

Example 1.5.50. Let $X = \operatorname{Spec}(A)$.

- $\dim(X) = \dim(A) = \sup\{n \mid \exists \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.
- For a prime ideal \mathfrak{p} , $\operatorname{codim}(V(\mathfrak{p}), X) = \operatorname{ht}(\mathfrak{p}) = \sup\{n \mid \exists \mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\}$

Theorem 1.5.51. Let A be a Noetherian ring.

- For every prime ideal \mathfrak{p} , $\operatorname{ht}(\mathfrak{p}) < \infty$.
- If A is local, then $\dim A < \infty$.

Warning 1.5.52. A Noetherian ring may have dimension ∞ (Nagata).

1.6 Morphisms and base change

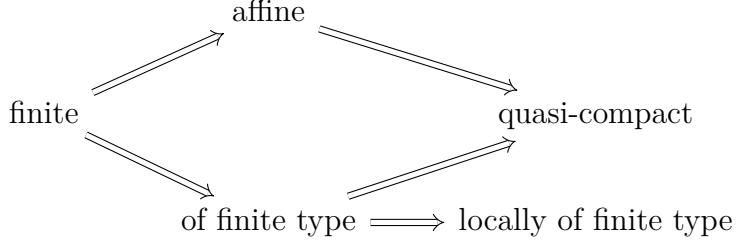
In this section, we talk about properties between morphisms of schemes.

Definition 1.6.1. Let $f : Y \rightarrow X$ be a morphism of schemes.

- f is called **locally of finite type** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine subset and for each i $f^{-1}(U_i) = \bigcup_j V_{ij}$ with each $V_{ij} = \operatorname{Spec}(B_{ij})$ open affine subset such that B_{ij} is a finitely generated A_i -algebra.
- f is called **quasi-compact** if $X = \bigcup_i U_i$ with each U_i open affine such that $f^{-1}(U_i)$ is quasi-compact.
- f is of **finite type** if f is locally of finite type and quasi-compact.
- f is called **affine** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine subset such that $f^{-1}(U_i)$ is also affine.
- f is called **finite** if $X = \bigcup_i U_i$ with each $U_i = \operatorname{Spec}(A_i)$ open affine and for each i $f^{-1}(U_i) = \operatorname{Spec}(B_i)$ such that B_i is a finite A_i -algebra. (Recall that an A -algebra B is **finite** if B is finitely generated as an A -module.)

Remark 1.6.2. In the definition above, the existence of an open affine cover can be replaced by “for every open affine cover”.

We clearly have the following implications:



Example 1.6.3. Let A be a DVR with fractional field K and residue field k . Then the natural morphisms $\text{Spec}(k) \rightarrow \text{Spec}(A)$ is finite but $\text{Spec}(K) \rightarrow \text{Spec}(A)$ is of finite type. Note that if π is a uniformizer of A , then $K = A[\pi^{-1}]$.

Example 1.6.4. $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z})$ is not locally of finite type.

Example 1.6.5. Let A be a ring. Then $\mathbb{A}_A^n \rightarrow \text{Spec}(A)$ and $\mathbb{P}_A^n \rightarrow \text{Spec}(A)$ are both of finite type.

Definition 1.6.6. Let k be a field.

- An **affine k -variety** is an integral scheme equipped with an affine morphism of finite type to $\text{Spec}(k)$.
- A **k -variety** is an integral scheme equipped with a separated morphism of finite type to $\text{Spec}(k)$.

It is clear that we have an equivalence of categories

$$\{\text{affine } k\text{-varieties}\} \cong \{\text{finitely generated } k\text{-algebras that are domains}\}^{\text{op}}.$$

Date: 10.6

We first supplement some results on dimension:

Fact 1.6.7. *Let A be a Noetherian ring.*

- $\dim A[x] = \dim A + 1$ ([AM, Exercise 11.7], [M2, Theorem 15.4]).
- *If A is a finitely generated k -algebra which is also a domain, then $\dim A = \text{tr.deg}(\text{Frac}(A)/k)$ [M2, Theorem 5.6].*
- *Krull's principal ideal theorem: Let $f \in A$. Then for each minimal prime \mathfrak{p} containing f , $\text{ht}(\mathfrak{p}) \leq 1$ [AM, Corollary 11.16]. Moreover, for $A \neq 0$, $\text{ht}(f) = 0$ if and only if f is a zero divisor [AM, Proposition 4.7].*

Repeatedly applying Krull's principal ideal theorem, we get that for each minimal prime \mathfrak{p} containing (f_1, \dots, f_r) , $\text{ht}(\mathfrak{p}) \leq r$. In particular, $\text{ht}(f_1, \dots, f_r) \leq r$.

Lemma 1.6.8. *Let X be a topological space and $Y \subseteq X$ a closed subset. Then $\dim X \geq \dim Y + \text{codim}(Y, X)$.*

Proof. Take $Z \subseteq Y$ irreducible closed. By definition,

$$\dim X \geq \dim Z + \text{codim}(Z, X) \geq \dim Z + \text{codim}(Y, X).$$

We conclude by taking supremum over $Z \subseteq Y$. □

Example 1.6.9. Let A be a DVR and let $\mathfrak{m} = (\pi)$ be the maximal ideal. Consider the ideal $\mathfrak{p} = (\pi x - 1)$ in $B = A[x]$. This is a maximal ideal, since $B/\mathfrak{p} = A[1/\pi] = \text{Frac}(A)$. From Fact 1.6.7, $\dim B = \dim A + 1 = 2$ and $\text{ht}(\mathfrak{p}) = 1$, hence $\text{ht}(\mathfrak{p}) + \dim B/\mathfrak{p} < \dim B$. In geometric form, we have $\dim \text{Spec}(B) > \dim \{\mathfrak{p}\} + \text{codim}(\{\mathfrak{p}\}, \text{Spec}(B))$.

Definition 1.6.10. Let X be a topological space.

- We call X **equidimensional** if all irreducible components have the same dimension.
- Assume that X is T_0 . We call X **equicodimensional** if all closed points have the same codimension.

Example 1.6.11. Suppose $\dim X = \dim Y + \text{codim}(Y, X)$ holds for all $Y \subseteq X$ closed.

- Take Y to be an irreducible component of X . Then $\text{codim}(Y, X) = 0$, hence $\dim Y = \dim X$. Thus X is equidimensional.
- Assume that X is T_0 . Take $Y = \{x\}$ to be a closed point. Then $\dim \{x\} = 0$, hence $\text{codim}(\{x\}, X) = \dim X$. Thus X is equicodimensional.

The example $B = A[x]$ in Example 1.6.9 is not equicodimensional.

By contrast, we have the following result.

Theorem 1.6.12. Let $S = \text{Spec}(k)$, k a field or let S be an integral Noetherian scheme of dimension 1 which has infinitely many points. For any equidimensional scheme X equipped with a finite type morphism $X \rightarrow S$, the equality $\dim X = \dim Y + \text{codim}(Y, X)$ holds for every closed subset $Y \subseteq X$. In particular, X is equicodimensional.

For a proof, see [G, IV 10.6.1].

Definition 1.6.13. Let $\phi: A \rightarrow B$ be a ring homomorphism.

- We call B a finite A -algebra if B is a finitely generated A -module.
- We call B an integral A -algebra if $\forall x \in B$, $\phi(A)[x]$ is a finitely generated A -module.

We have B is a finite A -algebra $\Leftrightarrow B$ is finitely generated and integral.

Definition 1.6.14. Let $f: Y \rightarrow X$ be a morphism of schemes. We say that f is **integral** if there exists an cover $X = \bigcup U_i$ with $U_i = \text{Spec}(A_i)$ affine open such that $f^{-1}(U_i) = \text{Spec}(B_i)$ and B_i integral over A_i .

For $f: Y \rightarrow X$, we have f finite $\Leftrightarrow f$ integral and locally of finite type.

Theorem 1.6.15. An integral morphism is a closed map.

Proof. Let $f: Y \rightarrow X$ be integral. Since a subset is closed if and only if its intersection with every member of an open cover is closed, we may assume $X = \text{Spec}(A)$ is affine. In this case $Y = \text{Spec}(B)$ is affine as well and f is induced by $\phi: A \rightarrow B$. Let $J \subseteq B$ be an ideal, $V(J) \subset \text{Spec}(B)$ a closed subset. Let $I = \phi^{-1}(J)$. We have $A/I \rightarrow B/J$ is integral as well. From the fact that every prime ideal in A/I is a contracted ideal [AM, Theorem 5.10] (which implies the going-up theorem), we have $f(V(J)) = V(I)$. Therefore, f is closed. \square

Fiber Products

Recall a fiber product of a diagram $X \xrightarrow{a} S \xleftarrow{b} S$ is an object $X \times_S Y$ equipped with two morphisms p, q indicated below, which satisfies the following universal property: For any object Z equipped with two morphisms f, g such that $af = bg$, there exists a unique morphism h such that $ph = f$, $qh = g$.

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow f & \downarrow h & \searrow g & \\
 X & \xleftarrow{p} & X \times_S Y & \xrightarrow{q} & Y \\
 & \searrow a & & \swarrow b & \\
 & S & & &
 \end{array}$$

Proposition 1.6.16. *Fiber products exist in the category of schemes.*

Proof. Let $a: X \rightarrow S$ and $b: Y \rightarrow S$ be given.

Case 1: $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $Y = \text{Spec}(C)$ are all affine.

Define $X \times_S Y = \text{Spec}(B \otimes_A C)$. For any scheme Z , we have $\text{Hom}(Z, X) \simeq \text{Hom}(B, \mathcal{O}_Z(Z))$ and similarly for Y and S . The universal property for $X \times_S Y$ translates into the universal property of $B \otimes_A C$ in the category of rings.

Case 2: $X = \bigcup X_i$ with $X_i \subseteq X$ open such that $X_i \times_S Y$ exists.

For any $U \subseteq X_i$, $U \times_S Y$ exists and can be identified with the inverse image of U along $X_i \times_S Y \rightarrow X_i$, as shown in the diagram with Cartesian squares

$$\begin{array}{ccccc} U \times_S Y & \longrightarrow & X \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & X & \longrightarrow & S \end{array}$$

Let $X_{ij} = X_i \cap X_j$. Then $X_{ij} \times_S Y$ exists and we can glue $X_i \times_S Y$ along $X_{ij} \times_S Y$ and get $X \times_S Y$.

Case 3: S and Y are affine and X is general.

Cover X by affine open subsets and apply Cases 1 and 2.

Case 4: S affine and X, Y general.

Cover X by affine open subsets and apply Cases 2 and 3 (with X and Y swapped).

Case 5: The general case.

Let $S = \bigcup S_i$ be an affine open cover. Let $X_i = a^{-1}(S_i)$, $Y_i = b^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists by Case 4. But we have $X_i \times_{S_i} Y_i \cong X_i \times_S Y$ as shown in the diagram below

$$\begin{array}{ccccc} X_i \times_{S_i} Y_i & \longrightarrow & Y_i & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow b \\ X_i & \longrightarrow & S_i & \longrightarrow & S \end{array}$$

Thus we can glue them to get $X \times_S Y$.

□

Warning 1.6.17. The natural map $\text{sp}(X \times_S Y) \rightarrow \text{sp}(X) \times_{\text{sp}(S)} \text{sp}(Y)$ is not injective in general.

Definition 1.6.18. Let $f: X \rightarrow S$ be a morphism of schemes and let $s \in S$. Define the fiber X_s of f at s

$$\begin{array}{ccc} X_s = X \times_S \text{Spec}(\kappa(s)) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

Proposition 1.6.19. *The map $X_s \rightarrow f^{-1}(s)$ is a homeomorphism.*

Proof. Without loss of generality, we may assume $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and f is induced by $\phi: A \rightarrow B$. Let $s \in S$ be defined by the prime ideal \mathfrak{p} . We have $\kappa(s) = \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Hence $X_s = \text{Spec}(B \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})) = \text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Elements of $\text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ correspond bijectively to primes of \mathfrak{q} of B such that $\mathfrak{q} \supseteq \phi(\mathfrak{p})$ and \mathfrak{q} does not intersect $\phi(A \setminus \mathfrak{p})$. This is equivalent to $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Thus the map $g: X_s \rightarrow f^{-1}(s)$ is a bijection. Since $\text{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \rightarrow \text{Spec}(B_{\mathfrak{p}}) \rightarrow \text{Spec}(B)$ are successive embeddings and $f^{-1}(s)$ is endowed with the subspace topology, g is a homeomorphism. \square

We may view a morphism $f: X \rightarrow S$ as a family of fibers X_s parameterized by $s \in S$.

Example 1.6.20. Consider $f: X = \text{Spec}(k[t, y, x]/(xy - t)) \rightarrow S = \text{Spec}(k[t])$. The fiber at a rational point $t = a$ of S is $\text{Spec}(k[x, y]/(xy - a))$. For $a \neq 0$, the fiber is a hyperbola isomorphic to $\text{Spec}(k[x, x^{-1}])$. For $a = 0$, the fiber is the union of the coordinate axes of the affine plane and, in particular, is not irreducible.

Definition 1.6.21. Let \mathcal{P} be a class of morphisms. We call \mathcal{P} **stable under base change** if for every $f: X \rightarrow S$ in \mathcal{P} and every morphism $Y \rightarrow S$, the base change $f \times_S Y: X \times_S Y \rightarrow Y$ belongs to \mathcal{P} .

Example 1.6.22. The following classes of morphisms are stable under base change

- locally of finite type
- quasi-compact
- affine
- integral
- of finite type
- finite

Lemma 1.6.23. *Surjective morphisms are stable under base change.*

Proof. Let $f: X \rightarrow S$ be a surjective morphism and $S' \rightarrow S$ a morphism. Let $X' = S \times_S S'$. Take $s' \in S'$. We need to show that the fiber $X'_{s'} \neq \emptyset$.

$$\begin{array}{ccccc}
 X'_{s'} & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow & & \downarrow f \\
 \text{Spec}(\kappa(s')) & \xrightarrow{\quad} & S' & \xrightarrow{\quad} & S \\
 & \searrow & \downarrow f' & \nearrow & \\
 & & \text{Spec}(\kappa(s)) & &
 \end{array}$$

Since f is surjective, $X_s \neq \emptyset$. We are thus reduced to showing that for any k -scheme $X \neq \emptyset$ and any field extension k'/k , we have $X \otimes_k k' := X \times_{\text{Spec}(k)} \text{Spec}(k') \neq \emptyset$. We may assume $X = \text{Spec}(A)$ is affine. In this case it suffices to observe that $A \otimes_k k' \neq 0$. \square

Warning 1.6.24. Injectivity and bijectivity are not stable under base change. For example, $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$ is bijective. After base change to \mathbb{C} , we have $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2 + 1)) = \mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C} \times \mathbb{C}$, which has two prime ideals. Thus $\text{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \rightarrow \text{Spec}(\mathbb{C})$ is not injective or bijective.

Warning 1.6.25. Closed morphisms are not stable under base change. For example, $\mathbb{A}_k^1 \rightarrow \text{Spec}(k)$ is closed but $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is not closed, since the image of $V(xy - 1)$ is the open subset $\mathbb{A}_k^1 \setminus \{0\}$, which is not closed.

Definition 1.6.26. Let f be a morphism of schemes.

- f is called **universally closed** if every base change of f is a closed mapping.
- f is called a **universal homeomorphism** if every base change of f is a homeomorphism.
- f is called **universally injective** or **radiciel** if every base change of f is injective.

Example 1.6.27. An integral morphism is universally closed.

Proposition 1.6.28. Let $f: X \rightarrow Y$ be a morphism of schemes. The following are equivalent

- f is radiciel.
- f is injective and $\forall x \in X, \kappa(x)/\kappa(f(x))$ is purely inseparable.
- For every field K , $f(K): X(K) \rightarrow Y(K)$ is injective, where $X(K) = \text{Hom}_{\text{Sch}}(\text{Spec}(K), X)$.

Note that $X(K)$ can be identified with the set of pairs (x, ι) , where $x \in X$ and $\iota: \kappa(x) \rightarrow K$ is a field embedding.

Proof. (a) \Rightarrow (c). Let $t_1, t_2 \in X(K)$ such that $f(K)(t_1) = f(K)(t_2)$. Consider the Cartesian square in the following diagram

$$\begin{array}{ccc} X \times_Y \text{Spec}(K) & \xrightarrow{\quad} & X \\ s \uparrow \downarrow f' & \nearrow t_1 & \downarrow f \\ \text{Spec}(K) & \xrightarrow{\quad} & Y \end{array}$$

Each t_i corresponds to a section s_i of f' by the universal property of fiber product. f' is injective, the image of s_1 coincides with the image of s_2 . For any morphism $g: Z \rightarrow \text{Spec}(K)$, sections s of g are uniquely determined by the image of s . Thus $s_1 = s_2$ and hence $t_1 = t_2$.

(c) \Rightarrow (a). For any $Y' \rightarrow Y$, if we write $X' = X \times_Y Y'$, then $X'(K) = X(K) \times_{Y(K)} Y'(K)$, which injects into $Y'(K)$.

$$\begin{array}{ccccc} & & \text{Spec}(K) & \longrightarrow & X' \longrightarrow X \\ & \nearrow & \downarrow f' & & \downarrow f \\ & & Y' & \longrightarrow & Y \end{array}$$

Therefore, it suffices to prove that f is injective itself. Let $x, x' \in X$ such that $f(x) = f(x') = y$. There exists a field K and field embeddings $\kappa(x) \xrightarrow{\iota} K \xleftarrow{\iota'} \kappa(x')$ making

$$\begin{array}{ccccc} & & K & & \\ & \swarrow \iota & & \searrow \iota' & \\ \kappa(x) & & & & \kappa(x') \\ & \nwarrow & & \nearrow & \\ & \kappa(y) & & & \end{array}$$

commutative. This defines $(x, \iota), (x', \iota') \in X(K)$ satisfying $f(K)(x, \iota) = f(K)(x', \iota')$. Hence $(x, \iota) = (x', \iota')$ and in particular $x = x'$.

For the equivalence $(b) \Leftrightarrow (c)$, recall that, in the category of fields, $k \rightarrow k'$ is an epimorphism if and only if k'/k is purely inseparable.

$(c) \Rightarrow (b)$. We have already proven that f is injective. It suffices to show that $\phi: \kappa(f(x)) \rightarrow \kappa(x)$ is an epimorphism of fields. Let $\iota, \iota': \kappa(x) \rightarrow K$ be field embeddings satisfying $\iota\phi = \iota'\phi$. Then $f(K)(x, \iota) = f(K)(x, \iota')$. Hence $(x, \iota) = (x, \iota')$, namely $\iota = \iota'$.

$(b) \Rightarrow (c)$. This is similar to the last step. Let $(x, \iota), (x', \iota') \in X(K)$ such that $f(K)(x, \iota) = f(K)(x', \iota')$. In other words, $f(x) = f(x')$ and $\iota\phi = \iota'\phi$. Since f is injective, we have $x = x'$. Since ϕ is an epimorphism of fields, we have $\iota = \iota'$. \square

Remark 1.6.29. We have integral + surjective + radiciel \Rightarrow universal homeomorphism. The converse also holds by a result of Deligne [G, IV 18.12.11].

Example 1.6.30. Let k'/k be a purely inseparable field extension. Then $\text{Spec}(k) \rightarrow \text{Spec}(k')$ is integral, surjective, radiciel, and hence a universal homeomorphism.

Definition 1.6.31. Let \mathcal{P} be a class of morphisms. We say \mathcal{P} is **stable under composition** if whenever $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f, g \in \mathcal{P}$, we have $gf \in \mathcal{P}$.

Example 1.6.32. The classes in Example 1.6.22 are stable under composition.

Lemma 1.6.33. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$.

- (1) If gf is locally of finite type, then so is f .
- (2) If gf is quasi-compact and f is surjective, then g is quasi-compact.

Proof. The first statement boils down to the following property of rings: if the composition $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is of finite type, then so is ψ .

For the second statement, let V be a quasi-compact open subset of Z . Then $(gf)^{-1}(V)$ is quasi-compact and $g^{-1}(V) = f((gf)^{-1}(V))$ is quasi-compact. \square

Date: 10.8

We first continue our discussion about topology.

Let $f: X \rightarrow Y$ be a continuous map. For $x, x' \in X$ and $x \rightsquigarrow x'$, we have $f(x) \rightsquigarrow f(x')$. Indeed, for every closed subset F of Y containing $f(x)$, we have $f^{-1}(F) \ni x$ and consequently $f^{-1}(F) \ni x'$ and $F \ni f(x')$.

Definition 1.6.34. Let $f: X \rightarrow Y$ be a continuous map.

- f is called **specilizing** if $\forall y \rightsquigarrow y' \in Y, \forall x \in f^{-1}(y), \exists x' \in f^{-1}(y')$ such that $x \rightsquigarrow x'$.
- f is called **generalizing** if $\forall y \rightsquigarrow y' \in Y, \forall x' \in f^{-1}(y'), \exists x \in f^{-1}(y)$ such that $x \rightsquigarrow x'$.

Example 1.6.35. f closed $\Rightarrow f$ specializing. This is easily deduced from $f(\overline{\{x\}}) \supseteq \overline{\{f(x)\}}$.

Let X be a scheme. Then $\text{Spec}(\mathcal{O}_{X,x})$ maps homeomorphically onto the subspace $\{x' \in X \mid x' \rightsquigarrow x\}$ of X . To see this, we may assume that $X = \text{Spec}(A)$ is affine. Let x correspond to a prime ideal \mathfrak{p} . Then

$$\{x' \in X \mid x' \rightsquigarrow x\} = \{\mathfrak{q} \in \text{Spec}(A) \mid \mathfrak{q} \subseteq \mathfrak{p}\} \simeq \text{Spec}(A_{\mathfrak{p}}).$$

From this, we deduce:

Lemma 1.6.36. A morphism of schemes $f: X \rightarrow Y$ is generalizing if and only if $\forall x \in X, \text{Spec}(\mathcal{O}_{X,x}) \rightarrow \text{Spec}(\mathcal{O}_{Y,f(x)})$ is surjective.

Definition 1.6.37. A morphism of schemes $f: X \rightarrow Y$ is called **flat** if $\forall x \in X, f_x^{\sharp}: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat.

Recall that a ring homomorphism $\phi: A \rightarrow B$ is flat $\Leftrightarrow \forall \mathfrak{q} \in \text{Spec}(B), A_{\phi^{-1}(\mathfrak{q})} \rightarrow B_{\mathfrak{q}}$ is flat.

Flat morphisms are stable under composition and base change.

Lemma 1.6.38. Every flat local homomorphism $\phi: A \rightarrow B$ of local rings is faithfully flat. In other words, ϕ induces a surjective map $\text{Spec}(\phi): \text{Spec}(B) \rightarrow \text{Spec}(A)$.

Proof. This follows from the fact that a flat homomorphism of rings $A \rightarrow B$ is faithfully flat if and only if for every maximal ideal \mathfrak{m} of A , we have $\mathfrak{m}B \subsetneq B$ ([AM, Exercise 3.16], [M2, Theorem 7.2]). \square

Corollary 1.6.39. Every flat morphism of schemes is generalizing.

Remark 1.6.40. Let $f: X \rightarrow Y$ be a morphism of schemes.

- (1) If f is generalizing, then every maximal point of X lies above a maximal point of Y .

(2) Assume that Y is irreducible with generic point η . We have an injective map

$$\begin{aligned} \text{IrrComp}(X_\eta) &\rightarrow \text{IrrComp}(X), \\ Z &\rightarrow \bar{Z} \end{aligned}$$

whose image consists precisely of the irreducible components intersecting X_η . In particular, if f is generizing, then the above map is a bijection.

In particular:

Lemma 1.6.41. *Let $f: X \rightarrow Y$ be a morphism of schemes. Suppose Y is irreducible with generic point η . Then*

- (1) X irreducible $\Rightarrow X_\eta$ irreducible or empty.
- (2) If f is generizing, then X irreducible $\iff X_\eta$ irreducible.

Consider k'/k a field extension, X/k a k scheme, denote $X \otimes_k k' = X \times_{\text{Spec}(k)} \text{Spec}(k')$.

Remark 1.6.42. Let k'/k be a field extension.

- (1) $X \otimes_k k'$ connected $\Rightarrow X$ connected.
- (2) $X \otimes_k k'$ irreducible $\Rightarrow X$ is irreducible.
- (3) $X \otimes_k k'$ reduced $\Rightarrow X$ reduced.
- (4) $X \otimes_k k'$ integral $\Rightarrow X$ integral.

(1) and (2) follow from the surjectivity of $X \otimes_k k' \rightarrow X$. To see (3), we may assume $X = \text{Spec}(A)$ is affine. Then $A \hookrightarrow A \otimes_k k'$ and the latter is assumed to be reduced. For (4), combine (2) and (3).

Definition 1.6.43. Let X be a scheme over a field k and let \bar{k} be an algebraic closure of k .

- X is called **geometrically connected** if $X \otimes_k \bar{k}$ is connected.
- X is called **geometrically irreducible** if $X \otimes_k \bar{k}$ is irreducible.
- X is called **geometrically reduced** if $X \otimes_k \bar{k}$ is reduced.
- X is called **geometrically integral** if $X \otimes_k \bar{k}$ is integral.

Remark 1.6.44. If k is separably closed, then

$$\begin{aligned} X \text{ connected} &\iff X \text{ geometrically connected} \\ X \text{ irreducible} &\iff X \text{ geometrically irreducible} \end{aligned}$$

Indeed, in this case \bar{k}/k is purely inseparable and $\text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ is a universal homeomorphism.

Proposition 1.6.45. *Let X/k be a scheme over a field. The following are equivalent.*

- (1) *For every finite separable extension k'/k , $X \otimes_k k'$ is irreducible.*
- (2) *X is geometrically irreducible.*
- (3) *X is irreducible with generic point η and the separable closure of k in $\kappa(\eta)$ is k .*

Proof. (2) \Rightarrow (1) is clear, since $X \otimes_k \bar{k} \rightarrow X \otimes_k k'$ is surjective.

(1) \Rightarrow (3). X is clearly irreducible. For every finite separable extension k'/k , since $X \otimes_k k'$ is irreducible, $(X \otimes_k k')_\eta = \text{Spec}(\kappa(\eta) \otimes_k k')$ is irreducible. Let $\alpha \in \kappa(\eta)$ be a separable algebraic element over k with minimal polynomial $P(x)$. Let $k' = k(\alpha)$. Then $\kappa(\eta) \otimes_k k' = (\kappa(\eta) \otimes_k k'[x]/(P(x))) = k(\eta)[x]/(P(x))$. We have $P(x) = (x - \alpha)Q(x)$ with $Q(x) \in k(\eta)[x]$. Then $k(\eta)[x]/(P(x)) = k(\eta)[x]/(x - \alpha) \oplus k(\eta)[x]/(Q(x))$, which implies $Q(x) = 1$ and $\alpha \in k$.

(3) \Rightarrow (2). The projection $X \otimes_k \bar{k} \rightarrow X$ is a base change of $\text{Spec}(\bar{k}) \rightarrow \text{Spec}(k)$ and hence is flat and generizing. Thus $X \otimes_k \bar{k}$ is irreducible if and only if $(X \otimes_k \bar{k})_\eta = \text{Spec}(\kappa(\eta) \otimes_k \bar{k})$ is irreducible. Let k^{sep} be a separable closure of k . Since \bar{k}/k^{sep} is purely inseparable, it suffices to show that $\text{Spec}(\kappa(\eta) \otimes_k k^{\text{sep}})$ is irreducible. By the lemma below applied to the Galois extension, k^{sep}/k $\kappa(\eta) \otimes_k k^{\text{sep}}$ is a field. \square

Lemma 1.6.46. *Let k'/k be a field extension and K/k a Galois extension. Assume $k' \cap K = k$ in the composite field $k' \cdot K$. Then $k' \otimes_k K$ is a field.*

Proof. Since K is a union of Galois extensions of k , we may assume K/k is a finite Galois extension of degree d . Consider the surjection $\phi: k' \otimes_k K \rightarrow k' \cdot K$ is surjective. Note that $k' \cdot K/k'$ is a Galois extension of Galois group $\text{Gal}(k' \cdot K/k') \cong \text{Gal}(K/k' \cap K) = \text{Gal}(K/k)$. Thus $\dim_{k'}(k' \cdot K) = d = \dim_{k'}(k' \otimes_k K)$. It follows that ϕ is an isomorphism. \square

We give some examples which are not geometrically irreducible.

Example 1.6.47. $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$ is not geometrically connected, since we have shown its base change to \mathbb{C} is two points. This can also be seen from criterion (3), since \mathbb{C}/\mathbb{R} is a separable algebraic extension.

Example 1.6.48. Let $A = \mathbb{R}[x, y]/(x^2 + y^2)$ and $X = \text{Spec}(A) \rightarrow \text{Spec}(\mathbb{R})$. Since $x^2 + y^2$ is irreducible in $\mathbb{R}[x, y]$, X is irreducible. Since $x^2 + y^2$ factors as $(x + iy)(x - iy)$ in \mathbb{C} , the base change of X to \mathbb{C} is $\mathbb{C}[x, y]/(x + iy)(x - iy)$, which is the union of two lines intersecting at a point. Thus X is geometrically connected but not geometrically irreducible.

Let η be the generic point of η . In $\kappa(\eta) = \text{Frac}(A)$, we have $(x/y)^2 + 1 = 0$. Thus the separable closure of \mathbb{R} in $\kappa(\eta)$ can be identified with \mathbb{C} .

Next, we study geometrically reduced schemes. We start with the case of field extensions.

Definition 1.6.49. A field extension K/k is said to be **separable** if $\text{Spec}(K)$ is geometrically reduced over $\text{Spec}(k)$.

Let \bar{k} be the algebraic closure of k . By definition, K/k is separable if and only if $K \otimes_k \bar{k}$ is reduced.

Since $K = \bigcup_{\alpha \in K} k(\alpha)$, K/k is separable if and only if $k(\alpha)/k$ is separable for all $\alpha \in K$. We have

$$k(\alpha) \otimes_k \bar{k} = \begin{cases} \bar{k}(\alpha) & \alpha \text{ transcendental} \\ \bar{k}[x]/(P(x)) & \alpha \text{ algebraic with minimal polynomial } P(x) \end{cases}$$

Note that $\bar{k}[x]/(P(x))$ is reduced if and only if $P(x)$ is a separable polynomial (namely, a polynomial with only simple roots in \bar{k}). We have proved the following.

Lemma 1.6.50. *K/k is separable if and only if $\forall \alpha \in K$, α is either transcendental over k or separable algebraic over k .*

Remark 1.6.51. In particular,

- (1) Definition 1.6.49 extends the usual notion of separable algebraic extensions.
- (2) Any purely transcendental extension is separable.
- (3) If k is a perfect field, then any field extension K/k is separable.

Lemma 1.6.52. *Let $L/K/k$ be a tower of field extensions.*

- (1) K/k is separable \iff for every finite field extension k'/k , $K \otimes_k k'$ is reduced.
- (2) L/K and K/k separable $\implies L/k$ separable.
- (3) L/k separable $\implies K/k$ separable.

Proof. (1) and (3) are trivial.

(2) For any finite field extension k'/k , $L \otimes_k k' = L \otimes_K (K \otimes_k k')$. Since K/k is separable, $K \otimes_k k'$ is a finite direct sum of finite field extensions of K . We conclude by the assumption that L/K is separable. \square

Warning 1.6.53. Unlike the case of separable algebraic extensions, for a tower $L/K/k$ of field extensions, L/k separable does **not** imply L/K separable. Here is an example: $L = k(x)$, $K = k(x^p)$, where $p = \text{char}(k) > 0$. Then L/k is separable but L/K is purely inseparable.

Definition 1.6.54. Let K/k be a separable extension. A **separating transcendence basis** is a transcendence basis B for K/k such that $K/k(B)$ is separable.

Lemma 1.6.55. *Let $K = k(x_1, \dots, x_n)/k$ be a finitely generated separable extension. Then K admits a separating transcendence basis contained in $\{x_1, \dots, x_n\}$.*

Proof. This is proved in [M2, Theorem 26.2]. Note that the definition there *a priori* differs from ours. We give a proof here for completeness.

We may assume $\text{char}(k) = p > 0$. We proceed by induction on n . We may assume that x_1, \dots, x_r is a transcendence basis. If $r = n$, we are done. Suppose $r < n$. Then x_1, \dots, x_{r+1} are algebraically dependent. There exists a nonzero $P \in k[X_1, \dots, X_{r+1}]$

with least degree such that $P(x_1, \dots, x_{r+1}) = 0$. The minimality of the degree implies that P is irreducible.

Let us prove $P \notin k[X_1^p, \dots, X_{r+1}^p]$. Assume otherwise. Then $P = Q^p$ with $Q \in k^{1/p}[X_1, \dots, X_{r+1}]$. Write $Q = \sum c_\alpha X^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_{r+1})$ and $X^\alpha = X_1^{\alpha_1} \cdots X_{r+1}^{\alpha_{r+1}}$. Let $I = \{\alpha \mid c_\alpha \neq 0\}$. Since $(\sum c_\alpha \otimes x^\alpha)^p = 0$ in $\bar{k} \otimes_k K$ and $\bar{k} \otimes_k K$ is reduced, we have $\sum_{\alpha \in I} c_\alpha \otimes x^\alpha = 0$ in $\bar{k} \otimes_k K$. This implies that $(x^\alpha)_{\alpha \in I}$ is linearly dependent over k . Thus there exists $R \in k[X_1, \dots, X_{r+1}]$ of degree $\leq \deg(Q) < \deg(P)$ such that $R(x_1, \dots, x_{r+1}) = 0$, a contradiction.

Thus we may assume $P \notin k[X_1^p, X_2, \dots, X_{r+1}]$. Then x_1 is separable over $k(x_2, \dots, x_{r+1})$, hence separable over $k(x_2, \dots, x_n)$. By assumption $k(x_2, \dots, x_n)$ has a separating transcendence basis $B \subseteq \{x_2, \dots, x_n\}$. Then B is a separating transcendence basis for K/k . \square

Warning 1.6.56. A separating transcendence basis does not exist in general. For example, for $\text{char}(k) = p > 0$, $K = \bigcup_{n \in \mathbb{N}} k(x^{1/p^n})$ is a separable extension of k of transcendence degree 1. However, for any $y \in K$ transcendental over k , $K/k(y)$ is not separable.

Now we come to the general case.

Proposition 1.6.57. *Let X/k be a k -scheme. The following are equivalent:*

- (1) $X \otimes_k k'$ is reduced for every finite purely inseparable extension k'/k .
- (2) X is geometrically reduced.
- (3) $X \times_k Y$ is reduced for every reduced k -scheme Y .
- (4) X is reduced and for every maximal point $x \in X$, $\kappa(x)/k$ is separable.

Recall that a maximal point of a scheme is the generic point of an irreducible component.

Proof. (3) \Rightarrow (2). Take $Y = \text{Spec}(\bar{k})$.

(2) \Rightarrow (4). X is clearly reduced. Let $x \in X$ be a maximal point. Since X is reduced, we have $\kappa(x) = \mathcal{O}_{X,x}$. Since $X \otimes_k \bar{k}$ is reduced, $\mathcal{O}_{X,x} \otimes_k \bar{k}$ is reduced. In other words, $\kappa(x)/k$ is separable.

(4) \Rightarrow (3). We may assume that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine. It suffices to show that $A \otimes_k B$ is reduced.

Since B is a union of finitely generated k -algebras, we may assume that B itself is finitely generated and reduced. In this case, B is Noetherian and has finitely many minimal prime ideals \mathfrak{q} . Since B is reduced, we have $B \hookrightarrow \prod_{\mathfrak{q}} B/\mathfrak{q} \hookrightarrow \prod_{\mathfrak{q}} \kappa(\mathfrak{q})$ and the product is finite. Tensoring with A , we get $A \otimes_k B \hookrightarrow \prod_{\mathfrak{q}} A \otimes_k \kappa(\mathfrak{q})$. We are reduced to proving that $A \otimes_k k'$ is reduced for any field extension k'/k .

Since A is reduced, we have

$$A \hookrightarrow \prod_{\mathfrak{p}} (A/\mathfrak{p}) \hookrightarrow \prod_{\mathfrak{p}} \kappa(\mathfrak{p}),$$

where the product is taken over all minimal prime ideals. Tensoring with k' , we get

$$A \otimes_k k' \hookrightarrow \left(\prod_{\mathfrak{p}} \kappa(\mathfrak{p}) \right) \otimes_k k' \hookrightarrow \prod_{\mathfrak{p}} (\kappa(\mathfrak{p}) \otimes_k k').$$

(To see the injectivity of the last map, take a k -linear basis of k' .)

Thus it suffices to show that for any separable extension K/k , $K \otimes_k k'$ is reduced. Since $K = \bigcup_{\alpha \in K} k(\alpha)$, we may assume $K = k(\alpha)$. If α is separable algebraic over k of minimal polynomial $P(x)$, then $k(\alpha) \otimes_k k' = k'[x]/(P(x))$ is reduced. If α is transcendental over k , then $k(\alpha) \otimes_k k'$ is a localization of $k'[x]$ and hence reduced.

(2) \Rightarrow (1). Clear.

(1) \Rightarrow (2). Let k^{perf} be the perfection of k . By assumption, $Y = X \otimes_k k^{\text{perf}}$ is reduced. Now \bar{k}/k^{perf} is separable. Applying (4) \Rightarrow (3) to $\text{Spec}(\bar{k})$, we get that $X \otimes_k \bar{k} \simeq Y \otimes_{k^{\text{perf}}} \bar{k}$ is reduced. \square

Corollary 1.6.58. *If k is a perfect field, then a k -scheme X is reduced if and only if X is geometrically reduced.*

Immersions

Recall that a morphism of schemes $f: Z \rightarrow X$ is an open immersion if and only if $\text{sp}(f)$ is an open embedding and $f^\sharp: f^{-1}\mathcal{O}_X \cong \mathcal{O}_Z$.

Definition 1.6.59. Let $f: Z \rightarrow X$ be a morphism of schemes.

- (1) f is called a **closed immersion** if f is a closed embedding and $f^\sharp: f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Z$ is epimorphism of Abelian sheaves, i.e. $\forall z \in Z$, $f_z^\sharp: \mathcal{O}_{X,f(z)} \rightarrow \mathcal{O}_{Z,z}$ is surjective.
- (2) f is called an **immersion** if f factorizes as $Z \rightarrow U \rightarrow X$ where $Z \rightarrow U$ is a closed immersion and $U \rightarrow X$ is an open immersion.

Lemma 1.6.60.

- A morphism of schemes $f: Z \rightarrow X$ is an immersion if and only if f is a locally closed embedding and $\forall z \in Z$, $f_z^\sharp: \mathcal{O}_{X,f(z)} \rightarrow \mathcal{O}_{Z,z}$ is surjective.
- Immersions are stable under composition.
- Immersions are monomorphisms.

Example 1.6.61. Let A be a ring, I an ideal. Then $\text{Spec}(A/I) \rightarrow \text{Spec}(A)$ induced by $A \rightarrow A/I$ is a closed immersion. Indeed, $\forall \mathfrak{p} \supseteq I$, $A_\mathfrak{p} \rightarrow A_\mathfrak{p}/IA_\mathfrak{p} \simeq (A/I)_\mathfrak{p}$ is surjective.

Definition 1.6.62. Let X be a scheme.

- A **closed subscheme** of X is an equivalence class of pairs (Z, f) , where Z is a scheme and $f: Z \rightarrow X$ is a closed immersion. Two pairs (Z, f) and (Z', f') are said to be equivalent if $\exists \varphi: Z \xrightarrow{\sim} Z'$ making

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow \varphi & \nearrow f' & \\ Z' & & \end{array}$$

commutative. φ is necessarily unique.

- A **subscheme** of X is an equivalence class of pairs (Z, f) , where Z is a scheme and $f: Z \rightarrow X$ is an immersion. Two pairs (Z, f) and (Z', f') are said to be equivalent if $\exists \varphi: Z \xrightarrow{\sim} Z'$ making

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow \varphi & \nearrow f' & \\ Z' & & \end{array}$$

commutative.

By Lemma 1.6.60, we get:

Lemma 1.6.63. *An immersion that is closed is a closed immersion.*

Warning 1.6.64. An immersion that is open is **not** an open immersion in general. For example, if A is a non-reduced ring, then $\text{Spec}(A/\sqrt{(0)}) \rightarrow \text{Spec}(A)$ is a homeomorphism and a closed immersion, but not an open immersion.

Warning 1.6.65. If $I \neq J$ are ideals of A such that $\sqrt{I} = \sqrt{J}$, then $\text{Spec}(A/I)$ and $\text{Spec}(A/J)$ have the same underlying subspace of $\text{Spec}(A)$, but are not the same as closed subscheme.

Warning 1.6.66. Let $f: Z \rightarrow X$ be an immersion. It is **not** possible in general to factorize f as $Z \rightarrow Y \rightarrow X$ where $Z \rightarrow Y$ is an open immersion and $Y \rightarrow X$ is a closed immersion. See [SP, 078B] for an example. For a positive result, see Lemma 1.9.27 later.

Warning 1.6.67. Not all monomorphisms are immersions. For example, the monomorphism $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z})$ is not an immersion since it is not locally closed. In the same vein, a subobject of a scheme is **not** a subscheme in general.

An important class of immersions is given by the diagonal construction.

Diagonals, Separation axioms

For any morphism of schemes $f: X \rightarrow Y$, consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_f} & X \times_Y X & \xrightarrow{p_2} & X \\ & \searrow & \downarrow p_1 & & \downarrow f \\ & & X & \xrightarrow{f} & Y \end{array}$$

We sometimes write Δ or $\Delta_{X/Y}$ for Δ_f .

Proposition 1.6.68. Δ_f is an immersion.

Before proving the proposition, we first consider an illuminating example.

Example 1.6.69. Let $X = \text{Spec}(B)$, $Y = \text{Spec}(A)$, and $f = \text{Spec}(\phi)$, where $\phi: A \rightarrow B$. Then $\Delta_f: X \rightarrow X \times_Y X$ corresponds to

$$\begin{aligned}\nabla_\phi: B \otimes_A B &\rightarrow B \\ b_1 \otimes b_2 &\mapsto b_1 b_2\end{aligned}$$

This is clearly surjective. Hence Δ_f is a closed immersion.

Proof of Proposition 1.6.68. Let $Y = \bigcup V_i$, V_i affine open subsets. Let $f^{-1}(U_i) = \bigcup U_{ij}$, U_{ij} affine open. Let $W_{ij} = p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) \simeq U_{ij} \times_{V_i} U_{ij}$ and $W = \bigcup W_{ij}$. We have $\Delta_f(U_{ij}) \subseteq p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = W_{ij}$. Thus Δ_f factorizes as $X \xrightarrow{\delta} W \subseteq X \times_Y X$. Now $\Delta_f^{-1}(W_{ij}) = \Delta_f^{-1}(p_1^{-1}(U_{ij})) \cap \Delta_f^{-1}(p_2^{-1}(U_{ij})) = U_{ij}$ and the restriction of δ to $U_{ij} \rightarrow W_{ij}$ can be identified with $\Delta_{f_{ij}}$, where $f_{ij}: U_{ij} \rightarrow V_i$ is the restriction of f . By the example above, each $\Delta_{f_{ij}}$ is a closed immersion. Thus δ is a closed immersion. It follows that $\Delta_f: X \rightarrow Y$ is an immersion. \square

Definition 1.6.70. Let $f: X \rightarrow Y$ be a morphism of schemes.

- f is called **separated** if Δ_f is a closed immersion.
- f is called **quasi-separated** if Δ_f is quasi-compact.

It is clear that we have affine \Rightarrow separated \Rightarrow quasi-separated.

The following graph construction will be very useful in the sequel. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow g \\ & & S \end{array}$$

be a morphism of S -schemes. The graph of f , denoted Γ_f , is defined as follows:

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & \curvearrowright & \searrow & \\ X & \xrightarrow{\Gamma_f} & X \times_S Y & \xrightarrow{\quad} & Y \\ & \searrow id_X & \downarrow & \downarrow & \\ & & X & \longrightarrow & S \end{array}$$

From the functorial point of view, we have $\Gamma_f(x) = (x, f(x))$.

We have a commutative diagram with Cartesian squares

$$\begin{array}{ccccccc} & & f & & & & \\ & \nearrow & \curvearrowright & \searrow & & & \\ X & \xrightarrow{\Gamma_f} & X \times_S Y & \xrightarrow{q} & Y & & \\ & \searrow f & \swarrow id_Y & \searrow p & \swarrow g & & \\ Y & \xrightarrow{\Delta_g} & Y \times_S Y & \xrightarrow{p} & X & \xrightarrow{gf} & S \end{array}$$

Thus, we get the following Lemma:

Lemma 1.6.71. *If \mathcal{P} is a class of morphisms that \mathcal{P} is stable under base change and composition, then $gf, \Delta_g \in \mathcal{P}$ implies $f \in \mathcal{P}$.*

In particular, we have

Corollary 1.6.72. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If gf is quasi-compact and g is quasi-separated, then f is quasi-compact.*

Definition 1.6.73. • A scheme X is said to be **separated** if the morphism $X \rightarrow \text{Spec}(\mathbb{Z})$ is separated.

- A scheme X is said to be **quasi-separated** if the morphism $X \rightarrow \text{Spec}(\mathbb{Z})$ is quasi-separated.

Corollary 1.6.74. *Let $f: X \rightarrow Y$ be a morphism of schemes with X quasi-compact and Y quasi-separated. Then f is quasi-compact.*

Proposition 1.6.75. *A scheme X is quasi-separated if and only if for all quasi-compact opens U and V of X , $U \cap V$ is quasi-compact.*

Proof. \implies . Since U is quasi-compact and X is quasi-separated, the open immersion $j: U \rightarrow X$ is quasi-compact by Corollary 1.6.74. Therefore, $U \cap V = j^{-1}(V)$ is quasi-compact.

\Leftarrow .

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X \times_{\mathbb{Z}} X & \xrightarrow{p_2} & X \\ & \searrow & \downarrow p_1 & & \downarrow \\ & & X & \longrightarrow & \text{Spec}(\mathbb{Z}) \end{array}$$

Let $X = \bigcup_i U_i$ be an affine open cover. Let $W_{ij} = p_1^{-1}(U_i) \cap p_2^{-1}(U_j)$. Then $\bigcup_{ij} W_{ij} = X \times_{\text{Spec}(\mathbb{Z})} X$. Each $W_{ij} \simeq U_i \times_{\text{Spec}(\mathbb{Z})} U_j$ is an affine scheme, and $\Delta^{-1}(W_{ij}) = U_i \cap U_j$ is quasi-compact. Thus Δ is a quasi-compact morphism. \square

Example 1.6.76. Let X be a scheme

- If the underlying space of X is locally Noetherian, then X is quasi-separated.
- Let X be a scheme and let $U \subseteq X$ be an open subset that is not closed. Let $Y = X \coprod_U X$ be the scheme obtained by gluing two copies of X along U . Then Y is not separated.

To see this, let j_0 and j_1 denote the two open immersions from X to Y . We have an immersion $f = (j_0, j_1): X \rightarrow Y \times_{\text{Spec}(\mathbb{Z})} Y$. Let $\Delta = \Delta_{Y/\text{Spec}(\mathbb{Z})}$. The inclusion $f(X) \cap \Delta(X) \subseteq f(X)$ can be identified with the inclusion $U \subseteq X$, which is not closed.

- Let X be a quasi-compact scheme and let $U \subseteq X$ be an open subset that is not quasi-compact (e.g. X is the Cantor set and U is the complement of a point). Then the same argument as above shows that $Y = X \coprod_U X$ is not quasi-separated.

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Warning 1.6.77. If α is transcendental over k and k'/k a field extension, then $k(\alpha) \otimes_k k'$ is the localization of $k'[\alpha]$ with respect to the multiplicative set $S = k[\alpha] \setminus \{0\}$. It is **not** a field in general.

We have seen that $f: X \rightarrow Y$ is separated if and only if $\Delta_f(X) \subseteq X \times_Y X$ is closed. This is analogous to the fact in general topology that a topological space X is Hausdorff if and only if $\Delta_X(X) \subseteq X \times X$ is closed.

1.7 Quasi-coherent sheaves

The properties of a ring A are often reflected by the category of A -modules. In order to better study a sheaf of rings \mathcal{O}_X , we now introduce the notion of \mathcal{O}_X -module.

Definition 1.7.1. Let (X, \mathcal{O}_X) be a ringed space.

- An \mathcal{O}_X -module or **sheaf of \mathcal{O}_X -modules** is consists of
 - a sheaf of sets \mathcal{F} on X ;
 - $\forall U \subseteq X$ open, a structure of $\mathcal{O}_X(U)$ -module on $\mathcal{F}(U)$

such that for all $U \subseteq V$, the restriction map

$$\mathcal{F}(V) \xrightarrow{\rho} \mathcal{F}(U)$$

is a homomorphism of $\mathcal{O}_X(V)$ -modules. Here $\mathcal{F}(U)$ is viewed as an $\mathcal{O}_X(V)$ -module via the restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$.

- A **morphism** of \mathcal{O}_X -modules $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of sets such that $\forall U \subseteq X$, $\phi_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules.

Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules. The sheaf of local homomorphisms, or “sheaf hom” for short, denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, is defined as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}_U, \mathcal{G}_U)$$

It is easy to see that this is a sheaf of \mathcal{O}_X -module.

The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined as the sheafification of

$$U \rightarrow \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

The tensor product is again a sheaf of \mathcal{O}_X -modules.

The following properties are easy to verify.

Lemma 1.7.2. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of \mathcal{O}_X -modules.

- (1) $\forall x \in X$, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.

$$(2) \text{ Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})).$$

Recall that a morphism of ringed spaces $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ consists of the following:

- a continuous map $f: X \rightarrow Y$;
- a morphism of sheaves of rings $f^\sharp: f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ (or, equivalently by adjunction, $f^\flat: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$).

For an \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is then naturally an $f_*(\mathcal{O}_X)$ -module. We regard $f_*(\mathcal{F})$ as an \mathcal{O}_Y -module via f^\flat .

For an \mathcal{O}_Y -module \mathcal{G} , $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$ -module. We define

$$f^*(\mathcal{G}) := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Then $f^*(\mathcal{G})$ is an \mathcal{O}_X -module.

Combining the adjunction $f^{-1} \dashv f_*$ and the adjunction between \otimes and $\mathcal{H}om$, we have

$$\text{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*(\mathcal{F})).$$

In other words, we have $f^* \dashv f_*$ between the categories of \mathcal{O} -modules.

Warning 1.7.3. f^* is **not** exact in general. f^* is exact if f is flat, i.e. $\forall x \in X$, $\mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat.

Definition 1.7.4. Let (X, \mathcal{O}_X) be a ringed space and \mathcal{F} an \mathcal{O}_X -module.

- \mathcal{F} is said to be **free** if it is isomorphic to a direct sum of copies of \mathcal{O}_X . \mathcal{O}_X^n is called a free \mathcal{O}_X -module of rank n . For I a set, we write $\mathcal{O}_X^{\oplus I} := \bigoplus_{i \in I} \mathcal{O}_X$.
- \mathcal{F} is said to be **locally free** if there is an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$ -module.
- \mathcal{F} is said to be **locally free of rank n** if there is an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$ -module of rank n .
- \mathcal{F} is said to be **invertible** if it is locally free of rank 1.

Remark 1.7.5. • For \mathcal{F} locally of rank n , its **dual** $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is also locally free of rank n .

- For \mathcal{F} is invertible, we have $\mathcal{F} \otimes \mathcal{F}^\vee \xrightarrow{\sim} \mathcal{O}_X$. We let $\text{Pic}(X)$ denote the set of isomorphism classes of invertible \mathcal{O}_X -modules. $(\text{Pic}(X), \otimes)$ is an Abelian group, called the **Picard group** of X .

Definition 1.7.6. An \mathcal{O}_X -module \mathcal{F} is said to be **quasi-coherent** if there exists an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is a cokernel of free $\mathcal{O}_X|_{U_i}$ modules. i.e.

$$\mathcal{F}|_{U_i} \cong \text{coker}(\mathcal{O}_X|_{U_i}^{\oplus I_i} \rightarrow \mathcal{O}_X|_{U_i}^{\oplus J_i})$$

Let A be a ring and M an A -module. Let $X = \text{Spec}(A)$ and $D(f) \subseteq X$, $f \in A$ be a principal open subset. Define $\widetilde{M}(D(f)) = M_f$, which is a module over $A_f = \mathcal{O}_X(D(f))$. If $D(f) \subseteq D(g)$, then the homomorphism $A_g \rightarrow A_f$ induces $M_g \rightarrow M_f$. Let \mathcal{B} be the partially ordered set $\{D(f) \mid f \in A\}$.

Lemma 1.7.7. *The functor*

$$\begin{aligned} \mathcal{B}^{\text{op}} &\rightarrow \text{Ab} \\ D(f) &\mapsto M_f \end{aligned}$$

extends uniquely to a sheaf of \mathcal{O}_X -module.

Proof. By Lemma 1.4.1, it suffices to verify the gluing property for a cover in \mathcal{B} of some $D(f)$. Up to replacing A by A_f , we may without loss of generality suppose that the cover has the form $X = \bigcup_{i \in I} D(f_i)$. Since X is quasi-compact, we may assume as in the proof of Proposition 1.4.2 that I is finite. It suffices to show that

$$M \rightarrow \bigoplus_i M_{f_i} \rightrightarrows \bigoplus_{i,j} M_{f_i f_j}$$

is an equalizer diagram. For this, one can repeat the arguments in either one of the two proofs of Proposition 1.4.2. \square

We have

$$\widetilde{M}_{\mathfrak{p}} = \underset{\mathfrak{p} \in D(f)}{\text{colim}} M_f = \underset{f \notin \mathfrak{p}}{\text{colim}} M_f = M_{\mathfrak{p}}$$

Proposition 1.7.8. *The functor*

$$\begin{aligned} F: A\text{-Mod} &\rightarrow \text{Shv}(X, \mathcal{O}_X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

is exact, fully faithful and left adjoint to $\Gamma(X, -)$.

Proof. Let \mathcal{F} be an \mathcal{O}_X -module and M an A -module. We consider the map

$$\Psi: \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \rightarrow \text{Hom}(M, \mathcal{F}(X))$$

carrying $\phi: \widetilde{M} \rightarrow \mathcal{F}$ to $\phi(X): M = \widetilde{M}(X) \rightarrow \mathcal{F}(X)$. For $\psi: M \rightarrow \mathcal{F}(X)$, we define $\phi = \Phi(\psi): \widetilde{M} \rightarrow \mathcal{F}$ by $\phi(D(f)): \widetilde{M}(D(f)) = M_f \xrightarrow{\psi_f} \mathcal{F}(X)_f \rightarrow \mathcal{F}(D(f))$ for each $f \in A$. One checks Φ and Ψ are inverse to each other. This shows $F \dashv \Gamma(X, -)$.

Since $\Gamma(X, \widetilde{M}) = M$, F is fully faithful. Finally, F is exact since the functor $M \mapsto M_f$ is exact $\forall f \in A$. \square

One checks the following properties:

Lemma 1.7.9. *Let $\phi: A \rightarrow B$ be a ring homomorphism, $X = \text{Spec}(B)$, $Y = \text{Spec}(A)$, and $f = \text{Spec}(\phi): X \rightarrow Y$.*

(1) *For A -modules M and M' , we have $\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N} \simeq (M \otimes_A N)^{\sim}$.*

(2) *For every A -module M , we have $f^*(\widetilde{M}) \simeq (M \otimes_A B)^{\sim}$.*

(3) For every B -module N , we have $f_*(\widetilde{N}) = \widetilde{AN}$, where AN is N considered as an A -module via ϕ .

Proof. (1) The canonical morphism $\widetilde{M} \otimes_{\mathcal{O}_Y} \widetilde{N} \rightarrow (M \otimes_A N)A^\sim$ is an isomorphism by taking stalks.

(2) Consider the canonical morphism $f^*(\widetilde{M}) \rightarrow (M \otimes_A B)^\sim$. Let \mathfrak{q} be a prime in B and $\mathfrak{p} = f(\mathfrak{q}) = \phi^{-1}(\mathfrak{p})$. The stalk of the morphism at \mathfrak{q} is $(f^*\widetilde{M})_{\mathfrak{q}} \simeq (f^{-1}\widetilde{M})_{\mathfrak{q}} \otimes_{f^{-1}\mathcal{O}_{Y,\mathfrak{q}}} \mathcal{O}_{X,\mathfrak{q}} \simeq M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \simeq (M \otimes_A B)_{\mathfrak{q}}^\sim$.

(3) Indeed, $\forall g \in B$, we have $f_*(\widetilde{N})(D(g)) = \widetilde{N}(D(\phi(g))) = N_{\phi(g)}$. \square

Proposition 1.7.10. *Let $X = \text{Spec}(A)$. An \mathcal{O}_X -module \mathcal{F} is quasi-coherent if and only if $\mathcal{F} \cong \widetilde{M}$ for some A -module M .*

More generally, we have the following characterization of quasi-coherent sheaves on schemes.

Proposition 1.7.11. *Let X be a scheme and \mathcal{F} an \mathcal{O}_X -module. Then the following are equivalent*

(a) \mathcal{F} is quasi-coherent.

(b) $\exists X = \bigcup U_i$ with $U_i = \text{Spec}(A_i)$ affine open, such that for every i , $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ for some A_i -module M_i .

(c) $\forall U = \text{Spec}(A) \subseteq X$ affine open, we have $\mathcal{F}|_U \cong \widetilde{M}$ for some A -module M .

Proof. (b) \Rightarrow (a). It suffices to show for every A -module M , \widetilde{M} quasi-coherent. There is a presentation of M using free modules:

$$A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0.$$

Taking the associated sheaves, we get an exact sequence

$$\exists \mathcal{O}_X^{\oplus I} \longrightarrow \mathcal{O}_X^{\oplus J} \longrightarrow \widetilde{M} \longrightarrow 0.$$

(c) \Rightarrow (b). Trivial.

(a) \Rightarrow (c). We may assume $X = \text{Spec}(A)$ is affine. Let $M = \mathcal{F}(X)$. It suffices to show that the canonical map $M_f \rightarrow \mathcal{F}(D(f))$ is an isomorphism for every $f \in A$, which gives an isomorphism $\widetilde{M} \xrightarrow{\sim} \mathcal{F}$. The following lemma is a generalization of this assertion. \square

Lemma 1.7.12. *Let X be a quasi-compact scheme, $f \in \mathcal{O}_X(X)$, and \mathcal{F} quasi-coherent \mathcal{O}_X -module.*

(1) *The map $\mathcal{F}(X)_f \rightarrow \mathcal{F}(X_f)$ is an injection.*

(2) *If X is quasi-separated, then $\mathcal{F}(X)_f \rightarrow \mathcal{F}(X_f)$ is bijective.*

Proof. Let $X = \bigcup_{i=1}^n U_i$ be an affine open cover with $U_i = \text{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i}$ is the cokernel of free module. Then $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some A_i -module M_i . Consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_f & \longrightarrow & \prod_{i=1}^n \mathcal{F}(U_i)_f & \xrightarrow{\varphi} & \prod_{i,j} \mathcal{F}(U_i \cap U_j)_f \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & \mathcal{F}(X_f) & \longrightarrow & \prod_{i=1}^n \mathcal{F}((U_i)_f) & \xrightarrow{\varphi'} & \prod_{i,j} \mathcal{F}((U_i)_f \cap (U_j)_f) \end{array}$$

where φ and φ' are differences of the restriction maps. $U_{i,f}$ means $U_i \cap X_f$. The rows are exact by the sheaf condition and by the exactness of localization.

(1) We have $\mathcal{F}(U_i)_f = (M_i)_f = \mathcal{F}((U_i)_f)$. Hence v is an isomorphism. This implies that u is injective.

(2) Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact and w is injective by (1). It follows that u is an isomorphism by a simple diagram chase. \square

Example 1.7.13. For an open immersion $j: U \hookrightarrow X$, $j_! \mathcal{O}_U$ is not a quasi-coherent \mathcal{O}_X -module in general. Recall

$$(j_!^{\text{psh}} \mathcal{O}_U)(V) = \begin{cases} \mathcal{O}_U(V) & V \subseteq U \\ 0 & V \subsetneq U \end{cases} \quad j_! \mathcal{O}_U = (j_!^{\text{psh}} \mathcal{O}_U)^+.$$

Indeed, if X is irreducible and V is an affine open satisfying $V \subsetneq U$, then $j_! \mathcal{O}_U(V) = 0$. This implies that $j_! \mathcal{O}_U$ is not quasi-coherent. Otherwise we would have $(j_! \mathcal{O}_U)|_V = 0$, which is absurd.

Corollary 1.7.14. Let X be a scheme. The full subcategory $\text{QCoh}(X) \subseteq \text{Shv}(X, \mathcal{O}_X)$ is stable under kernels and colimits.

Lemma 1.7.15. Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

- (1) For quasi-coherent \mathcal{O}_Y -modules \mathcal{F} and \mathcal{G} , $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}$ is also a quasi-coherent \mathcal{O}_Y -module.
- (2) For any quasi-coherent \mathcal{O}_Y -module \mathcal{F} , then $f^* \mathcal{F}$ is a quasi-coherent \mathcal{O}_X -module.

Proof. We leave (1) as an exercise. For (2), note that f^* is right exact and preserves cokernels of free modules. \square

Example 1.7.16. Let A be a DVR, $K = \text{Frac}(A)$, $X = \text{Spec}(A)$. Then $\mathcal{O}_X^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathcal{O}_X$ is **not** quasi-coherent. Indeed, $\mathcal{F}(X) = A^{\mathbb{N}}$ and $\mathcal{F}(\eta) = K^{\mathbb{N}}$, and the map $A^{\mathbb{N}} \otimes_A K \rightarrow K^{\mathbb{N}}$ is not an isomorphism.

Let $f: Y = \coprod_{n \in \mathbb{N}} X \rightarrow X$. Then $\mathcal{F} = f_*(\mathcal{O}_Y)$. This shows f_* does not preserve quasi-coherent sheaves in general.

Proposition 1.7.17. Let $f: X \rightarrow Y$ be a qcqs (quasi-coherent and quasi-separated) morphism of schemes and \mathcal{F} a quasi-coherent \mathcal{O}_X -module. Then $f_*(\mathcal{F})$ is a quasi-coherent \mathcal{O}_Y -module.

Proof. If $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$, the assertion follows from Lemma 1.7.9.

In general, we may assume that Y is affine. Then X is quasi-compact and quasi-separated (See Lemma 1.7.18 below). Let $X = \bigcup_{i=1}^n U_i$ be an affine open cover. Then $U_i \cap U_j$ is quasi-compact and we can write $U_i \cap U_j = \bigcup U_{ijk}$ with k finite and U_{ijk} affine open. Let $u_i: U_i \hookrightarrow X$ and $u_{ijk}: U_{ijk} \hookrightarrow X$ be the inclusions. The sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i u_{i*}(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} u_{ijk*}(\mathcal{F}|_{U_{ijk}})$$

is exact by sheaf condition. Applying f_* , we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i (fu_i)_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} (fu_{ijk})_*(\mathcal{F}|_{U_{ijk}}).$$

Since U_i is affine, $(fu_i)_*(\mathcal{F}|_{U_i})$ is quasi-coherent. Similarly $(fu_{ijk})_*(\mathcal{F}|_{U_{ijk}})$ is quasi-coherent. It follows that $f_*\mathcal{F}$ is quasi-coherent. \square

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We supplement some properties of quasi-separated morphisms.

Proposition 1.7.18.

- (1) *Quasi-separated morphisms are stable under composition and base change.*
- (2) *If $X \xrightarrow{f} Y \xrightarrow{g} S$ are morphisms such that gf is quasi-separated, then f is quasi-separated.*

Proof. For (1), we first consider base change. Suppose f is quasi-separated, and $g: Y' \rightarrow Y$ is another morphism. Form the Cartesian square on the left. Then the square on the right is also Cartesian.

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X' & \xrightarrow{\Delta_{f'}} & X' \times_{Y'} X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_f} & X \times_Y X \end{array}$$

Since quasi-compact morphisms are stable under base change, $\Delta_{f'}$ is quasi-compact.

For composition, let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Consider the diagram with pullback square:

$$\begin{array}{ccccc} & & \Delta_{gf} & & \\ & \nearrow \Delta_f & & \searrow & \\ X & \xrightarrow{\Delta_f} & X \times_Y X & \xrightarrow{\Delta'} & X \times_S X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_g} & Y \times_S Y & & \end{array}$$

Δ' is quasi-compact by base change. Thus Δ_{gf} is the composition of quasi-compact morphisms, and hence quasi-compact.

