Linear Algebra 2

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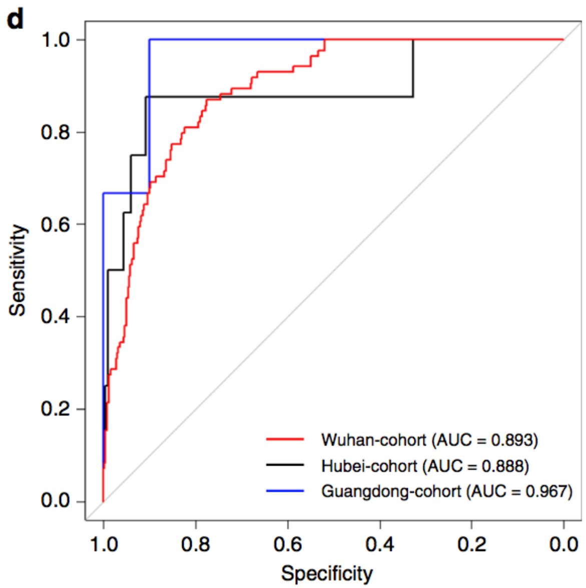
Early Triage of Critically-ill COVID-19 Patients using Deep Learning

W. Liang et al., Nature Communications, 2020

- a deep learning-based survival model
- predict the risk of COVID-19 patients developing critical illness
- using clinical characteristics at admission.
- validate the model on three separate cohorts from Wuhan, Hubei and Guangdong provinces consisting of 1393 patients
- Dataset: 1590 patients from 575 medical centers



Online assessment tool

https://aihealthcare.tencent.com/COVID19-Triage_en.html

2.4.1 Groups

- Consider a set $\mathcal{G}$ and an operation $\otimes$: $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a group if the following holds
  - Closure of $\mathcal{G}$ under $\otimes$: $\forall x, y \in G: x \otimes y \in G$
  - Associativity: $\forall x, y, z \in G, (x \otimes y) \otimes z = x \otimes (y \otimes z)$
  - Neutral element: $\exists e \in G, \forall x \in G, x \otimes e = x, e \otimes x = x$
  - Inverse element: $\forall x \in G, \exists y \in G, x \otimes y = e, y \otimes x = e$

    We often write x^{-1} to denote the inverse element of x
- Additionally, If $\forall x, y \in G, x \otimes y = y \otimes x$ (commutative), then $G := (\mathcal{G}, \otimes)$ is an Abelian group.
- Examples
- $(\mathbb{Z}, +)$ is a group and an Abelian group
  - …,-5, -4, -3, -2, -1, 0, 1, 2, 3,4, …

  ~~Closure~~
  Associativity $(x+y)+z = x+(y+z)$
  Neutral 0
  Inverse $x \in \mathbb{Z}, y = -x \in \mathbb{Z}$
- $(\mathbb{Z}, -)$ is not a group: it does not satisfy associativity, has no neutral element or inverse element

  $(x-y)-z \neq x-(y-z)$

  $x - 0 = x, \quad 0 - x \neq x$

- Examples
- $(\mathbb{R}^{m\times n},+)$, the set of $m\times n$-matrices is Abelian (component-wise addition).
  - Closure:
  - Associativity:
  - Neutral element
  - Inverse element
  - commutative

2.4.2 Vector spaces

- Definition
- A real-valued vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$+: V \otimes V \to V$$
$$\cdot: \mathbb{R} \otimes V \to V$$

- where
  - $(\mathcal{V}, +)$ is an Abelian group
  - Distributivity:

$$\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}: \quad \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$$
$$\forall \lambda, \varphi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V}: \quad (\lambda + \varphi) \cdot x = \lambda \cdot x + \varphi \cdot x$$

  - Associativity (operation $\cdot$):

$$\forall \lambda, \varphi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V}: \quad \lambda \cdot (\varphi \cdot x) = (\lambda \varphi) \cdot x$$

