ALGEBRAIC GEOMETRY 2020 PROBLEM ANSWERS

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Sometimes I only give a hint if I think the rest is something purely algebraic or topological. Every error or typo is due to me myself. You can contact with me by email: `qiucaiyong18@mails.ucas.ac.cn` or by Wechat(preferred): TsaiyonTschiu.

If possible, I will only quote online references which can be visit in China:

[Stacks]*Stacks Project* by de Jong. `stacks.math.columbia.edu`

I hope [Stacks] will be sufficient. But I want to point out that there is another important online resource that you can download and read for pleasure:

[EGA]*Eléments de Géométrie Algébrique*, `www.numdam.org/item/PMIHES`, Tome 4(EGA I (old version), *Le langage des schémas*), Tome 8(EGA II, *Étude globale élémentaire de quelques classes de morphismes*), Tome 11&17(EGA III, *Étude cohomologique des faisceaux cohérents*), Tome 20&24&28&32(EGA IV, *Étude locale des schémas et des morphismes de schémas*)

You can also buy a Chinese version for EGA I (new version) and EGAII.

The symbol $\subset$ means subset but not equal to.

1. Problem 1

1.1. Let A be a ring. Let U and V be quasi-compact open subsets of $\mathrm{Spec}(A)$. Show that $U \cap V$ is quasi-compact.

Proof. We claim that $V(I)^c$ is quasi-compact if and only if I is finitely generated ([Stacks]Lemma.10.28.1). This can be shown by the fact that $V(I)^c$ is a finite union of principal opens.

Suppose $U^c = V(\mathfrak{a}), V^c = V(\mathfrak{b})$, then $(U \cap V)^c = V(\mathfrak{a}\mathfrak{b})$, where $\mathfrak{a}\mathfrak{b}$ is obviously finitely generated.

□

1.2. An open subset of $\mathrm{Spec}(A)$ is called principal if it is of the form $D(f)$ for some $f \in A$.

(a) Find an open subset of $\mathrm{Spec}(\mathbb{Z}[X])$ that is not principal.

(b) Let A be a Dedekind domain whose ideal class group is torsion (e.g. A is the ring of integers of a number field). Show that every open subset of $\mathrm{Spec}(A)$ is principal.

Proof. We say an ideal I is radically principal if there exists $f \in A$ such that $\sqrt{I} = \sqrt{(f)}$. We say a ring is radically principal if every ideal is radically principal. This is equivalent to that every open subset of $\mathrm{Spec}(A)$ is principal.

(a) We claim that $(2, X)$ is not radically principal.

Suppose that we have $\sqrt{(2,X)} = \sqrt{(f)}$, then $2, X \in \sqrt{(f)}$ and we have $f = 1$, but $1 \notin \sqrt{(2,X)}$.

(b) By definition of Dedekind domains ([Stacks]Definition.10.119.14), any closed subset of $\mathrm{Spec}(A)$ is of the form $\bigcup_{i=1}^{n} V(\mathfrak{p}_i)$. But Note that $V(\mathfrak{p}_i^n) = V(\mathfrak{p}_i)$ for all $n \in \mathbb{N}_{>0}$. For some n, $\mathfrak{p}_i^n$ will be principal since $\mathrm{Cl}(A)$ is torsion. Finally, use the fact that $\bigcap_{i=1}^{n} D(f_i) = D(\prod_{i=1}^{n} f_i)$.

□

1.3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. We let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ denote the presheaf on X carrying an open subset $U \subseteq X$ to $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf on X.

Proof. Omitted. See [Stacks]Lemma.6.33.1.

□

1.4. Let X be a topological space, U an open subset, and $j : U \to X$ the inclusion map.

(a) (Extension by the empty set) Let $\mathcal{F}$ be a sheaf of sets on U. Show that the presheaf on X

$$j_!^{\mathbf{Set}} \mathcal{F} : V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ \emptyset & V \not\subseteq U \end{cases}$$

is a sheaf. Compute the stalks of $j_!^{\mathbf{Set}} \mathcal{F}$.

(b) (Extension by zero) Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $j_! \mathcal{F}$ be the sheafification of the presheaf on X

$$j_!^{\mathrm{psh}} \mathcal{F} : V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ \{0\} & V \not\subseteq U \end{cases}$$

Compute the stalks of $j_! \mathcal{F}$. Deduce that $j_! : \mathrm{Shv}(U, \mathbf{Ab}) \to \mathrm{Shv}(X, \mathbf{Ab})$ is an exact functor. Find an example for which $j_!^{\mathrm{psh}} \mathcal{F}$ is not a sheaf.

Proof. (a) Suppose V is an open subset and $V = \bigcup_i V_i$ is an open covering. We verify the sheaf axioms. If $V \subseteq U$ then there is nothing to check. So assume $Y \not\subseteq U$. Then axiom SH.I and SH.II is automatically true, since $\mathcal{F}(V)$ and at least one $\mathcal{F}(V_i)$ are empty. It is obvious that

$$(j_!^{\mathbf{Set}} \mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in U \\ \emptyset & x \notin U \end{cases}$$

(b) Sheafification doesn't change the stalk, so by the same reason we have

$$(j_! \mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in U \\ \{0\} & x \notin U \end{cases}$$

$j_!$ is exact since $(j_! \varphi)_x = \begin{cases} \varphi_x & x \in U \\ 0 & x \notin U \end{cases}$

□

2. Problem 2

2.1. (=5). (a) Show that a ring homomorphism $\phi : A \to B$ is a monomorphism if and only if ϕ is an injection.

(b) Let $f : Y \to X$ be an epimorphism of schemes. Show that $f^\flat_X : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is an injection and $f(Y)$ intersects with every nonempty closed subset Z of X.

(c) Use (b) to give an example of an injective ring homomorphism $\phi : A \to B$ such that $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not an epimorphism of schemes.

Proof. (a) We only need to show that if ϕ is a monomorphism, then $\ker\phi = 0$. Since $\mathrm{Hom}_{\mathrm{ring}}(\mathbb{Z}[X], A) \cong A$, we can choose $0 \neq a \in \ker\phi$ to get a morphism $\varphi_a : \mathbb{Z}[X] \to A, X \mapsto a$.

(b) Use the fact that $\mathrm{Mor}(X, \mathrm{Spec}(R)) \cong \mathrm{Hom}_{\mathrm{ring}}(R, \mathcal{O}_X(X))$ we know that $f^\flat_X$ is an injection. Use the hint, the second assertion is obvious.

(c) $\mathrm{Spec}\mathbb{Q} \to \mathrm{Spec}\mathbb{Z}$.

□

2.2. (=6). We say that a continuous map $f : Y \to X$ is dominant if $f(Y)$ is dense in X. We say that a morphism $f : Y \to X$ of schemes is scheme-theoretically dominant if $f^\flat_U : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ is an injection for every open subset U of X.

(a) Show that a ring homomorphism $\phi : A \to B$ is an injection if and only if $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is scheme-theoretically dominant.

(b) Show that a scheme-theoretically dominant morphism $f : Y \to X$ is dominant. Show moreover that the converse holds for X reduced.

(c) Show that a scheme-theoretically dominant morphism that is surjective (as a continuous map of topological spaces) is an epimorphism of schemes. Deduce that any surjective morphism of schemes $f : Y \to X$ with X reduced is an epimorphism.

Proof. (a) We only need to prove that, if ϕ is injection, then $\mathrm{Spec}(\phi)^\flat_U$ is injective for all U. Notice that for U is principal, this is indeed true since the localization of ϕ is again injective. The rest is easy.

(b) Suppose $U \subseteq X$ is an nonempty open subset such that $f^{-1}U = \emptyset$, then $f^\flat_U : \mathcal{O}_X(U) \to 0$ cannot be injective, hence f is dominant.

Now suppose X is reduced and f is dominant, but there exists an open subset $U \subseteq X$ such that $(\ker f^\flat)(U) \neq 0$, then we can find an affine open subset $\mathrm{Spec}(A) = U' \subseteq U$ such that $(\ker f^\flat)(U') \neq 0$. Notice that A is reduced since X is, which means that if $\mathfrak{a}$ is an ideal of A such that $\sqrt{\mathfrak{a}} = 0$ then $\mathfrak{a} = 0$.

According to the assumption, we know that $f^{-1}U' \xrightarrow{f} \mathrm{Spec}(A)$ factors through $\mathrm{Spec}A/\mathfrak{a}$ where $0 \neq \mathfrak{a} = (\ker f^\flat)(U')$. But since f is dominant, we have $\sqrt{\mathfrak{a}} = 0$, which is not possible.

(c) This is actually easy. Suppose we have $(g, g^\flat) : X \to Z$, then g is determined by gf since f is surjective. And $g^\flat_W$ is determined by $f^\flat_{g^{-1}W} \circ g^\flat_W$ since $f^\flat_{g^{-1}W}$ is injective. By (b), any surjective morphism of schemes $f : Y \to X$ with X reduced is an epimorphism.

□

2.3. (=7). (a) Show that a ring homomorphism $\phi : A \to B$ is an epimorphism if and only if $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a monomorphism of schemes.

(b) Let X be a scheme. Let $X' = \coprod_{x\in X} \mathrm{Spec}(\kappa(x))$, where $\kappa(x)$ denotes the residue field of $\mathcal{O}_{X,x}$ and let $f : X' \to X$ be the canonical morphism sending $x' = \mathrm{Spec}(\kappa(x))$ to x with $f^\sharp_{x'} : \mathcal{O}_{X,x} \to \kappa$ given by the projection. Show that f is a monomorphism of schemes.

(c) Use (b) and Problem.2.2(c) to give an example of a morphism of affine schemes that is a monomorphism of schemes, an epimorphism of schemes, and a bijection, but not an isomorphism of schemes.

Proof. (a) Abstract nonsense. Omitted.

(b) This is actually easy. Suppose we have $(g, g^\flat) : Y \to X'$, then g is determined by fg since f is injective. And $g^\sharp_y$ is determined by $g^\sharp_y \circ f^\sharp_{g(y)}$ since $f^\sharp_{g(y)}$ is surjective.

(c) Maybe consider $X = \mathrm{Spec}(A)$ where A is a reduced ring with finite spectrum? Such that $\prod_{\mathfrak{p}\in\mathrm{Spec}(A)} \kappa(\mathfrak{p})$ is not isomorphic to A? For example, A is a DVR. $\square$

2.4. (=8). Let $\mathcal{P}$ be the set of prime numbers and let $A \subseteq \prod_{p\in\mathcal{P}} \mathbb{F}_p$ be the subring consisting of $a = (a_p)$ such that $\mathrm{supp}(a) = \{p | a_p \neq 0\}$ is a finite subset or $\mathcal{P} - \mathrm{supp}(a)$ is a finite subset of $\mathcal{P}$. Let $\mathfrak{m}_p$ be the kernel of the projection $A \to \mathbb{F}_p$ and let $\mathfrak{m}_\infty = \bigoplus_{p\in\mathcal{P}} \mathbb{F}_p \subseteq A$.

