Society of Actuaries' Textbook
ON
LIFE CONTINGENCIES

BY

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PREFACE FOR THE SOCIETY OF ACTUARIES

The first edition, published in 1952, has admirably served the purpose for which it was commissioned—to provide a modern text on Life Contingencies, applicable to the benefits commonly offered by life insurance companies in the United States and Canada. That it has served so well is due to the fortunate combination of technical ability and expository talent of the author, C. Wallace Jordan, Professor of Mathematics at Williams College and Fellow of the Society of Actuaries, and to the members of the Society's Education and Examination Committee who assisted him, particularly Morton D. Miller, who determined in large measure the scope of the first edition and persuaded Professor Jordan to be its author, and David H. Harris, who devoted much time and thought to the organization and preparation of the text.

When it became evident that an updated textbook was desirable, Paul T. Rotter, then General Chairman of the Society's Education and Examination Committee, prevailed upon Professor Jordan to revise his book. Once again, Professor Jordan has applied to the task his many talents which, with his unfailing good humor, were so successful in producing the first edition. Zehman I. Mosesson, F.S.A., made a painstaking review of the revised manuscript and page proofs and worked closely with the author, as did John A. Fibiger, F.S.A.

The Board of Governors of the Society of Actuaries is most appreciative of the great amount of work and talent contributed by Professor Jordan toward the education of actuarial students, both past and to come. The usefulness of the second edition not only to those studying life contingencies but also to those engaged in actuarial practice will be a source of gratification to all concerned.

HAROLD R. LAWSON
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Chairman
Advisory Committee on Education and Examinations

August, 1967

EXCERPTS FROM THE AUTHOR'S PREFACE TO THE FIRST EDITION

This book was prepared for the Society of Actuaries as a text for students of life contingencies. The material included forms a survey of the fundamental principles of life contingencies with applications to American life insurance practice.

In designing the pedagogic approach, the author has been mindful of the needs of those many actuarial students who must acquire their knowledge of life contingencies without the assistance of an instructor. Since the text is their only teacher, these students deserve particularly full discussions of the basic concepts. Accordingly, a special effort has been made to give careful and meaningful explanations of such fundamental ideas as the mortality table, the force of mortality, reserves, and the various multiple-life statuses. Emphasis is constantly placed on a relatively small number of primary principles, and attention is drawn to the unity inherent in the different branches of the subject.

It is important for the student to test his comprehension of the text at frequent intervals by solving problems. Collections of exercises are placed at the end of each chapter, and the student should turn to the exercises as he completes each section of a chapter. Answers are given for all the exercises, either on the same page when the form of the answer gives no clue as to the method of solution, or at the end of the book.

The mathematical symbols used in the book are based on the International Actuarial Notation. A few new symbols are introduced; in particular, the following new usages should be noted: $(\bar{I}\bar{a})_x$ and $(\bar{I}\bar{A})_x$ for the present values of a continuous increasing annuity and insurance, $P^{\{m\}}$ for an apportionable premium payable m times a year, and $\ddot{s}_{x:\overline{n|}}$ for the value of a foreborne annuity-due in place of the older symbol ${}_nu_x$. No previous knowledge of the International Code is assumed here; each symbol is defined at its first appearance, and an Index to Notation shows the page on which each definition is given.

The material in this book has been obtained from many sources, including the author's own experience in practical actuarial work and in teaching. The author was first introduced to life contingencies through the textbook of the British Institute of Actuaries written by George King and revised by E. F. Spurgeon, and his present task has been greatly simplified by the existence of this excellent text. The importance of King's work in systematizing the subject of life contingencies cannot be overemphasized. Another writer whose work has been a more recent influence is Ernst Zwinggi.

The preparation of the new text has afforded the opportunity of consolidating in a single volume a large body of material which has previously been available to students only through scattered papers in the actuarial journals. The author thus counts as his collaborators a large and distinguished company of actuaries upon whose published contributions he has drawn. These original sources are listed in the appended bibliography. The names of the various actuarial journals are abbreviated as follows:

TSA	Transactions of the Society of Actuaries
TASA	Transactions of the Actuarial Society of America
RAIA	Record of the American Institute of Actuaries
JIA	Journal of the Institute of Actuaries (Great Britain)

Many persons have assisted in the preparation of the manuscript. The major contribution has been that of Mr. David H. Harris, F.S.A., who served as editorial consultant for the whole project on behalf of the Society of Actuaries. During the past three years, he has generously devoted many hours to a careful reading of each successive draft of the entire manuscript. His critical comments and creative suggestions have contributed greatly to the improvement of the literary style, the clarity of the exposition, and the form of the mathematical demonstrations. The author considers himself most fortunate to have had the assistance of so able a collaborator.

The author is also indebted to Mr. Harry Walker, F.S.A., and to Mr. Robert D. Acker, F.S.A., both of whom read the whole manuscript. Their valuable criticisms resulted in conspicuous improvements in the original text.

Finally, the author wishes to express his appreciation to the members of the Education and Examination Committee, and especially to Mr. F. Bruce Gerhard, Mr. Morton D. Miller, Mr. Charles A. Spoerl, and Mr. Bert A. Winter, not only for their efficient handling of many administrative details, but also for their warm interest and encouragement while the work was in progress.

AUTHOR'S PREFACE TO THE SECOND EDITION

The general plan of the first edition has been retained, but there has been some rearrangement of the material together with the addition of a new chapter on pension plans and disability benefits (Chapter 16). The chapter on population problems has been completely rewritten and expanded, and it is now included in Part I as Chapter 8. In Part II, the two short chapters on the Last-Survivor Status and the Generalized Multiple-Life Status have been combined and now appear as Chapter 10. In the other chapters, changes were made wherever it seemed possible to clarify the exposition and improve the form of the mathematical demonstrations. The exercise sets were also revised and expanded.

The Bibliography has been enlarged and up-dated to include the relevant literature of the past fifteen years. In order to avoid interrupting the text with references and footnotes, a section has been added at the end of each chapter which cites the more useful sources and relates them to the context of the chapter.

The task of reviewing the manuscript for the Society of Actuaries was undertaken by Zehman I. Mosesson, F.S.A., and it is a great pleasure to acknowledge the special contributions which he has made. His helpful suggestions and wise counsel are reflected in countless improvements throughout the book.

Valuable help was also received from John A. Fibiger, F.S.A., who worked closely with Mr. Mosesson in reviewing each successive draft of the entire manuscript. His comments, particularly on style and nomenclature, were an important influence.

Many other members of the Society of Actuaries were helpful either in reading drafts of individual chapters or in suggesting improvements in the treatment of certain topics. The following were particularly generous in their assistance: Dwight K. Bartlett, Murray L. Becker, M. David R. Brown, Edward F. Dalton, Carl H. Fischer, Harry Gershenson, Harold G. Ingraham, Jr., John H. Miller, James J. Olsen, Robert B. Shapland, Charles L. Trowbridge, Charles B. H. Watson, and Howard Young.

Thanks are due also to those members of the Education and Examination Committee who handled the administrative details. Paul T. Rotter, F. S. A., was influential in getting the project started. Robert J. Johansen, F.S.A., and Julius Vogel, F.S.A., assisted in many ways while the work was in progress. Their encouragement, support, and cooperation were important factors in the completion of the work.

Williamstown, Massachusetts C. WALLACE JORDAN
May 1, 1967

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Part I

SINGLE-LIFE FUNCTIONS

Chapter 1

THE MEASUREMENT OF MORTALITY

1. Introduction

The systematic analysis of the contingencies of human life forms the foundation of an actuary's work. In the solution of problems involving these contingencies, he requires some type of quantitative measure of their effects; and, in financial problems, he requires also a set of principles by which the measurements may be combined with interest functions to produce monetary values.

The actuary's first concern is with the contingencies of death and survival. In this chapter, the problem of giving quantitative expression to observed mortality patterns will be considered, and the various functions which have been used for this purpose will be defined and examined. In the chapters that follow, formulas will be developed in which mortality functions and interest functions are combined to produce monetary values for life annuities and insurances.

2. The survival function

The normal mortality pattern observed among human lives is generally familiar. The elimination of individual lives by death, rather rapid in infancy, slows down during childhood, then increases throughout adolescence and middle-life, accelerating as the end of the life-span is approached. ~~In seeking~~ a numerical measure of the extent of these effects, we begin by defining a fundamental probability function.

Let x represent the age in years of a human life. Then x may have any value from 0 to the upper limit of the life-span.

Consider now the probability that a new life, aged 0, will survive to attain age x. We may regard this probability as a function of x, and will refer to it as the *survival function*, $s(x)$.

What properties may we attribute to this function from our general knowledge of the normal mortality pattern described above? In the first place, it is clear that $s(x)$ is a decreasing function as x increases, since the probability of surviving to age x is greater than the probability of surviving to age $x + t$, for $t > 0$.

Secondly, since we are confining our attention to normal patterns of mortality, it is convenient and reasonable to assume that $s(x)$ is a continuous function of x. In addition, we know two specific values of $s(x)$. When $x = 0$, $s(x)$ has the value 1, and at the upper end of the life-span, $s(x) = 0$. If we denote by the symbol ω the smallest value of x for which $s(x)$ vanishes, we can express this last condition as $s(\omega) = 0$, where ω is some value probably near 100. Summarizing, $s(x)$ *is a continuous function of* x, *defined on the interval* $0 \leqslant x \leqslant \omega$, *which decreases from the value* $s(0) = 1$ *to the value* $s(\omega) = 0$.

The student should realize that the designation of a terminal age ω is merely a convenient simplifying device. No fact of experience supports the assumption that a life can survive for n years but not for n years and one second. On the other hand, the probabilities of survival to advanced ages are extremely small, and in practical work it is helpful to be able to assume that these probabilities become negligible at a certain specified age. It would therefore be more realistic to state that the values of $s(x)$ are negligible for $x \geqslant \omega$. However, we shall retain the more precise condition $s(\omega) = 0$ because of its convenience in the mathematical analysis that follows.

A typical $s(x)$ curve, derived from experience, is shown in Figure 1. It is difficult to find a mathematical expression, involving a small number of parameters, which will fit this curve closely over its entire range. If, however, we sacrifice the requirement of close fit, we can find many simple functions which satisfy the conditions on $s(x)$ summarized above. Let us then, for purposes of illustration, regard as a survival function any mathematical expression that satisfies the above analytic conditions, whether or not its graph closely follows the typical $s(x)$ curve of Figure 1. Our objective here is not so much to reproduce the normal mortality pattern as to illustrate through the use of simple functions the manner in which quantitative measurements of mortality may be obtained when a particular mathematical law of survival is assumed.

The simplest type of expression satisfying the required conditions on $s(x)$ is a linear function. Consider, for example,

$$s(x) = 1 - \frac{x}{105}, \qquad 0 \leqslant x \leqslant 105.$$

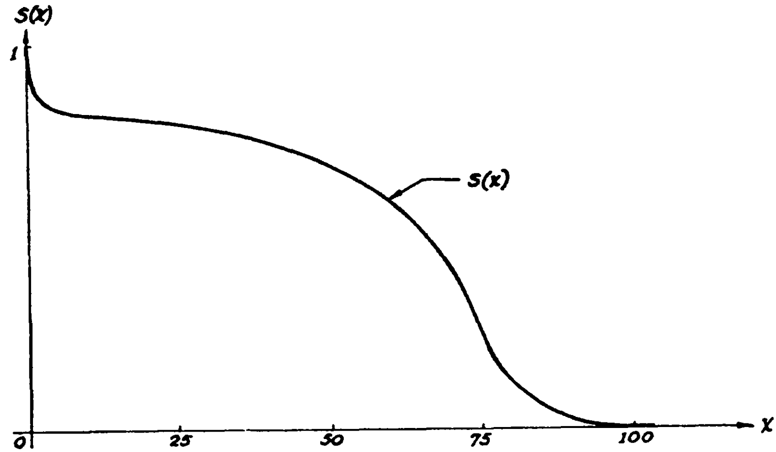

FIG. 1

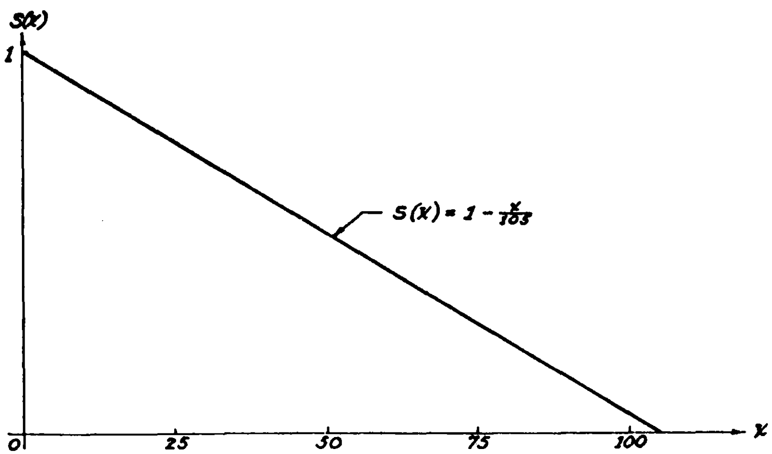

FIG. 2

We note first that the conditions for a survival function are satisfied, since this expression is continuous and decreasing, and has the value 1 for $x = 0$ and the value 0 for $x = 105$. The upper limit of life ω is thus 105 in this case. The graph is shown in Figure 2.

We proceed to obtain some sample mortality measurements from the above function. The probability that a life aged 0 will survive to age 15 is

$$s(15) = 1 - 15/105 = 1 - 1/7 = 6/7.$$

The probability of surviving from birth to age 42 is

$$s(42) = 1 - 42/105 = 1 - 2/5 = 3/5.$$

Then the probability that a life aged 0 will die between age 15 and age 42 will be

$$s(15) - s(42) = 6/7 - 3/5 = 9/35.$$

What is the probability p that a life aged 15 will survive to age 42? The probability of surviving from birth to age 42 is equal to the probability of the compound event of surviving to age 15 and then surviving from age 15 to age 42. Hence,

$$s(42) = s(15)\cdot p,$$

$$\text{and } p = \frac{s(42)}{s(15)} = 7/10.$$

Then the probability that a life aged 15 will die before attaining age 42 is $1 - 7/10 = 3/10$.

As a second example, let

$$s(x) = 1/10\sqrt{100 - x}, \qquad 0 \leqslant x \leqslant 100.$$

See Figure 3. Some sample probabilities computed from this function follow.

The probability of surviving from birth to age 36 is

$$s(36) = 1/10\sqrt{100 - 36} = 8/10 = 4/5.$$

The probability of surviving from birth to age 64 is

$$s(64) = 1/10\sqrt{100 - 64} = 6/10 = 3/5.$$

The probability that a life aged 36 will survive to age 64 is

$$\frac{s(64)}{s(36)} = 3/4.$$

The probability that a life aged 0 will die between ages 36 and 64 is $4/5 - 3/5 = 1/5$.

The probability that a life aged 36 will die before attaining age 64 is $1 - 3/4 = 1/4$.

It should be clear from these examples that numerical measures of the probabilities of living and dying are readily calculated in any instance where a survival function is known or can be assumed.

3. The mortality table

Suppose that we wish to exhibit the effect of the survival function

$$s(x) = \tfrac{1}{10}\sqrt{100 - x}$$

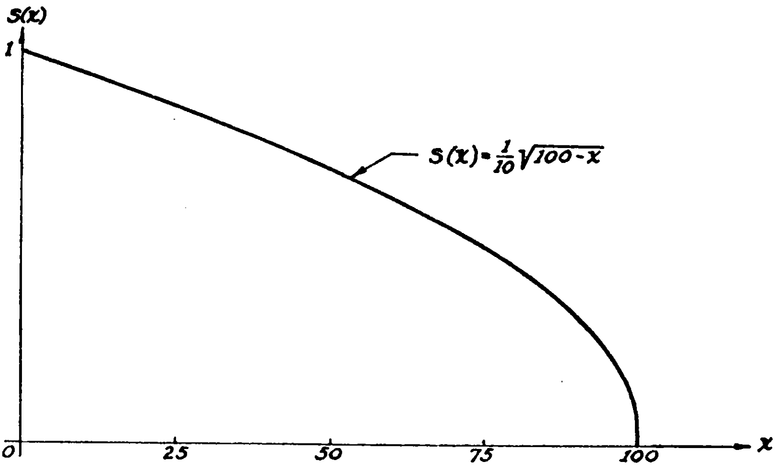

FIG. 3

on an assumed group of 100,000 newly-born lives. Now the number λ_1 of lives surviving to age 1 is a random variable with expected value

$$\begin{aligned} E(\lambda_1) &= 100{,}000\, s(1) \\ &= 100{,}000 \cdot \frac{\sqrt{99}}{10} \\ &= 99{,}499. \end{aligned}$$

We denote this expected number of survivors by l_1. We can now compute the expected number of lives dying in the first year of life as $100{,}000 - 99{,}499 = 501$. Similarly, the expected number of lives surviving to age 2 is

$$l_2 = 100{,}000\, s(2) = 100{,}000 \cdot \frac{\sqrt{98}}{10} = 98{,}995,$$

and the expected number of lives dying in the second year of life is $99{,}499 - 98{,}995 = 504$. Continuing this process, we have Table 1, where the figures in the l_x column can be thought of as representing survivors at each age x and the d_x figures as representing deaths in the year of age x to $x + 1$.

TABLE 1

SECTION OF HYPOTHETICAL MORTALITY TABLE WITH $s(x) = \frac{1}{10}\sqrt{100 - x}$, $k = 100{,}000$

Age x	l_x	d_x	Age x	l_x	d_x
0	100,000	501	6	96,954	517
1	99,499	504	7	96,437	520
2	98,995	506	8	95,917	523
3	98,489	509	9	95,394	526
4	97,980	512	10	94,868	528
5	97,468	514			

This device for exhibiting mortality data is known as the *mortality table* or the *life table*. The l_x and d_x functions have the following definitions:

$$l_x = k \cdot s(x), \text{ where } k \text{ is a positive constant,} \tag{1.1}$$

$$d_x = l_x - l_{x+1}\,. \tag{1.2}$$

The value of k is called the *radix* of the table. It corresponds to the value l_0 and is generally taken to be some large round number.