For (2), we apply Lemma 1.6.71. We have already proven the collection of quasi-separated morphisms are stable under base change and composition. Since Δ_g is an immersion and an immersion is clearly quasi-separated, we get that f is quasi-separated. \square

1.8 Relative spectrum

Definition 1.8.1. Let (X, \mathcal{O}_X) be a ringed space. An **\mathcal{O}_X -algebra** or a sheaf of \mathcal{O}_X -algebra consists of

- a sheaf of sets \mathcal{A} on X ;
- $\forall U \subseteq X$ open, a structure of $\mathcal{O}_X(U)$ -algebra on $\mathcal{A}(U)$

$\mathcal{A}(V) \longrightarrow \mathcal{A}(U)$
such that $\forall U \subseteq V$, \uparrow \uparrow commutes as ring homomorphisms. A
 $\mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U)$

morphism of \mathcal{O}_X -algebras $\phi: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of sheaves of sets such that $\forall U \subseteq X$, $\phi_U: \mathcal{A}(U) \rightarrow \mathcal{B}(U)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules.

We say that an \mathcal{O}_X -algebra \mathcal{A} is **quasi-coherent** \mathcal{O}_X -algebra if it is quasi-coherent as an \mathcal{O}_X -module.

Example 1.8.2. If $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, then $f_*(\mathcal{O}_X)$ is an \mathcal{O}_Y -algebra via f^* .

Example 1.8.3. Let $X = \text{Spec}(B)$. Then we have an equivalence of categories

$$\begin{aligned} B\text{-Alg} &\xrightarrow{\sim} \{\text{quasi-coherent } \mathcal{O}_X\text{-algebras}\} \\ A &\mapsto \tilde{A} \end{aligned}$$

Construction 1.8.4. Let S be a scheme and \mathcal{A} a quasi-coherent \mathcal{O}_S -algebra. We construct a scheme $X = \underline{\text{Spec}}(\mathcal{A})$ and an affine morphism $f: \underline{\text{Spec}}(\mathcal{A}) \rightarrow S$ as follows.

For $U \subseteq S$ affine open, we consider $f_U: \text{Spec}(\mathcal{A}(U)) \rightarrow \text{Spec}(\mathcal{O}(U)) \simeq U$. For any inclusion $U \subseteq V$ of affine open subsets, we have a Cartesian square

$$\begin{array}{ccc} \text{Spec}(\mathcal{A}(U)) & \longrightarrow & \text{Spec}(\mathcal{A}(V)) \\ f_U \downarrow & & \downarrow f_V \\ U & \longrightarrow & V. \end{array}$$

One verifies that these data glue to a scheme $X = \underline{\text{Spec}}(\mathcal{A})$ and an affine morphism $f: \underline{\text{Spec}}(\mathcal{A}) \rightarrow S$.

By construction, $(f_* \mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U)) = \mathcal{A}(U)$. Thus $\mathcal{A} \simeq f_* \mathcal{O}_X$.

Example 1.8.5. Let $\mathcal{F} = \mathcal{O}_S^n$ be a free \mathcal{O}_S -module. Let $\mathcal{A} = \text{Sym}_{\mathcal{O}_S}(\mathcal{F})$. If $U = \text{Spec}(B) \subseteq S$ is affine, then $\mathcal{A}(U) = B[x_1, \dots, x_n]$. We call $\underline{\text{Spec}}(\mathcal{A})$ the **affine n -space** of S . We have $\mathbb{A}_S^n \simeq \mathbb{A}_{\mathbb{Z}}^n \times_{\text{Spec}(\mathbb{Z})} S$.

In general, for any quasi-coherent \mathcal{O}_S -module \mathcal{F} , we call $\mathbb{V}(\mathcal{F}) = \underline{\text{Spec}}(\text{Sym}_{\mathcal{O}_S}(\mathcal{F}))$ the **vector bundle** over S associated to \mathcal{F} . If \mathcal{F} is locally free of rank n , $\mathbb{V}(\mathcal{F})$ is locally isomorphic to an affine n -space over S . For $n = 1$, we speak of line bundle instead of vector bundle.

Next we extend some properties of Spec to $\underline{\text{Spec}}$.

Proposition 1.8.6. Let $f: X \rightarrow S$ be a morphism of schemes and \mathcal{A} a quasi-coherent \mathcal{O}_S -algebra. Then we have a canonical bijection

$$\text{Hom}_S(X, \underline{\text{Spec}}(\mathcal{A})) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_S\text{-Alg}}(\mathcal{A}, f_*(\mathcal{O}_X)).$$

This is a relative analogue of the bijection

$$\text{Hom}_{\text{Sch}}(X, \text{Spec}(A)) \xrightarrow{\sim} \text{Hom}_{\text{Ring}}(A, \mathcal{O}_X(X)).$$

Proof. Let $Y = \underline{\text{Spec}}(\mathcal{A})$. The map is constructed as follows. To any morphism of S -schemes

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

we associate $\mathcal{A} \cong g_*\mathcal{O}_Y \xrightarrow{g_*h^\flat} g_*h_*\mathcal{O}_X \cong f_*\mathcal{O}_X$.

We will prove that for every open $U \subseteq S$,

$$(1.8.1) \quad \text{Hom}_U(f^{-1}(U), \underline{\text{Spec}}(\mathcal{A}|_U)) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_U\text{-alg}}(\mathcal{A}|_U, f_*(\mathcal{O}_X)|_U).$$

Since any morphism of S -schemes $f^{-1}(U) \rightarrow \underline{\text{Spec}}(\mathcal{A})$ factors through $\underline{\text{Spec}}(\mathcal{A}|_U)$, we have

$$\text{Hom}_U(f^{-1}(U), \underline{\text{Spec}}(\mathcal{A}|_U)) \cong \text{Hom}_S(f^{-1}(U), \underline{\text{Spec}}(\mathcal{A})).$$

Now both sides of (1.8.1) are sheaves when U runs through all open subsets of S . In order to show that the morphism of sheaves is an isomorphism, it suffices to check that (1.8.1) is an isomorphism for every affine open subset U .

Thus we may assume that $S = \text{Spec}(B)$ is affine and thus $\mathcal{A} = \tilde{A}$ with A a B -algebra. Then

$$\begin{aligned} \text{Hom}_S(X, \underline{\text{Spec}}(\mathcal{A})) &= \text{Hom}_{\text{Spec}(B)}(X, \text{Spec}(A)) \\ &\xrightarrow{\sim} \text{Hom}_{B\text{-alg}}(A, \mathcal{O}_X(X)) = \text{Hom}_{\mathcal{O}_S\text{-Alg}}(\tilde{A}, f_*(\mathcal{O}_X)). \end{aligned}$$

□

Consider the functor

$$\begin{aligned} \{\text{schemes qcqs over } S\} &\rightarrow \{\text{quasi-coherent } \mathcal{O}_S\text{-algebras}\}^{\text{op}} \\ (f: X \rightarrow S) &\mapsto f_*(\mathcal{O}_X). \end{aligned}$$

The above proposition shows $\underline{\text{Spec}}$ is a right adjoint of this functor. Moreover, $\underline{\text{Spec}}$ is fully faithful since $\mathcal{A} \simeq f_*\mathcal{O}_{\underline{\text{Spec}}(\mathcal{A})}$.

Corollary 1.8.7. *Let S be a scheme. There is an equivalence of categories*

$$\begin{aligned} \{\text{quasi-coherent } \mathcal{O}_S\text{-algebras}\}^{\text{op}} &\xrightarrow{\sim} \{\text{schemes affine over } S\} \\ \mathcal{A} &\mapsto \underline{\text{Spec}}(\mathcal{A}). \end{aligned}$$

Proof. It remains to check that for every affine morphism $f: X \rightarrow S$, the morphism $X \rightarrow \underline{\text{Spec}}(f_*\mathcal{O}_X)$ is an isomorphism. For this we may assume that S is affine and the assertion is then clear. □

Immersions

Definition 1.8.8. Let (X, \mathcal{O}_X) be a ringed space. An **ideal sheaf** I of \mathcal{O}_X is a \mathcal{O}_X -submodule of \mathcal{O}_X . This makes \mathcal{O}_X/I into an \mathcal{O}_X -algebra.

Proposition 1.8.9. *Let X be a scheme. There is an order-reserving bijection*

$$\begin{aligned}\Phi: \{\text{quasi-coherent ideal sheaves of } \mathcal{O}_X\} &\cong \{\text{closed subschemes of } X\} \\ I &\mapsto \underline{\text{Spec}}(\mathcal{O}_X/I) \\ I_Y = \text{Ker}(\mathcal{O}_X \rightarrow i_* \mathcal{O}_Y) &\leftrightarrow (i: Y \rightarrow X)\end{aligned}$$

I_Y is called the **ideal sheaf** of Y .

Proof. Let $\Psi: (i: Y \rightarrow X) \mapsto I_Y$. Since a closed immersion is qcqs, $i_* \mathcal{O}_Y$ is a sheaf of \mathcal{O}_X and I_Y is quasi-coherent ideal sheaf of \mathcal{O}_X .

It is clear that $\Psi\Phi = \text{id}$. Indeed, for $Y = \underline{\text{Spec}}(\mathcal{O}_X/I)$ and $i: Y \rightarrow X$, we have $i_* \mathcal{O}_Y = \mathcal{O}_X/I$. Thus Ψ is surjective.

It remains to prove that Ψ is injective. Since $\mathcal{O}_X \rightarrow i_* \mathcal{O}_Y$ is an epimorphism of sheaves of \mathcal{O}_X -modules, we have $\mathcal{O}_X/I_Y \xrightarrow{\sim} i_* \mathcal{O}_Y$. Since i is a closed imbedding, we have $\text{sp}(Y) = \text{supp}(i_* \mathcal{O}_Y) = \text{supp}(\mathcal{O}_X/I_Y)$. Thus $\text{sp}(Y)$ is uniquely determined by I_Y . Furthermore, $\mathcal{O}_Y \simeq i^{-1} i_* \mathcal{O}_Y$ is also uniquely determined by I_Y . \square

Corollary 1.8.10. *For $X = \text{Spec}(A)$, we have an order-reversing bijection*

$$\begin{aligned}\{\text{ideals of } A\} &\cong \{\text{closed subschemes of } \text{Spec}(A)\} \\ I &\mapsto \text{Spec}(A/I)\end{aligned}$$

This is not so obvious without using ideal sheaves.

Corollary 1.8.11. *Closed immersions are finite and stable under base change.*

Corollary 1.8.12. *Immersions are stable under base change.*

Proof. Both open immersions and closed immersions are stable under base change. \square

Proposition 1.8.13. (1) *Separated morphisms are stable under composition and base change.*

(2) *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. If gf is separated, then f is separated.*

Proof. The proof of (1) is similar to that Proposition 1.7.18. We use the stability of closed immersions under base change and composition.

For (2), we can apply Lemma 1.6.71 as before. Let us give a more direct proof. Consider the diagram with pullback square:

$$\begin{array}{ccccc} & & \Delta_{gf} & & \\ & \nearrow \Delta_f & & \searrow & \\ X & \xrightarrow{\Delta_f} & X \times_Y X & \xrightarrow{\Delta'} & X \times_S X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_g} & Y \times_S Y & & \end{array}$$

Since $\Delta_{gf}(X)$ is closed in $X \times_S X$, $\Delta_f(X)$ is closed in $X \times_Y X$. This shows that f is separated. \square

Proposition 1.8.14. *Let X be a scheme and $T \subseteq X$ a closed subset. Then there exists a unique reduced closed subscheme $Y \subseteq X$ whose underlying space is T .*

The closed subscheme structure of Y is called the **reduced induced closed subscheme structure** on T .

Proof. Existence. Let $I(U) = \{f \in \mathcal{O}_X(U) \mid f_x \in m_x, \forall x \in U \cap T\}$. Then $I \subseteq \mathcal{O}_X$ is clearly an ideal. We prove that I is quasi-coherent. Let $U = \text{Spec}(A) \subseteq X$ be an affine open subset. Then $T \cap U = V(J)$ with J radical. We have $I(U) = \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p} = J$. This remains true if we replace U by any principal open subset of U : $\forall f \in A$, $I(D(f)) = JA_f$. This shows $I|_U = \tilde{J}$ and I is quasi-coherent. Thus I gives rise to a closed subscheme $Y = \underline{\text{Spec}}(\mathcal{O}_X/I)$. The scheme Y is reduced, since for every affine open $U = \text{Spec}(A)$, $\mathcal{O}_Y(\overline{U}) = A/J$ is reduced.

Uniqueness. Let Y' be another reduced closed subscheme with underlying space T . To check $Y = Y'$, it suffices to do so on each affine open subset. Thus we may assume $X = \text{Spec}(A)$ is affine. In this case $Y = \text{Spec}(A/J)$ and $Y' = \text{Spec}(A/J')$ with J and J' radical and $V(J) = T = V(J')$. Thus $J = J'$. \square

Example 1.8.15. Taking $T = X$, we get a unique reduced closed subscheme $X_{\text{red}} \subseteq X$ whose underlying subspace is X . The scheme X_{red} is called the reduced scheme associated to X .

Normalization

Definition 1.8.16. A scheme X is said to be **normal** if for all $x \in X$, $\mathcal{O}_{X,x}$ is an integrally closed domain.

Proposition 1.8.17. [AM, Proposition 5.12] *Taking integral closure is compatible with localization: let $\phi: A \rightarrow B$ be a ring homomorphism and $S \subseteq A$ a multiplicative system. Let C be the integral closure of A in B . Then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Corollary 1.8.18. *Let A be a domain. Then A is integrally closed if and only if $\forall \mathfrak{p} \in \text{Spec}(A)$, $A_{\mathfrak{p}}$ is integrally closed.*

Corollary 1.8.19. *Let $A \rightarrow B$ be a morphism of ring homomorphism. Then $b \in B$ is integral over A if and only if $\forall \mathfrak{p} \in \text{Spec}(A)$, b is integral over $A_{\mathfrak{p}}$.*

Construction 1.8.20. Let X be an integral scheme, K its function field, L/K a field extension. Define

$$\mathcal{A}(U) = \begin{cases} \{f \in L \mid f \text{ integral over } \mathcal{O}_{X,x}, \forall x \in U\} & U \neq \emptyset \\ 0 & U = \emptyset \end{cases}$$

This is clearly an \mathcal{O}_X -algebra. For $U = \text{Spec}(A) \subseteq X$ affine open, $\mathcal{A}(U)$ is the integral closure A' of A in L by Corollary 1.8.19. For any principal open subset $D(f)$ of U , $\mathcal{A}(D(f)) = A'_f$. This shows that $\mathcal{A}|_U = \widetilde{A'}$. Thus \mathcal{A} is quasi-coherent.

The scheme $X' = \underline{\text{Spec}}(\mathcal{A})$ equipped with the morphism $X' \rightarrow X$ is called the **normalization** of X in L . If $L = K$, then $X' := \underline{\text{Spec}}(\mathcal{A})$ is called the **normalization** of X .

From the construction, we see X' is integral and normal and the canonical morphism $X' \rightarrow X$ is integral.

Example 1.8.21. $X = \text{Spec}(k[x, y]/(y^2 - x^3))$ has a cusp at the origin O . Since $(y/x)^2 = x$ in the function field, X is not normal. Let $z = y/x$. Then $X^\nu := \text{Spec}(k[x, z]/(z^2 - x)) \simeq \text{Spec}(k[z])$ is the normalization of X . In this case, $X^\nu \rightarrow X$ is a universal homeomorphism.

Example 1.8.22. $X = \text{Spec}(k[x, y]/(y^2 - x^2(x + 1)))$ has a node at O . Since $(y/x)^2 = x + 1$ in the function field, X is not normal. Let $y/x = z$. Then $X^\nu := \text{Spec}([x, z]/(z^2 - (x + 1)) \simeq \text{Spec}(k[z])$ is the normalization of X . The fiber of $X^\nu \rightarrow X$ at the origin consists of the two rational points $z = \pm 1$.

Definition 1.8.23. An integral scheme X is said to be **Japanese** if for every finite extension L of the function field K of X , the normalization X' of X in L is finite over X .

A scheme X is said to be **universally Japanese** if every integral scheme locally of finite type over X is Japanese.

Theorem 1.8.24. Let A be

- a field, or
- a Dedekind domain with fraction field K satisfying $\text{char}(K) = 0$, or
- a Noetherian complete local ring.

Then $\text{Spec}(A)$ is universally Japanese.

1.9 Valuative criterion

Definition 1.9.1. A ring A is called a **valuation ring** if it is a domain and $\forall x \in \text{Frac}(A)$, either $x \in A$ or $x^{-1} \in A$.

Definition 1.9.2. Let $f: X \rightarrow S$ be a morphisms of schemes.

- f is said to satisfy the existence part of the **valuation criterion** if for every valuation ring A with fraction field K , and all morphisms $i: \text{Spec}(K) \rightarrow X$ and $j: \text{Spec}(A) \rightarrow S$ making the following square commutative, there exists a morphism $t: \text{Spec}(A) \rightarrow X$ making the two triangles below commutative:

$$\begin{array}{ccc} \text{Spec}(K) & \xrightarrow{i} & X \\ \downarrow & \nearrow t & \downarrow f \\ \text{Spec}(A) & \xrightarrow{j} & S \end{array}$$

- f is said to satisfy the uniqueness part of the valuation criterion if whenever given i and j as above, there exists at most one t making the triangles commutative. That is, for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{i} & X \\ \downarrow & \nearrow t_1 & \downarrow f \\ \mathrm{Spec}(A) & \xrightarrow{j} & S \end{array}$$

where A is a valuation ring with fraction field K , we have $t_1 = t_2$.

Remark 1.9.3. $f: X \rightarrow S$ satisfies the existence part of valuation criterion if and only if for all valuation ring A with fraction field K , $X(A) \rightarrow X(K) \times_{S(K)} S(A)$ is surjective.

$f: X \rightarrow S$ satisfies the uniqueness part of valuation criterion if and only if for all valuation ring A with fraction field K , $X(A) \rightarrow X(K) \times_{S(K)} S(A)$ is injective.

Remark 1.9.4. If we are given a diagram as below,

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & X \times_S S' & \twoheadrightarrow & X \\ \downarrow & \nearrow t & \downarrow t' & \downarrow f' & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & S' & \longrightarrow & S \end{array}$$

then the dotted arrow t exists if and only if t' exists and t is unique if and only if t' is unique. This follows immediately from the universal property of fiber product.

Definition 1.9.5. A morphism of schemes $f: X \rightarrow S$ is **universally specializing** if every base change of f is specializing.

Theorem 1.9.6. Let $f: X \rightarrow S$ be a morphism of schemes.

- (1) f satisfies the existence part of valuation criterion $\iff f$ is universally specializing.
- (2) f satisfies the uniqueness part of valuation criterion $\iff \Delta_f$ is universally specializing.

Proof. (1) \implies (2). We prove f satisfies uniqueness $\iff \Delta_f$ satisfies existence.

Consider the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow c & \downarrow \Delta_f \\ \mathrm{Spec}(A) & \xrightarrow{(a,b)} & X \times_S X & \quad \quad \quad \begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow a & \downarrow f \\ \mathrm{Spec}(A) & \xrightarrow{b} & S \end{array} \end{array}$$

The morphism c exists if and only if $a = b$. Thus the existence of Δ_f corresponds to uniqueness of f . □

Date: 10.20

Definition 1.9.7. Let K be a field and let $A, B \subseteq K$ be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$. We say that B **dominates** A if $A \subseteq B$ and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$. We say that A is a **valuation ring of K** if A is a valuation ring and $\text{Frac}(A) = K$.

Fact 1.9.8. (1) [AM, Exercise 5.27] Let $A \subseteq K$ be a local domain. Then A is a valuation ring of K if and only if A is maximal for the dominance relation among local rings in K .

(2) [M2, Theorem 10.2] For any local ring $B \subseteq K$, there exists a valuation ring A of K dominating B .

Proof of Theorem 1.9.6 (1). \Leftarrow . Since f is universally specializing, we may pull back and reduce to the following lifting problem:

$$\begin{array}{ccc} \text{Spec}(K) & \xrightarrow{g} & X \\ \downarrow & \nearrow t & \downarrow f \\ \text{Spec}(A) & \xlongequal{\quad} & S \end{array}$$

Write $\text{Im}(g) = \{x'\}$, s the closed point of S . Since f is specializing, and $f(x') \rightsquigarrow s$, $\exists x' \rightsquigarrow x$ such that $f(x) = s$. Consider

$$\begin{array}{ccccc} & & \phi & & \\ & K & \xleftarrow{\kappa(x')} & \xleftarrow{\quad} & \mathcal{O}_{X,x} \\ & \swarrow & & & \uparrow A \\ & & & & \end{array}$$

Since $f(x) = s$, $A \rightarrow \mathcal{O}_{X,x}$ is a local homomorphism. Thus $\phi(\mathcal{O}_{X,x})$ dominates A . Since A is a valuation ring, it is maximal for the dominance relation, hence $\phi(\mathcal{O}_{X,x}) = A$. Let ψ be ϕ regarded as a map $\mathcal{O}_{X,x} \rightarrow A$. Then $\text{Spec}(A) \xrightarrow{\text{Spec}(\psi)} \text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ furnishes the desired morphism t .

\Rightarrow . It suffices to show f specializing. Let $x' \in X$ and $f(x') = s' \rightsquigarrow s$. We need to find $x' \rightsquigarrow x$ such that $f(x) = s$. Consider

$$\begin{array}{ccccc} & & \phi & & \\ & K = \kappa(x') & \xleftarrow{\quad} & \xleftarrow{\quad} & \mathcal{O}_{X,x'} \\ & \uparrow & & & \mathcal{O}_{S,s'} \xleftarrow{\quad} \mathcal{O}_{S,s} \\ & & & & \end{array}$$

Since $\phi(\mathcal{O}_{S,s})$ is a local ring in K , there is a valuation ring A of K dominating it. Thus we have

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & \text{Spec}(\mathcal{O}_{X,x'}) & \dashrightarrow & X \\ \downarrow & & \downarrow t & & \downarrow f \\ \text{Spec}(A) & \dashrightarrow & \text{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

Since f satisfies the existence part of valuation criterion, there exists t making the two triangles commutative. Let η and σ be the generic and closed points of $\text{Spec}(A)$, respectively. Let $x = t(\sigma)$. Then $f(x) = s$ and t maps the specialization $\eta \rightsquigarrow \sigma$ to $x' \rightsquigarrow x$ as desired. \square

Proposition 1.9.9. (1) *A closed map is specializing.*

(2) *Conversely, a specializing and quasi-compact morphism of schemes is closed.*

Proof. (1) This was shown in Example 1.6.35.

(2) Let $f: X \rightarrow S$ be specializing and quasi-compact morphism of schemes. Let $Y \subseteq X$ be closed subset. Equip Y with the induced reduced subscheme structure. We observe that the composition $Y \hookrightarrow X \xrightarrow{f} S$ is specializing and quasi-compact. We conclude that $f(Y)$ is closed by the following lemma. \square

Lemma 1.9.10. *Let $f: X \rightarrow S$ be a quasi-compact morphism of schemes. Then $f(X)$ contains every maximal point of $\overline{f(X)}$. In particular, $f(X)$ is closed if moreover $f(X)$ is closed under specialization.*

Proof. We may assume that $S = \text{Spec}(B)$ is affine. Then X is quasi-compact. Take a finite affine open covering $X = \bigcup_i U_i$, $U_i = \text{Spec}(A_i)$. Then $f(X) = \bigcup_i f(U_i) = \text{Im}(\text{Spec}(\prod_i A_i) \rightarrow \text{Spec}(B))$. Thus we may assume $X = \text{Spec}(A)$ is affine and $f = \text{Spec}(\phi)$ where $\phi: B \rightarrow A$ is a ring homomorphism.

Factor ϕ as $B \twoheadrightarrow B/I \hookrightarrow A$, where $I = \text{Ker}(\phi)$. Then $\overline{f(X)}$ is contained in $\text{Spec}(B/I)$. Thus may assume that ϕ is injective. This case is the content of next Lemma. \square

Lemma 1.9.11. *Let $\phi: B \rightarrow A$ is an injective ring homomorphism. Then the image of $f = \text{Spec}(\phi)$ contains all maximal points of $\text{Spec}(B)$.*

Compare with Lemma 1.2.11(2).

Proof. Let $\mathfrak{p} \in \text{Spec}(B)$ be a maximal point. Then $B_{\mathfrak{p}}$ has a unique prime ideal, $\mathfrak{p}B_{\mathfrak{p}}$. Since $B_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}$ remains injective by flatness, we have $A_{\mathfrak{p}} \neq 0$. Thus there exists a maximal ideal \mathfrak{m} of $A_{\mathfrak{p}}$. Then $\mathfrak{m} \cap B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$. \square

Corollary 1.9.12. *A universally specializing and quasi-compact morphism of schemes is universally closed.*

Example 1.9.13. $\coprod_p \text{Spec}(\mathbb{Z}/p) \rightarrow \text{Spec}(\mathbb{Z})$ is universally specializing but not closed. The image is the set of closed points of $\text{Spec}(\mathbb{Z})$.

Corollary 1.9.14. f is separated \iff f is quasi-separated and satisfies the uniqueness part of the valuative criterion.

Proof. f is separated \iff Δ_f is closed \iff Δ_f is quasi-compact and universally specializing \iff Δ_f is quasi-compact and satisfies the existence part of the valuative criterion \iff f is quasi-separated and satisfies the uniqueness part of the valuative criterion. \square

Definition 1.9.15. We say that a morphism of schemes is **proper** if it is separated, of finite type and universally closed.

Corollary 1.9.16. f is proper \iff f is of finite type, quasi-separated and satisfies both parts of the valuative criterion.

Proposition 1.9.17. f is integral \iff f is affine and universally closed.

Corollary 1.9.18. f is finite \iff f is affine and proper.

Proof of the proposition. We need only to prove \Leftarrow . Let $f: X \rightarrow S$ be an affine morphism that is universally closed. We may assume that $S = \text{Spec}(B)$ is affine. Then $X = \text{Spec}(A)$ is affine and $f = \text{Spec}(\phi)$, $\phi: A \rightarrow B$. In this case we have the following stronger result. \square

Lemma 1.9.19. Let $\phi: B \rightarrow A$ be a ring homomorphism such that $\text{Spec}(A[X]) \rightarrow \text{Spec}(B[X])$ is closed. Then ϕ is integral.

Proof. Let $a \in A$. We will show a is integral over B . Consider

$$\begin{aligned} I &= \text{Ker}(B[X] \rightarrow A) \\ X &\mapsto a \end{aligned}$$

and

$$\begin{aligned} J &= \text{Ker}(B[X] \rightarrow A[X]/(aX - 1) = A[a^{-1}]) \\ X &\mapsto X \end{aligned}$$

If $f = \sum_{i \geq 0} b_i X^i \in J$, then $f = (aX - 1)g$ where $g = \sum_{i \geq 0} a_i X^i \in A[X]$. Expand the coefficients we have $b_i = aa_{i-1} - a_i$. Thus for $n \geq \max\{\deg(f), \deg(g) + 1\}$, $h = \sum_i b_i X^{n-i} = \sum_i (aa_{i-1} - a_i) X^{n-i} = (a - x) \sum a_i X^{n-1-i} \in I$. The leading coefficient of h is b_0 . Thus, to show that a is integral, it suffices to find $f \in J$ with constant term in B^\times .

Note that J contains a polynomial with constant term in B^\times if and only if $J + XB[X] = B[X]$. Now consider the closed map $f = \text{Spec}(\phi[X]): \text{Spec}(A[x]) \rightarrow \text{Spec}(B[x])$. By Lemma 1.2.11, $\overline{f(V(aX - 1))} = V(J)$. Thus $f(V(aX - 1)) = V(J)$. It follows that $g: \text{Spec}(A[x]/(aX - 1)) \rightarrow \text{Spec}(B[x]/(J))$ is surjective. We have a Cartesian square

$$\begin{array}{ccc} \emptyset & \xrightarrow{g'} & \text{Spec}(B[X]/(J + XB[X])) \\ \downarrow & & \downarrow \\ \text{Spec}(A[x]/(aX - 1)) & \xrightarrow{g} & \text{Spec}(B[x]/(J)). \end{array}$$

Indeed, $A[X]/(X, aX - 1) = 0$. Thus g' is surjective. In other words, $\text{Spec}(B[x]/(J + XB[X])) = \emptyset$ and $J + XB[X] = B[X]$. \square

As usual, proper morphisms behave well under composition and base change:

Proposition 1.9.20. (1) Proper morphisms are stable under composition and base change.

(2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ gf proper and g separated $\implies f$ proper.

Definition 1.9.21. Let S be a scheme. We call $\mathbb{P}_S^n := \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec}(\mathbb{Z})} S$ the **projective n -space** over S .

For $S = \text{Spec}(A)$, we have $\mathbb{P}_A^n \simeq \mathbb{P}_{\text{Spec}(A)}^n$.

We have seen that finite morphisms are proper. Another nontrivial example is the following.

Proposition 1.9.22. $\mathbb{P}_S^n \rightarrow S$ is proper.

It suffices to show that $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \text{Spec}(\mathbb{Z})$ is proper. Recall that $\mathbb{P}_{\mathbb{Z}}^n = \bigcup_{i=0}^n U_i$, $U_i \simeq \text{Spec}(R_i)$, $R_i = \mathbb{Z}[x_j/x_i]_{j=0}^n$. Moreover, $U_i \cap U_j \simeq \text{Spec}(R_{ij})$, $R_{ij} = \mathbb{Z}[\{x_k/x_i, x_k/x_j\}_{k=0}^n]$. Since each R_i is a finite type \mathbb{Z} -algebra and $U_i \cap U_j$ is quasi-compact, $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \text{Spec}(\mathbb{Z})$ is of finite type and quasi-separated. It remains to prove both parts of the valuative criterion.

We now describe the functor represented by $\mathbb{P}_{\mathbb{Z}}^n$ in a special case. For a ring A , let $\mathbb{P}_{\mathbb{Z}}^n(A) = \text{Hom}_{\text{Sch}}(\text{Spec}(A), \mathbb{P}_{\mathbb{Z}}^n)$.

Lemma 1.9.23. Let A be a local domain and let $K = \text{Frac}(A)$. Then

$$\mathbb{P}_{\mathbb{Z}}^n(A) \cong W/K^\times$$

where $W = \{(a_0, \dots, a_n) \in K^{n+1} \setminus \{(0, \dots, 0)\} \mid \exists i, \forall j, a_j \in a_i A\}$ and K^\times acts on W by scalar multiplication.

We will give a description of $\mathbb{P}_{\mathbb{Z}}^n(S)$ for a general scheme S later. The class of (a_0, \dots, a_n) is denoted by $[a_0 : \dots : a_n]$.

Proof. Let $f: \text{Spec}(A) \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ and let s be the closed point of $\text{Spec}(A)$. Then there exists i such that $f(s) \in U_i$. It follows that $f: \text{Spec}(A) \rightarrow U_i$. Thus $\mathbb{P}_{\mathbb{Z}}^n(A) = \bigcup_i U_i(A)$. Consider the subset $W_i = \{(a_0, a_1, \dots, a_n) \in K^{n+1} \setminus \{(0, \dots, 0)\} \mid \forall j, a_j \in a_i A\} \subseteq W$. Note that $U_i(A) \simeq \text{Hom}_{\text{Ring}}(R_i \rightarrow A)$. The map $\phi_i: W_i/K^\times \rightarrow U_i(A)$ carrying $[a_0 : \dots : a_n]$ to the homomorphism $x_j/x_i \mapsto a_j/a_i$ is a bijection. Indeed, the inverse carries $g: R_i \rightarrow A$ to $[g(x_j/x_i)]_{0 \leq j \leq n}$. Similarly, $(U_i \cap U_j)(A) \simeq \text{Hom}_{\text{Ring}}(R_{ij}, A)$. The maps ϕ_i and ϕ_j restrict to $(W_i \cap W_j)/K^\times \xrightarrow{\sim} (U_i \cap U_j)(A)$. Since $W = \bigcup_{i=0}^n W_i$, the maps ϕ_i patch together to a bijection $W/K^\times \xrightarrow{\sim} \mathbb{P}_{\mathbb{Z}}^n(A)$. \square

We next discuss the valuation defined by a valuation ring.

Definition 1.9.24. Let Γ be a totally ordered abelian group ($a \leq b \implies a + c \leq b + c$) and let K be a field. A **valuation** $v: K^\times \rightarrow \Gamma$ is a group homomorphism satisfying the strong triangle inequality:

$$v(x + y) \geq \max(v(x), v(y)).$$

We extend v to $v(0) = \infty$.

If $v: K^\times \rightarrow \Gamma$ is a valuation, then $\{x \in K \mid v(x) \geq 0\}$ is a valuation ring of K . Conversely, if A is a valuation ring of K , then the quotient map $v: K^\times \rightarrow K^\times/A^\times$ is a valuation. Here the total order on K^\times/A^\times is defined as follows: $xA^\times \leq yA^\times$ if $x^{-1}y \in A$.

End of proof of Proposition 1.9.22. Since $\text{Spec}(\mathbb{Z})$ is a final object, it suffices to show for every valuation ring A of fraction field K , the map

$$\varphi: \mathbb{P}_{\mathbb{Z}}^n(A) \rightarrow \mathbb{P}_{\mathbb{Z}}^n(K)$$

is a bijection. By the description in Lemma 1.9.23, this map can be identified with the inclusion

$$W/K^\times \subseteq (K^{n+1} \setminus \{O\})/K^\times,$$

where $O = (0, \dots, 0)$. Let $v: K^\times \rightarrow \Gamma$ be the valuation given by v . Let $(a_0, \dots, a_n) \in K^{n+1} \setminus \{O\}$. We can find a (nonzero) a_i with the smallest valuation. Then $v(a_j/a_i) \geq 0$ for all $0 \leq j \leq n$. In other words, $a_j/a_i \in A$ for all j . This shows that $(a_0, \dots, a_n) \in W$. Thus $W = (K^{n+1} \setminus \{O\})/K^\times$ and φ is a bijection. \square

GAGA (Algebraic Geometry and Analytic Geometry). Let X/\mathbb{C} be a scheme of finite type. Any affine open $U \subseteq X$ is of the form $\text{Spec}(\mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_m))$. Then $U(\mathbb{C}) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid f_i(a_1, \dots, a_n) = 0\} \subseteq \mathbb{C}^n$. We equip $U(\mathbb{C})$ with the subspace topology induced from the usual topology on \mathbb{C}^n . One can show that this does not depend on the choice of the embedding $U \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$ and there exists a topology τ on $X(\mathbb{C})$ such that each $U(\mathbb{C})$ is an open subspace. The space $X^{\text{an}} = (X(\mathbb{C}), \tau)$ is called the **analytic space** associated to X .

Fact 1.9.25. • X separated $\iff X^{\text{an}}$ Hausdorff.

- X/\mathbb{C} proper $\iff X^{\text{an}}$ Hausdorff and quasi-compact.

• For $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & \text{Spec}(\mathbb{C}) & \end{array}$ where both $X \rightarrow \mathbb{C}$ and $Y \rightarrow \mathbb{C}$ are separated and of finite type, f is proper $\iff f^{\text{an}}: X^{\text{an}} \rightarrow Y^{\text{an}}$ is proper (i.e. for every quasi-compact subset $V \subseteq Y^{\text{an}}$, $(f^{\text{an}})^{-1}(V)$ is quasi-compact, or, equivalently, for every topological space Z , $f^{\text{an}} \times Z: X^{\text{an}} \times Z \rightarrow Y^{\text{an}} \times Z$ is closed).

Theorem 1.9.26 (Nagata compactification). *Let S be a quasi-compact quasi-separated scheme and let $f: X \rightarrow S$ be a separated morphism of finite type. Then there exists an open immersion $j: X \hookrightarrow \overline{X}$ and a proper morphism $\bar{f}: \overline{X} \rightarrow S$ such that $\bar{f}j = f$.*

Nagata proved the theorem for Noetherian schemes and Deligne proved the general case.

In a couple of simple cases, we already know the result of Nagata compactification.

Lemma 1.9.27. *Let $f: X \rightarrow S$ be a quasi-compact immersion. Then there exists an open immersion $j: X \rightarrow \overline{X}$ and a closed immersion $\bar{f}: \overline{X} \rightarrow S$ such that $\bar{f}j = f$.*

The quasi-compactness assumption cannot be dropped. See Warning 1.6.66.

Proof. It suffices to take $\overline{X} = \underline{\text{Spec}}(\mathcal{O}_X/\mathcal{I})$ with $\mathcal{I} = \text{Ker}(\mathcal{O}_S \rightarrow f_*\mathcal{O}_X)$. This is called the **scheme-theoretic closure** of X . \square

Example 1.9.28. Let $X = \text{Spec}(A)$, $S = \text{Spec}(B)$, $\phi: B \rightarrow A$ and $f = \text{Spec}(\phi)$. Assume that f is of finite type, namely A is a finitely generated B -algebra. Choosing a set of generators, we obtain a closed immersion $X \rightarrow \mathbb{A}_B^n$ over B . Choose an open immersion $\mathbb{A}_B^n \rightarrow \mathbb{P}_B^n$ over B .

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{A}_B^n & \longrightarrow & \mathbb{P}_B^n \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

Let \overline{X} be the scheme-theoretic closure of X in \mathbb{P}_B^n . Then $\overline{X} \rightarrow S$ is proper.

Date: 10.22

1.10 Homogeneous spectrum

Let k be a field. We have a bijection

$$\begin{aligned}\mathbb{P}^n(k) &\xrightarrow{\sim} \{\text{lines in } \mathbb{A}^{n+1} \text{ through } O\} \\ [a_0 : \cdots : a_n] &\mapsto V(a_i x_j - a_j x_i).\end{aligned}$$

Closed subsets of \mathbb{P}^n correspond to conical subsets of \mathbb{A}^{n+1} of the form $V(f_1, \dots, f_r)$, with each f_i homogeneous.

Example 1.10.1. The cone $V(x^2 - y^2 - z^2)$ in \mathbb{A}^3 corresponds to a curve in \mathbb{P}^2 .

For every graded ring R , we will construct a scheme $\text{Proj}(R)$, called the **homogeneous spectrum** of R . Recall that a **graded ring** is a ring equipped with a decomposition $R = \bigoplus_{d \geq 0} R_d$ as abelian groups, satisfying $R_d R_e \subseteq R_{d+e}$. In particular, $1 \in R_0$ and R_0 is both a sub-ring and a quotient ring of R . Elements in R_d are called **homogeneous of degree d** .

An ideal $I \subseteq R$ is said to be **homogeneous** if $I = \bigoplus_{d \geq 0} (I \cap R_d)$.

Example 1.10.2. $R_+ = \bigoplus_{d > 0} R_d$ is a homogenous ideal.

Lemma 1.10.3. Let $\mathfrak{p} \subseteq R$ be a homogeneous ideal. Then \mathfrak{p} is a prime ideal if and only if $\forall x, y \in R$ homogeneous, $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Proof. \Leftarrow . Let $x = \sum x_d$, $y = \sum y_e$, $x, y \notin \mathfrak{p}$. Then there exists a smallest d_0 such that $x_{d_0} \notin \mathfrak{p}$. Similarly there exists a smallest e_0 such that $y_{e_0} \notin \mathfrak{p}$. Expanding xy , we see that $x_{d_0}y_{e_0} \notin \mathfrak{p}$. Thus $xy \notin \mathfrak{p}$. \square

Definition 1.10.4. For a graded ring R , we define a subset $\text{Proj}(R) \subseteq \text{Spec}(R)$ by

$$\text{Proj}(R) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \text{ homogeneous and } \mathfrak{p} \not\supseteq R_+\} \subseteq \text{Spec}(R).$$

We equip $\text{Proj}(R)$ with the subspace topology.

Notation 1.10.5. For any subset $T \subseteq R$, we write $V_+(T) = V(T) \cap \text{Proj}(R)$. For $f \in R$, we write $V_+(f) = V_+((f))$. For $f \in R_+$ homogeneous, we write $D_+(f) = \text{Proj}(R) \setminus V_+(f)$.

Thus $V_+(T)$ is the set of homogeneous prime ideals of R satisfying $T \subseteq \mathfrak{p}$ and $R_+ \subsetneq \mathfrak{p}$. It is a closed subset of $\text{Proj}(R)$. If $p_d: R \rightarrow R_d$ denotes the projection, then the homogeneous ideal generated by T is $I = \bigcup_{d \geq 0} p_d(T)$. It is clear that $V_+(T) = V_+(I)$. Thus every closed subset of $\text{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal I .

Example 1.10.6. $V_+(R) = V_+(R_+) = \emptyset$, $V_+(0) = \text{Proj}(R)$.

Lemma 1.10.7. If $f \in R_0$, then

$$D(f) \cap \text{Proj}(R) = \bigcup_{g \in R_+ \text{ homogeneous}} D_+(fg)$$

Proof. \supseteq is clear. For \subseteq , suppose there exists $\mathfrak{p} \in D(f) \cap \text{Proj}(R)$ such that $\mathfrak{p} \notin \bigcup_g D_+(fg)$. Then $f \notin \mathfrak{p}$, but $\forall g \in R_+$ homogeneous, $fg \in \mathfrak{p}$, so that $g \in \mathfrak{p}$. It follows that $R_+ \subseteq \mathfrak{p}$, a contradiction. \square

This is the reason why we only consider $D_+(f)$ with f homogeneous of positive degree.

Definition 1.10.8. $D_+(f)$ with $f \in R_+$ homogeneous are called **standard open subsets**.

Standard open subsets form an open basis for $\text{Proj}(R)$ and $D_+(fg) = D_+(f) \cap D_+(g)$.

For any homogeneous $f \in R_+$, R_f is a \mathbb{Z} -graded ring. Let $R_{(f)}$ denote the degree 0 piece of R_f .

Lemma 1.10.9. *For f homogeneous of positive degree, we have*

$$D_+(f) \xleftarrow{\sim} \{\text{homogeneous primes of } R_f\} \xrightarrow{\sim} \text{Spec}(R_{(f)})$$

are homeomorphisms.

Proof. The isomorphism on the left is easy to establish. For the one on the right, we apply the following Lemma. \square

Lemma 1.10.10. *Let S be a \mathbb{Z} -graded ring such that there exists $f \in S_d \cap S^\times$ for some $d > 0$. Then we have a homeomorphism*

$$\begin{aligned} j: G = \{\mathbb{Z}\text{-graded primes of } S\} &\xrightarrow{\sim} \text{Spec}(S_0) \\ \mathfrak{p} &\mapsto \mathfrak{p} \cap S_0 \\ \sqrt{\mathfrak{p}_0 S} &\leftarrow \mathfrak{p}_0 \end{aligned}$$

We remark that in a \mathbb{Z} -graded ring S , S_0 is a subring but typically not a quotient.

Proof. We need to prove that $\sqrt{\mathfrak{p}_0 S}$ is a prime ideal. Let $a, b \in \sqrt{\mathfrak{p}_0 S}$ homogeneous. There exists $n \geq 1$ such that $(ab)^n \in \mathfrak{p}_0 S$. We have $(a^d b^d / f^{\deg(a)+\deg(b)})^n \in \mathfrak{p}_0$, hence $a^d / f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d / f^{\deg(b)} \in \mathfrak{p}_0$. This shows $a \in \sqrt{\mathfrak{p}_0 S}$ or $b \in \sqrt{\mathfrak{p}_0 S}$.

It is easy to check that j is a bijection. It is continuous. We prove that it is open. Consider the open subset $G \cap D(g)$, where $g = \sum_i g_i$, $g_i \in S_i$. Then $j(G \cap D(g)) = \bigcup_i D(g_i^d / f^{\deg(g_i)})$. Thus j is a homeomorphism. \square

We now proceed to equip $X = \text{Proj}(R)$ with a sheaf of rings \mathcal{O}_X . We take $\mathcal{O}_X(D_+(f)) = R_{(f)}$. The functoriality of this assignment is guaranteed by the following.

Lemma 1.10.11. *Assume $D_+(g) \subseteq D_+(f)$. Then there exist $n \geq 1$ such that $g^n = af$ with $a \in R$ homogeneous. Moreover, we have a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & R_f & \longleftarrow & R_{(f)} \\ & \searrow & \downarrow & & \downarrow \\ & & R_g & \longleftarrow & R_{(g)} \simeq (R_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \end{array}$$

Proof. We first show that $f^{\deg(g)}/g^{\deg(f)} \in R_{(g)}$ is invertible. If $f^{\deg(g)}/g^{\deg(f)} \in \mathfrak{p}_0 \in \text{Spec}(R_{(g)})$, then $f \in \sqrt{\mathfrak{p}_0 R_g} \cap R = \mathfrak{p} \in D_+(g) \subseteq D_+(f)$, a contradiction. Thus $f \frac{b}{g^m} = 1$ in $R_{(g)}$ with some $m \geq 1$. It follows that $af = g^n$ for some $n \geq 1$. The last part of the lemma is now clear. \square

Proposition 1.10.12. *For a graded ring R , the functor*

$$\begin{aligned} \{\text{standard open subsets of } \text{Proj}(R)\}^{\text{op}} &\rightarrow \text{Ring} \\ D_+(f) &\mapsto R_{(f)} \end{aligned}$$

extends to a sheaf \mathcal{O}_X on $X = \text{Proj}(R)$. Moreover, $(D_+(f), \mathcal{O}_X|_{D_+(f)}) \simeq \text{Spec}(R_{(f)})$ and (X, \mathcal{O}_X) is a scheme over $\text{Spec}(R)$.

Proof. We first verify the gluing property. Let $D_+(f) = \bigcup_i D_+(g_i)$. Since $D_+(f) \simeq \text{Spec}(R_{(f)})$ and

$$D_+(g_i) \simeq \text{Spec}(R_{(g_i)}) \simeq \text{Spec}(R_{(f)})_{g_i^{\deg(f)}} / f^{\deg(g_i)},$$

the gluing property for \mathcal{O}_X follows from the gluing property for $\mathcal{O}_{\text{Spec}(R_{(f)})}$. The last assertion is now clear. \square

Proposition 1.10.13. *For all $\mathfrak{p} \in \text{Proj}(R)$, we have $\mathcal{O}_{X,\mathfrak{p}} = R_{(\mathfrak{p})}$. Here $R_{(\mathfrak{p})}$ is the degree 0 piece of $T^{-1}R$, where $T = \{f \in R \setminus \mathfrak{p} \text{ homogeneous}\}$.*

Proof. We have

$$\mathcal{O}_{X,\mathfrak{p}} = \underset{f \in R_+ \setminus \mathfrak{p} \text{ homogeneous}}{\text{colim}} R_{(f)} = R_{(\mathfrak{p})}.$$

Here we used the fact that there exists $g \in R_+ \setminus \mathfrak{p}$ homogeneous and $a/f = \frac{ag}{fg}$ in $R_{(\mathfrak{p})}$. \square

Example 1.10.14. $\mathbb{P}_A^n \simeq \text{Proj}(R)$, where $R = A[x_0, \dots, x_n]$. Indeed, $\text{Proj}(R) = \bigcup D_+(x_i)$, $D_+(x_i) = R_{(x_i)} = A[x_j/x_i]_{j=0}^n$, $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j) = A[x_k/x_i, x_k/x_j]_{k=0}^n$. In particular, $\text{Spec}(A) = \mathbb{P}_A^0 \simeq \text{Proj}(A[x_0])$.

Example 1.10.15. Let $R = A[x_0, \dots, x_n]$ and let $d_0, \dots, d_n > 0$ be integers. We define a grading on R by $R_0 = A$ and $\deg(x_i) = d_i$. We call $\text{Proj}(R) := \mathbb{P}_A(d_0, \dots, d_n)$ the **weighted projective n -space** of weights (d_0, \dots, d_n) . It is clear that $\mathbb{P}_A(d_0, \dots, d_n) = \mathbb{P}_A(dd_0, \dots, dd_n)$ for any $d \geq 1$.

Lemma 1.10.16. *$\text{Proj}(R)$ is quasi-separated.*

Proof. $\text{Proj}(R) = \bigcup_f D_+(f)$ and $D_+(f) \cap D_+(g) = D_+(fg)$. \square

In fact, $\text{Proj}(R)$ is separated (exercise).

Proposition 1.10.17. *$\text{Proj}(R)$ is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \dots, f_r \in R_+$ such that $R_+ \subseteq \sqrt{(f_1, \dots, f_r)}$.*

Proof. $\text{Proj}(R)$ is quasi-compact if and only if a finite number of standard opens cover $\text{Proj}(R)$. In other words, there exist $f_1, \dots, f_r \in R_+$ homogeneous such that $V_+(f_1, \dots, f_r) = \emptyset$. We conclude by the next Lemma. \square

Lemma 1.10.18. *Let $I \subseteq R$ be a homogeneous ideal. Then $V_+(I) = \emptyset \iff R_+ \subseteq \sqrt{I}$.*

Proof. \Leftarrow . Clear.

\Rightarrow . Assume $R_+ \not\subseteq \sqrt{I}$. Then there exists $f \in R_+ \setminus \sqrt{I}$ homogeneous. We have $(R/I)_f \neq 0$, so that $(R/I)_{(f)} \neq 0$ (since it contains $1 \in (R/I)_f$). Thus $\text{Proj}(R/I) \neq \emptyset$. Then there exists a homogeneous prime \mathfrak{q} of R/I satisfying $\mathfrak{q} \not\supseteq (R/I)_+$. The pre-image of \mathfrak{q} in R is a homogeneous prime \mathfrak{p} of I satisfying $\mathfrak{p} \not\supseteq R_+$. Thus $\mathfrak{p} \in V_+(I)$, a contradiction. \square

Functionality

Let $\phi: R \rightarrow S$ be a homomorphism of graded rings. For $\mathfrak{q} \in \text{Proj}(S)$, $\phi^{-1}(\mathfrak{q})$ is a homogeneous prime ideal of R , but in general it may happen that $\phi^{-1}(\mathfrak{q}) \supseteq R_+$. Let

$$U(\phi) = \{\mathfrak{q} \in \text{Proj}(S) \mid \phi^{-1}(\mathfrak{q}) \not\supseteq R_+\}$$

In other words, $U(\phi) = \text{Proj}(S) \setminus f^{-1}(V(R_+))$, where $f = \text{Spec}(\phi)$.

$$\begin{array}{ccc} & U(\phi) & \\ & \downarrow & \\ \text{Proj}(R) & & \text{Proj}(S) \\ & \downarrow & \downarrow \\ \text{Spec}(R) & \xleftarrow[f]{\quad} & \text{Spec}(S), \end{array}$$

Lemma 1.10.19.

$$U(\phi) = \bigcup_{\text{homogenous } a \in R_+} D_+(\phi(a))$$

Proof. $\phi^{-1}(\mathfrak{q}) \in \text{Proj}(R) \iff \exists a \in R_+, a \notin \phi^{-1}(\mathfrak{q}) \iff \exists a \in R_+, \phi(a) \notin \mathfrak{q}$. \square

The natural morphisms of schemes $D_+(\phi(a)) \rightarrow D_+(a)$ given by $\phi_{(a)}: R_{(a)} \rightarrow S_{(\phi(a))}$ glue to a morphism of schemes $\text{Proj}(\phi): U(\phi) \rightarrow \text{Proj}(R)$.

We will give an example where $\text{Proj}(\phi)$ is defined on $\text{Proj}(S)$.

Let us start with a general remark on homogeneous localization. For $a \in R_+$ homogenous of degree d ,

$$R_{(a)} = \text{colim}(\ R_0 \xrightarrow{a} R_d \xrightarrow{a} R_{2d} \longrightarrow \dots)$$

can be computed using R_{nd} for n running through any unbounded subset of \mathbb{N} . In particular,

- If $\phi: R \rightarrow S$ is such that $\sup\{n \mid \phi_{nd} \text{ is surjective}\} = \infty$, then $\phi_{(a)}: R_{(a)} \rightarrow S_{(\phi(a))}$ is surjective.
- If $\phi: R \rightarrow S$ is such that $\sup\{n \mid \phi_{nd} \text{ is an isomorphism}\} = \infty$, then $\phi_{(a)}: R_{(a)} \rightarrow S_{(\phi(a))}$ is an isomorphism.

Proposition 1.10.20. *Let $\phi: R \rightarrow S$ be a graded homomorphism such that for all $d \geq 1$, there exists $n \geq 1$ such that ϕ_{nd} is surjective. Then $U(\phi) = \text{Proj}(S)$ and $\text{Proj}(\phi): \text{Proj}(S) \hookrightarrow \text{Proj}(R)$ is a closed immersion.*

Proof. Let $\mathfrak{q} \in \text{Proj}(S)$. There exists $b \in S_+$ homogeneous of degree $d > 0$, $b \notin \mathfrak{q}$. Then $b^n \notin \mathfrak{q}$ for all n . By assumption, there exists $n \geq 1$ such that ϕ_{nd} is surjective, and thus there exists $a \in R_+$ with $\phi(a) = b^n$. Thus $a \notin \phi^{-1}(\mathfrak{q})$ and $\phi^{-1}(\mathfrak{q}) \in \text{Proj}(R)$. This shows $U(\phi) = \text{Proj}(S)$. Moreover, for $a \in R_+$ homogeneous, $R_{(a)} \rightarrow S_{(\phi(a))}$ is surjective. Thus $\text{Proj}(\phi)$ is a closed immersion. \square

Example 1.10.21. For any homogeneous ideal $I \subseteq R$, $\text{Proj}(R/I) \hookrightarrow \text{Proj}(R)$ is a closed subscheme of image $V_+(I)$. We will give a partial converse later.

Proposition 1.10.22. *Let $\phi: R \rightarrow S$ be a graded homomorphism such that for all $d \geq 1$, there exists $n \geq 1$ such that ϕ_{nd} is an isomorphism. Then $\text{Proj}(\phi): \text{Proj}(S) \xrightarrow{\sim} \text{Proj}(R)$ is an isomorphism.*

Next we look at a different kind of functoriality.

Notation 1.10.23. For $d \geq 1$, we let $R^{(d)} := \bigoplus_n R_{nd}$ denote the graded ring with $R_n^{(d)} = R_{nd}$.

Proposition 1.10.24. *We have an isomorphism of schemes over $\text{Spec}(R_0)$*

$$\begin{aligned} \text{Proj}(R) &\xrightarrow{\sim} \text{Proj}(R^{(d)}) \\ \mathfrak{p} &\mapsto \mathfrak{p} \cap R^{(d)}. \end{aligned}$$

Proof. We write $R_{+, \text{homog}} = \bigcup_{i>0} R_i$. We have $\text{Proj}(R) = \bigcup_{f \in R_{+, \text{homog}}} D_{+,R}(f)$ and $\text{Proj}(R^{(d)}) = \bigcup_{f \in R_{+, \text{homog}}} D_{+,R^{(d)}}(f^d)$. Indeed, for $g \in R_{+, \text{homog}}^{(d)}$, $D_{+,R^{(d)}}(g) = D_{+,R^{(d)}}(g)$. Observe that the inclusion $R^{(d)} \subseteq R$ induces an isomorphism $R_{(f^d)}^{(d)} \rightarrow R_{(f)}$, with inverse given by $a/f^n \mapsto af^{n(d-1)}/f^{nd}$. This gives $D_{+,R}(f) \xrightarrow{\sim} D_{+,R^{(d)}}(f^d)$, which patches together to an isomorphism of schemes $\text{Proj}(R) \xrightarrow{\sim} \text{Proj}(R^{(d)})$ over $\text{Spec}(R_0)$. \square

Remark 1.10.25. The underlying homeomorphism $\iota: \text{Proj}(R) \xrightarrow{\sim} \text{Proj}(R^{(d)})$ is compatible with the continuous map $\text{Spec}(R) \rightarrow \text{Spec}(R^{(d)})$ induced by the inclusion $R^{(d)} \subseteq R$, which is not graded for $d > 1$. The inverse of ι carries \mathfrak{q} to $\mathfrak{p} = \{g \in R \mid g^d \in \mathfrak{q}\}$. To see this, we first need to show that \mathfrak{p} is an ideal. If $g^d \in \mathfrak{q}$, $h^d \in \mathfrak{q}$, then $(g+h)^{2d} \in \mathfrak{q}$, hence $(g+h)^d \in \mathfrak{q}$. Thus \mathfrak{p} is an ideal. It is graded, since otherwise, writing $g = \sum_i g_i$ with $g_i \in R_i$, there exists $g_i \notin \mathfrak{p}$ of lowest degree. Then $g^d \in \mathfrak{q}$ implies $g_i^d \in \mathfrak{q}$, a contradiction. It is clear that \mathfrak{p} is a prime and $\mathfrak{q} \mapsto \mathfrak{p}$ is an inverse of ι .

More trivially we can also define a graded ring $R^{(1/d)}$ where

$$R_n^{(1/d)} = \begin{cases} R_{n/d} & d|n \\ 0 & d \nmid n. \end{cases}$$

We also have $\text{Proj}(R^{(1/d)}) \xrightarrow{\sim} \text{Proj}(R)$.

Example 1.10.26. $R = A[x_0, \dots, x_n]$ with $\deg(x_i) = 1$ for all i . Then $R^{(d)} = A[M_0, \dots, M_N]$, where M_0, \dots, M_N are the monomials of degree d . We have $N = \binom{d+n}{n} - 1$. We have a surjective graded homomorphism $S = A[y_0, \dots, y_N] \rightarrow R^{(d)}$ sending y_i to M_i . Let I be the kernel. Taking Proj, we get a closed immersion $\mathbb{P}_A^n = \text{Proj}(R) \simeq \text{Proj}(R^{(d)}) \hookrightarrow \text{Proj}(S) = \mathbb{P}_A^N$. This is called the **d -uple embedding**. Here are some examples of low dimension and low degree.

- $n = 1, d = 2, \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. This is a conic. $R = A[u, v]$,

$$S = A[x, y, z] \rightarrow R^{(2)} = A[u^2, uv, v^2].$$

The kernel is $I = (y^2 - xz)$.

- $n = 1, d = 3, \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. This is a twisted cubic curve.

$$S = A[w, x, y, z] \rightarrow R^{(3)} = A[u^3, u^2v, uv^2, v^3].$$

We have $I = (x^2 - wy, y^2 - xz, wz - xy)$. For $A = k$ a field, the closed subscheme $C \subseteq \mathbb{P}^3$ given by the triple embedding has codimension 2, but it is easy to see that I cannot be generated by two elements. We say that C is not a complete intersection in \mathbb{P}^3 .

For example, for $J = (x^2 - wy, y^2 - xz)$, $V_+(J)$ is not irreducible, as it contains the line $V_+(x, y)$. For $I' = (x^2 - wy, y^3 - wz^2) \subsetneq I$, we have $\sqrt{I'} = \sqrt{I}$, so that $V_+(I') = V_+(I)$ as sets: C is a set-theoretic complete intersection.

- $n = 1, d = 4, \mathbb{P}^1 \hookrightarrow \mathbb{P}^4$. This is a twisted quartic curve. Consider

$$R' = A[u^4, u^3v, uv^3, v^4] \subseteq R^{(4)} = A[u^4, u^3v, u^2v^2, uv^3, v^4]$$

We have $R'_n = R_n^{(4)}$ for all $n \geq 2$. Thus $\text{Proj}(R') \xrightarrow{\sim} \text{Proj}(R^{(4)})$. We have $\text{Proj}(R') \hookrightarrow \mathbb{P}^3$. For A a domain, R' is not integrally closed, since $(u^2v^2)^2 \in R'$ but $u^2v^2 \notin R'$.

- $n = 2, d = 2, \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. This is called the **Veronese embedding**. $R = A[u, v, w]$,

$$S = A[y_0, y_1, y_2, y_3, y_4, y_5] \twoheadrightarrow R^{(2)} = A[u^2, v^2, w^2, uv, uw, vw]$$

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Example 1.10.27. $\mathbb{P}_A(1, 2, 3) = \text{Proj}(A[u, v, w])$, $\deg(u) = 1$, $\deg(v) = 2$, $\deg(w) = 3$. It is easy to see that $R^{(6)} = A[u^6, v^3, w^2, u^4v, u^3w, u^2v^2, uvw]$ is generated by $R_1^{(6)}$ over A . We have thus obtained a closed immersion $\mathbb{P}_A(1, 2, 3) \hookrightarrow \mathbb{P}_A^6$. This is a del Pezzo surface (for $A = k$ a field).

The same argument works for any finitely generated graded ring.

Lemma 1.10.28. *Let R be a graded ring, finitely generated over R_0 . Then there exists $d \geq 1$ such that $R^{(d)}$ is generated by finitely many elements in $R_1^{(d)}$ over R_0 . Moreover, if $R = R_0[f_1, \dots, f_r]$ with f_i homogeneous of degree $d_i \geq 1$ and $m = \text{lcm}(d_1, \dots, d_r)$, then we can take $d = sm$ where s is any integer $\geq \max\{r - 1, 1\}$.*

Proof. Consider $P = f_1^{e_1} \cdots f_r^{e_r}$ of total degree Nm , $N \geq r$. In other words, $\sum_i d_i e_i \geq Nm$. Then there exists i such that $e_i \geq \frac{m}{d_i}$ and we have $P = P_1 Q$, where $P_1 = f_i^{m/d_i}$ has degree m and Q is homogeneous of degree $(N - 1)m$. Thus if $N = nsm$, we obtain by induction a decomposition $P = P_1 \cdots P_{(n-1)sm} Q$, where $P_j \in R_m$ and $Q \in R_{sm}$. Thus P can be generated by $R_1^{(d)}$ over R_0 . The finiteness is clear. \square

Corollary 1.10.29. *Let R be a graded ring, finitely generated over R_0 . Then there exists a closed immersion $\text{Proj}(R) \hookrightarrow \mathbb{P}_{R_0}^n$ for some n .*

Proof. Let d be as in the previous lemma. Since $R^{(d)}$ is generated by finitely many elements of $R_1^{(d)}$ over R_0 , it is the quotient of the polynomial ring $R_0[X_0, \dots, X_n]$. This gives a closed immersion $\text{Proj}(R) \simeq \text{Proj}(R^{(d)}) \hookrightarrow \mathbb{P}_{R_0}^n$. \square

Definition 1.10.30. We say that a morphism $f: X \rightarrow Y$ is **projective** if it factors as

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}_S^n \\ & \searrow f & \downarrow p \\ & & S \end{array} = \mathbb{P}_{\mathbb{Z}}^n \times_{\mathbb{Z}} S$$

where i is a closed immersion and p is the projection.

Remark 1.10.31. Projective morphisms are proper.

Base change

Let $\phi: R \rightarrow S$ be a graded ring homomorphism. We have for any $a \in R_+$ homogeneous, a commutative diagram

$$\begin{array}{ccc} D_+(\phi(a)) & \longrightarrow & D_+(a) \\ \downarrow & & \downarrow \\ U(\phi) & \xrightarrow{r} & \text{Proj}(R) \\ \downarrow & & \downarrow \\ \text{Proj}(S) & & \\ \downarrow & & \downarrow \\ \text{Spec}(S_0) & \longrightarrow & \text{Spec}(R_0) \end{array}$$

The top most square is Cartesian and thus r is an affine morphism.

Proposition 1.10.32. *Let R be a graded ring and $R_0 \rightarrow S_0$ a ring homomorphism. Let $S = R \otimes_{R_0} S_0$ and $\phi: R \rightarrow S$. Then $U(\phi) = \text{Proj}(S)$ and we have a Cartesian square*

$$\begin{array}{ccc} \text{Proj}(S) & \longrightarrow & \text{Proj}(R) \\ \downarrow & & \downarrow \\ \text{Spec}(S_0) & \longrightarrow & \text{Spec}(R_0) \end{array}$$

Proof. Since $R_+S = S_+$, we have $U(\phi) = \text{Proj}(S) \setminus V(S_+) = \text{Proj}(S)$. Take $a \in R_+$ homogeneous. We need to check that

$$\begin{array}{ccc} D_+(\phi(a)) & \longrightarrow & D_+(a) \\ \downarrow & & \downarrow \\ \text{Spec}(S_0) & \longrightarrow & \text{Spec}(R_0) \end{array}$$

is Cartesian. This follows from the fact that tensor product commutes with localization: the diagram of rings

$$\begin{array}{ccc} S_{(\phi(a))} & \longleftarrow & R_{(a)} \\ \uparrow & & \uparrow \\ S_0 & \longleftarrow & R_0 \end{array}$$

is coCartesian. □

Now let R and S be graded rings satisfying $R_0 = S_0 = A$. We will determine the fiber product of the diagram.

$$\begin{array}{ccc} \text{Proj}(R) & & \\ \downarrow & & \\ \text{Proj}(S) & \longrightarrow & \text{Spec}(A) \end{array}$$

A first attempt is to consider $R \otimes_A S$ with grading given by $(R \otimes_A S)_d = \bigoplus_{i+j=d} R_i \otimes S_j$. But for homogeneous elements $a \in R_+$ and $b \in S_+$, the map $R_{(a)} \otimes S_{(b)} \rightarrow (R \otimes_A S)_{(a \otimes b)}$ is typically not surjective.

