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第 1 章　Sarksiov 纲领

1.1　下降法

首先回顾 Corti [3] 给出的三维终端奇点的 Sarkisov 纲领。令 $f : X \to S$ 和 $f' : X' \to S'$ 是双有理等价的两个有终端奇点的三维森纤维空间。S' 上的丰沛除子 A' 使得对某个 $\mu' > 0$ 有 X' 上的一般的丰沛除子 H' 满足 $H' \sim -\mu' K_{X'} + f'^* A'$，并令 H 是 H' 在 X 上的双有理变换 (birational transform)。取一个公共解消 $p : W \to X$ 和 $q : W \to X'$。

(1)　令 $\mu = \max\{c \in \mathbb{R} : K_X + \frac{1}{c}H在S上数值有效\}$；

(2)　令 $\lambda = \min\{c \in \mathbb{R} : (X, \frac{1}{c}H)有典范奇点\}$；

(3)　令 $e = (X, \frac{1}{\lambda}H)的无差异的例外除子的个数$。

如果 $\lambda \leqslant \mu$，在 X 运行相对于恰当基底的 $(K_X + \frac{1}{\mu}H)$-MMP；如果 $\lambda > \mu$ 则构造一个除子解压 (divisorial extraction) $p : Z \to X$，并运行相对于 S 的 $(K_Z + \frac{1}{\lambda}H_Z)$-MMP，这样得到第一个 Sarkisov 连接 $\psi_1 : X \dashrightarrow X_1$。这两种情况都是 2-ray games。用 X_1 和 $\Phi_1 = \Phi \circ \psi_1^{-1} : X_1 \dashrightarrow X'$ 替换 X 和 Φ，并重复这个过程，这样递归地构造一系列 Sarkisov 连接。在这个过程中不变量 (μ, λ, e) 将按字典序下降，最终得到 $\Psi_N : X_{N-1} \dashrightarrow X_N$，且 $X_N \cong X'$。这就是三维终端奇点代数簇的 Sarkisov 纲领。

对于 $\mathbb{Q}$-分解的有 klt 奇点的代数簇对，考虑 MMP-相关的森纤维空间 (X, B) 和 (X', B')。一个自然的想法是按如下定义 μ 和 λ:

(1)　令 $\mu = \max\{c \in \mathbb{R} : K_X + B + \frac{1}{c}H在S上数值有效\}$；

(2)　令 $\lambda = \min\{c \in \mathbb{R} : (X, B + \frac{1}{c}H)有典范奇点\}$；

(3)　令 $e = (X, B + \frac{1}{\lambda}H)的无差异的例外除子的个数$。

对 λ 的定义将导致一些困难。当 $\lambda > \mu$ 时，为了构造 Sarkisov 连接需要在一个除子解压 $p : Z \to X$ 上运行 $(K_Z + B_Z + \frac{1}{\lambda}H_Z)$-MMP。这个除子解压会解压出一个素除子 E，这个素除子 E 在 Z 的边界 B_Z 中的系数是 1。如果 E 是 (X', B') 的边界 B' 的一项，那么 E 在 B' 中的系数小于 1，这两者不匹配。另一方面，这是需要在具有 lc 奇点的代数簇对上运行 MMP，这比在具有 klt 奇点的代数簇对上的 MMP 有技术上的困难。除此之外，由于具有 klt 奇点的 Fano 代数簇对的有界性失效，在证明 Sarkisov 纲领的终结性也有困难。

Bruno 和 Matsuki 给出了 λ 的另一种定义 (见1.3)，且取决于特定的一个包含 (X, B) 和 (X', B') 的代数簇对的集合 C_θ，这个集合满足:

- 对任意两个 $\mathcal{C}_\theta$ 中的代数簇对 $(X,B),(X',B')$，存在 $\mathcal{C}_\theta$ 中的代数簇对 (W,B_W) 和算术公共解消 $p: W\to(X,B)$ 和 $q: W\to(X',B')$，使得 (W,B_W) 具有 klt 奇点且 $p_*B_W=B, q_*B_W=B'$。

- 在集合中的任意代数簇对 (X,B) 和 (Z,B_Z) 上可以运行 (K_X+B+cH)-MMP 和 $(K_Z+B_Z+cH_Z)$-MMP，并且所有结果都任然在集合 $\mathcal{C}_\theta$ 中；