  - Neutral element (w.r.t to operation $\cdot$):

$$\forall \boldsymbol{x} \in \mathcal{V}: \quad 1 \cdot x = x$$

2.4.2 Vector spaces

- Elements $\boldsymbol{x} \in \mathcal{V}$ are called vectors
- The neutral element of $(\mathcal{V}, +)$ is the zero vector $0 = [0, \cdots 0]^T$
- $+$ is called vector addition
- Elements $\lambda \in \mathbb{R}$ are called scalars
- Operation $\cdot$ is a multiplication by scalars

Vector spaces - example

- $\mathcal{V} = \mathbb{R}^n, n \in \mathbb{N}$ is a vector space. Its operations are defined as

  - Addition: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$x + y = (x_1, \dots x_n)^T + (y_1, \dots y_n)^T = (x_1 + y_1, \dots, x_n + y_n)^T$$

  - Multiplication by scalars: for $\boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$

$$\lambda x = \lambda(x_1, x_2, \dots x_n)^T = (\lambda x_1, \lambda x_2, \dots \lambda x_n)^T$$

- We usually write $\boldsymbol{x} \in \mathbb{R}^n$ in a column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

2.4.3 Vector Subspaces

- Sets contained in the original vector space
- "closed"
- When we perform vector space operations on elements within this subspace, we will never leave it

- $U = (\mathcal{U}, +, \cdot)$ is called vector subspace of $V = (\mathcal{V}, +, \cdot)$, if
- $\mathcal{U} \subseteq \mathcal{V}$,
- $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$
- Closure of U

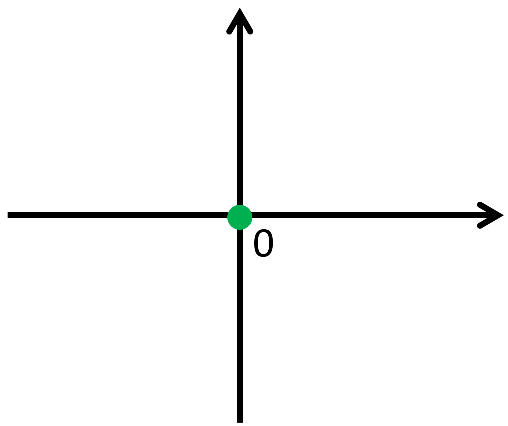
  - $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{U}, \boldsymbol{x} + \boldsymbol{y} \in \mathcal{U}$
  - $\forall \boldsymbol{x} \in \mathcal{U}, \lambda \in \mathbb{R}, \lambda \boldsymbol{x} \in \mathcal{U}$

2.4.3 Vector Subspaces

- Examples

R^2 R^2

- For every vector space V, the trivial subspaces are V itself and $\{\mathbf{0}\}$
- Is it a subspace of $\mathbb{R}^2$?



✓ Is it a subset of $\mathbb{R}^2$?

✓ Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$

✓ Does it satisfy closure?

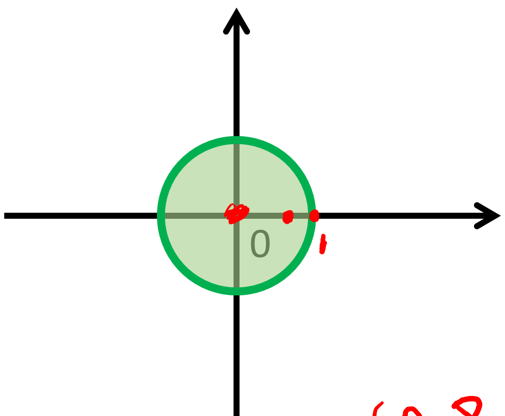
$\overset{0}{x} + \overset{0}{y} \in \{0\}$

$\lambda x \in \{0\}$, $\lambda \in R$, $x = 0$

2.4.3 Vector Subspaces

- Examples
- Is it a subspace of $\mathbb{R}^2$?