Instead we consider the subring $R \otimes_A S = \bigoplus_{d \geq 0} R_d \otimes_A S_d \subseteq R \otimes_A S$, with grading given by $(R \otimes_A S)_d = R_d \otimes_A S_d$. We have a Cartesian square

$$\begin{array}{ccc} \text{Proj}(R \otimes_A S) & \longrightarrow & \text{Proj}(R) \\ \downarrow & & \downarrow \\ \text{Proj}(S) & \longrightarrow & \text{Spec}(A) \end{array}$$

Indeed, for a and b as above, we have $R_{(a)} \otimes S_{(b)} \simeq (R \otimes_A S)_{(a \otimes b)}$.

The subring can be more complicated than the tensor product, as shown by the following example.

Example 1.10.33. Let $R = A[x_0, \dots, x_r]$, $S = A[y_0, \dots, y_s]$. Then $R \otimes_A S = A[x_0, \dots, x_r, y_0, \dots, y_s]$, but $R \otimes_A S = A[x_i y_j]_{\substack{0 \leq i \leq r \\ 0 \leq j \leq s}}$. We have a surjection

$$\begin{aligned} T &= A[z_{ij}] \rightarrow S \\ z_{ij} &\mapsto x_i y_j \end{aligned}$$

with kernel $I = (z_{ij} z_{i'j'} - z_{ij'} z_{i'j})_{i,j,i',j'}$. This gives a closed immersion $\mathbb{P}_A^r \times_{\text{Spec}(A)} \mathbb{P}_A^r \simeq \text{Proj}(R \otimes_A S) \hookrightarrow \text{Proj}(T) = \mathbb{P}_A^N$, where $N = (r+1)(s+1) - 1 = rs + r + s$. This is called the **Segre embedding**.

In the case $r = s = 1$, we have $N = 3$ and the image of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is the quadric surface defined by $xw - yz = 0$.

Proposition 1.10.34. *Projective morphisms are stable under base change and composition.*

Proof. The stability under base change follows from the fact that closed immersions are stable under base change: if $X \rightarrow S$ is a projective morphism and $S' \rightarrow S$ an arbitrary morphism, then we have a diagram with Cartesian squares

$$\begin{array}{ccccc} X \times_S S' & \longrightarrow & \mathbb{P}_{S'}^n & \longrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}_S^n & \longrightarrow & S \end{array}$$

For the stability under composition, let $X \rightarrow Y$ and $Y \rightarrow S$ be projective morphisms and consider the following commutative diagram with Cartesian squares:

$$\begin{array}{ccccccc} X & \hookrightarrow & \mathbb{P}_Y^n & \hookrightarrow & \mathbb{P}_{\mathbb{P}_S^m}^n & \longrightarrow & \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_{\mathbb{Z}}^m \xrightarrow{\text{Segre}} \mathbb{P}_{\mathbb{Z}}^N \\ \searrow & \downarrow & \downarrow & & \downarrow p & & \swarrow \\ Y & \hookrightarrow & \mathbb{P}_S^m & \longrightarrow & & & \text{Spec}(\mathbb{Z}) \end{array}$$

The square in the middle is Cartesian because $\mathbb{P}_{\mathbb{P}_S^m}^n = \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_S^m = \mathbb{P}_{\mathbb{Z}}^n \times \mathbb{P}_{\mathbb{Z}}^m \times S$ by definition. Since p is projective by the Segre embedding, its base change $\mathbb{P}_{\mathbb{P}_S^m}^n \rightarrow S$ is also projective. Thus $X \rightarrow S$ is projective. \square

Definition 1.10.35. Let \mathcal{P} be a class of schemes. \mathcal{P} is called **local** if

- $X \in \mathcal{P}, U \subseteq X$ open $\implies U \in \mathcal{P}$.
- $X = \bigcup_i U_i, U_i$ open and $U_i \in \mathcal{P}$ for all $i \implies X \in \mathcal{P}$.

Here are some local properties of schemes: reduced, normal, locally Noetherian, empty.

Non-local properties: affine, quasi-compact, separated, quasi-separated, irreducible, connected, integral, Noetherian.

Definition 1.10.36. Let \mathcal{P} be a class of morphisms. We say \mathcal{P} is **local on the source** if

- $(X \xrightarrow{f} Y) \in \mathcal{P}, U \xrightarrow{j} X$ open immersion $\implies f \circ j \in \mathcal{P}$.
- Given $f: X \rightarrow Y, X = \bigcup_i U_i, U_i$ open and $\forall i, (f|_{U_i}: U_i \rightarrow Y) \in \mathcal{P} \implies f \in \mathcal{P}$.

We say \mathcal{P} is **local on the target** if

- $(X \xrightarrow{f} Y) \in \mathcal{P}, V \subseteq Y$ open $\implies (f^{-1}(V) \xrightarrow{f_V} V) \in \mathcal{P}$.
- Given $f: X \rightarrow Y, Y = \bigcup_i V_i, V_i \subseteq Y$ open and $\forall i, (f^{-1}(V_i) \xrightarrow{f_{V_i}} V_i) \in \mathcal{P} \implies f \in \mathcal{P}$.

Local on the source and the target: locally of finite type, flat, open, generalizing.

Local on the target: quasi-compact, affine, closed, specializing, integral, finite, quasi-separated, separated, proper, immersion, surjective, injective.

Not local on the target: projective.

An example of Hironaka shows that projectiveness is **not** local on the target. See [H, Example B.3.4.2].

Quasi-coherent sheaves on $\text{Proj}(R)$

For every graded R -module M , we will construct a quasi-coherent sheaf \widetilde{M} on $\text{Proj}(R)$. Recall that a **graded R -module** is an R -module M equipped with a \mathbb{Z} -grading as abelian group $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $R_d M_e \subseteq M_{d+e}$.

Given a graded R -module M and $n \in \mathbb{Z}$, we define a graded R -module $M(n)$, called the twisted module, by $M(n)_d = M_{n+d}$. If we visualize a graded R -module by writing down its pieces sequentially, then $M(1)$ corresponds to a shift to the left.

Given graded R -modules M and N , the tensor product $M \otimes_R N$ is a graded R -module as follows. The R_0 -module $M \otimes_{R_0} N$ clearly admits a grading: $(M \otimes_{R_0} N)_d = \bigoplus_{i+j=d} M_i \otimes_{R_0} N_j$. Then $M \otimes_R N$ can be identified with the quotient of $M \otimes_{R_0} N$ by the graded submodule generated by $am \otimes n - m \otimes an$ with homogeneous elements $m \in M, n \in N, a \in R$.

Homomorphisms of graded modules are required to preserve degrees. We let $\text{GrHom}_R(M, N)_0$ denote the R_0 -module of graded homomorphisms $M \rightarrow N$. (One can define a graded R -module $\text{GrHom}_R(M, N) = \bigoplus_n \text{GrHom}(M, N(n))$ but this will not be used in the sequel.)

For $f \in R_+$ homogeneous, we let $M_{(f)}$ denote the degree 0 piece of M_f . The proof of the following is similar to Propositions 1.10.12 and 1.10.13.

Proposition 1.10.37. *The functor*

$$\begin{aligned} \{\text{standard open subsets of } \text{Proj}(R)\} &\rightarrow \{\text{abelian groups}\} \\ D_+(f) &\mapsto M_{(f)} \end{aligned}$$

extends to a quasi-coherent sheaf \widetilde{M} on $\text{Proj}(R) = (X, \mathcal{O}_X)$. We have $\widetilde{M}|_{D_+(f)} \simeq \widetilde{M}_{(f)}$ for all $f \in R_+$ homogeneous and $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$ for all $\mathfrak{p} \in \text{Proj}(R)$. Here $M_{(\mathfrak{p})}$ is the degree 0 piece of $T^{-1}M$, $T = \bigcup_{d \geq 0} R_d \setminus \mathfrak{p}$.

We obtain a functor

$$\begin{aligned} \text{GrMod}(R) &\rightarrow \text{Shv}(X, \mathcal{O}_X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

It is easy to see that this functor is exact and commutes with colimits. The canonical morphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \rightarrow (M \otimes_R N)^\sim$, given locally by $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \rightarrow (M \otimes_R N)_{(f)}$, is not an isomorphism in general.

We have $M_{(f)} = \text{colim}(\ M_0 \xrightarrow{f} M_d \xrightarrow{f} M_{2d} \xrightarrow{f} \dots)$, where $d = \deg(f)$. Thus if $Z \subseteq \mathbb{Z}$ with $\sup Z = +\infty$, then $M_{(f)}$ depends only on M_{nd} , $n \in Z$. This motivates the following.

Notation 1.10.38. For $d \geq 1$, let $U_d = \bigcup_{f \in R_d} D_+(f) \subseteq \text{Proj}(R)$.

We have $\text{Proj}(R) = \bigcup_{d \geq 1} U_d$ and $U_d \subseteq U_{dn}$ for all $n \geq 1$.

- If $\text{Proj}(R)$ is quasi-compact, then $\text{Proj}(R) = U_d$ for some d .
- If R is generated by R_1 over R_0 , then $X = U_1$.

Definition 1.10.39. We define the quasi-coherent sheaf $\mathcal{O}_X(n)$ to be $\widetilde{R(n)}$. We call $\mathcal{O}_X(1)$ the **twisting sheaf**.

Proposition 1.10.40. *Let $X = \text{Proj}(R)$. Let M and N be graded R -modules and let $n \in \mathbb{Z}$.*

- *On U_d , $\mathcal{O}_X(nd)$ is an invertible sheaf and the map*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(nd) \rightarrow \widetilde{M(nd)}$$

is an isomorphism when restricted to U_d . In particular, $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(nd)|_{U_d} \xrightarrow{\sim} \mathcal{O}_X(m+nd)|_{U_d}$, $\mathcal{O}_X(nd)|_{U_d}^\vee \simeq \mathcal{O}_X(-nd)|_{U_d}$.

- *The restriction of $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \xrightarrow{\sim} (M \otimes_R N)^\sim$ to U_1 is an isomorphism.*

This boils down to the following lemmas.

Lemma 1.10.41. *For $f \in R_d$, $d > 0$, we have an isomorphism $\mathcal{O}_X|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X(nd)|_{D_+(f)}$ given by*

$$\begin{aligned} R_{(f)} &\xrightarrow{\sim} R(nd)_{(f)} \\ a &\mapsto f^n a. \end{aligned}$$

Lemma 1.10.42. *For $f \in R_1$, the canonical map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \rightarrow (M \otimes_R N)_{(f)}$ is an isomorphism.*

Proof. We have $R_f = R_{(f)}[f, f^{-1}] \simeq R_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[X, X^{-1}]$. Thus $(M \otimes_R N)_f \simeq M_f \otimes_{R_f} N_f \simeq (M_{(f)} \otimes_{R_{(f)}} N_{(f)})[f, f^{-1}]$. Thus $M_{(f)} \otimes_{R_{(f)}} N_{(f)}$ is the degree 0 piece of $(M \otimes_R N)_{(f)}$. \square

Example 1.10.43. Consider $X = \mathbb{P}^n(d, \dots, d) = \text{Proj}(R)$, where $R = A[x_0, \dots, x_n]$ with $\deg(x_i) = d \geq 2$. For $d \nmid n$, $R(n)_{(f)} = 0$, $f \in R_d$, since the non-zero degrees of $R(n)$ are $\equiv -n \pmod{d}$. Thus $\mathcal{O}(n) = 0$ for $d \nmid n$ and $0 = \mathcal{O}(1) \otimes_{\mathcal{O}} \mathcal{O}(-1) \not\simeq \mathcal{O}$.

Date: 10.29

Recall we have defined for each graded module M over a graded ring R , a quasi-coherent sheaf \widetilde{M} over $X = \text{Proj}(R)$ satisfying $\widetilde{M}(D_+(f)) = M_{(f)}$. We want to study the behavior of \widetilde{M} under change of R .

Let $\phi: R \rightarrow S$ be a graded ring homomorphism. Let $X = \text{Proj}(R)$, $Y = \text{Proj}(S)$, and $r: U(\phi) \rightarrow X$ the morphism induced by ϕ .

$$\begin{array}{ccc} \text{Proj}(S) & & \text{Proj}(R) \\ \downarrow & \nearrow r & \\ U(\phi) & & \end{array}$$

- Let N be a graded S -module. Then $r_*(\widetilde{N}|_{U(\phi)}) = \widetilde{r_*N}$. Indeed, for each $a \in R_+$ homogeneous, $r_*(\widetilde{N}|_{U(\phi)})(D_+(a)) = \widetilde{N}(D_+(\phi(a))) = N_{(\phi(a))}$.
- Let M be a graded R -module. Then $\widetilde{M \otimes_R S}$ is a graded S -module and we have a natural morphism $r^*(\widetilde{M}) \rightarrow \widetilde{M \otimes_R S}|_{U(\phi)}$, locally defined by $M_{(a)} \otimes_{R_{(a)}} S_{(\phi(a))} \rightarrow (M \otimes_R S)_{(\phi(a))}$ on $D_+(\phi(a))$ for $a \in R_+$ homogeneous. This is not an isomorphism in general. However, for $d \geq 1$ and $n \in \mathbb{Z}$, $r^*(\mathcal{O}_X(nd))|_{r^{-1}(U_d)} \xrightarrow{\sim} \mathcal{O}_Y(nd)|_{r^{-1}(U_d)}$ and $r^*(\widetilde{M})|_{r^{-1}(U_1)} \xrightarrow{\sim} \widetilde{M \otimes_R S}|_{r^{-1}(U_1)}$.

Next consider $i: \text{Proj}(R) \cong \text{Proj}(R^{(d)})$. Let M be a graded R -module. Then $i^*(\widetilde{M^{(d)}}) \xrightarrow{\sim} \widetilde{M}$. Here $M^{(d)}$ is the graded $R^{(d)}$ -module defined by $(M^{(d)})_n = M_{dn}$. In particular, we have $i^*\mathcal{O}(n) = \mathcal{O}(dn)$.

The functor Γ_*

Since $M \rightarrow \widetilde{M}$ commutes with colimits, it admits a right adjoint by the adjoint functor theorem. We can describe the adjoint explicitly.

Notation 1.10.44. Given $X = \text{Proj}(R)$ and an \mathcal{O}_X -module \mathcal{F} (not necessarily quasi-coherent), we let

- $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n));$
- $\Upsilon_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-n), \mathcal{F}).$

We denote the degree n pieces of $\Gamma_*(\mathcal{F})$ and $\Upsilon_*(\mathcal{F})$ by $\Gamma_n(\mathcal{F})$ and $\Upsilon_n(\mathcal{F})$, respectively.

Each $a \in R_d$ induces a morphism of \mathcal{O}_X -modules $\mathcal{O}_X(n) \rightarrow \mathcal{O}_X(n+d)$. This makes $\Gamma_*(\mathcal{F})$ and $\Upsilon_*(\mathcal{F})$ into graded R -modules. The natural pairing

$$(\mathcal{F} \otimes \mathcal{O}_X(n)) \otimes \mathcal{O}_X(-n) \rightarrow \mathcal{F}$$

induces a homomorphism

$$\nu: \Gamma_*(\mathcal{F}) \rightarrow \Upsilon_*(\mathcal{F}).$$

If $X = U_d$, then ν_{dn} is an isomorphism for all $n \in \mathbb{Z}$.

We have defined functors Γ_* and Υ_* from $\text{Shv}(X, \mathcal{O}_X)$ to $\text{GrMod}(R)$.

Proposition 1.10.45. $\sim \dashv \Upsilon_*$.

Proof. We define the unit and counit

$$\begin{aligned}\phi: M &\longrightarrow \Upsilon_*(\widetilde{M}) \\ \psi: \Upsilon_*(\mathcal{F})^\sim &\longrightarrow \mathcal{F}\end{aligned}$$

as follows.

For $m \in M_d$,

$$\phi(m): \mathcal{O}_X(-d) \rightarrow \widetilde{M}$$

is defined by

$$R(-d)_{(a)} \xrightarrow{\times m} M_{(a)}$$

on $D_+(a)$, $a \in R_+$ homogeneous. Here $\times m$ denotes multiplication by m .

For $a \in R_d$, $d > 0$, we define

$$\begin{aligned}\Gamma(D_+(a), \psi): \Upsilon_*(\mathcal{F})_{(a)} &\rightarrow \Gamma(D_+(a), \mathcal{F}) \\ g/a^n &\mapsto g(a^{-n})\end{aligned}$$

where $g \in \Upsilon_{dn}(\mathcal{F}) = \text{Hom}(\mathcal{O}_X(-dn), \mathcal{F})$, $a^{-n} \in R(-nd)_{(a)} = \Gamma(D_+(a), \mathcal{O}_X(-nd))$.

One verifies that this gives the expected adjunction. \square

Proposition 1.10.46. *Assume that $X = \text{Proj}(R)$ is quasi-compact. For any quasi-coherent sheaf \mathcal{F} on X , $\Gamma_*(\mathcal{F})^\sim \xrightarrow[\sim]{\tilde{\nu}} \Upsilon_*(\mathcal{F})^\sim \xrightarrow[\sim]{\psi} \mathcal{F}$.*

Thus, for $\text{Proj}(R)$ quasi-compact, Υ_* induces a fully faithful functor from $\text{QCoh}(X)$ to the category of graded R -modules.

Proof. Since X is quasi-compact, we have $X = U_d$ for some $d > 0$. Then ν_{dn} is an isomorphism for all $n \in \mathbb{Z}$. It follows that $\tilde{\nu}$ is an isomorphism. Thus it suffices to show that for all $a \in R_+$ homogeneous, $\Gamma(D_+(a), \psi\tilde{\nu}): \Gamma_*(\mathcal{F})_{(a)} \rightarrow \Gamma(D_+(a), \mathcal{F})$ is an isomorphism. Up to replacing a by a^d , we may assume that $d \mid \deg(a) = m$. Note that X is quasi-compact and quasi-separated. It suffices to apply the lemma below to the invertible sheaf $\mathcal{O}_X(m)$ and the section defined by a . \square

Let X be a scheme, \mathcal{L} an invertible sheaf on X , and \mathcal{F} a quasi-coherent sheaf on X .

- For every $f \in \Gamma(X, \mathcal{L})$, define $X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x\}$, where \mathfrak{m}_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. This is an open subset of X .
- Define $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, $\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Then $\Gamma_*(X, \mathcal{L})$ is a graded ring and $\Gamma_*(X, \mathcal{L}, \mathcal{F})$ is a graded $\Gamma_*(X, \mathcal{L})$ -module.

Here, for $n < 0$, $\mathcal{L}^{\otimes n}$ denotes $(\mathcal{L}^\vee)^{\otimes n}$.

Lemma 1.10.47. *Assume that X is quasi-compact. Let $f \in \Gamma(X, \mathcal{L})$. Then the canonical map*

$$\alpha: \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(f)} \rightarrow \Gamma(X_f, \mathcal{F})$$

is injective. Moreover, if X is quasi-separated, then α is an isomorphism.

Here $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(f)}$ is simply

$$\text{colim}(\Gamma(X, \mathcal{F}) \xrightarrow{f} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}) \xrightarrow{f} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes 2}) \rightarrow \dots).$$

For $\mathcal{L} = \mathcal{O}_X$, we recover Lemma 1.7.12.

Proof. Cover X by a finite number of open affine subsets U_1, \dots, U_n such that $\mathcal{L}|_{U_i}$ is trivial, i.e. $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(f)} & \longrightarrow & \bigoplus_{i=1}^n \Gamma_*(U_i, \mathcal{L}, \mathcal{F})_{(f)} & \longrightarrow & \bigoplus_{i,j=1}^n \Gamma_*(U_i \cap U_j, \mathcal{L}, \mathcal{F})_{(f)} \\ & & \downarrow \alpha & & \downarrow \theta & & \downarrow \beta \\ 0 & \longrightarrow & \Gamma(X_f, \mathcal{F}) & \longrightarrow & \bigoplus_{i=1}^n \Gamma((U_i)_f, \mathcal{F}) & \longrightarrow & \bigoplus_{i,j=1}^n \Gamma((U_i \cap U_j)_f, \mathcal{F}) \end{array}$$

Since each U_i is affine and \mathcal{L} is trivial on U_i , θ is an isomorphism. This implies that α is injective. In the case where X is quasi-separated, $U_i \cap U_j$ is quasi-compact and β is injective by the previous case. It follows that α is an isomorphism in this case. \square

The bijectivity of ϕ is more complicated. We will limit our attention to the case $M = R$. In this case, we have a commutative diagram

$$\begin{array}{ccc} & & \Gamma_*(\mathcal{O}_X) \\ & \nearrow \varphi & \downarrow \nu \\ R & \xrightarrow{\phi} & \Upsilon_*(\mathcal{O}_X) \end{array}$$

Note that $\Gamma_*(\mathcal{O}_X)$ is a \mathbb{Z} -graded ring (in the notation above, $\Gamma_*(X, \mathcal{O}_X(1))$ is the degree ≥ 0 part of $\Gamma_*(\mathcal{O}_X)$) and φ is a homomorphism of \mathbb{Z} -graded rings. By contrast, there is no natural ring structure on $\Upsilon_*(\mathcal{O}_X)$ in general.

Proposition 1.10.48. *We have:*

- (1) ν is an isomorphism if $X = U_1$.
- (2) φ is an isomorphism if
 - $R = A[x_0, \dots, x_n]$, $n \geq 1$; or
 - R is a Noetherian normal ring and $\text{ht}(R_+) \geq 2$.

Part (1) of the proposition is clear since ν_n is an isomorphism for all $n \in \mathbb{Z}$ in the case $X = U_1$. To prove part (2), we will give an interpretation of $\Gamma_*(\mathcal{O}_X)$.

Lemma 1.10.49. *Let X be a quasi-compact scheme and $\{\mathcal{F}_i\}_{i \in I}$ a family of quasi-coherent sheaves. Then the natural map*

$$\epsilon: \bigoplus_{i \in I} \Gamma(X, \mathcal{F}_i) \rightarrow \Gamma(X, \bigoplus_{i \in I} \mathcal{F}_i)$$

is injective. Moreover, if X is quasi-separated, then ϵ is an isomorphism.

Proof. The map ϵ is an isomorphism for X affine. In general, proceed as in Lemma 1.10.47. \square

Remark 1.10.50. The first (resp. second) statement of the lemma holds in fact for any quasi-compact (resp. quasi-compact, quasi-separated, admitting a quasi-compact open basis) topological space X and any abelian sheaf \mathcal{F} on X .

In the case $X = \text{Proj}(R)$, consider $\epsilon: \Gamma_*(\mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{A})$, where $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. Note that \mathcal{A} is a quasi-coherent \mathcal{O}_X -algebra. Consider $f: \underline{\text{Spec}}(\mathcal{A}) \rightarrow \text{Proj}(R)$. The restriction of f to $D_+(a) \subseteq \text{Proj}(R)$ can be identified with $\text{Spec}(R_a) \rightarrow \text{Spec}(R_{(a)})$. Thus $\underline{\text{Spec}}(\mathcal{A})$ can be identified with the open subscheme $U = \bigcup_{a \in R_{+, \text{homog}}} D(a) = \text{Spec}(\bar{R}) \setminus V(R_+)$ of $\text{Spec}(R)$. The following is easy to check.

Lemma 1.10.51. $\epsilon\varphi: R \rightarrow \Gamma(X, \mathcal{A})$ can be identified with the restriction map $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_Y(U)$, where $Y = \text{Spec}(R)$.

Proof of Proposition 1.10.48(2). Note that in both cases $\text{Proj}(R)$ is quasi-compact, so that ϵ is an isomorphism. Thus it suffices to show that the restriction map $\mathcal{O}_Y(Y) \rightarrow \mathcal{O}_Y(U)$ is an isomorphism.

Case $R = A[x_0, \dots, x_n]$, $n \geq 1$. We have $U = \bigcup_{i=0}^n D(x_i)$, $D(x_i) = \text{Spec}(R_{x_i})$, $D(x_i) \cap D(x_j) = \text{Spec}(R_{x_i x_j})$. The relevant rings can be compatibly regarded as subrings $R_{x_0 \dots x_n}$ and $\mathcal{O}_Y(U) = \bigcap_{i=0}^n R_{x_i} = R$.

Case R Noetherian normal ring and $\text{ht}(R_+) \geq 2$. A Noetherian normal ring is finite product of Noetherian normal domains. The Proposition then follows immediately from the following Lemma. \square

Lemma 1.10.52. Let R be a Noetherian normal domain. Then $R = \bigcap_{\text{ht}(\mathfrak{p})=1} R_{\mathfrak{p}}$.

This is a consequence of Krull's principal ideal theorem. See [M2, Theorem 11.5].

Example 1.10.53. • $R = A[x]$, $\Gamma_*(\mathcal{O}_X) = A[x, x^{-1}]$. In this case $R \hookrightarrow \Gamma_*(\mathcal{O}_X)$ is not an isomorphism unless $A = 0$.

- $R = k[u^4, u^3v, uv^3, v^4]$, $\text{Proj}(R) \cong \mathbb{P}_k^1$. We have remarked that R is not integrally closed. The map $R \hookrightarrow \Gamma_*(\mathcal{O}_X)$ identifies $\Gamma_*(\mathcal{O}_X)$ with the integral closure of R (exercise).

The morphism $f: \text{Spec}(R) \setminus V(R_+) \rightarrow \text{Proj}(R)$ gives an interpretation of $\text{Proj}(R)$ as a quotient. We now give some indications towards this direction.

The affine line $\mathbb{A}^1 = \text{Spec}(\mathbb{Z}[x])$ is equipped with a multiplication $m: \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ and a unit morphism $e: \text{Spec}(\mathbb{Z}) \rightarrow \mathbb{A}^1$, making \mathbb{A}^1 a monoid scheme. The morphisms m and e are given by the following ring homomorphisms, called comultiplication and counit:

$$\begin{aligned} \mathbb{Z}[x] &\rightarrow \mathbb{Z}[y] \otimes \mathbb{Z}[z] & \mathbb{Z}[x] &\rightarrow \mathbb{Z} \\ x &\mapsto y \otimes z & x &\mapsto 1 \end{aligned}$$

Equipped with these homomorphisms, $\mathbb{Z}[x]$ is a bialgebra. The open subscheme $\mathbb{G}_m = \mathbb{A}^1 \setminus V(x) = \text{Spec}(\mathbb{Z}[x, 1/x])$ is a group scheme, called the **multiplicative**

group. It is equipped with the inverse morphism $i: \mathbb{G}_m \rightarrow \mathbb{G}_m$, which is defined by the antipode

$$\begin{aligned} \mathbb{Z}[x, x^{-1}] &\rightarrow \mathbb{Z}[x, x^{-1}] \\ x &\mapsto x^{-1} \end{aligned}$$

This makes $\mathbb{Z}[x, x^{-1}]$ into a Hopf algebra.

An action $\mathbb{A}^1 \curvearrowright X$ is a morphism $a: \mathbb{A}^1 \times X \rightarrow X$ compatible with m and e . If $X = \text{Spec}(R)$ is affine, then a is defined by a ring homomorphism

$$\begin{aligned} R &\rightarrow \mathbb{Z}[x] \otimes R = R[x] \\ r &\mapsto \sum_{d \geq 0} r_d x^d \end{aligned}$$

One checks that an action of \mathbb{A}^1 on $\text{Spec}(R)$ is equivalent to a grading on R . Similarly, an action of \mathbb{G}_m on $\text{Spec}(R)$ is equivalent to a \mathbb{Z} -grading on R . One can interpret $V(R_+)$ as the fixed point locus by the action of \mathbb{A}^1 on $\text{Spec}(R)$, and $\text{Proj}(R)$ as the quotient of $\text{Spec}(R) \setminus V(R_+)$ by the action of \mathbb{G}_m .

Proposition 1.10.54. *Let R be a graded ring such that $X = \text{Proj}(R)$ is quasi-compact and $\varphi: R \rightarrow \Gamma_*(\mathcal{O}_X)$ is an isomorphism. Then any closed subscheme of X is defined by a homogeneous ideal of R .*

Proof. Let $Z \subseteq X$ be a closed subscheme defined by a quasi-coherent ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$. Then $\Gamma_*(\mathcal{I}_Z) \hookrightarrow \Gamma_*(\mathcal{O}_X) \simeq R$ can be identified with a homogeneous ideal \mathfrak{a} of R . Thus $\tilde{\mathfrak{a}} \simeq \Gamma_*(\mathcal{I}_Z)^\sim \xrightarrow{\sim} \mathcal{I}_Z$. Since the ideal sheaf of the closed subscheme $\text{Proj}(R/\mathfrak{a}) \subseteq X$ is $\tilde{\mathfrak{a}}$, we have $Z = \text{Proj}(R/\mathfrak{a})$ as subscheme of X . \square

Corollary 1.10.55. *A morphism of schemes $f: X \rightarrow \text{Spec}(A)$ is projective if and only if there exists a graded ring R finitely generated over $R_0 = A$ such that $X = \text{Proj}(R)$ and f is the canonical morphism.*

Proof. \Leftarrow . This is Corollary 1.10.29.

\Rightarrow . Let $X \hookrightarrow \mathbb{P}_A^n \rightarrow \text{Spec}(A)$ be a factorization. Since X is a closed subscheme of \mathbb{P}_A^n , it is defined by a homogeneous ideal $I \subseteq R = A[x_0, \dots, x_n]$. In other words, $X = \text{Proj}(R/I)$. \square

Functor represented by $\text{Proj}(R)$

We are mainly interested in the functor represented by the open subscheme U_1 of $\text{Proj}(R)$. Let $\varphi: R \rightarrow \Gamma_*(U_1, \mathcal{O}(1)) = \bigoplus_{n \geq 0} \Gamma(U_1, \mathcal{O}(n))$ be the canonical homomorphism of graded rings.

Definition 1.10.56. Let (X, \mathcal{O}_X) be a ringed space. Let \mathcal{F} be a \mathcal{O}_X -module and $\Sigma \subseteq \Gamma(X, \mathcal{F})$ a subset. We say that \mathcal{F} is **generated** by Σ if

$$\bigoplus_{s \in \Sigma} \mathcal{O}_X \xrightarrow{(s)} \mathcal{F}$$

is an epimorphism. We say that \mathcal{F} is **globally generated** if \mathcal{F} is generated by $\Gamma(X, \mathcal{F})$.

Note that if X is a scheme (or a locally ringed space), an invertible sheaf \mathcal{L} is generated by Σ if and only if $\bigcup_{s \in \Sigma} X_s = X$. In particular, $\mathcal{O}_{\text{Proj}(R)}(dn)|_{U_d}$ is generated by $\varphi(R_{dn})$ for $d, n \geq 1$.

Example 1.10.57. On \mathbb{P}_A^n , $\Gamma(\mathbb{P}_A^n, \mathcal{O}(d)) = A[x_0, \dots, x_n]_d$ for $d \geq 0$ and $\Gamma(\mathbb{P}_A^n, \mathcal{O}(-d)) = 0$ for $d \geq 1$. In particular, $\mathcal{O}(-d)$ is not globally generated.

Proposition 1.10.58. *Let Y be a scheme and $X = \text{Proj}(R)$. Then there is a bijection*

$$\begin{aligned} \text{Hom}_{\text{Sch}}(Y, U_1) &\longrightarrow \left\{ (\mathcal{L}, \gamma) \mid \begin{array}{l} \mathcal{L} \text{ invertible sheaf on } Y, \\ \gamma: R \rightarrow \Gamma_*(Y, \mathcal{L}) \text{ homomorphism of graded rings} \end{array} \right\} / \cong \\ &\quad \text{such that } \mathcal{L} \text{ is generated by } \gamma(R_1) \\ (f: Y \rightarrow U_1) &\mapsto [(f^*(\mathcal{O}(1)|_{U_1}), R \xrightarrow{\varphi} \Gamma_*(U_1, \mathcal{O}(1)) \rightarrow \Gamma_*(Y, f^*(\mathcal{O}(1))))], \end{aligned}$$

where $(\mathcal{L}, \gamma) \cong (\mathcal{L}', \gamma')$ if there exists $c: \mathcal{L} \cong \mathcal{L}'$ rendering

$$\begin{array}{ccc} R & \xrightarrow{\gamma} & \Gamma_*(X, \mathcal{L}) \\ & \searrow \gamma' & \downarrow c \\ & & \Gamma_*(X, \mathcal{L}') \end{array}$$

commutative.

Proof. We construct the inverse $[(\mathcal{L}, \gamma)] \mapsto f$ as follows. For $a \in R_d$, $d > 0$ satisfying $D_+(a) \subseteq U_1$, we have a ring homomorphism

$$R_{(a)} \xrightarrow{\gamma} \Gamma_*(Y, \mathcal{L})_{\gamma(a)} \rightarrow \Gamma(Y_{\gamma(a)}, \mathcal{O}_Y).$$

This gives a morphism $Y_{\gamma(a)} \rightarrow D_+(a)$. Since \mathcal{L} is generated by $\gamma(R_1)$, we have $\bigcup_{a \in R_1} Y_{\gamma(a)} = Y$. Thus these morphisms glue to a morphism $f: Y \rightarrow U_1$. \square

Corollary 1.10.59. *For $X = \mathbb{P}_{\mathbb{Z}}^n = \text{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, we have a bijection*

$$\begin{aligned} \text{Hom}_{\text{Sch}}(Y, \mathbb{P}_{\mathbb{Z}}^n) &\longrightarrow \left\{ (\mathcal{L}, s_0, \dots, s_n) \mid \begin{array}{l} \mathcal{L} \text{ invertible sheaf on } Y, s_i \in \Gamma(Y, \mathcal{L}), \\ \mathcal{L} \text{ is generated by } s_0, \dots, s_n \end{array} \right\} / \cong \\ f &\mapsto (f^*(\mathcal{O}(1)), f^*x_0, \dots, f^*x_n) \end{aligned}$$

The functor represented by U_d can be described with the help of the isomorphism $\text{Proj}(R) \simeq \text{Proj}(R^{(d)})$. Indeed, this isomorphism restricts to $U_{d,R} \simeq U_{1,R^{(d)}}$.

Remark 1.10.60. Given a scheme Y and (\mathcal{L}, γ) , where \mathcal{L} is a line bundle on Y and $\gamma: R \rightarrow \Gamma_*(X, \mathcal{L})$ is a homomorphism of graded rings (without assumptions on generation by global sections), the construction in the proof above produces a morphism of schemes $f: Y_{\gamma} \rightarrow \text{Proj}(R)$, where $Y_{\gamma} = \bigcup_{a \in R_{+, \text{homog}}} Y_{\gamma(a)}$.

1.11 Ample invertible sheaves

Given a graded ring R , the opens $D_+(f) \cap U_1$ form a basis for the topology on $U_1 \subseteq \text{Proj}(R)$. Each $f \in R_d$, $d > 0$ gives rise to an element $\varphi(f) \in \Gamma(U_1, \mathcal{O}(d))$ and $D_+(f) \cap U_1 = (U_1)_{\varphi(f)}$. Thus the open subsets $(U_1)_s$, $s \in \bigcup_{d \geq 1} \Gamma(U_1, \mathcal{O}(d))$ form a basis for the topology on U_1 . We generalize this property to arbitrary invertible sheaves on schemes as follows.

Definition 1.11.1. Let X be a scheme and \mathcal{L} an invertible sheaf on X . We say that \mathcal{L} is **ample** if

- X is quasi-compact and
- $\{X_s \mid s \in \Gamma(X, \mathcal{L}^{\otimes d}), d \geq 1\}$ forms a basis for the topology on X .

Remark 1.11.2. Let $U = \text{Spec}(A) \subseteq X$ be open affine such that \mathcal{L} is trivial on U . Then for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, $X_s \cap U = \text{Spec}(A_s)$ is affine. In particular, if $X_s \subseteq U$, then X_s is affine. (Exercise: Show that assumption that \mathcal{L} is trivial can be removed.)

Lemma 1.11.3. Let X be an affine scheme. Then any invertible sheaf \mathcal{L} on X is ample.

Proof. Let $X = \text{Spec}(A)$. Then $\mathcal{L} \simeq \widetilde{M}$ for some A -module M . The opens X_{am} , $a \in A$, $m \in M$ form a basis for the topology on X . Indeed, $X_{am} \subseteq D(a)$ and $\bigcup_{m \in M} X_m = X$. \square

Lemma 1.11.4. Given $d \geq 1$, \mathcal{L} is ample $\iff \mathcal{L}^{\otimes d}$ is ample.

Proof. We have $X_s = X_{s^{\otimes d}}$. \square

Lemma 1.11.5. Let $i: Y \rightarrow X$ be a quasi-compact immersion. For any ample invertible sheaf \mathcal{L} on X , $i^*\mathcal{L}$ is ample on Y .

Proof. We have $Y_{i^*s} = Y \cap X_s$. \square

Theorem 1.11.6. Let X be a quasi-compact scheme and let \mathcal{L} be an invertible sheaf on X . Let $S = \Gamma_*(X, \mathcal{L})$. Then the following conditions are equivalent:

- (a) \mathcal{L} is ample.
- (b) $\{X_s \text{ affine} \mid s \in S_{+, \text{homog}}\}$ is a basis for X .
- (c) $\{X_s \text{ affine} \mid s \in S_{+, \text{homog}}\}$ covers X .
- (d) The morphism $X \hookrightarrow \text{Proj}(S)$ defined by $(\mathcal{L}, \text{id}: S \rightarrow S)$ is an open immersion.
- (e) There exists a graded ring R , an immersion $i: X \hookrightarrow U_1 \subseteq \text{Proj}(R)$ and $d \geq 1$ such that $\mathcal{L}^{\otimes d} \simeq i^*\mathcal{O}(1)$.
- (f) $\forall \mathcal{F}$ quasi-coherent sheaf on X , $\bigcup_{n \geq 1} \text{Im}(\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathcal{L}^{\otimes -n}) \rightarrow \mathcal{F} = \mathcal{F}$.
- (g) $\forall \mathcal{F}$ quasi-coherent ideal sheaf on X , the condition in (f) holds.

Proof. (a) \implies (b). This follows from Remark 1.11.2.

(b) \implies (c). Trivial.

(c) \implies (d). We first prove that X is quasi-separated. By assumption $X = \bigcup_{i=1}^n X_{s_i}$ with X_{s_i} affine. There exists an affine open covering $X = \bigcup_{k=1}^m U_k$ such that \mathcal{L} is trivial on each U_k . Then $X_{s_i} \cap X_{s_j} = X_{s_i \otimes s_j} = \bigcup_{k=1}^m (X_{s_i \otimes s_j} \cap U_k)$ is quasi-compact, since $X_{s_i \otimes s_j} \cap U_k$ is affine. (In fact $X_{s_i} \cap X_{s_j}$ is affine by the exercise mentioned in Remark 1.11.2.)

We can now apply Lemma 1.10.47 to see $S_{(s)} \xrightarrow{\sim} \Gamma(X_s, \mathcal{O}_X)$. For X_s affine, this implies $X_s \xrightarrow{\sim} D_+(s)$. Thus $X \hookrightarrow \text{Proj}(S)$ is an open immersion.

(d) \implies (e). Since X is quasi-compact, the image of the open immersion $j: X \hookrightarrow \text{Proj}(S)$ in (d) is contained in U_d for some d . We take $R = S^{(d)}$ and let $i: X \hookrightarrow U_{d,S} \simeq U_{1,R}$. Then $i^*(\mathcal{O}_{U_{1,R}}(1)) = j^*(\mathcal{O}_{U_{d,S}}(d)) = \mathcal{L}^{\otimes d}$.

(e) \implies (a). By the discussion at the beginning of the section, $\mathcal{O}(1)|_{U_1}$ is ample. By Lemma 1.11.4, $\mathcal{L}^{\otimes d} \simeq i^*(\mathcal{O}(1)|_{U_1})$ is ample, which implies that \mathcal{L} ample by Lemma 1.11.3.

(a) \implies (f). We have shown that (a) implies that X is quasi-separated. Thus, by Lemma 1.10.47, $\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \xrightarrow{\sim} \Gamma(X_s, \mathcal{F})$ for $s \in \Gamma(X, \mathcal{L}^{\otimes d})$, $d \geq 1$. Elements in $\Gamma(X_s, \mathcal{F})$ can be written as $a = b|_{X_s} \otimes s^{-n}$, $b \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes nd})$, $s^{-n} \in \Gamma(X_s, \mathcal{L}^{\otimes -nd})$. Thus a is in the image of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes nd}) \otimes_{\mathbb{Z}} \mathcal{L}^{\otimes -nd} \rightarrow \mathcal{F}$. Since X_s forms an open basis, \mathcal{F} equals the union as shown in (f).

(f) \implies (g). Trivial.

(g) \implies (a). Let $x \in U \subseteq X$ be an open neighborhood of x . It suffices to show that there exists $s \in S_+$ homogeneous such that $x \in X_s \subseteq U$. Let $Z = X \setminus U$ and equip it with the induced reduced closed subscheme structure. Let \mathcal{I}_Z be the corresponding ideal sheaf. Then $\mathcal{I}_Z|_U = \mathcal{O}_X|_U$. The assumption in (g) implies

$$\bigcup_{n \geq 1} \text{Im}(\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \otimes \mathcal{L}^{\otimes -n}) \rightarrow \mathcal{I}_Z = \mathcal{I}_Z.$$

In particular, there exists $n \geq 1$, $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n})$ such that $s_x \notin \mathfrak{m}_x(\mathcal{I}_Z \otimes \mathcal{L}^{\otimes n})_x = (\mathcal{L}^{\otimes n})_x$, where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. Let $i: Z \rightarrow X$ be the closed immersion. The exact sequence $0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$ induces an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(Z, i^*\mathcal{L}^{\otimes n}|_Z).$$

Regarding s as an element of $\Gamma(X, \mathcal{L}^{\otimes n})$, we have $x \in X_s$. The image of s in $\Gamma(Z, i^*\mathcal{L}^{\otimes n}|_Z)$ is zero, which implies that $X_s \cap Z = \emptyset$ and $X_s \subseteq U$. \square

Corollary 1.11.7. *Any scheme admitting an ample invertible sheaf is separated.*

Indeed, $\text{Proj}(R)$ is separated.

Date: 11.3

(Additional equivalent conditions have been inserted into Theorem 1.11.6.)

Definition 1.11.8. We say that a scheme X is **quasi-affine** if X is a quasi-compact open subset of an affine scheme.

Corollary 1.11.9. A scheme X is quasi-affine $\iff \mathcal{O}_X$ is ample.

Proof. \implies . If $j: X \hookrightarrow \text{Spec}(A)$ is a quasi-compact open immersion, then $\mathcal{O}_X = j^*\mathcal{O}_{\text{Spec}(A)}$ is ample.

\impliedby . We apply the theorem above with $S = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}_X) = A[x]$, where $A = \Gamma(X, \mathcal{O}_X)$. Then $j: X \rightarrow \text{Proj}(S) = \text{Spec}(A)$ is an open immersion. By assumption, X is quasi-compact. It follows that j is quasi-compact. \square

Definition 1.11.10. Let (X, \mathcal{O}_X) be a ringed space, \mathcal{F} an \mathcal{O}_X -module. We say that \mathcal{F} is **of finite type** if there exists an open cover $\{U_i\}$ of X , integers $n_i \geq 0$ and epimorphisms $\mathcal{O}_{U_i}^{n_i} \twoheadrightarrow \mathcal{F}|_{U_i}$.

Remark 1.11.11. Let \mathcal{F} be an \mathcal{O}_X -module of finite type.

- Every quotient of \mathcal{F} is of finite type.
- If X is a locally Noetherian scheme and \mathcal{F} is quasi-coherent, then every quasi-coherent subsheaf of \mathcal{F} is also of finite type.

Corollary 1.11.12. Let X be a scheme, \mathcal{L} an ample invertible sheaf on X , and \mathcal{F} a quasi-coherent sheaf of finite type on X . Then there exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated.

Remark 1.11.13. The tensor product of two globally generated \mathcal{O}_X -modules is globally generated.

Proof. By Theorem 1.11.6 (a) \Rightarrow (f) and the lemma below, for any quasi-coherent sheaf \mathcal{G} of finite type on X , there exists $e = e(\mathcal{G}, \mathcal{L}) \geq 1$ such that $\mathcal{G} \otimes \mathcal{L}^{\otimes e}$ is globally generated. Let $d = e(\mathcal{O}_X, \mathcal{L})$, so that $\mathcal{L}^{\otimes d}$ is globally generated. For $0 \leq i < d$, let $e_i = e(\mathcal{F} \otimes \mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes d})$, so that $\mathcal{F} \otimes \mathcal{L}^{\otimes de_i+i}$ is globally generated. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes de+i}$ is globally generated for $e \geq e_i$. Take $n_0 = \max_{0 \leq i < d} \{de_i + i\}$. Then for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. \square

Lemma 1.11.14. Let (X, \mathcal{O}_X) be a ringed space with X quasi-compact. Let \mathcal{F} be an \mathcal{O}_X -module of finite type.

- (1) Assume $\mathcal{F} = \text{colim}_{i \in I} \mathcal{F}_i$ with I filtered. Then there exists i such that the canonical morphism $\mathcal{F}_i \rightarrow \mathcal{F}$ is an epimorphism.
- (2) If \mathcal{F} is globally generated, then \mathcal{F} is generated by finitely many global sections.

Proof. (1) For any $x \in X$, there exist an open neighborhood U and an epimorphism $\mathcal{O}_U^n \twoheadrightarrow \mathcal{F}|_U$. Shrinking U if necessary, we can find i such that $\mathcal{O}_U^n \rightarrow \mathcal{F}_i|_U \rightarrow \mathcal{F}|_U$. Then $\mathcal{F}_i|_U \twoheadrightarrow \mathcal{F}|_U$. Since X is quasi-compact, we can find an $i \in I$ such that $\mathcal{F}_i|_U \twoheadrightarrow \mathcal{F}|_U$ holds for U running through an open cover of X . This shows $\mathcal{F}_i \twoheadrightarrow \mathcal{F}$.

(2) We have

$$\mathcal{F} = \bigcup_{\Sigma \subseteq \Gamma(X, \mathcal{F}) \text{ finite}} \text{Im}(\mathcal{O}_X^\Sigma \rightarrow \mathcal{F})$$

By (1), there exists $\Sigma \subseteq \Gamma(X, \mathcal{F})$ finite such that \mathcal{O}_X^Σ surjects onto \mathcal{F} . \square

Corollary 1.11.15. *Let X be a scheme, \mathcal{L} an ample invertible sheaf on X . Then for every quasi-coherent sheaf \mathcal{F} of finite type on X , there exists $n \geq 1, m \geq 0$ such that \mathcal{F} is a quotient of $(\mathcal{L}^{\otimes -n})^m$.*

Proof. There exists n such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated. By the lemma above, there exist m and an epimorphism $\mathcal{O}_X^m \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$. Thus $(\mathcal{L}^{\otimes -n})^m \twoheadrightarrow \mathcal{F}$. \square

Remark 1.11.16. If X is Noetherian, then the condition in the corollary is equivalent to the ampleness of \mathcal{L} . Indeed, in this case, every ideal sheaf is finitely generated. In fact, the equivalence holds as long as X is quasi-compact and quasi-separated, because in this case every quasi-coherent \mathcal{O}_X -module is the union of its submodules of finite type [SP, 01PG].

Relative ampleness

Definition 1.11.17. Let $f: X \rightarrow S$ be a morphism of schemes and let \mathcal{L} be an invertible sheaf on X .

- We say that \mathcal{L} is **f -ample** if f is quasi-compact and for every affine open $V \subseteq S$, $\mathcal{L}|_{f^{-1}(V)}$ is ample.
- We say that \mathcal{L} is **f -very ample** if there exists a decomposition

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}_S^n \\ & \searrow f & \downarrow \\ & & S \end{array}$$

where i is an immersion such that $\mathcal{L} \simeq i^* \mathcal{O}_{\mathbb{P}_S^n}(1)$. Here $\mathcal{O}_{\mathbb{P}_S^n}(1) := p^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)$, where $p: \mathbb{P}_S^n = \mathbb{P}_{\mathbb{Z}}^n \times S \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ is the projection.

Lemma 1.11.18. *Let $f: X \rightarrow Y$ be a quasi-compact morphism of schemes and let \mathcal{L} be an invertible sheaf on X .*

- (1) *If \mathcal{L} is ample, then \mathcal{L} is f -ample.*
- (2) *If \mathcal{L} is f -very ample, then \mathcal{L} is f -ample.*

Theorem 1.11.19. *Let $f: X \rightarrow S$ be a morphism locally of finite type and let \mathcal{L} be an ample invertible sheaf on X . Then there exists $d \geq 1$ such that $\mathcal{L}^{\otimes d}$ is f -very ample.*

Proof. Let $R = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. Since $\{X_s \text{ affine} \mid s \in R_d, d \geq 1\}$ forms a basis for the topology of X and X is quasi-compact, there exists a finite cover $X = \bigcup_{i=1}^n X_{s_i}$ with $X_{s_i} = \text{Spec}(B_i)$ such that $f(X_{s_i}) \subseteq V_i = \text{Spec}(A_i)$, where V_i is an affine open of S . Since f is locally of finite type, B_i is a finitely generated A_i -algebra, say $B_i = A_i[b_{i,1}, \dots, b_{i,n_i}]$. By Lemma 1.10.47, $R_{(s_i)} \simeq \Gamma(X_{s_i}, \mathcal{O}_X)$. Thus $b_{ij} = f_{ij}/s_i^{e_{ij}}$, with f_{ij} homogeneous of degree $e_{ij} \deg(s_i)$.

Take d such that $\deg(s_i) \mid d$ and $d \geq \deg(f_{ij})$ for all i, j . Let

$$\Sigma = \{s_i^{d/\deg(s_i)}, f_{ij}s_i^{d/\deg(s_i)-e_{ij}}\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n_i}} \subseteq R_d.$$

Then Σ generates $\mathcal{L}^{\otimes d}$ since $\bigcup_{s \in \Sigma} X_s \supseteq \bigcup_i X_{s_i} = X$. Let $T = \mathbb{Z}[x_i, x_{ij}]_{i,j}$ and consider the ring homomorphism

$$\begin{aligned} T &\rightarrow R \\ x_i &\mapsto s_i^{d/\deg(s_i)} \\ x_{ij} &\mapsto f_{ij}s_i^{d/\deg(s_i)-e_{ij}}. \end{aligned}$$

This gives a morphism of schemes $X \rightarrow \text{Proj}(T) = \mathbb{P}_{\mathbb{Z}}^N$, $N = \#\Sigma - 1$. This induces a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{r} & \mathbb{P}_S^N & \longrightarrow & \mathbb{P}_{\mathbb{Z}}^N \\ & \searrow f & \downarrow & & \downarrow \\ & & S & \longrightarrow & \text{Spec}(\mathbb{Z}). \end{array}$$

We have $r^{-1}(D_+(x_i) \times S) = X_{s_i}$ and the restriction of r is the composition

$$X_{s_i} \xrightarrow{v} D_+(x_i) \times V_i \xrightarrow{u} D_+(x_i) \times S,$$

where u is an open immersion and v is a morphism of affine schemes. The ring homomorphism corresponding to v

$$\begin{aligned} T_{(x_i)} \otimes_{\mathbb{Z}} A_i &\rightarrow B_i \\ x_{ij}/x_i &\mapsto b_{ij} \end{aligned}$$

is surjective, which implies that v is a closed immersion. Therefore, r is an immersion. By construction, $r^*\mathcal{O}(1) \simeq \mathcal{L}^{\otimes d}$. \square

Remark 1.11.20. The conclusion of the theorem can be strengthened to the existence of an integer d_0 such that for all $d \geq d_0$, $\mathcal{L}^{\otimes d}$ is f -very ample (exercise).

Corollary 1.11.21. *Let S be an affine scheme, $f: X \rightarrow S$ a morphism of finite type, \mathcal{L} an invertible sheaf on X . Then the following conditions are equivalent:*

- (a) \mathcal{L} is ample.
- (b) \mathcal{L} is f -ample.
- (c) there exists $d \geq 1$ such that $\mathcal{L}^{\otimes d}$ is f -very ample.

Definition 1.11.22. We say that a morphism of schemes $f: X \rightarrow S$ is **quasi-projective** if there exists a factorization

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}_S^n \\ & \searrow & \downarrow \\ & & S \end{array}$$

where i is a quasi-compact immersion.

Warning 1.11.23. Our definitions of f -very ampleness and quasi-projectiveness differ from the EGA. We will see later that being f -very ample in our sense is not local on S .

Example 1.11.24. $X = \mathbb{P}_A^n$, $A \neq 0$, $n > 1$. For $d > 0$, $\mathcal{O}(d)$ is very ample over A . Indeed, if $i_d: \mathbb{P}_A^n \hookrightarrow \mathbb{P}_Z^N$ denotes the d -uple embedding, then $i_d^*\mathcal{O}(1) \simeq \mathcal{O}(d)$. For $d < 0$, $\mathcal{O}(d)$ has no nonzero global sections. It follows that for $d \leq 0$, $\mathcal{O}_X(d)$ is not ample, because $\mathcal{O}_X(d)^{\otimes n} \otimes \mathcal{O}(-1) = \mathcal{O}(dn - 1)$ is not globally generated for any $n \geq 0$. In summary,

$$\mathcal{O}_X(d) \text{ is } \begin{cases} \text{very ample over } A & d > 0 \\ \text{not ample} & d \leq 0. \end{cases}$$

Example 1.11.25. Let

$$\begin{array}{ccc} X = \mathbb{P}_A^m \times_{\text{Spec}(A)} \mathbb{P}_A^n & \xrightarrow{p_1} & \mathbb{P}_A^m \\ & \downarrow p_2 & \\ & \mathbb{P}_A^n & \end{array}$$

with $A \neq 0$, $m, n \geq 1$. Let $\mathcal{L}_{a,b} = \mathcal{O}(a) \boxtimes_A \mathcal{O}(b) = p_1^*\mathcal{O}(a) \otimes_{\mathcal{O}_X} p_2^*\mathcal{O}(b)$. We have

$$\mathcal{L}_{a,b} \text{ is } \begin{cases} \text{very ample over } A & a, b > 0 \\ \text{not ample} & a \leq 0 \text{ or } b \leq 0. \end{cases}$$

For $a, b > 0$, let $i_a: \mathbb{P}_A^m \rightarrow \mathbb{P}_A^M$ and $i_b: \mathbb{P}_A^n \rightarrow \mathbb{P}_A^N$ be the a -uple and b -uple embeddings, respectively. Let $i: \mathbb{P}_A^M \times \mathbb{P}_A^N \rightarrow \mathbb{P}^r$ be the Segre embedding. Then

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & \curvearrowright & \searrow & \\ \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{i_a \times i_b} & \mathbb{P}^M \times \mathbb{P}^N & \xrightarrow{i} & \mathbb{P}^r \end{array}$$

and $j^*\mathcal{O}(1) \simeq (i_a \times i_b)^*(\mathcal{O}(1) \boxtimes_A \mathcal{O}(1)) \simeq \mathcal{O}(a) \boxtimes_A \mathcal{O}(b)$. In fact, on $\text{Proj}(R \otimes_A S) \simeq \text{Proj}(R) \times_A \text{Proj}(S)$, we have $M \otimes_A N \simeq \widetilde{M} \boxtimes_A \widetilde{N}$, where $(M \otimes N)_d = M_d \otimes_A N_d$.

For $a \leq 0$, we choose a section s of $\mathbb{P}_A^n \rightarrow \text{Spec}(A)$ satisfying $s^*\mathcal{O}(1) = \mathcal{O}$ and consider the pullback

$$\begin{array}{ccc} \mathbb{P}^m & \xleftarrow{t} & \mathbb{P}^m \times \mathbb{P}^n \\ \downarrow \lrcorner & & \downarrow p_2 \\ \text{Spec}(A) & \xleftarrow{s} & \mathbb{P}^n \end{array}$$

Then $t^*(\mathcal{O}(a) \boxtimes \mathcal{O}(b)) \simeq \mathcal{O}(a)$, which is not ample on \mathbb{P}^m . Thus $\mathcal{O}(a) \boxtimes \mathcal{O}(b)$ is not ample. The case $b \leq 0$ is similar.

Example 1.11.26. Let k be an algebraically closed field, C an integral normal k -scheme of dimension 1 and proper over k . Assume $C \not\simeq \mathbb{P}_k^1$. We will show later that the properness of C implies $\dim_k(\Gamma(C, \mathcal{O}_C)) < \infty$. Since $\Gamma(C, \mathcal{O}_C)$ is an integral finite-dimensional k -algebra, it is k itself. Let $P \in C$ be a closed point, corresponding

to the ideal sheaf \mathcal{I}_P . Then \mathcal{I}_P is an invertible sheaf. Let $\mathcal{L}(P) := \mathcal{I}_P^\vee$. We will show later that $\mathcal{L}(P)$ is ample. Let us show that $\mathcal{L}(P)$ is not very ample over k .

Let $i: P \rightarrow C$ be the inclusion. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_C \longrightarrow i_* k \longrightarrow 0$$

Tensoring the above sequence with $\mathcal{L}(P)$, we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{L}(P) \longrightarrow i_* k \longrightarrow 0$$

Taking global sections, we see that $\dim_k(\Gamma(X, \mathcal{L}(P))) \leq 2$. Suppose there exists an immersion $j: C \rightarrow \mathbb{P}(V) := \text{Proj}(\text{Sym}_k(V))$, where V is a finite-dimensional k -vector space, such that $i^* \mathcal{O}(1) \simeq \mathcal{L}(P)$. Then i corresponds to a k -linear map $\phi: V \rightarrow \Gamma(X, \mathcal{L}(P))$ whose image generates $\mathcal{L}(P)$. Let $W = \text{im}(\phi)$. Then $\dim_k(W) \leq 2$ and i factorizes through $i: C \rightarrow \mathbb{P}(W)$. Then i is a closed immersion. It follows that $C \simeq \mathbb{P}(W) \simeq \mathbb{P}^1$. Contradiction.

1.12 Relative homogeneous spectrum

Let S be a scheme and let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra. A **graded \mathcal{O}_S -algebra** \mathcal{R} is an \mathcal{O}_S -algebra \mathcal{R} equipped with a grading $\mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d$ such that $\mathcal{R}_d \mathcal{R}_e \subseteq \mathcal{R}_{d+e}$.

We define a scheme $\underline{\text{Proj}}(\mathcal{R})$ and a morphism $\pi: \underline{\text{Proj}}(\mathcal{R}) \rightarrow S$ by gluing. If $V' \subseteq V \subseteq S$ are affine open subsets, we have Cartesian squares

$$\begin{array}{ccccc} \underline{\text{Proj}}(\mathcal{R}(V')) & \longrightarrow & \underline{\text{Proj}}(\mathcal{R}(V)) & \longrightarrow & \underline{\text{Proj}}(\mathcal{R}) \\ \downarrow & & \downarrow & & \downarrow \pi \\ V' & \xhookrightarrow{\quad} & V & \xhookrightarrow{\quad} & S \end{array}$$

Remark 1.12.1. π is separated.

Example 1.12.2. $S = \text{Spec}(A)$, $\mathcal{R} = \tilde{R}$, where R is a graded A -algebra. Then $\underline{\text{Proj}}(\tilde{R}) = \underline{\text{Proj}}(R)$.

Example 1.12.3. Let \mathcal{A} be a quasi-coherent \mathcal{O}_S -algebra. Then $\underline{\text{Proj}}(\mathcal{A}[x]) = \underline{\text{Spec}}(\mathcal{A})$.

Example 1.12.4. $\underline{\text{Proj}}(\mathcal{O}_S[x_0, \dots, x_n]) \cong \mathbb{P}_S^n$.

Example 1.12.5. Let \mathcal{E} be a quasi-coherent \mathcal{O}_S -module. Then $\mathbb{P}(\mathcal{E}) := \underline{\text{Proj}}(\text{Sym}(\mathcal{E}))$ is called the **projective bundle** over S associated to \mathcal{E} . (In the literature, \mathcal{E} is sometimes assumed to be locally free.)

Date: 11.5

The following treatment of \mathcal{O} -modules is based on a method of Berthelot [SGA6, VI 2].

Quasi-coherent sheaves on $\underline{\mathrm{Spec}}(\mathcal{A})$

Let S be a scheme, \mathcal{A} a quasi-coherent \mathcal{O}_S -algebra, and $\pi: X = \underline{\mathrm{Spec}}(\mathcal{A}) \rightarrow S$. We have $\pi_*(\mathcal{O}_X) = \mathcal{A}$, which gives by adjunction a morphism of sheaves of rings $(\pi_\sharp)^\sharp: \pi^{-1}\mathcal{A} \rightarrow \mathcal{O}_X$ on X . The morphism $(\pi_\sharp)^\sharp$ is flat. To see this, let $V = \mathrm{Spec}(B)$ be an affine open of S and let $\mathcal{A}|_V = \tilde{A}$, where A is a B -algebra. Then, the stalk of $(\pi_\sharp)^\sharp$ at $\mathfrak{p} \in \mathrm{Spec}(A)$ is the localization $(\pi^{-1}\mathcal{A})_{\mathfrak{p}} \simeq A_{\mathfrak{q}} \rightarrow A_{\mathfrak{p}} \simeq \mathcal{O}_{X,\mathfrak{p}}$. Here $\mathfrak{q} = \mathfrak{p} \cap B$.

The morphism π , when regarded as a morphism of ringed spaces, can be decomposed into $(X, \mathcal{O}_X) \xrightarrow{\pi_\sharp} (S, \mathcal{A}) \rightarrow (S, \mathcal{O}_S)$. The morphism of ringed spaces π_\sharp induces a pair of functors $\mathrm{Shv}(X, \mathcal{O}_X) \xleftrightarrow[\pi_\sharp^*]{\pi_*} \mathrm{Shv}(S, \mathcal{A})$, where $\pi_\sharp^* \mathcal{M} = \pi^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{A}} \mathcal{O}_X$.

Proposition 1.12.6. (1) $\pi_\sharp^* \dashv \pi_*$ and π_\sharp^* is exact.

(2) The functors induce equivalences of categories

$$\mathrm{QCoh}(X, \mathcal{O}_X) \xleftrightarrow[\pi_\sharp^*]{\pi_*} \mathrm{QCoh}(S, \mathcal{A})$$

quasi-inverse to each other. Moreover, for $\mathcal{M} \in \mathrm{QCoh}(S, \mathcal{A})$ and $V \subseteq S$ an affine open, $\pi_\sharp^*(\mathcal{M})|_{\pi^{-1}V} = \widetilde{\mathcal{M}(V)}$.

Proof. (1) This holds for any flat morphism of ringed spaces.

(2) That π_* carries quasi-coherent \mathcal{O}_X -modules to quasi-coherent \mathcal{A} -modules follows from the lemma below. The proof of the other statements is similar, by choosing a presentation locally. \square

Lemma 1.12.7. An \mathcal{A} -module \mathcal{M} is quasi-coherent as \mathcal{A} -module $\iff \mathcal{M}$ is quasi-coherent as \mathcal{O}_S -module.

Proof. \implies . Locally, $\mathcal{M} \simeq \mathrm{Coker}(\mathcal{A}^{\oplus I} \rightarrow \mathcal{A}^{\oplus J})$. Since \mathcal{A} is a quasi-coherent \mathcal{O}_X -module, so is \mathcal{M} .

\impliedby . We may assume $S = \mathrm{Spec}(B)$. Then $\mathcal{M} = \widetilde{M}$, where M is a B -module, and $\mathcal{A} = \tilde{A}$, where A is a B -algebra. The \mathcal{A} -module structure on \mathcal{M} induces an A -module structure on M . Choose a presentation

$$A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0.$$

This induces an exact sequence

$$\mathcal{A}^{\oplus I} \longrightarrow \mathcal{A}^{\oplus J} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Thus \mathcal{M} is quasi-coherent as \mathcal{A} -module. \square

Quasi-coherent sheaves on $\underline{\text{Proj}}(\mathcal{R})$

Let S be a scheme and let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra. Let $\pi: X = \underline{\text{Proj}}(\mathcal{R}) \rightarrow S$ be the canonical morphism.

Definition 1.12.8. A **graded \mathcal{R} -module** is an \mathcal{R} -module \mathcal{M} equipped with a \mathbb{Z} -grading $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$ such that $\mathcal{R}_d \mathcal{M}_e \subseteq M_{d+e}$.

