Linear Algebra

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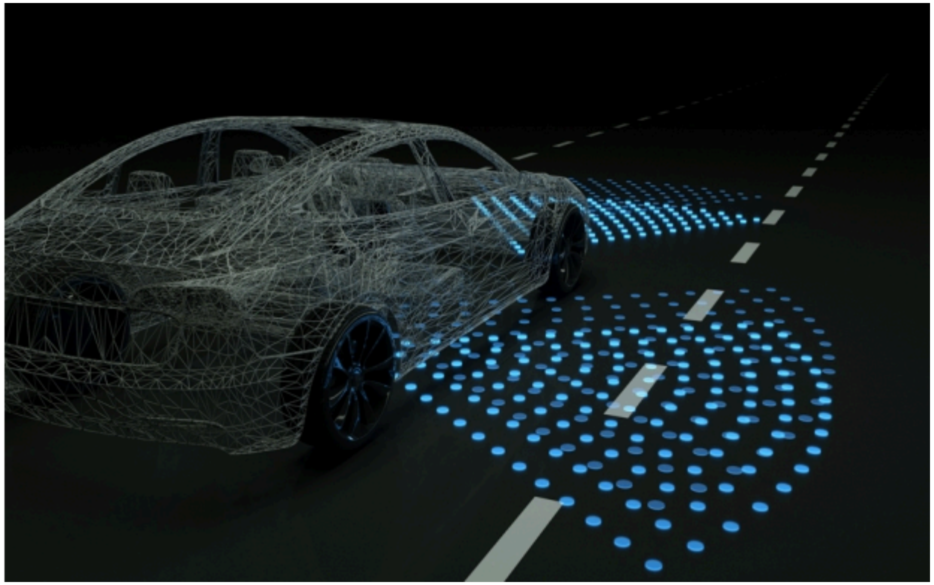
Latest development in machine learning

DriveSeg dataset developed by MIT and Toyota

20,100 video frames

12 classes of road objects



Simulation before testing on road

a photorealistic world with infinite steering possibilities, helping the cars learn to navigate a host of worse-case scenarios before cruising down real streets

Vectors

- A simple example of vector, an element of $\mathbb{R}^n$

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ \vdots \\ -1 \end{bmatrix} \in \underline{R^n} \qquad \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} \in R^3$$

- Adding two vectors

$a, b \in R^n$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$a + b = c \in R^n$$

- Multiplying $a \in R^n$ by $\lambda \in R$ results in a scaled vector:

$$\lambda a \in R^n$$

2.1 Systems of Linear Equations

- Examples

$$-x_1 + x_2 + 3x_3 = 3 \qquad (1)$$
$$x_1 + x_2 + 2x_3 = 2 \qquad (2)$$
$$2x_2 + 5x_3 = \underline{1} \qquad (3)$$

3 unknowns

x_1, x_2, x_3

Does it have solution? no.

$$0 + 2x_2 + 5x_3 = \underline{5} \qquad (4)$$

2.1 Systems of Linear Equations

- Examples

$$x_1 + x_2 = 1 \qquad (1)$$
$$x_1 - x_2 = 3 \qquad (2)$$

x_1, x_2

$(2, -1)$

Does it have solution? Yes, unique

$$x_1 + x_2 + \overset{2}{x_3} = 0 \qquad (1)$$
$$x_1 + x_2 + \overset{4}{2x_3} = 2 \qquad (2)$$
$$3x_3 = 6 \qquad (3)$$

$x_1 + x_2 = -2$

$\boxed{x_3 = 2}$

Does it have solution? Yes. infinitely ~~number~~ many

2.2 Matrices

- A rectangular scheme consisting of m rows and n columns:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \end{array} \qquad a_{ij} \in \mathbb{R}.$$

row col

ith row jth col

- By convention $(1, n)$-matrices are called rows and $(m, 1)$-matrices are called columns. These special matrices are also called row/column vectors.

