Proof that a perfect cuboid does not exist

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Introduction

An Euler brick is a rectangular cuboid characterized by having all edges and all face diagonals that are of integer length. A perfect cuboid that's also known as a perfect Euler brick is an Euler brick whose space diagonal is also of integer length [Kni08]. As per the Wikipedia on this date 6/27/2021 9:02:04 AM, neither has a perfect cuboid been found nor been proven nonexistent [Wik21]. Good news is that this hundreds of years old mathematics problem gracefully comes to a close here.

Reference cuboid

Consider the cuboid below with vertices **P**, **Q**, **R**, **S**, **T**, **U**, **V**, and **W**. For clarity, similar edges and similar diagonals are labelled alike where needed.

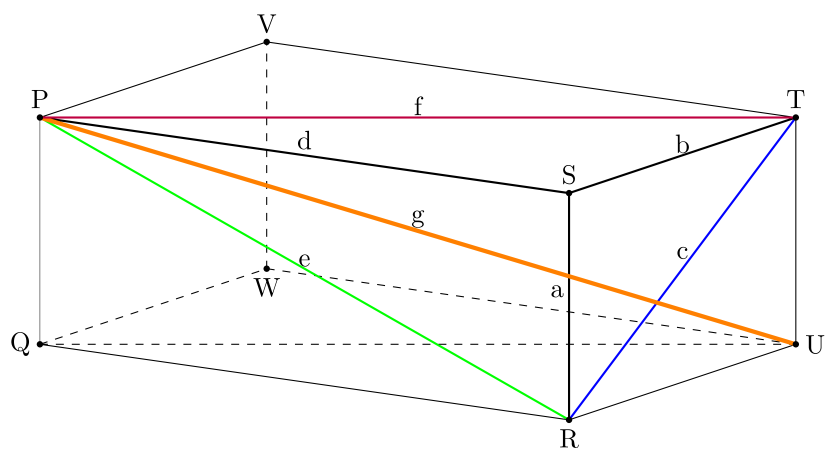

Figure 1: Reference cuboid

Observations and analysis

By definition, for any cuboid, all edges and all diagonals must be greater than zero. In the case of a perfect cuboid, all edges and all diagonals must also be integers [Kni08]. Therefore, it follows that, for any solution of a perfect cuboid, face diagonals **e** and **f** must intersect with edge **d** and space diagonal **g** at a point **P** such that every edge and every diagonal of the cuboid is an integer greater than zero. If you perceive **d**, **e**, **f**, and **g** as a system of vectors originating at point P, you notice that **d** is an eigenvector [Kre07]. However, this proof does not require any vector analysis.

1. Let edge **a** be of any length **n** where,

   $$\{n \in \mathbb{Z} \mid n > 0\}$$

   Thus, for (**a**, **b**, **c**) to be a Pythagorean triple [Mao19], **b** must be of any length **m** such that

   $$\{m \in \mathbb{Z} \mid m > 0\}$$

   and

   $$\left\{\sqrt{(n^2+m^2)} \in \mathbb{Z} \mid \sqrt{(n^2+m^2)} > 0\right\}$$

   where $\sqrt{(n^2+m^2)}$ is the length of face diagonal **c**.

2. Consider triangle (**a**, **d**, **e**) where **a** is of any length **n** where

   $$\{n \in \mathbb{Z} \mid n > 0\}$$

   as in the case of triangle (**a**, **b**, **c**). It follows that for (**a**, **d**, **e**) to be a Pythagorean triple [Mao19], **d** must be of integer length which can be expressed as $(\mathbf{n} \tan\theta)$ using Pythagoras Theorem [Mao19] where θ is the angle of intersection between **e** and **d** such that

   $$\{\tan\theta \in \mathbb{R} \mid \tan\theta > 0\}$$

   and **e** must be of length

   $$\sqrt{(n^2+(n\tan\theta)^2)}$$

   where

   $$\left\{\sqrt{(n^2+(n\tan\theta)^2)} \in \mathbb{Z} \mid \sqrt{(n^2+(n\tan\theta)^2)} > 0\right\}$$