Let $\text{QCohGr}(S, \mathcal{R})$ denote the category of quasi-coherent graded \mathcal{R} -modules. Consider the functor

$$\begin{aligned} \text{QCohGr}(S, \mathcal{R}) &\rightarrow \text{QCoh}(X) \\ \mathcal{M} &\mapsto \widetilde{\mathcal{M}} \end{aligned}$$

where $\widetilde{\mathcal{M}}$ is constructed by gluing: for every open affine subset $V \subseteq S$, $\widetilde{\mathcal{M}}|_{\pi^{-1}(V)} \simeq \widetilde{\mathcal{M}(V)}$. Note that $\mathcal{M}(V) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n(V)$.

Definition 1.12.9. $\mathcal{O}_X(n) = \widetilde{\mathcal{R}(n)}$.

We now proceed to extend the functor $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ to graded modules that are not necessarily quasi-coherent. We have a morphism of \mathbb{Z} -graded \mathcal{O}_S -algebras

$$\mathcal{R} \rightarrow \bigoplus_{n \in \mathbb{Z}} \pi_* \mathcal{O}_X(n)$$

given locally on an affine open $V \subseteq S$ by $\varphi_n: \mathcal{R}_n(V) \rightarrow \Gamma(\pi^{-1}(V), \mathcal{O}(n))$. Recall that $\pi^{-1}(V) \simeq \underline{\text{Proj}}(\mathcal{R}(V))$. By adjunction, we obtain a morphism of \mathbb{Z} -graded sheaf of rings

$$\pi^{-1} \mathcal{R} \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n),$$

which is flat as in the case of $\underline{\text{Spec}}(\mathcal{A})$.

We consider the following categories and functors:

$$\begin{array}{ccccc} & (\bullet)_l & & \pi^*_\sharp & \\ \text{Shv}(X, \mathcal{O}_X) & \xleftarrow{(\)_0} & \text{GrShv}(X, \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)) & \xleftarrow{\pi^*_\sharp} & \text{GrShv}(S, \mathcal{R}) \\ & (\bullet)_r & & \pi^\oplus_* & \end{array}$$

The functor $(\)_0$ are obvious. The functor π^\oplus_* is defined by $\pi^\oplus_*(\bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n) := \bigoplus_{n \in \mathbb{Z}} \pi_* \mathcal{F}_n$. For $\mathcal{M} \in \text{GrShv}(S, \mathcal{R})$,

$$\pi^*_\sharp \mathcal{M} := \pi^{-1} \mathcal{M} \otimes_{\pi^{-1} \mathcal{R}} \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n) \right).$$

For \mathcal{F} an \mathcal{O}_X -module,

$$\begin{aligned} \mathcal{F}(\bullet)_l &:= \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n), \\ \mathcal{F}(\bullet)_r &:= \bigoplus_{n \in \mathbb{Z}} \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{O}_X(-n), \mathcal{F}). \end{aligned}$$

We have adjunctions

$$(\bullet)_l \dashv (\)_0 \dashv (\bullet)_r \quad \pi_{\natural}^* \dashv \pi_*$$

In particular, we have a canonical natural transformation $\nu: \mathcal{F}(\bullet)_l \rightarrow \mathcal{F}(\bullet)_r$. For $d \geq 1$, let $U_d := \bigcup_{V,s} D_+(V, s)$ where V runs through affine opens of S and s runs through elements of $\mathcal{R}_d(V)$. Then ν_{dn} is an isomorphism on U_{dn} for all $n \in \mathbb{Z}$.

By composition, we obtain functors

$$\begin{array}{ccc} & \Gamma_* & \\ \text{Shv}(X, \mathcal{O}_X) & \xleftarrow{\sim} & \text{GrShv}(S, \mathcal{R}) \\ & \Upsilon_* & \end{array}$$

For $\mathcal{M} \in \text{GrShv}(S, \mathcal{R})$,

$$\widetilde{\mathcal{M}} := (\pi_{\natural}^* \mathcal{M})_0.$$

This extends the definition of \sim on $\text{QCohGr}(S, \mathcal{R})$. For $\mathcal{F} \in \text{Shv}(X, \mathcal{O}_X)$,

$$\begin{aligned} \underline{\Gamma}_*(\mathcal{F}) &:= \bigoplus_{n \in \mathbb{Z}} \pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)), \\ \underline{\Upsilon}_*(\mathcal{F}) &:= \bigoplus_{n \in \mathbb{Z}} \pi_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-n), \mathcal{F}). \end{aligned}$$

We denote by $v: \underline{\Gamma}_* \rightarrow \underline{\Upsilon}_*$ the natural transformation induced by $\nu: (\bullet)_l \rightarrow (\bullet)_r$.

Proposition 1.12.10. (1) $\sim \dashv \underline{\Upsilon}_*$ and \sim is exact.

(2) Suppose that π is quasi-compact. For $\mathcal{F} \in \text{QCoh}(X, \mathcal{O}_X)$, we have $\underline{\Gamma}_*(\mathcal{F}) \in \text{QCoh}(S, \mathcal{R})$ and

$$\underline{\Gamma}_*(\mathcal{F})^\sim \xrightarrow{\sim} \underline{\Upsilon}_*(\mathcal{F})^\sim \xrightarrow{\sim} \mathcal{F}$$

Proof. (1) is clear. For (2), the quasi-coherence is clear. The last statement follows from the corresponding result for Proj (Proposition 1.10.12). \square

Corollary 1.12.11. The functor $\underline{\Upsilon}_*: \text{QCoh}(X) \rightarrow \text{GrShv}(S, \mathcal{R})$ is fully faithful.

Proposition 1.12.12. Let \mathcal{E} be a locally free \mathcal{O}_S -module of rank ≥ 2 and let $\pi: \mathbb{P}(\mathcal{E}) \rightarrow S$. Then the morphism $\mathcal{R} \rightarrow \underline{\Gamma}_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})})$ is an isomorphism, where $\mathcal{R} = \text{Sym}(\mathcal{E})$. In other words, the morphism $\text{Sym}(\mathcal{E}) \rightarrow \bigoplus_{n \in \mathbb{Z}} \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n))$ is an isomorphism.

In particular, $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) = 0$ for $n < 0$, $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{O}_S$, and $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq \mathcal{E}$.

Proof. We reduce to the case where S is affine and \mathcal{E} is a free module. In this case, $\mathbb{P}(\mathcal{E})$ is a projective space and Proposition 1.10.48(2) applies. \square

Functor represented by $\text{Proj}(\mathcal{R})$

Let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra and let $U_1 \subseteq \underline{\text{Proj}}(\mathcal{R})$ be the open subscheme as above.

Proposition 1.12.13. *Let $f: Y \rightarrow S$ be a morphism of schemes. Then we have a bijection*

$$\text{Hom}_S(Y, U_1) \xleftrightarrow{1:1} \left\{ \begin{array}{l} (\mathcal{L}, \gamma) \mid \mathcal{L} \text{ invertible sheaf on } Y \\ \gamma: f^*\mathcal{R} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n} \text{ homomorphism of graded } \mathcal{O}_Y\text{-algebras} \\ \gamma_1: f^*\mathcal{R}_1 \twoheadrightarrow \mathcal{L} \end{array} \right\} / \simeq$$

$$g \mapsto (g^*\mathcal{O}(1), f^*\mathcal{R} \xrightarrow{g^*(\varphi|_{U_1})} \bigoplus_{n \in \mathbb{Z}} g^*\mathcal{O}(n))$$

where $(\mathcal{L}, \gamma) \simeq (\mathcal{L}', \gamma')$ if there exists $c: \mathcal{L} \simeq \mathcal{L}'$ rendering

$$\begin{array}{ccc} f^*\mathcal{R} & \xrightarrow{\gamma} & \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n} \\ & \searrow \gamma' & \downarrow c \\ & & \bigoplus_{n \geq 0} \mathcal{L}'^{\otimes n} \end{array}$$

commutative. Here $\varphi: \pi^*\mathcal{R} \rightarrow \bigoplus_{n \geq 0} \mathcal{O}_X(n)$ denotes the canonical morphism.

Corollary 1.12.14. *Let \mathcal{E} be a quasi-coherent \mathcal{O}_S -module. Then we have a bijection*

$$\text{Hom}_S(Y, \mathbb{P}(\mathcal{E})) \xleftrightarrow{1:1} \left\{ \begin{array}{l} (\mathcal{L}, \gamma_1) \mid \mathcal{L} \text{ invertible sheaf on } Y \\ \gamma_1: f^*\mathcal{E} \twoheadrightarrow \mathcal{L} \text{ homomorphism of } \mathcal{O}_Y\text{-modules} \end{array} \right\} / \simeq$$

Example 1.12.15. Let k be a field and let V be a k -vector space. Then we have bijections

$$\begin{aligned} \mathbb{P}(V)(k) &\xleftrightarrow{1:1} \{\text{quotients of } V \text{ of dimension 1}\} \\ &\xleftrightarrow{1:1} \{\text{hyperplanes of } V\} \end{aligned}$$

The functor represented by the projective bundle $\mathbb{P}(\mathcal{E})$ should be compared to the functor represented by the vector bundle $\mathbb{V}(\mathcal{E}) = \underline{\text{Spec}}(\text{Sym}(\mathcal{E}))$. For any morphism $f: Y \rightarrow S$, we have

$$\text{Hom}_S(Y, \mathbb{V}(\mathcal{E})) \simeq \text{Hom}_{\mathcal{O}_S}(\text{Sym}(\mathcal{E}), f_*\mathcal{O}_Y) \simeq \text{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E}, \mathcal{O}_Y).$$

In particular, for $Y = S$, we have

$$\text{Hom}_S(S, \mathbb{V}(\mathcal{E})) \simeq \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y).$$

If \mathcal{E} is locally free, then $\text{Hom}_S(S, \mathbb{V}(\mathcal{E})) \simeq \Gamma(S, \mathcal{E}^\vee)$. In words, sections of the morphism $\pi: \mathbb{V}(\mathcal{E}) \rightarrow S$ correspond to sections of the sheaf \mathcal{E}^\vee .

The S -scheme $\mathbb{P}(\mathcal{E})$ classifies quotients of \mathcal{E} locally free of rank 1. More generally, we have the following.

Theorem 1.12.16. Let \mathcal{E} be a quasi-coherent \mathcal{O}_S -module and let $r \geq 0$ be an integer. Then there exists an S -scheme $\text{Grass}_r(\mathcal{E})$, called the **Grassmannian**, equipped with a functorial bijection

$$\text{Hom}(Y, \text{Grass}_r(\mathcal{E})) \simeq \{\text{quotients of } f^*\mathcal{E} \text{ that are locally free of rank } r\}.$$

Moreover, the morphism

$$\begin{aligned} \text{Grass}_r(\mathcal{E}) &\rightarrow \mathbb{P}(\bigwedge^r \mathcal{E}) \\ (f^*\mathcal{E} \twoheadrightarrow \mathcal{F}) &\mapsto (f^*(\bigwedge^r \mathcal{E}) \twoheadrightarrow \bigwedge^r \mathcal{F}) \end{aligned}$$

is a closed immersion, called the **Plücker embedding**.

The proof is a good exercise. See [GD, 9.7, 9.8].

Remark 1.12.17. $\text{Grass}_1(\mathcal{E}) = \mathbb{P}(\mathcal{E})$.

Functoriality

Let $\phi: \mathcal{R} \rightarrow \mathcal{R}'$ be a morphism of quasi-coherent graded \mathcal{O}_S -algebra. Let $U(\phi) := \bigcup D_+(V, \phi(s))$, where the union is taken over all $V \subseteq S$ affine open and $s \in \mathcal{R}_+(V)$ homogeneous. We have a commutative diagram

$$\begin{array}{ccc} U(\phi) & \xrightarrow{r} & \underline{\text{Proj}}(\mathcal{R}) \\ \downarrow & & \downarrow \\ \underline{\text{Proj}}(\mathcal{R}') & \longrightarrow & S \end{array}$$

where r is an affine morphism.

Base change

Let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra and let $f: S' \rightarrow S$ be a morphism of schemes. Then we have a Cartesian square

$$\begin{array}{ccc} \underline{\text{Proj}}(f^*\mathcal{R}) & \xrightarrow{f'} & \underline{\text{Proj}}(\mathcal{R}) \\ \downarrow \lrcorner & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

For any quasi-coherent graded \mathcal{R} -module \mathcal{M} , we have $f'^*\widetilde{\mathcal{M}} = \widetilde{f^*\mathcal{M}}$.

Let \mathcal{R} and \mathcal{R}' be quasi-coherent graded \mathcal{O}_S -algebras. Then we have a Cartesian square

$$\begin{array}{ccc} \underline{\text{Proj}}(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}') & \xrightarrow{p} & \underline{\text{Proj}}(\mathcal{R}) \\ \downarrow p' \lrcorner & & \downarrow \\ \underline{\text{Proj}}(\mathcal{R}') & \longrightarrow & S \end{array}$$

Here $\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}' := \bigoplus_{d \geq 0} \mathcal{R}_d \otimes_{\mathcal{O}_S} \mathcal{R}'_d$. For a quasi-coherent graded \mathcal{R} -module \mathcal{M} and a quasi-coherent graded \mathcal{R}' -module \mathcal{M}' , we have

$$(\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{M}')^\sim \simeq \widetilde{\mathcal{M}} \boxtimes_S \widetilde{\mathcal{M}'} := p^* \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{X''}} p'^* \widetilde{\mathcal{M}'}.$$

Here $\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{M}' := \bigoplus_{d \in \mathbb{Z}} \mathcal{M}_d \otimes_{\mathcal{O}_S} \mathcal{M}'_d$ and $X'' = \underline{\text{Proj}}(\mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}')$.

Example 1.12.18. Let \mathcal{E} and \mathcal{E}' be quasi-coherent \mathcal{O}_S -modules. Then we have $\text{Sym}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}') \twoheadrightarrow \text{Sym}(\mathcal{E}) \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{E}')$, which induces a closed immersion $\mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}')$. This is called the **Segre embedding** and generalizes the Segre embedding for the product of two projective spaces.

Example 1.12.19. Let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra. Let \mathcal{L} be an invertible sheaf on S and let $\mathcal{R}' = \text{Sym}(\mathcal{L}) = \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$. Then $\mathcal{R}'' = \mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R}' = \bigoplus_{d \geq 0} \mathcal{R}_d \otimes \mathcal{L}^{\otimes d}$. We have a Cartesian square

$$\begin{array}{ccc} \underline{\text{Proj}}(\mathcal{R}'') & \xrightarrow{p} & \underline{\text{Proj}}(\mathcal{R}) \\ \downarrow \lrcorner & \searrow \pi'' & \downarrow \pi \\ \mathbb{P}(\mathcal{L}) & \xrightarrow{\pi'} & S \end{array}$$

Note that π' is an isomorphism, because it is so locally. Thus $p: \underline{\text{Proj}}(\mathcal{R}'') \rightarrow \underline{\text{Proj}}(\mathcal{R})$ is an isomorphism. We have $\mathcal{O}_{\mathbb{P}(\mathcal{L})}(d) = \pi'^* \mathcal{L}^{\otimes d}$ and $\mathcal{O}_{X''}(d) = p^* \mathcal{O}_X(d) \otimes \pi''^* \mathcal{L}^{\otimes d}$, where $X = \underline{\text{Proj}}(\mathcal{R})$, $X'' = \underline{\text{Proj}}(\mathcal{R}'')$.

Proposition 1.12.20. Let \mathcal{R} be a quasi-coherent graded \mathcal{O}_S -algebra generated by \mathcal{R}_1 over \mathcal{O}_S and let $\pi: X = \underline{\text{Proj}}(\mathcal{R}) \rightarrow S$. Suppose that \mathcal{R}_1 is a \mathcal{O}_S -module of finite type. Then

- (1) π is proper.
- (2) If S is quasi-compact and \mathcal{L} is an invertible sheaf on S such that $\mathcal{R}_1 \otimes_{\mathcal{O}_S} \mathcal{L}$ is generated by global sections, then $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \pi^* \mathcal{L}$ is π -very ample. In particular, if S admits an ample invertible sheaf, then π is projective.

Note that the condition on generation by \mathcal{R}_1 implies that the morphism $\mathcal{O}_S \rightarrow \mathcal{R}_0$ is an epimorphism of sheaves of sets.

Proof. (1) The problem being local on S , we may assume S affine. Then π is projective and thus proper.

(2) Let \mathcal{L} be an invertible sheaf on S such that $\mathcal{R}_1 \otimes \mathcal{L}$ is generated by globally sections. In the case where S admits an ample invertible sheaf \mathcal{M} , we can take \mathcal{L} to be $\mathcal{M}^{\otimes d}$ for some d . Since S is quasi-compact, there exists an epimorphism $\mathcal{O}_S^{n+1} \twoheadrightarrow \mathcal{R}_1 \otimes \mathcal{L}$. The morphisms of \mathcal{O}_S -algebras

$$\text{Sym}(\mathcal{O}_S^{n+1}) \rightarrow \text{Sym}(\mathcal{R}_1 \otimes \mathcal{L}) \rightarrow \mathcal{R} \otimes \text{Sym}(\mathcal{L})$$

are epimorphisms of \mathcal{O}_S -modules. The composition induces a closed embedding $i: \underline{\text{Proj}}(\mathcal{R}) \hookrightarrow \underline{\text{Proj}}(\mathcal{R}')$, where $\mathcal{R}' = \mathcal{R} \otimes \text{Sym}(\mathcal{L})$, that fits in the commutative

diagram

$$\begin{array}{ccccc} \underline{\text{Proj}}(R) & \xleftarrow[p]{\sim} & \underline{\text{Proj}}(\mathcal{R}') & \xrightarrow{i} & \mathbb{P}_S^n \\ & \searrow \pi & \downarrow \pi' & \swarrow & \\ & & S. & & \end{array}$$

We have $i^*\mathcal{O}_{\mathbb{P}_S^n}(1) \simeq \mathcal{O}_{\underline{\text{Proj}}(\mathcal{R}')} (1) \simeq p^*\mathcal{O}_{\underline{\text{Proj}}(\mathcal{R})}(1) \otimes \pi'^*\mathcal{L}$. Let $f = i \circ p^{-1}: \underline{\text{Proj}}(\mathcal{R}) \rightarrow \mathbb{P}_S^n$. Then $f^*\mathcal{O}_{\mathbb{P}_S^n}(1) \simeq \overline{\mathcal{O}_{\underline{\text{Proj}}(\mathcal{R})}(1)} \otimes \pi^*\mathcal{L}$. \square

Remark 1.12.21. Let $\pi: X \rightarrow S$ be a morphism of schemes. If \mathcal{L} is π -very ample, then \mathcal{L} is globally generated. Indeed, $\mathcal{O}_{\mathbb{P}_S^n}(1)$ is globally generated. Conversely, if π is an isomorphism, S is quasi-compact and \mathcal{L} is globally generated, then \mathcal{L} is π -very ample by the preceding proposition. Thus very ampleness is **not** local on S .

Remark 1.12.22. A morphism of schemes $f: X \rightarrow S$ is said to be **EGA projective** if there exists a decomposition

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}(\mathcal{E}) \\ & \searrow f & \downarrow \\ & & S \end{array}$$

where i is a closed immersion and \mathcal{E} is a quasi-coherent \mathcal{O}_S -module of finite type.

Blowing up

Definition 1.12.23. Let S be a scheme, $\mathcal{I} \subseteq \mathcal{O}_S$ a quasi-coherent ideal sheaf which defines a closed subscheme Z . Consider the quasi-coherent graded \mathcal{O}_S -module $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}^n$, where $\mathcal{I}^0 = \mathcal{O}_S$. Then

$$X = \underline{\text{Proj}}(\mathcal{R}) \xrightarrow{\pi} S$$

is called the **blowing up** of S along Z (or with **center** Z , or in \mathcal{I}). The closed subscheme $\pi^{-1}(Z) \subseteq X$ is called the **exceptional divisor**.

On an affine open $\text{Spec}(B) = V \subseteq S$, we have $\mathcal{I}|_V = \tilde{I}$ where $I \subseteq B$ an ideal, and $\pi^{-1}(V) = \underline{\text{Proj}}(\bigoplus_{n \geq 0} I^n)$, where $I^0 = B$. We have $\pi^{-1}(V) = \bigcup_{a \in I} D_+(V, a^{(1)})$, where for $a \in I$, $a^{(1)}$ denotes a viewed as an element of $\mathcal{R}_1(V) = I$. We have $D_+(V, a^{(1)}) = \text{Spec}(B[\frac{I}{a}])$, where $B[\frac{I}{a}] := (\bigoplus_{n \geq 0} I^n)_{(a^{(1)})}$ is called the affine blow up algebra. Elements of $B[\frac{I}{a}]$ are of the form x/a^n , $x \in I^n$ and $x/a^n = y/a^m$ if and only if there exists k such that $a^k(a^m x - a^n y) = 0$.

We will discuss divisors more thoroughly later in the course. Here we limit our attention to effective Cartier divisors.

Definition 1.12.24. An **effective Cartier divisor** on a scheme X is a closed subscheme $D \subseteq X$ whose sheaf of ideals \mathcal{I}_D is invertible.

Lemma 1.12.25. Let $D \subseteq X$ be a closed subscheme. Then D is an effective Cartier divisor if and only if every $x \in X$ admits an affine open neighborhood $x \in U = \text{Spec}(A)$ such that $U \cap D = \text{Spec}(A/(f))$, where $f \in A$ is a non zero-divisor.

Proof. An ideal I of A is free of rank 1 if and only if I is generated by a non zero-divisor of A . \square

Lemma 1.12.26. *Let $D \subseteq X$ be an effective Cartier divisor. Then $j: X \setminus D \hookrightarrow X$ is schematically dense. Namely, $j^\flat: \mathcal{O}_X \rightarrow j_*(\mathcal{O}_{X \setminus D})$ is a monomorphism.*

Proof. Locally, j corresponds to the ring homomorphism $A \rightarrow A_f$, where $f \in A$ is a non zero-divisor. The ring homomorphism is clearly injective. \square

Remark 1.12.27. Let D_1 and D_2 be two effective Cartier divisors on X with ideal sheaves \mathcal{I}_{D_1} , \mathcal{I}_{D_2} , respectively. Then the natural morphism $\mathcal{I}_{D_1} \otimes_{\mathcal{O}_X} \mathcal{I}_{D_2} \rightarrow \mathcal{I}_{D_1} \mathcal{I}_{D_2}$ is an isomorphism. (Indeed, it is by definition an epimorphism of sheaves of \mathcal{O}_X -modules and the morphism $\mathcal{I}_{D_1} \otimes \mathcal{I}_{D_2} \rightarrow \mathcal{O}_X$ is a monomorphism by flatness.) We define the sum of the two divisors $D_1 + D_2$ as the subscheme defined by $\mathcal{I}_{D_1} \mathcal{I}_{D_2}$. Then $\text{CaDiv}_+(X) = (\{\text{effective Cartier divisors on } S\}, +)$ is a commutative monoid.

Proposition 1.12.28. *Let $Z \subseteq S$ be a closed subscheme and let X be the blowing up of S along Z . Then*

- (1) $\pi|_{\pi^{-1}(S \setminus Z)}: \pi^{-1}(S \setminus Z) \rightarrow S \setminus Z$ is an isomorphism.
- (2) $E = \pi^{-1}(Z)$ is an effective Cartier divisor with $\mathcal{I}_E = \mathcal{O}_X(1)$.

Proof. (1) The construction being compatible with restriction to open subschemes, we may assume $Z = \emptyset$. Then $\mathcal{I}_Z = \mathcal{O}_X$ and $X = \underline{\text{Proj}}(\bigoplus_{n \geq 0} \mathcal{O}_S) = \mathbb{P}_S^0 \simeq S$.

(2) Let $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}^n$. We have

$$\begin{aligned} \mathcal{I}_E &= \widetilde{\mathcal{I}\mathcal{R}} = \left(\bigoplus_{n \geq 0} \mathcal{I}^{n+1} \right)^\sim, \\ \mathcal{O}(1) &= \widetilde{\mathcal{R}(1)} = \left(\bigoplus_{n \geq -1} \mathcal{I}^{n+1} \right)^\sim. \end{aligned}$$

They are isomorphic as sheaves. \square

Proposition 1.12.29 (Universal property of blowing up). *Let $Z \subseteq S$ be a closed subscheme and let X be the blowing up of S along Z . Let $f: Y \rightarrow S$ be a morphism of schemes such that $f^{-1}(Z)$ is an effective Cartier divisor on Y . Then there exists a unique $g: Y \rightarrow X$ such that $f = \pi g$.*

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & X \\ & \searrow f & \downarrow \pi \\ Z & \xrightarrow{\quad} & S \end{array}$$

Proof. Existence. Let \mathcal{I} be the ideal sheaf of Z and let $D = f^{-1}(Z)$. Then $\mathcal{I}_D = f^{-1}(\mathcal{I})\mathcal{O}_Y$. We have epimorphisms $\gamma_n: f^*\mathcal{I}^n \twoheadrightarrow \mathcal{I}_D^n = \mathcal{I}_D^{\otimes n}$, which induces a morphism of graded \mathcal{O}_Y -algebras $\gamma: f^*(\bigoplus_{n \geq 0} \mathcal{I}^n) \twoheadrightarrow \bigoplus_{n \geq 0} \mathcal{I}_D^{\otimes n}$. This corresponds to an S -morphism $g: Y \rightarrow X$.

Uniqueness. Suppose there are two S -morphisms $g, g': Y \rightrightarrows X$. Let E be their equalizer:

$$\begin{array}{ccc} E & \longrightarrow & Y \xrightarrow{\quad g \quad} X \\ & \searrow^{g'} \downarrow \pi & \downarrow (g, g') \\ & f & \downarrow \\ & S & X \xrightarrow{\Delta_S} X \times_S X \end{array}$$

Since π is separated, E is a closed subscheme of Y . Moreover, since π is an isomorphism on $\pi^{-1}(S \setminus Z)$, we have $E \supseteq f^{-1}(S \setminus Z)$ as subschemes. Since $Y \setminus f^{-1}(Z) \hookrightarrow Y$ is schematically dense in Y by Lemma 1.12.26, so is E . Therefore, $E = Y$ and $g = g'$. \square

Corollary 1.12.30. *Let $Z \subseteq S$ be a closed subscheme, $f: S' \rightarrow S$ a morphism of schemes, $\pi: X \rightarrow S$ the blowing up of S along Z , and $\pi': X' \rightarrow S'$ the blowing up of S' along $f^{-1}(Z)$. Then there exists a unique $g: X' \rightarrow X$ such that $\pi g = f \pi'$:*

$$\begin{array}{ccc} X' & \dashrightarrow^g & X \\ \downarrow \pi' & & \downarrow \pi \\ S' & \xrightarrow{f} & S \end{array}$$

Moreover,

- (a) If f is a closed immersion, so is g .
- (b) If f is flat, then the square is Cartesian.

Proof. The existence and uniqueness of g follows from the universal property of blowing up.

For (a), we may assume $S' = \underline{\text{Spec}}(\mathcal{O}_S/\mathcal{J})$, where $\mathcal{J} \subseteq \mathcal{O}_S$ is an ideal sheaf. Let $\mathcal{I} \subseteq \mathcal{O}_S$ be the ideal sheaf of Z . Then the ideal sheaf of $f^{-1}(Z)$ is $\mathcal{I}\mathcal{J}/\mathcal{J}$. Then the canonical morphism $\bigoplus_{n \geq 0} \mathcal{I}^n \rightarrow \bigoplus_{n \geq 0} \mathcal{I}^n \mathcal{J}/\mathcal{J}$ is an epimorphism as sheaves of \mathcal{O}_S -modules and the corresponding morphism $X' \rightarrow X$ is a closed immersion.

For (b), we need to show that the morphism $X' \rightarrow X \times_S S'$ is an isomorphism. Since f is flat, we have $f^*(\mathcal{I}^n) \simeq f^{-1}(\mathcal{I}^n)\mathcal{O}_{S'}$. Thus $X' = \text{Proj}(\bigoplus_{n \geq 0} f^{-1}(\mathcal{I}^n)\mathcal{O}_{S'}) \simeq \text{Proj}(\bigoplus_{n \geq 0} f^*(\mathcal{I}^n)) \simeq X \times_S S'$. \square

Definition 1.12.31. In case (a), X' is called the **strict transform** of S' .

Remark 1.12.32. The exceptional divisor of the blowing up of a scheme S along a closed subscheme Z defined by the ideal sheaf \mathcal{I} is $E = \text{Proj}(\mathcal{R}/\mathcal{I}\mathcal{R}) \simeq \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$. This is a closed subscheme of $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$ and is sometimes called the **projective normal cone** of $Z \subseteq S$. Here $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}^n$.

Example 1.12.33. Let $S = \mathbb{A}_A^n = \text{Spec}(B)$, where $B = A[x_1, \dots, x_n]$. Let Z be the closed subscheme defined by $I = (x_1, \dots, x_n)$. Let $X = \text{Bl}_Z(S) = \text{Proj}(R)$, where $R = \bigoplus_{n \geq 0} I^n$. We have the surjective homomorphism

$$\begin{aligned} B[y_1, \dots, y_n] &\rightarrow R \\ y_i &\mapsto x_i^{(1)}, \end{aligned}$$

which gives a closed immersion $X \hookrightarrow \mathbb{P}_{\mathbb{A}_A^n}^{n-1}$. We have $R \simeq B[y_1, \dots, y_n]/(x_i y_j - x_j y_i)$. The exceptional divisor $E \simeq \mathbb{P}_A^{n-1}$.

For $S = \mathbb{A}^2$, the strict transforms of lines ℓ through the origin are disjoint. The intersection of the strict transform of ℓ with the exceptional divisor is given by the slope of ℓ .

Let $B' = A[x, y]/(y^2 - x^2(x + 1))$. The blowing up of B' in (x, y) is $B[\frac{I}{x}] = A[x, z]/(z^2 - (x + 1))$, where $z = y/x$. By contrast, the blowing up of B' in (x) is B' , because $x \in B'$ is a non zero-divisor. We see that blowing up depends on the closed subscheme and not only on the closed subset.

Chapter 2

Cohomology of Quasi-coherent Sheaves

Date: 11.17

2.1 Homological algebra

We will give a brief introduction to derived categories and derived functors and refer to [GM] and [Z, Chapter 2] for more complete treatments.

Let \mathcal{A} and \mathcal{B} be abelian categories and let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor. For any short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{A} , we have, by the left exactness of F , an exact sequence

$$0 \rightarrow FX \rightarrow FY \rightarrow FZ$$

in \mathcal{B} . Under suitable conditions, we can define additive functors $R^n F: \mathcal{A} \rightarrow \mathcal{B}$, $i \geq 1$, called the **right derived functors** of F , such that the exact sequence in \mathcal{B} extends to a long exact sequence

$$\begin{aligned} 0 \rightarrow FX \rightarrow FY \rightarrow FZ \rightarrow R^1 FX \rightarrow R^1 FY \rightarrow R^1 FZ \rightarrow \cdots \\ \rightarrow R^n FX \rightarrow R^n FY \rightarrow R^n FZ \rightarrow \cdots . \end{aligned}$$

Roughly speaking, the right derived functors measure the lack of right exactness of F . The functors can be assembled into one single functor $RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ between **derived categories**.

Recall that an object I of \mathcal{A} is said to be **injective** if $\text{Hom}_{\mathcal{A}}(-, I): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$ is exact. Assume that \mathcal{A} admits enough injectives (namely, every object of \mathcal{A} can be embedded into an injective object of \mathcal{A}). Then every object X of \mathcal{A} admits an **injective resolution** of X , namely an exact sequence

$$0 \rightarrow X \rightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

with I^i injective. Then RFX is computed by the complex

$$FI: \cdots \rightarrow 0 \rightarrow FI^0 \xrightarrow{Fd^0} FI^1 \xrightarrow{Fd^1} \cdots$$

and $R^i FX$ is computed by the i -th cohomology of RFX : $\ker(Fd^i)/\text{im}(Fd^{i-1})$.

Definition 2.1.1. Let \mathcal{A} be an additive category. A **(cochain) complex** in \mathcal{A} consists of $X = (X^n, d^n)_{n \in \mathbb{Z}}$, where X^n is an object of \mathcal{A} , $d_X^n: X^n \rightarrow X^{n+1}$ is a morphism of \mathcal{A} (called **differential**) such that for any n , $d_X^{n+1}d_X^n = 0$. The index n in X^n is called the **degree**. A **(cochain) morphism** of complexes $X \rightarrow Y$ is a collection of morphisms $(f^n)_{n \in \mathbb{Z}}$ of morphisms $f^n: X^n \rightarrow Y^n$ in \mathcal{A} such that $d_Y^n f^n = f^{n+1} d_X^n$. We let $C(\mathcal{A})$ denote the category of complexes in \mathcal{A} .

Note that $C(\mathcal{A})$ is an additive category. We have $(X \oplus Y)^n = X^n \oplus Y^n$ and the zero complex 0 with $0^n = 0$ is a zero object of $C(\mathcal{A})$.

Let \mathcal{A} be an abelian category. Then $C(\mathcal{A})$ is an abelian category as well, with $\text{Ker}(f)^n = \text{Ker}(f^n)$ and $\text{coker}(f)^n = \text{coker}(f^n)$.

Definition 2.1.2. Let X be a complex in \mathcal{A} . We define

$$\begin{aligned} Z^n X &= \text{Ker}(d_X^n: X^n \rightarrow X^{n+1}), \\ B^n X &= \text{im}(d_X^{n-1}: X^{n-1} \rightarrow X^n), \\ H^n X &= \text{coker}(B^n X \hookrightarrow Z^n X), \end{aligned}$$

and call them the **cocycle**, **coboundary**, **cohomology** objects, of degree n .

The letter Z stands for German **Zyklus**, which means cycle. We get additive functors

$$Z^n, B^n, H^n: C(\mathcal{A}) \rightarrow \mathcal{A},$$

with Z^n left exact.

Definition 2.1.3. A complex X is said to be **acyclic** if $H^n X = 0$ for all n . A morphism of complexes $X \rightarrow Y$ is called a **quasi-isomorphism** if $H^n f: H^n X \rightarrow H^n Y$ is an isomorphism for all n .

We will soon define the derived category $D(\mathcal{A})$ of \mathcal{A} . Roughly speaking, $D(\mathcal{A})$ is $C(\mathcal{A})$ modulo quasi-isomorphisms. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then F induces $C(F): C(\mathcal{A}) \rightarrow C(\mathcal{B})$ (also denoted by F). If F is exact, then $C(F)$ preserves quasi-isomorphisms and induces a functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$. For the general case, it is convenient to introduce an intermediary between $C(\mathcal{A})$ and $D(\mathcal{A})$.

Let \mathcal{A} be an additive category. Let X and Y be complexes in \mathcal{A} . We let

$$\text{Ht}(X, Y) = \prod_n \text{Hom}_{\mathcal{A}}(X^n, Y^{n-1})$$

denote the abelian group of families of morphisms $h = (h^n: X^n \rightarrow Y^{n-1})_{n \in \mathbb{Z}}$. Given h , consider $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n: X^n \rightarrow Y^n$. We have

$$d_Y^n f^n = d_Y^{n-1}d_Y^n h^n + d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n = d_Y^n h^{n+1}d_X^n + h^{n+2}d_X^{n+1}d_X^n = f^{n+1}d_X^n.$$

Thus we get a morphism of complexes $f: X \rightarrow Y$. We get a homomorphism of abelian groups

$$(2.1.1) \quad \text{Ht}(X, Y) \rightarrow \text{Hom}_{C(\mathcal{A})}(X, Y).$$

Definition 2.1.4. We say that a morphism of complexes $f: X \rightarrow Y$ is **null-homotopic** if there exists $h \in \text{Ht}(X, Y)$ such that $f^n = d_Y^{n-1}h^n + h^{n+1}d_X^n$. We say that two morphisms of complexes $f, g: X \rightarrow Y$ are **homotopic** if $f - g$ is null-homotopic.

Lemma 2.1.5. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ be morphisms of complexes in \mathcal{A} . If f or g is null-homotopic, then gf is null-homotopic.

Proof. If $f = dh + hd$ for $h \in \text{Ht}(X, Y)$, then $gf = gdh + ghd = d(gh) + (gh)d$, where $gh \in \text{Ht}(X, Z)$. The other case is similar. \square

Definition 2.1.6. We define the **homotopy category of complexes in \mathcal{A}** , $K(\mathcal{A})$, as follows. The objects of $K(\mathcal{A})$ are objects of $C(\mathcal{A})$, that is, complexes in \mathcal{A} . For complexes X and Y , we put

$$\text{Hom}_{K(\mathcal{A})}(X, Y) = \text{coker}(\text{Ht}(X, Y) \xrightarrow{(2.1.1)} \text{Hom}_{C(\mathcal{A})}(X, Y)).$$

In other words, morphisms in $K(\mathcal{A})$ are homotopy classes of morphisms of complexes.

Remark 2.1.7. The category $K(\mathcal{A})$ is an additive category and the functor $C(\mathcal{A}) \rightarrow K(\mathcal{A})$ carrying a complex to itself and a morphism of complexes to its homotopy class is an additive functor.

Definition 2.1.8. Let \mathcal{A} be an abelian category. We call $D(\mathcal{A}) = K(\mathcal{A})[S^{-1}]$ the **derived category** of \mathcal{A} , where S is the collection of quasi-isomorphisms in $K(\mathcal{A})$.

By definition, objects of $D(\mathcal{A})$ are complexes in \mathcal{A} and morphisms are equivalence classes of zigzags of morphisms of $K(\mathcal{A})$

$$\rightarrow \cdots \rightarrow \leftarrow \cdots \leftarrow \rightarrow \cdots \rightarrow \cdots \leftarrow \cdots \leftarrow,$$

where each \leftarrow represents an element of S . One advantage of defining $D(\mathcal{A})$ as a localization of $K(\mathcal{A})$ instead of $C(\mathcal{A})$ is that left and right calculus of fractions holds:

$$\text{Hom}_{D(\mathcal{A})}(X, Y) \simeq \underset{(Y', s) \in S_{Y/}}{\text{colim}} \text{Hom}_C(X, Y') \simeq \underset{(X', s) \in S_{/X}^{\text{op}}}{\text{colim}} \text{Hom}_C(X', Y).$$

In general, $D(\mathcal{A})$ does not have small Hom sets, even if \mathcal{A} has small Hom sets. See however Remark 2.1.35 below.

The categories $K(\mathcal{A})$ and $D(\mathcal{A})$ admit an additional structure, making them **triangulated categories**. To introduce this structure, we need a couple of constructions.

Let \mathcal{A} be an additive category.

Definition 2.1.9. Let X be a complex and let k be an integer. We define a complex $X[k]$ by $X[k]^n = X^{n+k}$ and $d_{X[k]}^n = (-1)^k d_X^{n+k}$. For a morphism of complexes $f: X \rightarrow Y$, we define $f[k]: X[k] \rightarrow Y[k]$ by $f[k]^n = f^{n+k}$. The functor $[k]: C(\mathcal{A}) \rightarrow C(\mathcal{A})$ is called the **translation** (or shift) functor of degree k .

The sign in the definition of $X[k]$ will be explained after the following definition.

Definition 2.1.10. Let $f: X \rightarrow Y$ be a morphism of complexes in \mathcal{A} . We define the **mapping cone** of f to be the complex $\text{Cone}(f)^n = X[1]^n \oplus Y^n = X^{n+1} \oplus Y^n$ with differential

$$d_{\text{Cone}(f)}^n = \begin{pmatrix} d_X^n & 0 \\ f[1]^n & d_Y^n \end{pmatrix} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}.$$

Intuitively, for $\begin{pmatrix} x \\ y \end{pmatrix} \in X^{n+1} \oplus Y^n$, $d_{\text{Cone}(f)}^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d_X^{n+1}x \\ f^{n+1}x + d_Y^n y \end{pmatrix}$.

Note that the sign in the definition of the differential of $X[1]$ makes $\text{Cone}(f)$ a complex:

$$d_{\text{Cone}(f)}^n d_{\text{Cone}(f)}^{n-1} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \begin{pmatrix} -d_X^n & 0 \\ f^n & d_Y^{n-1} \end{pmatrix} = \begin{pmatrix} d_X^{n+1} d_X^n & 0 \\ d_Y^n f^n - f^{n+1} d_X^n & d_Y^n d_Y^{n-1} \end{pmatrix} = 0.$$

Example 2.1.11. If X and Y are concentrated in degree 0, then $\text{Cone}(f)$ can be identified with the complex $X^0 \xrightarrow{f^0} Y^0$ concentrated on degrees -1 and 0 .

Triangulated categories

Given a category \mathcal{D} equipped with a functor $X \mapsto X[1]$, diagrams of the form $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ are called **triangles**. It is sometimes useful to visualize such diagrams as

$$\begin{array}{ccc} & Z & \\ & \nearrow +1 & \swarrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

A morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ f \downarrow & & g \downarrow & & h \downarrow & & f[1] \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

Such a morphism is an isomorphism if and only if f, g, h are isomorphisms.

Definition 2.1.12 (Verdier). A **triangulated category** consists of the following data:

- (1) An additive category \mathcal{D} .
- (2) A **translation functor** $\mathcal{D} \rightarrow \mathcal{D}$ which is an equivalence of categories. We denote the functor by $X \mapsto X[1]$.
- (3) A collection of **distinguished triangles** $X \rightarrow Y \rightarrow Z \rightarrow X[1]$.

These data are subject to the following axioms:

(TR1)

- The collection of distinguished triangles is stable under isomorphism.

- Every morphism $f: X \rightarrow Y$ in \mathcal{D} can be extended to a distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1]$.

- For every object X of \mathcal{D} , $X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow X[1]$ is a distinguished triangle.

(TR2) A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if the (clockwise) rotated diagram $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a distinguished triangle.

(TR4) Given three distinguished triangles

$$\begin{aligned} X &\xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} X[1], \\ Y &\xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} Y[1], \\ X &\xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} X[1], \end{aligned}$$

with $h = gf$, there exists a distinguished triangle $U \xrightarrow{i} V \xrightarrow{i'} W \xrightarrow{i''} U[1]$ such that the following diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{h} & Z & \xrightarrow{g'} & W & \xrightarrow{i''} & U[1] \\ f \searrow & \nearrow g & h' \searrow & \nearrow i' & g'' \searrow & f'[1] \nearrow & \\ Y & & V & & & & Y[1] \\ f' \searrow & \nearrow i & h'' \searrow & \nearrow f[1] & & & \\ U & \xrightarrow{f''} & X[1] & & & & \end{array}$$

This notion was introduced by Verdier (see his 1967 thesis of **doctorat d'État** [V]). Some authors call the translation functor the suspension functor and denote it by Σ . (TR4) is sometimes known as the octahedron axiom, as the four distinguished triangles and the four commutative triangles can be visualized as the faces of an octahedron.

Date: 11.17

Remark 2.1.13. The original definition included an axiom (TR3): Given a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{i} & Y & \xrightarrow{j} & Z & \xrightarrow{k} & X[1] \\ f \downarrow & & g \downarrow & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' & \xrightarrow{k'} & X'[1] \end{array}$$

in which both rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative. Note that we do **not** require the dotted arrow to be unique.

May [M3, Section 2] observed that this axiom can be deduced from (TR1) and (TR4). Indeed, by (TR1), we may extend $gi = i'f$ to a distinguished triangle

$$X \xrightarrow{gi} Y' \xrightarrow{j''} Z'' \xrightarrow{k''} X[1].$$

Applying (TR1) to g and (TR4) to the distinguished triangles with bases g , i , and gi , we get a morphism $Z \xrightarrow{h'} Z''$ such that $h'j = j''g$ and $k = k''h'$. Similarly, applying (TR1) to f and (TR4) to the distinguished triangles with bases f , i' , and gi , we get $Z'' \xrightarrow{h''} Z$ such that $j' = h''j''$ and $f[1]k'' = k'h''$. It suffices to take $h = h''h'$.

Corollary 2.1.14. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ be a distinguished triangle. Then $gf = 0$.

Proof. By (TR1), $X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow X[1]$ is a distinguished triangle. By (TR3), there exists a morphism $0 \rightarrow Z$ such that the diagram

$$(2.1.2) \quad \begin{array}{ccccccc} X & \xrightarrow{\text{id}_X} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \text{id}_X \downarrow & & f \downarrow & & \downarrow & & \downarrow \text{id}_{X[1]} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & X[1] \end{array}$$

commutes. The commutativity of the square in the middle implies $gf = 0$. \square

Proposition 2.1.15. Let \mathcal{D} be a triangulated category. Let W be an object of \mathcal{D} and let $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ be a distinguished triangle. Then the sequences

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(W, X) &\rightarrow \text{Hom}_{\mathcal{D}}(W, Y) \rightarrow \text{Hom}_{\mathcal{D}}(W, Z), \\ \text{Hom}_{\mathcal{D}}(Z, W) &\rightarrow \text{Hom}_{\mathcal{D}}(Y, W) \rightarrow \text{Hom}_{\mathcal{D}}(X, W) \end{aligned}$$

are exact.

If \mathcal{D} has small Hom sets, then the proposition means that the functors

$$\text{Hom}_{\mathcal{D}}(W, -): \mathcal{D} \rightarrow \text{Ab}, \quad \text{Hom}_{\mathcal{D}}(-, W): \mathcal{D}^{\text{op}} \rightarrow \text{Ab}$$

are cohomological functors.

Proof. Let us show that the first sequence is exact, the other case being similar. Since $gf = 0$, the composition is zero. Thus it suffices to show that for $j: W \rightarrow Y$ satisfying $gj = 0$, there exists $i: W \rightarrow X$ such that $j = fi$. Applying (TR1), (TR2), (TR3), we get the following commutative diagram

$$\begin{array}{ccccccc} W & \longrightarrow & 0 & \longrightarrow & W[1] & \xrightarrow{-\text{id}_{W[1]}} & W[1] \\ \downarrow j & & \downarrow & & \downarrow i[1] & & \downarrow j[1] \\ Y & \xrightarrow{g} & Z & \longrightarrow & X[1] & \xrightarrow{-f[1]} & Y[1]. \end{array}$$

□

Corollary 2.1.16. *Let*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

be a morphism of distinguished triangles. If f and g are isomorphisms, so is the third one.

Thus triangles extending a morphism $X \rightarrow Y$ are unique up to **non-unique** isomorphisms.

Proof. Let W be any object of the triangulated category. Then we have a commutative diagram

$$\begin{array}{ccccccc} \text{Hom}(W, X) & \longrightarrow & \text{Hom}(W, Y) & \longrightarrow & \text{Hom}(W, Z) & \longrightarrow & \text{Hom}(W, X[1]) \longrightarrow \text{Hom}(W, Y[1]) \\ \downarrow \text{Hom}(W, f) & & \downarrow \text{Hom}(W, g) & & \downarrow \text{Hom}(W, h) & & \downarrow \text{Hom}(W, f[1]) \\ \text{Hom}(W, X') & \longrightarrow & \text{Hom}(W, Y') & \longrightarrow & \text{Hom}(W, Z') & \longrightarrow & \text{Hom}(W, X'[1]) \longrightarrow \text{Hom}(W, Y'[1]) \end{array}$$

with exact rows. By the five lemma, $\text{Hom}(W, h)$ is an isomorphism. Therefore h is an isomorphism by Yoneda's lemma. □

Corollary 2.1.17. *In a distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1]$, f is an isomorphism if and only if Z is a zero object.*

Proof. Applying Corollary 2.1.16 to the diagram (2.1.2), we see that f is an isomorphism if and only if h is an isomorphism. □

Definition 2.1.18. Let \mathcal{D} and \mathcal{D}' be triangulated categories. A **triangulated functor** consists of the following data:

- (1) An additive functor $F: \mathcal{D} \rightarrow \mathcal{D}'$.
- (2) A natural isomorphism $\phi_X: F(X[1]) \simeq (FX)[1]$ of functors $\mathcal{D} \rightarrow \mathcal{D}'$.

These data are subject to the condition that F carries distinguished triangles in \mathcal{D} to distinguished triangles in \mathcal{D}' . That is, for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in \mathcal{D} , $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\phi_Z \circ Fh} (FX)[1]$ is a distinguished triangle in \mathcal{D}' .

Let $(F, \phi), (F', \phi'): \mathcal{D} \rightarrow \mathcal{D}'$ be triangulated functors. A **natural transformation of triangulated functors** is a natural transformation $\alpha: F \rightarrow F'$ such that the following diagram commutes for all X :

$$\begin{array}{ccc} F(X[1]) & \xrightarrow{\phi_X} & (FX)[1] \\ \alpha(X[1]) \downarrow & & \downarrow \alpha(X)[1] \\ F'(X[1]) & \xrightarrow{\phi'_X} & (F'X)[1]. \end{array}$$

Derived categories

Let \mathcal{A} be an additive category. We equip $K(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.9. We say that a triangle in $K(\mathcal{A})$ is **distinguished** if it is isomorphic to a standard triangle, namely a triangle of the form $X \xrightarrow{f} Y \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} X[1]$, where i and p are the canonical morphisms. If \mathcal{A} is abelian, we equip $D(\mathcal{A})$ with the translation functor $X \mapsto X[1]$ in Definition 2.1.9 and we say that a triangle in $D(\mathcal{A})$ is **distinguished** if it is isomorphic to a standard triangle.

Theorem 2.1.19. *Let \mathcal{A} be an additive category.*

- (1) *$K(\mathcal{A})$ is a triangulated category.*
- (2) *If \mathcal{A} is abelian, then $D(\mathcal{A})$ is a triangulated category and the functor $Q: K(\mathcal{A}) \rightarrow D(\mathcal{A})$ (equipped with the trivial natural isomorphism $Q(X[1]) = (QX)[1]$) is a triangulated functor.*

For a proof, see for example [Z, Chapter 2].

We define **naive truncation** functors

$$\sigma^{\leq n}: C(\mathcal{A}) \rightarrow C(\mathcal{A}), \quad \sigma^{\geq n}: C(\mathcal{A}) \rightarrow C(\mathcal{A})$$

by $(\sigma^{\leq n}X)^m = X^m$ for $m \leq n$, $(\sigma^{\leq n}X)^m = 0$ for $m > n$ and $(\sigma^{\geq n}X)^m = X^m$ for $m \geq n$, $(\sigma^{\geq n}X)^m = 0$ for $m < n$.

Let \mathcal{A} be an abelian category. The morphisms $H^nX \rightarrow H^n\sigma^{\leq n}X$, $H^n\sigma^{\geq n}X \rightarrow H^nX$ are not isomorphisms in general. Moreover, if $f: X \rightarrow Y$ is a quasi-isomorphism, $\sigma^{\leq n}f: \sigma^{\leq n}X \rightarrow \sigma^{\leq n}Y$ and $\sigma^{\geq n}f: \sigma^{\geq n}X \rightarrow \sigma^{\geq n}Y$ are not quasi-isomorphisms in general. To remedy this problem, we introduce the following **truncation** functors.

Definition 2.1.20. Let X be a complex. We define

$$\begin{aligned} \tau^{\leq n}X &= (\cdots \rightarrow X^{n-1} \xrightarrow{d_X^{n-1}} Z^nX \rightarrow 0 \rightarrow \cdots), \\ \tau^{\geq n}X &= (\cdots \rightarrow 0 \rightarrow X^n/B^nX \xrightarrow{d_X^n} X^{n+1} \rightarrow \cdots). \end{aligned}$$

Here X^n/B^nX denotes $\text{coker}(d_X^{n-1})$.

We obtain functors

$$\tau^{\leq n}, \tau^{\geq n}: C(\mathcal{A}) \rightarrow C(\mathcal{A}),$$

with $\tau^{\leq n}$ left exact and $\tau^{\geq n}$ right exact.

Remark 2.1.21. The morphism $\tau^{\leq n}X \rightarrow X$ induces an isomorphism $H^m\tau^{\leq n}X \rightarrow H^mX$ for $m \leq n$ and $H^m\tau^{\leq n}X = 0$ for $m > n$. The morphism $X \rightarrow \tau^{\geq n}X$ induces an isomorphism $H^mX \rightarrow H^m\tau^{\geq n}X$ for $m \geq n$ and $H^m\tau^{\geq n}X = 0$ for $m < n$. The functors $\tau^{\leq n}$ and $\tau^{\geq n}$ preserve quasi-isomorphisms.

Remark 2.1.22. For $a \leq b$, we have $\tau^{\leq a}\tau^{\geq b}X \simeq \tau^{\geq b}\tau^{\leq a}X$ and we write $\tau^{[a,b]}X$ for either of them. We have $\tau^{[n,n]}X \simeq (H^nX)[-n]$.

The functor H^n is neither left exact nor right exact in general. However, it has the following important property.

Proposition 2.1.23. *Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a short exact sequence of complexes. Then we have a **long exact sequence***

$$\cdots \rightarrow H^nL \xrightarrow{H^nf} H^nM \xrightarrow{H^ng} H^nN \xrightarrow{\delta} H^{n+1}L \xrightarrow{H^{n+1}f} H^{n+1}M \xrightarrow{H^{n+1}g} H^{n+1}N \rightarrow \cdots,$$

which is functorial with respect to the short exact sequence.

The morphism δ is called the **connecting** morphism.

Proof. The sequence $\tau^{[n,n+1]}L \rightarrow \tau^{[n,n+1]}M \rightarrow \tau^{[n,n+1]}N$ provides a commutative diagram

$$\begin{array}{ccccccc} L^n/B^nL & \longrightarrow & M^n/B^nM & \longrightarrow & N^n/B^nN & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{n+1}L & \longrightarrow & Z^{n+1}M & \longrightarrow & Z^{n+1}N \end{array}$$

with exact rows. Applying the snake lemma, we obtain the desired exact sequence. \square

Corollary 2.1.24. *For every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $D(\mathcal{A})$, we have a long exact sequence $H^nX \xrightarrow{H^nf} H^nY \xrightarrow{H^ng} H^nZ \xrightarrow{H^nh} H^{n+1}X$.*

Proof. We may assume that the triangle is standard: $X \xrightarrow{f} Y \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} X[1]$. The short exact sequence of complexes

$$0 \rightarrow Y \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} X[1] \rightarrow 0.$$

induces a long exact sequence

$$\cdots \rightarrow H^{n-1}(X[1]) \xrightarrow{\delta} H^nY \xrightarrow{H^ni} H^n(\text{Cone}(f)) \xrightarrow{H^np} H^n(X[1]) \rightarrow \cdots.$$

It suffices to check that, via the isomorphism $H^{n-1}(X[1]) \simeq H^nX$, the connecting morphism can be identified with H^nf . The connecting morphism is constructed using the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccc} Y^{n-1}/B^{n-1}Y & \longrightarrow & C^{n-1}/B^{n-1}C & \longrightarrow & X^n/B^nX & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^nY & \longrightarrow & Z^nC & \longrightarrow & Z^{n+1}X, \end{array}$$

where $C = \text{Cone}(f)$. We reduce by the Freyd-Mitchell Theorem to the case of modules. Let $x \in Z^n X$. Then $\begin{pmatrix} x \\ 0 \end{pmatrix} + B^{n-1} C$ is a lifting of $x + B^n X$. We conclude by $d_C^{n-1} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f^n(x) \end{pmatrix}$. \square

Corollary 2.1.25. *Consider a short exact sequence of complexes $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$. Then the map $\phi = (0, g): \text{Cone}(f) \rightarrow Z$ is a quasi-isomorphism.*

In this case, we get a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{p\phi^{-1}} X[1]$ in $D(\mathcal{A})$.

Proof. We have a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} H^n X & \xrightarrow{H^n f} & H^n Y & \xrightarrow{H^n i} & H^n(\text{Cone}(f)) & \xrightarrow{H^n p} & H^{n+1} X & \xrightarrow{H^{n+1} f} & H^{n+1} Y \\ \parallel & & \parallel & & \downarrow H^n \phi & (*) & \parallel & & \parallel \\ H^n X & \xrightarrow{H^n f} & H^n Y & \xrightarrow{H^n g} & H^n Z & \xrightarrow{-\delta} & H^{n+1} X & \xrightarrow{H^{n+1} f} & H^{n+1} Y. \end{array}$$

Indeed, for the commutativity of the square $(*)$ we reduce by the Freyd-Mitchell Theorem to the case of modules, and it suffices to note that for $\begin{pmatrix} x \\ y \end{pmatrix} \in Z^n \text{Cone}(f)$, we have $f^n(x) + d^n y = 0$. By the five lemma, $H^n \phi$ is an isomorphism. \square

Definition 2.1.26. Let \mathcal{A} be an additive category. We say that a complex X is **bounded below** (resp. **bounded above**) if $X^n = 0$ for $n \ll 0$ (resp. $n \gg 0$). We say that X is **bounded** if it is bounded below and bounded above. For an interval $I \subseteq \mathbb{Z}$, we say that X is concentrated in degrees in I if $X^n = 0$ for $n \notin I$. We let $C^+(\mathcal{A})$, $C^-(\mathcal{A})$, $C^b(\mathcal{A})$, $C^I(\mathcal{A})$ denote the full subcategories of $C(\mathcal{A})$ consisting of complexes bounded below, bounded above, bounded, concentrated in I , respectively. We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$, $K^I(\mathcal{A})$ denote their respective images in $K(\mathcal{A})$.

For \mathcal{A} abelian, we let $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, resp. $D^b(\mathcal{A})$, resp. $D^I(\mathcal{A})$) denote the full subcategories of $D(\mathcal{A})$ consisting of complexes satisfying $H^n = 0$ for all $n \ll 0$ (resp. $n \gg 0$, resp. $|n| \gg 0$, resp. $n \notin I$).

Proposition 2.1.27. *The functor $H^0: D^{[0,0]}(\mathcal{A}) \rightarrow \mathcal{A}$ is an equivalence of categories.*

Proof. Consider the functor $F: \mathcal{A} \rightarrow D^{[0,0]}(\mathcal{A})$ carrying A to a complex X concentrated in degree 0 with $X^0 = A$. We have $H^0 F A \simeq A$. For any complex X concentrated in degree 0, $X \simeq \tau^{[0,0]} X \simeq F H^0 X$. \square

Derived functors

Let \mathcal{A} and \mathcal{B} be abelian categories and let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. We have remarked that F extends to an additive functor $C(F): C(\mathcal{A}) \rightarrow C(\mathcal{B})$, which induces a triangulated $K(F): K(\mathcal{A}) \rightarrow K(\mathcal{B})$. We have a commutative diagram

$$\begin{array}{ccc} C(\mathcal{A}) & \longrightarrow & K(\mathcal{A}) \\ C(F) \downarrow & & \downarrow K(F) \\ C(\mathcal{B}) & \longrightarrow & K(\mathcal{B}). \end{array}$$

Definition 2.1.28. Let $Q_{\mathcal{A}}: K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ and $Q_{\mathcal{B}}: K^+(\mathcal{B}) \rightarrow D^+(\mathcal{B})$ be the localization functors. A **right derived functor** of F is a pair (RF, ϵ) , where $RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is a triangulated functor, and $\epsilon: Q_{\mathcal{B}}(K^+F) \rightarrow (RF)Q_{\mathcal{A}}$ is a natural transformation of triangulated functors, such that for every such pair (G, η) , there exists a unique natural transformation of triangulated functors $\alpha: RF \rightarrow G$ such that $\eta = (\alpha Q_{\mathcal{A}})\epsilon$.

If RF exists, we put $R^n FK = H^n RFK \in \mathcal{B}$ for $K \in D^+(\mathcal{A})$ (sometimes called the **hypercohomology** of K with respect to RF). The functor $R^n F: \mathcal{A} \rightarrow \mathcal{B}$ is called the n -th right derived functor of F .

In the sequel, we will often abbreviate $C(F)$ and $K(F)$ to F . For F exact, we also let F denote the functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ given by F .

To show the existence of the right derived functor, we need resolutions.

Theorem 2.1.29. Let $\mathcal{J} \subseteq \mathcal{A}$ be a full additive subcategory. Assume that for every object X of \mathcal{A} , there exists a monomorphism $X \rightarrow Y$ with Y in \mathcal{J} .

- (1) For every $K \in C^{\geq n}(\mathcal{A})$, there exist $L \in C^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $f: K \rightarrow L$ such that $\tau^{\geq m} f$ is a monomorphism of complexes for each m .
- (2) The functor $K^+(\mathcal{J}) \rightarrow D^+(\mathcal{A})$ induces an equivalence of triangulated categories

$$K^+(\mathcal{J})[S^{-1}] \rightarrow D^+(\mathcal{A}),$$

where S is the collection of quasi-isomorphisms in $K^+(\mathcal{J})$.

Part (2) follows from part (1) and a general result on localization of triangulated categories (omitted).

Proof of (1). It suffices to construct $L_m = (\cdots \rightarrow L^m \rightarrow 0 \rightarrow \cdots) \in C^{[n,m]}(\mathcal{J})$ and a morphism $f_m: K \rightarrow L_m$ of complexes for each m such that f_m^i and $K^i/B^i K \rightarrow L^i/B^i L$ are monomorphisms for each $i \leq m$, $H^i f_m$ is an isomorphism for each $i < m$, $L_m = \sigma^{\leq m} L_{m+1}$ and f_m equals the composite $K \xrightarrow{f_{m+1}} L_{m+1} \rightarrow L_m$. We proceed by induction on m . For $m < n$, we take $L_m = 0$. Given L_m , we construct L_{m+1} as follows. Form the pushout square

$$\begin{array}{ccc} K^m/B^m K & \longrightarrow & L^m/B^m L \\ \downarrow & & \downarrow \\ K^{m+1} & \longrightarrow & X. \end{array}$$

By induction hypothesis, the upper horizontal arrow is a monomorphism. It follows that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^m/B^m K & \longrightarrow & L^m/B^m L & \longrightarrow & Z \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & K^{m+1} & \longrightarrow & X & \longrightarrow & Z \longrightarrow 0 \end{array}$$

with exact rows. By assumption, there exists a monomorphism $X \rightarrow L^{m+1}$ with L^{m+1} in \mathcal{J} . We define $f^{m+1}: K^{m+1} \rightarrow L^{m+1}$ and $d_L^m: L^m \rightarrow L^{m+1}$ by the obvious

compositions. Then f_{m+1} is a morphism of complexes. It is clear that f^{m+1} is a monomorphism. Applying the snake lemma to the above diagram, we see that $K^{m+1}/B^{m+1}K \rightarrow L^{m+1}/B^{m+1}L$ is a monomorphism and $H^m f$ is an isomorphism.

□

Definition 2.1.30. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ is an additive functor between abelian categories. A full additive subcategory $\mathcal{J} \subseteq \mathcal{A}$ is said to be **F -injective** if it satisfies the following conditions:

- (a) For every $X \in \mathcal{A}$, there exists a monomorphism $X \rightarrow Y$ with $Y \in \mathcal{J}$.
- (b) For every $L \in K^+(\mathcal{J})$ acyclic, FL is acyclic.

The terminology is not completely standard. Our definition here follows [KS, Definitions 10.3.2, 13.3.4]. Some authors replace (b) by the stronger condition (b') below.

Proposition 2.1.31. *Condition (b') below implies (b).*

- (b') Every monomorphism $X' \rightarrow X$ in \mathcal{A} with $X', X \in \mathcal{J}$ can be completed into a short exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

in \mathcal{A} with $X'' \in \mathcal{J}$ such that the sequence

$$0 \rightarrow FX' \rightarrow FX \rightarrow FX'' \rightarrow 0$$

is exact.

Proof. Let $L \in K^+(\mathcal{J})$ be an acyclic complex. Then L breaks into short exact sequences

$$0 \rightarrow Z^n L \rightarrow L^n \rightarrow Z^{n+1} L \rightarrow 0.$$

By (b), one shows by induction on n that $Z^n L$ is isomorphic to an object in \mathcal{J} and we have short exact sequences

$$0 \rightarrow F(Z^n L) \rightarrow F(L^n) \rightarrow F(Z^{n+1} L) \rightarrow 0,$$

so that $K^+(F)(L)$ is acyclic. □

Corollary 2.1.32. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories and let $\mathcal{J} \subseteq \mathcal{A}$ be an F -injective subcategory.*

- (1) *The right derived functor $(RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B}), \epsilon)$ of F exists and for $L \in K^+(\mathcal{J})$, $\epsilon_L: FL \xrightarrow{\sim} RFL$ is an isomorphism. Moreover, RF carries $D^{\geq n}(\mathcal{A})$ into $D^{\geq n}(\mathcal{B})$.*
- (2) *If F is left exact, then the morphism $FX \rightarrow R^0FX$ is an isomorphism for all $X \in \mathcal{A}$.*

Part (1) follows from the theorem.