2.2 Matrices

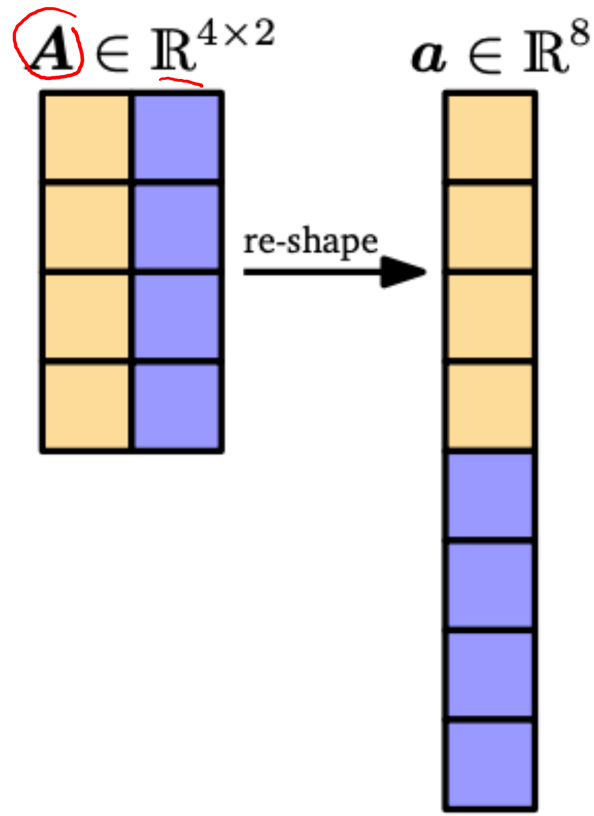
rows cols

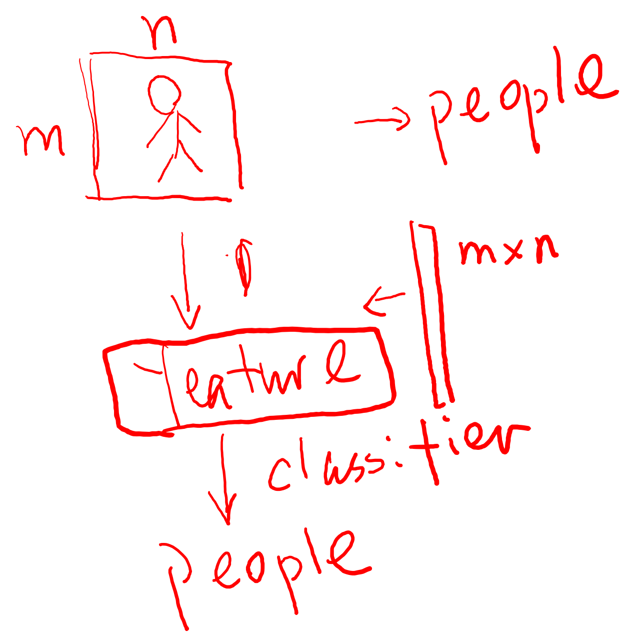
- $\mathbb{R}^{m \times n}$ is the set of all real-valued (m, n)-matrices.

space

R^3

R^n





Matrix - example

[0, 255]