Let $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$ be the one-point compactification of the discrete set $\mathcal{P}$. In other words, the open subsets of $\mathcal{P}^*$ are the complements of finite subsets of $\mathcal{P}^*$ and all the subsets of $\mathcal{P}$.

(a) Show that $\mathfrak{m}_p$ and $\mathfrak{m}_\infty$ are indeed maximal and the map $\mathcal{P}^* \to \mathrm{Spec}(A)$ sending p to $\mathfrak{m}_p$ and ∞ to $\mathfrak{m}_\infty$ is a homeomorphism.

(b) Let $\kappa_\infty = A/\mathfrak{m}_\infty$. Show that κ_∞ is a field of characteristic zero and the cardinality of κ_∞ is the continuum.

Proof. (a) Since $A \to \mathbb{F}_p$ is surjective, $\mathfrak{m}_p$ is maximal. Also, any element with $\mathrm{supp}(a) = \mathcal{P}$ is invertible, so $\mathfrak{m}_\infty$ cannot be properly contained in any proper ideal.

Now we describe the ideals of A. Suppose $\mathfrak{a}$ is an ideal of A and $a \in \mathfrak{a}$, if $p \in \mathrm{supp}(a)$ then we have $\mathbb{F}_p \subseteq \mathfrak{a}$. This tells us that if we define $\mathrm{supp}(\mathfrak{a}) = \{p \in \mathcal{P} | \exists a \in \mathfrak{a}, p \in \mathrm{supp}(a)\} = \bigcup_{a\in\mathfrak{a}} \mathrm{supp}(a)$, then $\bigoplus_{p\in\mathrm{supp}(\mathfrak{a})} \mathbb{F}_p \subseteq \mathfrak{a}$.

Suppose now $\mathfrak{p}$ is an prime ideal of A. Define $\mathrm{cosupp}(\mathfrak{p}) = \mathcal{P} - \mathrm{supp}(\mathfrak{p})$. Apparently we have $\#\mathrm{cosupp}(\mathfrak{p}) \leq 1$. If $\mathrm{cosupp}(\mathfrak{p}) = \{p_0\}$, then we have $\bigoplus_{p\neq p_0} \mathbb{F}_p \subseteq \mathfrak{p} \subseteq \mathfrak{m}_{p_0}$. Notice that $\bigoplus_{p\neq p_0} \mathbb{F}_p$ is not a prime ideal, so there exists an element in $\mathfrak{p}$ such that its support is not finite. This shows that $\mathfrak{p} = \mathfrak{m}_{p_0}$. If $\mathrm{cosupp}(\mathfrak{p})$ is empty, then automatically we have $\mathfrak{p} = \mathfrak{m}_\infty$.

Now we describe the topology $\mathcal{T}$(= all open subsets) of $\mathrm{Spec}(A)$. Denote e_p to be the element with $\mathrm{supp}(e_p) = \{p\}$ for all prime p.

First of all, if $p \in \mathcal{P}$, then $\{\mathfrak{m}_p\}$ is open since is complement is $V(e_p)$. So $2^{\mathcal{P}} \subseteq \mathcal{T}$. Next, we show that $U \supseteq \{\mathfrak{m}_\infty\}$ is open if and only if U is cofinite. That is to show that a subset Z such that $Z \subseteq \mathcal{P}$ is closed if and only if it is finite. When $Z = \{\mathfrak{m}_{p_1}, \ldots, \mathfrak{m}_{p_n}\}$ is finite, then $Z = V(1 - \sum_{i=1}^n e_{p_i})$, and when $Z = V(\mathfrak{a})$ is not finite, then $\mathrm{supp}(\mathfrak{a})$ is finite and we have $\mathfrak{m}_\infty \in V(\mathfrak{a}) = Z$, contradiction.

(b) Apparently, $(p, p, \ldots, p, \ldots) \notin \bigoplus_{p\in\mathcal{P}} \mathbb{F}_p$, where p is a prime number. So κ_∞ is of characteristic 0. For the cardinality, Cantor's diagonal trick works perfectly. $\square$

2.5. (=9). Let X be a quasi-compact scheme. Show that X has a closed point.

Proof. True for any nonempty quasi-compact Kolmogorov(=T_0) space ([Stacks]Lemma.5.12.8). □

2.6. (=10). (a) Let X be a quasi-compact scheme. Let $A = \mathcal{O}_X(X)$ and $f \in A$. Show that the restriction map $A \to \mathcal{O}_X(X_f)$ factors through an injective homomorphism $\phi : A_f \to \mathcal{O}_X(X_f)$.

(b) Let X be a scheme admitting a finite cover $\{U_i\}$ by open affines such that each intersection $U_i \cap U_j$ is quasi-compact. Show that $\phi : A_f \to \mathcal{O}_X(X_f)$ is an isomorphism.

(c) Let X be a scheme such that there exists $f_1, \dots, f_n \in \mathcal{O}_X(X) = A$ with $\sum_{i=1}^n f_i A = A$ and X_{f_i} affine for all i. Show that X is affine.

Proof. Omitted. We point out that this is called the Serre's criterion for affineness, for a demanding proof, see: [EGA II]Theorem4.5.2 &Theorem5.2.1, [EGA IV]1.7.17. □

2.7. (=11). Let $f : Y \to X$ be a morphism of schemes.

(a) Show that if f is locally of finite type, $U \cong \mathrm{Spec}(A)$ is an affine open of X and $V \cong \mathrm{Spec}(B)$ is an affine open of $f^{-1}(U)$, then B is a finitely-generated A-algebra.

(b) Show that if f is quasi-compact and U is a quasi-compact open subset of X, then $f^{-1}(U)$ is quasi-compact.

(c) Show that if f is affine and U is an affine open of X, then $f^{-1}(U)$ is an affine open of Y.

(d) Show that if f is finite and $U \cong \mathrm{Spec}(A)$ is an affine open of X, then $f^{-1}(U) \cong \mathrm{Spec}(B)$ with B a finite A-algebra.

Proof. Omitted. This is a standard example of the so-called LOCT (= local on the target) property. □

3. Problem 3

3.1. (=12). (a) Let A be a Noetherian local ring of dimension ≥ 1. Show that the maximal ideal $\mathfrak{m}$ is the union of prime ideals of A of height 1.

(b) Let A be a Noetherian ring of dimension ≥ 2. Deduce from (a) that there are infinitely many prime ideals of A of height 1.

(c) Deduce from (b) that every locally Noetherian scheme of dimension ≥ 2 has infinitely many points.

Proof. (a)

(b) By definition, there exists a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{ht}(\mathfrak{m}) \geq 2$. We prove that

$$\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \text{ prime}, \mathfrak{p} \subseteq \mathfrak{m}, \mathrm{ht}(\mathfrak{p})=1} \mathfrak{p}$$

Now $\forall a \in \mathfrak{m}$, consider $\frac{a}{1} \in A_{\mathfrak{m}}$, then there exists a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ such that $\frac{a}{1} \in \mathfrak{p}_{\mathfrak{m}} = \{\frac{\alpha}{\beta} | \alpha \in \mathfrak{p}, \beta \notin \mathfrak{m}\}$ and $\mathrm{ht}(\mathfrak{p}_{\mathfrak{m}})(= \mathrm{ht}(\mathfrak{p})) = 1$. And we can know that $a \in \mathfrak{p}$.

Now use the prime avoidance, if there is only finitely many prime ideals of A of height 1, then the union above is a finite union and we have $\mathfrak{m}$ equals to some prime ideal in the union, so $\mathrm{ht}(\mathfrak{m}) = 1$, a contradiction.

(c) Let $X = \bigcup_i \operatorname{Spec}(A_i)$ be a locally Noetherian scheme where each A_i is Noetherian. Then we have $\dim X = \sup_i\{\dim A_i\} \geq 2$, and X cannot be finite.

$\square$

3.2. (=13). (a) Show that a morphism $f : Y \to X$ in a category admitting fiber products is a monomorphism if and only if the first projection $Y \times_X Y \to Y$ is an isomorphism. Show moreover that monomorphisms are stable under base change.

(b) Let k be a field. Use (a) to show that a ring homomorphism $\phi : k \to B$ is an epimorphism if and only if $B = 0$ or ϕ is an isomorphism. Deduced that a morphisms of schemes $f : Y \to \operatorname{Spec}(k)$ is a monomorphism if and only if $Y = \emptyset$ or f is an isomorphism.

(c) Let $f : Y \to X$ be a monomorphism of schemes. Show that f is an injection and for every point $y \in Y$, the extension of residue fields $\kappa(y)/\kappa(f(y))$ is trivial.

Proof. (a) Omitted.

(b) We know that ϕ is an epimorphism if and only if $\operatorname{Spec}(B) \to \operatorname{Spec}(k)$ is monomorphism, if and only if $\operatorname{Spec}(B) \times_k \operatorname{Spec}(B) \to \operatorname{Spec}(B)$ is an isomorphism, if and only if $B \to B \otimes_k B$ is an isomorphism. So we have $\dim_k B = (\dim_k B)^2$.

Now suppose f is a monomorphism. Then for any affine open $U \subseteq Y$ we know that $U \to Y \to \operatorname{Spec}(k)$ is a monomorphism. That is, $\mathcal{O}_Y(U)$ is 0 or k. Hence Y is a topological sum of $\operatorname{Spec}(k)$, so $Y = \coprod_{i \in \mathcal{I}} \operatorname{Spec}(k)$. Since f is monomorphism, we have $\mathcal{I}$ has no more than one element.

(c) We make a base change of f by the canonical morphism $\pi : \operatorname{Spec}(\kappa(f(y))) \to X$, by (a), we know that $f_\pi : Y_{f(y)} \to \operatorname{Spec}(\kappa(f(y)))$ is again a monomorphism. So by (b), $Y_{f(y)}$ has exactly one element. In this case, f_π is an isomorphism.

In another words, the canonical $\operatorname{Spec}(\kappa(x)) \xrightarrow{\theta} X$ factors through Y where $x = f(y)$. Say $\theta = \left(\operatorname{Spec}(\kappa(x)) \xrightarrow{g} Y \xrightarrow{f} X\right)$. Then we have

$$\mathcal{O}_{X,x} \xrightarrow{f_y^\sharp} \mathcal{O}_{Y,y} \xrightarrow{g_*^\sharp} \kappa(x)$$

is the canonical quotient ring map. Hence the field extension $\kappa(y)/\kappa(f(y))$ given by $f_y^\sharp$ is trivial.

$\square$

3.3. (=14). Given a scheme X and a field K, we let $X(K)$ denote $\operatorname{Mor}(\operatorname{Spec}(K), X)$.

(a) Let $\phi : K \to L$ be a field embedding. Show that the induced map $X(\phi) : X(K) \to X(L)$ is an injection.

(b) Show that a morphism of schemes $f : X \to Y$ is surjective if and only if for every field K, there exists a field extension L/K such that $f(L) : X(L) \to Y(L)$ is a surjection.