3. To make a perfect cuboid with Pythagorean triples (**a**, **b**, **c**) and (**a**, **d**, **e**), its deduced that **f** must be of length

   $$\sqrt{(n\tan\theta)^2+(m^2)}$$

   where

   $$\left\{\sqrt{(n\tan\theta)^2+(m^2)} \in \mathbb{Z} \mid \sqrt{(n\tan\theta)^2+(m^2)} > 0\right\}$$

and **g** must be of length

$$\sqrt{\left(\sqrt{(n\tan\theta)^2+(m^2)}\right)^2+n^2}$$

where

$$\left\{\sqrt{\left(\sqrt{(n\tan\theta)^2+(m^2)}\right)^2+n^2}\in\mathbb{Z}\mid\sqrt{\left(\sqrt{(n\tan\theta)^2+(m^2)}\right)^2+n^2}>0\right\}$$

In summary

- a = n
- b = m
- c = $\sqrt{(n^2+m^2)}$
- d = (ntan θ)
- e = $\sqrt{(n^2+(n\tan\theta)^2)}$
- f = $\sqrt{(n\tan\theta)^2+(m^2)}$
- g = $\sqrt{\left(\sqrt{(n\tan\theta)^2+(m^2)}\right)^2+n^2}$

Where edge d, face diagonals e and f, and space diagonal g intersect

1. If **d** intersects with **e**, then **d** = **e** at the point of intersection

$$(n\tan\theta)=\sqrt{(n^2+(n\tan\theta)^2)} \tag{1}$$

$$n=0$$

2. If **d** intersects with **f**, then **d** = **f** at the point of intersection

$$(n\tan\theta)=\sqrt{(n\tan\theta)^2+(m^2)} \tag{2}$$

$$m=0$$

3. If **d** intersects with **g**, then **d** = **g** at the point of intersection

$$(n\tan\theta)=\sqrt{\left(\sqrt{(n\tan\theta)^2+(m^2)}\right)^2+n^2} \tag{3}$$

$$n=\sqrt{-(m^2)}$$

4. If **e** intersects with **f**, then **e** = **f** at the point of intersection

$$\sqrt{(n^2 + (n\tan\theta)^2)} = \sqrt{(n\tan\theta)^2 + (m^2)} \tag{4}$$

$$n = m$$

5. If **e** intersects with **g**, then **e** = **g** at the point of intersection

$$\sqrt{(n^2 + (n\tan\theta)^2)} = \sqrt{\left(\sqrt{(n\tan\theta)^2 + (m^2)}\right)^2 + n^2} \tag{5}$$

$$m = 0$$

6. If **f** intersects with **g**, then **f** = **g** at the point of intersection

$$\sqrt{(n\tan\theta)^2 + (m^2)} = \sqrt{\left(\sqrt{(n\tan\theta)^2 + (m^2)}\right)^2 + n^2} \tag{6}$$

$$n = 0$$

Results

Equations (1), (2), and (4) show that edge **d** and face diagonals **e** and **f** intersect only when $n = m = 0$. Equations (5) and (6) show that face diagonals **e** and **f** only intersect with space diagonal **g** when $n = m = 0$. Equation (3) shows that **d** intersects with **g** only when the square of **n** is negative. Such a real number whose square is negative does not exist [Wei]. These results can be visualized by graphing **d**,**e**, **f**, and **g** for any qualified **n**.

Discussion

When **d**, **e**, **f**, and **g** intersect at a point P, they create a three-dimensional space with vertices PQRSTU which when $n = m = 0$ is reduced to a point where the point is as defined in classical Euclidean geometry, and therefore PQRSTU does not exist when $n = m = 0$.

Conclusion

In the above analysis, the three-dimensional space PQRSTU and its mirror on the plane PQTU is a perfect cuboid which exists if and only if $n = m = 0$, which accordingly does not exist.

References

[Kre07] Erwin Kreyszig. *Advanced Engineering Mathematics 9th Edition with Wiley Plus Set*. John Wiley & Sons, 2007, p. 334.

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[Wik21] Wikipedia. *Euler brick — Wikipedia, The Free Encyclopedia*. `http://en.wikipedia.org/w/index.php?title=Euler\%20brick&oldid=1029720909`. [Online; accessed 27-June-2021]. 2021.

[Wei] Eric W. Weisstein. *Square Root*. URL: `https://mathworld.wolfram.com/SquareRoot.html`.