✗



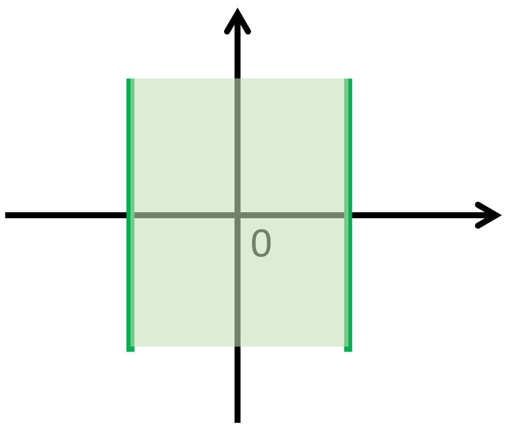
- ✓ Is it a subset of $\mathbb{R}^2$?
- ✓ Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$
- ✗ Does it satisfy closure?

$$\underline{(0.8, 0)} + \underline{(0.9, 0)} = \underline{(1.7, 0)}$$

2.4.3 Vector Subspaces

- Examples
- Is it a subspace of $\mathbb{R}^2$?

✗



✓ Is it a subset of $\mathbb{R}^2$?

✓ Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$

✗ Does it satisfy closure?

2.4.3 Vector Subspaces

- Examples

- The solution set of a homogeneous system of linear equations $\boldsymbol{Ax} = \mathbf{0}$ with n unknowns $\boldsymbol{x} = [x_1, \cdots, x_n]^{\mathrm{T}}$. Is it a subspace of $\mathbb{R}^n$?

$x = 0$

✓ Is it a subset of $\mathbb{R}^n$?
✓ Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\mathbf{0} \in \mathcal{U}$
✓ Does it satisfy closure?

$x, y \in G \quad Ax = 0, \quad Ay = 0.$

$x + y \in G \quad A(x+y) = Ax + Ay = 0$

$\lambda x \in G \quad A(\lambda x) = \lambda(Ax) = 0$

2.4.3 Vector Subspaces

- Examples

- The solution set of an inhomogeneous system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{b} \neq \boldsymbol{0}$. Is it a subspace of $\mathbb{R}^n$?

Is it a subset of $\mathbb{R}^2$?

Does it satisfy $\mathcal{U} \neq \emptyset$, in particular $\boldsymbol{0} \in \mathcal{U}$

Does it satisfy closure?

Linear combination

- Consider a vector space V and k vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k \in V$. For $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$, $\boldsymbol{v} \in V$ is called a linear combination of vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$, if

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V$$

2.5 Linear Independence

$A\,x = 0$

- Consider a system of linear functions $\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i = \boldsymbol{0}$
- If there is a non-trivial solution, $\lambda_1, \ldots, \lambda_k$, with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly dependent

$\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_k x_k = 0$

- If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$, then vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ are linearly independent
- Intuitively, a set of linearly independent vectors consists of vectors that have no redundancy, *i.e.,* if we remove any of those vectors from the set, we will lose something.

How to determine linear (in)dependence

- Write all vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$ as columns of a matrix $\boldsymbol{A}$
- Perform Gaussian elimination until the matrix is in row echelon form
- The pivot columns correspond to independent vectors

$$\begin{array}{ccc} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ x_1 \quad x_2 \quad x_3 \end{array}$$

$x_2 = 3x_1$

$x_1 \quad x_3$ in

- All column vectors are linearly independent if and only if all columns are pivot columns.
- If there is at least one non-pivot column, the columns are linearly dependent.