(c) Show that a morphism of schemes $f : X \to Y$ is radiciel if and only if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is surjective. Deduced that every radiciel morphism is separated.

Proof. (a) We only need to show that $\operatorname{Spec}(\phi) : \operatorname{Spec}(L) \to \operatorname{Spec}(K)$ is an epimorphism. We prove that $\operatorname{Spec}(\phi)$ is scheme-theoretically dominant and surjective, which is obvious.

(b) Suppose for every field K, there exists a field extension L/K such that $f(L) : X(L) \to Y(L)$ is a surjection. We choose $K = \kappa(y)$ and consider the canonical morphism $\operatorname{Spec}(\kappa(y)) \to Y$. Then there exists $\operatorname{Spec}(L) \to \operatorname{Spec}(\kappa(y))$ such that

$f(L) : X(L) \to Y(L)$ is surjective. Now consider $(\mathrm{Spec}(L) \to \mathrm{Spec}(\kappa(y)) \to Y) \in Y(L)$, then there exists $(\mathrm{Spec}(L) \to X) \in X(L)$ such that

$$(\mathrm{Spec}(L) \to \mathrm{Spec}(\kappa(y)) \to Y) = (\mathrm{Spec}(L) \to X \to Y)$$

Apparently, the image of $\mathrm{Spec}(L) \to X$ lies in the preimage of $y \in Y$.

For the converse, see [EGA I]I.3.5.3.

(c) Recall that a morphism f is radiciel if and only if every base change of f is injective as a continuous map. For a detailed discussion, see [Stacks]Section.29.10.

If f is radiciel, then the second projection $p_2 : X \times_Y X \to X$, as a base change of f, is again injective. But it is surjective since $p_2\Delta_f = \mathrm{id}_X$, so Δ_f must be surjective. In this case, the image of Δ_f is clearly closed, hence radiciel implies separated.

Now notice that, if we have $\theta : S \to Y$, denote $X \times_Y S$ by X_θ and the projection $X_\theta \to S$ by f_θ. Suppose we have two fibre product:

$$X \xleftarrow{p_1} X \times_Y X \xrightarrow{p_2} X$$

$$X_\theta \xleftarrow{q_1} X_\theta \times_X X_\theta \xrightarrow{q_2} X_\theta$$

Then we can find $X_\theta \times_X X_\theta \xrightarrow{\lambda} X \times_Y X$ such that for $i = 1, 2$.

$$X_\theta \times_X X_\theta \xrightarrow{q_i} X_\theta \to X = p_i\lambda$$

Now it is easy to verify that the right side square in the following is Cartesian.

$$\begin{tikzcd}
X \times_{(X \times_Y X)} (X_\theta \times_S X_\theta) \arrow[r, "\Delta_\lambda"] \arrow[d] & X_\theta \times_S X_\theta \arrow[r, "q_i"] \arrow[d, "\lambda"'] & X_\theta \arrow[d] \\
X \arrow[r, "\Delta"] & X \times_Y X \arrow[r, "p_i"] & X
\end{tikzcd}$$

So we know that $X \times_{(X \times_Y X)} (X_\theta \times_S X_\theta)$ is isomorphic to X_θ. After chosen an isomorphism $X_\theta \xrightarrow{\alpha} X \times_{(X \times_Y X)} (X_\theta \times_S X_\theta)$, we know that $\Delta_\lambda \circ \alpha$ is the diagonal of f_θ, which is again surjective since being sujective is closed under base change. SO WE ONLY NEED TO SHOW THAT IF Δ_f IS SURJECTIVE THEN f IS INJECTIVE. Notice that diagonal morphisms are always injective. So we only need to show that if Δ_f is bijective then f is injective. Notice that, as continuous maps, now we have $p_1^\natural = p_2^\natural : |X \times_Y X| \to |X|$.

Now if $f(x_1) = f(x_2)$, then consider

$$\phi_i : \mathrm{Spec}(\kappa(x_1)\kappa(x_2)) \to \mathrm{Spec}(\kappa(x_i)) \to X$$

Then ϕ_i factors through $X \times_Y X \xrightarrow{p_i} X$, so we have $x_1 = x_2$.

$\square$

4. Problem 4

4.1. (=15). (a) Let g and h be morphisms of schemes $X \to W$ and let E be their equalizer. Show that the morphism $E \to X$ is an immersion whose image is contained in the set-theoretic equalizer $E' = \{x \in X | g(x) = h(x)\}$.

(b) Deduce the following improvement of Problem 2.2.(c): a scheme-theoretically dominant morphism $f : Y \to X$ such that $f(Y)$ intersects with every nonempty closed subset Z of X is an epimorphism.

(c) Let A be a local domain of dimension ≥ 2 with fraction field K and residue field k. Use (b) and Problem2.3.(b) to show that $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is a monomorphism of schemes and an epimorphism of schemes, but not a surjection.

Proof. (a) First of all, we point out that, in a category with final objects and fibre products, every finite limit (for example, equalizers) exists ([Stacks]Lemma.4.8.14). In fact, we have (for example in **Sch**):

$$\mathrm{Eq}(g, h : X \to W) = j = \left(E = W \times_{(\Delta_W, W \times_{\mathrm{Spec}(\mathbb{Z})} W, (g,h))} X \xrightarrow{\mathrm{pr}} X\right)$$

being the base change of the absolute diagonal Δ_W, $E \xrightarrow{j} X$ is always an immersion. And we have $gj = p_1 \Delta_W \theta = \theta = p_2 \Delta_W \theta = hj$ where $\theta : E \to W$ is the canonical projection of fibre product. Hence $|E| \subseteq E'$.

(b) We only need to prove that, if $g_1 f = g_2 f$ then $g_1^\natural = g_2^\natural$. We can factor f through $E = \mathrm{Eq}(g_1, g_2) \xrightarrow{j} X$, say $f = jf_0$. Then $f(Y) = jf_0(Y) \subseteq j(E)$, so $j(E)$ intersects with every nonempty closed subset of X.

Since $j(E)$ is locally closed, there exists $U \subseteq X$ open such that $j(E)$ is closed in U. Denote $X - U$ by Z, then $j(E) \cap Z = \emptyset$. So $j(E)$ is closed in X. But f is dominant, so $f(Y)$ is dense in X and we have $j(E) = X$. Hence $E' = X$.

(c) The morphism is given by the inclusion $A \subseteq K$ and the quotient map $A \to k$. Since $\mathrm{Spec}(K \times k) = \mathrm{Spec}(\kappa(\{0\})) \coprod \mathrm{Spec}(\kappa(\mathfrak{m}))$ and $\coprod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Spec}(\kappa(\mathfrak{p})) \to X$ is monomorphism, we know that $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is a monomorphism. Since $A \to K \times k$ is injective, by Exercise 2.2.(a), $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is scheme-theoretically dominant. Notice that the image of $\mathrm{Spec}(K \times k) \to \mathrm{Spec}(A)$ is $\{(0), \mathfrak{m}\}$, which clearly intersects with any nonempty closed subset. But not a surjection since $\dim A \geq 2$.

□

4.2. (=16). (a) Let $i : Z \to X$ be a closed immersion of schemes. Let

$$W = \underline{\mathrm{Spec}}(\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X)$$

Show that the canonical morphism $X \coprod X \simeq \underline{\mathrm{Spec}}(\mathcal{O}_X \times \mathcal{O}_X) \to W$ is finite surjective. Describe the underlying topological space of W.

(b) Let $f : Y \to X$ be a quasi-compact morphism of schemes. Show that the ideal sheaf $\mathcal{I} = \ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$ is quasi-coherent and the closed subscheme Z of X defined by $\mathcal{I}$ is the smallest closed subscheme of X through which f factors. We call Z the scheme-theoretic image of f.

(c) Deduce that a quasi-compact morphism of schemes $f : Y \to X$ is an epimorphism if and only if f is scheme-theoretically dominant and $f(Y)$ intersects with every nonempty closed subset of X.

Proof. (a) First of all, the morphism of sheaves of rings $i^\flat : \mathcal{O}_X \to i_*\mathcal{O}_Z$ makes $i_*\mathcal{O}_Z$ an $\mathcal{O}_X$-algebra. And it is quasi-coherent as an $\mathcal{O}_X$-module since i is qcqs and $\mathcal{O}_X$ itself is quasi-coherent.

Now in the category $\mathbf{QCohAlg}(\mathcal{O}_X)$, we prove that

$$\Gamma(U, \mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X) = \{(f,g) \in \mathcal{O}_X(U) \times \mathcal{O}_X(U), i_U^\flat(f) = i_U^\flat(g)\}$$

First of all, this is a sheaf of sets. And also clearly an $\mathcal{O}_X$-algebra. It is quasi-coherent since if $U \subseteq X$ is an affine open subset where $U \cap Z = \mathrm{Spec}(A/I)$, then

$$(\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X)|_U = \widetilde{A \times_{A/I} A}$$

Now we only need to prove that $A \times_{A/I} A \subseteq A \times A$ is finite. This is ealy since $(1,0),(0,1)$ would be a set of generator. Now $A \times_{A/I} A \subseteq A \times A$ is injective, so $\mathrm{Spec}(A \times A) \to \mathrm{Spec}(A \times_{A/I} A)$ is dominant. It is closed since it is finite, hence surjective. We now claim that $X \coprod X \to W$ is a quotient map of topological spaces. This is because it is closed and surjective. Now we claim that $(A \times_{A/I} A) \cap (\mathfrak{p} \times A) = (A \times_{A/I} A) \cap (A \times \mathfrak{p})$ if and only if $I \subseteq \mathfrak{p}$. This is easy.

Notice that I don't think that I have described the topology of W very clear.??????

(b) If $U \subseteq X$ is an affine open subset, then we have

$$\mathcal{I}|_U = \ker\left(\mathcal{O}_U \to f_*(\mathcal{O}_{f^{-1}U})\right)$$

We only need to show that $\mathcal{I}|_U$ is quasi-coherent. Now let $f^{-1}U = \bigcup_{i=1}^n V_i$ where each V_i is affine open subset of $f^{-1}U$. Consider the morphism between AFFINE schemes:

$$\coprod_{i=1}^n V_i \xrightarrow{p} f^{-1}U \xrightarrow{f} U$$

Apply the sheaf axioms to $\mathcal{O}_{f^{-1}U}$ we know that $p^\flat$ is injective. Since f_* is left exact, we know that

$$\mathcal{I}|_U = \ker\left(\mathcal{O}_U \to f_*(p_*(\mathcal{O}_{\coprod_{i=1}^n V_i}))\right)$$

is quasi-coherent.

In order to show that $Z = \underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I})$ is the smallest closed subscheme of X through which f factors, we include a lemma:

Lemma: Let $f : Y \to X$ be a quasi-compact morphism of schemes and $i : Z \to X$ be a closed subscheme. Then f factors through i if and only if $\ker i^\flat \subseteq \ker f^\flat$.

For the proof, see [Stacks]Lemma.26.4.6.