The interpretation of l_x as a "number living" or "number surviving" and of d_x as a "number dying" is a convenient aid in visualizing many of the relations that follow. It should be borne in mind, however, that neither l_x nor d_x has any absolute meaning; the values of both are dependent on the value of the radix chosen in the construction of the table. It should also be remembered that l_x, from definition (1.1), is a continuous function of x, although tabulated values appear in mortality tables only for integral values of x.

Probabilities of death and survival may be obtained directly from the l_x and d_x columns of the mortality table, and the relations exhibited below may be readily verified on the basis of elementary probability concepts. The symbol (x) is used hereafter to denote "a life aged x."

The probability that (x) will survive at least 1 year, that is, to attain age $x + 1$, is denoted by p_x, and

$$p_x = \frac{l_{x+1}}{l_x}. \tag{1.3}$$

The probability that (x) will survive to age $x + n$ is denoted by ${}_np_x$, and

$${}_np_x = \frac{l_{x+n}}{l_x}. \tag{1.4}$$

The probability that (x) will die within 1 year is denoted by q_x, and

$$q_x = 1 - p_x = 1 - \frac{l_{x+1}}{l_x} = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}. \tag{1.5}$$

The probability that (x) will die within n years is denoted by ${}_nq_x$, and

$${}_nq_x = 1 - {}_np_x = 1 - \frac{l_{x+n}}{l_x} = \frac{l_x - l_{x+n}}{l_x} = \frac{1}{l_x}\sum_{y=x}^{x+n-1} d_y. \tag{1.6}$$

The probability that (x) will survive for n years and die in the $(n + 1)$-th year is denoted by ${}_{n|}q_x$, and

$${}_{n|}q_x = \frac{d_{x+n}}{l_x}. \tag{1.7}$$

The probability that (x) will die between ages $x + n$ and $x + n + m$ is denoted by ${}_{n|m}q_x$, and

$${}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}. \tag{1.8}$$

In formula (1.4), when $x = 0$, ${}_np_x$ becomes the survival function: ${}_np_0 = s(n)$.

It should be noted that the above probabilities can be evaluated from the mortality table only if x, n, and m have integral values.

Probabilities like ${}_{2\frac{1}{4}}p_{24}$ and ${}_{7}q_{21\frac{3}{4}}$ can be computed exactly from the survival function, but they can only be estimated from mortality table data.

Although we have defined the mortality table in terms of its underlying $s(x)$ law, in practice these tables are usually constructed on a purely empirical basis from statistical studies of mortality data. When this is done, the values of q_x are developed from a mortality investigation, and these values are used to construct the l_x and d_x columns of the mortality table by the following process.

An arbitrary value is chosen for l_0.
Then

$$\begin{aligned} d_0 &= l_0 \cdot q_0 , \\ l_1 &= l_0 - d_0 , \\ d_1 &= l_1 \cdot q_1 , \\ l_2 &= l_1 - d_1 , \end{aligned}$$

and generally,

$$\begin{aligned} d_x &= l_x \cdot q_x , \\ l_{x+1} &= l_x - d_x . \end{aligned}$$

It often happens in such studies that there is insufficient data to define q_x reliably for the younger ages, and in many instances the purpose for which the table is being prepared is such that there is no need to include values for these ages even if available. In such cases, it is customary to start the table at the earliest significant age, perhaps 10 or 20, and the arbitrarily chosen value of l_x at that age will be called the radix of the table.

When mortality tables are constructed in this way, it is not usually possible to find a simple mathematical expression for $s(x)$, and the mortality table itself then constitutes the entire definition of the mortality pattern. We shall nevertheless find it convenient in the further development of the theory to keep in mind the existence of an underlying continuous $s(x)$, even when we do not know its exact mathematical form.

An example of a table constructed from experience with $s(x)$ unknown is given in Table 2. This table is based on the mortality during the years 1959–61 of white male lives resident in the United

TABLE 2

Mortality Table for U. S. White Males 1959–61

x	q_x	l_x	d_x	x	q_x	l_x	d_x
0	.02592	100,000	2,592	40	.00332	92,427	306
1	.00153	97,408	149	41	.00368	92,121	339
2	.00101	97,259	99	42	.00409	91,782	376
3	.00081	97,160	78	43	.00454	91,406	415
4	.00069	97,082	67	44	.00504	90,991	458
5	.00062	97,015	60	45	.00558	90,533	505
6	.00057	96,955	55	46	.00617	90,028	556
7	.00053	96,900	52	47	.00686	89,472	613
8	.00049	96,848	47	48	.00766	88,859	681
9	.00045	96,801	43	49	.00856	88,178	754
10	.00042	96,758	40	50	.00955	87,424	835
11	.00042	96,718	40	51	.01058	86,589	916
12	.00047	96,678	46	52	.01162	85,673	995
13	.00059	96,632	56	53	.01264	84,678	1,071
14	.00075	96,576	73	54	.01368	83,607	1,144
15	.00093	96,503	90	55	.01475	82,463	1,216
16	.00111	96,413	107	56	.01593	81,247	1,295
17	.00126	96,306	121	57	.01730	79,952	1,383
18	.00139	96,185	134	58	.01891	78,569	1,486
19	.00149	96,051	143	59	.02074	77,083	1,598
20	.00159	95,908	153	60	.02271	75,485	1,714
21	.00169	95,755	162	61	.02476	73,771	1,827
22	.00174	95,593	167	62	.02690	71,944	1,935
23	.00172	95,426	163	63	.02912	70,009	2,039
24	.00165	95,263	157	64	.03143	67,970	2,136
25	.00156	95,106	149	65	.03389	65,834	2,231
26	.00149	94,957	141	66	.03652	63,603	2,323
27	.00145	94,816	137	67	.03930	61,280	2,409
28	.00145	94,679	137	68	.04225	58,871	2,487
29	.00149	94,542	141	69	.04538	56,384	2,559
30	.00156	94,401	147	70	.04871	53,825	2,621
31	.00163	94,254	154	71	.05230	51,204	2,678
32	.00171	94,100	161	72	.05623	48,526	2,729
33	.00181	93,939	170	73	.06060	45,797	2,775
34	.00193	93,769	180	74	.06542	43,022	2,815
35	.00207	93,589	194	75	.07066	40,207	2,841
36	.00225	93,395	210	76	.07636	37,366	2,853
37	.00246	93,185	229	77	.08271	34,513	2,855
38	.00270	92,956	251	78	.08986	31,658	2,844
39	.00299	92,705	278	79	.09788	28,814	2,821

States. It is one of a series of tables constructed in connection with the national census of 1960. Although the original table includes data through age 109, the values are shown in Table 2 only up through age 79. A mortality table covering the entire range of ages, the Commissioners 1958 Standard Ordinary Mortality table for male lives, may be seen in Table A of Appendix I. In referring to this table, we shall usually abbreviate the name to 1958 CSO table. The complete published table shows values for female lives also, but only the values for male lives are shown in Table A. We shall, unless otherwise specified, assume male mortality when reference is made, in the text or in the exercises, to either of the mortality tables described in this paragraph.

4. The force of mortality

The probability functions defined in the preceding section are useful for measuring mortality over specified *intervals* of time. The function q_x, for example, which measures the probability of dying within one year, may be regarded as an index of the *average* mortality effective over the year of age x to $x + 1$. In this sense, q_x is commonly known as the annual rate of mortality.

It is evident, however, that the intensity of mortality is varying at each *moment* of age, and it is important to have some way of measuring this *instantaneous* variation. The graph of Figure 1, with a change in the vertical scale, may be thought of as an l_x curve. Now the slope of such a curve at any point is related to the number of deaths near that point, for the steeper the curve the greater the number of deaths. The slope is measured by the derivative of the function, and it is therefore natural to study next the derivative of l_x.

We take first a concrete case with a known $s(x)$ so that the differentiation can be actually carried out.

Let $s(x) = \frac{1}{10}\sqrt{100 - x}$ and let the radix be 10,000 so that

$$l_x = 1000\sqrt{100 - x}.$$

Denoting the derivative of l_x with respect to x by Dl_x, we have

$$Dl_x = -\frac{500}{\sqrt{100 - x}},$$

and this measures the rate of decrease of l_x with respect to x. At age 51, for example,

$$Dl_x\Big]_{x=51} = -\frac{500}{\sqrt{100-51}} = -71,$$

to the nearest integer. This means that at age 51 l_x is decreasing at the rate of 71 lives per year.

As a measure of the death rate, this figure is unsatisfactory since it depends upon the number of lives which are subject to the risk of death at age 51, namely, $l_{51} = 7000$. By dividing 71 by 7000, we obtain at age 51 a mortality index of $71/7000 = .0101$, a result which is independent of the number of lives attaining age 51.

The mortality index just described is known as the *force of mortality* and is denoted by the symbol μ_x. Its definition is

$$\mu_x = -\frac{Dl_x}{l_x}. \tag{1.9}$$

An equivalent expression is

$$\mu_x = -\frac{Ds(x)}{s(x)}.$$

The following properties should be kept in mind: (1) μ_x is a measure of the mortality *at the precise moment of attaining age* x; (2) μ_x expresses this mortality in the form of an *annual rate.*

The values of μ_x are not confined to the interval $0 \leqslant \mu \leqslant 1$, in the way that probability functions are restricted, but may assume any positive value. It can be seen from formula (1.9) that μ_x will exceed unity whenever the numerical value of the slope of the l_x curve exceeds the corresponding value of l_x. This normally occurs at both ends of the life-span, in the first few days following birth, and again in the year of age preceding age ω. In fact, as x approaches age ω, l_x approaches 0 and μ_x becomes infinite.

It is now possible to write an expression for the value of l_x at a particular age y in terms of the function μ_x. It will be seen from formula (1.9) that an alternative expression for μ_x is

$$\mu_x = -D \log l_x. \qquad (1.10)^1$$

Replacing x by y and integrating both sides between the limits 0

[1] The symbol *log* is used here to denote a natural logarithm. Common logarithms will be denoted by $\log_{10}$.

and x,

$$\int_0^x \mu_y\,dy = -\int_0^x D\log l_y\,dy$$
$$= -\log l_y\Big]_0^x = -\log\frac{l_x}{l_0},$$

and hence we have

$$l_x = l_0 e^{-\int_0^x \mu_y\,dy}. \tag{1.11}$$

It is now easy to obtain an expression for ${}_np_x$. From (1.11), we have

$$l_{x+n} = l_0 e^{-\int_0^{x+n} \mu_y\,dy}.$$

Then

$${}_np_x = \frac{l_{x+n}}{l_x} = \frac{l_0 e^{-\int_0^{x+n} \mu_y\,dy}}{l_0 e^{-\int_0^x \mu_y\,dy}} = e^{-\int_x^{x+n} \mu_y\,dy}.$$

In the definite integral, the values of μ_y are needed only on the interval $x \leqslant y \leqslant x + n$, and this suggests the change of variable $y = x + t$, $0 \leqslant t \leqslant n$. We thus have

$${}_np_x = e^{-\int_0^n \mu_{x+t}\,dt}. \tag{1.12}$$

The probability ${}_nq_x$ may then be expressed as

$${}_nq_x = 1 - e^{-\int_0^n \mu_{x+t}\,dt}. \tag{1.13}$$

An alternative expression may be derived for this probability from formula (1.9). Writing the formula as

$$l_y\mu_y = -Dl_y,$$

and integrating between the limits x and $x + n$, we obtain

$$\int_x^{x+n} l_y\mu_y\,dy = -\int_x^{x+n} Dl_y\,dy$$
$$= -\,l_y\Big]_x^{x+n} = l_x - l_{x+n},$$

or, with a change of variable,

$$l_x - l_{x+n} = \int_0^n l_{x+t}\,\mu_{x+t}\,dt. \tag{1.14}$$

Dividing by l_x, we have the probability ${}_nq_x$:

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} = \frac{1}{l_x}\int_0^n l_{x+t}\mu_{x+t}\,dt = \int_0^n {}_tp_x\mu_{x+t}\,dt. \quad (1.15) $$

The probability ${}_{n|m}q_x$ may similarly be expressed in definite integral form as

$$ {}_{n|m}q_x = \int_n^{n+m} {}_tp_x\mu_{x+t}\,dt. \quad (1.16) $$

The form of the integrand in formulas (1.15) and (1.16) is easily remembered since the differential expression ${}_tp_x\mu_{x+t}\,dt$ may be thought of as the probability that (x) will survive for t years and then die at the instant of attaining age $x + t$. When this expression is integrated between limits for t, it gives the probability of dying within the corresponding period of time.

The role played by the force of mortality in the mathematics of life contingencies is analogous to that played by the force of interest in compound interest theory. This will be apparent in the duality exhibited below.

Interest	*Mortality*
An original sum of A_0 is subject to the incremental effect of the varying (or constant) force of interest δ.	An original group of l_x lives is subject to the decremental effect of the varying force of mortality μ.
Let A_t be the amount of the sum at time t.	Let l_{x+t} be the number of lives in the group at time t.
The rate of change of A_t is given by $$\frac{dA_t}{dt} = A_t\delta_t.$$	The rate of change of l_{x+t} is given by $$\frac{dl_{x+t}}{dt} = -l_{x+t}\mu_{x+t}.$$
The total increment to A produced in a year is $$\int_0^1 A_t\delta_t\,dt = A_t\Big]_0^1 = A_1 - A_0 = I,$$	The total decrement from l_x produced in a year is $$\int_0^1 l_{x+t}\mu_{x+t}\,dt = -l_{x+t}\Big]_0^1 = -l_{x+1} + l_x = d_x,$$
and the annual increment per unit of A_0 is $\frac{I}{A_0} = i$.	and the annual decrement per unit of l_x is $\frac{d_x}{l_x} = q_x$.

Just as i represents the effective rate of interest corresponding to the operation of the varying (or constant) force δ throughout a yearly period, so q_x may be interpreted as the effective annual rate of mortality corresponding to the operation of the varying force μ throughout the year of age x to $x + 1$.

Further insight into the nature of the force of mortality may be gained by analyzing formula (1.9) in terms of the definition of the derivative. The derivative of l_x may be expressed as

$$Dl_x = \lim_{h \to 0} \frac{l_{x+h} - l_x}{h},$$

and μ_x, from (1.9), may then be written

$$\mu_x = \lim_{h \to 0} \frac{l_x - l_{x+h}}{h \cdot l_x} = \lim_{h \to 0} \frac{{}_hq_x}{h}. \tag{1.17}$$

Now the expression $\frac{{}_hq_x}{h}$ may be regarded as an annual rate of mortality based upon the mortality during the age interval x to $x + h$. For example, if $h = \frac{1}{2}$, $\frac{{}_hq_x}{h}$ becomes $2 \cdot {}_{\frac{1}{2}}q_x$, which represents *twice* the probability that (x) will die within *half* a year. As h approaches 0, the limit of this expression, the force of mortality, may be described as the nominal annual rate of mortality based upon the intensity of mortality at the *instant* of attaining age x.

It is now easy to see why the values of μ_x normally exceed 1 at the beginning and end of the mortality table. Infant mortality is high in the period immediately following birth. Considering the first 24 hours of life, for example, the value of ${}_{\frac{1}{365}}q_0$ may exceed $1/365$ so that the ratio $\frac{{}_hq_0}{h}$ exceeds 1. As shorter intervals of time are taken, the ratio may increase still more. Now consider the upper end of the mortality table. Since there are no survivors at age ω, we may write ${}_{\omega-x}q_x = 1$, which is true for all x. Now if x is such an age that $\omega - x$ is less than 1, it will follow that

$$\frac{{}_{\omega-x}q_x}{\omega - x} > 1,$$

and hence values of μ_x exceeding 1 will occur in the year of age $\omega - 1$ to ω.

Values of μ_x greater than 1 are occasionally encountered when we are dealing with hypothetical survival functions not closely related to the normal pattern of mortality. For example, if

$$l_x = (100 - x)^{101}, \qquad 0 \leqslant x \leqslant 100,$$

it is easy to verify that μ_x is greater than 1 at all ages.

5. Estimation of μ_x from the mortality table

When l_x is defined in terms of a mortality table and the underlying mathematical law is unknown, values of μ_x can be determined only approximately.

From formula (1.12) with $n = 1$,

$$p_x = e^{-\int_0^1 \mu_{x+t}\, dt},$$

whence, taking logarithms, we have

$$\log p_x = -\int_0^1 \mu_{x+t}\, dt. \tag{1.18}$$

The definite integral represents the mean value of μ between the ages x and $x + 1$. If we approximate this mean value by $\mu_{x+\frac{1}{2}}$, we have[2]

$$\mu_{x+\frac{1}{2}} \fallingdotseq -\log p_x\,. \tag{1.19a}$$

If we integrate μ_{x+t} between $t = -1$ and $t = 1$, we find

$$\int_{-1}^1 \mu_{x+t}\, dt = -\log p_{x-1} - \log p_x$$

and this is twice the mean value of μ between the ages $x - 1$ and $x + 1$. This leads to the following approximation:

$$\begin{aligned} \mu_x &\fallingdotseq -\tfrac{1}{2}(\log p_{x-1} + \log p_x) \\ &= \tfrac{1}{2}(\log l_{x-1} - \log l_{x+1}). \end{aligned} \tag{1.19b}$$

Other approximations for μ_x can be derived by the use of numerical methods. A simple approximation for Df_x is

$$Df_x \fallingdotseq \frac{1}{2h}(f_{x+h} - f_{x-h}).$$

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[2] The sign $\fallingdotseq$ is used to denote an approximation.

Using $h = 1$,

$$\mu_x = -\frac{1}{l_x} Dl_x \doteqdot \frac{l_{x-1} - l_{x+1}}{2l_x} = \frac{d_{x-1} + d_x}{2l_x}. \tag{1.20}$$

The formula is exact if l_x is a polynomial of the second degree. If we assume that l_x is a polynomial of the fourth degree, the following formula can be derived:

$$\mu_x \doteqdot \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12l_x}. \tag{1.21}$$

Other formulas may be obtained by using the relation connecting the differential operator with the finite difference operator Δ:

$$D = \log(1 + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots.$$

For example,

$$\mu_x = -\frac{1}{l_x} Dl_x = -\frac{1}{l_x}(\Delta l_x - \tfrac{1}{2}\Delta^2 l_x + \tfrac{1}{3}\Delta^3 l_x - \cdots)$$

$$= \frac{1}{l_x}(d_x - \tfrac{1}{2}\Delta d_x + \tfrac{1}{3}\Delta^2 d_x - \cdots). \tag{1.22}$$

6. Methods for fractional ages

Consider the problem of evaluating the probability ${}_{1/3}p_{30}$ that a life aged 30 will survive at least 4 months. If the mathematical law for l_x is known, we can readily compute l_{30} and $l_{30\,1/3}$, and hence determine the required probability exactly as $\dfrac{l_{30\,1/3}}{l_{30}}$. However, if l_x is defined only by a mortality table, it becomes necessary to obtain a value for $l_{30\,1/3}$ by some approximation method. Using the data of Table 2, we note that $l_{30} = 94{,}401$ and $l_{31} = 94{,}254$. The simplest solution is to interpolate linearly between these two values, obtaining $l_{30\,1/3} \doteqdot l_{30} - \tfrac{1}{3}(l_{30} - l_{31}) = 94{,}352$.