Determine linear (in)dependence

- Consider three vectors in $\mathbb{R}^3$

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 \end{matrix}$$

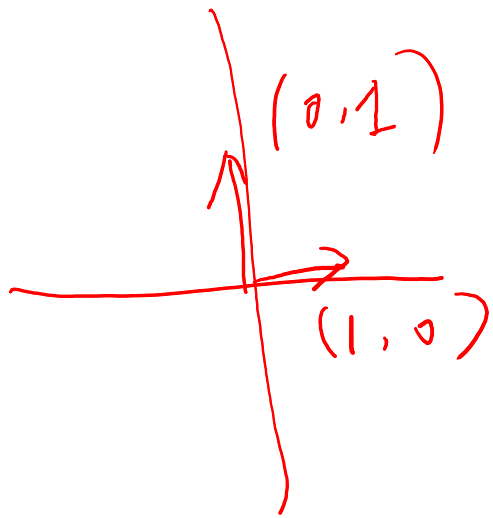
$$x_3 = x_1 + 2x_2$$

The Basis of a vector space

- A set of vectors $\{x_1, \cdots, x_k\}$ is said to form a basis for a vector space if

(1) The vectors $\{x_1, \cdots, x_k\}$ span the vector space

(2) The vectors $\{x_1, \cdots, x_k\}$ are linearly independent.



- Example
- In $\mathbb{R}^3$, the canonical/standard basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Different bases in $\mathbb{R}^3$ are

$(a, b, c)^T \in R^3$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

① linearly independent

② span(R^3).

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} \lambda_3 = c \\ \lambda_2 = b - c \\ \lambda_1 = a - b \end{cases}$$

- Another different basis in $\mathbb{R}^3$ is

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

- Another example

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

① lin In

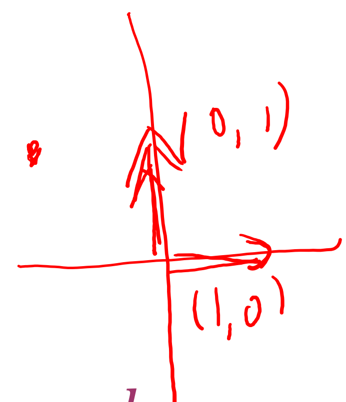
② span

is linearly independent, but not a basis of $\mathbb{R}^4$: For instance, the $[1,0,0,0]^\top$ cannot be obtained by a linear combination of elements in $\mathcal{A}$.

So, a couple of things about basis

- Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$ be a basis of V.
- $\mathcal{B}$ is a maximal linearly independent set of vectors in V, i.e., adding any other vector to this set will make it linearly dependent.
- Every vector $\boldsymbol{x} \in V$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., with

$$x = \sum_{i=1}^{k} \lambda_i \boldsymbol{b}_i = \sum_{i=1}^{k} \psi_i \boldsymbol{b}_i$$



and $\lambda_i, \psi_i \in \mathbb{R}, \boldsymbol{b}_i \in B$ it follows that $\boldsymbol{\lambda}_i = \psi_i$, $i = 1, \cdots, k$.

- Every vector space V possesses a basis $\mathcal{B}$.
- There can be many bases of a vector space.
- All bases possess the same number of elements, called the basis vectors

$\mathbb{R}^3$

- Dimension of (V): number of basis vectors of V. We write $\dim(V)$
- If $U \subseteq V$ is a subspace of V , then

$$B = \underbrace{\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}_{3}$$

- $\dim(U) = \dim(V)$ if and only if $U = V$

$\dim(U) \leq \dim(V)$

Determining a Basis

- Write the spanning vectors as columns of a matrix $\boldsymbol{A}$
- Determine the row-echelon form of $\boldsymbol{A}$.
- The spanning vectors associated with the pivot columns are a basis of U.