(c) We only need to prove that a quasi-compact epimorphism of schemes $f : Y \to X$ is scheme-theoretically dominant. Suppose f is not scheme-theoretically dominant, then $\mathcal{I} \neq 0$ is quasi-coherent. We consider

$$Y \to \underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I}) = Z \to X \xrightarrow{\alpha_1,\alpha_2} X \coprod X \xrightarrow{\pi} W = \underline{\mathrm{Spec}}(\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X)$$

This tells us that $\pi\alpha_1 = \pi\alpha_2$. By the construction of W, we know (by what??????) that $Z = X$.

□

4.3. (=17). Let k be an algebraically closed field. In each of the following cases, compute the normalization of $f : X^\nu \to X$ of X. Describe all fibers of f that are not geometrically irreducible of geometrically reduced. Is f a universal homeomorphism?

(a) $X = \mathrm{Spec}(k[x,y]/(y^7 - x^{2020}))$

(b) $X = \mathrm{Spec}(k[x,y,z]/(xy^2 - z^2))$

Proof. (a) Notice that

$$2020 \times 2 = 7 \times 577 + 1 = 2 \times (3 \times 577 + 289)$$

If we set $a = \frac{x^{289}}{y}, b = \frac{y^2}{x^{577}}$, then the equation is the same as $a = b^3$. And we have $b^7 = a^2 b = x, b^{2020} = \frac{(b^7)^{289}}{a} = \frac{x^{289}}{a} = y$. So the normalization is $\mathrm{Spec}(k[\frac{y^2}{x^{577}}]) \to X$ given by $x \mapsto b^7, y \mapsto b^{2020}$.

Now we compute the fibers of f.

(b) This is easy, set $t = \frac{z}{y}$, then we have $x = t^2, y = y, z = ty$. Hence the normalization is given by $\mathrm{Spec}(k[t, y]) \to X$ given by $x \mapsto t^2, y \mapsto y, z \mapsto ty$. □

5. Problem 5

5.1. (=18). (a) Show that an injective and closed morphism of schemes is affine.

(b) Deduce that an injective and universally closed morphism of schemes is integral.

Proof. (a) Since injective and closed are LOCT properties, we only need to prove that: if $f : X \to \mathrm{Spec}(A)$ is a homeomorphism onto a closed subset, then f is affine.

Choose any $x \in X$, there exists $a \in A$ such that $f(x) \in D(a)$ and $U \subseteq X$ is an affine open neighborhood of x. Remenber that we have formula

$$f^{-1}(Y_\beta) = X_{f_Y^\flat(\beta)}, \forall \beta \in \Gamma(Y, \mathcal{O}_Y)$$

we have $U \cap f^{-1}(D(a)) = D(f^\flat_{\mathrm{Spec}(A)}(a)|_U)$, which is affine.

(b) We only need to show that

$$\text{affine} + \text{uni.closed} \Rightarrow \text{integral}$$

The question is LOCT, so everything is affine now. We need an algebraic result:

Let $R \to A$ be a ring morphism such that $\mathbb{A}^1_A \to \mathbb{A}^1_R$ is closed. Then A is integral over R. □

5.2. (=19). (a) Show that a scheme X is separated if and only if there exists an affine open cover $\{U_i\}$ of X such that $U_i \cap U_j$ if affine and the canonical homomorphism

$$\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

is surjective for all i, j.

(b) Let R be a graded ring. Show that, for all $f, g \in R_+$ homogeneous, the canonical homomorphism $R_{(f)} \otimes R_{(g)} \to R_{(fg)}$ is surjective. Deduce that $\mathrm{Proj}(R)$ is separated.

Proof. (a) We need a result, which is easy but useful:

If we have a fibre product of schemes $X \xleftarrow{p} X \times_{(f,S,g)} Y \xrightarrow{q} Y$, and $U \subseteq X, V \subseteq Y, T \subseteq S$ affine open subsets such that $f(U), g(V) \subseteq T$. Then $p^{-1}U \cap q^{-1}V$ is the fibre product $U \times_T V$. In particular, the canonical morphism $U \times_T V \to X \times_S Y$ is an open immersion.

In particular, if Δ_X is the absolute diagonal of $X \to \mathrm{Spec}(\mathbb{Z})$, $U, V \subseteq X$ are affine open subsets. Then $U \cap V = \Delta_X^{-1}(U \times_{\mathrm{Spec}(\mathbb{Z})} V)$. And if X is separated, then

$$U \cap V \xrightarrow{\Delta_X} U \times_{\mathrm{Spec}(\mathbb{Z})} V = \mathrm{Spec}(\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V))$$

is a closed immersion. This proves one direction of (a).

The converse is similar, notice that if $X = \bigcup_i U_i$ is an affine open covering, then $X \times_{\mathrm{Spec}(\mathbb{Z})} X = \bigcup_{i,j} U_i \times_{\mathrm{Spec}(\mathbb{Z})} U_j$ is also an affine open covering. And if each

$$U_i \cap U_j \xrightarrow{\Delta_X} U_i \times_{\mathrm{Spec}(\mathbb{Z})} U_j = \mathrm{Spec}(\mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j))$$

are affine and closed immersion, then Δ_X is also a closed immersion.

(b) Trivial.

□

5.3. (=20). Let R be a graded ring.

(a) Show that for any prime ideal $\mathfrak{p}$ of R, $\bigoplus_{d\geq 0}(\mathfrak{p} \cap R_d)$ is a homogeneous prime ideal of R. Deduce that any minimal prime ideal of R is homogeneous.

(b) Let T be the set of maximal points of $\mathrm{Spec}(R)$. Show that $T \cap \mathrm{Proj}(R)$ is the set of maximal points of $\mathrm{Proj}(R)$.

(c) Show that $\mathrm{Proj}(R)$ is normal if R is an integrally closed domain.

Proof. (a) Omitted.

(b) Omitted.

(c) Recall that a scheme is normal if and only if all local rings $\mathcal{O}_{X,x}$ are integrally closed domains(=ICD).

We use the notation as [Hartshorne] Page.76. If R is an ICD, then so does $T^{-1}R$, then any element in $\mathrm{Frac}(R_{(\mathfrak{p})})$ which is integral over $R_{(\mathfrak{p})}$ is also integral over $T^{-1}R$, hence it lies in $R_{(\mathfrak{p})}$.

□

5.4. (=21). (a) Let A be a Noetherian ring and $\mathfrak{b}$ an ideal of A. We say that an ideal $\mathfrak{a}$ of A is $\mathfrak{b}$-saturated if $(\mathfrak{a} : \mathfrak{b}) = \mathfrak{a}$, where $(\mathfrak{a} : \mathfrak{b}) := \{x \in A | x\mathfrak{b} \subseteq \mathfrak{a}\}$. For any ideal $\mathfrak{a}$ of A, show that the sequence of ideals $(\mathfrak{a} : \mathfrak{b}^n), n \geq 0$ is stationary and $(\mathfrak{a} :^\infty \mathfrak{b}) := \bigcup_{n\geq 0}(\mathfrak{a} : \mathfrak{b}^n)$ is the smallest $\mathfrak{b}$-saturated ideal contaning $\mathfrak{a}$.

(b) For any primary ideal $\mathfrak{q}$ of A, show that

$$(\mathfrak{q} :^\infty \mathfrak{b}) = \begin{cases} \mathfrak{q} & \sqrt{\mathfrak{q}} \not\supseteq \mathfrak{b} \\ A & \sqrt{\mathfrak{q}} \supseteq \mathfrak{b} \end{cases}$$

Deduce that $(\sqrt{\mathfrak{a}} :^\infty \mathfrak{b}) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a}) - V(\mathfrak{b})} \mathfrak{p}$

(c) Let R be a graded ring. For any subset $Y \subseteq \mathrm{Proj}(R)$, let $I(Y) = \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$. Show that $V_+(I(Y)) = \overline{Y}$ is the closure of Y in $\mathrm{Proj}(R)$.

(d) Assume that R is Noetherian. For any homogeneous ideal $\mathfrak{a}$ of R, show that $I(V_+(\mathfrak{a})) = (\sqrt{a} :^\infty R_+)$. Deduce that the maps V_+ and I induce a one-to-one order-reversing correspondence between R_+-saturated radical homogeneous ideals of R and closed subsets of $\mathrm{Proj}(R)$.

Proof. (a) Since $x\mathfrak{b}^{n+1} \subseteq x\mathfrak{b}^n$ and A is Noetherian, the sequence of ideals $(\mathfrak{a} : \mathfrak{b}^n), n \geq 0$ is stationary. Let's say $(\mathfrak{a} :^\infty \mathfrak{b}) = (\mathfrak{a} : \mathfrak{b}^N)$. We prove that $(\mathfrak{a} :^\infty \mathfrak{b})$ is $\mathfrak{b}$-saturated.

In fact, we have

$$((\mathfrak{a} : \mathfrak{b}^N) : \mathfrak{b}) = \{x \in A | \forall y \in \mathfrak{b}, xy\mathfrak{b}^N \subseteq \mathfrak{a}\} \subseteq (\mathfrak{a} : \mathfrak{b}^{N+1}) = (\mathfrak{a} : \mathfrak{b}^N)$$

Now if $\mathfrak{a} \subseteq \mathfrak{c}$ such that $\{x \in A | x\mathfrak{b} \subseteq \mathfrak{c}\} \subseteq \mathfrak{c}$, we need to show that $(\mathfrak{a} : \mathfrak{b}^n) \subseteq \mathfrak{c}$ for all $n \geq 0$. Since $x \in (\mathfrak{a} : \mathfrak{b}^n) \subseteq (\mathfrak{c} : \mathfrak{b}^n)$.

So we have $x \prod_{i=1}^{n} \beta_i \in \mathfrak{c}$ for all $\beta_i \in \mathfrak{b}$. Hence $x \prod_{i=1}^{n-1} \beta_i \in \mathfrak{c}$ for all $\beta_i \in \mathfrak{b}$. Repeat this process and we have $x \in \{x \in A | x\mathfrak{b} \subseteq \mathfrak{c}\} \subseteq \mathfrak{c}$.

(b) Recall that an ideal $\mathfrak{q}$ is primary if and only if in $A/\mathfrak{q}$, the concept of zero-divisors and nilpotent elements are the same.

Now if $\mathfrak{b} \subseteq \sqrt{\mathfrak{q}}$. An important result in Noetherian ring is that, every ideal in a Noetherian ring contains a power of its radical ([Atiyah]Prop.7.14). Say $(\sqrt{\mathfrak{q}})^{n_0} \subseteq \mathfrak{q}$, then we have $\mathfrak{b}^{n_0} \subseteq \mathfrak{q}$ and $(\mathfrak{q} : \mathfrak{b}^{n_0}) = A$.