RGB image 5466

3244

5466x3244x3

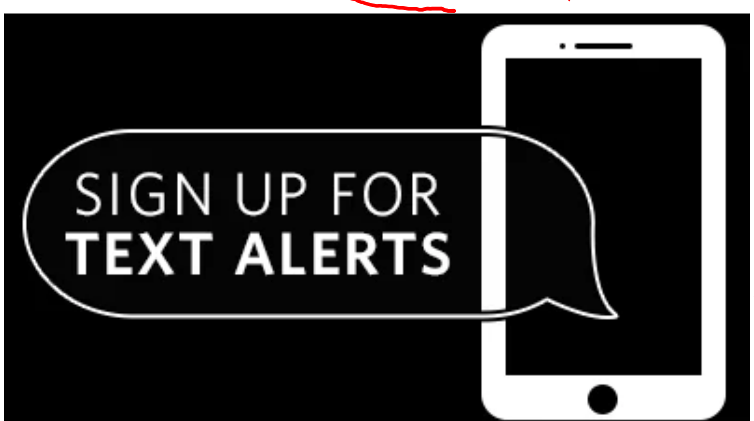
640 Gray scale image

427

640x427

Binary image 400

{0, 1}



400x255

255

Summary statistics table

	TREATMENT					
	A			B		
	N	Mean	SD	N	Mean	SD
LENGTH	50	176.500	5.9083	50	175.640	5.5467
WEIGHT	50	77.680	10.6492	50	76.400	8.4540
Body Mass Index	50	24.918	3.0644	50	24.763	2.4787

3

6

3x6

2.2.1 Matrix Addition

- The sum of two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{m \times n}$ is defined as the element-wise sum,

$$\boldsymbol{A} + \boldsymbol{B} := \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \in R^{m \times n}$$

- Example

For $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 3 & -2 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$, $\boldsymbol{B} = \begin{bmatrix} -5 & 0 \\ 1 & 1 \\ 0 & -4 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$, we obtain

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} -5 & 1 \\ 2 & 3 \\ 3 & -6 \end{bmatrix} \in R^{3 \times 2}$$

2.2.1 Matrix Multiplication

$-1 \times 1 + 1 \times 0$ $\quad\quad$ $-3+1$

- Example

For $\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{3\times2}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times3}$, we obtain

$$\boldsymbol{AB} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 4 & 2 \\ 1 & 8 & 6 \end{bmatrix} \in R^{3\times3}$$

$$\boldsymbol{BA} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 1 & 7 \end{bmatrix} \in R^{2\times2}$$

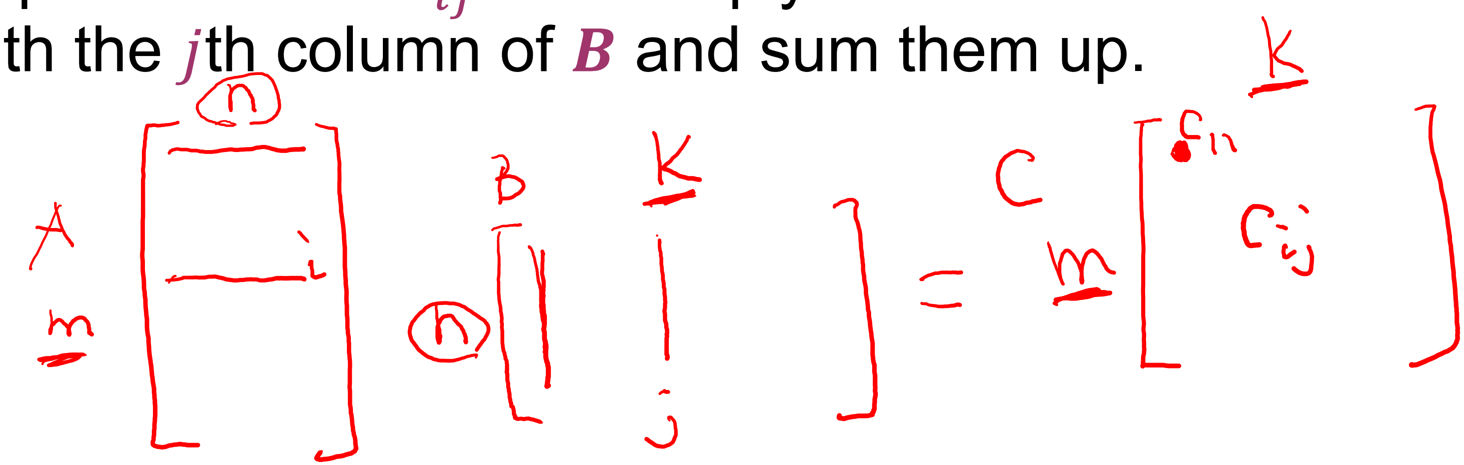
2.2.1 Matrix Multiplication

- For matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times k}$, the element c_{ij} of the product $\boldsymbol{C} = \boldsymbol{AB} \in \mathbb{R}^{m \times k}$ is defined as