- Example
- For a vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \in \mathbb{R}^5$$

Determining a Basis - Example

- Which vectors of $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are a basis for U?
- Check whether $\boldsymbol{x}_1, \dots, \boldsymbol{x}_4$ are linearly independent.

$$\sum_{i=1}^{4} \lambda_i \boldsymbol{x}_i = \boldsymbol{0}$$

- A homogeneous system of equations with matrix

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

- Through Gaussian Elimination, we obtain the row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4$ are linearly independent. Therefore, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4\}$ is a basis of U

2.6.2 Rank

n

m $\begin{bmatrix} \quad \end{bmatrix}$ A

2

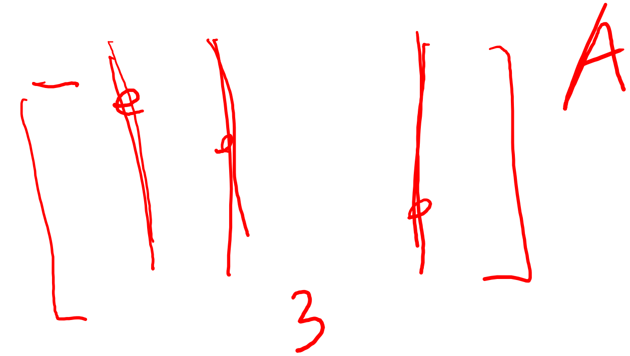
3 2

- The number of linearly independent columns of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is called the rank of $\boldsymbol{A}$, denoted by $\mathrm{rk}(\boldsymbol{A})$

- $\mathrm{rk}(\boldsymbol{A})$ also equals the number of linearly independent rows

- Rank gives us an idea of how much information a matrix contains

Important properties



- $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\mathbf{A}^{\mathrm{T}})$
- Columns and rows of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ are both subspaces of the same dimension $\mathrm{rk}(\boldsymbol{A})$
- The basis of the subspace spanned by columns (rows) can be found by Gaussian elimination to $\boldsymbol{A}$ $(\mathbf{A}^{\mathrm{T}})$ to identify the pivot columns.
- For all $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ it holds that $\boldsymbol{A}$ is regular (invertible) if and only if $\mathrm{rk}(\boldsymbol{A}) = n$.

- Example

- We use Gaussian elimination to determine the rank

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \end{array}$$

- 2 pivot columns. So $\mathrm{rk}(\boldsymbol{A}) = 2$

More properties

$r[A] = r(A|b)$

$=3 \quad =3$

- For all $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and all $\boldsymbol{b} \in \mathbb{R}^{m}$ it holds that the linear equation system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ can be solved if and only if $\mathrm{rk}(\boldsymbol{A}) = \mathrm{rk}(\boldsymbol{A}|\boldsymbol{b})$, where $\boldsymbol{A}|\boldsymbol{b}$ denotes the augmented matrix

$r([A]) = r([A \mid b])$

- For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ the subspace of solutions for $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{0}$ possesses dimension $n - \mathrm{rk}(\boldsymbol{A})$.

$n \times n \quad A \quad r(A) = n$

$Ax = 0$

$\rightarrow$

$x = 0$

less flexibility

More properties

- A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions.

$r(A) = 3$

- The rank of a full-rank matrix is the lesser of the number of rows and columns, i.e., $\mathrm{rk}(\boldsymbol{A}) = \min(m, n)$.

5×3

- A matrix is said to be rank deficient if it does not have full rank.

2.7 Linear Mappings

- For vector spaces V, W, a mapping $\Phi: V \to W$ is called a linear mapping if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V, \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y})$$

- It implies the following

$$\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + \Phi(\boldsymbol{y}) \qquad \Phi(\lambda \boldsymbol{x}) = \lambda\, \Phi(\boldsymbol{x})$$

Example

$c \in \mathbb{R}$

- The mapping $\Phi: \mathbb{R} \rightarrow \mathbb{R}, \Phi(x) = cx$, is a linear mapping:

$$\Phi(x_1 + x_2) = c(x_1 + x_2) = cx_1 + cx_2$$
$$= \Phi(x_1) + \Phi(x_2)$$

$$\Phi(\lambda x) = c\lambda x = \lambda cx = \lambda \Phi(x)$$

Example

cx affine

- The mapping $\Phi\colon \mathbb{R} \to \mathbb{R}, \Phi(x) = x + 1$, is not a linear mapping:

$$\Phi(x_1 + x_2) = x_1 + x_2 + 1$$
$$\Phi(x_1) + \Phi(x_2) = x_1 + x_2 + 2$$

$$\Phi(x_1 + x_2) \neq \Phi(x_1) + \Phi(x_2)$$

2.7 Linear Mappings

- For linear mappings $\Phi: V \to W$ and $\Psi: W \to X$, the mapping $\Phi \circ \Psi: V \to X$ is also linear