It is of course possible to improve this estimate by examining the differences of l_x in the neighborhood of l_{30} and adopting an appropriate numerical formula. The linear interpolation is widely used, however, and is considered to be sufficiently accurate for many practical purposes. Since we shall have frequent use for it in our later theory, let us examine the nature of the assumption that it involves.

The approximation which it produces may be expressed analytically as

$$l_{x+t} \doteq l_x - t(l_x - l_{x+1}) \qquad \begin{array}{l}\text{for integral } x \\ \text{and } 0 \leqslant t \leqslant 1.\end{array} \tag{1.23a}$$

Graphically, the assumption involves the replacement of the l_x curve by a polygonal graph consisting of straight-line segments joining the points at integral ages.

If we write (1.23a) as

$$l_{x+t} \doteq l_x - t \cdot d_x, \qquad 0 \leqslant t \leqslant 1, \tag{1.23b}$$

it is apparent that a t-th part of the total deaths d_x in the year of age x to $x + 1$ are assumed to occur in a t-th part of that year. In other words, we are making the assumption of a *uniform distribution of deaths throughout the year of age*. Actually, of course, the deaths in any year of age are not spread evenly throughout the year, and in any case where close accuracy is important, the size and direction of the error introduced by this assumption should be considered.

It is important to note the approximations for certain other functions that result from this assumption. The following relations follow directly from (1.23b):

$${}_tp_x \doteq 1 - t \cdot q_x \tag{1.23c}$$

and

$${}_tq_x \doteq t \cdot q_x \qquad \text{for} \qquad 0 \leqslant t \leqslant 1. \tag{1.23d}$$

A consistent approximation for the force of mortality at fractional ages may also be derived. We first write

$$\mu_{x+t} = -\frac{Dl_{x+t}}{l_{x+t}}.$$

Now under our assumption the derivative Dl_{x+t} has the constant value $-d_x$ for $0 < t < 1$, as may be seen by differentiating (1.23b) with respect to t. Then

$$\mu_{x+t} \doteq \frac{d_x}{l_{x+t}} = \frac{d_x}{l_x} \cdot \frac{l_x}{l_{x+t}} = \frac{q_x}{{}_tp_x} \doteq \frac{q_x}{1 - t \cdot q_x} \qquad \text{for} \quad 0 < t < 1. \tag{1.24a}$$

This approximation for μ_{x+t} is not uniformly good for all values of t in the interval $0 < t < 1$. It produces a good estimate when applied in the middle portion of the year of age where the slope of the assumed straight-line segment does not differ materially from the slope of the true l_x curve, but yields only a rough estimate near the ends of the interval where the slope of l_x is more greatly distorted by the nature of the underlying assumption. In fact, as t approaches 0 in (1.24a), we find that μ_{x+t} approaches q_x.

The expression ${}_tp_x\mu_{x+t}$ occurs frequently, and from (1.24a) we have

$$ {}_tp_x\mu_{x+t} \doteq q_x \quad \text{for} \quad 0 < t < 1. \tag{1.24b} $$

Another assumption which has proved valuable in dealing with mortality functions at fractional ages is the assumption that the reciprocal of l_{x+t} is a linear function of t for $0 \leqslant t \leqslant 1$:

$$ \frac{1}{l_{x+t}} \doteq \frac{1}{l_x} - t\left(\frac{1}{l_x} - \frac{1}{l_{x+1}}\right), \qquad 0 \leqslant t \leqslant 1. $$

Under this assumption, the probability ${}_{1-t}q_{x+t}$ takes on a particularly simple form:

$$ {}_{1-t}q_{x+t} \doteq (1 - t)q_x, \qquad 0 \leqslant t \leqslant 1. \tag{1.25} $$

Formula (1.25) is known as the Balducci hypothesis. It is especially useful in the theory underlying the construction of mortality tables. Its properties will be explored further in the exercises.

The student should realize that all the formulas of this section, although written as approximations, are exact mathematical relations under the assumptions specified. They have been designated as approximations because the assumptions themselves are only approximate descriptions of the usual behavior of mortality data.

7. Some famous laws of mortality

The mathematical expressions that we have been using as survival functions are admittedly artificial. Mathematicians have long been intrigued with the problem of constructing analytic functions which closely reproduce the typical l_x curve. It is a twisted, descending curve, and the presence of at least two points of inflection makes it a difficult curve to fit with any simple formula. We examine here some of the more important solutions that have been proposed.

The earliest and simplest proposal was that of Abraham de Moivre (1724) who suggested that l_x be represented by a straight line. De Moivre recognized that this was a very rough approximation. His objective, however, was the practical one of simplifying the calculation of life annuity values, which in those days was an arduous task. He recommended that his assumption be used only for the range of ages from about 12 to 86. The formula met with ready acceptance and was widely used for the purpose intended. It may be expressed as

$$l_x = k(\omega - x). \tag{1.26}$$

Figure 2 is based upon this law, with $\omega = 105$.

In 1825, Benjamin Gompertz, in a celebrated actuarial paper, examined the effect of assuming "the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals." Thus he assumed that man's power to resist death decreases at a rate proportional to itself. Since μ_x is a measure of man's susceptibility to death, Gompertz used the reciprocal $\frac{1}{\mu_x}$ to measure man's resistance to death. His assumption may be expressed as

$$D\left(\frac{1}{\mu_x}\right) = -h \cdot \frac{1}{\mu_x}$$

where h is the constant of proportionality.
Integrating,

$$\log\left(\frac{1}{\mu_x}\right) = -hx - \log B,$$

where $-\log B$ is the constant of integration.
Then, with $e^h = c$,

$$\mu_x = Bc^x. \tag{1.27}$$

It thus appears that Gompertz's hypothesis is equivalent to the assumption that the force of mortality increases in geometric progression. We may now derive Gompertz's expression for l_x, using formula (1.11).

First,

$$\int_0^x \mu_y\, dy = \int_0^x Bc^y\, dy = \frac{Bc^x}{\log c} - \frac{B}{\log c}$$
$$= -(c^x - 1)\log g = -\log g^{c^x-1},$$

where

$$\log g = -\frac{B}{\log c}.$$

Then $l_x = l_0 e^{-\int_0^x \mu_y\, dy} = l_0 e^{\log g^{c^x-1}} = l_0 g^{c^x-1}$.

This is usually written as

$$l_x = kg^{c^x}, \tag{1.28}$$

where $k = \dfrac{l_0}{g}$.

By suitable choice of the constants g and c, the formula can be made to follow closely the typical l_x curve over a considerable range of ages. In his original paper, Gompertz restricted the use of the formula to the period from about age 10 or 15 to age 55 or 60, and it is not possible to use the formula over the entire range of ages without making a change in the constants at some point, usually between ages 50 and 60. Indeed, it would not be expected that a single two-parameter formula would fit the twisted l_x curve over its entire range.

In presenting his formula, Gompertz stated, "It is possible that death may be the consequence of two generally coexisting causes: the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or increased inability to withstand destruction." In deriving his law of mortality, however, he took account of only the second of these causes. It remained for Makeham (1860) to combine the two causes. The effect of the first cause, chance, would be to make a constant addition to the Gompertz force of mortality. Hence, Makeham's assumption may be written as

$$\mu_x = A + Bc^x. \tag{1.29}$$

To derive the expression for l_x, we first evaluate

$$\int_0^x \mu_y\,dy = \int_0^x (A + Bc^y)\,dy = Ax + \frac{Bc^x}{\log c} - \frac{B}{\log c}$$
$$= -\log s^x - \log g^{c^x - 1},$$

where

$$\log s = -A \text{ and } \log g = -\frac{B}{\log c}.$$

Then $l_x = l_0 e^{-\int_0^x \mu_y\,dy} = l_0 e^{\log s^x + \log g^{c^x - 1}} = l_0 s^x g^{c^x - 1}.$

This is usually written as

$$l_x = ks^x g^{c^x}, \tag{1.30}$$

with $k = \dfrac{l_0}{g}$.

Makeham's law turned out to be a remarkable improvement on Gompertz's assumption, and the formula, with proper choice of the constants g, c, and s, can often be applied from about age 20 almost to the end of life.

Both Gompertz's and Makeham's laws possess properties which are of great practical importance in the simplification of compound probabilities involving the survival of more than one life, and these will be noted later. Because of their practical application in this connection, both laws continue to be used today. Gompertz's law was employed in the construction of the 1937 Standard Annuity Table, and Makeham's law was used in connection with the Commissioners 1941 Standard Ordinary Mortality Table and also with the Annuity Table for 1949.

Each of these laws involves a certain number of unspecified parameters, and hence each gives rise to an infinite number of different survival functions. These laws of mortality thus define only the *form* of the mathematical functions to be assumed and do not yield numerical measurements of mortality until appropriate values are chosen for the parameters. It will be found that the values of each parameter lie within a certain restricted range when the survival function follows the usual mortality pattern closely. For example, with Makeham's law,

$$\mu_x = A + Bc^x,$$

the parameters are usually confined to the ranges shown below:

$$\begin{aligned} .001 &< A < .003, \\ 10^{-6} &< B < 10^{-3}, \\ 1.08 &< c < 1.12. \end{aligned}$$

8. Select mortality tables

The mortality of a group of insured lives exhibits certain distinctive characteristics which derive from the special nature of such a group. Before an insurance policy is issued, the insurer must be satisfied that the applicant meets certain underwriting standards. Some applicants, because of health conditions or other factors, will not be offered insurance on a standard basis, and some others may be considered uninsurable. As a result of this selection process, a group of lives insured on a standard basis does not constitute a random group, but rather a select group all the members of which have initially satisfied certain criteria of insurability.

It follows that the mortality in such a group will vary not only by age, but also by the duration of the insurance. Thus, a group of lives just insured on a standard basis at age 30 will be subject to a lower rate of mortality during the following year than another group of lives aged 30 who were similarly insured a year ago and are now in the second year of insurance, and the mortality of both of these groups will be lower than that among a third group aged 30 who were insured two years ago at age 28.

If we write $q_{[x]+n}$ for the rate of mortality at attained age $x + n$ among a group of lives insured at age x, the tendency described above may be expressed mathematically by the series of inequalities

$$q_{[x]} < q_{[x-1]+1} < q_{[x-2]+2} < \cdots .$$

Normally the difference $q_{[x-n]+n} - q_{[x-n+1]+n-1}$ diminishes quite rapidly and becomes negligible for practical purposes after some years. Thus, there will probably be no appreciable variation in the mortality experienced by insured groups of the same age which have been insured for durations greater than 10 or 15 years. The period of time during which the effects of selection are still significant is called the *select period*.

This feature of the mortality among insured lives is one that must often be recognized in the construction of mortality tables. *Select mortality tables*, which show the mortality variation both by age and duration, are therefore prepared.

A completely select table would consist of a set of mortality tables, one for each age at issue. It is not usually necessary to prepare tables in this extensive form, however, since it is generally possible to assume a uniform select period for all ages at issue, and condense the table into a *select and ultimate* form. If it can be assumed, for example, that the effects of selection disappear in 3 years, the table can be shown in the select and ultimate form illustrated in Table 3. The first column, headed $l_{[x]}$, represents the number of lives insured at age x, the symbol $[x]$ denoting a life newly insured at age x. The second column, $l_{[x]+1}$, represents the number of survivors at age $x + 1$ of the $l_{[x]}$ lives insured at age x. The columns headed $l_{[x]+2}$ and l_{x+3} represent the corresponding numbers of survivors of the original $l_{[x]}$ lives at the two succeeding years of age. The select symbol $[x]$ is not used in the l_{x+3} column, since the effects of selection do not carry over into the fourth year, and l_{x+3} is therefore equally representative of the number of survivors of the $l_{[x]}$ lives insured 3 years previously, the $l_{[x-1]}$ lives insured 4 years previously, and so on, and this column constitutes an *ultimate* mortality table. The value of l_{x+4} is found directly below l_{x+3} in the ultimate column. The complete set of values for age at issue x is thus obtained by reading the select values horizontally starting with $l_{[x]}$ and the ultimate values vertically from l_{x+3}. At age 21, for example, there are 944,710 lives insured. The number of survivors at age 23 is found by reading $l_{[21]+2} = 941{,}916$ from the third column of the select portion of the table; and the number of survivors at age 25 is found by reading $l_{25} = 938{,}359$ from the ultimate column. Note that this same value for l_{25} also represents the number of survivors at age 25 of the $l_{[20]}$ lives insured at age 20 and of the $l_{[22]}$ lives insured at age 22.

In order to construct a select table in this form, it is first necessary to have values of $q_{[x]}$, $q_{[x]+1}$, and $q_{[x]+2}$ for each age at issue for the select portion, and a complete set of values of q_{x+3} for the ultimate section (assuming a 3-year select period). Suppose that the earliest age at issue is 20. A convenient radix is then assumed for $l_{[20]}$, and the complete mortality table for this age at issue is constructed using the values of $q_{[20]}$, $q_{[20]+1}$, $q_{[20]+2}$, q_{23}, q_{24}, etc.

TABLE 3

SECTION OF SELECT AND ULTIMATE TABLE

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
20	946,394	945,145	943,671	942,001	23
21	944,710	943,435	941,916	940,202	24
22	942,944	941,652	940,108	938,359	25
23	941,143	939,835	938,265	936,482	26
24	939,279	937,964	936,379	934,572	27
25	937,373	936,061	934,460	932,628	28
26	935,433	934,123	932,507	930,651	29
27	933,467	932,151	930,520	928,631	30
28	931,488	930,156	928,491	926,560	31
29	929,476	928,119	926,421	924,429	32
30	927,422	926,040	924,290	922,220	33
$[x]$	$d_{[x]}$	$d_{[x]+1}$	$d_{[x]+2}$	d_{x+3}	$x+3$
20	1,249	1,474	1,670	1,799	23
21	1,275	1,519	1,714	1,843	24
22	1,292	1,544	1,749	1,877	25
23	1,308	1,570	1,783	1,910	26
24	1,315	1,585	1,807	1,944	27
25	1,312	1,601	1,832	1,977	28
26	1,310	1,616	1,856	2,020	29
27	1,316	1,631	1,889	2,071	30
28	1,332	1,665	1,931	2,131	31
29	1,357	1,698	1,992	2,209	32
30	1,382	1,750	2,070	2,306	33
$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
20	.00132	.00156	.00177	.00191	23
21	.00135	.00161	.00182	.00196	24
22	.00137	.00164	.00186	.00200	25
23	.00139	.00167	.00190	.00204	26
24	.00140	.00169	.00193	.00208	27
25	.00140	.00171	.00196	.00212	28
26	.00140	.00173	.00199	.00217	29
27	.00141	.00175	.00203	.00223	30
28	.00143	.00179	.00208	.00230	31
29	.00146	.00183	.00215	.00239	32
30	.00149	.00189	.00224	.00250	33

This produces the entire ultimate column. The select values for ages at issue over 20 must now be filled in so that the ultimate values already obtained represent the correct number of survivors for each age at issue. This is done by working backwards from the ultimate values. To fill in the select values for age at issue 21, we compute $p_{[21]+2}$ and then obtain $l_{[21]+2} = \dfrac{l_{24}}{p_{[21]+2}}$. Similarly, $l_{[21]+1} = \dfrac{l_{[21]+2}}{p_{[21]+1}}$ and $l_{[21]} = \dfrac{l_{[21]+1}}{p_{[21]}}$.

When the complete l_x table has been constructed in this way. the d_x table is formed from the relations,

$$d_{[x]} = l_{[x]} - l_{[x]+1} ,$$

$$d_{[x]+1} = l_{[x]+1} - l_{[x]+2} ,$$

$$d_{[x]+2} = l_{[x]+2} - l_{x+3} ,$$

and in the ultimate column

$$d_{x+n} = l_{x+n} - l_{x+n+1} , \qquad n > 2.$$

Table 3 is based on the actual experience of a group of insured lives. The degree of selection inherent in this table can be seen by inspection. At age 25, for example, the select $q_{[25]}$ is seen to be just 70% of the ultimate rate q_{25}. Similarly, $q_{[25]+1}$ is 84% of q_{26}, and $q_{[25]+2}$ is 94% of q_{27}, and these ratios illustrate the way in which the influence of initial selection diminishes.

In evaluating probabilities of death or survival involving select lives, care should be taken to relate all select functions to the correct age at entry. Note the following examples based on the data of Table 3:

$${}_2p_{[22]} = \frac{l_{[22]+2}}{l_{[22]}} = \frac{940{,}108}{942{,}944}$$

$${}_5p_{[20]} = \frac{l_{25}}{l_{[20]}} = \frac{938{,}359}{946{,}394}$$

$$p_{[24]+1} = \frac{l_{[24]+2}}{l_{[24]+1}} = \frac{936{,}379}{937{,}964}$$

$${}_{2|}q_{[23]+1} = \frac{d_{26}}{l_{[23]+1}} = \frac{1{,}910}{939{,}835}$$

Although the table we have used here for illustration involves a select period of three years, it should not be inferred that this is the normal period during which the effects of selection are significant. The length of the select period depends on the nature of the underlying data, and the effects of selection may persist noticeably for many years. A number of standard American tables have 5-year select periods; one of the classical British tables was constructed with a 10-year select period; and the continuous mortality investigation carried on by the Society of Actuaries maintains a 15-year select period.

Although the above discussion has been concerned with the mortality of insured lives, the analysis is perfectly general and applies to any group in which mortality varies by duration as well as by age. Select mortality is found in the same form among annuitants as among insured lives, although in this case the effect is produced by self-selection on the part of the annuitants themselves, since usually only those persons who think themselves to be in good health will consider the purchase of an annuity.