Proof of (2). Choose a quasi-isomorphism $X \rightarrow L$ with $L \in K^{\geq 0}(\mathcal{J})$, corresponding to an exact sequence

$$0 \rightarrow X \rightarrow L^0 \rightarrow L^1 \rightarrow \cdots.$$

Applying F , we obtain an exact sequence

$$0 \rightarrow FX \rightarrow FL^0 \rightarrow FL^1.$$

Thus $R^0FX \simeq H^0FL \simeq FX$. \square

Corollary 2.1.33. *Let \mathcal{A} be an abelian category with enough injectives. We let \mathcal{I} denote the full subcategory of \mathcal{A} consisting of injective objects.*

- (1) *For every $K \in C^{\geq n}(\mathcal{A})$, there exist $L \in C^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $f: K \rightarrow L$.*
- (2) *The triangulated functor $K^+(\mathcal{I}) \rightarrow D^+(\mathcal{A})$ is an equivalence of triangulated categories.*
- (3) *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. Then \mathcal{I} is F -injective. In particular, the right derived functor $(RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B}), \epsilon)$ of F exists and for $L \in K^+(\mathcal{I})$, $\epsilon_L: FL \xrightarrow{\sim} RFL$ is an isomorphism.*

Proof. This follows from the theorem and Corollary 2.1.32(1). For (2), we need the following lemma. For (3), note that \mathcal{I} satisfies conditions (a) and (b'). Indeed, any short exact sequence of injective objects splits. \square

Lemma 2.1.34. *Let \mathcal{A} be an abelian category. We let \mathcal{I} denote the full subcategory of \mathcal{A} consisting of injective objects. Then any acyclic complex in $K^+(\mathcal{I})$ is isomorphic to zero in $K^+(\mathcal{I})$.*

Proof. Let $L \in K^+(\mathcal{I})$ be an acyclic complex. Then L breaks into short exact sequences

$$0 \rightarrow Z^n L \rightarrow L^n \rightarrow Z^{n+1} L \rightarrow 0.$$

One shows by induction on i that $Z^n L$ is injective and the sequence splits. Thus L^n can be identified with $Z^n \oplus Z^{n+1}$. Then $h^n: Z^n \oplus Z^{n+1} \rightarrow Z^n \rightarrow Z^{n-1} \oplus Z^n$ satisfies $hd + dh = \text{id}_X$. \square

Remark 2.1.35. By the preceding corollary, if \mathcal{A} has small Hom sets and admits enough injectives, then $D^+(\mathcal{A})$ has small Hom sets.

Proposition 2.1.36. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$, $G: \mathcal{B} \rightarrow \mathcal{C}$ be additive functors between abelian categories. Let $\mathcal{J} \subseteq \mathcal{B}$ be a G -injective subcategory. Assume that \mathcal{A} admits enough injectives and $FI \in \mathcal{J}$ for every injective object I of \mathcal{A} . Then the natural transformation $\eta_L: R(GF) \rightarrow (RG)(RF)$ given by the universal property of right derived functors is a natural isomorphism.*

This applies in particular to the case where \mathcal{B} admits enough injectives and F preserves injectives.

Proof. Let \mathcal{I} denote the full subcategory of \mathcal{A} consisting of injective objects. For $L \in K^+(\mathcal{I})$, the composite $(GF)L \xrightarrow{\epsilon_L} R(GF)L \xrightarrow{\eta_L} (RG)(RF)L$ and ϵ_L are both isomorphisms in $D^+(\mathcal{C})$, and hence so is η_L . \square

2.2 Derived direct image

Proposition 2.2.1. *Let (X, \mathcal{O}_X) be a ringed space. Then $\text{Shv}(X, \mathcal{O}_X)$ admits enough injectives.*

Proof. Let $\mathcal{F} \in \text{Shv}(X, \mathcal{O}_X)$. For $x \in X$, let $i_x: \{x\} \rightarrow X$ be the inclusion. Let us show that the canonical morphism $\mathcal{F} \rightarrow \prod_{x \in X} i_{x*} i_x^{-1} \mathcal{F}$ is a monomorphism. For every $y \in X$, we have a commutative diagram

$$\begin{array}{ccc} i_y^{-1} \mathcal{F} & \longrightarrow & i_y^{-1} \prod_{x \in X} i_{x*} i_x^{-1} \mathcal{F} \\ \downarrow \text{id} & \searrow & \downarrow \\ i_y^{-1} \mathcal{F} & \longleftarrow & i_y^{-1} i_{y*} i_y^{-1} \mathcal{F} \end{array}$$

It follows that the top horizontal arrow is injective at every stalk, and hence a monomorphism.

Each $i_x^{-1} \mathcal{F}$ is an $\mathcal{O}_{X,x}$ -module and can be embedded into an injective $\mathcal{O}_{X,x}$ -module $i_x^{-1} \mathcal{F} \hookrightarrow \mathcal{I}_x$. Then $\mathcal{F} \hookrightarrow \prod_{x \in X} i_{x*} i_x^{-1} \mathcal{F} \hookrightarrow \prod_{x \in X} i_{x*} \mathcal{I}_x$. Note that $i_{x*} \mathcal{I}_x$ is injective by the next lemma applied to the adjoint functors $i_x^{-1} \vdash i_{x*}$ with i_x^{-1} exact. We conclude by the fact that a product of injective sheaves is injective. \square

Lemma 2.2.2. *Let $\mathcal{A} \xrightleftharpoons[G]{F}$ be functors between abelian categories with $\mathcal{F} \dashv G$ and F exact. For $X \in \text{Ob}(\mathcal{B})$ injective, $G(X)$ is injective.*

Proof. In fact, $\text{Hom}_{\mathcal{A}}(-, G(X)) \simeq \text{Hom}_{\mathcal{B}}(F-, X)$ is exact. \square

Example 2.2.3. Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a flat morphism of ringed spaces. Then f^{-1} is exact. It follows that f_* sends injective sheaves to injective sheaves.

Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor $f_*: \text{Shv}(X, \mathcal{O}_X) \rightarrow \text{Shv}(Y, \mathcal{O}_Y)$ is left exact. It follows from the proposition that f_* admits a right derived functor Rf_* .

Example 2.2.4. $Y = \text{pt}$, $\mathcal{O}_Y = \mathbb{Z}$. Then

$$\begin{aligned} f_* &= \Gamma(X, -): \text{Shv}(X, \mathcal{O}_X) \rightarrow \text{Ab} \\ Rf_* &= R\Gamma(X, -): D^+(X, \mathcal{O}_X) \rightarrow D^+(\text{Ab}). \end{aligned}$$

For $L \in D^+(X, \mathcal{O}_X)$, we call $H^n(X, L) := R^n f_*(L)$ the i -th (hyper)cohomology of L .

If $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a flat morphism of ringed spaces, then for $M \in D^+(Y, \mathcal{O}_Y)$, there is a natural restriction morphism $R\Gamma(Y, M) \rightarrow R\Gamma(X, f^*M)$.

Proposition 2.2.5. *Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $L \in D^+(X, \mathcal{O}_X)$. Then there is a canonical isomorphism*

$$R^i f_* L \simeq a(V \mapsto H^i(f^{-1}(V), L|_{f^{-1}(V)})$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}(X, \mathcal{O}_X) & \xrightarrow{\iota} & \mathrm{PShv}(X, \mathcal{O}_X) \\ \downarrow f_* & & \downarrow f_*^{\mathrm{psh}} \\ \mathrm{Shv}(Y, \mathcal{O}_Y) & \xleftarrow{a} & \mathrm{PShv}(Y, \mathcal{O}_Y) \end{array}$$

Since a and f_*^{psh} are exact, we have $Rf_* = af_*^{\mathrm{psh}}R\iota$. We conclude by the next lemma, which computes $R\iota$. \square

Lemma 2.2.6. *For $L \in D^+(X, \mathcal{O}_X)$, $R^i\iota L: U \mapsto H^i(U, L|_U)$,*

Proof. Let $L \rightarrow I$ be a quasi-isomorphism with $I \in K^+(X, \mathcal{O}_X)$ and I^i injective for all i . Then $R^i\iota L \simeq H^n(\iota I)$. Since $\Gamma(U, -)$ is exact on presheaves, we have $(H^i\iota I)(U) \simeq H^i(I(U)) \simeq H^i(U, L|_U)$. In the last isomorphism we used the fact that $L|_U \rightarrow I|_U$ is a quasi-isomorphism and $I|_U^i$ is injective for all i . To see this last point, let $j: U \rightarrow X$ be the inclusion. Then j^* preserves injectives, because it has an exact left adjoint, $j_!$, and Lemma 2.2.2 applies. \square

Flabby sheaves

Definition 2.2.7. A sheaf \mathcal{F} on X is said to be **flabby** if for all $U \subseteq X$ open, the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective.

Remark 2.2.8. If \mathcal{F} is flabby, then for any inclusion $V \subseteq U$ of opens, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is surjective. In other words, if \mathcal{F} is flabby, then $\mathcal{F}|_U$ is flabby for every open $U \subseteq X$.

Proposition 2.2.9. *Consider a short exact sequence in $\mathrm{Shv}(X, \mathcal{O}_X)$:*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0.$$

- (1) *If \mathcal{F}' is flabby, then $0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$ is exact.*
- (2) *If \mathcal{F}' and \mathcal{F} are flabby, then so is \mathcal{F}'' .*
- (3) *Injective \mathcal{O}_X -modules are flabby.*

Proof. (1) Take $s \in \mathcal{F}''(X)$. Consider

$$\Omega = \left\{ (U, t) \quad \middle| \quad \begin{array}{l} U \subseteq X \text{ open} \\ t \in \mathcal{F}(U), \quad \psi(t) = s|_U \end{array} \right\}$$

Define a partial order on Ω by $(U, t) \leq (U', t')$ if $U \subseteq U'$ and $t'|_U = t$. By Zorn's lemma, there exists a maximal element (U, t) of Ω . If $U = X$, we are done. Assume $U \subsetneq X$. Take $x \in X \setminus U$. Since $\mathcal{F} \rightarrow \mathcal{F}''$ is surjective, there exists an open neighborhood $V \ni x$ and $r \in \mathcal{F}(V)$ such that $\psi(r) = s|_V$. Since $\psi(t|_{U \cap V}) = \psi(r|_{U \cap V}) = s|_{U \cap V}$, there exists $v \in \mathcal{F}'(U \cap V)$ such that $t|_{U \cap V} - r|_{U \cap V} = \phi(v)$. Since \mathcal{F}' is

flabby, there exists $\bar{v} \in \mathcal{F}'(V)$ such that $\bar{v}|_{U \cap V} = v$. By construction, $t \in \mathcal{F}(U)$ and $r + \phi(\bar{v}) \in \mathcal{F}(V)$ agree on $U \cap V$. Thus they define $\bar{t} \in \mathcal{F}(U \cup V)$ such that $\bar{t}|_U = t$, $\bar{t}|_V = r + \phi(\bar{v})$. Clearly $\psi(\bar{t}) = s|_{U \cup V}$. This shows $(U \cup V, \bar{t}) \in \Omega$ and contradicts the maximality of (U, t) .

(2) Consider the commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}''(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \twoheadrightarrow & \mathcal{F}''(U) \end{array}$$

The left vertical arrow is surjective since \mathcal{F} is flabby. The bottom horizontal arrow is surjective by (1). It follows that the right vertical arrow is surjective.

(3) Let $U \subseteq X$ be an open subset and let $j: U \rightarrow X$ be the inclusion. Since $j_! \mathcal{O}_U \hookrightarrow \mathcal{O}_X$ is a monomorphism and \mathcal{F} is injective, $\text{Hom}(\mathcal{O}_X, \mathcal{F}) \rightarrow \text{Hom}(j_! \mathcal{O}_U, \mathcal{F})$ is surjective. This can be identified with the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ via the adjunction $\text{Hom}(j_! \mathcal{O}_U, \mathcal{F}) \simeq \text{Hom}(\mathcal{O}_U, j^{-1} \mathcal{F}) \simeq \mathcal{F}(U)$. \square

Corollary 2.2.10. *Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then the full subcategory consisting of the flabby \mathcal{O}_X -modules is f_* -injective (Definition 2.1.30).*

Proof. Condition (b') of Proposition 2.1.31 follows from (1) and (2). Condition (a) Definition 2.1.30 follows from (3) and the existence of enough injectives. One can give a more direct proof of (a). For any \mathcal{O}_X -module \mathcal{F} , we have $\mathcal{F} \hookrightarrow \mathcal{G} = \prod_{x \in X} i_{x*} i_x^{-1} \mathcal{F}$, where \mathcal{G} is flabby because $\mathcal{G}(U) = \prod_{x \in U} i_{x*} i_x^{-1} \mathcal{F}$. \square

Remark 2.2.11. It is clear that f_* sends flabby sheaves to flabby sheaves.

Corollary 2.2.12. *Let $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then $R(gf)_* L \simeq Rg_* Rf_* L$ for all $L \in D^+(X, \mathcal{O}_X)$.*

Example 2.2.13. Consider the commutative diagram

$$\begin{array}{ccc} (X, \mathbb{Z}_X) & \xleftarrow{\mu_X} & (X, \mathcal{O}_X) \\ \downarrow f_0 & & \downarrow f \\ (Y, \mathbb{Z}_X) & \xleftarrow{\mu_Y} & (Y, \mathcal{O}_Y) \end{array}$$

Then $\mu_{Y*} Rf_* \simeq Rf_{0*} \mu_{X*}$. Here μ_{X*} is the functor forgetting the \mathcal{O}_X -module structure. In other words, the functor Rf_* does not depend on sheaf of rings.

Theorem 2.2.14 (Grothendieck). *Let X be a Noetherian topological space of finite dimension d . Then for any abelian sheaf \mathcal{F} on X , $H^i(X, \mathcal{F}) = 0$ for $i > d$.*

Remark 2.2.15. By a result of Spaltenstein [S], the derived functor $Rf_*: D(X, \mathcal{O}_X) \rightarrow D(Y, \mathcal{O}_Y)$ between unbounded derived categories exists. We refer to [KS, Chapters 14, 18] for more details.

2.3 Čech cohomology

Let X be a topological space. Let \mathcal{F} be an abelian presheaf on X and let $\{U_i\}_I = \mathfrak{U}$ be an open cover. The sheaf condition is the exactness of the sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_{ij})$$

$$(s_i) \longmapsto s_i|_{U_{ij}} - s_j|_{U_{ij}}$$

where $U_{ij} = U_i \cap U_j$. In Čech cohomology, we extend this sequence to the right.

Definition 2.3.1 (Čech complex). Let \mathcal{F} be a presheaf. The **Čech complex** $C^\bullet(\mathcal{U}, \mathcal{F}) \in C^{\geq 0}(\text{Ab})$ is defined by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0, \dots, i_p}), \quad U_{i_0, \dots, i_p} = \bigcap_{k=0}^p U_{i_k}$$

with the differential given by

$$(d^p s)_{i_0, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_0, \dots, \widehat{i_k}, \dots, i_p}|_{U_{i_0, \dots, i_{p+1}}}$$

for $s \in C^p((U), \mathcal{F})$. One can check $dd = 0$. We call $\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p C^\bullet(\mathcal{U}, \mathcal{F})$ the **Čech cohomology**.

Remark 2.3.2. The global section functor factors as

$$\begin{array}{ccc} \text{Shv}(X) & \xrightarrow{\iota} & \text{PShv}(X) \xrightarrow{\check{H}^0(\mathfrak{U}, -)} \text{Ab} \\ & & \curvearrowright \Gamma(X, -) \end{array}$$

We will show that $\check{H}^p(\mathfrak{U}, -)$ are the right derived functors of $H^0(\mathfrak{U}, -)$.

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Let X be a topological space. Recall for an open cover $\mathcal{U} = \{U_i\}$ of X and an abelian presheaf \mathcal{F} , we have defined the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$. We can extend it to a Čech complex of presheaves $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \in C^{\geq 0}(\mathrm{PShv}(X))$, $\Gamma(V, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})) = \mathcal{C}^\bullet(\mathcal{U} \cap V, \mathcal{F})$, where $\mathcal{U} \cap V = \{U_i \cap V\}$. In other words,

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} j_{i_0, \dots, i_p *} j_{i_0, \dots, i_p}^{-1} \mathcal{F}, \quad j_{i_0, \dots, i_p}: U_{i_0, \dots, i_p} \hookrightarrow X.$$

Let $f: \coprod_i U_i \rightarrow X$. Then $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \mathcal{C}^0(\mathcal{U}, \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})) = \underbrace{f_* f^{-1} \cdots f_* f^{-1}}_{p+1} \mathcal{F}$,

where $f_* f^{-1}$ appears $p+1$ in the expression.

Proposition 2.3.3. *Let \mathcal{F} be a sheaf on X . Then*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

is exact in $\mathrm{Shv}(X)$.

We will prove a more general form of the proposition.

Lemma 2.3.4. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ a functor between abelian categories admitting a right adjoint G . Let $A \in \mathcal{A}$. Consider the complex $L \in C^{\geq -1}(\mathcal{A})$:*

$$0 \longrightarrow A \xrightarrow{d^{-1}} GFA \xrightarrow{d^0} GFGFA \xrightarrow{d^1} \dots$$

where

$$L^p = \underbrace{GF \cdots GF}_{p+1} A$$

$$d^p = \sum_{k=0}^{p+1} (-1)^k \underbrace{GF \cdots GF}_k \epsilon \underbrace{GF \cdots GF}_{p+1-k} A$$

where $\epsilon: \mathrm{id} \rightarrow GF$ is the unit. Then $FL = 0$ in $K(\mathcal{B})$.

Proof. Define $h \in \mathrm{Ht}(FL, FL)$ as follows. Let $\eta: FG \rightarrow \mathrm{id}$ be the counit. We take

$$h^p = \eta F \underbrace{GF \cdots GF}_p A: FL^p \rightarrow FL^{p-1}$$

One checks that $dh + hd = \mathrm{id}$. □

Proposition 2.3.5. *Let $f: Y \rightarrow X$ be a surjective continuous map and let \mathcal{F} be a sheaf on X . Define $\mathcal{C}^p(f, \mathcal{F}) := \underbrace{f_* f^{-1} \cdots f_* f^{-1}}_{p+1} \mathcal{F}$. Then*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0(f, \mathcal{F}) \rightarrow \mathcal{C}^1(f, \mathcal{F}) \rightarrow \dots$$

is an exact sequence.

Proof. Take $F = f^{-1}$, $G = f_*$ in the lemma. Then $f^{-1}(L)$ is acyclic. Since f is surjective, L is acyclic. \square

In fact, in the lemma, L is acyclic if F is conservative.

Example 2.3.6. Let $f: Y = \coprod_{x \in X} x \rightarrow X$. Then $f^{-1}f_*\mathcal{F} = \prod_{x \in X} i_{x*}i_x^{-1}\mathcal{F}$. This is used in the proof of the existence of enough flabby sheaves. The flabby resolution given by the proposition is called **Godement resolution**. Note that every sheaf on Y is flabby and f_* preserves flabby sheaves.

Corollary 2.3.7. Let \mathcal{F} be a flabby sheaf on X . Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$ and open cover \mathcal{U} of X .

Proof. Let $f: \coprod_i U_i \rightarrow X$. Since f_* and f^{-1} both preserve flabby sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

is a flabby resolution of \mathcal{F} . Taking global sections, we get the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \dots$$

\square

Let \mathcal{F} be a sheaf. Then we can replace \mathcal{F} by $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ in $\mathcal{D}(\mathrm{Shv}(X))$. In general, $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is not flabby. We can choose a quasi-isomorphism $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow L^\bullet$ with $L \in K^+$ and L^p injective or flabby for all p . This gives a canonical homomorphism

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}),$$

which is an isomorphism for $p = 0$. We have the following criterion for the map to be an isomorphism.

Theorem 2.3.8 (Leray). *Let \mathcal{F} be a sheaf. Assume $H^n(U_{i_0, \dots, i_p}, \mathcal{F}) = 0$ for all $p \geq 0$, $(i_0, \dots, i_p) \in I^{p+1}$, $n \geq 1$. Then the canonical map $\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$ is an isomorphism for all $p \geq 0$.*

We will give a proof based on an interpretation of the Čech cohomology as derived Hom. For this we need more homological algebra.

Double complexes

Let \mathcal{A} be an additive category.

Definition 2.3.9. We define the category of **double complexes** in \mathcal{A} to be $C^2(\mathcal{A}) = C(C(\mathcal{A}))$. Thus a double complex consists of objects $X^{i,j}$ for $i, j \in \mathbb{Z}$ and differentials $d_I: X^{i,j} \rightarrow X^{i+1,j}$, $d_{II}: X^{i,j} \rightarrow X^{i,j+1}$ such that $d_I^2 = 0$, $d_{II}^2 = 0$, $d_Id_{II} = d_{II}d_I$.

Definition 2.3.10. Let X be a double complex in \mathcal{A} . We define two complexes in \mathcal{A} with $(\text{tot}_\oplus X)^n = \bigoplus_{i+j=n} X^{i,j}$ (if the coproducts exist) and $(\text{tot}_\Pi X)^n = \prod_{i+j=n} X^{i,j}$ (if the products exist), called **total complex** of X with respect to coproducts and products, respectively. The differentials are defined as follows. Let $i + j = n$. The composition $X^{i,j} \rightarrow (\text{tot}_\oplus X)^n \xrightarrow{d^n} (\text{tot}_\oplus X)^{n+1}$ is given by

$$(2.3.1) \quad d_I^{i,j} + (-1)^i d_{II}^{i,j}.$$

The composition $(\text{tot}_\Pi X)^{n-1} \xrightarrow{d^{n-1}} (\text{tot}_\Pi X)^n \rightarrow X^{i,j}$ is given by

$$(2.3.2) \quad d_I^{i-1,j} + (-1)^i d_{II}^{i,j-1}.$$

Remark 2.3.11. The sign in (2.3.1) and (2.3.2) ensures that $d^2 = 0$. If Y is the transpose of X defined by $Y^{i,j} = X^{j,i}$ and by swapping the two differentials, then we have an isomorphism $\text{tot}_\oplus X \simeq \text{tot}_\oplus Y$ given by $(-1)^{ij} \text{id}_{X^{i,j}}$. The same holds for tot_Π .

In the literature, a variant of Definition 2.3.9 with $d_I d_{II} + d_{II} d_I = 0$ is sometimes used. If we adopt this variant, then (2.3.1) can be simplified to $d = d_I + d_{II}$. The two definitions correspond to each other by multiplying $d_{II}^{i,j}$ by the sign $(-1)^i$.

Definition 2.3.12. We say that a double complex X is **biregular** if for every n , $X^{i,j} = 0$ for all but finitely many pairs (i, j) with $i + j = n$. We let $C_{\text{reg}}^2(\mathcal{A}) \subseteq C^2(\mathcal{A})$ denote full subcategory consisting of biregular double complexes. It is an additive subcategory.

If $X^{i,j} = 0$ for $i < a$ or $j < b$ (X concentrated in a (translated) first quadrant) or $X^{i,j} = 0$ for $i > a$ or $j > b$ (X concentrated in a (translated) third quadrant), then X is biregular. If $X^{i,j} = 0$ for $|i| \gg 0$ (concentrated in a vertical stripe) or $X^{i,j} = 0$ for $|j| \gg 0$ (concentrated in a horizontal stripe), then X is biregular.

Remark 2.3.13. If X is a biregular double complex, then $\text{tot}_\oplus X$ and $\text{tot}_\Pi X$ exist and we have $\text{tot}_\oplus X \xrightarrow{\sim} \text{tot}_\Pi X$. We will simply write $\text{tot} X$. We get an additive functor $\text{tot}: C_{\text{reg}}^2(\mathcal{A}) \rightarrow C(\mathcal{A})$.

Example 2.3.14. Let $f: L \rightarrow M$ be a morphism of complexes in \mathcal{A} . We define a double complex X by $X^{-1,j} = L^j$, $X^{0,j} = M^j$, $X^{i,j} = 0$ for $i \neq -1, 0$, $d_I^{-1,j} = f^j$, d_{II} given by d_L and d_M . Then $\text{tot} X = \text{Cone}(f)$.

Let \mathcal{A} be an abelian category. For a double complex X in \mathcal{A} , we put

$$H_I(X)^{i,j} = \text{Ker}(d_I^{i,j})/\text{im}(d_I^{i-1,j}), \quad H_{II}(X)^{i,j} = \text{Ker}(d_{II}^{i,j})/\text{im}(d_{II}^{i,j-1}).$$

The full additive subcategory $C_{\text{reg}}^2(\mathcal{A}) \subseteq C^2(\mathcal{A})$ is stable under subobjects and quotients. Thus $C_{\text{reg}}^2(\mathcal{A})$ is an abelian category and the inclusion functor is exact. The functor $\text{tot}: C_{\text{reg}}^2(\mathcal{A}) \rightarrow C(\mathcal{A})$ is exact.

Proposition 2.3.15. Let X be a biregular double complex such that $H_I^{i,\bullet}(X)$ is acyclic for every i . Then $\text{tot} X$ is acyclic.

A similar statement holds for H_{II} , which generalizes the fact that the cone of a quasi-isomorphism is acyclic.

Proof. For each m , there exists N such that $H^m(\text{tot}X) = H^m\text{tot}(\tau_I^{\leq n}X)$ for all $n \geq N$. It suffices to show that $H^m\text{tot}(\tau_I^{\leq n}X) = 0$ for all n . We proceed by induction on n (for a fixed m). For $n \ll 0$, $(\text{tot}(\tau_I^{\leq n}X))^m = 0$. Assume that $H^m\tau_I^{\leq n-1}(X) = 0$ and consider the short exact sequence of double complexes

$$0 \rightarrow \tau_I^{\leq n-1}X \rightarrow \tau_I^{\leq n}X \rightarrow Y \rightarrow 0,$$

where $Y = (B_I^{n,\bullet}X \xrightarrow{f} Z_I^{n,\bullet}X)$ is concentrated on the columns $n-1$ and n . Applying tot , we get an exact sequence of complexes

$$0 \rightarrow \text{tot}\tau_I^{\leq n-1}X \rightarrow \text{tot}\tau_I^{\leq n}X \rightarrow \text{tot}Y \rightarrow 0.$$

We have a quasi-isomorphism $\text{tot}(Y)[n] \simeq \text{Cone}((-1)^nf) \rightarrow H_I^{n,\bullet}(X)$. It follows $\text{tot}Y$ is acyclic. Taking long exact sequence, we get

$$H^m\text{tot}\tau_I^{\leq n}X \simeq H^m\text{tot}\tau_I^{\leq n-1}X = 0.$$

□

Corollary 2.3.16. *Let X be a biregular double complex such that $X^{\bullet,j}$ is acyclic for every j (namely, every row of X is acyclic). Then $\text{tot}X$ is acyclic.*

A similar statement holds for columns of X : if $X^{i,\bullet}$ is acyclic for every i , then $\text{tot}X$ is acyclic.

Corollary 2.3.17. *Let $f: X \rightarrow Y$ be a morphism of biregular double complexes such that $H_I^{i,\bullet}(f): H_I^{i,\bullet}(X) \rightarrow H_I^{i,\bullet}(Y)$ is a quasi-isomorphism for each i . Then $\text{tot}(f): \text{tot}(X) \rightarrow \text{tot}(Y)$ is a quasi-isomorphism.*

Proof. We let $W = \text{Cone}_{II}(f)$ with $W^{i,j} = X^{i,j+1} \oplus Y^{i,j}$. Then $H_I^{i,\bullet}(W) \simeq \text{Cone}(H_I^{i,\bullet}(f))$ is acyclic. By the proposition applied to W , $\text{tot}(W) \simeq \text{Cone}(\text{tot}(f))$ is acyclic. □

Corollary 2.3.18. *Let $f: X \rightarrow Y$ be a morphism of biregular double complexes such that $f^{\bullet,j}: X^{\bullet,j} \rightarrow Y^{\bullet,j}$ is a quasi-isomorphism for each j . Then $\text{tot}(f): \text{tot}(X) \rightarrow \text{tot}(Y)$ is a quasi-isomorphism.*

Derived Hom

Let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be additive categories. Let $F: \mathcal{A} \times \mathcal{A}' \rightarrow \mathcal{A}''$ be a functor that is additive in each variable. Then F extends to a functor $C^2(F): C(\mathcal{A}) \times C(\mathcal{A}') \rightarrow C^2(\mathcal{A}'')$ additive in each variable. For $X \in C(\mathcal{A}), Y \in C(\mathcal{A}')$, the double complex $C^2(F)(X, Y)$ is defined by $C^2(F)(X, Y)^{i,j} = F(X^i, Y^j)$, with $d_I^{i,j} = F(d_X^i, \text{id}_{Y^j})$, $d_{II}^{i,j} = F(\text{id}_{X^i}, d_Y^j)$.

Example 2.3.19. Let \mathcal{A} be an additive category with small Hom sets. The functor $\text{Hom}_{\mathcal{A}}: \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Ab}$ is additive in each variable. We have an isomorphism $C(\mathcal{A})^{\text{op}} \simeq C(\mathcal{A}^{\text{op}})$, carrying (X, d) to $((X^{-n}), (-1)^n d^{-n-1})$. Thus $\text{Hom}_{\mathcal{A}}$ extends to a functor

$$\text{Hom}_{\mathcal{A}}^{\bullet\bullet}: C(\mathcal{A})^{\text{op}} \times C(\mathcal{A}) \rightarrow C^2(\text{Ab}),$$

additive in each variable. For $X, Y \in C(\mathcal{A})$, $\text{Hom}_{\mathcal{A}}^{\bullet\bullet}(X, Y)^{i,j} = \text{Hom}_{\mathcal{A}}(X^{-j}, Y^i)$, with

$$d_I^{i,j} = \text{Hom}_{\mathcal{A}}(X^{-j}, d_Y^i), \quad d_{II}^{i,j} = \text{Hom}_{\mathcal{A}}((-1)^j d_X^{-j-1}, Y^i).$$

We define $\text{Hom}_{\mathcal{A}}^\bullet$ as the composite functor

$$C(\mathcal{A})^{\text{op}} \times C(\mathcal{A}) \xrightarrow{\text{Hom}_{\mathcal{A}}^{\bullet\bullet}} C^2(\text{Ab}) \xrightarrow{\text{tot}_{\Pi}} C(\text{Ab}).$$

We have

$$\text{Hom}_{\mathcal{A}}^\bullet(X, Y)^n = \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(X^j, Y^{n+j}),$$

and for $f = (f^j) \in \text{Hom}_{\mathcal{A}}^\bullet(X, Y)^n$,

$$(d^n f)^j = d_Y^{j+n} f^j + (-1)^{n+1} f^{j+1} d_X^j.$$

Proposition 2.3.20. *We have*

$$\begin{aligned} Z^0 \text{Hom}_{\mathcal{A}}^\bullet(X, Y) &\simeq \text{Hom}_{C(\mathcal{A})}(X, Y), \\ B^0 \text{Hom}_{\mathcal{A}}^\bullet(X, Y) &\simeq \text{im}(\text{Ht}(X, Y) \rightarrow \text{Hom}_{C(\mathcal{A})}(X, Y)), \\ H^0 \text{Hom}_{\mathcal{A}}^\bullet(X, Y) &\simeq \text{Hom}_{K(\mathcal{A})}(X, Y). \end{aligned}$$

Proof. We have $d^0(f) = df - fd$, so that $d^0(f) = 0$ if and only if $f: X \rightarrow Y$ is a morphism of complexes. We have $\text{Ht}(X, Y) = \text{Hom}_{\mathcal{A}}^\bullet(X, Y)^{-1}$, and for $h \in \text{Ht}(X, Y)$, $d^{-1}(h) = dh + hd$. \square

Definition 2.3.21. Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be triangulated categories. A **triangulated bifunctor** is a functor $F: \mathcal{D} \times \mathcal{D}' \rightarrow \mathcal{D}''$ equipped with natural isomorphisms $F(X[1], Y) \simeq F(X, Y)[1]$, $F(X, Y[1]) \simeq F(X, Y)[1]$, such that the following diagram anticommutes

$$\begin{array}{ccc} F(X[1], Y[1]) & \longrightarrow & F(X, Y[1])[1] \\ \downarrow & & \downarrow \\ F(X[1], Y)[1] & \longrightarrow & F(X, Y)[2] \end{array}$$

and such that F is triangulated in each variable.

Note that Hom^\bullet factorizes through a triangulated bifunctor $K(\mathcal{A})^{\text{op}} \times K(\mathcal{A}) \rightarrow K(\text{Ab})$.

Proposition 2.3.22. *Assume that \mathcal{A} admits enough injectives. Then the triangulated bifunctor*

$$\text{Hom}_{\mathcal{A}}^\bullet: K(\mathcal{A})^{\text{op}} \times K^+(\mathcal{A}) \rightarrow K(\text{Ab})$$

admits a right derived bifunctor

$$R \text{Hom}_{\mathcal{A}}: D(\mathcal{A})^{\text{op}} \times D^+(\mathcal{A}) \rightarrow D(\text{Ab})$$

such that, for $M \in K^+(\mathcal{A})$ with injective components and $L \in K(\mathcal{A})$, we have

$$\text{Hom}_{\mathcal{A}}^\bullet(L, M) \xrightarrow{\sim} R \text{Hom}_{\mathcal{A}}(L, M).$$

Sketch of proof. We need to show that for $L \in K(\mathcal{A})$, $M \in K^+(\mathcal{A})$, M^n injective for all n , with L or M acyclic, then $\text{Hom}_{\mathcal{A}}^\bullet(L, M)$ is acyclic. Indeed,

$$H^n \text{Hom}_{\mathcal{A}}^\bullet(L, M) \simeq \text{Hom}_{K(\mathcal{A})}(L, M[n]) \simeq \text{Hom}_{D(\mathcal{A})}(L, M[n]) = 0.$$

□

Remark 2.3.23. Assume that \mathcal{A} has enough injectives. For $L \in D(\mathcal{A})$, $M \in D^+(\mathcal{A})$, we have

$$H^n R \text{Hom}_{\mathcal{A}}(L, M) \simeq H^n \text{Hom}_{\mathcal{A}}^\bullet(L, M') \simeq \text{Hom}_{K(\mathcal{A})}(L, M'[n]) \simeq \text{Hom}^n(L, M[n]),$$

where we have taken a quasi-isomorphism $M \rightarrow M' \in K^+(\mathcal{A})$ such that M' has injective components. In particular, for $X \in \mathcal{A}$, $\text{Hom}_{D(\mathcal{A})}(X, -[n])$ is the n -th right derived functor of $\text{Hom}(X, -)$.

Dually, we have the following.

Proposition 2.3.24. *Assume that \mathcal{A} admits enough projectives. Then the triangulated bifunctor*

$$\text{Hom}_{\mathcal{A}}^\bullet: K^-(\mathcal{A})^{\text{op}} \times K(\mathcal{A}) \rightarrow K(\text{Ab}).$$

admits a right derived bifunctor

$$R \text{Hom}_{\mathcal{A}}: D^-(\mathcal{A})^{\text{op}} \times D(\mathcal{A}) \rightarrow D(\text{Ab})$$

such that for $L \in K^-(\mathcal{A})$ with projective components and $M \in K(\mathcal{A})$, we have

$$\text{Hom}_{\mathcal{A}}^\bullet(L, M) \xrightarrow{\sim} R \text{Hom}_{\mathcal{A}}(L, M).$$

Remark 2.3.25. In the case where \mathcal{A} admits enough injectives and enough projectives, the functors $R \text{Hom}$ defined in Propositions 2.3.22 and 2.3.24 are isomorphic when restricted to $D^-(\mathcal{A})^{\text{op}} \times D^+(\mathcal{A})$. Indeed, for $L \in D^-(\mathcal{A})$ and $M \in D^+(\mathcal{A})$, $R \text{Hom}(L, M)$ can be computed by finding quasi-isomorphisms $L' \rightarrow L$ and $M \rightarrow M'$ such that L' has projective components and M' has injective components and taking $\text{Hom}^\bullet(L, M)$.

Back to Čech cohomology

Let \mathcal{F} be a presheaf. We have

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0, \dots, i_p}) \simeq \text{Hom}(\mathcal{C}_p(\mathcal{U}), \mathcal{F}),$$

where

$$\mathcal{C}_p(\mathcal{U}) = \bigoplus_{i_0, \dots, i_p \in I^{p+1}} j_{i_0, \dots, i_p}^{\text{psh}} \mathbb{Z}_{U_{i_0, \dots, i_p}}^{\text{psh}}, \quad j_{i_0, \dots, i_p}: U_{i_0, \dots, i_p} \hookrightarrow X.$$

Here $\mathbb{Z}_{U_{i_0, \dots, i_p}}^{\text{psh}}$ denotes the constant presheaf. There is a complex $\mathcal{C}_\bullet(\mathcal{U})$ in $C^{\leq 0}$ with $\mathcal{C}_\bullet(\mathcal{U})^{-p} = \mathcal{C}_p(\mathcal{U})$ satisfying $C^\bullet(\mathcal{U}, \mathcal{F}) \simeq \text{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), \mathcal{F})$. To specify the differentials and to study this complex, it is convenient to consider the functor

$$f_!^{\text{psh}} \mathcal{G} = \bigoplus_{i \in I} j_i^{\text{psh}}(\mathcal{G}|_{U_i})$$

between categories of presheaves, where $f: \coprod_{i \in I} U_i \rightarrow X$. This functor is left adjoint to f^{-1} , and we have the counit $\eta_{\mathcal{U}}: f_!^{\text{psh}} f^{-1} \rightarrow \text{id}$. Note that $(f_!^{\text{psh}} f^{-1} \mathcal{F})(U) \simeq \bigoplus_{U \subseteq U_i} \mathcal{F}(U_i)$. Moreover,

$$\mathcal{C}_p(\mathcal{U}) \simeq \underbrace{f_!^{\text{psh}} f^{-1} \dots f_!^{\text{psh}} f^{-1}}_{p+1} \mathbb{Z}_X^{\text{psh}}.$$

We define the differentials of $\mathcal{C}_{\bullet}(\mathcal{U})$ by

$$d^{-p} = \sum_{k=0}^p (-1)^{k+1} \underbrace{f_!^{\text{psh}} f^{-1} \dots f_!^{\text{psh}} f^{-1}}_k \eta_{\mathcal{U}} \underbrace{f_!^{\text{psh}} f^{-1} \dots f_!^{\text{psh}} f^{-1}}_{p-k}.$$

Lemma 2.3.26. (1) *The sequence*

$$(*) \quad \dots \longrightarrow \mathcal{C}_1(\mathcal{U}) \longrightarrow \mathcal{C}_0(\mathcal{U}) \xrightarrow{\eta_{\mathcal{U}}} \mathbb{Z}_X^{\text{psh}}$$

is exact.

(2) $\mathcal{C}_p(\mathcal{U})$ *is projective for each $p \geq 0$.*

Proof. (2) $\text{Hom}(\mathbb{Z}_X^{\text{psh}}, -) = \Gamma(X, -)$ is exact, which implies that $\mathbb{Z}_X^{\text{psh}}$ is projective. Moreover, since $f_!^{\text{psh}} \dashv f^{-1} \dashv f_*$ and $f_!$ and f_* are exact, the functors $f_!^{\text{psh}}$ and f^{-1} preserve projectives.

(1) $f^{-1}(*)$ is exact by Lemma 2.3.4. Thus $(*)|_{U_i}$ is exact for every $i \in I$. Let $U \subseteq X$ be an open subset. If there exists an $i \in I$ such that $U \subseteq U_i$, then $\Gamma(U, (*))$ is exact. Otherwise, $\Gamma(U, (*)) = (\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z})$, which is exact. \square

From the Lemma, we see $\mathcal{C}_{\bullet}(\mathcal{U})$ is a projective resolution of $\text{im}(\eta_{\mathcal{U}})$, and hence

$$C^{\bullet}(\mathcal{U}, \mathcal{F}) \simeq \text{Hom}^{\bullet}(\mathcal{C}_{\bullet}(\mathcal{U}), \mathcal{F}) \simeq R \text{Hom}(\text{Im}(\eta_{\mathcal{U}}), \mathcal{F}) \simeq R\check{\Gamma}(\mathcal{U}, \mathcal{F}),$$

where $R\check{\Gamma}(\mathcal{U}, -)$ denotes the right derived functor of $\check{H}^0(\mathcal{U}, -)$. For the last isomorphism we note

$$\text{Hom}(\text{im}(\eta_{\mathcal{U}}), \mathcal{F}) \simeq \check{H}^0(\mathcal{U}, \mathcal{F}),$$

which implies that for $L \in D^+(\text{PShv}(X))$, we have

$$R \text{Hom}(\text{im}(\eta_{\mathcal{U}}), L) \simeq R\check{\Gamma}(\mathcal{U}, L)$$

Consider

$$\begin{array}{ccc} & \text{PShv}(X) & \\ \iota \nearrow & & \searrow \check{H}^0(\mathcal{U}, -) \\ \text{Shv}(X) & \xrightarrow{\Gamma(X, -)} & \text{Ab} \end{array}$$

Lemma 2.3.27. $R\check{\Gamma}(\mathcal{U}, R\iota L) \simeq R\Gamma(X, L)$, $\forall L \in D^+(\text{Shv}(X))$.

Proof. Since $a \dashv \iota$ and a is exact, ι preserves injective objects. \square

For a sheaf \mathcal{F} , the canonical morphism $\iota\mathcal{F} \rightarrow R\iota\mathcal{F}$ induces

$$R\check{\Gamma}(\mathcal{U}, \iota\mathcal{F}) \rightarrow R\check{\Gamma}(\mathcal{U}, R\iota\mathcal{F}) \simeq R\Gamma(X, \mathcal{F}),$$

which in turn induces the maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$ by taking cohomology.

We are now in a position to prove Leray's theorem.

Proof of Leray's theorem. Let $L = R\iota\mathcal{F}$. For $p \geq 0$, consider the morphism of complexes

$$\begin{array}{ccc} \mathrm{Hom}^{p,\bullet}(\mathcal{C}_\bullet(\mathcal{U}), \iota\mathcal{F}) & \longrightarrow & \mathrm{Hom}^{p,\bullet}(\mathcal{C}(\mathcal{U}), L) \\ \| & & \| \\ \prod_{(i_0, \dots, i_p)} \Gamma(U_{i_0, \dots, i_p}, \iota\mathcal{F}) & & \prod_{(i_0, \dots, i_p)} \Gamma(U_{i_0, \dots, i_p}, L) \end{array}$$

By assumption, $H^n(U_{i_0, \dots, i_p}, \mathcal{F}) = 0$, $n \geq 1$, which means

$$\Gamma(U_{i_0, \dots, i_p}, \iota\mathcal{F}) \rightarrow \Gamma(U_{i_0, \dots, i_p}, L)$$

is a quasi-isomorphism. Thus

$$\mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), \iota\mathcal{F}) \rightarrow \mathrm{Hom}^\bullet(\mathcal{C}_\bullet(\mathcal{U}), L)$$

is also a quasi-isomorphism. Therefore,

$$R\check{\Gamma}(\mathcal{U}, \iota\mathcal{F}) \simeq R\check{\Gamma}(\mathcal{U}, L).$$

□

Proposition 2.3.28. *Let \mathcal{F} be a sheaf.*

(1) *We have an exact sequence*

$$0 \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, R^1\iota\mathcal{F}).$$

(2) *We have*

$$\mathrm{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, R^1\iota\mathcal{F}) = 0,$$

where \mathcal{U} runs through open covers of X . In particular,

$$\mathrm{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F}).$$

Proof. (1) This follows from Lemma 2.3.29 applied to $R\Gamma(X, -)$.

(2) Let $\mathcal{F} \xrightarrow{\sim} L$ be a quasi-isomorphism with $L \in K^+$ and L^i injective. Then in $\mathrm{PShv}(X)$,

$$R^q\iota\mathcal{F} = \ker(\iota L^q \rightarrow \iota L^{q+1}) / \mathrm{im}(\iota L^{q-1} \rightarrow \iota L^q).$$

For $q > 0$, $aR^q\iota\mathcal{F} \simeq \mathcal{H}^q\mathcal{F} = 0$. Here \mathcal{H}^q denotes the q -th cohomology sheaf. We conclude by Lemma 2.3.30 below. □

Lemma 2.3.29. *Let $F: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ be a triangulated functor carrying $D^{\geq 0}(\mathcal{A})$ into $D^{\geq 0}(\mathcal{B})$. Let $X \in D^{\geq 0}(\mathcal{A})$. We have an isomorphism $H^0 FH^0 X \simeq H^0 FX$ and an exact sequence*

$$0 \rightarrow H^1 FH^0 X \rightarrow H^1 FX \rightarrow H^0 FH^1 X \rightarrow H^2 FH^0 X \rightarrow H^2 FX.$$

We leave this as an exercise.

Lemma 2.3.30. *Let \mathcal{F} be a presheaf. Then the canonical map*

$$\operatorname{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}) \rightarrow \Gamma(X, a\mathcal{F})$$

is injective.

Proof. By definition, $a\mathcal{F} = (\mathcal{F}')'$, where

$$\mathcal{F}'(U) = \operatorname{colim}_{\mathcal{V}} \check{H}^0(\mathcal{V}, \mathcal{F}),$$

with \mathcal{V} running through open covers of U . Since \mathcal{F}' is separated, $\mathcal{F}' \rightarrow (\mathcal{F}')' = a\mathcal{F}$ is a monomorphism of presheaves. In particular, $\Gamma(X, \mathcal{F}') = \operatorname{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}) \rightarrow \Gamma(X, a\mathcal{F})$ is injective. \square

Remark 2.3.31. The map

$$\operatorname{colim}_{\mathcal{U}} \check{H}^2(\mathcal{U}, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$$

is injective but not bijective in general. Using hypercovers one can get isomorphisms to H^q all q .

Remark 2.3.32 (Alternating Čech complex). The following variant of the Čech complex is very useful. For a presheaf \mathcal{F} on X and an open cover $\mathcal{U} = \{U_i\}_{i \in I}$, we define a subcomplex $C_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}) \subseteq C^\bullet(\mathcal{U}, \mathcal{F})$, called the **alternating Čech complex**. An element $s = (s_{i_0, \dots, i_p}) \in \prod_{i_0, \dots, i_p \in I} \mathcal{F}(U_{i_0, \dots, i_p}) = C^p(\mathcal{U}, \mathcal{F})$ is said to be **alternating** if

$$\begin{cases} s_{i_0, \dots, i_p} = 0 & \text{if } i_j = i_k, \\ s_{i_\sigma(0), \dots, i_\sigma(p)} = \operatorname{sgn}(\sigma) s_{i_0, \dots, i_p} & \text{for } \sigma \in \operatorname{Aut}\{0, \dots, p\}. \end{cases}$$

We let $C_{\text{alt}}^p(\mathcal{U}, \mathcal{F}) \subseteq C^p(\mathcal{U}, \mathcal{F})$ denote the abelian subgroup consisting of the alternating elements. If we choose a total order on I , then $C_{\text{alt}}^p(\mathcal{U}, \mathcal{F}) \simeq \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots, i_p})$.

There are natural chain morphisms

$$C_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}) \xrightleftharpoons[r]{i} C^\bullet(\mathcal{U}, \mathcal{F})$$

where i is the inclusion and r is given by projection. We have $ri = \operatorname{id}$ and one can check that $ir - \operatorname{id} = dh + hd$ for some homotopy h . Thus i is a homotopy equivalence and we have

$$H^q C_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} \check{H}^q(\mathcal{U}, \mathcal{F}).$$

In particular, for $p \geq \#I$, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$, since $C_{\text{alt}}^p(\mathcal{U}, \mathcal{F}) = 0$.

Date: 11.26

2.4 Serre's theorem on affine schemes

Theorem 2.4.1 (Serre). *Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module on an affine scheme X . Then $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$.*

Lemma 2.4.2. *Let X be an affine scheme and let \mathcal{U} be a finite affine open cover. Let \mathcal{F} be a quasi-coherent sheaf. Then $\check{H}^q(\mathcal{U}, \mathcal{F}) = 0$, for all $q \geq 1$.*

Proof. We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots.$$

Each $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = f_* f^{-1} \cdots f_* f^{-1} \mathcal{F}$ is quasi-coherent, where $f: \coprod_i U_i \rightarrow X$. The functor $\Gamma(X, -)$ carries exact sequences of quasi-coherent sheaves to exact sequences of modules. Therefore,

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

is exact. □

Lemma 2.4.3. *Let $\mathcal{J} \subseteq \text{Shv}(X, \mathcal{O}_X)$ be the full subcategory consisting of \mathcal{O}_X -modules \mathcal{F} such that for every affine over subset $U \subseteq X$ and every finite affine open cover \mathcal{V} of U , we have $\check{H}^q(\mathcal{V}, \mathcal{F}) = 0$ for all $q \geq 1$. Then \mathcal{J} is $\Gamma(X, -)$ -injective.*

Proof. We check the axioms (a) and (b').

(a) It suffices to show that every injective \mathcal{O}_X -module \mathcal{F} belongs to \mathcal{J} . For every open subset $U \subseteq X$, $\mathcal{F}|_U$ is flabby. It follows that we have $\check{H}^q(\mathcal{V}, \mathcal{F}) = 0$ for all \mathcal{V} and all $q \geq 1$ by Corollary 2.3.7. Thus \mathcal{F} belongs to \mathcal{J} .

(b') Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence of \mathcal{O}_X -modules with $\mathcal{F}, \mathcal{G} \in \mathcal{J}$. Let $U \subseteq X$ be an affine open. We have

$$H^1(U, \mathcal{F}) \simeq \operatorname{colim}_{\mathcal{V}} \check{H}^1(\mathcal{V}, \mathcal{F}) = 0.$$

Here \mathcal{V} runs through finite affine open covers of U and we used the fact that every open cover of U can be refined by a cover \mathcal{V} . Thus the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{Q}(U) \longrightarrow 0$$

is exact. In particular, the sequence

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G}) \rightarrow \Gamma(X, \mathcal{Q}) \rightarrow 0$$

is exact. Moreover, we have a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{G}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{Q}) \longrightarrow 0,$$

which induces the exact sequence

$$\begin{array}{ccccccc} \check{H}^q(\mathcal{U}, \mathcal{G}) & \longrightarrow & \check{H}^q(\mathcal{V}, \mathcal{Q}) & \longrightarrow & \check{H}^{q+1}(\mathcal{V}, \mathcal{F}) \\ \parallel & & & & \parallel \\ 0 & & & & 0 \end{array}$$

for $q \geq 1$. Thus $\mathcal{Q} \in \mathcal{J}$. \square

Proof of Theorem 2.4.1. By Lemma 2.4.2, $\mathrm{QCoh}(X) \subseteq \mathcal{J}$. By Lemma 2.4.3, for all $\mathcal{F} \in \mathcal{J}$, we have $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$. \square

We have the following converse of Theorem 2.4.1.

Theorem 2.4.4 (Serre). *Let X be a quasi-compact scheme. Suppose that for all quasi-coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, we have $H^1(X, \mathcal{I}) = 0$. Then X is affine.*

We will prove this later as a consequence of Theorem 2.5.9.

Combine Theorem 2.4.1 and Leray's theorem, we obtain:

Corollary 2.4.5. *Let X be a scheme and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X such that U_{i_0, \dots, i_p} is affine for all $p \geq 0$. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then, for all q , the canonical map $\check{H}^q(\mathcal{U}, \mathcal{F}) \rightarrow H^q(X, \mathcal{F})$ is an isomorphism.*

Remark 2.4.6. If the diagonal $\Delta_X: X \rightarrow X \times X$ is affine (for example if X is separated), then for all affine open $U, V \subseteq X$, $U \cap V$ is affine. Indeed

$$\begin{array}{ccc} U \cap V & \longrightarrow & X \\ \downarrow & & \downarrow \Delta_X \\ U \times V & \longrightarrow & X \times X \end{array}$$

is an Cartesian square. In this case, the corollary applies to every affine open cover of X .

Corollary 2.4.7. *Let X be a scheme and let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

be an exact sequence of \mathcal{O}_X -modules. If $\mathcal{F}, \mathcal{Q} \in \mathrm{QCoh}(X)$, then $\mathcal{G} \in \mathrm{QCoh}(X)$.

Proof. We may assume that X is affine. Then the long exact sequence of cohomology has the form

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{Q}(X) \longrightarrow H^1(X, \mathcal{F}) = 0.$$

Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)^\sim & \longrightarrow & \mathcal{G}(X)^\sim & \longrightarrow & \mathcal{Q}(X)^\sim \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{Q} \longrightarrow 0. \end{array}$$

The vertical arrow in the middle is an isomorphism by the five lemma. \square

We let $D_{\text{qcoh}}^+(X, \mathcal{O}_X) \subseteq D^+(X, \mathcal{O}_X)$ denote the full subcategory consisting of objects L such that $\mathcal{H}^i L \in \text{QCoh}(X)$ for all i , where \mathcal{H}^i denotes the i -th cohomology sheaf. The corollary implies that $D_{\text{qcoh}}^+(X, \mathcal{O}_X) \subseteq D^+(X, \mathcal{O}_X)$ is triangulated subcategory. Indeed, if $L \rightarrow M \rightarrow N \rightarrow L[1]$ is a distinguished triangle with $L, M \in D_{\text{qcoh}}^+$, then, by the long exact sequence

$$\mathcal{H}^i L \longrightarrow \mathcal{H}^i M \longrightarrow \mathcal{H}^i N \longrightarrow \mathcal{H}^{i+1} L \longrightarrow \mathcal{H}^{i+1} M$$

and the corollary, we have $N \in D_{\text{qcoh}}^+$. The inclusion functor $\varphi: \text{QCoh}(X) \subseteq \text{Shv}(X, \mathcal{O}_X)$ is exact and induces a triangulated functor $\varphi: D^+\text{QCoh}(X) \rightarrow D_{\text{qcoh}}^+(X, \mathcal{O}_X)$.

Theorem 2.4.8. (1) (Gabber) $\text{QCoh}(X)$ admits enough injectives.

(2) Assume either

- X is Noetherian, or
- X is quasi-compact and Δ_X is affine.

Then the functor

$$\varphi: D^+(\text{QCoh}(X)) \rightarrow D_{\text{qcoh}}^+(X, \mathcal{O}_X)$$

is an equivalence of category. Moreover, for $L \in D^+(\text{QCoh}(X))$,

$$R\Gamma(X, \varphi-)(L) \simeq R\Gamma(X, \varphi L).$$

Remark 2.4.9. If X is Noetherian, then φ preserves injectives. In general, even for X affine, φ does not necessarily sends injectives to flabby sheaves.

We refer to [SP, 077P], [SGA6, II 3.5, Appendice I] and [TT, Propositions B.8, B.16] for more details.

Applications to Rf_*

Corollary 2.4.10. Let $f: X \rightarrow S$ be an affine morphism and let $\mathcal{F} \in \text{QCoh}(X)$. Then

- (1) $R^q f_* \mathcal{F} = 0$ for all $q \geq 1$.
- (2) $H^q(X, \mathcal{F}) \simeq H^q(S, f_* \mathcal{F})$ for all q .

Proof. Recall that $R^q f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^q(f^{-1}(V), \mathcal{F})$. For V affine, $f^{-1}(V)$ is affine and $H^q(f^{-1}(V), \mathcal{F}) = 0$ for $q \geq 1$. It follows that $f_* \mathcal{F} \simeq Rf_* \mathcal{F}$ and $R\Gamma(X, \mathcal{F}) \simeq R\Gamma(S, Rf_* \mathcal{F}) \simeq R\Gamma(S, f_* \mathcal{F})$. \square

Proposition 2.4.11. Let $f: X \rightarrow S$ be a quasi-compact and quasi-separated morphism of schemes. For all $\mathcal{F} \in \text{QCoh}(X)$ and all q , $R^q f_* \mathcal{F} \in \text{QCoh}(S)$.

Lemma 2.4.12 (Mayer-Vietoris). *Let $f: X \rightarrow S$ be a continuous map between topological spaces and let $X = U_1 \cup U_2$ be an open cover of X . Let $U = U_1 \cap U_2$. Let $f_i: U_i \rightarrow S$ and $g: U \rightarrow S$ denote the restrictions of f . For $L \in D^+(\mathrm{Shv}(X))$, we have a distinguished triangle*

$$Rf_*L \longrightarrow Rf_{1*}L \oplus Rf_{2*}L \longrightarrow Rg_*L \longrightarrow Rf_*L[1]$$

Proof. Up to replacing L by an injective resolution, we may assume $L \in K^+$ with L^i injective. It suffices to show the exactness of the sequence

$$0 \longrightarrow f_*L \longrightarrow f_{1*}L \oplus f_{2*}L \longrightarrow g_*L \longrightarrow 0.$$

Taking sections on an open subset $V \subseteq S$, we get

$$0 \rightarrow L^i(f^{-1}(V)) \rightarrow L^i(f_1^{-1}(V)) \oplus L^i(f_2^{-1}(V)) \xrightarrow{\alpha} L^i(f_1^{-1}(V) \cap f_2^{-1}(V)) \rightarrow 0.$$

The surjectivity of α follows from the fact that L^i is flabby and the remaining part of the exactness follows from the sheaf condition. \square

Proof of Proposition 2.4.11. We may assume that S is affine. Since X is quasi-compact, X can be covered by n affine opens for some n .

Case X separated. We proceed by induction on n . The case $n = 0$ is trivial. For $n > 0$, we have $X = U_1 \cup U_2$ with U_1 affine and U_2 covered by $n - 1$ affine opens. In the notation of the lemma above, we have a distinguished triangle

$$Rf_*\mathcal{F} \longrightarrow Rf_{1*}\mathcal{F} \oplus Rf_{2*}\mathcal{F} \longrightarrow Rg_*\mathcal{F} \longrightarrow Rf_*\mathcal{F}[1].$$

By Corollary 2.4.10, $Rf_{1*}(\mathcal{F}) \simeq f_{1*}\mathcal{F}$ is quasi-coherent. Moreover, $Rf_{2*}\mathcal{F} \in D_{\mathrm{qcoh}}^+(S)$ by induction hypothesis. Since X is separated, $U_1 \cap U_2$ can be covered by $n - 1$ affine opens, and consequently $Rg_*\mathcal{F} \in D_{\mathrm{qcoh}}^+(S)$ by induction hypothesis. It follows that $Rf_*\mathcal{F} \in D_{\mathrm{qcoh}}^+(S)$.

General Case. We proceed again by induction on n . The case $n = 0$ is trivial. For $n > 0$, write $X = U_1 \cap U_2$ with U_1 affine and U_2 covered by $n - 1$ affine opens. Proceed as in separated case except that $Rg_*\mathcal{F} \in D_{\mathrm{qcoh}}^+(S)$ is deduced from the separated case applied to $U = U_1 \cap U_2 \subseteq U_1$. Note that U is quasi-compact and separated. \square

Flat base change

Given a commutative diagram of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \downarrow f' & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array} \quad (*)$$

and an \mathcal{O}_X -module \mathcal{F} on X , we have a base change morphism

$$g^*f_* \rightarrow f'_*h^*$$

given equivalently by

$$g^*f_* \rightarrow g^*f_*h_*h^* \xrightarrow{\sim} g^*g_*f'_*h^* \rightarrow f'_*h^*$$

or

$$g^*f_* \rightarrow f'_*f'^*g^*f_* \xrightarrow{\sim} f'_*h^*f^*f_* \rightarrow f'_*h^*.$$

We will give sufficient conditions for the base change morphism to be an isomorphism in the case of quasi-coherent sheaves on schemes.

Lemma 2.4.13. *Assume that $(*)$ is a Cartesian square of schemes and f is affine. Then for $\mathcal{F} \in \mathrm{QCoh}(X)$, the base change map $g^*f_*\mathcal{F} \rightarrow f'_*h^*\mathcal{F}$ is an isomorphism.*

Proof. We may assume $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B \otimes_A A')$. Assume $\mathcal{F} = \widetilde{M}$ for a B -module M . Then the left hand side is $(M \otimes_A A')^\sim$ and the right hand side is $M \otimes_B (B \otimes_A A')^\sim$ and the base change map is the canonical isomorphism. \square

Proposition 2.4.14 (flat base change). *Assume that $(*)$ is a Cartesian square of schemes, f is quasi-compact and quasi-separated, and g is flat. Then for $\mathcal{F} \in \mathrm{QCoh}(X)$, the base change map $g^*Rf_*\mathcal{F} \rightarrow (Rf'_*)h^*\mathcal{F}$ is an isomorphism.*

Since g^* is exact, it induces a functor $g^*: D^+(S, \mathcal{O}_S) \rightarrow D^+(S', \mathcal{O}_{S'})$. The same holds for h^* . The base change map is given by

$$g^*Rf_* \simeq R(g^*f_*) \rightarrow R(f'_*h^*) \rightarrow (Rf'_*)h^*$$

Proof. We first prove the case where g is an open immersion. We replace \mathcal{F} by a resolution $L \in K^+$ with L^i injective. Then h^*L^i is injective and

$$g^*Rf_*L = g^*f_*L \xrightarrow{\sim} f'_*h^*L = (Rf'_*)h^*L$$

Having established the proposition for open immersions, we may assume that S is affine. Since f is quasi-compact, X is quasi-compact and can be covered by n affine open subsets. We then proceed by induction on n and apply Mayer-Vietoris as in Proposition 2.4.11 to reduce to the case where X is affine, which has been proved in Lemma 2.4.13. \square

2.5 Cohomology of projective space

Theorem 2.5.1. *Let A be a ring, $X = \mathbb{P}_A^d = \mathrm{Proj}(R)$, where $R = A[x_0, \dots, x_d]$, $d \geq 1$. We regard $\Gamma(X, -)$ as a functor $\mathrm{Shv}(X, \mathcal{O}_X) \rightarrow \mathrm{Mod}_A$.*

- $H^q(X, \mathcal{O}_X(n)) = 0$, for $q \neq 0, d$ and for all n .
- $R \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ as graded A -modules.
- $H^d(X, \mathcal{O}_X(n))$ is a free A -module with basis $\{x_0^{k_0} \cdots x_d^{k_d} \mid k_i < 0, \sum k_i = n\}$.

In particular, for $n \geq 0$, $H^0(X, \mathcal{O}_X(n))$ and $H^d(X, \mathcal{O}_X(-n - d - 1))$ are both free of rank $\binom{n+d}{d}$.

Recall that $\mathcal{O}_X(n) = \widetilde{R(n)}$. For the proof it is convenient to use derived tensor products of multiple complexes. We will not develop the theory in full generality but concentrate on what is necessary for the proof of theorem.

Definition 2.5.2. Let \mathcal{A} be an additive category. The category $C^m(\mathcal{A})$ of m -tuple complexes is defined recursively by $C^0(\mathcal{A}) = \mathcal{A}$ and $C^m(\mathcal{A}) = C(C^{m-1}(\mathcal{A}))$ for $m \geq 1$. We consider the total complex functor with respect to coproducts $\text{tot}_{\oplus}: C^m(\mathcal{A}) \rightarrow C(\mathcal{A})$ defined by

$$\begin{aligned} (\text{tot}_{\oplus} L)^n &= \bigoplus_{i_1 + \dots + i_m = n} L^{i_1, \dots, i_m} \\ d^{i_1, \dots, i_m} &= \sum_{j=1}^n (-1)^{i_1 + \dots + i_{j-1}} d_j^{i_1, \dots, i_m} \end{aligned}$$

The tensor product functor extends to the category of complexes:

$$\begin{aligned} C^-(\text{Mod}_A) \times C^-(\text{Mod}_A) &\rightarrow C^2(\text{Mod}_A) \xrightarrow{\text{tot}_{\oplus}} C(\text{Mod}_A) \\ (L, M) &\mapsto L \otimes M \end{aligned}$$

where $(L \otimes M)^{ij} = L^i \otimes M^j$.

Lemma 2.5.3. Let $L, M \in C^-(\text{Mod}_A)$. Assume that L^i is flat for all i and L or M is acyclic. Then $\text{tot}(L \otimes M)$ acyclic.

Proof. Case where M is acyclic. Then $L^i \otimes M$ is acyclic for each i and hence $\text{tot}(L \otimes M)$ is acyclic.

Case where L is acyclic. The complex L decomposes into short exact sequences

$$0 \longrightarrow Z^i L \longrightarrow L^i \longrightarrow Z^{i+1} L \longrightarrow 0.$$

By descending induction on i , one shows that $Z^i L$ is flat. Thus

$$0 \longrightarrow Z^i L \otimes M \longrightarrow L^i \otimes M \longrightarrow Z^{i+1} L \otimes M \longrightarrow 0$$

is exact, which implies $H_I^{i,\bullet}(L \otimes M) = 0$. Thus $\text{tot}(L \otimes M)$ is acyclic. \square

Proof of Theorem 2.5.1. Consider the cover $\mathcal{U} = \{U_i\}_{i=0}^d$ of X , where $U_i = D_+(x_i)$. Note that $U_{i_0, \dots, i_p} = D(x_{i_0} \cdots x_{i_p})$ is affine. By Leray's theorem, we have $\check{H}^q(\mathcal{U}, \mathcal{O}(n)) \simeq H^q(X, \mathcal{O}(n))$.