If $\mathfrak{b} \nsubseteq \sqrt{\mathfrak{q}}$, then we prove that $(\mathfrak{q} : \mathfrak{b}^n) \subseteq \mathfrak{q}$ for all $n \geq 0$. Now if $x \in (\mathfrak{q} : \mathfrak{b}^n) = \{x \in A | x\mathfrak{b}^n \subseteq \mathfrak{q}\}$, choose $\beta \in \mathfrak{b}$ but $\beta \notin \sqrt{\mathfrak{q}}$, then we have

$$\overline{x}\overline{\beta}^n = 0 \in A/\mathfrak{q}$$

if $\overline{x} \neq 0 \in A/\mathfrak{q}$, then $\overline{\beta}$ would be nilpotent, a contradiction.

Finally, we prove that $(\sqrt{\mathfrak{a}} :^{\infty} \mathfrak{b}) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a}) - V(\mathfrak{b})} \mathfrak{p}$

We know that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$, choose

$$N > \max\{\min\{n | (\sqrt{\mathfrak{a}} : \mathfrak{b}^n) = (\sqrt{\mathfrak{a}} :^{\infty} \mathfrak{b})\}, \max\{\min\{n | (\mathfrak{p} : \mathfrak{b}^n) = (\mathfrak{p} :^{\infty} \mathfrak{b})\} | \mathfrak{p} \supseteq \mathfrak{a}\}\}$$

then we have

$$(\sqrt{\mathfrak{a}} :^{\infty} \mathfrak{b}) = (\mathfrak{a} : \mathfrak{b}^N) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} (\mathfrak{p} : \mathfrak{b}^N) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} (\mathfrak{p} :^{\infty} \mathfrak{b}) = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}, \mathfrak{p} \not\supseteq \mathfrak{b}} \mathfrak{p}$$

(c) It is trivial that $\mathfrak{p} \in V(\bigcap_{\mathfrak{p} \in Y} \mathfrak{p})$ for all $\mathfrak{p} \in Y$, so we have $Y \subseteq V(\bigcap_{\mathfrak{p} \in Y} \mathfrak{p}) \cap \mathrm{Proj}(R) = V_+(I(Y))$. Now if we have a closed subset $V_+(I)$ contains Y, then $I \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in Y$. So we have $I \subseteq \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$ and

$$V_+(I(Y)) = V(\bigcap_{\mathfrak{p} \in Y} \mathfrak{p}) \cap \mathrm{Proj}(R) \subseteq V(I) \cap \mathrm{Proj}(R) = V_+(I)$$

(d) Omitted.

□

5.5. (=22). Let A be a ring and let $a, b \geq 1$ be integers. Show that the weighted projective line $\mathbb{P}_A(a, b)$ is canonically isomorphic to $\mathbb{P}^1_A$.

Proof. WLOG, asssume $\gcd(a, b) = 1$. Then use the affine open covering

$$\mathbb{P}_A(a, b) = \mathrm{Spec}(A[\frac{y^a}{x^b}]) \cup \mathrm{Spec}(A[\frac{x^b}{y^a}])$$

□

6. Problem 6

6.1. (=23). Let A be a ring and let $d \geq 2$ be an integer. Let $I \subseteq R = A[x_0, \ldots, x_d]$ denote the homogeneous ideal of the d-uple embedding $\mathbb{P}^1_A \hookrightarrow \mathbb{P}^d_A$. (That is, $I = \ker(A[x_0, \ldots, x_d] \xrightarrow{x_i \mapsto y^i z^{d-i}} A[y, z])$, by the course notes Example.1.10.26)

(a) Show that $I \cap R_2$ is a free A-module of rank $\binom{d}{2}$. Deduce that I cannot be generated by less than $\binom{d}{2}$ elements unless $A = 0$.

(b) Show that I is generated by $I \cap R_2$.

(c) Assume $d=3$. Let $J=(x_1^2-x_0x_2, x_2^3-x_0x_3^2)\subseteq I$. Check that $\sqrt{J}=\sqrt{I}$.

Proof. (a) Suppose $\sum_{i,j} a_{(i,j)}x_ix_j\in I$, then we have $\sum_{i+j=k} a_{(i,j)}=0$ for all k. Hence $\{x_ix_{k-i}-x_{\lfloor\frac{k}{2}\rfloor}x_{k-\lfloor\frac{k}{2}\rfloor}|x_ix_{k-i}-x_{\lfloor\frac{k}{2}\rfloor}x_{k-\lfloor\frac{k}{2}\rfloor}\neq 0\}$ is a A-basis for $I\cap R_2$. Count the number (which is not so trivial), we get

$$\sum_{k=2}^{d-1}(k+\frac{(-1)^k-1}{2})+\frac{d+\frac{(-1)^d-1}{2}}{2}=\binom{d}{2}$$

Notice that I is homogeneous, now if $\{f_1,\dots,f_b\}$ is a generator of I then the 2-degree part of f_i also generates $I\cap R_2$, so I cannot be generated by less than $\binom{d}{2}$ elements unless $A=0$.

(b) Omitted. Use Gröbner basis.

(c) Omitted. Use Gröbner basis.

□

6.2. (=24). We say that a scheme X is locally integral if $\mathcal{O}_{X,x}$ is a domain for every $x\in X$. Show that the irreducible component of a locally integral scheme are disjoint. Deduce that a locally integral scheme with finitely many irreducible components is a finite coproduct(= sum) of integral schemes.

Proof. Hint: if we consider the canonical $\mathrm{Spec}(\mathcal{O}_{X,x})\to X$, then there is a bijection between

$$\mathrm{Spec}(\mathcal{O}_{X,x})\leftrightarrow\{\text{irreducible closed subset which contains } x\}$$

given by taking closure ([EGA I]2.4.2, or you can prove this by describe the underlying topological space of $\mathrm{Spec}(\mathcal{O}_{X,x})$ and notice that schemes are sober). Notice that in an integral domain, there is only one minimal prime ideal.

The rest is easy.

□

6.3. (=25). Let k be a field.

(a) let A be a finitely generated k-algebra that is a domain. Assume that $A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$ of height 1. Show that the integral closure of A is $\bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through height 1 prime ideals.

(b) Let R be a graded Noetherian domain generated by R_1 over R_0. Assume that R_+ has height ≥ 2 and $X=\mathrm{Proj}(R)$ is normal. Show that the canonical map $R\xrightarrow{\varphi}\Gamma_*(\mathcal{O}_X):=\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{O}_X(n))$ identifies $\Gamma_*(\mathcal{O}_X)$ with the integral closure of R.

Proof. (a) The original question is wrong. Omitted.

(b) Consider the natural map $\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{O}_X(n))\xrightarrow{\epsilon}\Gamma(X,\bigoplus_{n\in\mathbb{Z}}\mathcal{O}_X(n))$. Since $X=\mathrm{Proj}(R)$ is qcqs and $\mathcal{O}_X(n)$ is quasi-coherent, we have ϵ is an isomorphism by Lemma.1.10.49 in the course notes.

We are left to show that $\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)\setminus V(R_+))=\bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through height 1 prime ideals.

□

6.4. (=26). Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Show that for any affine open U of X, $X_s \cap U$ is affine.

Proof. Recall that $X_s = \{x \in X | f(x) \neq 0 \in \mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x = \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)\}$ is an open subset of X (see the course notes). We now prove that the open immersion $X_s \to X$ is affine.

Now for any $x \in X$, there exists an affine open neighborhood U and an isomorphism $\pi : \mathcal{L}|_U \to \mathcal{O}_X|_U$, denote $\pi_U(f|_U)$ by $g \in \mathcal{O}_X(U)$.

Then we have $f(x) \neq 0$ if and only if $g_x = \pi_x(f_x)$ is invertible in $\mathcal{O}_{X,x}$, if and only if $x \in U_g$. So we have $X_s \cap U = U_g$ is affine. So $X_s \to X$ is affine. $\square$

6.5. (=27). Let A be a ring. For an A-module M, we let $\mathbb{P}_A(M)$ denote the projective bundle $\mathrm{Proj}(\mathrm{Sym}_A(M))$. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of A-modules.

(a) Show that g induces a closed immersion $\mathbb{P}_A(g) : \mathbb{P}_A(M'') \to \mathbb{P}_A(M)$ and f induces an affine morphism $\mathbb{P}_A(f) : \mathbb{P}_A(M) \setminus \mathrm{im}(\mathbb{P}_A(g)) \to \mathbb{P}_A(M')$.

(b) Assume that the exact sequence splits. Show that $\mathbb{P}_A(g)$ can be identified with the projection $\mathbb{V}(\mathcal{O}_Y(-1) \otimes_A M'') \to Y$. Here $Y := \mathbb{P}_A(M')$, and for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$, $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}(\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F}))$.

Proof. (a) Since $\mathrm{Sym}_A(M) \to \mathrm{Sym}_A(M'')$ is also surjective, so $\mathbb{P}_A(g)$ is a closed immersion by Prop.1.10.20 in the course notes.

Now notice that the (first type) functoriality gives us an affine morphism:

$$\mathrm{Proj}(\mathrm{Sym}_A(f)) : \mathrm{Proj}(\mathrm{Sym}_A(M)) \setminus V(f(M')) \to \mathbb{P}_A(M')$$

We only need to show that actually we have $V(f(M')) = \mathrm{im}(\mathbb{P}_A(g))$. Which is equivalent to the fact that $\ker(\mathrm{Sym}_A(g)) = f(M')\mathrm{Sym}_A(M)$. This is true by Bourbaki Algebra Chapter III.

(b) $\square$

7. Problem 7

7.1. (=28). Let $f : X \to Y$ be a morphism of schemes with X quasi-compact. Let $\mathcal{L}$ and $\mathcal{L}'$ be invertible sheaves on X and $\mathcal{M}$ an invertible sheaf on Y.

(a) Show that $X = \bigcup_{s \in S_{+,\mathrm{homog}}} X_s$ if and only if $\mathcal{L}^{\otimes n}$ is globally generated for some $n \geq 1$. Here $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. In this case we say that $\mathcal{L}$ is semiample.

(b) Show that if $\mathcal{L}$ is ample and $\mathcal{L}'$ is semiample, then $L \otimes_{\mathcal{O}_X} \mathcal{L}'$ is ample.

(c) Show that if $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then for $n \gg 0$, $\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M}^{\otimes n}$ is ample.

(d) Show that if $\mathcal{L}$ is f-very ample and $\mathcal{L}'$ is globally generated, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is f-very ample.

(e) Show that if f is locally of finite type and $\mathcal{L}$ is ample, then there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is f-very ample for all $n \geq n_0$.

Proof. $\square$

7.2. (=29). (a) Let $f : X \to S$ be a separated morphism of schemes. Show that every section s of f is a closed immersion.

(b) Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $f : \mathbb{V}(\mathcal{E}) \to S$ and let $s : S \to \mathbb{V}(\mathcal{E})$ be the zero section of f, namely the section induced by $0 : \mathcal{E} \to \mathcal{O}_S$. Let $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{V}(\mathcal{E})}$ be ideal sheaf corresponding to s. Show that $s^*\mathcal{I} \simeq \mathcal{E}$.