Values of the force of mortality at select ages may be estimated from mortality table data by using the methods of Section 5. Using the analogue of formula (1.21), for example,

$$\mu_{[x]+2} \doteq \frac{8(l_{[x]+1} - l_{[x]+3}) - (l_{[x]} - l_{[x]+4})}{12 l_{[x]+2}}$$

(assuming the select period to be greater than 4 years). Note that this formula could not be used to approximate $\mu_{[x]+1}$, since it would then involve the meaningless symbol $l_{[x]-1}$. The analogue of formula (1.22) could be used in this case.

A mortality table which is constructed from records of insured lives without regard to age at issue, all durations from zero on being included, and all lives of the same attained age being grouped together, is called an *aggregate* table. Another type of mortality table is the *ultimate* table, which is constructed from records of insured lives with the experience of the first years after issue omitted in order to eliminate the effects of selection. The ultimate column of any select and ultimate table constitutes in itself an ultimate mortality table. The American Experience table, one of the earliest tables to be used extensively for insurance purposes in the United States, is a well-known example of an ultimate table.

9. The International Actuarial Notation

The International Congress of Actuaries adopted in 1898 a system of notation for actuarial literature. In addition to prescribing definite symbols for the common actuarial functions, this code indicates the notational principles to be followed in adopting symbols for new functions that may be needed. In this way, it provides for the development of a consistent actuarial notation that will always be intelligible to anyone familiar with the code.

At the end of each chapter the notational ideas first introduced in that chapter will be summarized. These summaries will consist of three parts: (1) the fundamental principles, (2) the application of these principles to the specific types of function discussed in the current chapter, (3) consideration of any exceptional points.

Fundamental notational principles used in Chapter 1

A. Actuarial functions are represented by a *principal symbol* with associated *prefixes and suffixes* which may be either *subscripts* or *superscripts*. The principal symbol expresses the general nature of the function. The prefixes and suffixes each limit some phase of the generalization, so that the completed symbol constitutes a precise definition of the function.

B. A *suffixed subscript* (appearing at the lower right-hand corner of the principal symbol) indicates the conditions relative to ages and the order of succession of the events.

C. A *prefixed subscript* (appearing at the lower left-hand corner) indicates the conditions relative to the duration of the operations and to their position with regard to time.

D. A vertical bar with a prefixed subscript indicates a period of deferment.

Application to mortality table functions

1. The following principal symbols are used in this chapter:

l, a number living;
d, a number dying;
p, a probability of living;
q, a probability of dying;
μ, a force of mortality.

2. With the above principal symbols, a suffixed subscript indicates the age of the life involved (Principle B); e.g., l_x. If the sub-

script x is enclosed in square brackets, it indicates select mortality with duration measured from age x; e.g., $d_{[x]}$. To this may be added, outside the brackets, the number of years which have elapsed since age x, so that the total suffix denotes the attained age; e.g., $d_{[x]+t}$. The symbol ω is used to indicate the first age at which there are no survivors; i.e., ω is the smallest x for which $l_x = 0$.

3. With the principal symbols p and q, a prefixed subscript denotes the time interval over which the probability is taken (Principles C and D); e.g., ${}_np_x$ and ${}_{m|n}q_x$. When $n = 1$ in these symbols, it is customarily omitted: p_x, ${}_{m|}q_x$.

4. The special symbol (x) denotes "a life aged x."

Note: The symbol $s(x)$ is not defined in the International Notation. In demographic literature, it is called the "survival factor" and is denoted by $l(x)$ or $p(x)$.

References

In addition to the general texts on life contingencies by King (1902), Spurgeon (1932), and Hooker and Longley-Cook (1953, 1957), there are numerous articles in the actuarial journals which bear on the material of this chapter. A few of these will be cited as being of particular interest either as original sources or modern interpretations. The comments below are arranged according to the sections of this chapter to which they refer. The complete reference will be found in the Bibliography at the end of the book.

1, 2. The biometric background, including the survival function, is given by Lotka (1956), Chapter 9.

3. The data for Table 2 and for Table B in the Appendix is taken from "Life Tables for 1959–61," Volume I, No. 1, National Center for Health Statistics, Public Health Service, U.S. Department of Health, Education, and Welfare. The general characteristics of these tables are discussed by Myers and Bayo (1964). The Commissioners 1958 Standard Ordinary Mortality Table is described in TSA 10 (1958), pp. 686–706. The technical methods used in the construction of such tables are treated in the text by Spiegelman (1955).

4. A mathematical treatment of the force of mortality is given by Smith (1948) and by Brillinger (1961). The relationship between the theories of compound interest and life contingencies is developed more fully by Allen (1907).

5, 6. Techniques of numerical differentiation are treated in all standard texts on finite differences and numerical analysis; e.g., Freeman (1960), Fröberg (1965). The approximations for fractional ages are examined in detail by Mereu (1961).

7. The original papers on the laws of mortality are Gompertz (1825) and Makeham (1860, 1889). Other mathematical laws are surveyed by Elston (1923). See also Wolfenden (1942), p. 77. A modern interpretation, from the point of view of statistical life testing, is given by Brillinger (1961).

8. Discussions of the effects of selection may be found in Thompson (1934), Williamson (1942), and Jenkins (1943).

9. An outline of the International Actuarial Notation, with revisions, appears in TASA 48 (1947), p. 166. It is also given by Hooker and Longley-Cook (1953, 1957).

EXERCISES

1. Introduction; 2. The survival function

1. Verify that the function $\dfrac{20{,}000 - 100x - x^2}{20{,}000}$ satisfies the conditions on $s(x)$. What is ω?

Use the function to calculate the following probabilities:

(a) the probability of surviving from birth to age 20; (Ans. .88)

(b) the probability that a life aged 20 will survive to age 40; (Ans. $9/11$)

(c) the probability that a life aged 20 will die between 30 and 40. (Ans. $17/176$)

2. Given $s(x) = 1/10\sqrt{100 - x}$, $0 < x < 100$.

(a) Find the probability that a life aged 0 will die between age 19 and age 36. (Ans. $1/10$)

(b) Find the probability that a life aged 19 will die before age 36. (Ans. $1/9$)

3. The mortality pattern of a certain population may be described as follows: Out of every 98 lives born together one dies annually until there are no survivors.

(a) Find a simple function that can be used as $s(x)$ for this population.

(b) What is the probability that a life aged 30 will survive to attain age 35? (Ans. $63/68$)

4. Given the function $\varphi(x) = -\dfrac{d}{dx}\, s(x)$. Show that

(a) $\displaystyle\int_0^\omega \varphi(y)\, dy = 1;$

(b) $\int_x^{x+n} \varphi(y)\,dy$ is the probability that a life aged 0 will die between age x and age $x + n$;

(c) $s(x) = 1 - \int_0^x \varphi(y)\,dy$.

(d) If $\varphi(x) = \frac{3}{45} x^{-1/3}$, find $s(x)$. What is the value of ω?

3. The mortality table

5. Given $s(x) = 1 - .005x - .00005x^2$, construct the l_x and d_x columns of the corresponding mortality table for ages 0–2, using a radix of 100,000.

6. The following values of q_x have been derived from a mortality experience:

x	q_x
0	.011
1	.005
2	.003

Construct the corresponding l_x and d_x columns, using a radix of 10,000.

7. Use Table 2 to calculate the following probabilities (answers may be left in fractional form):
   (a) that (35) will live at least 30 years;
   (b) that (35) will die within 30 years;
   (c) that (35) will die in the thirtieth year hence;
   (d) that (35) will die between ages 55 and 65;
   (e) that (35) will not die between 55 and 65.

8. Show that
   (a) ${}_{n|m}q_x = {}_np_x - {}_{n+m}p_x$;
   (b) ${}_{n|}q_x = {}_np_x \cdot q_{x+n}$;
   (c) ${}_{n+m}p_x = {}_np_x \cdot {}_mp_{x+n}$.

Justify each one by general reasoning.

4. The force of mortality

9. Obtain an expression for μ_x if

$$l_x = ks^x w^{x^2} g^{c^x}.$$

Is formula (1.9) or (1.10) more convenient in this case?

10. Obtain an expression for l_x if

$$\mu_x = \frac{1}{100 - x}.$$

11. Show that

(a) $$\int_0^{\omega - x} l_{x+t}\mu_{x+t}\,dt = l_x,$$

(b) $$\int_0^{\omega - x} {}_tp_x\mu_{x+t}\,dt = 1,$$

(c) $$\int_0^1 (\mu_{x+t} + \delta)\,dt = -\log vp_x,$$

where δ is the force of interest,

(d) $$\frac{\partial}{\partial x}\,{}_tp_x = {}_tp_x(\mu_x - \mu_{x+t}),$$

(e) $$\frac{\partial}{\partial t}\,{}_tp_x = -{}_tp_x\mu_{x+t}.$$

12. A life aged 40 is subject to an extra hazard during the year of age 40 to 41. If the normal probability of survival from age 40 to 41 is .992 and if the extra risk may be expressed by an addition to the normal force of mortality which decreases uniformly from .06 at the beginning of the year to 0 at the end of the year, find the probability that the life will not survive to age 41.

5. Estimation of μ_x from the mortality table

13. If $l_x = 1000\sqrt{100 - x}$, compute μ_{51} exactly and compare with the value given by formula (1.20). (Ans. $1/98$)

14. Show that (1.21) can be written as

$$\mu_x \doteq \frac{7(d_{x-1} + d_x) - (d_{x-2} + d_{x+1})}{12l_x}.$$

15. Would you expect (1.22) to produce a satisfactory value for μ_0? Explain. What additional data would you require in order to produce a better estimate?

6. Methods for fractional ages

16. The table below gives expressions for various mortality functions on the basis of two different assumptions. Verify the expressions in the table, assuming $0 < t < 1$.

	Uniform distribution of deaths	Balducci hypothesis
$_tq_x$	$t\cdot q_x$	$\dfrac{t\cdot q_x}{1-(1-t)q_x}$
$_{1-t}q_{x+t}$	$\dfrac{(1-t)q_x}{1-t\cdot q_x}$	$(1-t)q_x$
μ_{x+t}	$\dfrac{q_x}{1-t\cdot q_x}$	$\dfrac{q_x}{1-(1-t)q_x}$
$_tp_x\mu_{x+t}$	q_x	$\dfrac{q_x(1-q_x)}{[1-(1-t)q_x]^2}$

17. Assuming a uniform distribution of deaths, calculate from Table 2 the values of

(a) $_{1/4}q_{25}$,

(b) $_{5\frac{1}{2}}p_{40}$,

(c) $\mu_{50\ 1/3}$.

18. For a certain value of x it is known that $_tq_x = kt$ over the interval $0 < t < 3$, where k is a constant. Express μ_{x+2} in terms of k.

19. Given $l_{40} = 7{,}746$ and $l_{41} = 7{,}681$.

Compute $\mu_{40\ 1/4}$

(a) assuming a uniform distribution of deaths, (Ans. .00841)

(b) using the Balducci hypothesis, (Ans. .00844)

(c) using the fact that the given values are obtained from $l_x = 1000\sqrt{100-x}$. (Ans. .00837)

7. Some famous laws of mortality

20. Show that under de Moivre's law the probability $_{n|}q_x$ is independent of n.

21. For a mortality table based on Gompertz's law, evaluate

$$\int_x^{x+1} \frac{\mu_y - \mu_{y+1}}{\log p_y}\, dy$$

in terms of the parameters g and c.

22. Makeham derived a Second Law from the assumption that the force of mortality consists of three elements, one constant, another increasing in arithmetical progression throughout life, and the third increasing in geometrical progression throughout life. Show that this implies that $\mu_x = A + Hx + Bc^x$, and derive the expression $l_x = ks^x w^{x^2} g^{c^x}$, giving the relations between the constants s, w, g, c and A, H, B.

23. Find l_x if $\mu_x = \dfrac{Ac^x}{1 + Bc^x}$.

8. Select mortality tables

24. Write an expression for ${}_{2|5}q_{[30]+2}$ in terms of l, assuming a select period of 5 years.

25. Explain why the following probabilities will differ on the basis of a select table:

(a) the probability that a newly insured life aged 23 will die before attaining age 27;

(b) the probability that a life aged 23 which was insured three years ago will die before attaining age 27.

Calculate both probabilities using Table 3.

26. Consider the three following mortality tables:

Table A, a select and ultimate table;

Table B, an ultimate table consisting of the ultimate column of Table A; and

Table C, an aggregate table constructed from the data on which Table A is based.

What will be the relative magnitudes of the following functions on the basis of the three tables?

(a) ${}_np_{[x]}$ in Table A compared with ${}_np_x$ in B;

(b) $q_{[x]}$ in Table A compared with q_x in B and C;

(c) μ_x .

9. The International Actuarial Notation

27. Express each of the following by a single symbol:

(a) the probability that a life now aged 50, insured 3 years ago, will die in his 59th year, i.e., between ages 58 and 59 (assuming a 5-year select period);

(b) the probability that a new life aged 0 will die between ages 67 and 72;

(c) the number of deaths occurring between age 29 and age 30 in the third year of insurance (assuming a 3-year select period).

Miscellaneous problems

28. q_x has been defined as "the complement of the reciprocal of the antilog of the arithmetic mean of the death rates operating at each instant throughout the age interval x to $x + 1$." (Weck, 1947) Demonstrate mathematically the accuracy of the definition.

29. The probability ${}_nq_x$ may be expressed in the following different ways:

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} \tag{1.6} $$

$$ = 1 - e^{-\int_0^n \mu_{x+t}\,dt} \tag{1.13} $$

$$ = \int_0^n {}_tp_x\mu_{x+t}\,dt. \tag{1.15} $$

Assuming the survival function $s(x) = \frac{1}{100}(100 - x)$, show that each of the above formulas leads to the result $_nq_x = \dfrac{n}{100 - x}$.

30. From a standard mortality table, a second table is prepared by doubling the force of mortality of the standard table. Is the rate of mortality q_x' at any given age under the new table more than double, less than double, or exactly double the mortality rate q_x of the standard table?

31. Find μ_{36} to four significant figures, given that

$$\mu_x = k \log x, \qquad \log {}_4p_2 = -.00536426,$$

$$\log 2 = .69315, \qquad \log 3 = 1.09861.$$

(Ans. .003584

Chapter 2

LIFE ANNUITIES

1. Pure endowments

The student will recall from his study of probability that if p is the chance that a person will receive a certain payment of value K, the product Kp is said to be his expectation. If the payment is deferred for n years, the present value of the expectation is $K \cdot v^n \cdot p$, where v^n is the usual discount factor computed at an appropriate rate of interest.

Suppose now that (x) is to receive a payment of K at the end of n years if he is then living. It will be seen that this situation is similar to the one described above. In this case the probability involved is the chance that (x) will survive n years, ${}_np_x$, and the present value can be expressed as $K \cdot v^n \cdot {}_np_x$. This type of deferred payment is called an *n-year pure endowment of K*.

The present value at age x of an n-year pure endowment of 1 is denoted by ${}_nE_x$. Since ${}_nE_x = v^n {}_np_x = \dfrac{v^n l_{x+n}}{l_x}$, the value can be calculated directly from mortality and interest tables. However, the computation involved in such calculations can be reduced by the use of *commutation functions*. We first write

$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x} . $$

This produces a symmetry in numerator and denominator which suggests the definition of a commutation function

$$ D_x = v^x l_x . $$

We can then write

$$ {}_nE_x = \frac{D_{x+n}}{D_x} . \tag{2.1} $$

Note that the computation of ${}_nE_x$ from the basic mortality data involves a multiplication and a division $\left(\dfrac{v^n \cdot l_{x+n}}{l_x}\right)$, and that this is reduced by the use of commutation functions to a single division $\left(\frac{D_{x+n}}{D_x}\right)$. It will be found that commutation functions produce even greater simplifications when used to evaluate functions which are more complex in form than ${}_nE_x$. Values of D_x, and of other commutation functions which will be defined as the need for them arises, are tabulated for all values of x for all mortality tables that form the basis of extensive monetary calculations.

The importance of commutation functions as a labor-saving device has been diminished to some extent by the wide-spread use of digital computers. When an entire series of calculations can be carried out automatically at high speed, the additional time saved by the use of commutation functions is sometimes unimportant. Aside from their computational advantages, however, commutation functions are useful in simplifying the construction and manipulation of actuarial formulas.

${}_nE_x$ may be called the *net single premium* at age x for an n-year pure endowment of 1. It represents the single payment which (x) should make in return for an insurer's promise to pay him the sum of 1 at the end of n years if he is then alive. The word *net* indicates that no provision is being made for the expense involved in the transaction.

Example

What is the net single premium at age 35 for a 20-year pure endowment of \$1000 according to the Commissioners 1958 Standard Ordinary Mortality Table with 3% interest?

Referring to the D_x columns for 3% in Table A of Appendix I, we find

$$1000\ {}_{20}E_{35} = \frac{1000D_{55}}{D_{35}} = \frac{1000(1{,}639{,}329.7)}{3{,}331{,}295.4}$$
$$= \$492.10.$$

It is illuminating to check this result from elementary principles. Referring to the 1958 CSO table, we note that l_{35} equals 9,373,807. Suppose that each of 9,373,807 lives at age 35 contributes \$492.10 to a fund which is left at 3% interest until the end of 20 years, at which time the accumulation is divided equally among the survivors at age 55. If our previous calculation is correct, the proceeds should be sufficient to provide $1000 for each of the survivors.

Original lives at age 35	9,373,807
Contribution of each member	× 492.10
Total amount of original fund	4,612,850,425
20-year accumulation factor	× $(1.03)^{20}$
Total amount after 20 years	8,331,319,894
Survivors at age 55	÷ 8,331,317
Share of each survivor	1,000.00

This calculation illustrates what it means to accumulate a fund *with benefit of survivorship*. The share of each individual who survives to age 55 includes a portion of the shares which are forfeited by those who die before attaining age 55, so that the share of a survivor includes a greater increment than it would receive from interest alone. At 3% interest, a sum of $492.10 accumulates in 20 years to $888.79. With benefit of interest *and survivorship*, $492.10 at age 35 accumulates in 20 years to $1000, using the 1958 CSO 3% basis.

Generalizing, we may say that a sum of ${}_nE_x$ at age x accumulates in n years with benefit of interest and survivorship to the amount of 1. This may be expressed alternatively by saying that ${}_nE_x$ is the present value at age x of a payment of 1 at the end of n years discounted with interest and survivorship. Thus, we may use ${}_nE_x$ as a discount factor for interest and survivorship, corresponding to v^n for interest alone. Similarly, we may use $\frac{1}{{}_nE_x}$ as an accumulation factor for interest and survivorship, corresponding to $(1 + i)^n$ for interest alone.