- 所有 $\mathcal{C}_\theta$ 中的代数簇对都具有 δ-lc 奇点，其中 δ 是取决于 $\mathcal{C}_\theta$ 的正实数。

1.1.1 前置知识

令 $K=K(X)$ 是双有理等价类的有理函数域 (注意到双有理等价的代数簇有相同的有理函数域) 令 $\Sigma=\{\nu\}$ 是有理函数域的离散赋值的集合。

定义 1.1. [4, Definition 3.5] 取一个函数 $\theta:\Sigma\to[0,1)_{\mathbb{Q}}$，那么可以定义关于 的集合 $\mathcal{C}_\theta$，包含满足下列条件的具有 klt 奇点的代数簇对 $(X,B=\sum a_iB_i)$：

(1) $a_i=\theta(B_i)$；

(2) 对所有 X 上的例外除子 E 有 $a(E;X,B)>-\theta(E)$。

注. 例如取 $\theta\equiv 0$ 为常值函数，那么 $\mathcal{C}_\theta$ 是所有和 X 双有理等价的具有终端奇点的代数簇 Y (不带有边界)。

根据这个集合可以定义 θ-差异数 (θ-discrepancy)：

定义 1.2 (θ-差异数)**.** 令 $\mathcal{C}_\theta$ 为上述代数簇对的集合，且 (X,B) 是有理函数域满足 $K(X)=K$ 的代数簇对。令 $f:Y\to X$ 是 (X,B) 的一个算术奇点解消，有分歧等式：

$$K_Y+B_Y+C=f^*(K_X+B)$$

其中 $B_Y=f_*^{-1}B+\sum_{E_i\ \mathrm{exc}}\theta(E_i)E_i$。则 X 的例外除子 E_i 的 θ-差异数定义为

$$a_\theta(E_i;X,B)=-\operatorname{mult}_{E_i}C.$$

或等价的，可以定义为

$$a_\theta(E_i;X,B)=a(E_i;X,B)+\theta(E_i).$$

如果 (X,B) 上的所有例外除子 E 满足 $a_\theta(E;X,B)\geqslant 0$ (对应的，$a_\theta(E;X,B)>0$)，则称代数簇对 (X,B) 具有 θ-典范奇点 (对应的，θ-终端奇点) 。

注. θ-典范代数簇对并不总在集合 $\mathcal{C}_\theta$ 中。

Bruno 和 Matsuki 的 [4, Lemma 3.6] 构造了运行 Sarkisov 纲领所需要的集合 $\mathcal{C}_\theta$：

命题 1.1. 令 $f:(X,B)\to S$ 和 $f':(X',B')\to S'$ 是两个 *MMP*-相关的具有 *klt* 奇点的 $\mathbb{Q}$-分解森纤维空间，有双有理映射 Φ:

$$\begin{tikzcd}
(X,B) \arrow[r, dotted, "\Phi"] \arrow[d, "f"'] & (X',B') \arrow[d, "f'"] \\
S & S'
\end{tikzcd}$$

假设 $B=\sum_i b_iB_i+\sum_j d_jD_j$ 和 $B'=\sum_j d'_jD_j+\sum_k b'_kB'_k$，其中 B_i 是在 X 上但不在 X' 上的除子，B'_k 是在 X' 上但不在 X 上的除子，而 D_j 是在 X 和 X' 上的除子。由引理**??**，有 $d_j=d'_j$。取一个有理数 ϵ 满足

$$-\operatorname{totdiscrep}(X,B),-\operatorname{totdiscrep}(X',B')<\epsilon<1$$

并按如下定义函数 $\theta:\{v\}\to[0,1)_{\mathbb{Q}}$：

- 对于边界 B,B' 的除子，有 $\theta(B_i)=b_i,\theta(D_j)=d_j,\theta(B'_k)=b'_k$；
- 如果 E 是 X 和 X' 上的例外除子，则 $\theta(E)=\epsilon$；
- 如果 D 是 X 和 X' 上的除子，但不是 B 或 B' 的部分，则 $\theta(D)=0$。