Proof. □

7.3. (=30). Let S be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module. Let $P = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_S)$. Let Z_P and 0_P denote the closed subschemes defined respectively by the closed immersions $\mathbb{P}(\mathcal{E}) \to P$ and $\mathbb{P}(\mathcal{O}) \to P$ given by the projections $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{E}$ and $\mathcal{E} \oplus \mathcal{O}_S \to \mathcal{O}_S$. We call Z_P the infinity locus and 0_P the zero section of $P \to S$.

(a) Show that Z_P is an effective Cartier divisor of P and that $P \setminus Z_P$ can be identified with $\mathbb{V}(\mathcal{E})$. We call P the projective closure of $\mathbb{V}(\mathcal{E})$.

(b) Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E})})$. Let Z_X and 0_X denote respectively the infinity locus and zero sections of $X \to \mathbb{P}(\mathcal{E})$. Construct an S-morphism $\pi : X \to P$ identifying X with the blowing up of P at 0_P such that $\pi^{-1}(0_P) = 0_X$ and $\pi^{-1}(Z_P) = Z_X$ as subschemes of X. Describe π in terms of the functors $(\mathbf{Sch}_S)^{\mathrm{op}} \to$ **Set** that X and P represent.

(c) Deduce that $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq \mathrm{Bl}_{0_P}(\mathbb{V}(\mathcal{E}))$ and $\mathbb{V}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)) \simeq P \setminus 0_P$.

Proof. □

7.4. (=31). Let k be a field of characteristic $\neq 2$ and let $S = \mathrm{Spec}(k[x,y]/(y^2 - x^4))$. find blowings up $S' \to S$ and $S'' \to S'$ such that S'' is normal.

Proof. □

8. Problem 8

8.1. (=32). Show that, in a triangulated category, the direct sum of two distinguished triangles is a distinguished triangle.

Proof. [Stacks]Lemma.13.4.10.

□

8.2. (=33). Let $\mathcal{D}$ be a triangulated category.

(a) Show that for objects X and Y in $\mathcal{D}$, the triangle $X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$, where i and p are the canonical morphisms, is a distinguished triangle.

(b) Conversely, show that every distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathcal{D}$ with $h = 0$ is isomorphic to the distinguished triangle in (1).

Proof. [Stacks]Lemma.13.4.11.

□

8.3. (=34). Let $\mathcal{A}$ be an abelian category. For every $L \in \mathsf{D}(\mathcal{A})$ and $n \in \mathbb{Z}$, construct a distinguished triangle $\tau^{\leq n} L \to L \to \tau^{\geq n+1} L \xrightarrow{h} (\tau^{\leq n} L)[1]$ in $\mathsf{D}(\mathcal{A})$. Show that $\mathrm{H}^i h = 0$ for all i. Given an example with h nonzero in $\mathsf{D}(\mathcal{A})$.

Proof. [Stacks]Lemma.13.12.4 gives us the distinguished triangle.

Since $\mathrm{H}^i(\tau^{\geq n+1} L) = 0$ when $i \leq n$ and $\mathrm{H}^i((\tau^{\leq n} L)[1]) = \mathrm{H}^{i+1}(\tau^{\leq n} L) = 0$ when $i \geq n$. Hence $\mathrm{H}^i h = 0$ for all i.

If $h = 0$, then $L \simeq (\tau^{\leq n} L) \oplus (\tau^{\geq n+1} L)$, it's easy to give an example such that this is not true.

$\square$

8.4. (= 35). Recall that $\mathcal{I}$ is F-injective if and only if: (i) For every $X \in \mathcal{A}$, there exists a monomorphism $X \to Y$ with $Y \in \mathcal{I}$. (ii) For every $L \in K^+(\mathcal{I})$ acyclic, FL is acyclic.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories admitting an F-injective subcategory $\mathcal{J} \subseteq \mathcal{A}$. We say that $X \in \mathcal{A}$ is F-acyclic if $\mathsf{R}^n FX = 0$ for all $n \geq 1$. We let $\mathcal{I}$ denote the full subcategory of $\mathcal{A}$ spanned by F-acyclic objects.

(a) Show that $\mathcal{I}$ is F-injective.

In the rest of this problem, assume that there exists $N > 0$ such that $\mathsf{R}^N FX = 0$ for all $X \in \mathcal{A}$.

(b) Show that $\mathsf{R}^n FX = 0$ for all $X \in \mathcal{A}$ and $n \geq N$.

(c) Show that for every exact sequence $X_{N-1} \to \cdots \to X_1 \to Y \to 0$ in $\mathcal{A}$ with $\mathsf{R}^j FX_i = 0$ for all $j \geq i$, Y is F-acyclic.

(d) Deduce that for every $L \in \mathbf{Comp}(\mathcal{I})$ acyclic, FL is acyclic.

Proof. (a) We first claim that actually we have $\mathcal{J} \subseteq \mathcal{I}$. Now if we have $L^\bullet \in K^+(\mathcal{J})$, then $\epsilon_{L^\bullet} \in \mathrm{Mor}_{\mathsf{D}^+(\mathcal{B})}(FL^\bullet, (\mathsf{R}F)L^\bullet)$ is an isomorphism by Corollary.2.1.32 in the course notes. Consider the special case: $X' \in \mathcal{J}$ and

$$L^\bullet = (\cdots \to 0 \to X' \to 0 \to \cdots)$$

Then we compute $\mathsf{R}^n FX' = (\mathsf{R}^n F)(L^\bullet) = \mathrm{H}^n((\mathsf{R}F)(L^\bullet)) = \mathrm{H}^n(FL^\bullet) = 0$ when $n \geq 1$. Hence every object in $\mathcal{J}$ is F-acyclic and $\mathcal{J} \subseteq \mathcal{I}$. Now we know that, for every $X \in \mathcal{A}$, it can embedded into an object in $\mathcal{I}$. Thus, in order to show that $\mathcal{I}$ is F-injective, by the proposition.2.1.31 in the course notes, we only need to show that:

Every monomorphism $X \to X'$ with $X, X' \in \mathcal{I}$ can be completed into a short exact sequence $0 \to X \to X' \to Y \to 0$ with $Y \in \mathcal{I}$ such that $0 \to FX \to FX' \to FY \to 0$ is also exact. This is something standard.

(b) Notice that every object is a subobject of an F-acyclic object. The rest is a standard exercise of dimension shifting and long exact sequence.

(c) Let's write this more explicitly and complete:

$$0 \to X_N = \ker(d_{N-1}) \xrightarrow{d_N} X_{N-1} \xrightarrow{d_{N-1}} X_{N-2} \xrightarrow{d_{N-2}} \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} Y \to 0$$

and we have these fundamental exact sequences:

$$0 \to Z_n \to X_n \to Z_{n-1} \to 0$$

use long exact sequences and we're done.

(d) In order to show that $\mathrm{H}^n(FL) = 0$, consider the truncation $\tau^{\leq n} L$:

$$L^{n-(N-1)} \to \cdots \to L^{n-1} \to B^n(L) \to 0$$

By the definition of $\mathcal{I}$, we know that $\mathsf{R}^j FL^{n-i} = 0$ for all $j \geq i$. So we know that $B^n(L)$ is F-acyclic. We conclude by dimension shifting.

$\square$

8.5. (=36, Serre's Criterion). Let X be a quasi-compact scheme. Assume that $\mathrm{H}^1(X, \mathcal{I}) = 0$ for every quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_X$. Proceed in the following steps to show that X is affine.

(a) show that for every closed point $x \in X$, there exists $f \in \mathcal{O}_X(X)$ such that $x \in X_f$ and X_f is affine.

(b) Use Problem.2.5 to deduce that there exist $f_1, \dots, f_n \in \mathcal{O}_X(X)$ with $X = \bigcup_{i=1}^n X_{f_i}$ and X_{f_i} affine.

(c) Show that $f_1, \dots, f_n$ generate the unit ideal in $\mathcal{O}_X(X)$. Conclude that X is affine.

Proof. (a) Suppose $x \in X$ is a closed point and $U = \mathrm{Spec}(A)$ is an affine neighborhood of $x = \mathfrak{m} \in \mathrm{Spec}(A)$. Let $j : Z = X - U \to X, j' : Z' = (X - U) \cup \{x\} \to X$ be the reduced closed subscheme of X with $\mathcal{I} = \ker j^\flat, \mathcal{I}' = \ker j'^\flat$. It is obvious that $\mathcal{I}, \mathcal{I}'$ are quasi-coherent ideals, and we have $\mathcal{I}' \subseteq \mathcal{I}$ by our course notes Prop.1.8.14.

By assumption and the canonical exact sequence we have

$$0 \to \Gamma(X, \mathcal{I}') \to \Gamma(X, \mathcal{I}) \xrightarrow{\varphi} \Gamma(X, \mathcal{I}/\mathcal{I}') \to 0$$

First we claim that $\Gamma(X, \mathcal{I}/\mathcal{I}') \simeq \kappa(x)$. This is easy: notice that if $y \in X$ such that $y \neq x$, then there exists an affine open neighborhood V of y such that $V \cap Z = V \cap Z'$, which means that $\mathcal{I}|_V = \mathcal{I}'|_V$. Hence $\mathcal{I}/\mathcal{I}'$ is supported in $\{x\}$, hence skyscraper, and we have $\Gamma(X, \mathcal{I}/\mathcal{I}') \simeq \Gamma(U, \mathcal{I}/\mathcal{I}') \simeq \kappa(x)$.

We choose $f \in \Gamma(X, \mathcal{I}) \subseteq \mathcal{O}_X(X)$ such that $\varphi(f) = 1 \in \kappa(x)$. To be precise, we mean that after the canonical ring map

$$A = \mathcal{I}(U) \to (\mathcal{I}/\mathcal{I}')(U) = \kappa(x)$$

$f|_U$ becomes 1. So we have $x \in X_f$. By the definition of $\mathcal{I}$ (see course notes Prop.1.8.14.), we have $X_f \cap Z = \emptyset$. This shows that $X_f = X_f \cap U$ is affine.

(b),(c) Consider the union of all X_f where $f \in \mathcal{O}_X(X)$ and X_f is affine. It contains all closed points. By Problem.2.5, it equals to X. We can make it into a finite union $X = \bigcup_{i=1}^n X_{f_i}$.

Since any morphism of $\mathcal{O}_X$-modules $\phi \in \mathrm{Mor}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$ is determined by $\phi(1_X) \in \mathcal{F}(X)$, we have a short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{O}_X^n \xrightarrow{(\times f_1, \dots, \times f_n)} \mathcal{O}_X \to 0$$

We now prove that $\mathrm{H}^1(X, \mathcal{F}) = 0$, and consequently we will have $\mathcal{O}_X(X) = (f_1, \dots, f_n)$. Consider $\mathcal{F}_k = \mathcal{F} \cap \mathcal{O}_X^k$, then $\mathcal{F}_k/\mathcal{F}_{k-1}$ is a quasi-coherent module by definition. Consider the canonical morphism $\mathcal{F}_k \to \mathcal{O}_X^k/\mathcal{O}_X^{k-1}$ we know that $\mathcal{F}_k/\mathcal{F}_{k-1}$ is an $\mathcal{O}_X$-ideal. By assumption we have $\mathrm{H}^1(X, \mathcal{F}_k/\mathcal{F}_{k-1}) = 0$. This tells us that $\mathrm{H}^1(X, \mathcal{F}_k) \simeq \mathrm{H}^1(X, \mathcal{F}_{k-1})$ for all k.