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \qquad c_{ij} \neq a_{ij} b_{ij}$$

$$i = 1, \dots, m. \; j = 1, \dots, k$$

- To compute element c_{ij} we multiply the elements of the ith row of $\boldsymbol{A}$ with the jth column of $\boldsymbol{B}$ and sum them up.



2.2.1 Matrix Addition and Multiplication

- One property that is unique to matrices is the dimension property. This property has two parts:

$$\underset{n \times k}{A} \; \underset{k \times m}{B} = \underset{n \times m}{C}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Identity Matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n \times n} \in \mathbb{R}^{n \times n}$$

2.2.1 Matrix Addition and Multiplication

- Properties of matrices
- Associativity

$$\forall \boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times q}: \quad (AB)C = A(BC)$$

- Distributivity

$$\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n\times p}: \quad (A+B)C = AC + BC$$

$$A(C+D) = AC + AD$$

- Multiplication with the identity matrix:

$$\forall A \in \mathbb{R}^{m\times n}, \quad I_m A = A I_n = A$$

2.2.2 Inverse and Transpose

- **Inverse**: consider a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Let matrix $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ have the property that $\boldsymbol{AB} = \boldsymbol{I}_n = \boldsymbol{BA}$. $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$ and denoted by $\boldsymbol{A}^{-1}$.

$B = A^{-1}$

$AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

- Example

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

The matrices are inverse to each other, because

$$\boldsymbol{AB} = \boldsymbol{I}_2 = \boldsymbol{BA}$$

$BA = I_2$

2.2.2 Inverse and Transpose

- Transpose: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the matrix $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is called the transpose of $\boldsymbol{A}$. We write $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 5 & 6 \\ 0 & 1 & 3 \end{bmatrix}}_{A} \Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & -5 & 1 \\ 2 & 6 & 3 \end{bmatrix}}_{A^T}$$

$$B = A^T$$

$$(AB)\cdot(B^{-1}A^{-1}) \overset{?}{=} I\ ?$$

$$= A\,\overset{I}{I}\,A^{-1} = AA^{-1} = I$$

- Important properties of inverses and transposes:

$$(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$$

$$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$$

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$$

$$\left(\boldsymbol{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{A}$$

$$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

definition of matrix multiplication

$$(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}^{\mathrm{T}}$$

$$(\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$$

2.2.2 Inverse and Transpose

- Symmetric: A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} \quad A = A^T$$

- The sum of symmetric matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ is always symmetric

$$A + B \overset{?}{=} (A+B)^T$$

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

- The product of two symmetric matrics is generally not symmetric

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

2.2.3 Multiplication by a Scalar

- A scalar $\lambda \in \mathbb{R}$
- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. Then $\lambda \boldsymbol{A} = \boldsymbol{K}$, where $k_{ij} = \lambda a_{ij}$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \end{bmatrix} \qquad \lambda = 1.5$$

$$\lambda A = \begin{bmatrix} 1.5 & 0 & 4.5 \\ 3 & 0 & -1.5 \end{bmatrix}$$

2.2.3 Multiplication by a Scalar

- For $\lambda, \varphi \in \mathbb{R}$, there following properties hold:
- Associativity

$$(\lambda\varphi)\boldsymbol{C} = \lambda(\varphi\boldsymbol{C}), \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

$$\lambda(\boldsymbol{BC}) = (\lambda\boldsymbol{B})\boldsymbol{C} = \boldsymbol{B}(\lambda\boldsymbol{C}) = (\boldsymbol{BC})\lambda, \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times k}$$

