Algebra III (UCAS)

Mid-term Exam 2021/06/29 (1h)

In the following, A denotes a commutative ring with unity.

1. Let k be a field and $f = X^m Y^n$ be a monomial in $k[X, Y]$, with $m, n \geq 0$ and $(m, n) \neq (0, 0)$. Answer the following questions :

(a) for which values of m, n, the ideal $\langle f \rangle$ is prime ?

(b) for which values of m, n, the ideal $\langle f \rangle$ is radical ?

(c) for which values of m, n, the ideal $\langle f \rangle$ is primary ?

(d) for every (m, n), give a minimal primary decomposition of $\langle f \rangle$.

2. Let A be a noetherian ring and $\mathfrak{p}$ be a prime ideal in A. Let $f : A \to A_{\mathfrak{p}}$ be the natural morphism sending a to $a/1$. For $n \geq 1$, let $\mathfrak{p}^{(n)} = (\mathfrak{p}^n)^{ec}$, an ideal in A.

(a) Prove that $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

(b) Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Prove that $\mathfrak{p}^{(n)} \subset \mathfrak{q}$ for some $n \geq 1$.

(c) Assume moreover that A is an integral domain. Prove that

$$\bigcap_{\mathfrak{q}:\ \mathfrak{p}\text{-primary}} \mathfrak{q} = (0).$$

3. Let $A \subset B$ be an integral extension and assume that B is a finitely generated A-algebra. Prove that : for any prime ideal $\mathfrak{p}$ in A, there exist only finitely many prime ideals in B lying over $\mathfrak{p}$. (**Attention** : A is not assumed to be noetherian.)

4. Prove that a semilocal Dedekind domain is a PID. (Recall : "semilocal" means that there exist only finitely many maximal ideals.)