2. Life annuities with annual payments

A *life annuity* of 1 payable annually to (x) is a series of annual payments of 1 commencing at the end of one year if (x) is then living and continuing throughout his lifetime. Its present value, denoted by a_x, may be expressed as the sum of a series of pure endowment values:

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \cdots + {}_{\omega-x-1}E_x$$
$$= \sum_{t=1}^{\omega-x-1} {}_tE_x .$$

The summation may also be written in the two forms:

$$a_x = \sum_{t=1}^{\omega-x-1} v^t {}_tp_x = \sum_{t=1}^{\omega-x-1} \frac{D_{x+t}}{D_x} .$$

By defining a new commutation function,

$$N_x = \sum_{t=0}^{\omega-x-1} D_{x+t} ,$$

the formula becomes

$$a_x = \frac{N_{x+1}}{D_x} . \tag{2.2}$$

Commutation functions here confer a very real advantage, for without their use it would be necessary to compute the value of a_x from the unwieldy expression $a_x = \sum_{t=1}^{\omega-x-1} \frac{v^t l_{x+t}}{l_x}$. In formula (2.2), by the use of commutation functions, this calculation is reduced to a single division. Complete columns of N_x are easily tabulated for this purpose by summing the D_x values from the oldest age back to the youngest age. This convenient continued process is based on the relation $N_x = N_{x+1} + D_x$. Once this preliminary tabulation has been performed, the value of a_x can readily be determined for any desired age.

Another form of life annuity has payments limited to a specified maximum period. This is the *n-year temporary life annuity*, which provides payments at the end of each year for n years if (x) survives. If the annual payment is 1, its present value, denoted by $a_{x:\overline{n}|}$ (or by ${}_na_x$), may be expressed as follows:

$$a_{x:\overline{n}|} = \sum_{t=1}^{n} {}_tE_x = \sum_{t=1}^{n} \frac{D_{x+t}}{D_x} .$$

Since $\sum_{t=1}^{n} D_{x+t}$ is the difference between N_{x+1} and N_{x+n+1}, we have

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x} . \tag{2.3}$$

A third type is the *n-year deferred life annuity*. This is a life annuity with the first n payments omitted. Thus, an n-year deferred life annuity to (x) provides payments commencing at age $x + n + 1$. Its present value, denoted by ${}_{n|}a_x$, may be expressed as follows:

$$ {}_{n|}a_x = \sum_{t=n+1}^{\omega-x-1} {}_tE_x = \frac{1}{D_x} \sum_{t=n+1}^{\omega-x-1} D_{x+t} = \frac{N_{x+n+1}}{D_x} . \tag{2.4} $$

Formula (2.4) may be obtained in other ways. The series of payments provided by the n-year deferred life annuity is easily seen to be equivalent to the difference between the life annuity and the n-year temporary life annuity. Hence,

$$ {}_{n|}a_x = a_x - a_{x:\overline{n|}} = \frac{N_{x+1}}{D_x} - \frac{N_{x+1} - N_{x+n+1}}{D_x} = \frac{N_{x+n+1}}{D_x} . $$

A third derivation of (2.4) depends on the use of ${}_nE_x$ as a discount factor. At age $x + n$, the series of payments under the n-year deferred life annuity has the present value a_{x+n}. Discounting to age x with interest and survivorship, we have

$$ {}_{n|}a_x = {}_nE_x \cdot a_{x+n} = \frac{D_{x+n}}{D_x} \cdot \frac{N_{x+n+1}}{D_{x+n}} = \frac{N_{x+n+1}}{D_x} . $$

A series of annual payments to (x) which commences at age $x + n + 1$ and continues for m years if (x) survives is called an *n-year deferred m-year temporary life annuity*. Its present value is denoted by ${}_{n|m}a_x$, and

$$ \begin{aligned} {}_{n|m}a_x &= \sum_{t=n+1}^{n+m} {}_tE_x = \frac{1}{D_x} \sum_{t=n+1}^{n+m} D_{x+t} \\ &= \frac{N_{x+n+1} - N_{x+n+m+1}}{D_x} . \end{aligned} \tag{2.5} $$

The life annuities defined thus far provide that the first payment be made at the *end* of a payment period. Any annuity of this type is called an *immediate* annuity. When the series of payments commences at the *beginning* of the payment period, the annuity is called an *annuity-due*, and its present value is distinguished by a double dot (dieresis) over the a, for example $\ddot{a}_x$.

When payments are made annually, the payments of the annuity-due commence one year earlier than those of the corresponding immediate annuity. The following formulas may be readily verified:

$$\ddot{a}_x = \frac{N_x}{D_x}, \tag{2.6}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}, \tag{2.7}$$

$$_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x}, \tag{2.8}$$

$$_{n|m}\ddot{a}_x = \frac{N_{x+n} - N_{x+n+m}}{D_x}. \tag{2.9}$$

The following relations between immediate annuities and annuities-due will also be evident:

$$\ddot{a}_x = 1 + a_x, \tag{2.10}$$

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}, \tag{2.11}$$

$$_{n|}\ddot{a}_x = {}_{n-1|}a_x, \tag{2.12}$$

$$_{n|m}\ddot{a}_x = {}_{n-1|m}a_x. \tag{2.13}$$

Life annuities are occasionally encountered which provide that payments be made at the end of a fraction $\frac{1}{k}$ of a year and annually thereafter. These may be regarded as annuities-due deferred for $\frac{1}{k}$ years. The present value, $_{1/k|}\ddot{a}_x$, of such an annuity may be approximated by interpolating linearly between the present value $\ddot{a}_x$ of the annuity-due (first payment deferred 0 years) and the present value a_x of the immediate annuity (first payment deferred 1 year). Thus,

$$_{1/k|}\ddot{a}_x \doteq \left(1 - \frac{1}{k}\right)\ddot{a}_x + \frac{1}{k}\,a_x,$$

or, since

$$a_x = \ddot{a}_x - 1,$$

$$_{1/k|}\ddot{a}_x \doteq \ddot{a}_x - \frac{1}{k}.$$

Formulas for other types of life annuity may be obtained by a similar interpolation.

It is sometimes necessary to use a function which is obtained by examining the temporary life annuity from a somewhat different point of view. Suppose that each member of a group of lives aged x agrees to make a payment of 1 into a fund at the end of each year, provided he is then alive. If these payments are accumulated in the fund for n years, the share of each survivor in the total fund at the end of that time is known as an n-year *forborne* life annuity, denoted by $s_{x:\overline{n}|}$.

A formula for $s_{x:\overline{n}|}$ may be derived by summing the amounts to which each of the individual payments of 1 will accumulate at the end of n years with benefit of interest and survivorship. The first payment of 1 made at age $x + 1$ will accumulate to $\dfrac{1}{{}_{n-1}E_{x+1}}$ at age $x + n$; the second payment will accumulate to $\dfrac{1}{{}_{n-2}E_{x+2}}$; and so on. Thus, the total accumulation for a life surviving at age $x + n$ is

$$s_{x:\overline{n}|} = \sum_{t=1}^{n} \frac{1}{{}_{n-t}E_{x+t}} = \sum_{t=1}^{n} \frac{D_{x+t}}{D_{x+n}} = \frac{N_{x+1} - N_{x+n+1}}{D_{x+n}}. \qquad (2.14)$$

It will be seen that $s_{x:\overline{n}|}$ represents the accumulated value at the end of n years of the series of payments for which $a_{x:\overline{n}|}$ denotes the present value at age x, and, if this present value is accumulated for n years, the value of $s_{x:\overline{n}|}$ is obtained.

$$\begin{aligned} s_{x:\overline{n}|} &= a_{x:\overline{n}|} \cdot \frac{1}{{}_nE_x} \\ &= \frac{N_{x+1} - N_{x+n+1}}{D_x} \cdot \frac{D_x}{D_{x+n}} = \frac{N_{x+1} - N_{x+n+1}}{D_{x+n}}. \end{aligned}$$

The accumulated value of n annuity-due payments may be similarly obtained. The formula is

$$\ddot{s}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_{x+n}}. \qquad (2.15)$$

The older symbol ${}_nu_x$ has been commonly used in place of $\ddot{s}_{x:\overline{n}|}$.

It should be noted that all of the annuity formulas derived in this section are of the general form

$$\frac{N_y - N_{y+n}}{D_x}, \tag{2.16}$$

where y is the age at which the first payment is due,
n is the number of payments, and
x is the age at which the value of the payments is desired.

It is thus unnecessary to memorize individual formulas for each different type. The present value, or net single premium, for a 10-year deferred 15-year temporary life annuity to (36), for example, may be immediately written down by formula (2.16). Noting that the first annuity payment will be due at age 47, we have $y = 47$, $n = 15$, and $x = 36$, so that

$$_{10|15}a_{36} = \frac{N_{47} - N_{62}}{D_{36}}.$$

If the accumulated value of the annuity at age 62 is required, we simply write

$$\frac{N_{47} - N_{62}}{D_{62}}.$$

In calculating life annuity values from select tables, it is convenient to use select commutation functions. The values of D can be computed in the usual way. If the select period is k years, the formulas are

$$D_{[x]+t} = v^{x+t}l_{[x]+t} \qquad \text{for} \qquad 0 \leqslant t < k,$$

and
$$D_{x+t} = v^{x+t}l_{x+t} \qquad \text{for} \qquad t \geqslant k.$$

The N columns are then obtained by summing the D values for each age at issue:

$$N_{[x]+t} = \sum_{r=t}^{k-1} D_{[x]+r} + \sum_{r=k}^{\omega-x-1} D_{x+r} \qquad \text{for} \qquad 0 \leqslant t < k,$$

and
$$N_{x+t} = \sum_{r=t}^{\omega-x-1} D_{x+r} \qquad \text{for} \qquad t \geqslant k.$$

Formulas for select annuity values may then be conveniently expressed in commutation symbols. For example, consider an n-year deferred life annuity to $[x]$:

$$
\begin{aligned}
{}_{n|}a_{[x]} &= \frac{N_{[x]+n+1}}{D_{[x]}} \qquad \text{for} \qquad n < k - 1 \\
&= \frac{N_{x+n+1}}{D_{[x]}} \qquad \text{for} \qquad n \geqslant k - 1.
\end{aligned}
$$

In the following pages, the word *annuity* will normally refer to a *life* annuity. Whenever life contingencies are not involved, the term *annuity-certain* will be used.

3. Annuities payable more frequently than once a year

In practice, life annuities are often payable on a semi-annual, quarterly, or monthly basis. The present value of an immediate life annuity of 1 payable m times a year to a life aged x is denoted by $a_x^{(m)}$. This series of payments, contingent on the survival of (x), consists of a first payment of $\frac{1}{m}$ at age $x + \frac{1}{m}$ followed by similar payments at intervals of $\frac{1}{m}$ years. The present value may be written

$v^{\frac{x+t}{m}}$ $l_{\frac{x+t}{m}}$

$$
a_x^{(m)} = \frac{1}{m} \sum_{t=1}^{\infty} {}_{t/m}E_x = \frac{1}{mD_x} \sum_{t=1}^{\infty} D_{x+t/m}.
$$

The upper limit of $m(\omega - x)$ has been written as ∞. This is a convenient device which is often used to indicate that a summation extends to the upper limit of the life-span. The value of the summation is not affected, since $D_{x+t/m} = 0$ for $t \geqslant m(\omega - x)$.

If it is necessary to evaluate $a_x^{(m)}$ in terms of mortality table data for which the mathematical form of l_x is unknown, as is usually the case in practice, a difficulty arises in that $D_{x+t/m}$ is not defined when $x + \frac{t}{m}$ is fractional. However, the expression $\sum_{t=1}^{\infty} D_{x+t/m}$ can be approximated in several ways. Using Woolhouse's formula, for example, we may write

$$
\frac{1}{mD_x} \sum_{t=1}^{\infty} D_{x+t/m}
$$

$$
= \frac{1}{D_x} \left[\sum_{t=1}^{\infty} D_{x+t} + \frac{m-1}{2m} D_x + \frac{m^2 - 1}{12m^2} \cdot \frac{dD_x}{dx} - \cdots \right].
$$

To simplify the expression, we must find the derivative of D_x. Using elementary calculus,

$$\frac{dD_x}{dx} = v^x \frac{dl_x}{dx} + l_x \frac{dv^x}{dx}.$$

Since the derivative of l_x is $-l_x\mu_x$ and the derivative of v^x is $v^x \log v = -v^x\delta$, we have

$$\begin{aligned} \frac{dD_x}{dx} &= -v^x l_x \mu_x - v^x l_x \delta \\ &= -v^x l_x(\mu_x + \delta) \\ &= -D_x(\mu_x + \delta). \qquad (2.17) \end{aligned}$$

With this substitution, the first three terms of the Woolhouse expression for $a_x^{(m)}$ reduce to

$$a_x^{(m)} \doteq a_x + \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta). \qquad (2.18)$$

In practice, the approximation given by the first two terms of (2.18) is usually regarded as sufficiently accurate, and the resulting simple formula

$$a_x^{(m)} \doteq a_x + \frac{m-1}{2m} \qquad (2.19)$$

is almost universally used. Formula (2.19) can also be derived[1] under the simple assumption that D_{x+t} is a linear function for $0 \leqslant t \leqslant 1$.

In deriving formulas for deferred and temporary annuities payable m times a year, we will consider only the practical approximations which correspond to formula (2.19) for $a_x^{(m)}$.

For the deferred annuity payable m times a year,

$$_{n|}a_x^{(m)} = {}_nE_x \cdot a_{x+n}^{(m)} \doteq {}_nE_x\left(a_{x+n} + \frac{m-1}{2m}\right),$$

whence

$$_{n|}a_x^{(m)} \doteq {}_{n|}a_x + \frac{m-1}{2m} \cdot {}_nE_x. \qquad (2.20)$$

For the temporary annuity payable m times a year,

$$a_{x:\overline{n|}}^{(m)} = a_x^{(m)} - {}_{n|}a_x^{(m)} \doteq \left(a_x + \frac{m-1}{2m}\right) - \left({}_{n|}a_x + \frac{m-1}{2m} \cdot {}_nE_x\right),$$

[1] See Exercise 9.

whence $$a_{x:\overline{n}|}^{(m)} \doteq a_{x:\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x). \tag{2.21}$$

The corresponding annuity-due formulas may now be easily derived. The annuity-due payable m times a year differs from the immediate annuity $a_x^{(m)}$ only by the amount of the initial payment of $\frac{1}{m}$. Thus, $\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m} \doteq a_x + \frac{m+1}{2m}$, or, since $a_x = \ddot{a}_x - 1$,

$$\ddot{a}_x^{(m)} \doteq \ddot{a}_x - \frac{m-1}{2m}. \tag{2.22}$$

For the deferred annuity-due payable m times a year,

$${}_{n|}\ddot{a}_x^{(m)} = {}_nE_x \cdot \ddot{a}_{x+n}^{(m)} \doteq {}_nE_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right),$$

whence $${}_{n|}\ddot{a}_x^{(m)} \doteq {}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x. \tag{2.23}$$

For the temporary annuity-due payable m times a year,

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_{n|}\ddot{a}_x^{(m)} \doteq \left(\ddot{a}_x - \frac{m-1}{2m}\right) - \left({}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x\right),$$

whence $$\ddot{a}_{x:\overline{n}|}^{(m)} \doteq \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x). \tag{2.24}$$

In terms of commutation functions, these formulas appear as follows:

Immediate Annuities

(2.19) $$a_x^{(m)} \doteq \frac{N_{x+1} + \frac{m-1}{2m} D_x}{D_x}$$

(2.20) $${}_{n|}a_x^{(m)} \doteq \frac{N_{x+n+1} + \frac{m-1}{2m} D_{x+n}}{D_x}$$

(2.21) $$a_{x:\overline{n}|}^{(m)} \doteq \frac{N_{x+1} - N_{x+n+1} + \frac{m-1}{2m}(D_x - D_{x+n})}{D_x}$$

Annuities-Due

(2.22)
$$\ddot{a}_x^{(m)} \doteq \frac{N_x - \frac{m-1}{2m} D_x}{D_x}$$

(2.23)
$$_{n|}\ddot{a}_x^{(m)} \doteq \frac{N_{x+n} - \frac{m-1}{2m} D_{x+n}}{D_x}$$

(2.24)
$$\ddot{a}_{x:\overline{n|}}^{(m)} \doteq \frac{N_x - N_{x+n} - \frac{m-1}{2m}(D_x - D_{x+n})}{D_x}$$

4. Continuous annuities

When the frequency of payment m becomes infinite, the resulting annuity is called a *continuous* annuity. We define

$$\bar{a}_x = \lim_{m \to \infty} a_x^{(m)},$$

and call $\bar{a}_x$ the present value of a continuous life annuity of 1 to (x). We may regard the continuous annuity as being payable momently throughout the year in such a way that the total annual payment is 1. This is admittedly an artificial concept. It has, however, considerable theoretical interest, and is useful in affording an approximation to annuities payable as often as weekly or daily.

Since $\bar{a}_x = \lim_{m \to \infty} a_x^{(m)}$, we may write

$$\bar{a}_x = \lim_{m \to \infty} \frac{1}{m} \sum_{t=1}^{\infty} {}_{t/m}E_x .$$

Since the right-hand side is equivalent to a definite integral, we have

$$\bar{a}_x = \int_0^\infty {}_tE_x \, dt \tag{2.25a}$$

and this may be written in the alternative forms

$$\bar{a}_x = \frac{1}{D_x} \int_0^\infty D_{x+t} \, dt \tag{2.25b}$$

$$= \int_0^\infty v^t {}_tp_x \, dt. \tag{2.25c}$$

It will be recalled that the upper limit of ∞ is used to indicate that the summation extends to the end of the life-span. Thus the symbol ∞ is used as an abbreviation for a more complicated finite expression, and the integrals in such formulas as (2.25) need not be regarded as improper integrals.