- Transpose

$$(\lambda\boldsymbol{C})^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\lambda^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\lambda = \lambda\boldsymbol{C}^{\mathrm{T}}, \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

- Distributivity

$$(\lambda + \varphi)\boldsymbol{C} = \lambda\boldsymbol{C} + \varphi\boldsymbol{C}, \quad \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

$$\lambda(\boldsymbol{B} + \boldsymbol{C}) = \lambda\boldsymbol{B} + \lambda\boldsymbol{C}, \quad \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m\times n}$$

2.2.4 Compact Representations of Systems of Linear Equations

- Consider the system of linear equations,

$$2x_1 + 3x_2 + 5x_3 = 1$$
$$4x_1 - 2x_2 - 7x_3 = 8$$
$$9x_1 + 5x_2 - 3x_3 = 2$$

$x_1,\ x_2\ x_3$

- Using matrix multiplication, we can write it into a compact form

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

2.3 Solving Systems of Linear Equations

- Now we have a general form of an equation system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

- 2.3.1 Particular and General Solution

Step 1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$

Step 2. Find all solutions to $\boldsymbol{Ax} = \boldsymbol{0}$

Step 3. Combine the solutions from steps 1. and 2. to the general solution

We use Gaussian elimination to solve the equation system

2.3.2 Elementary Transformations

- Elementary transformations keep the solution set the same, but transform the equation system into a simpler form.

- Elementary transformations include:
- Exchange of two equations
- Multiplication of an equation (row) with a constant $\lambda \in \mathbb{R}\backslash\{0\}$
- Addition of two equations (rows)

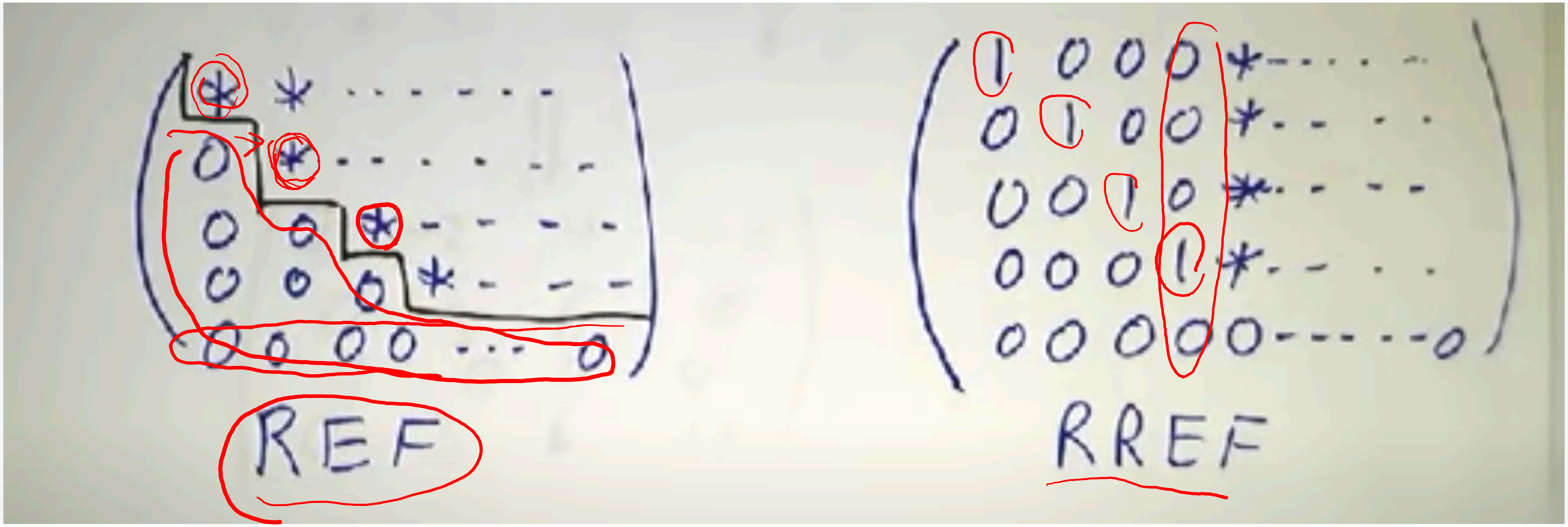
×3

Eq₁ + Eq₂

Eq₂

Eq₃

Row-echelon form (REF) and reduced row-echelon form (RREF)