We will compute the Čech cohomology. We have

$$C_{\text{alt}}^p(\mathcal{U}, \mathcal{O}(n)) = \prod_{i_0 < \dots < i_p} (R_{x_{i_0} \cdots x_{i_p}})_n.$$

Let

$$C_{\text{alt}}^p(\mathcal{U}, \mathcal{O}(\bullet)) := \bigoplus_{n \in \mathbb{Z}} C_{\text{alt}}^p(\mathcal{U}, \mathcal{O}(n)) = \bigoplus_{i_0 < \dots < i_p} (R_{x_{i_0} \cdots x_{i_p}})$$

and let K^\bullet be $\bigoplus_{n \in \mathbb{Z}} C_{\text{alt}}^\bullet(\mathcal{U}, \mathcal{O}(n))$:

$$\begin{array}{ccccccc} K^\bullet = (R \longrightarrow C_{\text{alt}}^0(\mathcal{U}, \mathcal{O}(\bullet)) \longrightarrow C_{\text{alt}}^1(\mathcal{U}, \mathcal{O}(\bullet)) \longrightarrow \cdots \longrightarrow C_{\text{alt}}^d(\mathcal{U}, \mathcal{O}(\bullet))) \\ \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \\ \bigoplus_i R_{x_i} \qquad \qquad \qquad \bigoplus_{i < j} R_{x_i x_j} \qquad \qquad \qquad R_{x_0 \dots x_d} \end{array}$$

with R placed at degree 0. We have

$$R_{x_{i_0} \dots x_{i_p}} = \left(\bigotimes_{i \in \{i_0, \dots, i_p\}} A[x_i]_{x_i} \right) \otimes \left(\bigotimes_{i \notin \{i_0, \dots, i_p\}} A[x_i] \right)$$

and

$$K^\bullet = \text{tot} \left(\bigotimes_{i=0}^d (A[x_i] \hookrightarrow A[x_i]_{x_i}) \right)$$

with $A[x_i]$ in degree 0. Since

$$(A[x_i] \rightarrow A[x_i]_{x_i}) \rightarrow \bigoplus_{k_i < 0} x_i^{k_i} A[-1]$$

is a quasi-isomorphism, we have a quasi-isomorphism

$$K^\bullet \rightarrow \bigotimes_{i=0}^d \bigoplus_{k_i < 0} x_i^{k_i} A[-d-1]$$

by Lemma 2.5.3. The theorem follows. \square

Definition 2.5.4 (Koszul complex). Let A be a ring, F an A -module, and $v: A \rightarrow F$ a homomorphism of A -modules (determined by $v(1) \in F$). Define $K^\bullet(v) \in C^{\geq 0}(\text{Mod}_A)$ by $K^p(v) = \Lambda_A^p(F)$,

$$\begin{aligned} d^p: \bigwedge^p F &\rightarrow \bigwedge^{p+1} F \\ x &\mapsto v(1) \wedge x \end{aligned}$$

For an A -module M , we define $K^\bullet(v, M) := K(v) \otimes_A M$.

For $F = A^r$ and $v(1) = f \in A^r$, we write $K^\bullet(f)$ for $K^\bullet(v)$.

Example 2.5.5. Let $f_1, \dots, f_r \in A$. Let $X = \text{Spec}(A)$. Then $\mathcal{U} = \{D(f_i)\}_i$ is an affine open cover of $U = \bigcup_{i=1}^r D(f_i) \subseteq X$. We have

$$\begin{aligned} K^\bullet(A \rightarrow \bigoplus_{i=1}^r A_{f_i}, M) &= \left(M \rightarrow \bigoplus_{i=1}^r M_{f_i} \rightarrow \bigoplus_{0 \leq i < j \leq r} M_{f_i f_j} \rightarrow \cdots \rightarrow M_{f_1 \dots f_r} \right) \\ &= \left(\Gamma(X, \widetilde{M}) \rightarrow C_{\text{alt}}^0(\mathcal{U}, \widetilde{M}) \rightarrow C_{\text{alt}}^1(\mathcal{U}, \widetilde{M}) \rightarrow \cdots \rightarrow C_{\text{alt}}^r(\mathcal{U}, \widetilde{M}) \right), \end{aligned}$$

where M is placed at degree 0.

Date: 12.1

Finiteness and vanishing theorems

Let X be a locally Noetherian scheme.

Definition 2.5.6. An \mathcal{O}_X -module \mathcal{F} is said to be **coherent** if it is quasi-coherent and of finite type. We let $\text{Coh}(X) \subseteq \text{QCoh}(X)$ denote the full subcategory consisting of all coherent \mathcal{O}_X -modules.

Theorem 2.5.7 (Serre). *Let A be a Noetherian ring, $S = \text{Spec}(A)$, $f: X \rightarrow S$ a projective morphism, and \mathcal{F} a coherent sheaf on X .*

- (1) (*finiteness*) *For all q , $H^q(X, \mathcal{F})$ is a finitely generated A -module.*
- (2) (*vanishing*) *Let \mathcal{L} be an ample invertible sheaf. Then there exists $n_0 \geq 0$ such that $H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $n \geq n_0$ and $q \geq 1$.*

Note that $H^q(X, \mathcal{F}) = 0$ for $q \gg 0$ (independently of \mathcal{F}) by Grothendieck's theorem (Theorem 2.2.14) or the proposition below.

Proposition 2.5.8. *Let X be a scheme and $\mathcal{U} = \{U_i\}_{i=1}^d$ an open cover of X such that each U_{i_0, \dots, i_p} is affine. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then $H^q(X, \mathcal{F}) = 0$ for all $q > d$.*

Proof. By Leray's theorem $H^q(X, \mathcal{F}) \simeq \check{H}_{\text{alt}}^q(\mathcal{U}, \mathcal{F}) = 0$ for $q > d$. □

Proof of Theorem 2.5.7. (1) Since f is projective, it factors through a closed immersion $i: X \hookrightarrow \mathbb{P}_A^d$. Then $H^q(X, \mathcal{F}) = H^q(\mathbb{P}_A^d, i_* \mathcal{F})$ and $i_* \mathcal{F}$ is a coherent $\mathcal{O}_{\mathbb{P}_A^d}$ -module. Up to replacing X by \mathbb{P}_A^d , we may assume $X = \mathbb{P}_A^d$.

In this case, we proceed by descending induction on q . For $q > d$, $H^q(X, \mathcal{F}) = 0$. Assume that the assertion is proved for $q + 1$. By the ampleness of $\mathcal{O}_X(1)$, there exists an epimorphism $\mathcal{O}_X(-m)^r \rightarrow \mathcal{F}$, which extends to a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(-m)^r \longrightarrow \mathcal{F} \longrightarrow 0$$

with \mathcal{G} coherent. Taking cohomology, we get the exact sequence

$$H^q(X, \mathcal{O}(-m)^r) \longrightarrow H^q(X, \mathcal{F}) \longrightarrow H^{q+1}(X, \mathcal{G}).$$

Since $H^q(X, \mathcal{O}(-m)^r)$ is a finitely generated A -module by Theorem 2.5.1 and $H^{q+1}(X, \mathcal{G})$ is a finitely generated A -module by induction hypothesis, $H^q(X, \mathcal{F})$ is a finitely generated A -module. (Here we used the assumption that A is Noetherian.)

(2) Case \mathcal{L} very ample. By assumption, we have a closed embedding $i: X \hookrightarrow \mathbb{P}_A^n$ with $\mathcal{L} \simeq i^* \mathcal{O}(1)$. Consequently, $i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \simeq i_* \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n} = i_* \mathcal{F} \otimes \mathcal{O}(n)$ and

$$H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^q(\mathbb{P}_A^n, i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})) \simeq H^q(\mathbb{P}_A^n, i_* \mathcal{F} \otimes \mathcal{O}(n)).$$

Since $i_* \mathcal{F}$ is a coherent sheaf, we may assume, up to replacing X by \mathbb{P}_A^d , that $X = \mathbb{P}_A^d$ and $\mathcal{L} = \mathcal{O}(1)$.

In this case, we proceed by descending induction on q . For $q > d$, $H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) = 0$. Assume the assertion proved for $q + 1$. As in (1), we have a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X(-m)^r \longrightarrow \mathcal{F} \longrightarrow 0,$$

which induces a short exact sequence

$$0 \longrightarrow \mathcal{G} \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}_X(n-m)^r \longrightarrow \mathcal{F} \otimes \mathcal{O}(n) \longrightarrow 0.$$

Taking cohomology, we get the exact sequence

$$H^q(X, \mathcal{O}(n-m)^r) \longrightarrow H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) \longrightarrow H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(n)).$$

Since $H^q(X, \mathcal{O}(n-m)^r) = 0$ for $n > m$ by Theorem 2.5.1 and $H^{q+1}(X, \mathcal{G} \otimes \mathcal{O}(n)) = 0$ for $n \gg 0$ by induction hypothesis, $H^q(X, \mathcal{F} \otimes \mathcal{O}(n)) = 0$ for $n \gg 0$.

General case. There exists $m \geq 1$ such that $\mathcal{L}^{\otimes m}$ is very ample. We apply the very ample case to $(\mathcal{F} \otimes \mathcal{L}^{\otimes i}, \mathcal{L}^{\otimes m})$, $0 \leq i \leq m-1$. For each i , there exists N_i such that $H^q(X, \mathcal{F} \otimes \mathcal{L}^{\otimes mn+i}) = 0$ for $n \geq N_i$ and $q \geq 1$. Therefore, it suffices to take $n_0 = \max_{0 \leq i < m} \{mN_i + 1\}$. \square

The vanishing theorem has the following converse.

Theorem 2.5.9. *Let X be a quasi-compact scheme, \mathcal{L} an invertible \mathcal{O}_X -module. Assume that for every quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, there exists $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$. Then \mathcal{L} is ample.*

In the case where X is Noetherian, every quasi-coherent ideal is coherent.

Proof. Let $x \in X$ be a closed point. There exists an affine open neighborhood $U = \text{Spec}(A) \ni x$ on which \mathcal{L} is trivial. Let $Z = X \setminus U$ and $Z' = Z \cup \{x\}$, equipped with induced reduced closed subscheme structure. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_Z/\mathcal{I}_{Z'} \longrightarrow 0,$$

where $\mathcal{I}_Z/\mathcal{I}_{Z'} \simeq i_*\kappa(x)$, $i: \{x\} \hookrightarrow X$. By assumption, there exists $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$. Twisting the short exact sequence by $\mathcal{O}(n)$ and taking cohomology, we get the exact sequence

$$\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \longrightarrow \kappa(x) \longrightarrow H^1(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) = 0.$$

Let $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n})$ be a pre-image of $1 \in \kappa(x)$. We may regard s as a section of $\mathcal{L}^{\otimes n}$ via the map $\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}) \hookrightarrow \Gamma(X, \mathcal{L}^{\otimes n})$. Then $X_s \subseteq X \setminus U = Z$. Since s is mapped to $1 \in \kappa(x)$, we have $x \in X_s$. Choose a trivialization $\mathcal{L}|_U \simeq \mathcal{O}_U$ and consider the induced map

$$\begin{aligned} \Gamma(U, \mathcal{L}^{\otimes n}) &\xrightarrow{\sim} \Gamma(U, \mathcal{O}_U) \\ s &\mapsto f. \end{aligned}$$

Then $X_s = \text{Spec}(A_f)$ is affine.

Let $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. Then $Y = \bigcup_{S_{+, \text{homog}}} X_s$ contains all closed points of X by the above. If $Y \neq X$, then $X \setminus Y$, which is a closed subset of X , contains at least one closed point. Thus $Y = X$. In other words, \mathcal{L} is ample. \square

Corollary 2.5.10. *Let A be a Noetherian ring, $f: X \rightarrow \text{Spec}(A)$ a proper morphism, and \mathcal{L} an invertible \mathcal{O}_X -module. Then \mathcal{L} ample if and only if for every quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, there exists $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$.*

Proof of Theorem 2.4.4. By Theorem 2.5.9, \mathcal{O}_X is ample. In other words, $X = \bigcup_{i=1}^n X_{f_i}$ with $f_i \in A = \Gamma(X, \mathcal{O}_X)$. The morphism

$$\mathcal{O}_X^n \xrightarrow{(f_1, \dots, f_n)} \mathcal{O}_X$$

is an epimorphism of sheaves of abelian groups, because it is so on each X_{f_i} . Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Let \mathcal{F}_i be the intersection of \mathcal{F} with the direct sum of the first i summands on \mathcal{O}_X^n . Then $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a quasi-coherent ideal sheaf. It follows that $H^1(X, \mathcal{F}) = 0$ and $\Gamma(X, \mathcal{O}_X^n) \rightarrow \Gamma(X, \mathcal{O}_X)$. Thus f_1, \dots, f_n generate the unit ideal of A . Therefore, X is affine (exercise). \square

The finiteness theorem has the following generalization.

Theorem 2.5.11. *Let X and S be locally Noetherian schemes and $f: X \rightarrow S$ a proper morphism. Let $\mathcal{F} \in \text{Coh}(X)$. Then $R^q f_*(\mathcal{F}) \in \text{Coh}(S)$ for all q .*

By contrast, for f affine, f_* does not preserve coherent sheaves in general.

Remark 2.5.12. If $S = \text{Spec}(A)$ and $R^q f_*(\mathcal{F})$ is quasi-coherent, then $R^q f_*(\mathcal{F}) = \Gamma(S, R^q f_* \mathcal{F})^\sim = H^q(X, \mathcal{F})^\sim$. Indeed, $\Gamma(S, R^q f_* \mathcal{F}) = H^q(X, \mathcal{F})$ is implied by the following Lemma.

Lemma 2.5.13. *If $F: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ a triangulated functor and $F(D^{\geq 0}) \subseteq D^{\geq 0}$. If there is a full subcategory $J \subseteq \mathcal{A}$ such that $F(J) \subseteq \mathcal{B}$. Then $\forall L \in D^+(\mathcal{A}), \forall q, H^q L \in J \implies \forall q, H^q F L = F H^q L$.*

Proof. exercise \square

From the Remark, we see Theorem 2.5.11 is true for f projective by Theorem 2.5.7.

To tackle the general case, we need Chow's Lemma.

Lemma 2.5.14 (Chow). *Let S be quasi-compact and quasi-separated scheme, $f: X \rightarrow S$ be separated and of finite type. Suppose X has only finitely many irreducible components. Then $\exists \pi: Y \rightarrow X$ projective such that $f\pi$ is quasi-projective and $\exists U \subseteq X$ dense open such that $\pi: \pi^{-1}(U) \xrightarrow{\sim} U$.*

$$\begin{array}{ccc} \pi^{-1}(U) & \hookrightarrow & Y \\ \downarrow \simeq & & \downarrow \pi \\ U & \hookrightarrow & X \\ & & \downarrow f \\ & & S \end{array}$$

Remark 2.5.15. If f is proper, $f\pi$ is proper, hence projective.

Proof. The proof proceeds in several steps:

- (1) Claim: $X = \bigcup_{i=1}^n U_i$, U_i affine open and $\bigcap_{i=1}^n U_i$ is dense in X .

Proof. Let $X = \bigcup_{j=1}^r X_j$ where X_j are irreducible components of X with generic point η_j . It suffices to show $\forall x \in X, \exists x \in U_x \subseteq X$ affine open such that $\eta_1, \dots, \eta_r \in U_x$.

Take $x \in V \subseteq X$ an arbitrary open affine subset. $\forall j, \exists$ an open affine subset V_j such that $\eta_j \in V_j$ and $\eta_{j'} \notin V_j$ for $j' \neq j$. Thus $V_j \cap X_{j'} = \emptyset$. We can take $U_x = V \sqcup \coprod_{\eta_j \notin V} V_j$. It is a disjoint union by construction, hence affine. \square

- (2) Consider $U \hookrightarrow \widetilde{X} \hookrightarrow X$ where $\widetilde{X} \hookrightarrow X$ is the schematic image of U . Since $\widetilde{X} \hookrightarrow X$ is a closed immersion, hence projective, we may replace X by \widetilde{X} and assume U is schematically dense in X .

Since U_i is of finite type over S , and \mathcal{O}_{U_i} is ample, hence very ample over S , we can embed U_i into $U_i \subseteq \mathbb{P}_S^{n_i}$. Let Z_i be the schematic closure of U_i in $\mathbb{P}_S^{n_i}$.

$$\begin{array}{ccc} U_i & \hookrightarrow & Z_i \\ & \searrow & \downarrow \text{proj} \\ & & S \end{array}$$

Let $U = \bigcap_{i=1}^n U_i$. Then $U \hookrightarrow U_i \hookrightarrow Z_i$ induces $U \hookrightarrow Z_1 \times_S \cdots \times_S Z_n$. Let Z be the schematic closure of U in $Z_1 \times_S \cdots \times_S Z_n$.

$$\begin{array}{ccccc} U & \xrightarrow{\quad} & V_i & \xrightarrow{\quad} & Z \\ & \searrow & \downarrow \pi_i & \nearrow & \downarrow p_i \\ & & U_i & \xrightarrow{\quad} & Z_i \end{array}$$

Form the pullback diagram to get $V_i = p_i^{-1}(U_i)$. Since each $Z_i \rightarrow S$ is projective, $Z_1 \times_S \cdots \times_S Z_n \rightarrow S$ is projective, hence p_i is projective, π_i is projective.

Let $Y = V_1 \cup \cdots \cup V_n \subseteq Z$. Then Y/S is quasi-projective. We want to glue $\pi_i: V_i \rightarrow U_i$ to a morphism $\pi: Y \rightarrow X$, hence we check compatibility of π .

Indeed, $\pi_i|_U = \pi_j|_U \implies \pi_i|_{V_i \cap V_j} = \pi_j|_{V_i \cap V_j}$ since $U \subseteq V_i \cap V_j$ is schematically dense.

Thus π_i glues to $\pi: Y \rightarrow X$ and $\pi|_{V_i} = \pi_i$.

$$\begin{array}{ccc} V_i & \longrightarrow & \pi^{-1}(U_i) \\ & \searrow & \downarrow \\ & & U_i \end{array}$$

Since $V_i \hookrightarrow \pi^{-1}(U_i)$ is proper and schematically dense, it is an isomorphism. Thus $\pi^{-1}U \xrightarrow{\sim} U$. Since $\pi^{-1}(U_i) = V_i \rightarrow U_i$ is proper, π is proper. Since each $Y \rightarrow S$ is quasi-projective, π is quasi-projective. This shows π is projective.

□

We introduce the concept of support to support further study.

Definition 2.5.16. Let X be a topological space, \mathcal{F} a sheaf, its **support** is defined as

$$\text{supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}$$

If (X, \mathcal{O}_X) is a ringed space, \mathcal{F} an \mathcal{O}_X -module, the **annihilator** of \mathcal{F} , $\text{Ann}(\mathcal{F}) \subseteq \mathcal{O}_X$, is a sheaf of ideal defined by

$$\text{Ann}(\mathcal{F})(U) = \{a \in \mathcal{O}(U) \mid a\mathcal{F}|_U = 0\}$$

Lemma 2.5.17. If X is a scheme, \mathcal{F} a quasi-coherent \mathcal{O}_X -module of finite type. Then $\text{supp}(\mathcal{F})$ is a closed subset and $\text{Ann}(\mathcal{F})$ is a quasi-coherent ideal sheaf which corresponds to a closed subscheme Z with underlying space $\text{supp}(\mathcal{F})$.

Proof. The question is local and we may assume $X = \text{Spec}(A)$ is affine, $\mathcal{F} = \widetilde{M}$, M an A -module.

We first show $\text{Ann}(\mathcal{F})$ is quasi-coherent. This means for $f \in A$, $\text{Ann}(\mathcal{F})(D(f)) = \text{Ann}(\mathcal{F})(X)_f$. Since $\text{Ann}(\mathcal{F})(X) = \text{Ann}_A(M)$, this amounts to $\text{Ann}_{A_f}(M_f) = \text{Ann}_A(M)_f$.

Clearly, there is a morphism $\text{Ann}_A(M)_f \rightarrow \text{Ann}_{A_f}(M_f)$. Suppose M is generated by $m_i, 1 \leq i \leq n$. If $a/f^n \in \text{Ann}_{A_f}(M_f)$, then for each i , $\exists r_i$ such that $f^{r_i}am_i = 0$. Take $r = \max\{r_i\}$, $f^r am_i = 0$, hence $a/f^n = af^r/f^{m+r}$ and $af^r \in \text{Ann}_A(M)$.

Next we show $\text{supp}(\mathcal{F})$ coincides with the closed subscheme defined by $\text{Ann}(\mathcal{F})$. Denote the latter by Z .

$$\text{Ann}(\mathcal{F})|_{X \setminus Z} = \mathcal{O} \implies \mathcal{F}|_{X \setminus Z} = 0 \implies \text{supp}(\mathcal{F}) \subseteq Z.$$

Conversely if $\mathfrak{p} \notin \text{supp}(\mathcal{F})$, then $M_{\mathfrak{p}} = 0$. By definition, $\exists f_i \in A \setminus \mathfrak{p}$ such that $f_i m_i = 0$. Let $f = f_1 \cdots f_n \in A \setminus \mathfrak{p}$, then $fM = 0$. Thus $\text{Ann}_A(M) \not\subseteq \mathfrak{p}$, i.e. $\mathfrak{p} \notin Z$.

□

Let $D_{coh}^+(X) = \{L \in D^+(X, \mathcal{O}_X) \mid \mathcal{H}^q L \in \text{Coh}(X)\} \subseteq D_{qch}^+$ is triangulated subcategory.

$$\text{Then finiteness theorem } \iff Rf_*(D_{coh}^+) \subseteq D_{coh}^+.$$

proof of theorem. 2.5.11 General case. We may assume $S = \text{Spec}(A)$, A Noetherian ring. We do Noetherian induction on $\text{supp}(\mathcal{F})$.

If $\text{supp}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$.

Suppose $\text{supp}(\mathcal{F}) \neq \emptyset$ and for any $\mathcal{G} \in \text{Coh}(X)$, $\text{supp}(\mathcal{G}) \subsetneq \text{supp}(\mathcal{F})$, $Rf_* \mathcal{G} \in D_{coh}^+$. Consider $\text{Ann}(\mathcal{F}) = \mathcal{I}_Z$, $i: Z \hookrightarrow X$. Then $\mathcal{F} = i_* \mathcal{F}_0$, $\mathcal{F}_0 \in \text{Coh}(Z)$.

Apply Chow's Lemma to $Z \rightarrow S$, there exists

$$\begin{array}{ccc} \pi^{-1}U & \longrightarrow & Y \\ \downarrow & & \downarrow \pi \\ U & \longrightarrow & Z \xrightarrow{i} X \\ & & \downarrow f \\ & & S \end{array}$$

with π projective, $fi\pi$ projective (f proper) and $U \subseteq Z$ dense open with $\pi^{-1}U \xrightarrow{\sim} U$.

We have $\mathcal{F}_0 \rightarrow \pi_*\pi^*\mathcal{F}_0 \rightarrow R\pi_*\pi^*\mathcal{F}_0$. Complete it into a distinguished triangle

$$\mathcal{F}_0 \longrightarrow R\pi_*\pi^*\mathcal{F}_0 \longrightarrow L \longrightarrow \mathcal{F}_0[1]$$

\mathcal{F}_0 is coherent and $R\pi_*\pi^*\mathcal{F}_0 \in D_{coh}^+(Z)$ by the Case π projective. Thus $L \in D_{coh}^+(Z)$. Since π is an isomorphism on U , $supp(\mathcal{H}^q(L)) \subseteq Z \setminus U \subsetneq Z$. Apply Rf_*i_* to the above distinguished triangle, we have

$$Rf_*i_*\mathcal{F}_0 \longrightarrow Rf_*i_*R\pi_*\pi^*\mathcal{F}_0 = R(f\pi)_*\pi^*\mathcal{F}_0 \longrightarrow Rf_*i_*L \longrightarrow Rf_*i_*\mathcal{F}_0[1]$$

By induction $Rf_*i_*L \in D_{coh}^+$. By case $fi\pi$ projective, $R(f\pi)_*\pi^*\mathcal{F}_0 \in D_{coh}^+$. Thus $Rf_*i_*\mathcal{F}_0 = Rf_*\mathcal{F} \in D_{coh}^+$. \square

Date: 12.3

Hilbert polynomials

Let A an Artinian ring, $X \rightarrow \text{Spec}(A)$ a proper morphism, $\mathcal{F} \in \text{Coh}(X)$. Then $H^i(X, \mathcal{F})$ is a finitely generated A -module, hence of finite length $l_A H^i(X, \mathcal{F}) < \infty$. Denote the alternating sum

$$\chi(X, \mathcal{F}) = \sum_{i \in \mathbb{Z}} (-1)^i l_A H^i(X, \mathcal{F}) \in \mathbb{Z}$$

It is called the **Euler characteristic**.

The alternating sum behaves better then a single length since, if given a long exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'').$$

Without further mentioning, A will always be an Artinian ring.

Theorem 2.5.18 (Hilbert-Serre). *Let X be a projective scheme over $\text{Spec}(A)$, $\mathcal{F} \in \text{Coh}(X)$. Let \mathcal{L} be a very ample invertible sheaf. Then there exists a $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ such that for any $d \in \mathbb{Z}$,*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = P_{\mathcal{F}}(d)$$

We call $P_{\mathcal{F}}(t)$ the **Hilbert polynomial** of \mathcal{F} (w.r.t \mathcal{L}). If $\mathcal{F} = \mathcal{O}_X$, we denote it by $P_X(t)$.

Proof. Since an Artinian ring A has a filtration by fields, we may assume $A = k$ is a field. We may also assume $X = \mathbb{P}_k^n$, $\mathcal{L} = \mathcal{O}(1)$. Denote $R = k[x_0, \dots, x_n]$. Then \mathcal{F} can be represented by \widetilde{M} with M a finitely generated graded R -module.

We can take a projective resolution of M as follows

$$0 \longrightarrow P^{-n} \longrightarrow \dots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

By Hilbert syzygy theorem, it terminates at degree $-n$. In addition, each P^{-i} is of the form $\bigoplus_{j=1}^r R(m_j)$.

By additive property of Euler characteristic, we may assume $\mathcal{F} = \mathcal{O}_X$. Define

$$P_{\mathcal{O}}(t) = \binom{t+n}{n} := \frac{1}{n!} (t+n)(t+n-1) \cdots (t+1)$$

By previous calculation,

- $d \geq 0$,

$$\chi(X, \mathcal{O}(d)) = \dim_k H^0(X, \mathcal{O}(d)) = \binom{d+n}{n} = P_{\mathcal{O}}(d)$$

- $-n-1 < d < 0$, for all i , $H^i(X, \mathcal{O}(d)) = 0$. $\chi(X, \mathcal{O}(d)) = 0 = P_{\mathcal{O}}(d)$.

- $d \leq -n-1$,

$$\chi(X, \mathcal{O}(d)) = (-1)^n \dim_k H^n(X, \mathcal{O}(d)) = (-1)^n \binom{-d-1}{n} = \binom{d+n}{n}$$

In conclusion, $\chi(X, \mathcal{O}(d)) = P_{\mathcal{O}}(d)$ is a polynomial. \square

Remark 2.5.19. • By vanishing theorem, $P_{\mathcal{F}}(d) = l_A H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d})$ for $d \gg 0$. This can be used to compute the Hilbert polynomial.

For another view, write $X = \text{Proj}(R)$, R a finitely generated graded A -algebra, $\mathcal{F} = \widetilde{M}$, M a finitely generated graded R -module, $\mathcal{L} = \mathcal{O}_X(1)$. Then $H^0(X, \mathcal{F} \otimes \mathcal{O}(d)) \xrightarrow{\sim} M_d$ for $d \gg 0$ and $P_{\mathcal{F}}(d) = l_A M_d$ for $d \gg 0$. This is the usual definition of Hilbert polynomial employed in commutative algebra.

- $\deg(P_{\mathcal{F}}(t)) = \dim \text{supp}(\mathcal{F})$.

Example 2.5.20. Let $R = k[x_0, \dots, x_n]$, $X = \mathbb{P}_k^n = \text{Proj}(R)$. Take a degree d polynomial $f \in R_d$ and define the hyper surface $i: X_d \hookrightarrow \mathbb{P}_k^n$, where $X_d = \text{Proj}(R/(f))$. Let us compute the Hilbert polynomial of $i_*(\mathcal{O}_{X_d})$.

Define $R(-d) \xrightarrow{\times f} R$ with cokernel denoted by Q . Then

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_{X_d} \longrightarrow 0$$

By additivity of Euler characteristics,

$$P_{X_d}(t) = P_X(t) - P_X(t-d) = \binom{t+n}{n} - \binom{t-d+n}{n} = \frac{dt^{n-1}}{(n-1)!} + \text{lower terms}$$

We read out the degree and dimension from the first term of its Hilbert polynomial.

Chapter 3

Differential calculus and Grothendieck-Serre duality

3.1 Kähler differentials

Definition 3.1.1. Let A be a ring, B an A -algebra, M an A -module. An A -derivation of B with values in M is an A -linear map $D: B \rightarrow M$ satisfying Leibniz rule: for $x, y \in B$,

$$D(xy) = xD(y) + yD(x)$$

Denote all A -derivations of B with values in M by $\text{Der}_A(B, M)$. It is a B -module.

For $a \in A$,

$$\begin{aligned} D(a) &= D(1 \cdot a) \stackrel{\text{Leibniz}}{=} 1 \cdot D(a) + aD(1) \\ &\stackrel{A\text{-linear}}{=} aD(1) \end{aligned}$$

This implies $D(a) = 0$. In fact A -linear \iff linear + $D(A) = 0$.

Definition 3.1.2. Let $\nabla_{B/A}: B \otimes_A B \rightarrow B$ be the multiplication map with kernel $I \subseteq B \otimes_A B$.

The B -module

$$\Omega_{B/A} := I/I^2$$

is called **module of (Kähler) differentials**

The module of differentials $\Omega_{B/A}$ inherits a natural derivation structure. Define

$$d_{B/A}: B \rightarrow \Omega_{B/A}, \quad b \mapsto 1 \otimes b - b \otimes 1 \mod I^2$$

The Leibniz rule reads

$$\begin{aligned} d(xy) &= 1 \otimes xy - xy \otimes 1 \\ &= (1 \otimes x)(1 \otimes y - y \otimes 1) + (y \otimes 1)(1 \otimes x - x \otimes 1) \\ &= xdy + ydx \end{aligned}$$

Theorem 3.1.3. *For any B -module M , there is a bijection*

$$\begin{aligned}\mathrm{Hom}_B(\Omega_{B/A}, M) &\xrightarrow{\sim} \mathrm{Der}_A(B, A) \\ u &\mapsto u \circ d_{B/A}\end{aligned}$$

Let us first deal with injectivity.

Lemma 3.1.4. $\Omega_{B/A} = Bd_{B/A}(B)$ i.e. the image of B under $d_{B/A}$ generates $\Omega_{B/A}$ as B -module.

Proof. In fact, each element in $\Omega_{B/A}$ can be written as $\sum_i b_i \otimes x_i$ with $b_i, x_i \in B$. By definition of $\Omega_{B/A}$, $\sum_i b_i x_i = 0$.

$$\begin{aligned}\sum b_i \otimes x_i &= \sum (b_i \otimes 1)(1 \otimes x_i - x_i \otimes 1) + \sum b_i x_i \otimes 1 \\ &= \sum (b_i \otimes 1)(dx_i)\end{aligned}$$

Therefore, it is in $Bd_{B/A}(B)$. \square

Next we treat surjectivity by introducing another equivalently useful description of derivation.

Definition 3.1.5. Let B be an A -algebra and M a B -module. Define the B -algebra $D_B(M) = B \oplus M$ with multiplication defined as $(b, m)(b', m') = (bb', bm' + b'm)$ (think of (b, m) as $b + m$ and $mm' = 0$). There is a natural B -algebra projection

$$p: D_B(M) \rightarrow B, \quad (b, m) \mapsto b$$

We reinterpret elements in $\mathrm{Der}_A(B, M)$ as A -algebra sections of p . Explicitly,

$$\begin{aligned}\mathrm{Der}_A(B, M) &\simeq \{B \rightarrow D_B(M) \mid A\text{-algebra sections of } p\} \\ D &\mapsto f_D, \quad f_D(x) = (x, D(x)) \\ D_f &\leftarrow f, \quad D_f(x) = f(x) - x\end{aligned}$$

For $D \in \mathrm{Der}_A(B, M)$, define $\varphi(x \otimes y) = xf_D(y)$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B \otimes_A B & \xrightarrow{\nabla} & B \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & D_B(M) & \xrightarrow{p} & B \longrightarrow 0 \end{array}$$

Then $p\varphi = 0$ and the dashed arrow gives a morphism $u: I/I^2 \rightarrow M$. We check that

$$u \circ d_{B/A}(b) = u(1 \otimes b - b \otimes 1) = f_D(b) - b = D(b)$$

This completes the proof of Theorem 3.1.3

As a first application, we have

Proposition 3.1.6. *If E is an A -module, $B = \mathrm{Sym}(E)$, then $\Omega_{B/A} \simeq B \otimes_A E$.*

Proof. For any B -module M ,

$$\begin{aligned}\mathrm{Hom}_B(\Omega_{B/A}, M) &\simeq \mathrm{Der}_A(B, M) \simeq \{B \rightarrow D_B(M) \mid A\text{-algebra sections of } p\} \\ &\simeq \{E \rightarrow B \oplus M \mid A\text{-homomorphisms inducing } E \rightarrow B\} \\ &\simeq \mathrm{Hom}_A(E, M) \simeq \mathrm{Hom}_B(B \otimes_A E, M)\end{aligned}$$

□

We move on the sheaf of differentials.

Definition 3.1.7. Suppose

$$\begin{array}{ccc} X & \xhookrightarrow{\quad} & Z \\ & \searrow^i & \nearrow j \\ & U & \end{array}$$

is an immersion with i closed and j open. Let \mathcal{I} be the ideal sheaf of X in U . Define $\mathcal{N}_{X/Z} \in \mathrm{QCoh}(X)$ such that $i_*(\mathcal{N}_{X/Z}) = \mathcal{I}/\mathcal{I}^2$. It is called the **conormal sheaf** of the immersion. The definition does not depend on the choice of U .

For any morphism of schemes $X \rightarrow Y$, we call

$$\Omega_{X/Y} := \mathcal{N}_{X/X \times_Y X} \in \mathrm{QCoh}(X)$$

the **sheaf of differentials (cotangent sheaf)**.

Example 3.1.8. $\Omega_{\mathrm{Spec}(B)/\mathrm{Spec}(A)} = \widetilde{\Omega_{B/A}}$

Example 3.1.9. If $X \rightarrow Y$ is a monomorphism, then $X \xrightarrow{\sim} X \times_Y X$, hence $\Omega_{X/Y} = 0$.

Definition 3.1.10. If $\mathcal{M} \in \mathrm{Shv}(X, \mathcal{O}_X)$, an Y -derivation $D: \mathcal{O}_X \rightarrow \mathcal{M}$ is an $f^{-1}\mathcal{O}_Y$ -linear map satisfying Leibniz rule:

$$D(xy) = xD(y) + yD(x), \quad x, y \in \mathcal{O}_X(U)$$

where U is an open subset of X . Denote all Y -derivations with values in \mathcal{M} by $\mathrm{Der}_Y(\mathcal{O}_X, \mathcal{M})$.

There is a natural Y derivation on $\Omega_{X/Y}$ defined as

$$d_{X/Y}: \mathcal{O}_X \rightarrow \Omega_{X/Y}, \quad b \mapsto 1 \otimes b - b \otimes 1 \mod \mathcal{I}^2$$

Theorem 3.1.11. $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{X/Y}, \mathcal{M}) \xrightarrow{\sim} \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{M})$, $u \mapsto u \circ d_{X/Y}$.

Proof. Be careful when dealing with \mathcal{M} not quasi-coherent. □

Example 3.1.12. $\Omega_{\mathbb{A}_S^n/S}$ is a free $\mathcal{O}_{\mathbb{A}_S^n}$ -module with basis dx_1, \dots, dx_n .

3.1.1 Functorality

For a commutative diagram

$$(*) \quad \begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{h} & Y \end{array}$$

The commutativity of

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow \Delta & & \downarrow \Delta \\ X' \times_{Y'} X' & \xrightarrow{g \times g'} & X \times_Y X \end{array}$$

induces $g^* \Omega_{X/Y} \rightarrow \Omega_{X'/Y'}$.

Proposition 3.1.13. *If $(*)$ is Cartesian, then $g^* \Omega_{X/Y} \xrightarrow{\sim} \Omega_{X'/Y'}$.*

Proof. We may assume they are all affine.

$$\begin{array}{ccc} B' & \longleftarrow & B \\ \uparrow & & \uparrow \\ A' & \longleftarrow & A \end{array}$$

The short exact sequence below is indeed split exact with splitting given by $B \rightarrow B \otimes_A B, b \mapsto b \otimes 1$.

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\nabla} B \longrightarrow 0$$

Tensoring with A' , it is still exact

$$0 \longrightarrow I \otimes_A A' = I' \longrightarrow B' \otimes_{A'} B' \xrightarrow{\nabla} B' \longrightarrow 0$$

From the diagram

$$\begin{array}{ccccccc} I^2 \otimes_A A' & \longrightarrow & I \otimes_A A' & \longrightarrow & I/I^2 \otimes_A A' & \longrightarrow & 0 \\ \downarrow & & \downarrow \simeq & & \downarrow \alpha & & \\ 0 & \longrightarrow & I'^2 & \longrightarrow & I' & \longrightarrow & I'/I'^2 \longrightarrow 0 \end{array}$$

and with the help of snake lemma, we conclude α is an isomorphism. \square

Proposition 3.1.14. *Given a commutative diagram,*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

Then the following sequence is exact.

$$f^* \Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Proof. Reducing to affine case, we have

$$\begin{array}{ccc} C & \xleftarrow{\quad} & B \\ \swarrow & & \uparrow \\ A & & \end{array}$$

For any C -module M , the exactness of

$$\mathrm{Hom}_C(C \otimes_B \Omega_{B/A}, M) \longrightarrow \mathrm{Hom}_C(\Omega_{C/A}, M) \longrightarrow \mathrm{Hom}_C(\Omega_{C/B}, M) \longrightarrow 0$$

is, on noting that $\mathrm{Hom}_C(C \otimes_B \Omega_{B/A}, M) = \mathrm{Hom}_B(\Omega_{B/A}, M)$, equivalent to the exactness of the following sequence, which can be checked by hand

$$\mathrm{Der}_A(B, M) \longrightarrow \mathrm{Der}_A(C, M) \longrightarrow \mathrm{Der}_B(C, M) \longrightarrow 0$$

□

Corollary 3.1.15. *If $X \rightarrow Y$ is locally of finite type, then $\Omega_{X/Y}$ is an \mathcal{O}_X -module of finite type.*

Proof. Locally X has an embedding $i: X \hookrightarrow \mathbb{A}_Y^n$. Since $i^*\Omega_{\mathbb{A}_Y^n/Y} \twoheadrightarrow \Omega_{X/Y}$ and $\Omega_{\mathbb{A}_Y^n/Y}$ is finite type, $\Omega_{X/Y}$ is of finite type. □

There is a more general construction than the cotangent sheaf which we call the **cotangent complex** $L_{X/Y} \in D_{qch}^{\leq 0}(X, \mathcal{O}_X)$. The cotangent sheaf is its 0-cohomology

$$\mathcal{H}^0 L_{X/Y} = \Omega_{X/Y}$$

Moreover, if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

Then

$$Lf^*L_{Y/S} \longrightarrow L_{Y/S} \longrightarrow L_{X/Y} \longrightarrow Lf^*L_{Y/S}[1]$$

is a distinguished triangle, where Lf^* is the left derived functor of f^* . All exact sequences discussed above can be derived from it.

Definition 3.1.16. If

$$\begin{array}{ccc} X & \xhookrightarrow{\quad} & Z \\ \searrow i & & \nearrow j \\ & U & \end{array}$$

is an immersion with i closed and j open. Let \mathcal{I} be the ideal sheaf of X in U . Then $\mathrm{Spec}(\mathcal{O}_U/\mathcal{I}^2) = Z_1$ is called the **first infinitesimal neighborhood** of X in Z .

The definition of the first infinitesimal neighborhood does not depend on the choice of U . In fact, X and Z_1 has the same underlying space.

Proposition 3.1.17. *Given a diagram with i an immersion.*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \downarrow & & \swarrow \\ S & & \end{array}$$

Then

- (1) *We have a short exact sequence*

$$\mathcal{N}_{X/Z} \xrightarrow{\delta} i^*\Omega_{Z/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

where δ is given by $d_{X/Z}: \mathcal{O}_Z \rightarrow \Omega_{Z/X}$.

- (2) *If $X \hookrightarrow Z_1$ admits a retraction where Z_1 is the first infinitesimal neighborhood of X , then*

$$0 \longrightarrow \mathcal{N}_{X/Z} \longrightarrow i^*\Omega_{Z/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

is split exact.

Proof. Reducing to affine case, we have

$$\begin{array}{ccc} C & \leftarrow B \\ \nwarrow & & \uparrow \\ & A & \end{array}$$

with $C = B/I$.

- (1). We have to prove a C -module exact sequence

$$I/I^2 \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

Choose an C -module M and consider

$$0 \longrightarrow \text{Hom}_C(\Omega_{C/A}, M) \longrightarrow \text{Hom}_C(\Omega_{B/A} \otimes_B C, M) \longrightarrow \text{Hom}_C(I/I^2, M)$$

Since M is an $C = B/I$ -module, $IM = 0$. Therefore $\text{Hom}_C(I/I^2, M) = \text{Hom}_B(I, M)$. Use the interpretation of Ω we are reduced to prove

$$0 \longrightarrow \text{Der}_A(C, M) \longrightarrow \text{Der}_A(B, M) \longrightarrow \text{Hom}_B(I, M)$$

which is easy to check.

- (2) Let $r: Z_1 \rightarrow X$ be a retraction. Look at the following diagram

$$\begin{array}{ccccc} & & i & & \\ & \nearrow & \curvearrowright & \searrow & \\ X & \xleftarrow[r]{i_1} & Z_1 & \xrightarrow{} & Z \\ \downarrow & & \swarrow & & \\ S & & & & \end{array}$$

We have a ladder of exact sequences

$$\begin{array}{ccccccc} \mathcal{N}_{X/Z} & \longrightarrow & i^*\Omega_{Z/S} & \longrightarrow & \Omega_{X/S} & \longrightarrow & 0 \\ \parallel & & \downarrow & & \parallel & & \\ \mathcal{N}_{X/Z_1} & \longrightarrow & i_1^*\Omega_{Z_1/S} & \longrightarrow & \Omega_{X/S} & \longrightarrow & 0 \end{array}$$

We only need to construct a retraction $i_1^*\Omega_{Z_1/S} \rightarrow \mathcal{N}$. Indeed, a retraction corresponds to $\Omega_{Z_1/S} \rightarrow i_*\mathcal{N}$, which further corresponds to an S -derivation of $\mathcal{O}_{Z_1} \rightarrow i_*\mathcal{N}$. This can be defined by $id - (i_1r)^*$.

□

Date: 12.8

Example 3.1.18. Suppose k is a field and X is a k -scheme. Given a k point $i: \{x\} \hookrightarrow X$, consider the diagram

$$\begin{array}{ccccc} & & i & & \\ & \text{Spec}(k) & \xrightarrow{i_1} & X_1 & \longrightarrow X \\ & & \searrow & \downarrow & \\ & & & & \text{Spec}(k) \end{array}$$

The first infinitesimal neighborhood X_1 admits a retraction because it is a k -scheme. The conormal sheaf of $\text{Spec}(k)$ in X is easily calculated as $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. We have a split short exact sequence according to Proposition 3.1.17

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{k/k} = 0$$

This identifies $i^*\Omega_{X/k}$ with $\mathfrak{m}/\mathfrak{m}^2$ as k -vector spaces. Its dual is the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ as mentioned before.

Proposition 3.1.19. Suppose $S = \text{Spec}(A)$ with A a ring. Let $X = \mathbb{P}_A^n$ be the projective space. Then we have a short exact sequence

$$0 \longrightarrow \Omega_{X/S} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proof. Let $R = A[x_1, \dots, x_n]$. Define $E = R(-1)^{n+1} = \bigoplus_{i=0}^n Re_i$, where $\deg(e_i) = 1$. Consider the map $E \rightarrow R$, $e_i \mapsto x_i$. Its cokernel is A and kernel denoted by M . Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow R \longrightarrow A \longrightarrow 0$$

(We could use Koszul complex to show M is generated by $x_i e_j - x_j e_i$). Taking \sim , we have

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow \widetilde{A} = 0$$

We are going to define $\Omega_{X/S} \rightarrow \widetilde{M}$ locally.

Recall on $U_i = D_+(x_i)$, we have $R_{x_i} = A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$. First write M_{x_i} as

$$M_{x_i} = \bigoplus_{j \neq i} R_{x_i} (x_i e_j - x_j e_i)$$

Then

$$\widetilde{M}(U_i) = M_{(x_i)} = \bigoplus_{j \neq i} R_{(x_i)} \frac{1}{x_i^2} (x_i e_j - x_j e_i)$$

Define

$$\begin{aligned} \varphi_i: \Omega_{X/S}|_{U_i} &\rightarrow \widetilde{M}|_{U_i} \\ d\left(\frac{x_j}{x_i}\right) &\mapsto \frac{1}{x_i^2} (x_i e_j - x_j e_i) \end{aligned}$$

We need to check compatibility of φ_i and φ_j . On $U_i \cap U_j$, $\frac{x_k}{x_i} = \frac{x_j}{x_i} \cdot \frac{x_k}{x_j}$, hence

$$d\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i} d\left(\frac{x_k}{x_j}\right) + \frac{x_k}{x_j} d\left(\frac{x_j}{x_i}\right)$$

We calculate that

$$\begin{aligned} & \varphi_i \left(d\left(\frac{x_j}{x_i}\right) - \frac{x_k}{x_j} d\left(\frac{x_j}{x_i}\right) \right) \\ &= \frac{1}{x_i x_j} (x_j e_k - x_k e_j) \\ &= \varphi_j \left(\frac{x_j}{x_i} d\left(\frac{x_k}{x_j}\right) \right) \end{aligned}$$

This verifies the compatibility and defines an isomorphism $\varphi: \Omega_{X/S} \xrightarrow{\sim} \widetilde{M}$. \square

3.1.2 De Rham cohomology

Definition 3.1.20. Let $f: X \rightarrow S$ be a morphism. Define $\Omega_{X/S}^i = \Lambda_{\mathcal{O}_X}^i \Omega_{X/S}$. We have $\Omega_{X/S}^0 = \mathcal{O}_X$, $\Omega_{X/S}^1 = \Omega_{X/S}$.

The **de Rham** complex is

$$0 \longrightarrow \Omega_{X/S}^0 \xrightarrow{d^0} \Omega_{X/S}^1 \xrightarrow{d^1} \Omega_{X/S}^2 \longrightarrow \dots$$

where $\Omega_{X/S}^0$ is placed at degree 0. The differential is given by for $U \subseteq X$ open, take $a \in \Omega^i(U)$, $b \in \Omega^j(U)$, $a \wedge b \in \Omega^{i+j}(U)$ and

$$d(a \wedge b) = da \wedge b + (-1)^i a \wedge db$$

The hyper cohomology of $\Omega_{X/S}^\bullet$ is called the **de Rham cohomology** and denoted by $H_{dR}^i(X/S) = H^i(X, \Omega_{X/S}^\bullet)$.

3.2 Smooth morphisms and unramified morphisms

3.2.1 Morphisms locally of finite presentation

Definition 3.2.1. Let A be a ring, A -algebra B is said to be **finitely presented** if $B = A[x_1, \dots, x_n]/I$ where I is a finitely generated ideal.

We have a categorical interpretation of finitely presented algebras.

Let (C_i) be a filtered inductive system of A -algebras with natural projections $C_i \rightarrow \text{colim}_i C_i$. For any A -algebra B , we have a canonical morphism

$$\varphi: \text{colim}_i \text{Hom}_{A-\text{alg}}(B, C_i) \rightarrow \text{Hom}_{A-\text{alg}}(B, \text{colim}_i C_i)$$

One can show that φ is an isomorphism for any filtered system $(C_i) \iff B$ is finitely presented A -algebra.

In categorical language, the functor from the category of A -algebras to sets

$$\mathrm{Hom}_{A\text{-}alg}(B, -) : A\text{-Alg} \rightarrow \mathrm{Set}$$

preserves filtered colimits. We call B a **compact** object.

In fact, any algebra can be written as a filtered colimit of finitely presented algebras.

Definition 3.2.2. A morphism of schemes $f: S \rightarrow T$ is said to be **locally of finite presentation** if there exists an affine open cover $X = \bigcup U_i$ with $U_i = \mathrm{Spec}(A_i)$, and an affine open cover $f^{-1}(U_i) = \bigcup_j V_{ij}$ with $V_{ij} = \mathrm{Spec}(B_{ij})$ such that B_{ij} is an A_i -algebra of finite presentation.

Remark 3.2.3. Let X/S be a scheme.

- For any filtered projective system (T_i) of affine schemes over S , the natural morphism

$$\mathrm{colim}_i \mathrm{Hom}_S(T_i, X) \xrightarrow{\sim} \mathrm{Hom}_S(\lim_i T_i, X)$$

is an isomorphism $\iff X/S$ is locally of finite presentation.

- locally of finite presentation are stable under composition and base change.

Lemma 3.2.4. If f is locally of finite type, then Δ_f is locally of finite presentation.

Proof. Reduce to $f: \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$. We want to prove B is finitely presented $B \otimes_A B$ -algebra. Suppose B is generated by b_1, \dots, b_n . Then the kernel I of $\nabla: B \otimes_A B \rightarrow B$ is finitely generated by $1 \otimes b_i - b_i \otimes 1$. \square

3.2.2 A lifting property

Definition 3.2.5. A closed immersion $T_0 \hookrightarrow T$ with ideal sheaf \mathcal{I} is called a **first order thickening** if $\mathcal{I}^2 = 0$.

Definition 3.2.6. Let $f: X \rightarrow Y$ be a morphism of schemes. Consider diagrams of the following type where $T_0 \hookrightarrow T$ is a first order thickening.

$$\begin{array}{ccc} T_0 & \longrightarrow & X \\ \downarrow & \nearrow g & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

f is said to be **formally unramified** (resp. **smooth**, **étale**) if for any diagrams as above, there exists locally at most (resp. at least, exactly) one g making the diagram commutative.

Definition 3.2.7. Let $f: X \rightarrow Y$ be a morphism of schemes.

- (1) f is called **unramified** if it is formally unramified and locally of finite type.
- (2) f is called **smooth** if it is formally smooth and locally of finite presentation.

(3) f is called **étale** if it is formally étale and locally of finite presentation.

Remark 3.2.8. We see from definition that

- (formally) étale \iff (formally) smooth + (formally) unramified.
- All six morphisms are stable under base change and composition.
- In the definition of formally unramified/formally étale morphisms, we may remove "locally".
- All monomorphism of schemes are formally unramified.
- f formally unramified (resp. smooth, étale) \iff

$$\mathcal{H}om_Y(T, X) \rightarrow \mathcal{H}om_Y(T_0, X)$$

is a monomorphism (resp epimorphism, isomorphism), where $\mathcal{H}om_Y(T, X)$ is sheaf of sets on T defined by

$$\mathcal{H}om(T, X)(U) = \text{Hom}_Y(U, X)$$

for $U \subseteq T$ open.

- f is formally unramified $\iff \Delta_f$ is formally smooth.

Let us first study uniqueness of liftings.

Proposition 3.2.9. *Let $f: X \rightarrow Y$ be a morphism of schemes. Consider the following diagram*

$$\begin{array}{ccc} T_0 & \xrightarrow{g_0} & X \\ \downarrow i & \nearrow g & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

where i is a first order thickening with ideal sheaf \mathcal{I} . Suppose g is a given morphism making it commutative. Then there is a bijection between all liftings of g_0 and all Y -derivations $\mathcal{O}_X \rightarrow g_{0*}\mathcal{I}$.

$$\begin{aligned} \{g': T \rightarrow X \mid Y\text{-morphisms, } g'i = g_0\} &\rightarrow \text{Der}_Y(\mathcal{O}_X, g_{0*}\mathcal{I}) \\ g' &\mapsto g'^b - g^b \end{aligned}$$

where $g^b: \mathcal{O}_X \rightarrow g_*\mathcal{O}_T$. The right hand side has other descriptions as $\text{Der}_Y(\mathcal{O}_X, g_{0*}\mathcal{I}) = \text{Hom } \mathcal{O}_X(\Omega_{X/Y}, g_{0*}\mathcal{I}) = \text{Hom}_{\mathcal{O}_{T_0}}(g_0^*\Omega_{X/Y}, \mathcal{I})$.

Proof. Reduce to affine case, we have

$$\begin{array}{ccccc} C_0 = C/I & \longleftarrow & B & & \\ \uparrow & & \nearrow g & & \uparrow \\ C & \xleftarrow{\quad} & A & \xleftarrow{\quad} & \end{array}$$

Since $g'i = gi$, we see $g' - g$ indeed maps to I . Let $\varphi = g'^\flat - g^\flat$. For $x, y \in B$,

$$\begin{aligned} & \varphi(xy) - x\varphi(y) - y\varphi(x) \\ &= g'(xy) - g(x)g(y) - g(x)(g'(y) - g(y)) - g'(y)(g'(x) - g(x)) \\ &= g'(xy) - g'(x)g'(y) \end{aligned}$$

The calculation illustrates that φ is a derivation $\iff g^\flat$ is a ring homomorphism, showing therefore the bijection. \square

Corollary 3.2.10. f is formally unramified $\iff \Omega_{X/Y} = 0$

Proof. \Leftarrow Apply above proposition.

\Leftarrow . First proof. Consider

$$\begin{array}{ccc} X & \xrightarrow{\quad \text{---} \quad} & X \\ \downarrow & \nearrow g & \downarrow \\ \underline{\text{Spec}}(D_{\mathcal{O}_X}(\Omega_{X/Y})) & \longrightarrow & Y \end{array}$$

where $D_{\mathcal{O}_X}(\Omega_{X/Y})$ is defined in Definition 3.1.5. There is a natural morphism g induced by $\mathcal{O}_X \hookrightarrow \mathcal{O}_X \oplus \Omega_{X/Y}$. The ideal sheaf of X in $\underline{\text{Spec}}(D_{\mathcal{O}_X}(\Omega_{X/Y}))$ is just $\Omega_{X/Y}$, thus we get $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/Y}, \Omega_{X/Y}) = 0$, which implies $\Omega_{X/Y} = 0$.

Second proof. Consider

$$\begin{array}{ccc} X & \xrightarrow{\quad \text{---} \quad} & X \\ \downarrow i & \swarrow \pi_1 \quad \swarrow \pi_2 & \downarrow \\ (X \times_Y X)_1 & & \\ \downarrow & & \downarrow \\ X \times_Y X & \longrightarrow & Y \end{array}$$

Since $X \hookrightarrow (X \times_Y X)_1$, the first infinitesimal neighborhood, is a first order thickening, we must have $\pi_1 = \pi_2$. But the equalizer of π_1 and π_2 is X , thus X equals its first infinitesimal neighborhood, which amounts to say the conormal sheaf is 0, hence $\Omega_{X/Y} = 0$. \square

Proposition 3.2.11. Suppose $f: X \rightarrow Y$ is locally of finite type. Then the followings are equivalent

- (a) f is unramified.
- (b) For all $y \in Y$, the fiber $X_y \rightarrow \text{Spec}(\kappa(y))$ is unramified.
- (c) For all $y \in Y$, the fiber $X_y = \coprod \text{Spec}(k_i)$ with k_i being finite separable extension of $\kappa(y)$.

Proof. (a) \implies (b) is true by base change.

(b) \implies (a) Since $f_y^*\Omega_{X/Y} = \Omega_{X_y/\kappa(y)} = 0$, for each point $x \in X$, write $y = f(x)$, we have $\Omega_{X/Y} \otimes \kappa(x) = (f_y i_x)^*\Omega_{X/Y} = 0$. By assumption, $\Omega_{X/Y}$ is of finite type, hence Nakayama's Lemma tells us $\Omega_{X/Y} = 0$.

(c) \iff (b). We may assume $Y = \text{Spec}(k)$. Again by Nakayama's Lemma, we may further assume k is algebraically closed. We show the equivalence under these assumptions.

(c) \implies (b). This is clear since $X = \coprod \text{Spec}(k)$.

(b) \implies (c). Take any point $x \in X$. Since k is algebraically closed and f locally of finite type, we have $\kappa(x) = k$ and $\kappa(x)$ defines a k -point. According to Example 3.1.18, we have $\mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega_{X/k} \otimes \kappa(x) = 0$. By Nakayama's lemma, $\mathfrak{m}_x = 0$. This shows $\mathcal{O}_{X,x}$ is a field and in particular, x is a closed point. We conclude by observing that on a variety over algebraically closed field, every point is closed implies $X = \coprod \text{Spec}(k)$. \square

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Example 3.2.12. $\mathbb{A}_S^n \rightarrow S$ is smooth, since for each first order thickening $T_0 \rightarrow T$, $\text{Hom}_S(T, \mathbb{A}_S^n) = \text{Hom}_{\mathbb{Z}}(T, \mathbb{A}_{\mathbb{Z}}^n) = \text{Hom}_{\text{ring}}(\mathbb{Z}[x_1, \dots, x_n], \mathcal{O}_T(T))$. If T is affine, $\text{Hom}_S(T, \mathbb{A}_S^n) \rightarrow \text{Hom}_S(T_0, \mathbb{A}_S^n)$ equals $\Gamma(T, \mathcal{O}_T)^n \rightarrow \Gamma(T_0, \mathcal{O}_{T_0})^n$.

Example 3.2.13. \mathbb{P}_S^n/S is smooth.

Example 3.2.14. Let $S = \text{Spec}(A)$. Then $\text{Spec}(A[x]/(x^2)) \rightarrow \text{Spec}(A)$ is not smooth. Consider the natural morphisms,

$$\begin{array}{ccc} A[x]/(x^2) & \longleftarrow & A[x]/(x^2) \\ \uparrow & \swarrow f & \uparrow \\ A[x]/(x^3) & \longleftarrow & A \end{array}$$

Suppose there is a lifting f , $f(x) = x + ax^2$. Then $f(x)^2 = x^2$ is non-zero in $A[x]/(x^3)$.

Theorem 3.2.15. Suppose i is an immersion and the diagram commutes. Consider the following conditions:

$$\begin{array}{ccc} X & \xhookrightarrow{i} & Z \\ \downarrow f & \nearrow g & \\ Y & & \end{array}$$

(a) f is formally smooth.

(b) $0 \longrightarrow \mathcal{N}_{X/Z} \xrightarrow{\delta} i^*\Omega_{Z/Y} \longrightarrow \Omega_{X/Y} \longrightarrow 0$ is exact and locally splits.

Then

(1) (a) \implies (b).

(2) (b) and g formally smooth \implies (a).

Proof. (1) Let $i_1: X \hookrightarrow Z_1$ be the first infinitesimal neighborhood of i . Since i_1 is a first order thickening, f formally smooth implies locally i_1 admits a retraction. Thus (b) is exact and locally split by Proposition 3.1.17.

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow & \nearrow & \downarrow f \\ Z_1 & \longrightarrow & Y \end{array}$$

One can give a direct proof of (2). We will instead deduce (2) from a general exact sequence after some preparations. \square

Corollary 3.2.16. If $f: X \rightarrow Y$ is smooth, then $\Omega_{X/Y}$ is locally free \mathcal{O}_X -module of finite rank.

Proof. We may assume there is an immersion i

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{A}_Y^n \\ \downarrow & \searrow & \\ Y & & \end{array}$$

Then $i^*\Omega_{\mathbb{A}_Y^n/Y} \twoheadrightarrow \Omega_{X/Y}$ locally splits. \square

Corollary 3.2.17 (Zariski's Jacobi criterion). *Suppose*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \downarrow f & \swarrow g & \\ Y & & \end{array}$$

with i closed immersion of ideal sheaf \mathcal{I} and g smooth. For $x \in X$, the following statements are equivalent

- (a) f is smooth at x , i.e. there exists an open neighborhood U of x such that $f|_U$ is smooth.
- (b) There exists an open neighborhood U of x and sections $s_1, \dots, s_r \in \Gamma(U, \mathcal{I})$ such that $(s_1)_x, \dots, (s_r)_x$ generates \mathcal{I}_x . and that

$$d_{Z/Y}(s_i) \otimes 1 \in (\Omega_{Z/Y})_x \otimes \kappa(x)$$

for $i = 1, \dots, r$ are $\kappa(x)$ -linearly independent.

In particular, $f: X \rightarrow Y$ is smooth at $x \in X \iff$ locally around x , $X = \text{Spec}(\mathcal{O}_Y[t_1, \dots, t_n]/(s_1, \dots, s_r))$ such that

$$\text{rank} \left(\frac{\partial s_i}{\partial t_j}(x) \right)_{1 \leq i \leq r, 1 \leq j \leq n} = r$$

Proof. From Nakayama's Lemma, $(s_1)_x, \dots, (s_r)_x$ generates $\mathcal{I}_x \iff s_1 \otimes \kappa(x), \dots, s_r \otimes \kappa(x)$ generates $\mathcal{I}_x \otimes \kappa(x) = (\mathcal{I}/\mathcal{I}^2)_x \otimes \kappa(x)$. Clearly (b) is equivalent to

$$(I/I^2) \otimes \kappa(x) \hookrightarrow \Omega_{X/Y} \otimes \kappa(x)$$

is injective. By the following Lemma, it is equivalent to $\delta: \mathcal{N}_{X/Z} \rightarrow i^*\Omega_{Z/Y}$ is a split monomorphism around x . We conclude by Theorem 3.2.15. \square

Lemma 3.2.18. *Let A be a local ring with residue field k . Let M be a finitely generated A -module, P be a projective A -modules. Given a morphism $u: M \rightarrow P$. Then $M \otimes k \rightarrow P \otimes k$ is injective $\iff u$ admits a retraction.*

Proof. \Leftarrow is clear.

\implies We know $M \otimes_A k \rightarrow P \otimes_A k$ admits a retraction v_0 . Since $M \rightarrow M \otimes_A k$ is surjective and P is projective, there is a morphism v such that

$$\begin{array}{ccc} M & \xleftarrow{v} & P \\ \downarrow & & \downarrow \\ M \otimes_A k & \xleftarrow{v_0} & P \otimes_A k \end{array}$$

Then vu is identity in $\text{Hom}_k(M \otimes_A k, M \otimes_A k)$. If M is free, we see $\det(vu)$ is invertible in k , hence vu is an isomorphism and $(vu)^{-1}v$ gives a retraction.

In general, let e_1, \dots, e_n be in M whose image in $M \otimes_A k$ is a k -basis. Let $w: A^n \rightarrow M$ be defined by e_i . Then w is surjective by Nakayama's Lemma. By previous case, uw admits a retraction, so is u . \square

Definition 3.2.19. Let $f: X \rightarrow Y$ be a morphism of schemes, $\mathcal{M} \in \text{QCoh}(X)$. An Y -extension by \mathcal{M} is a commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow i & \searrow f & \\ X' & \longrightarrow & Y \end{array}$$

where i the first order thickening of ideal sheaf isomorphic to \mathcal{M} .

A morphism between two Y -extensions $i: X \rightarrow X'$ and $j: X \rightarrow X''$ is a morphism $a: X' \rightarrow X''$ making the diagram commutative

$$\begin{array}{ccccc} X & \xrightarrow{i} & X' & \longrightarrow & Y \\ & \searrow j & \downarrow a & \swarrow & \\ & & X'' & & \end{array}$$

Let $\text{Ext}_Y(X, \mathcal{M})$ be the isomorphism classes of Y -extensions.

An Y -extension by \mathcal{M} is equivalent to a $f^{-1}\mathcal{O}_Y$ -module exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and a morphism between extensions is the same as a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow 0 \\ & & \searrow & \uparrow a^\sharp & \nearrow & & \\ & & & \mathcal{O}_{X''} & & & \end{array}$$

Example 3.2.20. Let $X' = \underline{\text{Spec}}(D_{\mathcal{O}_X}(\mathcal{M}))$. It is called the trivial Y -extension by \mathcal{M} and denote it by 0 in $\text{Ext}_Y(X, \mathcal{M})$. It is easy to see an Y -extension $i: X \rightarrow X'$ is trivial $\iff i$ admits a retraction.