- If $\Phi: V \to W$ and $\Psi: V \to W$ are both linear mappings, then $\Phi + \Psi$ and $\lambda\Phi, \lambda \in \mathbb{R}$ are also linear.

Coordinates of a vector

- Consider a vector space V and an ordered basis $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ of V. For any $\boldsymbol{x} \in V$ we obtain a unique representation

$$\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \ldots + \alpha_n \boldsymbol{b}_n$$

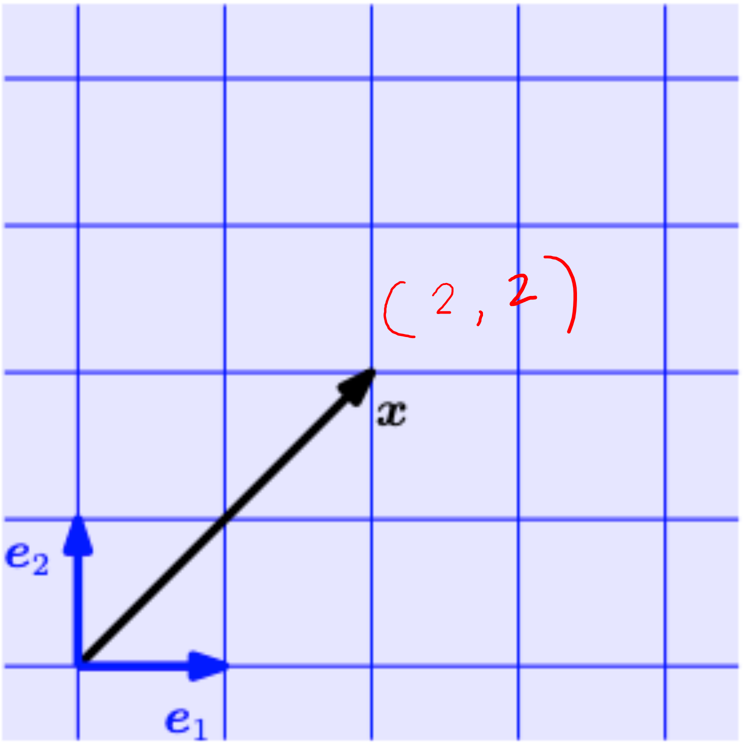
of $\boldsymbol{x}$ with respect to B. Then $\alpha_1, \cdots, \alpha_n$ are the coordinates of $\boldsymbol{x}$ with respect to B, and the vector

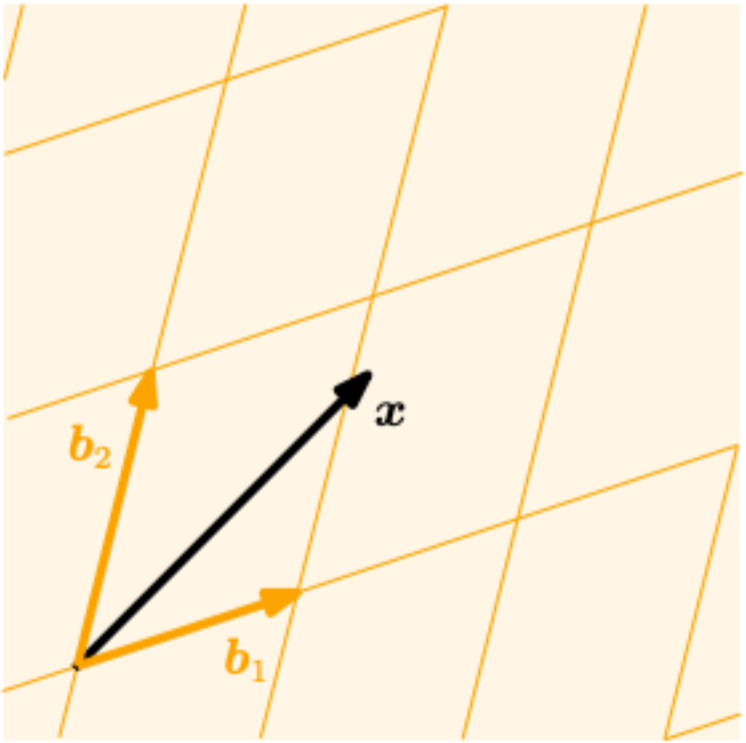
$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