From Lorenzo A. Sadun's teaching video

- All rows with 0s only are at the bottom
- A pivot is always strictly to the right of the pivot of the row above it

- Row Echelon Form
- Every pilot is 1
- The pivot is the only non-zero entry in its column

pivots

Gaussian Elimination - example

$$
\begin{aligned}
x_1 &+ x_2 - x_3 = 9 \\
& x_2 + 3x_3 = 3 \\
-x_1 & - 2x_3 = 2
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & 1 & -1 & 9 \\
0 & 1 & 3 & 3 \\
-1 & 0 & -2 & 2
\end{array}\right]
$$

augmented matrix

$$
\begin{bmatrix}
1 & 1 & -1 & 9 \\
0 & 1 & 3 & 3 \\
0 & 1 & -3 & 11
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -1 & 9 \\
0 & 1 & 3 & 3 \\
0 & 0 & -6 & 8
\end{bmatrix}
$$

REF

Gaussian Elimination - example

- Seek all solutions to the following system of equations

$$
\begin{array}{rcrcrcrcl}
2x_1 & + & 3x_2 & - & 2x_3 & + & 5x_4 & = & 1 \\
x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 2 \\
-x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 4
\end{array}
$$

$$
\begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 1 & 2 & -1 & 3 & 2 \\ -1 & -2 & 1 & -1 & 4 \end{bmatrix}
\rightarrow
\begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 3 & -2 & 5 & 1 \\ -1 & -2 & 1 & -1 & 4 \end{bmatrix} \end{array}
\begin{array}{l} \\ R_2 - 2R_1 \\ R_1 + R_3 \end{array}
$$

$$
\rightarrow
\begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -1 & 0 & -1 & -3 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix}
\quad \text{REF}
$$

How to find the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$

Step 1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$

Step 2. Find all solutions to $\boldsymbol{Ax} = \boldsymbol{0}$

Step 3. Combine the solutions from steps 1. and 2. to the general solution

Finding a particular solution to $Ax = b$

Let free variables be 0, calculate the value of basic variables

$$
\begin{array}{ccccc}
 & & & & \\
\end{array}
\left[\begin{array}{cccc|c}
1 & 2 & -1 & 3 & 2 \\
0 & -1 & 0 & -1 & -3 \\
0 & 0 & 0 & 2 & 6
\end{array}\right]
$$

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & b
\end{array}
$$

$A \quad x = b$

x_3: free

$x_1 \ x_2 \ x_4$: basic

$2x_4 = 6$

$-1x_2 - x_4 = -3$

$-x_2 - 3 = -3$

$x_1 + 0 - x_3 + 9 = 2$

$x_1 - x_3 = -7$

$$\begin{bmatrix} -7 \\ 0 \\ 6 \\ 3 \end{bmatrix}$$

Find all solutions to $\boldsymbol{Ax} = \mathbf{0}$

Let one free variables be 1, and the rest free variables be 0, calculate the value of basic variables

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{c} 2 \\ -3 \\ 6 \end{array} \quad b \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$Ax = b$