那么定义*1.1* 构造的集合 C_θ 满足：

(1) (X,B) 和 (X',B') 在集合 C_θ 中；

(2) 对 C_θ 中任意有限多个具有 *klt* 奇点的代数簇对 $\{(X_l,B_l)\}$，有 $(Z,B_Z)\in C_\theta$ 射影双有理态射 $Z\to X_l$ 使得 X_l 是相对于 X_l 的 (K_Z+B_Z)-*MMP* 的输出，因此也是相对于 $\operatorname{Spec}\mathbb{C}$ 的 (K_Z+B_Z)-*MMP* 结果；

(3) 任何从 C_θ 中一个元素出发的 $(K+B)$-*MMP*，其结果依然落入 C_θ。如果对任何 $c\in\mathbb{Q}_{>0}$ 和无基点的除子 H 给出的 $(K+B+cH)$-*MMP* 也成立。

注. 令 $\delta=1-\epsilon$，那么所有 C_θ 中的代数簇对都是具有 δ-lc。

使用命题 1.1中的假设和记号，可以定义 Sarkisov 次数（Sarkisov degree）。取 S' 上的非常丰沛的除子 A' 和足够大和可除的整数 $\mu'>1$ 使得

$$\mathcal{H}'=|-\mu'(K_{X'}+B')+f'^*A'|$$

是 X' 在 $\operatorname{Spec}\mathbb{C}$ 上的非常丰沛的完全线性系。令 (W,B_W) 是 X 和 X' 在 C_θ 中的公共算术解消，有射影态射 $\sigma:W\to X$ 和 $\sigma':W\to X'$ 满足 $\sigma_*B_W=B,\sigma'_*B_W=B'$。令 Let $\mathcal{H}_W:=\sigma'^*\mathcal{H}'$，那么 $\mathcal{H}:=\Phi_*^{-1}\mathcal{H}'=\sigma_*\mathcal{H}_W$。进一步，如果 $\mathcal{H}$ 不是无基点的，那么

$$\sigma^*\mathcal{H}=\mathcal{H}_W+F$$

其中 $F=\sum f_l F_l \geqslant 0$ 是固定部分 (fixed part)。取线性系 $\mathcal{H}'$ 中的一个一般除子 H' 使得 $H_W := \sigma'^* H' = \sigma_*'^{-1} H' \in \mathcal{H}_W$，并记 $H := \Phi_*^{-1} H' = \sigma_* H_W$。那么 H 是 f-丰沛的，且 $\sigma^* H = H_W + F$。通过取进一步的解消，不妨设 H_W 与 σ 和 σ' 的例外除子各部分光滑且互相横截相交 (即 $(W, H_W + \operatorname{Exc}\sigma + \operatorname{Exc}\sigma')$ 是算术光滑的)。

接下来定义在 C_θ 中关于 H' (或 $\mathcal{H}'$) 的 Sarkisov 次数：

定义 1.3. [4, Definition 3.8] C_θ 中关于 H' (或 $\mathcal{H}'$) 的 Sarkisov 次数是一个按字典序排序的三元组 (μ, λ, e)，其中：

- **数值有效阈值** μ：令 $C \subset X$ 是被 f 压缩的曲线，那么
$$\mu := -\frac{H.C}{(K_X + B).C}$$
即 $K_X + B + \frac{1}{\mu} H \equiv_S 0$；
- **θ-典范阈值** $\frac{1}{\lambda}$：若 $\mathcal{H}$ 无基点则定义 $\lambda = 0$；否则定义
$$\frac{1}{\lambda} := \max\{t : a_\theta(E; X, B + tH) \geqslant 0, \forall\ X\text{上例外除子}E\}$$
- **$(K_X + B_X + \frac{1}{\mu} H)$-无差别除子个数**: $e = 0$ 若 $\mathcal{H}$ 无基点 (此时 $\lambda = 0$) 则定义 $e = 0$；否则定义
$$e = \#\{E; E\ \text{是}\sigma\text{-例外除子，且} a_\theta(E; X, B + \frac{1}{\lambda} H) = 0\}$$