is the coordinate vector/coordinate representation of $\boldsymbol{x}$ with respect to the ordered basis B.

Coordinates of a vector

- [Left] A Cartesian coordinate system in two dimensions, which is spanned by the canonical basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$.

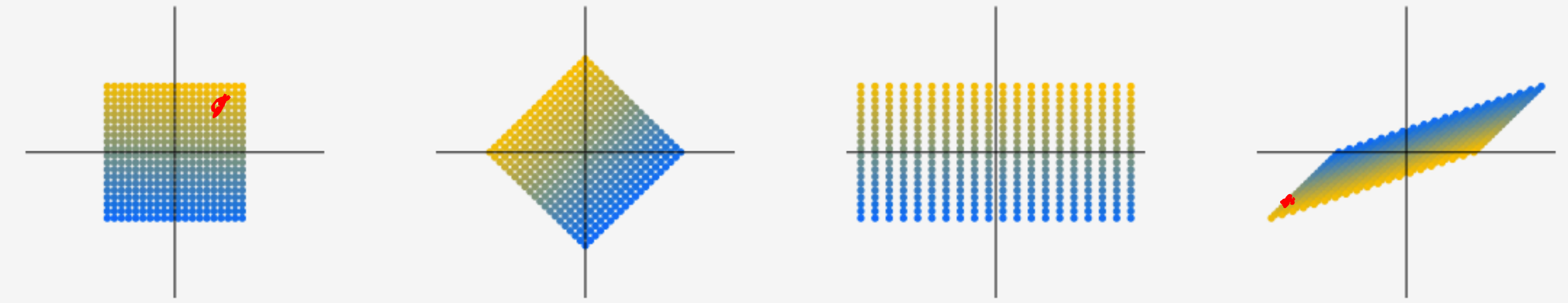




- The same vector $\boldsymbol{x}$ may have different coordinates under different basis.

2.7.1 Matrix Representation of Linear Mappings

- Example - Linear Transformations of Vectors



(a) Original data. (b) Rotation by 45°. (c) Stretch along the horizontal axis. (d) General linear mapping.

- The following three linear transformations are used

$$A_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad A_3 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

- Consider vector spaces V, W with corresponding bases $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \cdots, \boldsymbol{c}_m)$. We consider a linear mapping $\Phi: V \to W$. For $i \in \{1. \cdots . n\}$.

is the unique representation of $\Phi(\boldsymbol{b}_j)$ with respect to C. Then, we call the $m \times n$-matrix $\boldsymbol{A}_\Phi$ the transformation matrix of Φ , whose elements are given by