(3) $\left\{ x \in \mathbb{R}^4 : x = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$

$$
\begin{aligned}
-2x_1 &+ 4x_2 &- 2x_3 &- x_4 &+ 4x_5 &= -3\\
4x_1 &- 8x_2 &+ 3x_3 &- 3x_4 &+ x_5 &= 2\\
x_1 &- 2x_2 &+ x_3 &- x_4 &+ x_5 &= 0\\
x_1 &- 2x_2 & &- 3x_4 &+ 4x_5 &= a
\end{aligned}
$$

$$
\begin{bmatrix} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{bmatrix}
\rightarrow
\begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ -2 & 4 & -2 & -1 & 4 & -3 \\ 1 & -2 & 0 & -3 & 4 & a \end{bmatrix}
\begin{matrix} R_2 - R_1 \times 4 \\ R_3 + 2R_1 \\ R_4 - R_1 \end{matrix}
$$

$$
\rightarrow
\begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{bmatrix}
R_4 - R_2
\rightarrow
\begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & -3 & 6 & a-2 \end{bmatrix}
\rightarrow R_4 - R_3
$$

$$
0 = a+1 \quad a = -1
\rightarrow
\begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{bmatrix}
\text{REF}
$$

Finding a particular solution to $\boldsymbol{Ax = b}$

Let free variables be 0, calculate the value of basic variables

$$
\begin{array}{c}
\left[\begin{array}{ccccc|c}
1 & -2 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 3 & -2 \\
0 & 0 & 0 & \underline{1} & \underline{-2} & \underline{1} \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}
\end{array}
\quad \text{REF}
$$

$$
\begin{array}{c}
\left[\begin{array}{c} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right]
\end{array}
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}
$$

$$x_4 - 2\underline{x_5} = 1$$

$$\underset{}{x_3} - \underset{1}{x_4} + 3\underset{0}{x_5} = -2$$

$$x_1 - 2\underset{0}{x_2} + \underset{-1}{x_3} - \underset{1}{x_4} + \underset{0}{x_5} = 0$$

Find all solutions to $\boldsymbol{Ax = 0}$

- Let one free variables be 1, and the rest free variables be 0, calculate the value of basic variables

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & x_2 & & & x_5 & \end{array} \end{array}$$

$$\lambda_1 \begin{array}{c} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array}$$

$$x_3 - x_4 + 3x_5 = 0$$

How to find the general solution to $\boldsymbol{Ax} = \boldsymbol{b}$

- Step 3. Combine the solutions from steps 1. and 2. to the general solution

Step 1:

Ax=b

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Step 2:

Ax=0

$$\left\{ \mathbf{x} \in R^5 : \boldsymbol{x} = \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \lambda_1, \lambda_2 \in R \right\}$$

General solution:

$$\left\{ \mathbf{x} \in R^5 : \boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \lambda_1, \lambda_2 \in R \right\}$$

Proof

- Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with $m < n$, then $\boldsymbol{Ax} = \boldsymbol{0}$ has infinitely many solutions
- Proof
- This system always has at least one solution since homogeneous
  - Consider $\boldsymbol{A0} = \boldsymbol{0}$ always holds

    $x = 0$

- Matrix $\boldsymbol{A}$ brought in ~~reduced~~ row echelon form contains at most m pivots.

For example, $\begin{bmatrix} \boldsymbol{1} & -2 & 0 & 0 & -2 \\ 0 & 0 & \boldsymbol{1} & 0 & 1 \\ 0 & 0 & 0 & \boldsymbol{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 5}$

RO

$m \times n$

$m < n$

- There will have $n - m \geq 1$ non-pivot columns, or free variables.It means we can find at least one solution $\boldsymbol{x}^* \neq \boldsymbol{0}$. Then, $\lambda \boldsymbol{x}^*$, $\lambda \in \mathbb{R}$ are solutions to $\boldsymbol{Ax} = \boldsymbol{0}$.