We could endow $\text{Ext}_Y(X, \mathcal{M})$ with a structure of abelian group. Given

$$e_1: 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

$$e_2: 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Then

$$0 \longrightarrow \mathcal{M} \oplus \mathcal{M} \longrightarrow \mathcal{O}_{X_1} \times_{\mathcal{O}_X} \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

is exact and we can do the push out diagram from $\mathcal{M} \oplus \mathcal{M} \rightarrow \mathcal{M}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M} \oplus \mathcal{M} & \longrightarrow & \mathcal{O}_{X_1} \times_{\mathcal{O}_X} \mathcal{O}_{X_2} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{O}_{X_3} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \end{array}$$

The last extension is called the sum of e_1 and e_2 .

Definition 3.2.21. $\mathcal{E}xt_Y(U, \mathcal{M}) = a(U \mapsto \text{Ext}_U(U, \mathcal{M}))$ is an abelian sheaf on X .

We could reinterpret formally smooth in terms of $\mathcal{E}xt$:

Lemma 3.2.22. Let $f: X \rightarrow Y$ be a morphism of schemes. Then the following are equivalent:

- (1) f is formally smooth,
- (2) For any $U \subseteq X$ open with the induced subscheme structure and any first order thickening $U \rightarrow U'$,

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow g & \downarrow \\ U' & \longrightarrow & Y \end{array}$$

Lifting g exists locally on U' .

- (3) $\mathcal{E}xt_Y(U, \mathcal{M}) = 0$ for all $\mathcal{M} \in \text{QCoh}(U)$ and $U \subseteq X$ open.

Proof. (a) \implies (b) obvious by definition.

(b) \implies (c). The lifting g is a retraction, hence all Y -extensions are trivial.

(c) \implies (a) We may assume all schemes concerned are affine and $C \rightarrow C_0$ is a first order thickening with ideal I .

$$\begin{array}{ccccc} & & C_0 & \longleftarrow & B \\ & \uparrow & & \nearrow g & \uparrow \\ & C & \longleftarrow & A & \end{array}$$

The goal is to construct a lifting g . Let R be the pullback as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & B \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & C_0 \longrightarrow 0 \end{array}$$

Since $\text{Spec}(B) \rightarrow \text{Spec}(R)$ is a first order thickening and all extensions are trivial by assumption, there exists a retraction $B \rightarrow R$, which defines a lifting g . \square

Lemma 3.2.23. *Given a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \swarrow g \\ S & & \end{array}$$

If f is affine and $\mathcal{M} \in \text{QCoh}(X)$, then there exists a long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Der}_Y(\mathcal{O}_X, \mathcal{M}) & \longrightarrow & \text{Der}_S(\mathcal{O}_X, \mathcal{M}) & \longrightarrow & \text{Der}_S(\mathcal{O}_X, f_*\mathcal{M}) \\ & & & & \nearrow \partial & & \\ & & \text{Ext}_Y(X, \mathcal{M}) & \xleftarrow{\quad} & \text{Ext}_S(X, \mathcal{M}) & \longrightarrow & \text{Ext}_S(Y, f_*\mathcal{M}) \end{array}$$

where ∂ is defined as follows. Take $D \in \text{Der}_S(\mathcal{O}_X, f_*\mathcal{M})$ and let $X' = \text{Spec}(D_{\mathcal{O}_X}(\mathcal{M}))$ with the $f^{-1}\mathcal{O}_Y$ structure $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_{X'} \oplus \mathcal{M}$ defined via the adjunction of

$$\mathcal{O}_Y \xrightarrow{(f^\flat, D)} f_*\mathcal{O}_X \oplus f_*\mathcal{M}$$

Lemma 3.2.24. Suppose $i: X \hookrightarrow Z$ is an immersion, $\mathcal{M} \in \text{QCoh}(X)$. Then

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{N}_{X/Z}, \mathcal{M}) \simeq \text{Ext}_Z(\mathcal{O}_X, \mathcal{M})$$

The isomorphism is given as follows. Let $X \rightarrow Z_1$ be the first infinitesimal neighborhood. If $a \in \text{Hom}_{\mathcal{O}_X}(\mathcal{N}_{X/Z}, \mathcal{M})$, construct

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_{X/Z} & \longrightarrow & \mathcal{O}_{Z_1} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \end{array}$$

where $\mathcal{O}_{X'}$ is the push-out of \mathcal{O}_{Z_1} along a . The second row is the extension that corresponds to a .

proof of Theorem 3.2.15. (2): (b)+ g formally smooth $\implies f$ formally smooth

Up to replacing Z by an open subset, we may assume i is a closed immersion. It suffices to show that for all $\mathcal{M} \in \text{QCoh}(X)$, $\forall e \in \text{Ext}_Y(X, \mathcal{M})$, e is trivial locally.

The long exact sequence in Lemma 3.2.23 applied to $f = gi$ yields

$$\text{Der}_Y(\mathcal{O}_Z, i_*\mathcal{M}) \xrightarrow{\partial} \text{Ext}_Z(X, \mathcal{M}) \longrightarrow \text{Ext}_Y(X, \mathcal{M}) \longrightarrow \text{Ext}_Y(Z, i_*\mathcal{M})$$

Calculation shows that the following diagram is commutative

$$\begin{array}{ccc} \text{Der}_Y(\mathcal{O}_Z, i_*\mathcal{M}) & \xrightarrow{\partial} & \text{Ext}_Z(X, \mathcal{M}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_{\mathcal{O}_X}(i^*\mathcal{O}_{Z/Y}, \mathcal{M}) & \xrightarrow{\text{Hom}(\delta, \mathcal{M})} & \text{Hom}_{\mathcal{O}_X}(\mathcal{N}_{X/Z}, \mathcal{M}) \end{array}$$

Since δ is a split monomorphism by assumption, ∂ is surjective. As a result, $\text{Ext}_Y(X, \mathcal{M}) \rightarrow \text{Ext}_Y(Z, i_*\mathcal{M})$ is injective. Since g is formally smooth, e is locally trivial in $\text{Ext}_Y(Z, i_*\mathcal{M})$, hence locally trivial in $\text{Ext}_Y(X, \mathcal{M})$. \square

We now come back to schemes over a field.

Definition 3.2.25. A Noetherian local ring A is said to be **regular** if $\dim(m/m^2) = \dim A$.

A scheme is said to be **regular** if it is locally Noetherian and $\forall x \in X$, $\mathcal{O}_{X,x}$ is regular.

Theorem 3.2.26. Let $f: X \rightarrow \text{Spec}(k)$ be a morphism locally of finite type, k a field. Then the followings are equivalent

- (a) f is smooth.
- (b) For any field extension k'/k , $X \otimes_k k'$ is regular.
- (c) There exists a field extension k'/k with k' perfect such that $X \otimes_k k'$ is regular.

Theorem 3.2.27. A morphism $f: X \rightarrow Y$ is smooth \iff it is locally of finite presentation, flat and $\forall y \in Y$, $X_y \rightarrow \text{Spec}(\kappa(y))$ is smooth.

Corollary 3.2.28. A morphism $f: X \rightarrow Y$ is étale \iff f is locally of finite presentation, flat and unramified.

There are some topological consequences of smoothness. Here we review some facts:

Theorem 3.2.29. • Every flat morphism is universally generizing.

- Every generizing morphism locally of finite presentation is open.

This applies, in particular, to smooth morphisms.

Definition 3.2.30. Suppose $f: X \rightarrow Y$ is smooth. Define for each $x \in X$, $\dim_x(f) = \text{rank}_x(\Omega_{X/Y})_x$. Then $x \mapsto \dim_x(f)$ is a locally constant function on X .

Remark 3.2.31. Let $f: X \rightarrow Y$ be a smooth morphism.

- For $x \in X$, let $y = f(x)$ and $x \in U \subseteq X_y$ be a connected open subset. Then $\dim_x(f) = \dim(U)$.
- f is étale \iff it is smooth of dimension 0.

There is an important invariant derived from smoothness.

Definition 3.2.32. Let $f: X \rightarrow Y$ be a smooth morphism. The **canonical sheaf** of f is $\wedge^{\dim(f)} \Omega_{X/Y} = \omega_{X/Y}$. It is an invertible sheaf.

Example 3.2.33. Let $X = \mathbb{P}_S^n/S$. Recall we have established the exact sequence in Proposition 3.1.19

$$0 \longrightarrow \Omega_{X/S} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Take the top exterior power, we get $\omega_{X/S} \otimes \mathcal{O}_X = \wedge^{n+1} \mathcal{O}_X(-1)^{n+1} = \mathcal{O}_X(-n-1)$. Thus

$$\omega_{\mathbb{P}_S^n/S} = \mathcal{O}_{\mathbb{P}_S^n}(-n-1)$$

Example 3.2.34. Suppose i is an immersion and f, g are smooth.

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ \downarrow f & \swarrow g & \\ S & & \end{array}$$

We know $\mathcal{N}_{X/Z}$ is a locally free \mathcal{O}_X module of finite rank. Denote $\text{codim}_x(i) = \text{rank}(\mathcal{N}_{X/Z})_x$. Then $x \rightarrow \text{codim}_x(i)$ is also a locally constant function on X . We have in addition

$$\dim_x(f) = \text{codim}_x(i) = \dim_{i(x)}(g)$$

Take top exterior power of the short exact sequence

$$0 \longrightarrow \mathcal{N}_{X/Z} \longrightarrow i^*\Omega_{Z/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

we have

$$i^*\omega_{Z/S} = \omega_{X/S} \otimes \bigwedge^{\text{codim}(i)} \mathcal{N}_{X/Z}$$

Definition 3.2.35. Suppose X/k is proper and smooth with k a field. The **geometric genus** of X is $\dim_k \Gamma(X, \omega_{X/k}) = g(X)$.

Example 3.2.36. For projective spaces \mathbb{P}_k^n with $n \geq 0$, we have

$$g(\mathbb{P}_k^n) = \dim_k \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(-n-1)) = 0$$

Example 3.2.37. Suppose $n \geq 2$. Let $X_d \subseteq X = \mathbb{P}_k^n$ be a smooth hypersurface of degree d with ideal sheaf $\mathcal{O}_X(-d)$. Then $\mathcal{O}_{X_d}(-n-1) = \omega_{X_d/k} \otimes \mathcal{O}_X(-d)$ implies $\omega_{X_d/k} = \mathcal{O}_{X_d}(d-n-1)$.

Since

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_{X_d} \longrightarrow 0$$

is exact, twist by m we have

$$0 \longrightarrow \mathcal{O}_X(m-d) \longrightarrow \mathcal{O}_X(m) \longrightarrow i_* \mathcal{O}_{X_d}(m) \longrightarrow 0$$

Take cohomology and note that $H^1(X, \mathcal{O}(m-d-1)) = 0$,

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X(m-d)) \longrightarrow \Gamma(X, \mathcal{O}_X(m)) \longrightarrow \Gamma(X_d, \mathcal{O}_{X_d}(m)) \longrightarrow 0$$

If $m = 0$, then $\Gamma(X_d, \mathcal{O}_{X_d}) \simeq k$. This shows X_d is geometrically connected. Since $X_d \otimes_k \bar{k}$ is smooth, and in particular locally integral, it is also geometrically irreducible.

If $m = d-n-1$, then $\Gamma(X, \mathcal{O}_X(d-n-1)) = \Gamma(X_d, \omega_{X_d/k})$. Thus $g(X_d) = \binom{d-1}{n}$.

$$\begin{cases} d < n+1, & g(X_d) = 0 \\ d \geq n+1, & g(X_d) \geq 1 \end{cases}$$

Moreover, if $d, d' \geq n+1$ and $d \neq d'$, then $g(X_d) \neq g(X_{d'})$.

We can construct smooth X_d for each d .

Example 3.2.38. The Fermat hypersurfaces

$$\sum_{i=0}^n x_i^d = 0$$

is smooth if $\text{char}(k) = p \nmid d$.

The Gabber hypersurface

$$x_0^d + \sum_{i=0}^{n-1} x_i x_{i+1}^{d-1} = 0$$

is smooth if $p|d \geq 3$.

For $d = 2$. If $n = 2m - 1$ odd, take

$$\sum_{i=0}^{m-1} x_{2i} x_{2i+1} = 0$$

If $n = 2m$ even, take

$$x_{2m}^2 + \sum_{i=0}^{m-1} x_{2i} x_{2i+1} = 0$$

We briefly mention Bertini's theorem, which says at least in characteristic 0, hypersurfaces in \mathbb{P}_k^n of degree d in general position are smooth.

3.2.3 Rational equivalences

Definition 3.2.39. Given schemes X and Y . A **rational map** from X to Y , denoted by $X \dashrightarrow Y$, is an equivalence class of pairs (U, f) , where $U \subseteq X$ is dense open and $f: U \rightarrow Y$ is a morphism of schemes. Two pairs (U, f) and (U', f') are said to be equivalent if there is an open dense subset $V \subseteq U \cap U'$ such that $f|_V = f'|_V$.

A **rational function** on X is a rational map $X \dashrightarrow \mathbb{A}_{\mathbb{Z}}^1$.

Example 3.2.40. Rational functions on X can be expressed as

$$\operatorname{colim}_U \operatorname{Hom}_{\operatorname{Sch}}(U, \mathbb{A}_{\mathbb{Z}}^1) = \operatorname{colim}_U \mathcal{O}_X(U)$$

where U runs through all open dense subsets of X . If X is irreducible with generic point η , then it is identified with $\mathcal{O}_{X,\eta}$. It is called the **field of rational functions** or **function field** and denoted by $R(X)$.

Definition 3.2.41. Suppose X and Y are irreducible schemes. A rational map $f: X \dashrightarrow Y$ is called **dominant** if there is a representative (U, f) such that $f(U)$ is dense. If η_X and η_Y are generic points of X and Y respectively, then it is equivalent to $f(\eta_X) = \eta_Y$.

In this way, we can define composition of dominant rational maps between irreducible schemes and make the category \mathcal{C} that consists of irreducible schemes as objects and dominant rational maps as morphisms.

Definition 3.2.42. A rational map is said to be **birational** if it is an isomorphism in \mathcal{C} .

Two irreducible schemes are said to be **rationally equivalent** if they are isomorphic in \mathcal{C} .

We can also define birational maps over a base S .

Proposition 3.2.43. Let k be a field. There is an equivalence of categories between the category of varieties over k with dominant rational maps and the opposite category of finitely generated fields K over k .

$$\begin{aligned} \{\text{Varieties}\} &\xrightarrow{\sim} \{\text{Fields}\}^{\text{op}} \\ X &\mapsto R(X) \end{aligned}$$

We shall see that geometric genus is preserved by birational maps.

Proposition 3.2.44. Suppose X, X' are proper smooth varieties over k and are birationally equivalent. Then $g(X) = g(X')$. i.e. g is a birational invariant.

Proof. Let $f: X \dashrightarrow X'$ be a birational map. According to Zorn's Lemma we can choose a representative (U, f) that is maximal among all representatives. The morphism $f^*\Omega_{X'/k}|_{U \cap X'} \rightarrow \Omega_{X/k}|_U$ induces $f^*\omega_{X'/k}|_U \rightarrow \omega_{X/k}|_U$. We show $\Gamma(X', \omega_{X'/k}) \rightarrow \Gamma(U, \omega_{X/k})$ is injective. But X and X' are irreducible and f is birational, hence on some open dense subset $V \subseteq U \subseteq X$, $f|_V: V \rightarrow f(V)$ is an isomorphism. Thus $\Gamma(X', \omega_{X'/k})$ and $\Gamma(U, \omega_{X/k})$ both injects into $\Gamma(V, \omega_{V/k})$. If we show, as in the following Lemma, that $\Gamma(X, \omega_{X/k}) \simeq \Gamma(U, \omega_{X/k})$, then $g(X') \leq g(X)$ and by symmetry, we will get $g(X) = g(X')$. \square

Lemma 3.2.45. Let X be a Noetherian normal integral scheme, \mathcal{L} an invertible sheaf on X . Then for each $\emptyset \neq U \subseteq X$ open, $\Gamma(X, \mathcal{L}) \xrightarrow{\sim} \Gamma(U, \mathcal{L})$.

Proof. Indeed, $\text{codim}(X \setminus U, X) \geq 2$, and X is normal, hence the global sections is irrelevant to codimension 1 points. \square

Definition 3.2.46. A variety over k is called **rational** if it is birationally equivalent to \mathbb{P}_k^n .

We come back to discuss hypersurfaces $X_d \subseteq \mathbb{P}_k^n$. From previous discussion, we have

$$\begin{cases} d \geq n+1 & X_d \text{ irrational} \\ d, d' \geq n+1, d \neq d' & X_d, X_{d'} \text{ not birationally equivalent} \end{cases}$$

A natural question is when $d < n+1$, will X_d be rational? Let us Assume k algebraically closed for simplicity.

For $d = 2$. Take $x \in X_d(k)$. We can define, for each line l passing through x , a unique intersection $l \cap X_d$. This defines, given that \mathbb{P}_k^w parameterizes all lines, a birational map $\mathbb{P}_k^{n-1} \dashrightarrow X_d$.

For $n = 2$, take $xy - z^2 = 0$ and we have 2-uple embedding of $X_2 \simeq \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$.

For $n = 3$, take $xy - zw$ and we have Segre embedding $X_3 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$. This shows a rational X_d is not necessarily isomorphic to \mathbb{P}^n .

For $n = 3$, $d = 3$, we know that all cubic surfaces are rational.

For $n = 4$, $d = 3, 4$, we know, at least for $k = \mathbb{C}$, cubic or quartic three-folds are irrational. $d = 3$ is done by Clemens-Griffiths and $d = 4$ is done by Iskovskikh-Manin.

For $n = 5$, $d = 3$, we know that some cubic four-folds are rational.

An **Open Question** is to prove that some cubic four-folds are irrational.

Date: 12.15

Let X/k be a proper smooth scheme. The **Hodge number** is defined as $h^{p,q} = \dim H^q(X, \Omega_{X/k}^p)$. One can show $h^{p,q}$ is birational invariant for smooth proper varieties over k .

3.2.4 Resolution of singularities

If X is an integral Noetherian Japanese scheme of dimension less than or equal to 1. Then the normalization $f: X' \rightarrow X$ is a rational morphism. In this case, X' is normal, hence regular, and we get a smooth X' birational to X . In general, we have

Theorem 3.2.47 (Hironaka). *If k is a field of characteristic 0, X/k is a variety, then there exists a birational and proper map $\pi: X' \rightarrow X$ such that X'/k is smooth and quasi-projective. Birational here means there exists an open dense subset $U \subseteq X$ such that $\pi^{-1}U \simeq U$ is an isomorphism.*

The positive characteristic case is much more complicated. For application, we have

Theorem 3.2.48 (De Jong). *Let k be a field, X a variety over k . Then there exists $\pi: X' \rightarrow X$ an alteration with X' regular and quasi-projective over k . Alteration here means π is proper, surjective and $\exists U \subseteq X$ dense open such that $\pi^{-1}U \rightarrow U$ is finite.*

Let us come back to the lifting property. Let $f: X \rightarrow Y$ be a morphism, $T_0 \rightarrow T$ a first order thickening with ideal sheaf \mathcal{I} .

$$\begin{array}{ccc} T_0 & \xrightarrow{g_0} & X \\ \downarrow & \nearrow g & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

Define a sheaf \mathcal{G} on T by $\mathcal{G}(U) = \text{all liftings of the following type}$

$$\begin{array}{ccc} U_0 & \xrightarrow{g_0} & X \\ \downarrow & \nearrow g & \downarrow f \\ U & \longrightarrow & Y \end{array}$$

where $U_0 = U \cap T_0$.

We have seen that if lifting exists locally, then \mathcal{G} is a torsor under

$$\mathcal{H}om_{\mathcal{O}_{T_0}}(g_0^*\Omega_{X/Y}, \mathcal{I})$$

Let $c \in H^1(T_0, \mathcal{H}om_{\mathcal{O}_{T_0}}(g_0^*\Omega_{X/Y}, \mathcal{I}))$ be the class defined by \mathcal{G} . Thus $c = 0 \iff$ lifting exists globally.

Definition 3.2.49. Let \mathcal{M} be an \mathcal{O}_X -module. It is said to be **locally projective** if $\forall x \in X$, $\exists x \in U \subseteq X$ with $U = \text{Spec}(A)$ affine open such that $\mathcal{M}|_U = \widetilde{M}$ and M is projective A -module.

This implies for any $U = \text{Spec}(A) \subseteq X$ affine open, $\mathcal{M}|_U = \widetilde{M}$ and M is projective A -module.

Lemma 3.2.50. If $f: X \rightarrow Y$ is formally smooth, then $\Omega_{X/Y}$ is locally projective.

Proof. We may assume $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine. Let $A[x_i]_{i \in I} \twoheadrightarrow B$ be a surjective ring homomorphism where I is not necessarily finite. Let $Z = \text{Spec}(A[x_i]_{i \in I})$. Then

$$0 \longrightarrow \mathcal{N}_{X/Z} \longrightarrow i^*\Omega_{Z/Y} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

expresses $\Omega_{X/Y}$ as a direct summand of a free sheaf on X . \square

Corollary 3.2.51. If f is formally smooth, and T affine, then

$$H^1(T_0, \mathcal{H}\text{om}_{\mathcal{O}_{T_0}}(g_0^*\Omega_{X/Y}, \mathcal{I})) = 0$$

hence lifting exists locally implies it exists globally.

Proof. Since $g_0^*\Omega_{X/Y}$ is locally projective, we can embed it into $\bigoplus_{i \in J} \mathcal{O}_{T_0}$. We are reduced to prove

$$H^1(T_0, \mathcal{H}\text{om}_{\mathcal{O}_{T_0}}(\bigoplus_{i \in J} \mathcal{O}_{T_0}, \mathcal{I})) = H^1(T_0, \prod_{i \in J} \mathcal{I}) = 0$$

One can show that

$$H^1(T_0, \prod_{i \in J} \mathcal{I}) \hookrightarrow \prod_{i \in J} H^1(T_0, \mathcal{I}) = 0$$

The right hand side is 0 due to \mathcal{I} being quasi-coherent on T_0 affine. \square

Remark 3.2.52. The cotangent complex explains the whole story.

- If f is smooth, then $L_{X/Y} = \Omega_{X/Y}$.
- If f is formally smooth, then $H^{-1}L_{X/Y} = 0$. Moreover, given a diagram with i immersion and f smooth

$$\begin{array}{ccc} X & \dashrightarrow^i & Z \\ & \downarrow f & \swarrow g \\ & Y & \end{array}$$

the familiar exact sequence

$$0 \longrightarrow \mathcal{N}_{X/Z} \longrightarrow i^*\Omega_{Z/Y} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

is derived from

$$0 = H^{-1}L_{X/Y} \longrightarrow \mathcal{H}^{-1}L_{X/Z} \longrightarrow i^*\mathcal{H}^0L_{Z/Y} \longrightarrow \mathcal{H}^0L_{X/Y} \longrightarrow 0$$

In addition, if $\mathcal{M} \in \mathrm{QCoh}(X)$, then $\mathrm{Ext}_Y(X, \mathcal{M}) = \mathrm{Ext}_{\mathcal{O}_X}^1(L_{X/Y}, \mathcal{M})$. There exists a long exact sequence

$$0 \longrightarrow H^1(T_0, \mathcal{H}\mathbf{om}(g_0^*\Omega_{X/Y}, \mathcal{I})) \longrightarrow \mathrm{Ext}_{\mathcal{O}_{T_0}}^1(Lg_0^*L_{X/Y}, \mathcal{I}) \longrightarrow H^0(T_0, \mathcal{E}\mathbf{xt}_{\mathcal{O}_{T_0}}^1(Lg_0^*L_{X/Y}, \mathcal{I}))$$

and an obstruction $c \in \mathrm{Ext}_{\mathcal{O}_{T_0}}^1(Lg_0^*L_{X/Y}, \mathcal{I})$ governing the existence of lifting. The image of c is 0 \iff lifting exists locally while $c = 0 \iff$ lifting exists globally.

3.3 Four operations

Let (X, \mathcal{O}_X) be a ringed space, $D(X) = D(X, \mathcal{O}_X)$. We are going to construct four operations as follows

$$\begin{aligned} - \otimes^L - &: D(X) \times D(X) \rightarrow D(X) \\ R\mathcal{H}\mathbf{om}(-, -) &: D(X)^{\mathrm{op}} \times D(X) \rightarrow D(X) \end{aligned}$$

If $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces.

$$\begin{aligned} Lf^* &: D(Y) \rightarrow D(X) \\ Rf_* &: D(X) \rightarrow D(Y) \end{aligned}$$

We start to extend derived functors to unbounded complexes.

3.3.1 Unbounded derived functors

Let \mathcal{A} be an abelian category. Denote by $C(\mathcal{A})$ the category of cochain complexes, $K(\mathcal{A})$ the homotopy category of cochain complexes and $D(\mathcal{A})$ the derived category.

Definition 3.3.1. $I \in C(\mathcal{A})$ is said to be **homotopically injective** or **K-injective** if for any acyclic complex $M \in C(\mathcal{A})$, $\mathrm{Hom}^\bullet(M, I)$ is acyclic.

Since $H^n \mathrm{Hom}^\bullet(M, I) = \mathrm{Hom}_{K(\mathcal{A})}(M, I[n])$, we see I is K -injective \iff for any acyclic complex M , $\mathrm{Hom}_{K(\mathcal{A})}(M, I) = 0$.

In particular, if I is K -injective and acyclic, then $\mathrm{Hom}_{K(\mathcal{A})}(I, I) = 0$, hence $I = 0$ in $K(\mathcal{A})$.

Lemma 3.3.2. I is K -injective \iff for any complex M , $\mathrm{Hom}_{K(\mathcal{A})}(M, I) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(M, I)$.

Proof. \Leftarrow obvious by definition.

\Rightarrow . Since

$$\mathrm{Hom}_{D(\mathcal{A})}(M, I) = \underset{M' \rightarrow M \text{ qis}}{\mathrm{colim}} \mathrm{Hom}_{K(\mathcal{A})}(M', I)$$

we only need to show if $f: M' \rightarrow M$ is quasi-isomorphism, then $\mathrm{Hom}_{K(\mathcal{A})}(M, I) \simeq \mathrm{Hom}_{K(\mathcal{A})}(M', I)$. We have a distinguished triangle $M' \rightarrow M \rightarrow C \rightarrow M'[1]$ where C is the mapping cone of f . Then $C(f)$ is acyclic. Apply $\mathrm{Hom}_{K(\mathcal{A})}(-, I)$, we get

$$0 = \mathrm{Hom}_{K(\mathcal{A})}(C, I) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(M, I) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(M', I) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(C[1], I) = 0$$

□

An efficient way of producing enough K -injective objects is to check that \mathcal{A} is a Grothendieck category.

Definition 3.3.3. A **Grothendieck category** is an abelian category \mathcal{A} satisfying (AB5) and admitting a generator.

(AB5) says \mathcal{A} admits small filtered colimits and such colimits are exact.

A **generator** of \mathcal{A} is an object A of \mathcal{A} such that for any object B , $\text{Hom}_{\mathcal{A}}(A, B) = 0 \implies B = 0$.

Example 3.3.4. $\mathcal{A} = \text{Shv}(X, \mathcal{O}_X)$ is a Grothendieck category. A generator is

$$\bigoplus_{U \subseteq X \text{ open}} j_{U!} \mathcal{O}_U$$

where $j_U: U \hookrightarrow X$. Indeed, for any sheaf \mathcal{F} ,

$$\text{Hom}_{\mathcal{O}_X}\left(\bigoplus_{U \subseteq X} j_{U!} \mathcal{O}_U, \mathcal{F}\right) = \prod_U \text{Hom}_{\mathcal{O}_X}(j_{U!} \mathcal{O}_U, \mathcal{F}) = \prod_U \text{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \prod_U \mathcal{F}(U)$$

Theorem 3.3.5. If \mathcal{A} is a Grothendieck category, then for any $M \in C(\mathcal{A})$, there exists a K -injective complex M' and a quasi-isomorphism $M \rightarrow M'$.

In particular, there exists an equivalence of category $K_{hi}(\mathcal{A}) \simeq D(\mathcal{A})$, where $K_{hi}(\mathcal{A})$ is the full subcategory of $K(\mathcal{A})$ consisting of homotopically injective objects.

Corollary 3.3.6. Let \mathcal{A} be a Grothendieck category, \mathcal{B} an abelian category. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then $RF: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ exists and $\forall I \in K_{hi}(\mathcal{A}), FI \xrightarrow{\sim} RFI$.

Corollary 3.3.7. If \mathcal{A} is a Grothendieck category, then the derived Hom exists

$$R\text{Hom}: D(\mathcal{A})^{\text{op}} \times D(\mathcal{A}) \rightarrow D(\mathcal{A})$$

and $\forall I \in K_{hi}(\mathcal{A}), M \in K(\mathcal{A}), \text{Hom}^\bullet(M, I) \xrightarrow{\sim} R\text{Hom}(M, I)$.

Proof. We need to prove if M or I is acyclic, then

$$H^n(\text{Hom}^\bullet(M, I)) = \text{Hom}_{K(\mathcal{A})}(M, I[n]) = 0$$

Indeed, if M is acyclic, then it is 0 by definition. If I is acyclic, the $I = 0$ in $K(\mathcal{A})$. \square

Remark 3.3.8. For any $M, N \in C(\mathcal{A})$, if I is a K -injective complex and $N \rightarrow I$ is a quasi-isomorphism, then

$$\begin{aligned} H^n R\text{Hom}(M, N) &= H^n \text{Hom}^\bullet(M, I) \\ &= \text{Hom}_{K(\mathcal{A})}(M, I[n]) = \text{Hom}_{D(\mathcal{A})}(M, I[n]) \\ &= \text{Hom}_{D(\mathcal{A})}(M, N[n]) \end{aligned}$$

Denote $\text{Ext}^n(M, N) = \text{Hom}_{D(\mathcal{A})}(M, N[n])$.

3.3.2 Continue four operations

The above discussion shows the right derived functor $Rf_*: D(X) \rightarrow D(Y)$ exists.

Next we consider derived $\mathcal{H}\mathbf{om}$.

$$\mathcal{H}\mathbf{om}: \mathrm{Shv}(X, \mathcal{O}_X)^{\mathrm{op}} \times \mathrm{Shv}(X, \mathcal{O}_X) \rightarrow \mathrm{Shv}(X, \mathcal{O}_X)$$

induces

$$\begin{array}{ccc} C(X)^{\mathrm{op}} \times C(X) & \xrightarrow{\mathcal{H}\mathbf{om}^{\bullet\bullet}} & C^2(X) \\ & \searrow_{\mathcal{H}\mathbf{om}^\bullet} & \downarrow \mathrm{tot}_{\sqcap} \\ & & C(X) \end{array}$$

hence

$$\mathcal{H}\mathbf{om}: K(X)^{\mathrm{op}} \times K(X) \rightarrow K(X)$$

Recall

$$\begin{aligned} \mathcal{H}\mathbf{om}^{\bullet\bullet}(M, N)^{i,j} &= \mathcal{H}\mathbf{om}(M^{-j}, N^i) \\ d_I^{i,j} &= \mathcal{H}\mathbf{om}(M^{-j}, d_N^i) \\ d_{II}^{i,j} &= \mathcal{H}\mathbf{om}((-1)^j d_M^{-i-1}, N^j) \\ \mathcal{H}\mathbf{om}^\bullet(M, N)^n &= \prod_{j \in \mathbb{Z}} \mathcal{H}\mathbf{om}(M^j, N^{j+n}) \end{aligned}$$

Theorem 3.3.9. $R\mathcal{H}\mathbf{om}: D(X)^{\mathrm{op}} \times D(X) \rightarrow D(X)$ exists. If $M \in K(X)$, $I \in K_{hi}(X)$, then $\mathcal{H}\mathbf{om}^\bullet(M, I) \xrightarrow{\sim} R\mathcal{H}\mathbf{om}(M, I)$.

Proof. Let $M, I \in K(X)$. For $U \subseteq X$ open, $\Gamma(U, \mathcal{H}\mathbf{om}^\bullet(M, I)) = \mathrm{Hom}^\bullet(M|_U, I|_U)$. If one of them is acyclic, then $\mathrm{Hom}^\bullet(M|_U, I|_U) = \mathrm{Hom}^\bullet(j_{U!}M|_U, I)$ is acyclic. \square

Denote $\mathcal{E}\mathbf{xt}^i(M, N) = \mathcal{H}^i R\mathcal{H}\mathbf{om}(M, N)$.

Next we move to left derived functors

$$- \otimes_{\mathcal{O}_X} -: \mathrm{Shv}(X, \mathcal{O}_X) \times \mathrm{Shv}(X, \mathcal{O}_X) \rightarrow \mathrm{Shv}(X, \mathcal{O}_X)$$

induces

$$\begin{array}{ccc} C(X) \times C(X) & \longrightarrow & C^2(X) \\ & \searrow & \downarrow \mathrm{tot}_{\oplus} \\ & & C(X) \end{array}$$

Recall

$$\begin{aligned} (M \otimes N)^{i,j} &= M^i \otimes N^j \\ \mathrm{tot}_{\oplus}(M \otimes N)^n &= \bigoplus_{i+j=n} (M^i \otimes N^j) \end{aligned}$$

It factors through $K(X) \times K(X) \rightarrow K(X)$.

Definition 3.3.10. An \mathcal{O}_X -module \mathcal{F} is said to be **flat** if $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is exact.

A complex $F \in K(X)$ is said to be **K -flat** if for all acyclic complex $M \in K(X)$, $\mathrm{tot}_{\oplus}(F \otimes_{\mathcal{O}_X} M)$ is acyclic.

Remark 3.3.11. We can check flatness locally.

- \mathcal{F} is a flat \mathcal{O}_X -module $\iff \forall x \in X, \mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$ -module.
- A complex F is K -flat $\iff \forall x \in X, F_x$ is a K -flat $\mathcal{O}_{X,x}$ -module complex.

Theorem 3.3.12. $-\otimes_{\mathcal{O}_X}^L - : D(X) \times D(X) \rightarrow D(X)$ exists. For a K -flat $F \in K(X)$ and $M \in K(X)$, $\text{tot}_{\oplus}(F \otimes M) \xrightarrow{\sim} F \otimes^L M$.

Lemma 3.3.13. If $\mathcal{F} \in \text{Shv}(X, \mathcal{O}_X)$, then there exists a flat \mathcal{F}' and a surjective morphism $\mathcal{F}' \rightarrow \mathcal{F}$.

Proof. Let

$$\mathcal{F}' = \bigoplus_{U \subseteq X \text{ open}, s \in \mathcal{F}(U)} j_{U!} \mathcal{O}_X$$

It is flat. The morphism $j_{U!} \mathcal{O}_X \rightarrow \mathcal{F}$ with index s is defined by $\mathcal{O}_U \rightarrow \mathcal{F}|_U$ sending $1 \rightarrow s$. \square

Lemma 3.3.14. $\forall M \in C(X)$, there exists a quasi-isomorphism $F \rightarrow M$ with F K -flat. In fact, we can take $F = \text{Cone}(P \rightarrow Q)$ where P, Q are direct sums of objects in $C^-(\mathcal{F})$ and $\mathcal{F} = \{\text{flat } \mathcal{O}_X\text{-modules}\}$.

Lemma 3.3.15. If $F \in C^-(X)$, then F^i flat $\implies F$ is K -flat.

Proof. We need to show if $M \in C(X)$, then $\text{tot}_{\oplus}(F \otimes M)$ is acyclic. Since

$$\tau^{\leq n} M \rightarrow \tau^{\leq n+1} M \rightarrow \cdots \rightarrow M$$

we have $\text{colim}_n \tau^{\leq n} M = M$. Since tensor product commutes with colimits, we may assume $M \in C^-(M)$. In this case $F \otimes M$ is biregular. Since F^i is flat, $F^i \otimes M^\bullet$ is acyclic, hence its total complex is acyclic. \square

proof of Theorem 3.3.12. If F or M is acyclic, we need to show $\text{tot}_{\oplus}(F \otimes M)$ is acyclic. If M is acyclic, this is true by definition. If F is acyclic, let $M' \rightarrow M$ be a quasi-isomorphism with M' K -flat. Since F is K -flat, $\text{tot}_{\oplus}(M' \otimes F) \rightarrow \text{tot}_{\oplus}(M \otimes F)$ is a quasi-isomorphism. But F is acyclic and M' is K -flat implies $\text{tot}_{\oplus}(M \otimes F)$ is acyclic. \square

Denote $\mathcal{T}\text{or}_i^{\mathcal{O}_X}(M, N) = \mathcal{H}^{-i}(M \otimes_{\mathcal{O}_X}^L N)$.

Proposition 3.3.16. $Lf^* : D(Y) \rightarrow D(X)$ exists and for $F \in K(Y)$ K -flat, $f^* F \xrightarrow{\sim} Lf^* F$.

Proof. We need to show if F is K -flat and acyclic, then $f^* F$ is acyclic. But $f^* F = f^{-1} F \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$ and $f^{-1} F$ is K -flat, hence it is acyclic. \square

Date: 12.17

Recall for $f: X \rightarrow Y$, we have defined Lf^* , Rf_* , $- \otimes^L -$, $R\mathcal{H}\text{om}(-, -)$. Let us clarify their relations.

- $Lf^*(M \otimes^L N) \simeq Lf^*M \otimes^L Lf^*N$.
- $(L \otimes^L M) \otimes^L N \simeq L \otimes^L (M \otimes^L N)$.
- $M \otimes^L N \simeq N \otimes^L M$.
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $Lf^* \circ Lg^* \simeq L(gf)^*$.

Proposition 3.3.17. *We have $- \otimes^L M \dashv R\mathcal{H}\text{om}(M, -)$, i.e.*

$$\text{Hom}_{D(X)}(L \otimes^L M, N) \simeq \text{Hom}_{D(X)}(L, R\mathcal{H}\text{om}(M, N))$$

Proof. We may replace M by a K -flat complex M' and N by a K -injective complex N' . Then the statement follows from taking the H^0 of the equation below as long as we have shown $\mathcal{H}\text{om}^\bullet(M', N')$ is K -injective, which is the content of next Lemma.

$$\text{Hom}^\bullet(\text{tot}_\oplus(L \otimes M'), N') = \text{Hom}^\bullet(L, \mathcal{H}\text{om}^\bullet(M', N'))$$

□

Lemma 3.3.18. *If M is K -flat, N is K -injective. Then $\mathcal{H}\text{om}^\bullet(M, N)$ is K -injective.*

Proof. Let L be acyclic. Then

$$\text{Hom}^\bullet(L, \mathcal{H}\text{om}^\bullet(M, N)) = \text{Hom}^\bullet(\text{tot}_\oplus(L \otimes M), N)$$

Since M is K -flat, $\text{tot}_\oplus(L \otimes M)$ is K -flat. Since N is K -injective, $\text{Hom}^\bullet(L, \mathcal{H}\text{om}^\bullet(M, N))$ is acyclic. □

Proposition 3.3.19. *For $M \in D(X)$, $N \in D(Y)$, we have*

$$\text{Hom}_{D(X)}(Lf^*N, M) \simeq \text{Hom}_{D(Y)}(N, Rf_*(M))$$

Proof. We may assume N is K -flat, M is K -injective. Then

$$\text{Hom}_{D(X)}(N, f_*M) = \underset{N' \xrightarrow{\text{qis}} N}{\text{colim}} \text{Hom}_{K(Y)}(N', f_*M)$$

where $N' \rightarrow N$ is quasi-isomorphism and N' is, without loss of generality, K -flat. Therefore

$$\begin{aligned} \text{Hom}_{K(Y)}(N', f_*M) &= H^0 \text{Hom}^\bullet(N', f_*M) \\ &= H^0 \text{Hom}^\bullet(f^*N', M) \quad (\text{since } f^*N' \xrightarrow{\text{qis}} f^*N) \\ &= H^0 \text{Hom}^\bullet(f^*N, M) \\ &= \text{Hom}_{D(X)}(f^*N, M) \end{aligned}$$

□

Proposition 3.3.20. $R\text{Hom}(M, N) = R\Gamma(X, R\mathcal{H}\text{om}(M, N))$

Proof. We may assume N is K -injective, M is K -flat. Then

$$R\text{Hom}(M, N) = \text{Hom}(M, N) = \Gamma(X, \mathcal{H}\text{om}^\bullet(M, N)) = R\Gamma(X, R\mathcal{H}\text{om}(M, N))$$

due to $\mathcal{H}\text{om}(M, N)$ being K -injective. □

3.4 Grothendieck-Serre duality

Let $f: X \rightarrow Y$ be quasi-compact and quasi-separated. We have shown $Rf_*: D_{qch}^+(X) \rightarrow D_{qch}^+(Y)$. In fact it extends to $Rf_*: D_{qch}(X) \rightarrow D_{qch}(Y)$, where $D_{qch}(X) = \{L \in D(X) \mid \mathcal{H}^i L \in \mathrm{QCoh}(X), \forall i \in \mathbb{Z}\}$.

Proposition 3.4.1 (Projection formula). *The morphism*

$$Lf^*(N \otimes^L Rf_* M) \xrightarrow{\sim} Lf^* N \otimes^L Lf^* Rf_* M \rightarrow Lf^* N \otimes^L M$$

induces by adjunction, an isomorphism

$$N \otimes^L Rf_* M \xrightarrow{\sim} Rf_*(Lf^* N \otimes^L M)$$

where $M \in D_{qch}(X)$, $N \in D_{qch}(Y)$.

Proof. Reduce to $N = \mathcal{O}_Y$ □

Theorem 3.4.2. *Let X, Y be quasi-compact, quasi-separated schemes, $f: X \rightarrow Y$ quasi-compact and quasi-separated. Then $Rf_*: D_{qch}(X) \rightarrow D_{qch}(Y)$ admits a right adjoint*

$$f^\times: D_{qch}(Y) \rightarrow D_{qch}(X)$$

which is necessarily a triangulated functor.

If f is proper, we write $f^!$ instead of f^\times .

The adjunction means, more explicitly, for $M \in D_{qch}(X)$, $N \in D_{qch}(Y)$,

$$\mathrm{Hom}_Y(Rf_* M, N) \simeq \mathrm{Hom}_X(M, f^\times N)$$

Corollary 3.4.3. *We have*

- (1) $R\mathcal{H}\mathbf{om}_Y(Rf_* M, N) \simeq Rf_* R\mathcal{H}\mathbf{om}_X(M, f^\times N)$.
- (2) $R\mathrm{Hom}_Y(Rf_* M, N) \simeq R\mathrm{Hom}_X(M, f^\times N)$.

Proof. (2) is a consequence of (1) by taking $R\Gamma(Y, -)$.

For (1), we apply Yoneda's Lemma. Let $L \in D(Y)$.

$$\begin{aligned} \mathrm{Hom}(L, R\mathcal{H}\mathbf{om}_Y(Rf_* M, N)) &= \mathrm{Hom}(L \otimes^L Rf_* M, N) \\ &= \mathrm{Hom}(Rf_*(Lf^* L \otimes^L M), N) \quad \text{by projection formula} \\ &= \mathrm{Hom}(Lf^* \otimes^L M, f^\times N) \\ &= \mathrm{Hom}(Lf^* L, R\mathcal{H}\mathbf{om}(M, f^\times N)) \\ &= \mathrm{Hom}(L, Rf_* R\mathcal{H}\mathbf{om}(M, f^\times N)) \end{aligned}$$

□

Theorem 3.4.4 (Grothendieck). *Let $f: X \rightarrow Y$ be a flat and proper morphism of Noetherian scheme*

- For $N \in D_{qch}^+$, $f^! N = Lf^* N \otimes_{\mathcal{O}_X}^L f^! \mathcal{O}_Y$

- If moreover f is smooth, then $f^! \mathcal{O}_Y = \omega_{X/Y}[d]$ where $d = \dim(f)$.

Corollary 3.4.5 (Serre). Let k be a field, X a proper smooth scheme over k of pure dimension d . Then

$$H^j(X, M)^\vee \simeq \mathrm{Ext}^{d-j}(M, \omega_{X/k})$$

In particular, if M is locally free of finite rank, then

$$H^j(X, M)^\vee \simeq H^{d-j}(X, M^\vee \otimes \omega_{X/k})$$

Proof. Let $Y = \mathrm{Spec}(k)$, $N = \mathcal{O}_Y = k$. By Corollary 3.4.3 (2), we have

$$R\mathrm{Hom}(R\Gamma(X, M), k) \simeq R\mathrm{Hom}(M, \omega_{X/k}[d])$$

Take H^{-j} , we get the first formula. Moreover, if M is locally free, we have

$$\mathcal{H}\mathrm{om}(M, \omega_{X/k}) \simeq M^\vee \otimes \omega_{X/k}$$

□

Example 3.4.6. Let $M = \Omega_{X/k}^i$. There is a perfect pairing

$$\Omega_{X,k}^i \times \Omega_{X/k}^{d-i} \rightarrow \omega_{X/k}$$

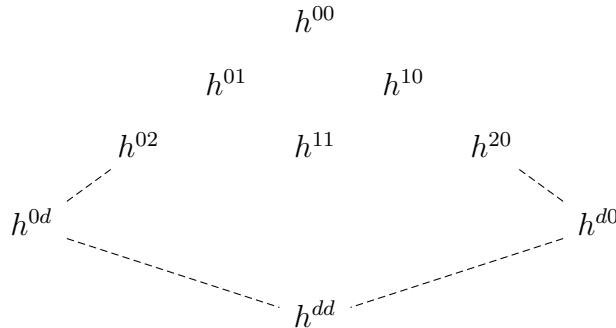
Then

$$(\Omega_{X/k}^i)^\vee = \Omega_{X/k}^{d-i} \otimes \omega_{X/k}^\vee$$

Serre duality takes the form

$$H^j(X, \Omega_{X/k}^i)^\vee \simeq H^{d-j}(X, \Omega_{X/k}^{d-i})$$

Moreover, the Hodge number $h^{i,j} = H^j(X, \Omega_{X/k}^i)$ satisfies $h^{i,j} = h^{d-i, d-j}$. There is a **Hodge diamond**, which is invariant under rotating 180°.



Recall $\Omega_{X/k}^\bullet$ is the de Rham complex. We have a spectral sequence

$$(*) \quad E_1^{i,j} = H^j(X, \Omega^i X/k) \implies H_{dR}^{i+j}(X/k) := H^{i+j}(X, \Omega_{X/k}^\bullet)$$

This is called **Hodge to de Rham spectral sequence**.

This spectral sequence implies

$$\dim_k H_{dR}^n(X/k) \leq \sum_{i+j=n} h^{i,j}$$

and equality holds \iff $(*)$ degenerates at E_1 .

Theorem 3.4.7 (Hodge, Deligne). *When $\text{char } k = 0$, we have*

- $h^{i,j} = h^{j,i}$ (Hodge symmetry).
- (*) degenerates at E_1 (Hodge decomposition).

Theorem 3.4.8 (Deligne-Illusie). *When $\text{char } k > 0$. If there is a pull-back diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow f \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(W_2(k)) \end{array}$$

where X_1 is a first order thickening and f is proper smooth, then (*) degenerates at E_1 .

An example of $W_2(k)$ is $W_2(\mathbb{F}_p) = \mathbb{Z}/p^2$.

Corollary 3.4.9 (to Serre duality 3.4.5). *Let k be algebraically closed, X/k be a smooth projective variety of $\dim X \geq 2$. Suppose $D \subseteq X$ is an effective Cartier divisor of ideal \mathcal{I} such that \mathcal{I}^\vee is ample. Then D is connected.*

Proof. Consider $i: nD \rightarrow X$, whose ideal sheaf is \mathcal{I}^n . We have a short exact sequence

$$0 \longrightarrow \mathcal{I}^n \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_{nD} \longrightarrow 0$$

Take cohomology, we see

$$0 \longrightarrow H^0(X, \mathcal{I}^{\otimes n}) \longrightarrow H^0(X, \mathcal{O}_X) = k \longrightarrow H^0(nD, i_* \mathcal{O}_{nD}) \longrightarrow H^1(X, \mathcal{I}^{\otimes n})$$

If $H^0(X, \mathcal{I}^{\otimes n})$ is non-zero, $H^0(X, \mathcal{I}^{\otimes n})$ contains $1 \in k$, hence $\mathcal{I}^{\otimes n} = \mathcal{O}_X$, which is not ample, a contradiction. Thus $H^0(X, \mathcal{I}^{\otimes n}) = 0$. We want to prove $H^0(nD, i_* \mathcal{O}_{nD}) = k$. By Serre duality, $H^1(X, \mathcal{I}^{\otimes n}) = H^{d-1}(X, \mathcal{I}^{\otimes -n} \otimes \omega_{X/k})$. This vanishes for $n \gg 0$ since \mathcal{I}^\vee is ample and $d \geq 2$. Thus D , whose underlying space coincides with nD , is connected. \square

To prove the existence of f^\times , we need some preparation from category theory.

3.4.1 Adjoint functor theorem

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If F admits a right adjoint, then it preserves small colimits. Conversely, if \mathcal{C} has enough colimits which are preserved by F , then F admits a right adjoint. This is the content of adjoint functor theorem, which can be found in books of category theory. In addition, if \mathcal{C} is a Grothendieck category. Then F admits a right adjoint $\iff F$ preserves small colimits. We are going to study it in the setting of triangulated categories.

Definition 3.4.10. Let \mathcal{D} be a triangulated category admitting small coproduct.

- An object $M \in \mathcal{D}$ is said to be **compact** if $\text{Hom}_{\mathcal{D}}(M, -)$ preserves small coproducts.

- A set $\mathcal{S} \subseteq \text{Ob}(\mathcal{D})$ is said to be **generating** if $\text{Hom}_{\mathcal{D}}(M, N) = 0$ for all $M \in \mathcal{S}$ implies $N = 0$.
- \mathcal{D} is said to be **compactly generated** if $\exists \mathcal{S} \subseteq \text{Ob}(\mathcal{D})$ a generating set of compact objects.

Theorem 3.4.11 (Brown representability, Nieman 1996). *Let $F: \mathcal{D} \rightarrow \mathcal{D}'$ be a triangulated functor between triangulated categories. Suppose \mathcal{D} admits small coproducts while F preserves such coproducts and \mathcal{D} is compactly generated. Then F admits a triangulated right adjoint.*

We are going to describe compact object in $D(X)$.

Definition 3.4.12. Let (X, \mathcal{O}_X) be a ringed space, $M \in D(X)$ is said to be **perfect** if $\forall x \in X$, there exists an open neighborhood U of x such that $M|_U \simeq N$ in $D(U)$ with $N \in C^b(X)$ and that for each i , N^i is a direct summand of a free \mathcal{O}_X -module of finite rank.

Lemma 3.4.13. *If M is perfect, then $R\text{Hom}(M, N) \simeq M^\vee \otimes^L N$, where $M^\vee = R\mathcal{H}\text{om}(M, \mathcal{O}_X)$.*

Example 3.4.14. If X is a quasi-compact and quasi-separated scheme, $M \in D(X)$ perfect. Then $\text{Hom}_{D(X)}(M, -) = \Gamma(X, M^\vee \otimes^L -)$, which preserves small coproducts.

In fact, $M \in D_{qch}(X)$ is compact \iff it is perfect.

Theorem 3.4.15 (Bondal, van den Bergh). *Suppose X is a quasi-compact and quasi-separated scheme. Then $\exists M \in D(X)$ perfect such that $\{M[n] \mid n \in \mathbb{Z}\}$ generates $D_{qch}(X)$.*

Remark 3.4.16. If f is quasi-compact and quasi-separated, then $Rf_*: D_{qch}(X) \rightarrow D_{qch}(Y)$ preserves small coproducts.

The above discussion shows the existence of f^\times .

We can say something about f^\times when f is a regular immersion. Let us start with regular sequences.

3.4.2 Koszul complex

Let A be a ring, E an A -module. Let $u: E \rightarrow A$ be a homomorphism of A -modules. The **Koszul complex** $K_\bullet(u) \in C^{\leq 0}(\text{Mod}_A)$ is defined as $K_\bullet(u)^{-n} = \bigwedge^n E$.

$$d^{-n}: \bigwedge^n E \rightarrow \bigwedge^{n-1} E$$

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^n (-1)^{i-1} u(x_i) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n$$

$$\cdots \longrightarrow \bigwedge^3 E \longrightarrow \bigwedge^2 E \longrightarrow E \longrightarrow 0$$

If M is an A -module, we set $K(u, M) = K(u) \otimes_A M$

Example 3.4.17. $E = A^r$, $u: A^r \rightarrow A$, $u(a_1, \dots, a_n) = \sum_i a_i f_i$. Denote $K_\bullet(u) = K_\bullet(f_1, \dots, f_r)$. We have

$$K_\bullet(u) = \text{tot}_\oplus \left(\bigotimes_{i=1}^r (A \xrightarrow{f_i} A) \right)$$

$$0 \longrightarrow A = \bigwedge^r A^r \longrightarrow \bigwedge^{r-1} A^r \longrightarrow \dots \longrightarrow A^r \longrightarrow A \longrightarrow 0$$

It is clear in this case,

$$H^0(K_\bullet(u, M)) = M/(f_1, \dots, f_r)M$$

Definition 3.4.18. Let $f_1, \dots, f_r \in A$, M an A -module.

- A sequence (f_1, \dots, f_r) is said to be **M -regular** if $\forall i > 0$,

$$M/\sum_{j < i} f_j M \xrightarrow{\times f_i} M/\sum_{j < i} f_j M$$

is injective.

- A sequence (f_1, \dots, f_r) is said to be **Koszul- M -regular** if $K(f_1, \dots, f_r, M) \rightarrow M/(f_1, \dots, f_r)M$ is a quasi-isomorphism. If $M = A$, then it is called **Koszul regular**.

Theorem 3.4.19 (Serre). *Consider the following conditions*

- (1) (f_1, \dots, f_r) is M -regular.
- (2) (f_1, \dots, f_r) is Koszul- M -regular.
- (3) $H^{-1}K(f_1, \dots, f_r, M) = 0$.

We have $(a) \implies (b) \implies (c)$

Moreover, if A is Noetherian, M a finitely generated A -module, $f_1, \dots, f_r \in \text{rad}(A)$, then $(c) \implies (a)$.

We first observe that for $L \in C(\text{Mod}_A)$, $x \in A$, $K_\bullet(x) = 0 \rightarrow A \xrightarrow{x} A \rightarrow 0$. Then

$$\text{tot}_\oplus(K_\bullet(x) \otimes_A L) = \text{Cone}(L \xrightarrow{\times x} L)$$

It induces a long exact sequence

$$(*) \quad H^q(L) \xrightarrow{\times x} H^q(L) \longrightarrow H^q(\text{tot}_\oplus(K_\bullet(x) \otimes_A L)) \longrightarrow H^{q+1}(L) \xrightarrow{\times x} H^{q+1}(L)$$

Use the exactness at the middle term, we have a short exact sequence

$$(**) \quad 0 \longrightarrow H^0(K(x, H^q(L))) \longrightarrow H^q(\text{tot}_\oplus(K_\bullet(x) \otimes_A L)) \longrightarrow H^{-1}(K(x, H^{q+1}(L))) \longrightarrow 0$$

Proof. (a) \implies (b). Let us do induction on r .

$r = 0$ is clear. For $r > 0$. Let $L = K(f_1, \dots, f_{r-1})$. Then $H^q(K(f_1, \dots, f_r, M)) = H^q(\text{tot}_{\oplus}(K_{\bullet}(f_r) \otimes_A L))$ and our goal is to prove it vanishes at $q \leq -1$.

Look at the long exact sequence (*). Since $L \rightarrow M/(f_1, \dots, f_{r-1})M$ is a quasi-isomorphism, we have for $q \leq -2$, $H^q(L) = H^{q+1}(L) = 0$, which implies $H^q(\text{tot}_{\oplus}(K_{\bullet}(f_r) \otimes_A L)) = 0$. For $q = -1$, we have

$$0 \longrightarrow H^{-1}(\text{tot}_{\oplus}(K_{\bullet}(x) \otimes_A L)) \longrightarrow H^0(L) \xrightarrow{\times f_r} H^0(L)$$

But $H^0(L) = M/(f_1, \dots, f_{r-1})M$ and multiplication by f_r is injective by assumption, hence $H^{-1}(\text{tot}_{\oplus}(K_{\bullet}(x) \otimes_A L)) = 0$. This shows $H^q(\text{tot}_{\oplus}(K_{\bullet}(x) \otimes_A L)) = 0$ for $q \leq -1$ and concludes the assertion.

(b) \implies (c) is clear.

(c) \implies (a). Let $L = K(f_1, \dots, f_{r-1})$. By substituting $q = -1$ in (**) we see $H^{-1}K(f_1, \dots, f_r, M) = 0$ implies multiplication by f_r on $M/(f_1, \dots, f_{r-1})M$ is injective. Therefore, the assertion holds once we prove $H^{-1}K(f_1, \dots, f_r, M) = 0$ implies $H^{-1}K(f_1, \dots, f_{r-1}, M) = 0$.

By substituting $q = -1$ in (*), we see multiplication by f_r on $H^{-1}(L)$ is surjective. Since $f_r \in \text{rad}(A)$ and M is finitely generated, we must have $H^{-1}(L) = 0$ due to Nakayama's Lemma. \square

Definition 3.4.20. An immersion $i: X \rightarrow Y$ is called **regular** (resp. **Koszul regular**) if $\forall x \in X$, there exists an affine open neighborhood $U = \text{Spec}(A) \subseteq Y$ of x such that $X \cap U = \text{Spec}(A/I)$ with I generated by a regular (resp. **Koszul regular**) sequence f_1, \dots, f_r .

It is clear regular immersion \implies Koszul regular. If Y is locally Noetherian, then the converse holds.

Example 3.4.21. If $D \subseteq X$ is an effective Cartier divisor, then the immersion is regular.

Example 3.4.22. If f and g are smooth, then i is a regular immersion.

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow f & \swarrow g & \\ S & & \end{array}$$

Proposition 3.4.23. If $i: X \rightarrow Y$ is a regular immersion, then $i^! \mathcal{O}_Y \simeq \wedge^i \mathcal{N}_{X/Y}[-c]$ where $c = \text{codim}(i)$.

Remark 3.4.24. If $f: X \rightarrow Y$ is proper and Y is regular, then $K_{X/Y} = f^! \mathcal{O}_Y$ is called the **dualizing complex**. Set $D_{X/Y} = R\mathcal{H}\text{om}(-, f^! \mathcal{O}_Y)$. Then

$$D_{X/Y}: D_{coh}^b(X)^{\text{op}} \rightarrow D_{coh}^b(X)$$

is an equivalence of categories satisfying for $M \in D_{coh}^b(X)$, $M \simeq D_{X/Y} D_{X/Y} M$.

There are remarkable books explaining the concept dualizing complex. Hartshorne: Residues and duality. Conrad wrote a book verifying commutativity of diagrams. Also Lipman. In all, Stacks project is a good reference.

Chapter 4

Divisors

Date: 12.22

4.1 Cartier divisors

Recall we have defined effective Cartier divisor on X as a closed subscheme $D \subseteq X$ whose ideal sheaf \mathcal{I}_D is invertible. If D_1, D_2 are effective Cartier divisors, then $D_1 + D_2$ is an effective Cartier divisor defined by $\mathcal{I}_{D_1}\mathcal{I}_{D_2} = \mathcal{I}_{D_1} \otimes_{\mathcal{O}_X} \mathcal{I}_{D_2}$. The collection of effective Cartier divisors CaDiv_+ is a commutative monoid.

Definition 4.1.1. Let \mathcal{L} be an invertible \mathcal{O}_X -module. A section $s \in \Gamma(X, \mathcal{L})$ is said to be **regular** if $\mathcal{O}_X \xrightarrow{\times s} \mathcal{L}$ is a monomorphism.

Consider the equivalence class of pairs $\{(\mathcal{L}, s)\}/\sim$ where \mathcal{L} is an invertible sheaf and $s \in \Gamma(X, \mathcal{L})$ is a regular section. Two elements $(\mathcal{L}, s) \sim (\mathcal{L}', s')$ if $\exists \phi: \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\phi(s) = s'$. Define $(\mathcal{L}, s) \otimes (\mathcal{L}', s') = (\mathcal{L} \otimes \mathcal{L}', s \otimes s')$. This makes $\{(\mathcal{L}, s)\}/\sim$ into a commutative monoid.

Proposition 4.1.2. *There is an isomorphism of monoids given as follows*

$$\text{CaDiv}_+ \xrightleftharpoons[\psi]{\phi} \{(\mathcal{L}, s)\}/\sim$$

For $D \in \text{CaDiv}_+$, the dual of $\mathcal{I}_D \hookrightarrow \mathcal{O}_X$ gives $\mathcal{O}_X \rightarrow \mathcal{I}_D^\vee$. Denote $\mathcal{O}(D) = \mathcal{I}_D$. Then $\phi(D) = (\mathcal{O}(D), \mathcal{O}_X \rightarrow \mathcal{O}(D))$.

For (\mathcal{L}, s) , s defines $\mathcal{O}_X \rightarrow \mathcal{L}$. Its dual is $\mathcal{L}^\vee \rightarrow \mathcal{O}_X$. Then $\psi(\mathcal{L}, s)$ is defined as $\underline{\text{Spec}}(\text{coker}(\mathcal{L}^\vee \rightarrow \mathcal{O}_X))$.

4.1.1 Sheaf of meromorphic functions

Warning 4.1.3. EGA and Hartshorne gives misleading definitions. See Kleiman, Misconceptions about K_X

Definition 4.1.4. We define the **sheaf of regular functions** $\mathcal{S} \subseteq \mathcal{O}_X$ as follows. For $U \subseteq X$ open, $\mathcal{S}(U) = \{s \in \mathcal{O}_X(U) \text{ regular}\}$, i.e. $\forall x \in U$, $s_x \in \mathcal{O}_{X,x}$ is a non-zero divisor.

If $s \in \mathcal{S}(U)$, then s is a non-zero divisor in $\mathcal{O}_X(U)$. Indeed, $\mathcal{O}_U \xrightarrow{s} \mathcal{O}_X$ is a monomorphism, hence $\mathcal{O}_X(U) \xrightarrow{s} \mathcal{O}_X(U)$ is injective.

Warning 4.1.5. $\mathcal{S}(U) \subsetneq \{\text{non-zero divisors in } \mathcal{O}_X(U)\}$ in general.

Remark 4.1.6. If $U = \text{Spec}(A)$, then $\mathcal{S}(U) = \{\text{non-zero divisors in } A\}$.

Definition 4.1.7. We define the **sheaf of meromorphic functions** $K_X = a(U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U))$

It is clear $\mathcal{O}_X(U) \hookrightarrow \mathcal{S}(U)^{-1}\mathcal{O}_X(U) \hookrightarrow K_X(U)$.

Warning 4.1.8. $\mathcal{S}(U)^{-1}\mathcal{O}_X(U) \hookrightarrow K_X(U)$ is not an isomorphism in general even for U affine.

Definition 4.1.9. Let X be a scheme.

- Let \mathcal{F} be an \mathcal{O}_X -module. A meromorphic section of \mathcal{F} is a global section of $\mathcal{F} \otimes_{\mathcal{O}_X} K_X = a(U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U))$.
- If \mathcal{L} is an invertible sheaf, then $\mathcal{L} \hookrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} K_X$. A section $s \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} K_X)$ is said to be **regular** if $\mathcal{O}_X \xrightarrow{s} \mathcal{L} \otimes_{\mathcal{O}_X} K_X$ is a monomorphism, i.e. $K_X \xrightarrow{s} \mathcal{L} \otimes_{\mathcal{O}_X} K_X$ is a monomorphism

Definition 4.1.10. A **Cartier divisor** on X is a global section of $K_X^\times/\mathcal{O}_X^\times$. The collection of Cartier divisors is denoted by $\text{CaDiv}(X)$.

Proposition 4.1.11. *There are isomorphisms of abelian groups*

$$\Gamma(X, K_X^\times/\mathcal{O}_X^\times) \xrightarrow[\phi]{\sim} \{\text{invertible sub sheaves of } K_X\} \xrightarrow[\psi]{\sim} \{(\mathcal{L}, s)\}/\sim$$

where the last set consists of pairs (\mathcal{L}, s) in which \mathcal{L} is an invertible sheaf and s is regular meromorphic section of \mathcal{L} . The isomorphism ϕ and ψ are given as follows.