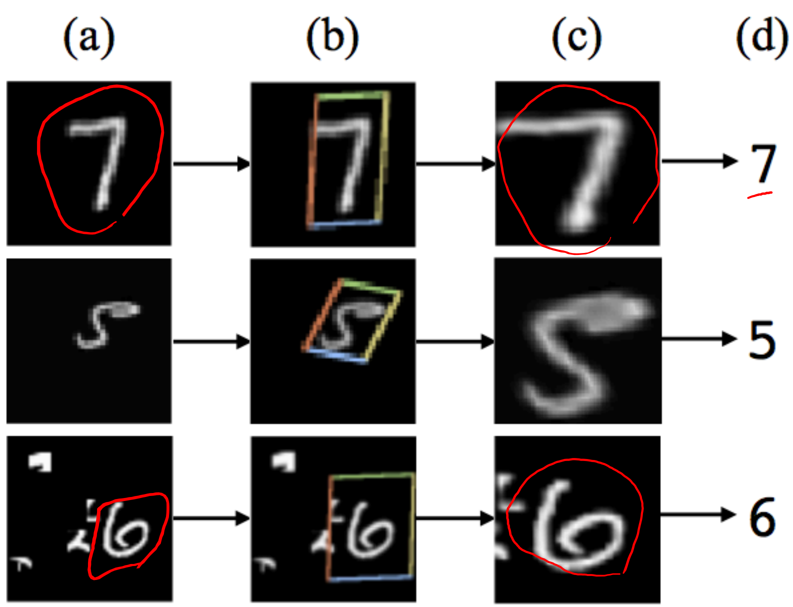
If $\hat{\boldsymbol{x}}$ is the coordinate vector of $\boldsymbol{x} \in V$ with respect to B, and $\hat{\boldsymbol{y}}$ the coordinate vector of $\boldsymbol{y} = \Phi(\boldsymbol{x}) \in W$ with respect to C, then

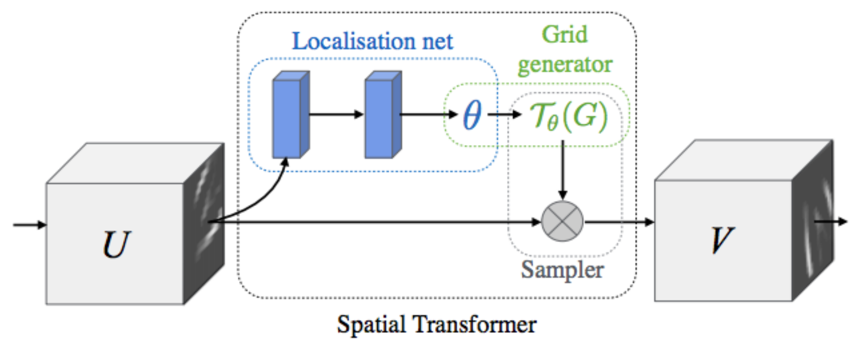
$$\hat{\boldsymbol{y}} = \boldsymbol{A}_\Phi \hat{\boldsymbol{x}}$$

Spatial Transformer Network (Jaderberg et al., NIPS 2015)

$$\begin{pmatrix} x_i^s \\ y_i^s \end{pmatrix} = \mathcal{T}_\theta(G_i) = \mathtt{A}_\theta \begin{pmatrix} x_i^t \\ y_i^t \\ 1 \end{pmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{bmatrix} \begin{pmatrix} x_i^t \\ y_i^t \\ 1 \end{pmatrix}$$

affine transformation





Check your understanding

- Which of the followings are correct

T
(A) In a vector space, any vector can be represented as a linear combination of a certain set of vectors in this space.

F
(B) The dimension of a vector equals the dimension of the space it is in.

$\forall$ $\exists$

F
(C) U is a vector subspace of V. Then vectors in U have lower dimension than vectors in V.

T
(D) $\mathrm{U} = \{(x, y) \colon x = y, x \in \mathbb{R}, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$

F
(E) The set $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$

T
(F) The vector $\mathbf{0}$ is linearly dependent with any vector in the same vector space. $\forall x \in \mathbb{R}^n$

$\underset{\substack{\shortparallel \\ 1}}{\lambda_1} \vec{0} + \underset{\substack{\shortparallel \\ 0}}{\lambda_2} x = 0$ $\quad \exists \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases}$