注. (1) Sarkisov 次数取决于 A', H' 和 θ 的选取。

(2) 取公共算术解消 $(W, B_W) \in C_\theta$，其中 $B_W = \sum \theta(E) E$，并且有射影双有理态射 $\sigma : W \to X, \sigma' : W \to X'$。由于 $\sigma^* \mathcal{H} = \mathcal{H}_W + \sum f_l F_l$，所以有分歧等式：
$$K_W + B_W + t H_W = \sigma^*(K_X + B + tH) + \sum (a_l - t f_l) E_l$$
其中 $\sum a_l E_l$ 是有效除子且支撑在 $\operatorname{Exc}\sigma$ 上。那么 $\lambda := \max\{\frac{f_l}{a_l}\}$。如果 $\mathcal{H}$ 是无基点的，那么 $\sum f_l F_l = 0$ 且 $\lambda = 0$。

(3) e 是公式
$$K_W + B_W + \frac{1}{\lambda} H_W = \sigma^*(K_X + B + \frac{1}{\lambda} H) + \sum (a_l - \frac{1}{\lambda} f_l) E_l.$$
中系数 $\sum (a_l - \frac{1}{\lambda} f_l) E_l$ 为 0 的部分的个数。这样的素除子 $E_1, \ldots, E_e$ 称作 $(K_X + B + \frac{1}{\lambda} H)$-$\theta$-无差别的。

需要构造在集合 C_θ 中的解压态射：

引理 1.2. 使用定义*1.3*中的记号，并假设 $\lambda \neq 0$，那么存在压缩态射 $f : Z \to X$ 满足：

- $(Z, B_Z) \in C_\theta$ *and* $(Z, B_Z + \frac{1}{\lambda} H_Z)$ *is* θ*-terminal and* $\mathbb{Q}$*-factorial;*
- $\rho(Z) = \rho(X) + 1$*;*
- f *is* $(K_X + B + \frac{1}{\lambda} H)$*-crepant, that is*

$$K_Z + B_Z + \frac{1}{\lambda} H_Z = f^*(K_X + B + \frac{1}{\lambda} H).$$

证明. We follow the idea of the proof in [4, Proposition 1.6]. Let $(W, B_W) \in C_\theta$ and $\sigma : W \to X, \sigma' : W \to X'$ be the common resolution as in Definition 1.3, and suppose $E_1, \dots, E_e$ are $(K_X + B + \frac{1}{\lambda} H)$-$\theta$-crepant divisors after renumbering. Then we have

$$K_W + B_W + \frac{1}{\lambda} H_W = \sigma^*(K_X + B + \frac{1}{\lambda} H) + \sum_{l=1}^{e} 0 \cdot E_l + \sum_{l>e} (a_l - \frac{1}{\lambda} f_l) E_l.$$

We run the $(K_W + B_W + \frac{1}{\lambda} H_W)$-MMP over X with scaling of some ample divisor, then the MMP ends with a minimal model $p : (Y, B_Y + \frac{1}{\lambda} H_Y) \to X$ for $(W, B_W + \frac{1}{\lambda} H_W)$ over X and the exceptional locus of p is exactly $\cup_{i=1}^{e} E_i$ and p is crepant:

$$K_Y + B_Y + \frac{1}{\lambda} H_Y = p^*(K_X + B + \frac{1}{\lambda} H).$$

Then we run the $(K_Y + B_Y)$-MMP over X with scaling of some ample divisor. This ends with the minimal model (X, B) of (Y, B_Y) over X. Let $f : Z \to X$ be the last contraction in the MMP, and f is the required extraction map. □

1.1.2 Flowchart for the Sarkisov program

We follow [4, §1] in this subsection.

If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu} H$ is nef, the two Mori fibre spaces are isomorphic by following Theorem and the program stops:

定理 1.3. *(Noether-Fano-Iskovskikh Criterion): Notations as in the definition of Sarkisov degree, then*

(1) $\mu \geqslant \mu'$*;*

(2) If $\mu \geqslant \lambda$ *and* $(K_X + B + \frac{1}{\mu} H)$ *is nef, then* Φ *is an isomorphism of Mori fibre spaces. That is, we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\sim", "\Phi"'] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ S \arrow[r, "\sim"] & S' \end{tikzcd}$$

证明. We follow the ideas of the proofs in [6, Claim 13.20], [7, Theorem 5.1] and [3, Theorem 4.2]:

(1) We only need to show $(K_X + B + \frac{1}{\mu'}H)$ is f-nef. Let $\sigma : W \to X$ and $\sigma' : W \to X'$ be a common resolution. Consider the ramification formulas:

$$\begin{aligned} K_W + B_W + \frac{1}{\mu'}H_W =& \sigma'^*(K_{X'} + B' + \frac{1}{\mu'}H') + \sum e'_j E_j + \sum g'_k G'_k \\ =& \sigma^*(K_X + B + \frac{1}{\mu'}H) + \sum g_i G_i + \sum e_j E_j \end{aligned}$$

Here $\{G_i\}, \{E_j\}$ are σ-exceptional divisors, and $\{E_j\}, \{G'_k\}$ are σ'-exceptional divisors. Since $H_W = \sigma'^* H'$, $g'_k > 0$ or there is no such G'_k. Then take a general curve $C \subset X$ contracted by f, such that its strict transform $\tilde{C}$ on W is disjoint from G_i, E_j, and is not contained in G'_k. Then we have:

$$\begin{aligned} C.\left(K_X + B + \frac{1}{\mu'}H\right) =& \tilde{C}.\left(\sigma^*\left(K_X + B + \frac{1}{\mu'}H\right) + \sum g_i G_i + \sum e_j E_j\right) \\ =& \tilde{C}.\left(\sigma'^*\left(K_{X'} + B' + \frac{1}{\mu'}H'\right) + \sum e'_j E_j + \sum g'_k G'_k\right) \\ =& \tilde{C}.\sigma'^* f'^* A' + \tilde{C}.\left(\sum g'_k G'_k\right) \geqslant 0. \end{aligned}$$

This implies $(K_X + B + \frac{1}{\mu'}H)$ is f-nef and $\mu \geqslant \mu'$;

(2) First we show that $\mu = \mu'$. By (1), we only need to show $(K_{X'} + B' + \frac{1}{\mu}H')$ is f'-nef. Indeed, same as (1), we can take a general curve C' on X' contracted by f', such that its strict transform $\tilde{C}'$ on W is disjoint from G'_k, E_j, and is not contained in G_i and $C'.\left(K_{X'} + B' + \frac{1}{\mu}H'\right) \geqslant 0$.

Then we show they are isomorphic. Take a very ample divisor D on X and let D' be its strict transform on X'. Then D' is f'-ample, thus there exists $0 < d \ll 1$ such that the following holds:

- $K_X + B + \frac{1}{\mu}H + dD$ is ample;
- $K_{X'} + B' + \frac{1}{\mu}H' + dD'$ is ample.

Therefore, X and X' are both log canonical models of $(W, B_W + \frac{1}{\mu}H_W + dD_W)$, hence $X \cong X'$. Furthermore, f and f' are contractions of the same numerical curve class, thus the two log Mori fibre spaces are isomorphic.

□

Otherwise, if the condition of the Noether-Fano-Iskovskikh Criterion does not hold:

(1) If $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef, then there is a contraction $f : X \to T$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type III or IV;

(2) If $\lambda > \mu$, then there is a divisorial extraction $p : Z \to X$ and a Sarkisov link $\psi_1 : X \dashrightarrow X_1$ of type I or II.

证明. (1) By assumption, $\lambda \leqslant \mu$ and $K_X + B + \frac{1}{\mu}H$ is not nef. Suppose f is the contraction of a $(K_X + B)$-negative extremal ray $R = \overline{\mathrm{NE}}(X/S)$, then $(K_X + B + \frac{1}{\mu}H).R = 0$ by definition of μ. There is an extremal ray $P \subset \overline{\mathrm{NE}}(X)$ such that $(K_X + B + \frac{1}{\mu}H).P < 0$ and $F := P + R$ is an extremal face (see [3, 5.4.2] for the details). Take $0 < \delta \ll 1$ such that $(K_X + B + (\frac{1}{\mu} - \delta)H).P < 0$, then $(K_X + B + (\frac{1}{\mu} - \delta)H).R < 0$ since H is f-ample. Therefore, F is a $(K_X + B + (\frac{1}{\mu} - \delta)H)$-negative extremal face. Since $(X, B + (\frac{1}{\mu} - \delta)H)$ is klt, there is a contraction $g : X \to T$ with respect to F factoring through $f : X \to S$. Since $(X, B + \frac{1}{\mu}H)$ is klt, and $\rho(X/T) = 2$, we can run the $(K_X + B + \frac{1}{\mu}H)$-MMP over T with scaling of some ample divisor. Since $B + \frac{1}{\mu}H$ is relatively big, the MMP terminates. There are the following cases:

(1).1 After finitely many flips $X \dashrightarrow Z$, the first non-flip contraction is a divisorial contraction $p : Z \to X_1$, which is then followed by a log Mori fibre space $f_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to S_1$. The contraction f_1 is also a log Mori fibre space of (X_1, B_1). This is a link of type III.