The values of $\bar{a}_x$ cannot be directly calculated from the above formulas unless we are given a survival function which is simple enough to permit the integration in (2.25c) to be carried out. This is not usually the case, and an approximate method must be used. The simplest device is to let m become infinite in the formulas already derived for $a_x^{(m)}$. From (2.18) and (2.19) we thus obtain

$$\bar{a}_x \doteq a_x + \tfrac{1}{2} - \tfrac{1}{12}(\mu_x + \delta) \tag{2.26}$$

and

$$\bar{a}_x \doteq a_x + \tfrac{1}{2}, \tag{2.27}$$

formula (2.27) being the usual practical approximation.

The calculation of continuous annuities is facilitated by the preparation of special commutation columns. We define

$$\bar{D}_x = \int_0^1 D_{x+t}\,dt \tag{2.28}$$

and

$$\bar{N}_x = \sum_{t=0}^{\infty} \bar{D}_{x+t}. \tag{2.29a}$$

It will then be seen that $\bar{N}_x = \displaystyle\int_0^{\infty} D_{x+t}\,dt,$ (2.29b)

so that we may write

$$\bar{a}_x = \frac{\bar{N}_x}{\bar{D}_x}. \tag{2.30}$$

In computing the values of $\bar{D}_x$ and $\bar{N}_x$, it is necessary to adopt an approximation for formula (2.28). If we assume that D_{x+t} is linear for $0 \leqslant t \leqslant 1$, we have

$$\int_0^1 D_{x+t}\,dt \doteq D_{x+1/2} \doteq \tfrac{1}{2}(D_x + D_{x+1}),$$

and

$$\bar{D}_x \doteq \tfrac{1}{2}(D_x + D_{x+1}).$$

Then $\bar{N}_x \doteq \frac{1}{2}(N_x + N_{x+1}) = N_x - \frac{1}{2}D_x = N_{x+1} + \frac{1}{2}D_x$.
This assumption produces values of $\bar{a}_x$ which are consistent with formula (2.27).

The various types of continuous annuity can now be expressed in both integral and commutation form:

$$\bar{a}_x = \int_0^\infty v^t\, {}_tp_x\, dt = \frac{\bar{N}_x}{D_x}; \tag{2.31}$$

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t\, {}_tp_x\, dt = \frac{\bar{N}_x - \bar{N}_{x+n}}{D_x}; \tag{2.32}$$

$$_{n|}\bar{a}_x = \int_n^\infty v^t\, {}_tp_x\, dt = \frac{\bar{N}_{x+n}}{D_x}. \tag{2.33}$$

We shall frequently need the derivatives of actuarial functions which are expressed as integrals. It is helpful to recall the following calculus principle:

$$\text{If}\quad F(x) = \int_a^x f(y)\, dy, \quad \text{then}\quad \frac{dF(x)}{dx} = f(x). \tag{2.34a}$$

$$\text{If}\quad F(x) = \int_x^b f(y)\, dy, \quad \text{then}\quad \frac{dF(x)}{dx} = -f(x). \tag{2.34b}$$

An application can be made to the function $\bar{N}_x$. Since formula (2.29b) can be written as

$$\bar{N}_x = \int_x^\infty D_y\, dy, \tag{2.35}$$

we use (2.34b) to find

$$\frac{d\bar{N}_x}{dx} = -D_x. \tag{2.36}$$

We can now find the derivative of $\bar{a}_x$:

$$\frac{d\bar{a}_x}{dx} = \frac{d}{dx}\left(\frac{\bar{N}_x}{D_x}\right) = \frac{1}{(D_x)^2}\left(D_x \cdot \frac{d\bar{N}_x}{dx} - \bar{N}_x \cdot \frac{dD_x}{dx}\right).$$

Substituting from (2.17) and (2.36) and simplifying, we have

$$\frac{d\bar{a}_x}{dx} = \bar{a}_x(\mu_x + \delta) - 1. \tag{2.37}$$

5. Varying annuities

Annuity forms are frequently encountered under which the rate of payment varies from time to time. These are known as *varying* annuities.

The present value of a varying annuity may usually be expressed as a combination of the present values of certain level annuities. Consider, for example, a varying annuity to (x) which provides annual payments of 1 for n years followed by payments of 2 for the balance of the lifetime of (x). Its present value is obviously equal to the sum of the present values of a life annuity of 1 and an n-year deferred life annuity of 1, $a_x + {}_{n|}a_x$; or it may be expressed as the sum of an n-year temporary annuity of 1 plus an n-year deferred annuity of 2, $a_{x:\overline{n}|} + 2\,{}_{n|}a_x$.

As a second example, let us suppose that a varying annuity-due provides annual payments of 1 for the first n years, payments of 2 for the next m years, and payments of 1 again for the balance of the lifetime of (x). This series of payments is evidently the same as would be provided by a life annuity-due of 1 plus an n-year deferred m-year temporary annuity-due of 1, $\ddot{a}_x + {}_{n|m}\ddot{a}_x$.

Special symbols have been assigned to certain common forms of varying annuity in which the payment pattern follows some regular law. The case in which the annual payments increase or decrease in arithmetical progression is of particular importance.

The symbol $(Ia)_x$ denotes the present value at age x of an *increasing life annuity* which provides payments of 1 at age $x + 1$, 2 at age $x + 2$, 3 at age $x + 3$, increasing by 1 for each year that (x) survives. Thus,

$$(Ia)_x = \sum_{t=1}^{\infty} tv^t{}_tp_x . \tag{2.38}$$

This pattern of payments is clearly equivalent to that provided by the following series of level annuities: an annuity of 1 (a_x), an annuity of 1 deferred 1 year $({}_{1|}a_x)$, an annuity of 1 deferred 2 years $({}_{2|}a_x)$, and so on. From this point of view,

$$\begin{aligned}(Ia)_x &= \sum_{t=0}^{\infty} {}_{t|}a_x \\ &= \sum_{t=0}^{\infty} \frac{N_{x+t+1}}{D_x} .\end{aligned}$$

If now we define the new commutation function,

$$S_x = \sum_{t=0}^{\infty} N_{x+t} ,$$

the formula may be written $(Ia)_x = \dfrac{S_{x+1}}{D_x}$. (2.39)

For the *n-year temporary increasing life annuity* under which the increasing payments cease with the n-th payment of n, the value may similarly be expressed as a sum of n deferred temporary annuities, each ceasing with the payment due at age $x + n$:

$$(Ia)_{x:\overline{n|}} = \sum_{t=0}^{n-1} {}_{t|n-t}a_x = \sum_{t=0}^{n-1} \frac{N_{x+t+1} - N_{x+n+1}}{D_x}$$

$$= \frac{S_{x+1} - S_{x+n+1} - nN_{x+n+1}}{D_x}. \tag{2.40}$$

Another type of increasing annuity provides increases of 1 each year until the n-th year with payments remaining constant at n for the remainder of the lifetime of (x). The present value, denoted by $(I_{\overline{n|}}a)_x$, may be expressed as the sum of the values of n deferred annuities:

$$(I_{\overline{n|}}a)_x = \sum_{t=0}^{n-1} {}_{t|}a_x = \sum_{t=0}^{n-1} \frac{N_{x+t+1}}{D_x}$$

$$= \frac{S_{x+1} - S_{x+n+1}}{D_x}. \tag{2.41}$$

A *temporary decreasing life annuity* which provides a payment of n at age $x + 1$ decreasing by 1 each year and ceasing with the n-th payment of 1 may be expressed as a sum of n level temporary annuities:

$$(Da)_{x:\overline{n|}} = \sum_{t=1}^{n} a_{x:\overline{t|}} = \sum_{t=1}^{n} \frac{N_{x+1} - N_{x+t+1}}{D_x}$$

$$= \frac{nN_{x+1} - (S_{x+2} - S_{x+n+2})}{D_x}. \tag{2.42}$$

The formulas for the corresponding varying annuities-*due* are easily obtained from the above immediate annuity formulas by simply making each numerator refer to an age one year younger. The two following examples may be readily verified:

$$(I\ddot{a})_x = \frac{S_x}{D_x}$$

$$(D\ddot{a})_{x:\overline{n}|} = \frac{nN_x - (S_{x+1} - S_{x+n+1})}{D_x}.$$

Although the formulas are useful in certain special cases, it should be noted that the formula for any varying annuity, with regular or irregular variations, may be developed from first principles by expressing the benefit as a sum of pure endowments.

When varying annuities are payable more frequently than once a year, two different cases arise depending on whether the *rate* of payment is constant or varies during each year of age. If the rate of payment during the year is constant, the total annual payment is made in m *equal* instalments with increases or decreases occurring only at the beginning or the end of the year. A varying annuity of this type may be expressed as a sum of a series of level annuities, and it is consequently possible to obtain approximate formulas for varying annuities payable m times a year by an extension of the process used in approximating the values of level annuities payable m times a year. For example, an increasing annuity payable m times a year is equivalent to a series of successively deferred level annuities payable m times a year. Thus,

$$(Ia)_x^{(m)} = \sum_{t=0}^{\infty} {}_{t|}a_x^{(m)} \doteq \sum_{t=0}^{\infty} \frac{N_{x+t+1} + \frac{m-1}{2m} D_{x+t}}{D_x}$$

$$= \frac{S_{x+1} + \frac{m-1}{2m} N_x}{D_x}. \tag{2.43}$$

The second type of varying annuity with m payments per year provides that the *rate* of payment increase or decrease m times a year. Suppose that an increasing annuity is payable at the rate of $\frac{1}{m}$ per annum at the end of the first $\frac{1}{m}$ years, $\frac{2}{m}$ per annum at the end of the second $\frac{1}{m}$ years, and so on. If this annuity is issued at age x, the first payment due at age $x + \frac{1}{m}$ will then be $\frac{1}{m^2}$, since this payment covers a period of $\frac{1}{m}$ years and the rate of payment during this first period is $\frac{1}{m}$ per annum. Then the second payment at age $x + \frac{2}{m}$ will be $\frac{2}{m^2}$, and so on. Denoting the present value of such an annuity by $(I^{(m)}a)_x^{(m)}$, we have

$$(I^{(m)}a)_x^{(m)} = \frac{1}{D_x}\sum_{t=1}^{\infty}\frac{t}{m^2}D_{x+\frac{t}{m}}.$$

Expanding this summation by means of Woolhouse's formula as far as the term involving the first derivative, we obtain

$$(I^{(m)}a)_x^{(m)} = \frac{1}{D_x}\left\{\sum_{t=1}^{\infty} tD_{x+t} + \frac{m-1}{2m}[tD_{x+t}]_{t=0} + \frac{m^2-1}{12m^2}\left[\frac{d}{dt}(tD_{x+t})\right]_{t=0} - \cdots\right\}.$$

Since

$$\left[\frac{d}{dt}(tD_{x+t})\right]_{t=0} = [D_{x+t} - tD_{x+t}(\mu_{x+t} + \delta)]_{t=0} = D_x,$$

this simplifies to

$$(I^{(m)}a)_x^{(m)} \doteq (Ia)_x + \frac{m^2-1}{12m^2}. \tag{2.44}$$

When the frequency of payment m becomes infinite in the above formulas, expressions for *continuous* varying annuities are obtained. Corresponding to the increasing annuity under which the rate of payment is constant during each year of age, we have the continuous increasing annuity, with present value $(I\bar{a})_x$, which provides for payments at the rate of 1 per annum at each moment of time during the first year of age x to $x + 1$, at the rate of 2 per annum during the second year, and so on. Thus,

$$(I\bar{a})_x = \lim_{m\to\infty}(Ia)_x^{(m)}.$$

This function may be regarded as a sum of successively deferred level continuous annuities. Then

$$\begin{aligned}(I\bar{a})_x &= \sum_{t=0}^{\infty} {}_{t|}\bar{a}_x \\ &= \sum_{t=0}^{\infty}\frac{\bar{N}_{x+t}}{D_x},\end{aligned}$$

and, letting

$$\sum_{t=0}^{\infty} \bar{N}_{x+t} = \bar{S}_x ,$$

we have

$$(I\bar{a})_x = \frac{\bar{S}_x}{D_x}. \tag{2.45}$$

An approximate formula for $(I\bar{a})_x$ may be obtained by letting m become infinite in formula (2.43) for $(Ia)_x^{(m)}$. We thus obtain

$$\begin{aligned}(I\bar{a})_x &\risingdotseq \frac{S_{x+1} + \frac{1}{2}N_x}{D_x} = (Ia)_x + \tfrac{1}{2}\ddot{a}_x \\ &= \frac{S_x - \frac{1}{2}N_x}{D_x} = (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x.\end{aligned}$$

This result indicates that approximate values of $\bar{S}_x$ may be calculated from

$$\bar{S}_x \risingdotseq S_x - \tfrac{1}{2}N_x .$$

Corresponding to the type of increasing annuity under which the rate of payment increases during each year of age, we have the continuous increasing annuity, with present value $(\bar{I}\bar{a})_x$, which provides for payments at the rate of t per annum at the moment of attaining age $x + t$. The present value may be expressed in the following ways:

$$(\bar{I}\bar{a})_x = \lim_{m\to\infty} (I^{(m)}a)_x^{(m)} \tag{2.46a}$$

$$= \int_0^\infty tv^t{}_tp_x\, dt \tag{2.46b}$$

$$= \frac{1}{D_x}\int_0^\infty tD_{x+t}\, dt. \tag{2.46c}$$

By letting m become infinite in formula (2.44) for $(I^{(m)}a)_x^{(m)}$, the following approximate relation is obtained:

$$(\bar{I}\bar{a})_x \risingdotseq (Ia)_x + \tfrac{1}{12}. \tag{2.47}$$

6. The effect of variations in interest and mortality

We normally regard a_x as a function of x defined by a particular mortality table and rate of interest. Suppose that we change our point of view and think of the different values of a_x produced for a given x when changes in mortality and interest occur. It is important to be able to estimate the effect of such variations.

The effect of a change in interest alone can be estimated by considering the derivative of a_x with respect to i. We write

$$a_x = \sum_{t=1}^{\infty} v^t {}_tp_x ,$$

and then
$$\frac{d}{di} a_x = \frac{d}{di}\left[\sum_{t=1}^{\infty} v^t {}_tp_x\right] = \frac{d}{di}\left[\sum_{t=1}^{\infty} (1+i)^{-t} {}_tp_x\right]$$
$$= \sum_{t=1}^{\infty} -t(1+i)^{-t-1} {}_tp_x = -v \sum_{t=1}^{\infty} tv^t {}_tp_x ,$$

or
$$\frac{d}{di} a_x = -v(Ia)_x . \tag{2.48}$$

The negative sign reflects the fact that a_x decreases as i increases. Formula (2.48) may be expressed in differential form as

$$da_x = -v(Ia)_x\, di,$$

and, since the differential of a function is an approximation to the change in the function corresponding to a given change in the independent variable, we may write

$$\Delta a_x \doteq -v(Ia)_x \cdot \Delta i.$$

It must be remembered that the differential provides a satisfactory estimate only when the change in the independent variable (Δi) is small. This produces the following approximate formula for values of a_x at a new rate of interest j when functions are tabulated at a closely neighboring rate i:

$$a_x^j \doteq a_x^i - \frac{j-i}{1+i} (Ia)_x^i . \tag{2.49}$$

There are a number of other practical devices for approximating annuity values. If values are tabulated at several rates of interest, an interpolation may produce a satisfactory estimate of the value at an intermediate rate. More elaborate methods are also available, but these are not often applied in practice and need not be considered here.

Changes in mortality assume many different forms and we can note here only a few simple results.

The effect on a_x of a change in the rate of mortality at a single age, say $x + n$, is easily determined. We write

$$a_x = a_{x:\overline{n}|} + v^n {}_np_x a_{x+n}$$

or, since $a_{x+n} = vp_{x+n}\ddot{a}_{x+n+1}$

and $\quad p_{x+n} = 1 - q_{x+n}$,

$$a_x = a_{x:\overline{n}|} + v^{n+1} {}_np_x(1 - q_{x+n})\ddot{a}_{x+n+1} .$$

If now q_{x+n} is replaced by $q_{x+n} + c$, we can see that the change in a_x will be $-cv^{n+1} {}_np_x\ddot{a}_{x+n+1}$.

Let us next examine the effect of a constant change in the force of mortality. Since the annuity value a_x can be expressed as the sum of pure endowment values, it will be sufficient to study the effect of this change on ${}_nE_x$. Writing

$${}_nE_x = v^n {}_np_x ,$$

and substituting $\quad v^n = e^{-\delta n} = e^{-\int_0^n \delta\, dt}$

and $\quad {}_np_x = e^{-\int_0^n \mu_{x+t}\, dt}$,

we have $\quad {}_nE_x = e^{-\int_0^n (\mu_{x+t}+\delta)\, dt}. \qquad (2.50)$

If now a constant change c is made in μ_x so that a new force of mortality, μ_x' , is formed equal to $\mu_x + c$ for all x, the corresponding pure endowment ${}_nE_x'$ may be expressed as follows:

$${}_nE_x' = e^{-\int_0^n (\mu_{x+t}'+\delta)\, dt} = e^{-\int_0^n (\mu_{x+t}+c+\delta)\, dt}$$

$$= e^{-\int_0^n (\mu_{x+t}+\delta')\, dt} \qquad \text{where} \qquad \delta' = \delta + c.$$

Comparing this result with (2.50), we note that ${}_nE_x'$ can be computed using the values of μ on the original mortality basis if the force of interest is adjusted by the constant addition c. For the pure endowment, then, a constant change in the force of mortality is equivalent to the same change in the force of interest and it follows that this will also be true for annuity values.

When the mortality follows a mathematical law, the effects of changes in the various parameters can often be easily expressed.

For example, with Makeham's law,

$$\mu_x = A + Bc^x,$$

a change in the constant A, producing a constant change in the force of mortality, is equivalent in its effect on annuity values to a change in the rate of interest. A change in B may be seen to be equivalent to a constant change in the age x. For, if B assumes the new value Bc^h, then the new force of mortality μ'_x, being equal to $A + Bc^{x+h}$, is the same as μ_{x+h}. If c is changed to $c' = c^k$, we have $\mu'_x = A + Bc^{kx} = \mu_{kx}$, a result which can be visualized in terms of an expansion or contraction in age.

7. Summary of notation

Principles A through D of Chapter 1, Section 9, should be reviewed.

E. A suffixed subscript may consist of a compound symbol of the form $x{:}\overline{n|}$. This indicates that the function depends upon the joint duration of a life aged x and a term-certain of n years.

F. A suffixed superscript (appearing at the upper right-hand corner of the principal symbol) indicates the periodicity of events. It is always enclosed in parentheses or brackets.