In particular, $(\mathcal{O}_X(X))^{\oplus n} \xrightarrow{(\times f_1, \dots, \times f_n)} \mathcal{O}_X(X)$ is surjective.

□

9. Exercise 9

9.1. (=37). Let $F : \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{B})$ be a triangulated functor carrying $\mathsf{D}^{\geq 0}(\mathcal{A})$ to $\mathsf{D}^{\geq 0}(\mathcal{B})$. Let $X \in \mathsf{D}^{\geq 0}(\mathcal{A})$. Prove the existence of an isomorphism $\mathrm{H}^0 F \mathrm{H}^0 X \simeq \mathrm{H}^0 F X$ and an exact sequence

$$0 \to \mathrm{H}^1 F \mathrm{H}^0 X \to \mathrm{H}^1 F X \to \mathrm{H}^0 F \mathrm{H}^1 X \to \mathrm{H}^2 F \mathrm{H}^0 X \to \mathrm{H}^2 F X$$

Proof. We have distinguished

$$\tau^{\leq -1} X \to X \to \tau^{\geq 0} X \to (\tau^{\leq -1} X)[1]$$

$$\tau^{\leq 0} X \to X \to \tau^{\geq 1} X \to (\tau^{\leq 0} X)[1]$$

It's easy to see that $\mathrm{H}^\bullet(\tau^{\leq -1} X) = 0^\bullet$, so $X \to \tau^{\geq 0} X$ is an isomorphism in $\mathsf{D}(\mathcal{A})$ and $\tau^{\leq 0} X \simeq \tau^{\leq 0} \tau^{\geq 0} X = \mathrm{H}^0 X$. We denote $\mathrm{H}^0 X$ by Y, $\tau^{\geq 1} X$ by Z. Since F is triangulated, we have distinguished

$$FY \to FX \to FZ \to F(Y[1])$$

$$F(\tau^{\leq 1} Z)(= F\mathrm{H}^1 X) \to FZ \to F(\tau^{\geq 2} Z)(= F(\tau^{\geq 2} X)) \to (F(\tau^{\leq 1} Z))[1]$$

And two long exact sequence:

$$0 \to \mathrm{H}^0 FY \to \mathrm{H}^0 FX \to 0 \to \mathrm{H}^1 FY \to \mathrm{H}^1 FX \to \mathrm{H}^1 FZ \to \mathrm{H}^2 FY \to \mathrm{H}^2 FX$$

$$\mathrm{H}^0 F(\tau^{\geq 2} X) \to \mathrm{H}^1 F \mathrm{H}^1 X \to \mathrm{H}^1 FZ \to \mathrm{H}^1 F(\tau^{\geq 2} X)$$

We're left to show that $\mathrm{H}^q F(\tau^{\geq 2} X) = 0$ if $q < 2$, which is true since a triangulated functor commutes with suspension and F carries $\mathsf{D}^{\geq 0}(\mathcal{A})$ to $\mathsf{D}^{\geq 0}(\mathcal{B})$. $\square$

9.2. Let $\mathcal{G}$ be a sheaf of groups on a topological space X. A sheaf $\mathcal{F}$ of sets on X equipped with a (left) action of $\mathcal{G}$ is called a $\mathcal{G}$-torsor if

(1) For every open subset U of X and every pair of sections $s, t \in \mathcal{F}(U)$, there exists a unique $g \in \mathcal{G}(U)$ such that $gs = t$.

(2) $\mathcal{F}_x \neq \emptyset$ for all $x \in X$.

A morphism of $\mathcal{G}$-torsors is a morphism of sheaves preserving the $\mathcal{G}$-action.

(a) Show that every morphism of $\mathcal{G}$-torsors is an isomorphism. Let $\mathrm{Tors}(\mathcal{G})$ denote the set of isomorphism classes of $\mathcal{G}$-torsors.

(b) In the case with $\mathcal{G}$ abelian, establish a bijection between $\mathrm{Tors}(\mathcal{G})$ and $\mathrm{H}^1(X, \mathcal{G})(= \varinjlim_{\mathfrak{U}} \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{G}))$. For every open cover $\mathfrak{U}$ of X, describe the collection of $\mathcal{G}$-torsors corresponding to the image of the map $\mathrm{H}^1(\mathfrak{U}, \mathcal{G}) \to \mathrm{H}^1(X, \mathcal{G})(= \varinjlim_{\mathfrak{U}} \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{G}))$.

(c) Let $\mathcal{O}_X$ be a sheaf of rings on X. Let $\mathrm{Loc}_n(\mathcal{O}_X)$ denote the set of isomorphism classes of locally free $\mathcal{O}_X$ modules of rank n. Establish a bijection between $\mathrm{Loc}_n(\mathcal{O}_X)$ and $\mathrm{Tors}(\mathrm{GL}_n(\mathcal{O}_X))$, where $\mathrm{GL}_n(\mathcal{O}_X)$ denotes the sheaf of groups $U \mapsto \mathrm{GL}_n(\mathcal{O}_X(U))$.

(d) By (b),(c), we have $\mathrm{Pic}(X, \mathcal{O}_X) \simeq \mathrm{H}^1(X, \mathcal{O}_X^\times)$

Proof. (a) Suppose we have $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ a morphism of $\mathcal{G}$-torsors, first we prove that ϕ_U is bijective for all $U \subseteq X$ open. Suppose $\phi_U(x_1) = \phi_U(x_2)$, where we can write $x_2 = gx_1$, then $\phi_U(x_2) = g\phi_U(x_1) = \phi_U(x_1)$ hence $g = 1$ and $x_1 = x_2$. Next we prove that ϕ_U is surjective, this is easy if $\mathcal{F}(U)$ is nonempty, but if $\mathcal{F}(U)$ is empty, we have to use some other tricks.

Notice that, since $\mathcal{F}_x \neq \emptyset$ for all $x \in U$, we can cover U by $\bigcup_i U_i$ such that $\mathcal{F}(U_i)$ is not empty. And we know that ϕ_{U_i} are isomorphisms, hence so does ϕ_U.

Now it is trivial to show that $(\phi_U^{-1})_U$ is the inverse of ϕ.

(b) We only give a remark, to explain why torsors may have something to do with the first cohomology. Suppose $\mathcal{F}$ is a $\mathcal{G}$-torsor, such that $\mathcal{F}(U)$ is not empty. Let's make $\mathcal{F}(U)$ a pointed set, that is, we choose an element $\theta \in \mathcal{F}(U)$. Then for any open subset $V \subseteq U$, the mapping

$$\mathcal{G}(V) \xrightarrow{g \mapsto g\cdot(\theta|_V)} \mathcal{F}(V)$$

is bijective. The proof is similar to (a).

In fact, the structure of $\mathcal{G}$ actions is all determined, as long as $\mathcal{F}(U)$ is not empty. Since we know that there exists an open covering $X = \bigcup_i U_i$ such that each $\mathcal{F}(U_i)$ is not empty, the structure of $\mathcal{F}$ is determined by the relation between these opens. This reminds us something like cocycle or Cech cohomology. But since the open covering $X = \bigcup_i U_i$ is not fixed, we have to take some sort of limit, that's why we will get sheaf cohomology.

Using cocycle, we then construct $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{G}) \to \mathrm{Tors}(\mathcal{G})$, and then check that these are all bijective (that is, every torsor is of this form up to an isomorphism) and we can take colimit. Details are omitted.

(c) Omitted.

□

9.3. (=39). Let X be a quasi-compact topological space such that: 1. X is quasi-compact and quasi-compact open subsets form a basis. 2. the intersection of any two quasi-compact opens are again quasi-compact. (For example, X is a qcqs scheme.) Show that $\mathrm{H}^q(X, -)$ commutes with filtered colimit.

Proof. Omitted, this is actually true for ringed spaces and cohomology of $\mathcal{O}_X$-modules, see [Stacks]Lemma.20.19.1 for a complete proof.

□

9.4. (=40, Finite thickening preserves affineness). (a) Let X be a scheme. Let $\mathcal{I}$ be a quasi-coherent ideal sheaf of $\mathcal{O}_X$ such that $\mathcal{I}^n = 0$. Assume that the closed subscheme X_0 of X defined by $\mathcal{I}$ is affine. Show that X is affine.

(b) Deduce that a Notherian scheme X such that X_{red} is affine is affine.

(c) Show that a reduced scheme X admitting a finite cover by affine closed subschemes is affine.

Proof. (a) A closed immersion of schemes $i : X_0 \to X$ is called a n-th order thickening if $\mathcal{I}^{n+1} = 0$ where $\mathcal{I} = \ker(i^\flat)$. Now suppose we have a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, then we have a chain of quasi-coherent $\mathcal{O}_X$-submodules (why submodule? why quasi-coherent? you can think about it by recall some definitions):

$$0 = \mathcal{I}^{n+1}\mathcal{F} \subseteq \mathcal{I}^n\mathcal{F} \subseteq \cdots \subseteq \mathcal{I}\mathcal{F} \subseteq \mathcal{F}$$

Notice that any consecutive quotient $\mathcal{F}_k = \mathcal{I}^k\mathcal{F}/\mathcal{I}^{k+1}\mathcal{F}$ is annihilated by $\mathcal{I}$. So we have $\mathcal{F}_k \simeq i_*i^*\mathcal{F}_k$. ([Stacks]29.4.1) and $\mathrm{H}^1(X, \mathcal{F}_k) = \mathrm{H}^1(X_0, i^*\mathcal{F}_k) = 0$. By standard homological algebra, we have $\mathrm{H}^1(X, \mathcal{F}) = 0$.

(b) By definition, $X_{\mathrm{red}} = (X, \mathcal{O}_X/\mathcal{N}_X)$. And in a Noetherian scheme, the nilradical $\mathcal{N}_X$ is nilpotent since the nilradical of a Noetherian ring is nilpotent and a Noetherian scheme is quasi-compact.

(c) Same filtration technique as (a). Consider $\mathcal{F}_k = \mathcal{F}\prod_{i=1}^{k}\mathcal{I}_i$.

$\square$

10. Exercise 10

10.1. (=41=Chevalley's Theorem). Let X be an affine scheme, Y be an Noetherian scheme. Suppose we have a morphism $f : X \to Y$ such that f is finite and surjective, then Y is also affine.

Remark: still true without the Noetherian assumption! You can memorize it as: if X is finite surjective over Y, then X is affine if and only if Y is affine.

Proof. We explain that why we only need to consider the case of integral schemes.

Step.1:

Consider the property of being affine. If we can prove that, for every closed subset Y' of Y, (every proper closed subset Y'' of Y' is affine $\Rightarrow$ Y' is affine) (see the remark below). Then the Noetherian induction will gives us that Y is affine.