(1).2 After finitely many flips $X \dashrightarrow X_1$, the first non-flip contraction is a log Mori fibre space $f_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to S_1$. The contraction f_1 is also a log Mori fibre space of (X_1, B_1). This is a link of type IV.

(1).3 After finitely many flips $X \dashrightarrow Z$, the first non-flip contraction is a divisorial contraction $p : Z \to X_1$ with

$$K_Z + B_Z + \frac{1}{\mu}H_Z = p^*(K_{X_1} + B_1 + \frac{1}{\mu}H_1) + eE$$

where $e > 0$ and $E = \mathrm{Exc}\, p$ and $f_1 : (X_1, B_1 + \frac{1}{\mu}H_1) \to T$ is a log minimal model of $(X, B + \frac{1}{\mu}H)$ over T. In fact the only ray of $\overline{\mathrm{NE}}(X_1/T)$ is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and hence is $(K_{X_1} + B_1)$-negative. Therefore, $f_1 : (X_1, B_1) \to T$ is a log Mori fibre space. Take $S_1 = T$. This is a link of type III.

(1).4 After finitely many flips $X \dashrightarrow X_1$, the $(K_X + B + \frac{1}{\mu}H)$-MMP ends with a log minimal model $(X_1, B_1 + \frac{1}{\mu}H_1)$ over T. Then there is an extremal ray R of $\overline{\mathrm{NE}}(X_1/T)$, which is $(K_{X_1} + B_1 + \frac{1}{\mu}H_1)$-trivial and $(K_{X_1} + B_1)$-negative.

Let $f_1 : X_1 \to S_1$ be the contraction with respect to R. This is a link of type IV.

(2) By assumption, $\lambda > \mu$. Take an extraction $p : (Z, B_Z + \frac{1}{\lambda} H_Z) \to (X, B + \frac{1}{\lambda} H)$ as in Lemma 1.2. That is, (Z, B_Z) is θ-terminal and $p^*(K_X + B + \frac{1}{\lambda} H) = K_Z + B_Z + \frac{1}{\lambda} H_Z$ where $B_Z = \sum \theta(E_v) E_v$. Then we run the $(K_Z + B_Z + \frac{1}{\lambda} H_Z)$-MMP over S with scaling of some ample divisor. Since Z is covered by $(K_Z + B_Z + \frac{1}{\lambda} H_Z)$-negative curves, $(K_Z + B_Z + \frac{1}{\lambda} H_Z)$ is not relatively pseudo-effective. Hence, this MMP ends with a log Mori fibre space by Theorem **??**. There are two cases:

(2).1 After finitely many flips $Z \dashrightarrow Z'$, the first non-flip contraction is a divisorial contraction $q : Z' \to X_1$, which is then followed by a log Mori fibre space $f_1 : (X_1, B_1 + \frac{1}{\lambda} H_1) \to S$. Let $S_1 = S$, then the contraction f_1 is also a log Mori fibre space of (X_1, B_1). This is a link of type II.

(2).2 After finitely many flips $Z \dashrightarrow X_1$, the first non-flip contraction is a log Mori fibre space $f_1 : (X_1, B_1 + \frac{1}{\lambda} H_1) \to S_1$. Since $(K_{X_1} + B_1 + \frac{1}{\lambda} H_1)$ is anti-ample over S_1 and H_1 is f_1-ample, $(K_{X_1} + B_1)$ is anti-ample over S_1. Therefore, $f_1 : (X_1, B_1) \to S_1$ is a log Mori fibre space. This is a link of type I.

□

We replace (X, B) with (X_1, B_1) and Φ with $\Phi \circ \psi_1^{-1}$, and repeat the above process.

注. The Sarkisov degree decreases in the flowchart of the Sarkisov program:

(1) (1).1 For the case 1a and 1b, since $K_{X_1} + B_1 + \frac{1}{\mu} H_1$ is anti-ample over S_1, we have $\mu_1 < \mu$.