G. When the periodicity of events becomes infinite, a horizontal bar over the principal symbol replaces the use of the suffixed superscript.

H. The dieresis above an annuity symbol is used to distinguish annuities-due from immediate annuities.

Application to pure endowment and annuity benefits

1. The following principal symbols are used:

   E, the present value of a pure endowment of 1;

   a, the present value of an annuity of 1;

   s, the accumulated amount of an annuity of 1;

   (Ia), the present value of an increasing annuity;

   (Da), the present value of a decreasing annuity.

2. The horizontal bar over an annuity symbol means that payments are made momently (Principle G). When the bar is used over the I or D of a varying annuity symbol, it indicates that the increases or decreases occur momently.

Comments. It will occasionally be found that two different symbols will specify the same function. For example, the temporary annuity is expressed by Principle E as $a_{x:\overline{n}|}$. Using Principle C, however, it may equally well be denoted by ${}_na_x$, and this symbol is occasionally encountered.

The symbol $(I^{(m)}a)$ is not defined in the International Code, and the notation for the corresponding continuous benefits is accordingly not standardized. In earlier actuarial literature the function which is here denoted by $(\bar{I}\bar{a})_x$ is represented by $(I\bar{a})_x$. To avoid confusion, the student should note carefully the definitions of these symbols when they are encountered elsewhere.

Commutation functions

$$D_x = v^x l_x \qquad \bar{D}_x = \int_0^1 v^{x+t} l_{x+t}\, dt = \int_0^1 D_{x+t}\, dt$$

$$N_x = \sum_{t=0}^{\infty} D_{x+t} \qquad \bar{N}_x = \sum_{t=0}^{\infty} \bar{D}_{x+t} = \int_0^{\infty} D_{x+t}\, dt$$

$$S_x = \sum_{t=0}^{\infty} N_{x+t} \qquad \bar{S}_x = \sum_{t=0}^{\infty} \bar{N}_{x+t}$$

$$= \sum_{t=0}^{\infty} (t+1) D_{x+t} \qquad = \sum_{t=0}^{\infty} (t+1) \bar{D}_{x+t}$$

References

2. When mortality is given by a mathematical law, annuity values can sometimes be computed directly without recourse to a mortality table. For the case of de Moivre's law, see Exercise 6 following. The case of Makeham's law is treated by Mereu (1962).

3, 4. The Woolhouse formula is derived in standard texts on numerical methods, e.g., Freeman (1960), p. 193, or Fröberg (1965), p. 214.

The errors in the approximation methods have been investigated by Mereu (1961).

5. A useful method which simplifies the derivation of formulas for complicated varying benefits has been given by Menge (1935).

6. Much ingenuity has been expended in developing methods for computing annuity values at new rates of interest when commutation columns are not available. Among the classical methods are those of Allen (1907) and Lever (1920) for ordinary annuities, and of Lidstone (1893) and Evans (1929) for varying annuities. Lidstone's method is the basis of Exercise 21. Interpolation methods are discussed by Baughman (1965). Experiments in obtaining annuity values by Monte Carlo methods using a digital computer have been described by Boermeester (1956).

EXERCISES

1. Pure endowments

1. If $s(x) = \frac{1}{10}\sqrt{100 - x}$, what is the present value at 3% of a 17 year pure endowment of 1 on a life aged 19?
(Given $v^{17} = .6050$ at 3%.) (Ans. .5378)

2. Find by the 1958 CSO table with interest at 3% the present value of a sum of \$1000 to be received by a person now aged 20 if he is alive at age 40. (Ans. \$529.41)

3. What is the value of \$1000 at age 50 accumulated to age 65 with interest at 3% and with survivorship given by the 1958 CSO table? (Ans. \$2,007.40)

2. Life annuities with annual payments

4. Show that

(a) $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$

(b) $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - {}_nE_x$.

5. A man wishes to purchase an annuity of \$1000 payable to his son, now aged 10, at the end of each year for life, with first payment at the end of his twenty-first year. What sum must he provide, assuming 3% interest and 1958 CSO mortality? (Ans. \$18,226.79)

6. Show that when de Moivre's law is assumed

$$a_x = \frac{n - \ddot{a}_{\overline{n}|}}{ni} \quad \text{where} \quad n = \omega - x.$$

7. Prove algebraically, and explain by general reasoning:

$$a_x = \sum_{t=1}^{\infty} {}_{t|}q_x \cdot a_{\overline{t}|} .$$

3. Annuities payable more frequently than once a year

8. Find the present value of an annuity providing \$500 at the end of every 3 months for life to a person now aged 50, assuming 1958 CSO mortality and 3% interest. Use the standard approximation. (Ans. \$32,065.87)

9. Show that $a_x^{(m)}$ may be expressed as

$$a_x^{(m)} = \frac{1}{mD_x} \sum_{t=0}^{\infty} \sum_{i=1}^{m} D_{x+t+\frac{i}{m}}.$$

Making the assumption $D_{x+t+\frac{i}{m}} \doteq D_{x+t} + \frac{i}{m}(D_{x+t+1} - D_{x+t})$, derive the formula $a_x^{(m)} \doteq a_x + \frac{m-1}{2m}$.

10. Give a formula in commutation functions for the net single premium for an annuity of \$10 per month to (30), first payment at age 40 and payable monthly thereafter for 10 years, using the standard approximation.

4. Continuous annuities

11. Show that $\bar{a}_{x:\overline{n}|} \doteq a_{x:\overline{n}|} + \frac{1}{2}(1 - {}_nE_x)$

$$= \tfrac{1}{2}(\ddot{a}_{x:\overline{n}|} + a_{x:\overline{n}|}).$$

12. If deaths in the year of age x to $x + 1$ are uniformly distributed over that year of age, show that

(a) $$\bar{D}_x = \left(\frac{\delta - d}{\delta^2}\right) D_x + \left(\frac{i - \delta}{\delta^2}\right) D_{x+1}$$

(b) $$\bar{a}_{x:\overline{1}|} = \frac{1}{\delta} - \frac{d}{\delta^2} - p_x\left(\frac{v}{\delta} - \frac{d}{\delta^2}\right)$$

(c) $$p_x = \frac{\bar{a}_x - \dfrac{1}{\delta} + \dfrac{d}{\delta^2}}{v\bar{a}_{x+1} - \dfrac{v}{\delta} + \dfrac{d}{\delta^2}}$$

13. If the force of mortality has the constant value μ between ages x and $x + n$, prove that

$$\bar{a}_{x:\overline{n}|} = \frac{1 - v^n {}_np_x}{\mu + \delta} = \bar{a}_{\overline{n}|}$$

at a force of interest equal to $\mu + \delta$.

14. Show that

$$\frac{d}{dx}\bar{a}_{x:\overline{n}|} = (\mu_x + \delta)\bar{a}_{x:\overline{n}|} + {}_nE_x - 1.$$

5. Varying annuities

15. Express in terms of commutation functions, using standard approximations where necessary:

(a) $\sum_{t=1}^{n} tD_{x+t}$ (b) $(I\ddot{a})_{x:\overline{n}|}$ (c) $(D\ddot{a})^{(m)}_{x:\overline{n}|}$

16. Express in commutation symbols the present value at age x of a series of 25 payments commencing at 1 at age x and increasing annually by 0.1 for 5 years, the sixth and subsequent payments being of uniform amount.

17. Express in commutation symbols the net single premium for a varying immediate annuity to a life aged x providing for annual payments of 100 the first year, 400 the second year, 700 the third year, increasing by 300 each year until 1600 is reached, after which the payments decrease by 400 per year until 400 is reached, thereafter remaining constant.

18. A varying temporary annuity to (x) provides for an initial payment of h at age y, and successive annual payments increasing by k each year until n payments have been made. Show that the value at age x is

$$\frac{hN_y + k(S_{y+1} - S_{y+n+1}) - (h + nk)N_{y+n}}{D_x}.$$

How would the formula be modified if the benefit were decreasing by k each year?

How would the formula be modified if the payments remained constant after n years?

19. Show that (2.46c) can be expressed as

$$(\bar{I}\bar{a})_x = \frac{1}{D_x}\sum_{n=0}^{\infty}\int_0^1 (t + n)D_{x+t+n}\,dt.$$

Assuming linearity for D_{x+t+n} over each year of age, show that

$$(\bar{I}\bar{a})_x \doteq (Ia)_x + \tfrac{1}{6}.$$

6. The effect of variations in interest and mortality

20. Show that $\dfrac{d\bar{a}_x}{di} = -v(\bar{I}\bar{a})_x$. What is $\dfrac{d\bar{a}_x}{d\delta}$, where δ is the force of interest?

21. (a) Show that

$$(Ia)_x = -(1 + i)\,\frac{da_x}{di}$$

$$= -\frac{(1 + i)}{\Delta i}\left(\Delta a_x - \tfrac{1}{2}\Delta^2 a_x + \tfrac{1}{3}\Delta^3 a_x - \cdots\right),$$

where the differences of a_x are taken over the interval Δi.

(b) Write the formula that you would use to estimate the value of $(Ia)_{40}$ at 2% being given the values of a_{40} at 2%, $2\tfrac{1}{2}$%, and 3%.

22. Estimate the value of a_{30} on the 1958 CSO table at 3.1% interest using only the data of Table A in Appendix I. (Ans. 22.0967)

23. Show that for annuity values a constant change of c in μ_x is equivalent to a change in the rate of interest from i to $e^c(1 + i) - 1$.

24. A life aged 40 is subject to 1958 CSO mortality except at age 45 where it will be exposed to an extra hazard producing an addition of .01 to the CSO rate of mortality. Calculate the value at age 40 of an immediate annuity of 1 per annum to this life at 3% interest. (Ans. 19.22465)

25. The values of p_x' in a new mortality table are related to the p_x values of a standard table by $p_x' = (1 + r)p_x$ for all x. Show that annuity values on the new table at rate of interest i are equal to those on the standard table at rate of interest $i' = \dfrac{i - r}{1 + r}$.

7. Summary of notation

26. Write a single symbol for each of the following:

(a) the present value at age 35 of an annuity of 1 per annum, first payment at age 42;

(b) the present value at age 35 of a 15-year temporary annuity-due of 10 per month;

(c) the present value at age 40 of an annuity of 1 payable every 6 months, first payment at the end of 3 months;

(d) the present value at age 50 of an immediate annuity payable monthly with first payment 1, second payment 2, increasing by 1 each month;

(e) the present value at age 20 of a 15-year deferred 10-year temporary continuous annuity of 1 per annum.

Miscellaneous problems

27. Write expressions for evaluating each of the following in terms of the commutation functions D, N, and S, using standard approximations where necessary:

(a) ${}_{5|}\ddot{a}_{[30]}$, assuming a 5-year select period;

(b) ${}_{7|12}a_{34}^{(4)}$

(c) ${}_{15|}\bar{a}_{20}$

(d) ${}_{1/3|}\ddot{a}_{35}$

(e) $(I\ddot{a})_{25}^{(2)}$

(f) $(\bar{I}\bar{a})_{50}$.

28. Given the following values at 3% interest, obtain an approximate value for the force of mortality at age 40:

x	a_x
39	17.711
40	17.384
41	17.052

(Ans. .00793)

29. Assuming a mortality table with a 5-year select period, derive a formula for the present value of a life annuity-due of 1 per annum, the life being select and aged 30, based on interest of 2% during the select period

and 3% thereafter. Commutation functions are available both at 2% and at 3%.

30. A substandard mortality table is derived by taking the force of mortality μ'_x to be $(1 + k)\mu_x$, where μ_x is given by a standard Gompertz table. It is found that the values of the substandard life annuities a'_x can be obtained by rating up the age in a table of the standard life annuities, thus: $a'_x = a_{x+r}$. Find an expression for the constant addition r.

CHAPTER 3

LIFE INSURANCE

1. Introduction

The functions described in the preceding chapter relate to payments which are contingent on the survival of a life. The functions now to be considered relate to payments which are contingent on death rather than survival; such payments are provided by *insurances*.

An insurance provides a payment of a specified amount upon the death of a given life, known as the *insured*. Values of insurances have traditionally been calculated on the assumption that the payment is made at the *end* of the year of death, even though the usual practice of insurers is to make payment as soon as possible after the occurrence of the death. This assumption is a convenient one in that the probabilities of death and survival to the end of a year can be calculated exactly from mortality table data. As will be seen later, there are various methods of adjusting values based on this assumption to take account of the fact that in practice the payment is made immediately; some insurers now make these adjustments directly in the original formulas and thus avoid the prior calculation of values based on the traditional assumption.

2. Insurances payable at the end of the year of death

An insurance which provides a payment only if death occurs within a limited period is known as a *term* insurance. The present value or net single premium at age x for such an insurance with term period of n years and amount 1 is denoted by $A^{1}_{x:\overline{n}|}$ and we may write

$$A^{1}_{x:\overline{n}|} = vq_x + v^2{}_{1|}q_x + v^3{}_{2|}q_x + \cdots + v^n{}_{n-1|}q_x .$$

Each term in this expression represents the probability that death will occur in a particular year multiplied by the present value of a payment of 1 at the end of that year. The expression may be written

$$A^{1}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^{t+1}{}_{t|}q_x = \frac{1}{l_x}\sum_{t=0}^{n-1} v^{t+1}\, d_{x+t} . \tag{3.1}$$

If the term period is extended to the end of the mortality table so that the insurance is payable whenever death occurs, the value of n becomes $\omega - x$, and we have the *whole life* insurance with present value A_x :

$$A_x = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_x = \frac{1}{l_x} \sum_{t=0}^{\infty} v^{t+1} d_{x+t}, \tag{3.2}$$

the symbol ∞ being used for $\omega - x - 1$ in this case.

If the payment is made only if death occurs after the expiration of a period of n years, we have the *deferred* insurance with present value

$$_{n|}A_x = \sum_{t=n}^{\infty} v^{t+1} {}_{t|}q_x = \frac{1}{l_x} \sum_{t=n}^{\infty} v^{t+1} d_{x+t}. \tag{3.3}$$

It is obvious that $A_x = A^1_{x:\overline{n}|} + {}_{n|}A_x$. In practice, the deferred insurance is not encountered alone, but $_{n|}A_x$ sometimes appears in combination with other insurance functions.

The expressions above are simplified by the introduction of the commutation functions

$$C_x = v^{x+1} d_x$$

and

$$M_x = \sum_{t=0}^{\infty} C_{x+t}.$$

Then

$$A^1_{x:\overline{n}|} = \frac{1}{l_x} \sum_{t=0}^{n-1} v^{t+1} d_{x+t} = \frac{1}{v^x l_x} \sum_{t=0}^{n-1} v^{x+t+1} d_{x+t}$$

$$= \frac{1}{D_x} \sum_{t=0}^{n-1} C_{x+t} = \frac{M_x - M_{x+n}}{D_x}, \tag{3.4}$$

$$A_x = \frac{M_x}{D_x}, \tag{3.5}$$

and

$$_{n|}A_x = \frac{M_{x+n}}{D_x}. \tag{3.6}$$

An n-year *endowment* insurance of 1 provides a payment of 1 at the end of the year of death if the insured dies within n years or a payment of 1 at the end of n years if the insured is then living. This is clearly a combination of an n-year term insurance with an n-year pure endowment. Hence, its value $A_{x:\overline{n}|}$ can be expressed as $A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$ or

$$A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{3.7}$$

The pure endowment value is frequently given an insurance symbol, $A_{x:\overline{n}|}^{\,1}$. Then, $A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\,1}$.

3. Relations between insurances and annuities

Since $C_x = v^{x+1}\,d_x = v^{x+1}(l_x - l_{x+1}) = vD_x - D_{x+1}$, it follows that $M_x = vN_x - N_{x+1}$; and, dividing by D_x,

$$\frac{M_x}{D_x} = v\,\frac{N_x}{D_x} - \frac{N_{x+1}}{D_x},$$

or

$$A_x = v\ddot{a}_x - a_x. \tag{3.8}$$

It is interesting to verify this relation between the life annuity and the whole life insurance by general reasoning. The expression $v\ddot{a}_x$ represents the present value of an annuity-due of v per annum to (x), and such an annuity has the same value as an annuity of 1 payable at the end of each year that (x) enters upon. The expression a_x represents the value of an annuity of 1 payable at the end of each year that (x) completes. The difference between these two annuities is evidently a single payment of 1 at the end of the year that (x) enters upon but does not complete. Since this amounts to a payment of 1 at the end of the year of death, it has the same value as the whole life insurance, A_x.

For the n-year term insurance, the corresponding relation is

$$A^{1}_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}. \tag{3.9}$$

For the n-year endowment,

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} + {}_nE_x,$$

but

$${}_nE_x = a_{x:\overline{n}|} - a_{x:\overline{n-1}|},$$

and hence

$$A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}. \tag{3.10}$$

The student should formulate verbal interpretations for relations (3.9) and (3.10).

Formula (3.8) can be written in another form which is often useful. Since

$$v\ddot{a}_x - a_x = v\ddot{a}_x - (\ddot{a}_x - 1) = 1 - (1 - v)\ddot{a}_x = 1 - d\ddot{a}_x,$$

we have

$$A_x = 1 - d\ddot{a}_x\,. \tag{3.11}$$

This equation can be interpreted as follows: A_x is the present value of a payment of 1 at the end of the year of death of (x). If this payment were made now, its value would be 1; but since the payment is deferred, we must deduct the value of the interest which it earns between now and the end of the year of death. The value at the beginning of the year of the interest on 1 is d, and the present value of the interest for each year that (x) enters upon is $d\ddot{a}_x$. Hence, $A_x = 1 - d\ddot{a}_x$.

A similar relation for endowment insurance is $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$. There is no corresponding formula for term or deferred insurances, since in those cases a payment is not certain to be made.

Multiplying by $(1 + i)$ in formula (3.11), we have

$$(1 + i)A_x = (1 + i) - i\ddot{a}_x\,, \qquad \text{since } (1 + i)\,d = i,$$

or, since $\ddot{a}_x = 1 + a_x$,

$$1 = ia_x + (1 + i)A_x\,. \tag{3.12}$$

This reflects the fact that a unit now is equivalent to the present value of a unit at the end of the year of death, A_x, plus the present value of interest on the unit at the end of each year in the meantime (including the end of the year of death), $ia_x + iA_x$. Formula (3.12) is the usual basis of estate and inheritance tax statutes, which define ia_x as the "life estate" and $(1 + i)A_x$ as the "remainder," $(1 - ia_x)$.