Remark: You may think that a subset is affine does not make sense. But notice that any closed subscheme of Y is again Noetherian, so the affineness of this subscheme can be safely viewed as a property on the set of closed subsets.

We know that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is also finite ([Hartshorne]Ex.II.4.8.(f)) and surjective. So from now on, X and Y will be reduced schemes. Not only that, we will show that, we only need to consider the case $Y' = Y$:

Suppose we have a closed subset $Y' \xrightarrow{j} Y$, here we view it as the reduced closed subscheme of Y, we consider the base change of f by j:

$$X \times_Y Y' \xrightarrow{f_{(j)}} Y'$$

This is again a finite surjective morphism, and $X \times_Y Y'$ is an affine scheme since it is a closed subscheme of X. Having seen this, we know that we only have to show that:

$(*)$ If $f : X \to Y$ is a finite surjective morphism between reduced schemes where X is affine and Y is Noetherian such that every proper closed subscheme of Y is affine. Then Y is affine.

Step.2:

We want to use the Serre's criterion. Let $\mathcal{F}$ be any coherent $\mathcal{O}_Y$-module, we will show that $\mathrm{H}^1(Y, \mathcal{F}) = 0$. Denote the support of $\mathcal{F}$ by Z, which is a closed subset of Y. Consider three cases:

Case.1: $Z \neq Y$. We need a lemma:

LEMMA.1.([EGA.I.9.3.5]) Suppose Y is a scheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type. Then there exists a closed subscheme $j : Z \to Y$ with the underlying topological space $Z = \mathrm{Supp}(\mathcal{F})$ such that the canonical morphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism.

Using this lemma, we immediately have $\mathrm{H}^1(Y, \mathcal{F}) \simeq \mathrm{H}^1(Z, j^*\mathcal{F}) = 0$

Case.2: $Z = \mathrm{Supp}(\mathcal{F}) = Y$ and Y is reducible. Let $Y' \xrightarrow{j} Y$ be a irreducible component which is viewed as the reduced closed subscheme with generic point η. Now $j_*j^*\mathcal{F}$ is again coherent and we have a canonical $\rho : \mathcal{F} \to j_*j^*\mathcal{F}$. If we can prove that

$$\mathrm{Supp}(\ker(\rho)) = \{y \in Y | \ker(\rho_y) \neq 0\} \neq Y$$

$$\mathrm{Supp}(\mathrm{im}(\rho)) \subseteq \mathrm{Supp}(j_* j^* \mathcal{F}) \neq Y$$

Then the long exact sequence will gives us $\mathrm{H}^1(Y, \mathcal{F}) = 0$.

Case.3: $Z = \mathrm{Supp}(\mathcal{F}) = Y$ and Y is irreducible. Hence Y is integral. Choose an irreducible component X' of X such that $X' \cap f^{-1}(\{\eta_Y\}) \neq \emptyset$. Now $X' \to X \to Y$ is closed, hence surjective. So we only need to show that:

$(**)$ If $f : X \to Y$ is a finite surjective morphism between integral schemes where X is affine and Y is Noetherian such that every proper closed subscheme of Y is affine. Then Y is affine.

□

10.2. (=42). Let X be a scheme proper over a field k. Assume that X is geometrically connected and geometrically reduced over k. Show that the canonical map $k \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

Proof. We need two results. The flat base change lemma and the finiteness theorem for proper morphisms. Both of them appear in our course note, but here I record a more convenient version.

[Stacks]30.5.2.(2): Let $A \to B$ be a flat ring homomorphism, X be a scheme which is qcqs over A, $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Denote the projection $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \to X$ by v. Then we have for all $i \geq 0$,

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_A B \simeq \mathrm{H}^i(X_B, v^* \mathcal{F})$$

[EGA.III]3.2.3: Let X be a scheme which is proper over a Noetherian ring A. Then for any $\mathcal{F} \in \mathbf{Coh}(\mathcal{O}_X)$ and $p \geq 0$, $\mathrm{H}^p(X, \mathcal{F})$ is a finitely generated A-module.

Since proper $\Rightarrow$ qcqs and every k-module is flat. Use the lemma and we have

$$\Gamma(X, \mathcal{O}_X) \otimes_k k^{\mathrm{alg}} \simeq \Gamma(X_{k^{\mathrm{alg}}}, \mathcal{O}_{X_{k^{\mathrm{alg}}}})$$

Denote the k^{alg}-algebra $\Gamma(X_{k^{\mathrm{alg}}}, \mathcal{O}_{X_{k^{\mathrm{alg}}}})$ by B, we only need to show that it has k^{alg}-dimension 1. This is because $X_{k^{\mathrm{alg}}}$ is connected and reduced.

□

10.3. (=43). Let S be a scheme and let X and Y be schemes over S.

(a) Assume that X is integral and Y is of finite type over S. Let $s \in S$ be a point and let $x \in X$ and $y \in Y$ be points above s. Let $\phi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a homomorphism of $\mathcal{O}_{S,s}$-algebras. Show that there exists an open neighborhood $U \subseteq X$ of x and a morphism $f : U \to Y$ over S such that $f(x) = y$ and $f_x^\sharp = \phi$.

(b) Assume that X is Noetherian normal of dimension 1 and Y is proper over S. Let $U \subseteq X$ be a dense open subset. Show that every S-morphism $U \to Y$ extends uniquely to an S-morphism $X \to Y$.

(c) Deduce from Chow's Lemma that a normal scheme of dimension 1 and proper over k is projective over k.

Proof. (a) Choose an affine open neighborhood W of $s \in S$. Choose an affine open neighborhood U' (V') of $x \in X$ $(y \in Y)$ which is contained in the preimage of W. Notice that U is also integral and $V' \to W$ is also of finite type. So we can assume $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), S = \mathrm{Spec}(R)$ are affine, where A is a domain and B is a finitely generated R-algebra. Denote x, y, s by $\mathfrak{P} \in \mathrm{Spec}(A), \mathfrak{Q} \in \mathrm{Spec}(B), \mathfrak{p} \in \mathrm{Spec}(R)$. Now we have $\phi : B_{\mathfrak{Q}} \to A_{\mathfrak{P}}$ a homomorphism of $R_{\mathfrak{p}}$-algebras. Suppose

B can be generated by $\beta_1, \dots, \beta_n$ as an R-algebra. Denote $\phi(\frac{\beta_i}{1})$ by $\frac{a_i}{\alpha_i} \in \mathrm{Frac}(A)$, and $\alpha = \prod_{i=1}^n \alpha_i$. Then we can define $\varphi : B \to A_\alpha$ by $\beta_i \mapsto \frac{a_i \prod_{j \neq i} \alpha_j}{\alpha}$.

(b)

(c) We use this version of Chow's lemma:

Let $X \to S$ be proper with S Noetherian (so X is also Noetherian). Then we have a projective morphism $\pi : Y \to X$ such that Y is projective over S and there exists a dense open subset $U \subseteq X$ such that $\pi^{-1}U \xrightarrow{\pi} U$ is an isomorphism.

Now in our case, we have a k-morphism $g : X \to Y$ such that $(\pi g)|_U = \mathrm{id}$. Since X is reduced and separated over k and U is dense, we have $\pi g = \mathrm{id}$ by Hartshorne.Ex.II.4.2 and we're done.

□

10.4. (=44). Suppose we have a fibre product $X \xleftarrow{p} X \times_S Y \xrightarrow{q} Y$, prove that:

$$p^*(\Omega_{X/S}) \oplus q^*(\Omega_{Y/S}) \to \Omega_{(X \times_S Y)/S}$$

is a canonical isomorphism.

Proof. We have the following exact sequence, called the first exact sequence:

$$p^*(\Omega_{X/S}) \to \Omega_{(X \times_S Y)/S} \to \Omega_{(X \times_S Y)/X} \to 0$$

$$q^*(\Omega_{Y/S}) \to \Omega_{(X \times_S Y)/S} \to \Omega_{(X \times_S Y)/Y} \to 0$$

And we have the following isomorphisms

$$\alpha = (p^*(\Omega_{X/S}) \to \Omega_{(X \times_S Y)/S} \to \Omega_{(X \times_S Y)/Y})$$

$$\beta = (q^*(\Omega_{Y/S}) \to \Omega_{(X \times_S Y)/S} \to \Omega_{(X \times_S Y)/X})$$

So these first exact sequences are short split exact and we're done.

□

11. Exercise 11

11.1. (=45). Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Consider the condition $(*)$: the sequence $0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ is exact and locally splits.

(a) Show that if f is formally smooth, then (*) holds.

(b) Show that if (*) holds and gf is formally smooth, then f is formally smooth.

11.2. (=46). (a) Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Show that if B is regular, then so is A.

(b) Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes, locally of finite presentation. Show that if f is flat and surjective and gf is smooth, then g is smooth.

11.3. (=47).

11.4. (=48).

12. Exercise 12

12.1. (=49). Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Show that we have isomorphisms

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*N, M) \simeq \mathsf{R}\mathcal{H}om_Y(N, \mathsf{R}f_*M)$$

$$\mathsf{R}\mathrm{Hom}_X(\mathsf{L}f^*N, M) \simeq \mathsf{R}\mathrm{Hom}_Y(N, \mathsf{R}f_*M)$$

functorial in $M \in \mathsf{D}(X)$ and $N \in \mathsf{D}(Y)$.

Proof. □

12.2. (=50). (a) Let $f_i : X_i \to S, i = 1, 2$ be qcqs morphisms of schemes. Let $X := X_1 \times_S X_2$ and $f := f_1 \times_S f_2 : X \to S$. Assume that f_1 is flat. Prove the Künneth formula.

$$\mathsf{R}f_{1*}M_1 \otimes^{\mathsf{L}} \mathsf{R}f_{2*}M_2 \simeq \mathsf{R}f_* \left(\mathsf{L}p_1^*M_1 \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathsf{L}p_2^*M_2\right)$$

for $M_i \in \mathsf{D}_{\mathrm{qco}}(X_i)$ and $p_i : X \to X_i$ is the projection.

(b) Let X_1 and X_2 be proper smooth schemes over a field k. Express the Hodge numbers $h^{p,q}$ of $X : X_1 \times_k X_2$ in terms of those of X_1 and X_2.

12.3. (=51, Bott vanishing).

13. Exercise 13, Merry Christmas

13.1. (=52). Let k be an algebraically closed field and let X be a smooth projective curve over k of genus g. The gonality of X, denoted $\mathrm{gon}(X)$, is defined to be the least integer $d \geq 1$ such that there exists a morphism $X \to \mathbb{P}^1_k$ over k of degree d.

(a) Show that $\mathrm{gon}(X) = \min\{\deg(\mathcal{L}) | h^0(\mathcal{L}) \geq 2\}$.

(b) Show that $\mathrm{gon}(X) \leq g + 1$.

13.2. (=53). Let k be an algebraically closed field.

(a) Let X be a smooth projective curve of genus g over k.

13.3. (=54).

13.4. (=55).