Proof

- A system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ either has no solutions, a unique solution or infinitely many solutions
- Proof
- Let's assume the system $\boldsymbol{Ax} = \boldsymbol{b}$ has two solutions $\boldsymbol{p}$ and $\boldsymbol{q}$.
- We have

$$\boldsymbol{Ap} = \boldsymbol{b} \qquad \boldsymbol{Aq} = \boldsymbol{b}$$

v different from p q

- Consider

$$\boldsymbol{v} = \boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p}), t \in \mathbb{R}$$

- We have

$$\boldsymbol{Av} = \boldsymbol{A}\big(\boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p})\big) = \underset{\overset{\|}{b}}{\boldsymbol{Ap}} + t(\underset{\overset{\|}{b}}{\boldsymbol{Aq}} - \underset{\overset{\|}{b}}{\boldsymbol{Ap}}) = \boldsymbol{b} + t(\boldsymbol{b} - \boldsymbol{b}) = \boldsymbol{b}$$

- We thus have infinitely many solutions (by varying t)

Calculating the Inverse with Gaussian Elimination

- To compute the inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$,
- We need to find a matrix $\boldsymbol{X}$ that satisfies $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}_n$.
- Then, $\boldsymbol{X} = \boldsymbol{A}^{-1}$.
- We can write this down as a set of simultaneous linear equations $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}_n$, where we solve for $\boldsymbol{X} = [\boldsymbol{x}_1 | \cdots | \boldsymbol{x}_n]$
- We use the augmented matrix notation, and use Gaussian Elimination.

$$[\boldsymbol{A} \mid \boldsymbol{I}_n] \rightsquigarrow \cdots \rightsquigarrow [\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}]$$

Calculating the Inverse with Gaussian Elimination

$$[\boldsymbol{A} \mid \boldsymbol{I}_n] \rightsquigarrow \cdots \rightsquigarrow [\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}]$$

Calculating the Inverse with Gaussian Elimination

- Example: determine the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

- First, write down the augmented matrix

$$\boldsymbol{A} = \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

A $\qquad\qquad$ I_n

Calculating the Inverse with Gaussian Elimination

RREF

- Use Gaussian elimination to bring it into reduced row-echelon form

REF →

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \cdots \rightsquigarrow A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right]$$

A^{-1}

- The desired inverse is given as its right-hand side

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Calculating Reduced Row-echelon form - example

$$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{bmatrix} \xrightarrow{\substack{R_2-R_1 \\ R_3+2R_1 \\ R_4-2R_1}} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{REF} \rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_1+R_3 \\ R_2-3R_3}}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{RREF}$$

not insersible

matrix inverse

$I_4 \quad \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & 1 \end{bmatrix}_{4\times 4}$

Check your understanding

- Which of the following are correct?

True

is still a vector

(A) A vector, when multiplied by a scaler, ~~becomes another vector~~.

True

(B) For a system of linear equations with n variables, it is possible that none of them are free variables. unique solution.

False

(C) For a system of linear equations with n variables, the maximum number of pivots in the REF is $n-1$. n

False

(D) A matrix, when added by an identity matrix, stays as is.

False

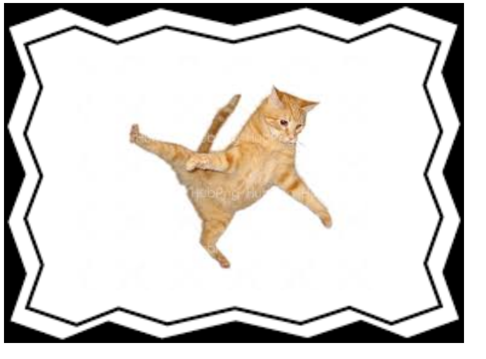
(E) We can use matrix transpose in Gaussian Elimination.

(F) Two arbitrary matrices can be multiplied False

(G) Two arbitrary matrices can be added. False

(H) An image with black borders is not a matrix.

False



Check your understanding

- Let A, B, C be 2x2 matrices.
- Which of the following are equivalent to A(B+C)?
  - AB+AC ✓
  - BA+CA ✗
  - A(C+B) ✓
  - (B+C)A ✗

- Which of the following expressions are equivalent to $I_2(A\cancel{+}B)$?
  - AB ✓
  - BA ✗
  - $(AB)I_2$ ✓
  - $(BA)I_2$ ✗