(1).2 For the case 1c and 1d, since $(K_{X_1} + B_1 + \frac{1}{\mu} H_1)$ is trivial on the ray $R = \overline{\mathrm{NE}}(X_1/S_1)$ for both cases, we have $\mu_1 = \mu$. Notice that $(X_1, B_1 + \frac{1}{\mu} H_1)$ stays θ-canonical, we have $\lambda_1 \leqslant \mu = \mu_1$, thus next link stays in the case 1. Furthermore, for case 1c we have $\rho(X_1) = \rho(X) - 1$.

(2) For the case 2, we have $\mu_1 \leqslant \mu$ and $\lambda_1 \leqslant \lambda$ and if $\lambda_1 = \lambda$, then $e_1 < e$.

1.1.3 Termination

The original method needs the following to prove the termination:

(1) the discreteness of nef thresholds μ;

(2) the termination of flips;

(3) the ascending chain condition of log canonical thresholds;

(4) the finiteness of local log canonical thresholds for the Sarkisov program for terminal varieties, and the finiteness of local θ-canonical thresholds for the Sarkisov program for the klt pairs.

Suppose there is an infinite sequence, that is, there are infinitely many X_i and birational maps obtained from the program:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X'$$

(1) Discreteness of nef thresholds holds for all dimensions, by the boundedness of δ-lc Fano varieties ([5, Theorem 1.1]). Therefore, we may assume μ_i is constant, that is, $\mu = \mu_0 = \mu_i$ for all i.

(2) We can now suppose μ_i is constant. If there is a Sarkisov link ψ_i of type III or IV in the sequence, then any the Sarkisov link $\psi_j, j > i$ is of type III or IV by Remark 1.1.2. There are only finitely many Sarkisov links of type III since the Picard numbers drop. The case of ψ_j, $j \gg 0$ being of type IV contradicts the termination of flips. But the termination of flips only holds for threefolds and pesudo-effective fourfolds.

(3) Suppose all the links are of type I and II. The ascending chain condition of log canonical thresholds holds for all dimensions [**?**]. Therefore, there is a positive number α such that $(X_i, B_i + \alpha H_i)$ are klt for $i \gg 0$, and every Sarkisov link $\psi_i, i \gg 0$ comes from the $(K_{Z_i} + B_{Z_i} + \alpha H_{Z_i})$-MMP over S_i. This is a contradiction to the finiteness of local θ-canonical thresholds ([4, Claim 2.2]).

参考文献

[1] Sarkisov V G. BIRATIONAL AUTOMORPHISMS OF CONIC BUNDLES [J/OL]. Mathematics of the USSR-Izvestiya, 1981, 17(1): 177-202. DOI: 10.1070/IM1981v017n01ABEH001326.

[2] Sarkisov V G. On conic bundle structures [J]. Izv. Akad. Nauk SSSR Ser. Mat., 1982, 46(2): 371-408, 432.

[3] Corti A. Factoring birational maps of threefolds after Sarkisov [J]. Journal of Algebraic Geometry, 1995, 1995(4): 223-254.

[4] Bruno A, Matsuki K. Log Sarkisov program [J/OL]. Internat. J. Math., 1997, 8(4): 451-494. https://doi.org/10.1142/S0129167X97000238.

[5] Birkar C. Singularities of linear systems and boundedness of Fano varieties [J/OL]. Annals of Mathematics, 2021, 193(2): 347 - 405. https://doi.org/10.4007/annals.2021.193.2.1.

[6] Hacon C D. The Minimal model program for Varieties of log general type [J/OL]. Wiadomości Matematyczne, 2012, 48(2): 49. DOI: 10.14708/wm.v48i2.317.

[7] Liu J. Sarkisov program for generalized pairs [J]. Osaka J. Math., 2021, 58(4): 899-920.

[8] Hacon C, McKernan J. The Sarkisov program [J/OL]. Journal of Algebraic Geometry, 2012, 22(2): 389-405. DOI: 10.1090/S1056-3911-2012-00599-2.

[9] Miyamoto K. The sarkisov program on log surfaces [J]. arXiv: Algebraic Geometry, 2019, 2019: 114-114.