A Cartier divisor $f \in \Gamma(X, K_X^\times/\mathcal{O}_X^\times)$ is given by a collection of data $\{(U_i, f_i)\}$ with $f_i \in \Gamma(U_i, K_X^\times)$. Then $f_i \mathcal{O}_{U_i} \subseteq K_X|_{U_i}$ defines an invertible sheaf $\phi(f) = \mathcal{I}$. Conversely, an invertible sheaf $\mathcal{I} \subseteq K_X^\times/\mathcal{O}_X^\times$ is defined locally by $\mathcal{I}|_{U_i} = f_i \mathcal{O}_{U_i}$ with $f_i \in K_X(U_i)$. These f_i glues to a global section $\phi^{-1}\mathcal{I} = f \in \Gamma(X, K_X^\times/\mathcal{O}_X^\times)$.

For an invertible sheaf $\mathcal{I} \subseteq K_X$, $\psi(\mathcal{I}) = (\mathcal{I}^\vee, \mathcal{O}_X \hookrightarrow \mathcal{I}^\vee \otimes K_X)$. For a pair (\mathcal{L}, s) , $s: \mathcal{O}_X \hookrightarrow \mathcal{L} \otimes K_X$ gives $\psi^{-1}\mathcal{L} = \mathcal{L}^\vee \xrightarrow{s \otimes \mathcal{L}^\vee} K_X$.

Remark 4.1.12. The definitions of Cartier divisors agree in the sense $\text{CaDiv}_+(X) \hookrightarrow \text{CaDiv}(X)$.

Consider the short exact sequence

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow K_X^\times \longrightarrow K_X^\times/\mathcal{O}_X^\times \longrightarrow 1$$

Take global sections,

$$1 \longrightarrow \Gamma(X, \mathcal{O}_X^\times) \longrightarrow \Gamma(X, K_X^\times) \longrightarrow \Gamma(X, K_X^\times/\mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^\times) = \text{Pic}(X)$$

Definition 4.1.13. $f \in \Gamma(X, K_X^\times)$ gives $\bar{f} \in \Gamma(X, K_X^\times/\mathcal{O}_X^\times)$ is called a **principal Cartier divisor** defined by f . Define

$$\text{CaCl}(X) := \text{coker}(\Gamma(X, K_X^\times) \rightarrow \Gamma(X, K_X^\times/\mathcal{O}_X^\times))$$

We have $\text{CaCl}(X) \hookrightarrow \text{Pic}(X)$.

Proposition 4.1.14. *Suppose X is an integral scheme. Then $\text{CaCl}(X) \simeq \text{Pic}(X)$.*

Proof. From the exact sequence

$$\Gamma(X, K_X^\times) \longrightarrow \Gamma(X, K_X^\times/\mathcal{O}_X^\times) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, K_X^\times)$$

it is enough to show $H^q(X, K_X^\times) = 0$ for all $q > 0$. Since X is integral, $K_X^\times(U) = R(X)^\times$ where $R(X)$ is the function field of X . Let η be the generic point of X , $i: \eta \rightarrow X$, then $K_X^\times = i_*R(X)^\times$ and i_* is exact. Thus $H^q(X, K_X^\times) = H^q(X, i_*R(X)^\times) = H^q(\eta, R(X)^\times) = 0$, for all $q > 0$.

Another way to see is that K_X^\times is flabby. \square

Remark 4.1.15. $\text{CaCl}(X) \simeq \text{Pic}(X)$ also hold for X Noetherian and admitting an ample invertible sheaf.

4.1.2 Pullback

Let $f: X \rightarrow Y$ be a morphism of schemes. If \mathcal{L} is an invertible \mathcal{O}_Y -module, then $f^*\mathcal{L}$ is an invertible \mathcal{O}_X -module. We get $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$. But f^* does not preserves Cartier divisors since the pullback of $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is $f^*\mathcal{I} \rightarrow \mathcal{O}_Y$ which may not be a monomorphism in general. However, if f is flat, $f^*: \text{CaDiv}(Y) \rightarrow \text{CaDiv}(X)$ is defined.

Proposition 4.1.16. *Let k be a field, X/k a projective scheme.*

- *If \mathcal{L} is an ample invertible sheaf, $s \in \Gamma(X, \mathcal{L})$. Then X_s is affine.*
- *If D is an effective Cartier divisor, $\mathcal{O}(D)$ is ample, then $X \setminus D$ is affine.*

Proof. Since for large m , $\mathcal{L}^{\otimes m}$ is very ample and $X_s = X_{s^{\otimes m}}$, we may assume \mathcal{L} is very ample.

Let $i: X \hookrightarrow \mathbb{P}_k^n$ be an immersion such that $i^*\mathcal{O}(1) \simeq \mathcal{L}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

Twist by d and take cohomology, we have

$$\Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}^n}) \longrightarrow \Gamma(X, \mathcal{L}^{\otimes d}) \longrightarrow H^1(\mathbb{P}^n, \mathcal{I}(d))$$

By Serre's vanishing theorem, $H^1(\mathbb{P}^n, \mathcal{I}(d)) = 0$ for $d \gg 0$. Thus there exists $t \in \Gamma(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}^n})$ which maps to $s^{\otimes d} \in \Gamma(X, \mathcal{L}^{\otimes d})$. Thus $X_s = i^{-1}(\mathbb{P}_k^n)_t = \text{Spec}(R_{(t)})$ is affine, $R = k[x_0, \dots, x_n]$. Since i is a closed immersion, X_s is affine.

Another way to see is that let $S = \bigoplus_{m \in \mathbb{N}} \Gamma(X, \mathcal{L}^{\otimes m})$, $i: X \hookrightarrow \text{Proj}(S)$ is a closed immersion with $X_s = i^{-1}D_+(s)$. \square

4.2 Weil divisors

Throughout this section, X is supposed to be a Noetherian integral scheme.

Definition 4.2.1. Let X be as above.

- A **prime divisor** of X is an integral and closed subscheme $Z \subseteq X$ of codimension 1.
- A **Weil divisor** of X is a formal sum $\sum_Z n_Z Z$ where Z runs through all prime divisors and $n_Z = 0$ for all but finitely many Z .

Denote $\text{Div}(X) = \bigoplus_Z \mathbb{Z}$ where Z runs through all prime divisors. A Weil divisor $\sum_Z u_Z Z$ is **effective** if $u_Z \geq 0$ for all Z .

Let A be a Noetherian local domain of dimension 1. For $a \in A \setminus \{0\}$, the order function is $\text{ord}_A(a) = \lg_A(A/aA)$ where \lg indicates the length. It is finite since A/aA is Artinian. For $a, b \in A \setminus \{0\}$, we have an exact sequence

$$0 \longrightarrow A/aA \xrightarrow{\times b} A/abA \longrightarrow A/bA \longrightarrow 0$$

Thus $\text{ord}_A(ab) = \text{ord}_A(a) + \text{ord}_A(b)$. This formula allows us to extend $\text{ord}_A: K^\times \rightarrow \mathbb{Z}$, $K = \text{Frac}(A)$.

Let $Z \subseteq X$ be a prime divisor with generic point ξ . Define $\text{ord}_Z: R(X)^\times \rightarrow \mathbb{Z}$ as the order function induced by $A = \mathcal{O}_{X,\xi} \subseteq R(X)$. It is called **order of vanishing along Z** . For $f \in R(X)^\times$, $\text{div}(f) = \sum_Z \text{ord}_Z(f)$ is called **principal divisor** defined by f .

Lemma 4.2.2. $\#\{Z \mid \text{ord}_Z(f) \neq 0\}$ is finite.

Proof. There exists an open subset $U \subseteq X$ such that $f \in \mathcal{O}(U)^\times$. If $\text{ord}_Z(f) \neq 0$, then the generic point ξ of Z is not in U , hence $Z \subseteq X \setminus U$. Since Z is of codimension 1, it is an irreducible component of $X \setminus U$, hence there are only finitely many of them. \square

We get a homomorphism $\text{div}: R(X)^\times \rightarrow \text{Div}(X)$.

Definition 4.2.3. Two Weil divisors $D, D' \in \text{Div}(X)$ are said to be **rationally equivalent** or **linearly equivalent** if $D - D' = \text{div}(f)$ for some $f \in R(X)^\times$. Denote $D \sim D'$. The **Weil divisor class group** is $\text{Cl}(X) = \text{Div}(X)/\sim$.

We have an exact sequence

$$R(X)^\times \xrightarrow{\text{div}} \text{Div}(X) \longrightarrow \text{Cl}(X) \longrightarrow 0$$

4.2.1 Cartier divisors vs Weil divisors

Let X be a Noetherian integral scheme. Consider (\mathcal{L}, s) where \mathcal{L} is an invertible \mathcal{O}_X -module, s a regular meromorphic section. For a prime divisor $Z = \overline{\{\xi\}}$, $\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{X,\xi}(s/s_\xi)$ where s_ξ is a generator of \mathcal{L}_ξ . Define $\text{div}_{\mathcal{L}}(s) = \sum_Z \text{ord}_{Z,\mathcal{L}}(s)Z \in \text{Div}(X)$.

We have an exact ladder

$$\begin{array}{ccccccc} 1 & \longrightarrow & R(X)^\times/\mathcal{O}(X)^\times & \longrightarrow & \text{CaDiv}(X) & \longrightarrow & \text{Pic}(X) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ & & R(X)^\times/\mathcal{O}(X)^\times & \longrightarrow & \text{Div}(X) & \longrightarrow & \text{Cl}(X) \longrightarrow 0 \end{array}$$

Lemma 4.2.4. *If X is Noetherian normal integral scheme, (\mathcal{L}, s) is a Cartier divisor. Then $\text{div}_{\mathcal{L}}(s) \geq 0 \iff s \in \Gamma(X, \mathcal{L})$. In this case, $\Gamma(X, \mathcal{L}) = \{0\} \cup \{f \in R(X)^\times \mid \text{div}(f) \geq -\text{div}_{\mathcal{L}}(s)\}$.*

Proof. \Leftarrow . For a prime divisor $Z = \overline{\{\xi\}}$, $\text{ord}_{Z,\mathcal{L}}(s) = \text{ord}_{X,\xi}(s/s_\xi)$ where s_ξ is a generator of \mathcal{L}_ξ . If $s \in \Gamma(X, \mathcal{L})$, $s/s_\xi \in \mathcal{O}_{X,\xi}$, hence the order is greater or equal to 0.

\Rightarrow . We want to show s factors through \mathcal{L} .

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{s} & \mathcal{L} \otimes_{\mathcal{O}_X} K \\ \downarrow & \nearrow & \\ \mathcal{L} & & \end{array}$$

This is a local question hence we may assume $X = \text{Spec}(A)$ and $\mathcal{L} = \mathcal{O}_X$. Since A is normal, $A = \cap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$. Consequently, for $s \in K = \text{Frac}(A)$, $\text{ord}_{\mathfrak{p}}(s) \geq 0$ for each height one prime \mathfrak{p} implies $s \in A$.

For another $t \in \Gamma(X, \mathcal{L})$, (\mathcal{L}, t) is a Cartier divisor with $\text{div}_{\mathcal{L}}(t) \geq 0$. Thus $f = t/s \in R(X)^\times$ and $\text{div}(f) = \text{div}(t) - \text{div}(s) \geq -\text{div}(s)$. \square

Proposition 4.2.5. *If X is normal, then $\text{CaDiv}(X) \rightarrow \text{Div}(X)$ is a monomorphism and $\text{CaDiv}_+(X) = \text{CaDiv}(X) \cap \text{Div}_+(X)$.*

Proof. Let $(\mathcal{L}, s) \in \text{CaDiv}(X)$ whose image $\text{div}_{\mathcal{L}}(s) \in \text{Div}(X)$ is 0. By previous Lemma, $\text{div}_{\mathcal{L}}(s) = 0$ implies $s \in \Gamma(X, \mathcal{L})$, $s^{-1} \in \Gamma(X, \mathcal{L}^\vee)$, hence $s: \mathcal{O}_X \rightarrow \mathcal{L}$ is an isomorphism. \square

Remark 4.2.6. In this case, the second row is exact

$$0 \longrightarrow R(X)^\times/\mathcal{O}(X)^\times \longrightarrow \text{Div}(X) \longrightarrow \text{Cl}(X) \longrightarrow 0$$

Proposition 4.2.7. *Let X be a Noetherian integral scheme. Then all local rings of X are UFDs $\iff X$ is normal and $\text{Pic}(X) = \text{Cl}(X)$.*

Lemma 4.2.8. *A Noetherian domain A is UFD \iff every height one prime ideal is principal.*

Corollary 4.2.9. *Let A be a Noetherian domain. Then A is UFD $\iff A$ is normal and $\text{Cl}(A) = \text{Cl}(\text{Spec}(A)) = 0$.*

Proof. \implies It is clear a UFD is normal. Lemma 4.2.8 tells us all prime divisors are principal hence $\text{Cl}(A) = 0$.

\impliedby Use Lemma 4.2.8, we need to show each height one prime ideal $\mathfrak{p} \subseteq A$ is principal. By assumption, $\mathfrak{p} = \text{div}(f)$ for some $f \in K^\times$, $K = \text{Frac}(A)$. Since A is normal and $\text{ord}_A(f) \geq 0$, we see $f \in A$. Since $\text{ord}_{\mathfrak{p}}(f) = 1$, we have $f \in \mathfrak{p}$. For any $g \in \mathfrak{p}$, $\text{div}(g/f) \geq 0$, hence $g/f \in A$. This shows $\mathfrak{p} = fA$. \square

proof of Proposition 4.2.7. \impliedby There is a commutative diagram

$$\begin{array}{ccc} \text{Pic}(X) & \xrightarrow{\sim} & \text{Cl}(X) \\ \downarrow & & \downarrow \\ \text{Pic}(\mathcal{O}_{X,x}) & \longrightarrow & \text{Cl}(\mathcal{O}_{X,x}) \end{array}$$

where $\text{Cl}(X) \rightarrow \text{Cl}(\mathcal{O}_{X,x})$ is surjective (just take the prime divisor corresponding to x).

Since $\mathcal{O}_{X,x}$ is local, $\Gamma(\mathcal{O}_{X,x}, \mathcal{F}) = \mathcal{F}_x$ is exact for any abelian sheaf \mathcal{F} , hence $\text{Pic}(\mathcal{O}_{X,x}) = H^1(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^\times) = 0$, hence $\text{Cl}(\mathcal{O}_{X,x}) = 0$. By Corollary 4.2.9, $\text{Cl}(\mathcal{O}_{X,x}) = 0$ implies $\mathcal{O}_{X,x}$ is a UFD.

\implies Since $\mathcal{O}_{X,x}$ are UFDs, hence normal, X is normal. Since $\text{Pic}(X) \rightarrow \text{Cl}(X)$ is injective, we show it is surjective. It suffices to show each prime divisor D comes from a Cartier divisor. Take $x \in D$, $\text{Spec}(\mathcal{O}_{X,x} \cap D)$ is defined by a height one prime ideal. Since $\mathcal{O}_{X,x}$ is UFD, the prime ideal is generated by a $f_x \in R(X)^\times$. Then there is a neighborhood U_x of x such that $\text{div}_{U_x}(f_x) = D|_{U_x}$. Thus \mathcal{I}_D , the ideal sheaf of D is locally defined by f_x , hence invertible. Therefore, $\{(U_x, f_x)\}$ defines an effective Cartier divisor \mathcal{I}_D corresponding to D . \square

Date: 12.24

Example 4.2.10. Let k be a field, $\mathbb{A}_k^n = \text{Spec}(k[x_1, \dots, x_n])$. Since $k[x_1, \dots, x_n]$ is UFD, $\text{Cl}(\mathbb{A}_k^n) = \text{Pic}(\mathbb{A}_k^n) = 0$.

Example 4.2.11. Consider \mathbb{P}_k^n where k is a field. Then $\text{Cl}(\mathbb{P}_k^n) = \text{Pic}(\mathbb{P}_k^n) = \mathbb{Z}$ with a generator $[H] \in \text{Cl}(\mathbb{P}_k^n)$ for $H = \{x_0 = 0\}$ a hyperplane or $\mathcal{O}(1) \in \text{Pic}(\mathbb{P}_k^n)$.

Consider $\deg: \text{Div}(\mathbb{P}_k^n) \rightarrow \mathbb{Z}$, $\sum n_Z Z \mapsto \sum n_Z \deg(Z)$, where $\deg(Z)$ is the degree of the polynomial defining Z . For $f \in R(X)^\times$, $f = g/h$ with g, h homogeneous polynomial of the same degree, hence $\deg(\text{div}(f)) = \deg(g) - \deg(h) = 0$. Therefore, \deg factors through $\text{Cl}(\mathbb{P}_k^n) \rightarrow \mathbb{Z}$.

Next, we show $\text{Cl}(\mathbb{P}_k^n)$ is generated by $[H]$. Let Z be a prime divisor of \mathbb{P}_k^n defined by $g \in R_d$. Then $Z - dH = \text{div}(g/x_0^d)$ is a principal divisor, hence $Z = dH$. Since $\deg(H) = 1$, we have $\text{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}$. Note that $\mathcal{O}(H) = \mathcal{O}(1)$, hence $\deg(\mathcal{O}(1)) = 1$ and $\mathcal{O}(1)$ is a generator of $\text{Pic}(\mathbb{P}_k^n)$.

Example 4.2.12. Let $R = \text{Sym}(V)$, V is a k -vector space of dimension $n+1$. Then $\text{Aut}_k(\mathbb{P}(V)) \simeq \text{PGL}_k(V)$ where $\text{PGL}_k(V) = \text{GL}_k(V)/k^\times$.

It is clear there is a morphism $\text{PGL}(V) \rightarrow \text{Aut}_k(\mathbb{P}(V))$. Let $g: \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ be an automorphism. Let $\mathcal{L} = g^*\mathcal{O}(1)$. Then g defines a linear map $\alpha: V \rightarrow \Gamma(\mathbb{P}(V), \mathcal{L})$ whose image generates \mathcal{L} . Since g is an automorphism, $g^*\mathcal{O}(1)$ should be a generator of $\text{Pic}(\mathbb{P}(V))$. Thus $g^*(\mathcal{O}(1)) = \mathcal{O}(\pm 1)$. But $\mathcal{O}(-1)$ has no global sections, hence $g^*\mathcal{O}(1) = \mathcal{O}(1)$. Then $\alpha: V \rightarrow \Gamma(\mathbb{P}(V), \mathcal{O}(1)) = V$ is an invertible linear transformation.

Example 4.2.13. Consider $A = k[x, y]/y^2 - x^2(x + 1)$, $X = \text{Spec}(A)$. It has a node at $(0, 0)$. Let $D = \text{Spec}(A/(y - x))$, $D' = \text{Spec}(A/y + x)$, then D and D' are effective Cartier divisors. But D, D' both maps to $3O$ as Weil divisors, hence $\text{CaDiv}(X) \rightarrow \text{Div}(X)$ is not injective.

Lemma 4.2.14. Let X be a Noetherian integral scheme, $Z \subsetneq X$ a closed subscheme, Z_1, \dots, Z_n be irreducible subschemes of Z of codimension 1 in X . Equip Z_i with reduced subscheme structure, we have an exact sequence

$$\mathbb{Z}^n \xrightarrow{\phi} \text{Cl}(X) \xrightarrow{\psi} \text{Cl}(X \setminus Z) \longrightarrow 0$$

where $\phi(a_1, \dots, a_n) = \sum a_i Z_i$ and $\psi(\sum n_i Y_i) = \sum n_i(Y_i \setminus Z)$. If $Y_i \subseteq Z$, then $Y_i \setminus Z = \emptyset$. In particular, if $\text{codim}(Z, X) \geq 2$, then $\text{Cl}(X) \simeq \text{Cl}(X \setminus Z)$.

Example 4.2.15. Let $A = k[x, y, z]/(z^2 - xy)$, $X = \text{Spec}(A)$. Consider $\text{div}(y) = 2Y$, where $Y = \text{Spec}(A/(y, z))$. We have an exact sequence

$$\mathbb{Z} \longrightarrow \text{Cl}(X) \longrightarrow \text{Cl}(X \setminus Y) \longrightarrow 0$$

$X \setminus Y = \text{Spec}(k[z, y, y^{-1}])$ since we can solve $x = z^2/y$. Since $k[z, y, y^{-1}]$ is UFD, $\text{Cl}(X \setminus Y) = 0$. Thus $Y \in \text{Cl}(X)$ is a generator with $2Y = \text{div}(y) = 0$. We shall show A is not UFD, hence $Y \neq 0$ and $\text{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$. Since A is normal, $\text{Pic}(X) \hookrightarrow \text{Cl}(X)$ and is not an isomorphism. Thus $\text{Pic}(X) = 0$.

Let $\mathfrak{m} = (x, y, z) \subseteq A$. Claim $A_{\mathfrak{m}}$ is not a UFD. It suffices to show $\text{Cl}(A_{\mathfrak{m}}) \neq 0$. Let $\mathfrak{p} = (y, z) \subseteq A$, we show \mathfrak{p} is not principal. It is easy to see $\dim_k \mathfrak{p} + \mathfrak{m}^2/\mathfrak{m}^2 = \mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 = 2$, hence \mathfrak{p} cannot be principal.

4.3 Linear systems

Let k be a field, X a scheme over k , \mathcal{L} an invertible \mathcal{O}_X -module. Let $V \subseteq \Gamma(X, \mathcal{L})$ be a finite dimensional k -subspace. Consider $\varphi: V \otimes \mathcal{L}^\vee \rightarrow \mathcal{O}_X$. We know φ is surjective $\iff V$ generates \mathcal{L} .

Definition 4.3.1. Define the **base locus** of V to be $\text{Bs}(V) = \underline{\text{Spec}}(\text{coker } \varphi)$. It is a closed subscheme.

We say V is **base point free** if $\text{Bs}(V) = \emptyset$.

We have a morphism $f: X \setminus \text{Bs}(V) \rightarrow \mathbb{P}(V)$ with $f^*\mathcal{O}(1) = \mathcal{L}$. Choose $s \in V \setminus 0$, then $H_s = \mathbb{P}(V/ks) \hookrightarrow \mathbb{P}(V)$ is a hyperplane and $f^{-1}(H_s) = \text{Bs}(ks) \setminus \text{Bs}(V)$. If s is regular, then $f^{-1}H_s = D_{\mathcal{L}, s} \setminus \text{Bs}(V)$, where $D_{\mathcal{L}, s}$ is the effective Cartier divisor defined by (\mathcal{L}, s) .

Definition 4.3.2. Notation as above, $|V| = (V \setminus \{0\})/k^\times = \mathbb{P}(V^\vee)(k)$ is called a **linear system**. If $\Gamma(X, \mathcal{L})$ is of finite dimension, then $|\mathcal{L}| = |\Gamma(X, \mathcal{L})|$ is called a **complete linear system**.

Assume X/k is proper and geometrically integral. Then

$$\begin{aligned}\Gamma(X, \mathcal{L}) \setminus \{0\} &= \{\text{regular sections of } \mathcal{L}\} \\ \Gamma(X, \mathcal{O}_X^\times) &= k^\times\end{aligned}$$

Thus $|\mathcal{L}|$ corresponds to effective Cartier divisors such that $\mathcal{O}(D) \simeq \mathcal{L}$. We have

$$\text{Bs}(|V|) = \bigcap_{D \in |V|} \text{supp}(D)$$

4.4 Locally free sheaves of finite rank

Proposition 4.4.1. Let X be an integral scheme, \mathcal{E} a locally free \mathcal{O}_X -module of finite rank. Then there exists an integral scheme X' and $\pi: X' \rightarrow X$ proper birational such that $\pi^*\mathcal{E}$ admits a filtration as $\mathcal{O}_{X'}$ -modules whose all graded pieces are invertible sheaves.

Proof. Let us do induction on $\text{rank}(\mathcal{E}) = r$.

For $r = 1$, \mathcal{E} is invertible sheaf itself.

Suppose the proposition holds for $r - 1$, $r \geq 1$. Consider $p: \mathbb{P}(\mathcal{E}) \rightarrow X$. By definition, there exists a surjective morphism $p^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let $U \subseteq X$ open such that \mathcal{E} is trivial. Consider

$$\begin{array}{ccc}\mathbb{P}_U^{r-1} & \longrightarrow & \mathbb{P}(\mathcal{E}) \\ \nwarrow & & \downarrow p \\ U & \longrightarrow & X\end{array}$$

We can write \mathbb{P}_U^{r-1} since \mathcal{E} is trivial on U . There is a section $U \rightarrow \mathbb{P}_U^{r-1}$, hence a section $U \rightarrow \mathbb{P}(\mathcal{E})$. Let \widetilde{X} be the schematic closure of U in $\mathbb{P}(\mathcal{E})$ and denote

$\pi: \widetilde{X} \rightarrow X$. We still have $\varphi: \pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}_{\widetilde{X}}(1)$ where $\mathcal{O}_{\widetilde{X}}(1)$ is understood to be $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ restricted to \widetilde{X} . Then $\tilde{\mathcal{E}} = \text{Ker}(\varphi)$ is a locally free sheaf of rank $r - 1$ and $\widetilde{X} \rightarrow X$ is proper birational (with U identified). Thus the induction hypotheses can be applied to $(\widetilde{X}, \tilde{\mathcal{E}})$. \square

Corollary 4.4.2. *Let X be a Noetherian regular integral scheme of dimension 1, \mathcal{E} locally free of finite rank. Then \mathcal{E} admits a filtration with invertible graded pieces.*

Proof. Let $\pi: X' \rightarrow X$ be as in above. Then valuative criterion says π has a section s . Thus the filtration on X' can be pulled back by s^* to X . \square

Theorem 4.4.3 (Grothendieck). *Let k be a field, $X = \mathbb{P}_k^1$. Then Every locally free \mathcal{O}_X -module of finite rank is of the form $\bigoplus_{i=1}^r \mathcal{O}(n_i)$.*

Chapter 5

Curves

5.1 Curves and divisors

Recall varieties over k are separated and finite type integral schemes over k .

Definition 5.1.1. Let k be a field. A **curve** X/k is a variety of dimension 1.

Definition 5.1.2. A morphism of varieties over k is **nonconstant** if the image is not a point.

Proposition 5.1.3. Let k be a field. The following categories are equivalent

- (1) The category of proper regular curves with non-constant morphisms.
- (2) The category of curves with dominant rational maps.
- (3) The opposite category of finitely generated field extensions K/k with transcendence degree 1.

The functor from (2) to (3) is given by $X \mapsto R(X)$ where $R(X)$ is the function field of X .

Proof. We have seen that (2) and (3) are equivalent for general dominant rational maps. Let us establish $(1) \simeq (2)$.

Fully faithfulness: Suppose X/k regular, Y/k proper, $\emptyset \neq U \subseteq X$, then $\text{Hom}_k(U, Y) \simeq \text{Hom}_k(X, Y)$ by valuative criterion of properness. (each closed point on X defines a DVR)

Essentially surjective: Let X be a curve over k . We may take regular locus of X to assume X is regular. We may further assume X affine, hence there is an embedding $X \hookrightarrow \mathbb{P}_k^n$. Let \bar{X} be the schematic closure of X . Then \bar{X} is proper and birational to X . Let X' be the normalization of \bar{X} , then X' is still a variety with extra properties as proper, regular and birational to X . \square

Proposition 5.1.4. Let X, Y be proper curves over k , $f: X \rightarrow Y$ morphism over k . Then f is nonconstant $\iff f$ is finite $\iff f$ is surjective.

Proof. f surjective $\implies f$ nonconstant.

f finite \implies surjective. We know $f(X)$ is either a point or Y . But f finite implies $f(X)$ cannot be a point, hence f is surjective.

f nonconstant $\implies f$ finite. It is clear f is surjective, hence we can consider $R(X)/R(Y)$, which is a finite field extension. Let X^ν be the normalization of X and \tilde{Y} be the normalization of Y in $R(X)$. Then we have

$$\begin{array}{ccc} X^\nu & \xrightarrow{f'} & \tilde{Y} \\ \downarrow g & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The existence of g is owe to the equivalence in Proposition 5.1.3. By the same reason, $R(X^\nu) = R(\tilde{Y})$ implies g is an isomorphism. We quote a result from homework that if fg is affine and g is finite surjective, then f is finite. Thus $fg: X^\nu \rightarrow Y$ is affine and $g: X^\nu \rightarrow X$ finite surjective implies f is affine. It is then finite since $X^\nu \rightarrow Y$ is finite. \square

Proposition 5.1.5. *Let $f: X \rightarrow Y$ be a morphism of integral schemes with generic points η_X and η_Y . If Y is regular of dimension 1, then f is flat $\iff f(\eta_X) = \eta_Y$.*

Proof. If f is flat, then it is generizing (Corollary 1.6.39), hence $f(\eta_X) = \eta_Y$.

If $f(\eta_X) = \eta_Y$, then f is dominant. Locally, this is induced by a ring map $\varphi: A \rightarrow B$ with A a Dedekind domain. Then B is flat $\iff B$ is torsion free by the following Lemma. \square

Lemma 5.1.6. *Let A be a Dedekind domain. Then an A -module M is flat $\iff M$ is torsion free.*

Proof. A localized at maximal primes are PID. \square

Remark 5.1.7. If X is regular of dimension 1, \mathcal{E} locally free of finite rank, then its submodules are locally free of finite rank.

Remark 5.1.8. Let X be a curve. Then X is proper regular over k \iff it is projective regular over k . (by chow's Lemma)

Let X be a projective regular curve over k . Since X is normal, Weil divisors coincides with Cartier divisors. Recall a Weil divisor on a curve is of the form $\sum n_x x$ where x are closed points and $n_x = 0$ for all but finitely many x . We will mainly use Weil divisors since they are of simpler forms.

Definition 5.1.9. Let $D = \sum n_x x$, $\deg(D) = \sum n_x \deg(x)$ where $\deg(x) = [\kappa(x) : k]$.

Lemma 5.1.10. *If $D \subseteq X$ is an effective Cartier divisor, then*

$$\deg(D) = \dim_k \Gamma(D, \mathcal{O}_D)$$

Proof. If $D = \sum n_x x$, then D is a finite union of closed points, hence $\mathcal{O}_D = \bigoplus \mathcal{O}_{n_x x}$. Thus we may assume $D = nx$ is supported on $x \in X$. Then $D = \text{Spec}(\mathcal{O}_{X,x}/\mathfrak{m}^n)$ and $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq \kappa(x)$, hence

$$\deg(D) = n \deg(x) = n[\kappa(x) : k] = \dim_k \mathcal{O}_{X,x}/\mathfrak{m}^n = \dim_k \Gamma(D, \mathcal{O}_D)$$

□

Proposition 5.1.11. *Let $f: X \rightarrow Y$ be a nonconstant morphism between projective smooth curves over k . For $D \in \text{Div}(Y)$, $\deg(f^*D) = \deg(f) \deg(D)$ where $\deg(f) := [R(X) : R(Y)]$.*

*f^*D is justified since f is flat.*

Proof. We may assume $D = y \in Y$ is a closed point. Denote $A = \mathcal{O}_{Y,y}$ with maximal ideal m . By Proposition 5.1.4 f is finite. Consider $\text{Spec}(B) = X \times_Y \text{Spec}(A)$.

$$\begin{array}{ccccc} f^{-1}y & \longrightarrow & \text{Spec}(B) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ y & \longrightarrow & \text{Spec}(A) & \longrightarrow & Y \end{array}$$

Since A is DVR, $B \subseteq R(X)$ is a finite A -module, hence free module of rank $\deg(f)$. Thus $f^{-1}y = \text{Spec}(B \otimes_A A/m)$ and

$$\deg(f^*D) = \dim_k (B \otimes_A A/m) = \deg(f) \dim_k A/m = \deg(f) \deg(D)$$

□

Corollary 5.1.12. *Suppose X is a regular projective curve over k , $f \in R(X)^\times$, then $\deg \text{div}(f) = 0$.*

Proof. f defines $X \dashrightarrow \mathbb{A}_k^1$, hence it extends to $f: X \rightarrow \mathbb{P}_k^1$ by Proposition 5.1.3. We have $\text{div}(f) = f^*(\{0\} - \{\infty\})$, hence $\deg(\text{div}(f)) = \deg(f) \deg(\{0\} - \{\infty\}) = 0$. □

Guaranteed by the Corollary, we can define

$$\deg: \text{Pic}(X) \rightarrow \mathbb{Z}$$

For $d \in \mathbb{Z}$, define

$$\text{Pic}^d(X) = \{\mathcal{L} \in \text{Pic}(X) \mid \deg(\mathcal{L}) = d\}$$

We have an exact sequence

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

Definition 5.1.13. Let \mathcal{E} be a locally free \mathcal{O}_X -module of finite rank r . Define

$$\begin{aligned} \det(\mathcal{E}) &= \bigwedge^r \mathcal{E} \\ \deg(\mathcal{E}) &= \deg(\det(\mathcal{E})) \end{aligned}$$

5.2 Riemann-Roch theorem

Assume k is algebraically closed. Let X be a projective smooth curve over k . Let \mathcal{F} be a coherent sheaf on X , denote $h^i(\mathcal{F}) = \dim_k H^i(X, \mathcal{F})$. We know $\omega_{X/k} = \Omega_{X/k}$ is invertible sheaf and $g = h^0(\omega_{X/k})$ is the **genus** of X .

Theorem 5.2.1 (Riemann-Roch). *Let \mathcal{E} be a locally free \mathcal{O}_X -module of finite rank. Then*

$$h^0(\mathcal{E}) - h^0(\mathcal{E}^\vee \otimes \omega_{X/k}) = (1 - g)rk(\mathcal{E}) + \deg(\mathcal{E})$$

Remark 5.2.2. By Serre duality, $h^0(\mathcal{E}^\vee \otimes \omega_{X/k}) = h^1(\mathcal{E})$. Riemann Roch takes the form

$$\chi(\mathcal{E}) = (1 - g)rk(\mathcal{E}) + \deg(\mathcal{E})$$

Lemma 5.2.3. *If*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of locally free sheaves of finite rank, then

$$\begin{aligned} \det(\mathcal{E}) &= \det(\mathcal{E}') \otimes \det(\mathcal{E})' \\ \deg(\mathcal{E}) &= \deg(\mathcal{E}') + \deg(\mathcal{E}'') \end{aligned}$$

proof of Theorem. By Corollary 4.4.2, \mathcal{E} admits a filtration with invertible graded pieces, hence the above Lemma reduces the problem to $\mathcal{E} = \mathcal{L}$ invertible sheaf.

We have

$$\chi(\mathcal{O}) = h^0(\mathcal{O}) - h^1(\mathcal{O}) = h^0(\mathcal{O}) - h^0(\omega) = 1 - g$$

If D is an effective Cartier divisor, then

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow i_*\mathcal{O}_D \longrightarrow 0$$

Twist by \mathcal{L}' , we have

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L}' \otimes \mathcal{O}(D) \longrightarrow i_*\mathcal{O}_D \longrightarrow 0$$

Thus

$$\chi(\mathcal{L}' \otimes \mathcal{O}(D)) = \chi(\mathcal{L}') + \chi(i_*\mathcal{O}(D)) = \chi(\mathcal{L}') + \deg(D)$$

We will apply this with $\mathcal{L}' = \mathcal{O}(-D')$.

For general \mathcal{L} , we can write $\mathcal{L} = \mathcal{O}(D - D')$ with D, D' effective. Then

$$\chi(\mathcal{L}) = \chi(\mathcal{O}(D - D')) = \chi(\mathcal{O}(-D')) + \deg(D) = \chi(\mathcal{O}) + \deg(D - D') = 1 - g + \chi(\mathcal{L})$$

□

Corollary 5.2.4. $\deg(\omega_{X/k}) = 2g - 2$.

Proof. $\chi(\omega) = h^0(\omega) - h^0(\mathcal{O}) = 1 - g + \deg(\omega)$. □

5.2.1 Riemann-Roch problem

Given \mathcal{L} , what is $h^0(\mathcal{L})$?

Lemma 5.2.5. *Let X be a smooth projective curve over algebraically closed field k .*

- If $\deg(\mathcal{L}) < 0$, then $h^0(\mathcal{L}) = 0$.
- If $\deg(\mathcal{L}) = 0$ and $\mathcal{L} \neq \mathcal{O}$, then $H^0(\mathcal{L}) = 0$.

Proof. If $h^0(\mathcal{L}) > 0$, then there exists $0 \neq s \in \Gamma(X, \mathcal{L})$. Thus (\mathcal{L}, s) either defines an effective Cartier divisor D such that $\deg(\mathcal{L}) = \deg(D) > 0$, or $\text{div}_{\mathcal{L}}(s) = 0$, in which case $\mathcal{L} = \mathcal{O}$. \square

Corollary 5.2.6. *Notation as above.*

- If $\deg(\mathcal{L}) > 2g - 2$, then $h^0(\mathcal{L}) = 1 - g + \deg(\mathcal{L})$.
- If $\deg(\mathcal{L}) = 2g - 2$, then

$$h^0(\mathcal{L}) = \begin{cases} g & \mathcal{L} = \omega \\ g - 1 & \mathcal{L} \neq \omega \end{cases}$$

Remark 5.2.7. For $0 \leq \deg(\mathcal{L}) \leq 2g - 2$, $h^0(\mathcal{L})$ is not determined by $\deg(\mathcal{L})$.

Corollary 5.2.8. *The following are equivalent:*

- (a) $X \simeq \mathbb{P}_k^1$.
- (b) $g(X) = 0$
- (c) $\text{Pic}^0(X) = 0$
- (d) $\exists x, y \in X$ closed point such that $x \sim y$.

Proof. (a) \implies (b) $g(\mathbb{P}_k^1) = h^0(\omega) = h^0(\mathcal{O}(-2)) = 0$.

(b) \implies (c) Let $\mathcal{L} \in \text{Pic}^0(X)$. Then $\deg(\mathcal{L}) > 2g - 2$, hence $h^0(\mathcal{L}) = 1$ by Corollary 5.2.6, hence $\mathcal{L} = \mathcal{O}$ by Lemma 5.2.5.

(c) \implies (d) Since $\mathcal{O}(x - y) = \mathcal{O}$, $x \sim y$.

(d) \implies (a) $\exists f \in R(X)^\times$, $\text{div}(f) = x - y$. Then $f: X \rightarrow \mathbb{P}_k^1$ with $\deg(f) = 1$, hence an isomorphism by Proposition 5.1.3 \square

Corollary 5.2.9. *Suppose $g(X) = 1$ and fix $x_0 \in X(k)$. Then there are one-to-one correspondences*

$$X(k) \xrightarrow{\varphi} \text{Pic}^1(X) \xrightarrow{\psi} \text{Pic}^0(X)$$

where $\varphi(x) = x$, $\psi(D) = D - x_0$.

Proof. Let $\mathcal{L} \in \text{Pic}^1(X)$. Since $\deg \mathcal{L} = 1 > 2g - 2 = 0$, $h^0(\mathcal{L}) = 1 - g + \deg(\mathcal{L}) = 1$. Then there exists a unique effective Cartier divisor D with $\mathcal{L} = \mathcal{O}(D)$. Since $\deg(D) = 1$, D is a closed point $x \in X$, hence $\mathcal{L} = \mathcal{O}(x)$. This shows φ is bijective. The statement for ψ is trivial. \square

Definition 5.2.10. Suppose X is a smooth projective curve over algebraically closed field k and $g(X) = 1$. Choose $x_0 \in X(k)$. Then (X, x_0) is called an **elliptic curve**. $(X(k), x_0)$ is an abelian group by previous Corollary.

Remark 5.2.11. There is a more general construction

$$\begin{aligned} X^d(k) &\rightarrow \text{Pic}^d(X) \\ (x_1, \dots, x_d) &\mapsto \sum x_i \end{aligned}$$

It factors through $X^d(X)/\Sigma_d$ where Σ_d is the permutation group acting on $X^d(k)$. For $d = 0$, $\text{Pic}^0(X) \simeq \text{Jac}_X(k)$ is called **Jacobian variety** of X . It is projective smooth of dimension g .

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Let k be algebraically closed, X smooth projective curve over k . For an invertible sheaf \mathcal{L} , Riemann-Roch theorem takes the form

$$\begin{aligned} h^0(\mathcal{L}) &= h^0(\mathcal{L}^\vee \otimes \omega_{X/k}) + 1 - g + \deg(\mathcal{L}) \\ &\geq 1 - g + \deg(\mathcal{L}) \end{aligned}$$

We have a Riemann-Roch line $h = 1 - g + d$ when $d \geq g - 1$ and $h = 0$ when $d \leq g - 1$. The h and d represent possible $h^0(\mathcal{L})$ and $\deg(\mathcal{L})$.

In general, the $h^0(\mathcal{L})$ will satisfy the Riemann-Roch line.

For $\deg(\mathcal{L}) \leq 0$ and $\deg(\mathcal{L}) \geq 2g - 2$, we see from Lemma 5.2.5 and Corollary 5.2.6 that most of them sit on the line.

For $0 < d = \deg(\mathcal{L}) \leq g - 1$. If $h^0(\mathcal{L}) > 0$, then $\mathcal{L} = \mathcal{O}(D)$ for $D \geq 0$. Since D is a positive linear combination of closed points, it is in the image of

$$\begin{aligned} X^d(k) &\rightarrow \text{Pic}^d(X) \\ (x_1, \dots, x_d) &\mapsto \sum x_i \end{aligned}$$

Since $\dim X^d(k) = d < g = \dim \text{Pic}^d(X)$, these \mathcal{L} are rare.

Let us give an upper bound for $h^0(\mathcal{L})$.

Theorem 5.2.12 (Clifford). *For $-2 \leq \deg(\mathcal{L}) \leq 2g$, we have*

$$h^0(\mathcal{L}) - 1 \leq \frac{1}{2} \deg(\mathcal{L})$$

This is called **Clifford line**.

Lemma 5.2.13. *Consider $\mathcal{L}, \mathcal{M} \in \text{Pic}(X)$, we have*

$$\begin{aligned} |\mathcal{L}| \times |\mathcal{M}| &\rightarrow |\mathcal{L} \otimes \mathcal{M}| \\ (D, D') &\mapsto D + D' \end{aligned}$$

The fibre of the map is finite. If $|\mathcal{L}|, |\mathcal{M}| \neq \emptyset$, then $\dim |\mathcal{L}| + \dim |\mathcal{M}| \leq \dim |\mathcal{L} \otimes \mathcal{M}|$.

Proof. Consider

$$\begin{array}{ccc} H^0(X, \mathcal{L}) \times H^0(X, \mathcal{M}) & \longrightarrow & H^0(X, \mathcal{L} \otimes \mathcal{M}) \\ \downarrow & \nearrow & \\ H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{M}) & & \end{array}$$

The above map is

$$f: Q = \mathbb{P}(H^0(X, \mathcal{L})^\vee) \times \mathbb{P}(H^0(X, \mathcal{M})^\vee) \rightarrow P = \mathbb{P}(H^0(X, \mathcal{L} \otimes \mathcal{M})^\vee)$$

For any k'/k , k' algebraically closed, $Q(k') \rightarrow P(k')$ is of finite fibre. In particular, the fibre of generic point of P has finite fibre. This implies $\dim Q \leq \dim P$. \square

proof of Clifford's theorem. Let $\mathcal{M} = \mathcal{L}^\vee \otimes \omega$. Then

$$\dim |\mathcal{L}| + \dim |\mathcal{L}^\vee \otimes \omega| \leq \dim |\omega| = g - 1$$

and Riemann-Roch

$$\dim |\mathcal{L}| - \dim |\mathcal{L}^\vee \otimes \omega| = 1 - g + \deg(\mathcal{L})$$

Take sum, we get $2 \dim |\mathcal{L}| \leq \deg(\mathcal{L})$. \square

Remark 5.2.14. Clifford gave equivalent condition for when the equality holds.

Brill-Noether theory says for X general (in the sense of moduli space)

$$H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}^\vee \otimes \omega) \rightarrow H^0(X, \omega)$$

is injective. Thus if $h = h^0(\mathcal{L})$, then $h(h + 1 - g + d) \leq g$.

5.3 Embeddings in projective space

Let k be algebraically closed.

Proposition 5.3.1. *Let X, Y be finite type over k , $f: X \rightarrow Y$ a proper morphism.*

(1) f is injective $\iff f(k): X(k) \rightarrow Y(k)$ is injective.

(2) f is a closed immersion $\iff f(k[\epsilon]): X(k[\epsilon]) \rightarrow Y(k[\epsilon])$ is injective where $\epsilon^2 = 0$ in $k[\epsilon]$.

Proof. (1) \implies is clear. For \iff . Suppose $x \neq x' \in X$ but $f(x) = f(x') = y$. Observe that the closure of $\{x\}, \{x'\}$ are not the same, we may assume there exists a closed point $w \in X$ such that $x \rightsquigarrow w$ but $x' \not\rightsquigarrow w$. Since f is a closed map, it is specializing. Since $f(x) = f(x') = y$ specializes to $f(w)$, we can find $w' \in X$ such that $x' \rightsquigarrow w'$ and $f(w') = f(w)$. Further specialization of w' , say to a closed point, will have the same image since $f(w)$ is a closed point in Y . Thus we may assume w' is a closed point, hence $f(w') = f(w)$ implies $w' = w$, contradicting $x' \not\rightsquigarrow w$.

(2) \implies is clear since f is a monomorphism. For \iff . The natural map $k \rightarrow k[\epsilon] \rightarrow k$ gives

$$\begin{array}{ccc} X(k) & \longrightarrow & Y(k) \\ \downarrow & & \downarrow \\ X(k[\epsilon]) & \longrightarrow & Y(k[\epsilon]) \\ \downarrow & & \downarrow \\ X(k) & \longrightarrow & Y(k) \end{array}$$

hence $X(k) \rightarrow Y(k)$ is injective. By (1), f is injective. Since f is proper, it is a closed embedding of topological spaces. We are left to show $f^*: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is an epimorphism of sheaves of abelian groups, i.e. $\forall x \in X$, $\mathcal{O}_{Y,x} \rightarrow \mathcal{O}_{X,x}$ is surjective.

It suffices to check this for all closed points of X since sheaves concerned are quasi-coherent, if their stalks surject at a point, they will surject in an open neighborhood. Fix $x \in X(k)$, the fibre of x under $X(k[\epsilon]) \rightarrow X(k)$ is identified with the Zariski tangent space at x , i.e. $(m_{X,x}/m_{X,x}^2)^\vee$ where $m_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. Then $X(k[\epsilon]) \rightarrow Y(k[\epsilon])$ is injective says $(m_{X,x}/m_{X,x}^2)^\vee \rightarrow (m_{Y,x}/m_{Y,x}^2)^\vee$ is injective for all $x \in X(k)$. In other words, $m_{Y,x}/m_{Y,x}^2 \rightarrow m_{X,x}/m_{X,x}^2$ is surjective. The following Lemma shows $\mathcal{O}_{Y,x} \rightarrow \mathcal{O}_{X,x}$ is surjective. \square

Lemma 5.3.2. *Let $\varphi: (A, m) \rightarrow (B, n)$ be a local homomorphism of Noetherian local rings. If $A/m = B/n$, $m/m^2 \rightarrow n/n^2$ is surjective, then φ is surjective.*

Proof. We have $mB \subseteq n$. Since $mB/n^2 \simeq n/n^2$, $mB = n$ by Nakayama's Lemma applied to B -modules mB and n . Since $B/mB = B/n = A/m$, $1 \in B$ generates B/mB as A -module, hence $A = B$ by Nakayama's Lemma. \square

Corollary 5.3.3. *Let k algebraically closed, X/k proper, V a finite dimensional k -vector space. Given $f: X \rightarrow \mathbb{P}(V)$, it corresponds to $\varphi: V \rightarrow \Gamma(X, \mathcal{L})$ with $\mathcal{L} = f^*\mathcal{O}(1)$. Let $W = \text{Im}(\varphi)$.*

Then f is a closed immersion \iff

- W separates points, i.e. $\forall x \neq y \in X(k)$, $\exists s \in W$ such that $s_x \in m_x\mathcal{L}_x$ but $s_y \notin m_y\mathcal{L}_y$.
- W separates tangent vectors, i.e. $\forall x \in X$, $W_x = \{s \in W \mid s_x \in m_x\mathcal{L}_x\}$ spans $m_x\mathcal{L}_x/m_x^2\mathcal{L}_x$.

Proof. The morphism f induces

$$\begin{aligned} X(k) &\rightarrow \mathbb{P}(V)(k) \\ x &\mapsto \varphi_x = [V \xrightarrow{\varphi} \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_x/m_x\mathcal{L}_x \simeq \mathcal{O}_{X,x}/m_x = k] \end{aligned}$$

We see φ_x is uniquely determined by $\text{Ker}(\varphi_x) = V_x$, hence $x \rightarrow \varphi_x$ is injective $\iff V$ (actually W) separates points.

On the tangent space, f induces

$$\begin{aligned} X(k[\epsilon]) &\rightarrow \mathbb{P}(V)(k[\epsilon]) \\ (x, t) &\mapsto [V \xrightarrow{\varphi} \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_x/m_x^2\mathcal{L}_x \simeq \mathcal{O}_{X,x}/m_x^2 \xrightarrow{t} k[\epsilon]] \end{aligned}$$

It is determined by φ_x and $V_x \rightarrow m_x/m_x^2 \xrightarrow{t} k$ where we identify the last k with $k\epsilon \subseteq k[\epsilon]$. Thus the map on tangent space is injective $\iff V_x \rightarrow k$ is determined by $m_x/m_x^2 \rightarrow k$, i.e. $V_x \rightarrow m_x/m_x^2$ is surjective. \square

Remark 5.3.4. Assume X integral, W separates points $\iff \forall x \neq y \in X(k)$, $\exists D \in |W|$ such that $x \in \text{supp}(D)$ but $y \notin \text{supp}(D)$.

W separates tangent vectors $\iff \forall x \in X(k)$, $\forall 0 \neq t \in T_x(X) := (m_x/m_x^2)^\vee$, $\exists D \in |W|$, $x \in \text{supp}(D)$ and $t \notin T_x(D)$. To see this, note first that D is effective, it can be viewed as a closed subscheme of X and $T_x(D) \subseteq T_x(X)$. Locally at x , D is defined by $f \in m_x$, hence $T_x(D) = (m_x/m_x^2 + f)^\vee$ and $t \notin T_x(D) \iff t(f) \neq 0$.

We now turn to study of curves.

Let k be algebraically closed, X/k smooth projective curve, \mathcal{L} invertible sheaf on X .

Proposition 5.3.5. (a) \mathcal{L} globally generated $\iff \forall x \in X$ closed point,

$$h^0(\mathcal{L} \otimes \mathcal{O}(-x)) = h^0(\mathcal{L}) - 1.$$

(b) \mathcal{L} very ample $\iff \forall x, y \in X$ closed points (possibly equal),

$$h^0(\mathcal{L} \otimes \mathcal{O}(-x-y)) = h^0(\mathcal{L}) - 2.$$

Lemma 5.3.6. $h^0(\mathcal{L} \otimes \mathcal{O}(-x)) = h^0(\mathcal{L})$ or $h^0(\mathcal{L}) - 1$.

Proof. Let $i: \{x\} \hookrightarrow X$. The following sequences are exact

$$0 \longrightarrow \mathcal{O}(-x) \longrightarrow \mathcal{O} \longrightarrow i_* k \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}(-x) \longrightarrow \mathcal{L} \longrightarrow i_* k \longrightarrow 0$$

Taking global sections, the result follows from the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}(-x)) \longrightarrow H^0(X, \mathcal{L}) \longrightarrow k$$

□

proof of Proposition 5.3.5. One can give a direct proof. We will work instead with linear systems.

(a)

$$\begin{aligned} h^0(\mathcal{L} \otimes \mathcal{O}(-x)) &= h^0(\mathcal{L}) - 1 \\ \iff H^0(X, \mathcal{L} \otimes \mathcal{O}(-x)) &\rightarrow H^0(X, \mathcal{L}) \text{ is not surjective} \\ \iff |\mathcal{L} \otimes \mathcal{O}(-x)| &\rightarrow |\mathcal{L}|, \quad D \mapsto D + x \text{ is not surjective} \\ \iff x &\notin \text{Bs } |\mathcal{L}| \end{aligned}$$

The last equivalent is because $\text{Bs } |\mathcal{L}| = \bigcap_{D \in |\mathcal{L}|} \text{supp}(D)$ and $x \in \text{supp}(D) \iff D - x$ is effective.

(b) First, observe that both sides imply \mathcal{L} is globally generated. Indeed, very ample implies globally generated and the right hand side forces $h^0(\mathcal{L} \otimes \mathcal{O}(-x)) = h^0(\mathcal{L} \otimes \mathcal{O}(-y)) = h^0(\mathcal{L}) - 1$. Thus we may assume \mathcal{L} is globally generated. Therefore, we get a morphism $i: X \rightarrow \mathbb{P}(V)$, $V = H^0(X, \mathcal{L})$. By Remark 5.3.4 $|\mathcal{L}|$ separates points $\iff \forall x \neq y \in X$ closed points, $\exists D \in |\mathcal{L}|$, $x \in \text{supp}(D)$ but $y \notin \text{supp}(D)$. Note that the last condition is equivalent to $y \notin \text{supp}(D - x)$, hence the result.

$|\mathcal{L}|$ separated tangent vectors $\iff \forall x \in X$ closed points and $0 \neq t \in m_x/m_x^2$, $\exists D \in |\mathcal{L}|$ such that $x \in \text{supp}(D)$ and $t \notin T_x(D)$. Since $T_x(D)$ is of dimension 1, $t \notin T_x(D) \iff T_x(D) = 0$, namely $x \notin \text{Bs}(\mathcal{L} \otimes \mathcal{O}(-x))$.

Finally, observe that $\forall x, y \in X$ (possibly equal), $y \notin \text{Bs}(\mathcal{L} \otimes \mathcal{O}(-x)) \stackrel{(a)}{\iff} h^0(\mathcal{L} \otimes \mathcal{O}(-y - x)) = h^0(\mathcal{L} \otimes \mathcal{O}(-x)) - 1$. □

Corollary 5.3.7. • If $\deg(\mathcal{L}) \geq 2g$, \mathcal{L} is globally generated.

- If $\deg(\mathcal{L}) \geq 2g + 1$, \mathcal{L} is very ample.

Proof. This is a combination of Riemann-Roch and the previous Proposition. \square

Corollary 5.3.8. \mathcal{L} ample $\iff \deg(\mathcal{L}) > 0$.

Proof. \Leftarrow . Let $d = \deg(\mathcal{L}) > 0$, then $\exists n > 0$ such that $\deg(\mathcal{L}^{\otimes n}) \geq 2g + 1$, hence $\mathcal{L}^{\otimes n}$ is very ample, \mathcal{L} is ample.

\Rightarrow \mathcal{L} is globally generated, hence $h^0(\mathcal{L}) > 0$ and $\mathcal{L} \neq \mathcal{O}$, hence $\deg(\mathcal{L}) > 0$ by Lemma 5.2.5. \square

Example 5.3.9. • $g = 0$, \mathcal{L} ample $\iff \mathcal{L}$ very ample $\iff \deg(\mathcal{L}) > 0$.
(More generally on \mathbb{P}_k^n , \mathcal{L} ample $\iff \mathcal{L}$ very ample $\iff \mathcal{L} = \mathcal{O}(n)$ for $n > 0$).

- $g = 1 \wedge \mathcal{L}$ very ample $\deg(\mathcal{L}) \geq 3$.

\Leftarrow is implied by above Corollary.

$\Rightarrow \dim |\mathcal{L}| \geq 2$, i.e. $h^0(\mathcal{L}) \geq 3$ thus $\deg(\mathcal{L}) \geq 3$.

In this case, $\deg(\mathcal{L}) = 1, 2$ implies \mathcal{L} is ample but **not** very ample. When $\deg(\mathcal{L}) = 3$, $h^0(\mathcal{L}) = 3$, hence it defines a cubic curve $X \hookrightarrow \mathbb{P}_k^2$.

Conversely, smooth cubic curves have genus 1.

Recall smooth curves in \mathbb{P}_k^n of degree d have genus $g = \frac{(d-1)(d-2)}{2}$. They are called triangular numbers.

Remark 5.3.10. If $g \geq 2$, \mathcal{L} very ample $\Rightarrow \dim |\mathcal{L}| \geq 2 \Rightarrow \deg(\mathcal{L}) \geq 4$.

Definition 5.3.11. Let $f: X \rightarrow \mathbb{P}_k^n$ a finite morphism, define degree of f to be $\deg(f^*\mathcal{O}(1))$.

Remark 5.3.12. $g = 2$, \mathcal{L} very ample $\iff \deg(\mathcal{L}) \geq 5$.

\Leftarrow implied by Corollary.

$\Rightarrow \dim |\mathcal{L}| \geq 3 \Rightarrow h^0(\mathcal{L}) \geq 4 \Rightarrow \deg(\mathcal{L}) \geq 5$.

$\deg(\mathcal{L}) = 5 \Rightarrow h^0(\mathcal{L}) = 4$, $X \hookrightarrow \mathbb{P}_k^3$ is a quintic curve.

Theorem 5.3.13. Let X a smooth projective variety over k of dimension n . Then $\exists X \hookrightarrow \mathbb{P}_k^{2n+1}$ closed immersion.

In fact, for $X \hookrightarrow \mathbb{P}(V)$ closed immersion, $\dim(V) > 2n + 1$, $\exists W \subseteq V$, $\dim \mathbb{P}(W) = 2n + 1$ and a projection $\mathbb{P}(V) \setminus \mathbb{P}(V/W) \rightarrow \mathbb{P}(W)$ such that the composition gives a closed immersion $X \hookrightarrow \mathbb{P}(W)$.

$$\begin{array}{ccc} X & \xhookrightarrow{\quad} & \mathbb{P}(V) \setminus \mathbb{P}(V/W) \\ & \searrow & \downarrow \\ & & \mathbb{P}(W) \end{array}$$

Proof. It suffices to show $\exists W \subseteq V$ a hyperplane corresponding to $x \in \mathbb{P}(V)(k)$ such that $x \notin X$,

$$\begin{array}{ccc} X & \xhookrightarrow{\quad} & \mathbb{P}(V) \setminus \{x\} \\ & \searrow f & \downarrow \\ & & \mathbb{P}(W) \end{array}$$

$f(k)$ is injective $\iff x$ is not on a secant line of X .

$f(k[\epsilon])$ is injective $\iff x$ is not on the tangent space of X at any $y \in X(k)$.

Let $Sec(X)$ be the union of secant lines of X . Locally, $Sec(X)$ is contained in the image of $(X \times X \setminus \Delta) \times \mathbb{P}^1$ which is of dimension $2n+1$. Let $Tan(X)$ be the union of tangent spaces of X at $y \in X(k)$. Locally $Tan(X)$ is contained in the image of $X \times \mathbb{P}_k^n$ which is of dimension $2n$. Thus if $\dim(\mathbb{P}(V)) \geq 2n+2$, $\exists x \in \mathbb{P}(V)(k) \setminus (Sec(X) \cup Tan(X))$. \square

Corollary 5.3.14. *If X is a smooth projective curve over k , then there exists a closed immersion $X \hookrightarrow \mathbb{P}_k^3$.*

Remark 5.3.15. We cannot remove smoothness assumption. Let $A = k[t^d, t^{d+1}, \dots, t^{2d-1}]$, $\text{Spec}(A) \hookrightarrow C$ an open immersion with C projective over k . Let $m = (t^d, t^{d+1}, \dots, t^{2d-1})$ be a maximal ideal of A , then $\dim_k(m/m^2) = d$, hence a closed embedding $C \hookrightarrow \mathbb{P}^n$ must have $n \geq d$.

5.4 Riemann-Hurwitz formula

Throughout this section, k is assumed to be algebraically closed.

Let $f: X \rightarrow Y$ be a nonconstant morphism of smooth projective curves over k . We will compare $g(X)$ and $g(Y)$. There is an exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{df} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

Definition 5.4.1. f is **separable** if $R(X)/R(Y)$ is a separable field extension.

Lemma 5.4.2. (1) *If f is separable, df is a monomorphism.*

(2) *If f is inseparable, $df = 0$.*

Proof. Since $f^*\Omega_{Y/k}$ is an invertible sheaf, $\text{Ker}(df)$ is either 0 or $f^*\Omega_{X/Y}$. Let η_X be the generic point of X .

(1) Suffices to show $(df)_{\eta_X}$ is a monomorphism, or equivalently $(df)_{\eta_X}$ is an epimorphism. This is equivalent to $(\Omega_{X/Y})_{\eta_X} = \Omega_{R(X)/R(Y)} = 0$ by the exact sequence.

(2) Suffices to show $(df)_{\eta_X} = 0$, i.e. $(\Omega_{X/Y})_{\eta_X} = \Omega_{R(X)/R(Y)} \neq 0$. \square

Assume f separable. Then

$$0 \longrightarrow f^*\Omega_{Y/k} \xrightarrow{df} \Omega_{X/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

is exact. Thus $\Omega_{X/Y}$ is supported on a finite closed subset of X . $\forall x \in X$, $\Omega_{X/Y,x}$ is a principal $\mathcal{O}_{X,x}$ -module of finite length.

Definition 5.4.3. The **ramification divisor (different)** is $R = \sum_x lg(\Omega_{X/Y,x})x$ where lg is the length function. It is an effective divisor.

Let $i: R \hookrightarrow X$. We have $\Omega_{X/Y} = i_*\mathcal{O}_R$. We have exact sequences

$$0 \longrightarrow \mathcal{O}(-R) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_R \longrightarrow 0$$

$$0 \longrightarrow \Omega_{X/k} \otimes \mathcal{O}(-R) \longrightarrow \Omega_{X/k} \longrightarrow i_*\mathcal{O}_R \longrightarrow 0$$

where $\Omega_{X/k} \otimes \mathcal{O}(-R) \simeq f^*\Omega_{Y/k}$. Taking degrees, we get

Theorem 5.4.4 (Riemann-Hurwitz).

$$2g(X) - 2 = (2g(Y) - 2)\deg(f) + \deg(R)$$

Corollary 5.4.5. $g(X) \geq g(Y)$. Equality holds $\iff g(X) = g(Y) = 0$ or $g(Y) = 1$ and f unramified or f isomorphism.

Corollary 5.4.6. If $f: X \rightarrow \mathbb{P}_k^1$ finite etale and X connected, then f is an isomorphism. This means \mathbb{P}_k^1 is 'simply connected'.

5.4.1 Ramification index

Definition 5.4.7. Let $f: X \rightarrow Y$ nonconstant morphism, $x \in X$ a closed point, $y = f(x)$. Then f induces $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$. Both rings are DVR, we can take a uniformizer $\pi_y \in \mathcal{O}_{Y,y}$, $e_x := v_x(\pi_y)$ is called **ramification index** at x .

It is clear $e_x = 1 \iff f$ is unramified at x .

We have

$$X_y = \coprod_{x \in f^{-1}(y)} \text{Spec}(\mathcal{O}_{X,x}/m_x^{e_x})$$

Let $\pi_x \in \mathcal{O}_{X,x}$ be a uniformizer. We can write $\pi_y = u\pi_x^{e_x}$, $u \in \mathcal{O}_{X,x}^\times$.

$$\frac{d\pi_y}{d\pi_x} = \pi_x^{e-1} \left(\frac{du}{d\pi_x} \pi_x + e_x u \right)$$

Let $p = \text{char}(k)$. Then $lg(\Omega_{X/Y,x}) \geq e_x - 1$ and equality holds $\iff p \nmid e_x$.

Definition 5.4.8. We say f is **tamely ramified** at x if $p \nmid e_x$. Otherwise, f is said to be **wildly ramified**.

Corollary 5.4.9. Assume f is tamely ramified. Then $2g(X) - 2 = (2g(Y) - 2)\deg(f) + \sum_x(e_x - 1)$.

Example 5.4.10. Assume $\text{char}(k) \neq 2$. Let $X \rightarrow \mathbb{P}_k^2$ be a morphism of degree 2. Then $2g(X) = -2 + \#\{\text{ramified points}\}$.

Let $P(t) \in k[t]$ be a separable polynomial of degree $d \geq 1$. Let $C^0 = \text{Spec}(k[y,t]/y^2 - P(t)) \hookrightarrow C$ an open immersion with C/k projective. Let $X = C^\nu$ be the normalization of C , then C^0 , which is smooth, factors through X . The natural morphism

$C^0 \rightarrow \mathbb{A}_k^1 = \text{Spec}(k[t])$ induces $X \rightarrow \mathbb{P}_k^1$. Use Riemann-Hurwitz formula, we see the number of ramified points is even. Therefore if d is odd, ∞ is ramified. If d is even, ∞ is unramified. Thus $g(X) = \lceil \frac{d}{2} \rceil - 1$ where $\lceil x \rceil$ is the largest integer not exceeding x . In this way, we have constructed curves of all possible genus (at least in $\text{char}(k) \neq 2$).

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