4. Insurances payable at the moment of death

In practice, the payment of the amount of an insurance is not deferred to the end of the year of death. If the payment under a whole life insurance were due at the end of the $1/m$-th part of a year in which (x) dies, the expression for the net single premium would take the form

$$\begin{aligned} A_x^{(m)} &= \frac{1}{l_x}\,[v^{1/m}(l_x - l_{x+1/m}) + v^{2/m}(l_{x+1/m} - l_{x+2/m}) + \cdots] \\ &= -\frac{1}{l_x}\sum_{t=1}^{\infty} v^{t/m}\Delta l_{x+\frac{t-1}{m}}\,, \end{aligned}$$

where the interval of differencing is $1/m$. As m becomes infinitely large, this expression approaches the net single premium for a whole life insurance payable at the moment of death instead of at the end of the year of death. Denoting this value by $\bar{A}_x$, we have

$$\bar{A}_x = \lim_{m\to\infty} A_x^{(m)} = \lim_{m\to\infty}\left[-\frac{1}{l_x}\sum_{t=1}^{\infty} v^{t/m}\Delta l_{x+\frac{t-1}{m}}\right]$$

$$= -\frac{1}{l_x}\int_0^{\infty} v^t\, dl_{x+t}\,,$$

or, since $\qquad dl_{x+t} = -l_{x+t}\mu_{x+t}\,dt,$

$$\bar{A}_x = \frac{1}{l_x}\int_0^{\infty} v^t l_{x+t}\mu_{x+t}\,dt, \tag{3.13a}$$

or

$$\bar{A}_x = \int_0^{\infty} v^t\,{}_tp_x\mu_{x+t}\,dt. \tag{3.13b}$$

The formula may be seen to be plausible by general reasoning, since ${}_tp_x\mu_{x+t}\,dt$ can be thought of as the probability that (x) will die at the moment of attaining age $x + t$, and v^t is the discounted value of the amount of 1 then payable.

The integral in (3.13a) can be evaluated by using the technique of integration by parts:

$$\int y\,dz = yz - \int z\,dy.$$

In (3.13a), we take v^t as y and $l_{x+t}\mu_{x+t}\,dt$ as dz. Then dy is $v^t \log v = -v^t\delta$ and z is $-l_{x+t}$. We thus find

$$\bar{A}_x = \frac{1}{l_x}\left([-v^t l_{x+t}]_0^{\infty} - \delta\int_0^{\infty} v^t l_{x+t}\,dt\right),$$

which reduces to

$$\bar{A}_x = 1 - \delta\bar{a}_x\,. \tag{3.14}$$

The analogy with $A_x = 1 - d\ddot{a}_x$ is clear.

The corresponding relations for the n-year term insurance payable at the moment of death are

$$\bar{A}^{1}_{x:\overline{n|}} = \int_0^{n} v^t\,{}_tp_x\mu_{x+t}\,dt = 1 - {}_nE_x - \delta\bar{a}_{x:\overline{n|}}\,. \tag{3.15}$$

For the n-year endowment,

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + {}_nE_x = 1 - \delta\bar{a}_{x:\overline{n|}}\,. \tag{3.16}$$

When a uniform distribution of deaths is assumed, there is a particularly simple relationship between the insurance payable at the moment of death and the insurance payable at the end of the year of death. This may be seen most easily in the case of a one-year term insurance. We have

$$\bar{A}^{1}_{x:\overline{1|}} = \int_0^1 v^t\,{}_tp_x\mu_{x+t}\,dt.$$

Making the assumption of a uniform distribution of deaths, we note from (1.24b) that the expression ${}_tp_x\mu_{x+t}$ in this integral may be replaced by q_x. Then

$$\bar{A}^{1}_{x:\overline{1|}} \doteq \int_0^1 v^t q_x\,dt = q_x \int_0^1 v^t\,dt.$$

The integral $\int_0^1 v^t\,dt$ will be recognized as the continuous annuity-certain

$$\bar{a}_{\overline{1|}} = \frac{1-v}{\delta} = \frac{iv}{\delta}.$$

Hence
$$\bar{A}^{1}_{x:\overline{1|}} \doteq \frac{i}{\delta} v q_x = \frac{i}{\delta} A^{1}_{x:\overline{1|}}.$$

From a practical standpoint, it is highly convenient that a constant factor $\frac{i}{\delta}$ may be used at all ages as the approximate adjustment for changing the value of an insurance payable at the end of the year of death to an insurance payable at the moment of death.

Another approximation can be derived by general reasoning. If, instead of assuming that deaths are uniformly distributed throughout each year of age, we regard the total deaths as being concentrated at the middle of the year of age, we find that the insurance payment is made a half-year earlier on the average when paid at the moment of death than when paid at the end of the year of death. In other words, an immediate payment of 1 is equivalent on the average to a payment of $(1+i)^{\frac{1}{2}}$ at the end of the year of death. It follows that

$$\bar{A}^{1}_{x:\overline{1|}} \doteq (1+i)^{\frac{1}{2}} A^{1}_{x:\overline{1|}}$$

or, introducing a further approximation,

$$\bar{A}^{1}_{x:\overline{1}|} \doteqdot \left(1 + \frac{i}{2}\right) A^{1}_{x:\overline{1}|}\,.$$

The values of the three factors $\frac{i}{\delta}$, $(1 + i)^{\frac{1}{2}}$, and $1 + \frac{i}{2}$ differ very little from one another at the usual rates of interest. For example, when i is 3%, $\frac{i}{\delta} = 1.0149261$, $(1 + i)^{\frac{1}{2}} = 1.0148892$, and $1 + \frac{i}{2} = 1.015$.

These approximations are not restricted to the one-year term insurance. Since $\bar{A}^{1}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t\,{}_tp_x \bar{A}^{\;1}_{x+t:\overline{1}|}$, and $\bar{A}^{\;1}_{x+t:\overline{1}|} \doteqdot \frac{i}{\delta} A^{\;1}_{x+t:\overline{1}|}$, it follows that

$$\bar{A}^{1}_{x:\overline{n}|} \doteqdot \frac{i}{\delta} A^{1}_{x:\overline{n}|}\,. \tag{3.17}$$

An important special case is

$$\bar{A}_x \doteqdot \frac{i}{\delta} A_x. \tag{3.18}$$

For endowment insurance, only the value of the term insurance portion is adjusted, since the pure endowment remains payable at the end of the n-th year if (x) survives. Thus,

$$\bar{A}_{x:\overline{n}|} \doteqdot \frac{i}{\delta} A^{1}_{x:\overline{n}|} + {}_nE_x. \tag{3.19}$$

The factors $(1 + i)^{\frac{1}{2}}$ and $1 + \frac{i}{2}$ may be used in place of $\frac{i}{\delta}$ in all these forms.

Special commutation functions may be defined for use with insurances payable at the moment of death. The formula

$$\bar{A}_x = \int_0^\infty v^t\,{}_tp_x\mu_{x+t}\,dt = \frac{1}{l_x}\int_0^\infty v^t l_{x+t}\mu_{x+t}\,dt$$
$$= \frac{1}{v^x l_x}\int_0^\infty v^{x+t} l_{x+t}\mu_{x+t}\,dt$$

suggests the following definitions:

$$\bar{C}_x = \int_0^1 v^{x+t} l_{x+t}\mu_{x+t}\, dt = \int_0^1 D_{x+t}\mu_{x+t}\, dt,$$

$$\bar{M}_x = \sum_{t=0}^{\infty} \bar{C}_{x+t} = \int_0^{\infty} D_{x+t}\mu_{x+t}\, dt.$$

We may then write $\bar{A}_x = \dfrac{\bar{M}_x}{D_x}$,

$$\bar{A}^{1}_{x:\overline{n}|} = \frac{\bar{M}_x - \bar{M}_{x+n}}{D_x},$$

$$_{n|}\bar{A}_x = \frac{\bar{M}_{x+n}}{D_x},$$

and

$$\bar{A}_{x:\overline{n}|} = \frac{\bar{M}_x - \bar{M}_{x+n} + D_{x+n}}{D_x}.$$

In practice, $\bar{C}_x$ is computed by a formula consistent with the usual approximation:

$$\bar{C}_x \doteq \frac{i}{\delta} C_x \qquad \text{or} \qquad (1+i)^{\frac{1}{2}} C_x \qquad \text{or} \qquad \left(1 + \frac{i}{2}\right) C_x.$$

5. Varying insurances

An insurance which provides an increasing or decreasing death benefit is called a *varying insurance*. There are several standard types for which simple formulas may be developed. The commutation symbol R_x is used:

$$R_x = \sum_{t=0}^{\infty} M_{x+t} = \sum_{t=0}^{\infty} (t+1) C_{x+t}.$$

An *increasing whole life insurance* provides a death benefit of 1 in the first year, 2 in the second year, and so on, increasing by 1 each year to the end of life. Its value is given by the expression

$$\begin{aligned}(IA)_x &= vq_x + 2v^2{}_{1|}q_x + 3v^3{}_{2|}q_x + \cdots \\ &= \frac{1}{l_x}\sum_{t=0}^{\infty} (t+1) v^{t+1} d_{x+t} \\ &= \frac{1}{D_x}\sum_{t=0}^{\infty} (t+1) C_{x+t} \\ &= \frac{R_x}{D_x}. \end{aligned} \tag{3.20}$$

The *increasing term insurance* provides a death benefit of 1 in the first year, increasing by 1 each year, but with no payment if death occurs after n years. Its value is easily seen to be

$$(IA)^{1}_{x:\overline{n}|} = \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}. \tag{3.21}$$

A whole life insurance providing a death benefit which increases for n years only and then remains constant at n has the net single premium

$$(I_{\overline{n}|}A)_x = \frac{R_x - R_{x+n}}{D_x}. \tag{3.22}$$

The *decreasing term insurance* provides an initial death benefit of n decreasing by 1 each year, with no payment if death occurs after n years:

$$(DA)^{1}_{x:\overline{n}|} = \frac{nM_x - (R_{x+1} - R_{x+n+1})}{D_x}. \tag{3.23}$$

The relationship that exists between the increasing annuity and the increasing insurance is easily derived. Since $M_x = vN_x - N_{x+1}$, it follows that $R_x = vS_x - S_{x+1}$, and, dividing by D_x, we obtain

$$\frac{R_x}{D_x} = \frac{vS_x}{D_x} - \frac{S_{x+1}}{D_x},$$

or

$$(IA)_x = v(I\ddot{a})_x - (Ia)_x. \tag{3.24}$$

Since $(Ia)_x = (I\ddot{a})_x - \ddot{a}_x$, the relation may be transformed to

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x. \tag{3.25}$$

Formulas (3.24) and (3.25) are analogous to formulas (3.8) and (3.11) for level insurance.

If the payment under the increasing insurance is made at the moment of death, the net single premium $(I\bar{A})_x$ may be expressed as follows:

$$(I\bar{A})_x = \frac{1}{D_x}\sum_{t=0}^{\infty}(t+1)\bar{C}_{x+t} = \frac{1}{D_x}\sum_{t=0}^{\infty}\bar{M}_{x+t}.$$

Defining the commutation symbol $\bar{R}_x$ as

$$\bar{R}_x = \sum_{t=0}^{\infty}\bar{M}_{x+t},$$

we have
$$(I\bar{A})_x = \frac{\bar{R}_x}{D_x}. \tag{3.26}$$

There is another type of increasing insurance that appears occasionally in combination with annuities or other insurances. It provides increases in the death benefit at intervals which are fractions of a year. Let the death benefit be $\frac{1}{m}$ during the first $\frac{1}{m}$ years, $\frac{2}{m}$ during the second $\frac{1}{m}$ years, and so on, increasing by $\frac{1}{m}$ each $\frac{1}{m}$ years. Although the increases occur at fractions of a year, let us consider first the case where the claim payment is deferred to the end of the year of death. The net single premium at age x is then denoted by $(I^{(m)}A)_x$, where the superscript (m) indicates that the increases occur at intervals of $\frac{1}{m}$ years. An approximation for the premium may be derived by noting that the average payment in the year of age $x + t$ to $x + t + 1$ is $t + \frac{m+1}{2m}$, and we therefore have

$$\begin{aligned}
(I^{(m)}A)_x &\doteq \frac{1}{D_x}\sum_{t=0}^{\infty}\left(t + \frac{m+1}{2m}\right)C_{x+t} \\
&= \frac{R_{x+1} + \dfrac{m+1}{2m}M_x}{D_x} \\
&= \frac{R_x - \dfrac{m-1}{2m}M_x}{D_x} = (IA)_x - \frac{m-1}{2m}A_x.
\end{aligned} \tag{3.27}$$

The reasoning here is equivalent to the assumption of a uniform distribution of deaths, and the formula is exact under that assumption. In applying the formula, it must be remembered that an initial amount of $\frac{1}{m}$ is assumed. For an insurance which is initially 1 and increases by 1 every $\frac{1}{m}$ years, the net single premium is $m(I^{(m)}A)_x$.

We consider next the case where the claim payment is made at the moment of death. The net single premium at age x is then denoted by $(I^{(m)}\bar{A})_x$, and we have from (3.27)

$$(I^{(m)}\bar{A})_x \doteq \frac{\bar{R}_x - \dfrac{m-1}{2m}\bar{M}_x}{D_x} \doteq (I\bar{A})_x - \frac{m-1}{2m}\bar{A}_x. \quad (3.28)$$

If the increases in insurance occur *momently* instead of at the end of $1/m$ years, so that the death benefit is t at the moment of attaining age $x + t$, the present value, denoted by $(\bar{I}\bar{A})_x$, may be expressed as

$$\begin{aligned}(\bar{I}\bar{A})_x &= \int_0^\infty tv^t\,{}_tp_x\mu_{x+t}\,dt \\ &= \frac{1}{D_x}\int_0^\infty tD_{x+t}\mu_{x+t}\,dt.\end{aligned} \quad (3.29)$$

This may be approximated by letting m become infinite in (3.28):

$$(\bar{I}\bar{A})_x \doteq \frac{\bar{R}_x - \frac{1}{2}\bar{M}_x}{D_x} = (I\bar{A})_x - \tfrac{1}{2}\bar{A}_x. \quad (3.30)$$

It is quite easy to see that this result is a reasonable one. During the first year, the payment under $(\bar{I}\bar{A})_x$ increases from 0 to 1 and may be regarded as having an average value during that year of $\frac{1}{2}$. Similarly, the average payment during the second year is $1\frac{1}{2}$; and so on. Comparing with $(I\bar{A})_x$, which assumes a payment of 1 during the first year, 2 during the second year, and so on, it is evident that $(\bar{I}\bar{A})_x$ is less than $(I\bar{A})_x$ by approximately the value of a level insurance of $\frac{1}{2}$ payable at the moment of death. We thus have

$$(\bar{I}\bar{A})_x \doteq (I\bar{A})_x - \tfrac{1}{2}\bar{A}_x, \quad (3.31)$$

as above.

6. Summary of notation

Principles A–H of Chapters 1 and 2 should be reviewed.

I. When a compound symbol is used as a suffixed subscript, as $x:\overline{n}|$, a 1 written above one element of the subscript indicates that the event is determined upon the prior failure (or expiration) of that element. If the 1 is not used, it indicates that the event is determined upon the first failure of either element.

Applications to insurances

1. The following principal symbols are used:
   A, the present value (or net single premium) of an insurance of 1;
   (IA), the present value of an increasing insurance;
   (DA), the present value of a decreasing insurance.
2. The horizontal bar over the A of a principal insurance symbol means that the payment is made at the moment of death (Principle G); e.g., $\bar{A}_x$, $(I\bar{A})_x$.
3. The horizontal bar over the I of an increasing insurance symbol means that the increases occur momently (Principle G); e.g., $(\bar{I}\bar{A})_x$.

Comments. The symbols used here for the continuous increasing insurances are not in the International Notation. In the earlier literature, the function $(\bar{I}\bar{A})_x$ was sometimes denoted by $(I\bar{A})_x$.

Commutation functions

$$C_x = v^{x+1}\, d_x \qquad\qquad \bar{C}_x = \int_0^1 v^{x+t} l_{x+t}\mu_{x+t}\, dt = \int_0^1 D_{x+t}\mu_{x+t}\, dt$$

$$M_x = \sum_{t=0}^{\infty} C_{x+t} \qquad\qquad \bar{M}_x = \sum_{t=0}^{\infty} \bar{C}_{x+t} = \int_0^{\infty} D_{x+t}\mu_{x+t}\, dt$$

$$R_x = \sum_{t=0}^{\infty} M_{x+t} = \sum_{t=0}^{\infty} (t+1)C_{x+t} \qquad\qquad \bar{R}_x = \sum_{t=0}^{\infty} \bar{M}_{x+t} = \sum_{t=0}^{\infty} (t+1)\bar{C}_{x+t}$$

References

4, 5. The errors in the approximations have been studied by Mereu (1961).

A useful technique for deriving formulas for the more complicated varying insurances and annuities has been described by Menge (1935).

The technique of integration by parts will often be used in the chapters that follow. A paper by G. J. L. and S. E. M. (1910) gives practical hints on the application of this method to life contingency problems. The student will find this an interesting reference when he has completed more of the work in this book.

EXERCISES

1, 2. Insurances payable at the end of the year of death

1. Calculate $A^{1}_{30:\overline{10|}}$ assuming that mortality follows the law $l_x = 100 - x$ and that interest is at 4% per annum. Given v^{10} at 4% = .6756.

(Ans. .116)

2. Show that

(a) $M_x = D_x - dN_x$

(b) $$\frac{1}{D_x}\left[\sum_{t=0}^{n-1} C_{x+t}\, v^{n-t-1} + D_{x+n}\right] = v^n$$

(c) $$\sum_{t=1}^{\infty} l_{x+t}A_{x+t} = l_x a_x$$

3. Obtain a formula for the net single premium at age x for an insurance of 1 payable at the end of 20 years from the date of the policy if death occurs within that period or at the end of the year of death if death occurs after 20 years.

3. Relations between insurances and annuities

4. Find the rate of interest if $a_x = 13.257$ and $A_x = .19304$. (Ans. 6%)

5. (a) Show that $a_{x:\overline{n|}} = \dfrac{v - A_{x:\overline{n+1|}}}{d}$.

(b) Show that $\dfrac{1 - ia_{x:\overline{t-1|}}}{1+i} = \dfrac{M_x - M_{x+t} + D_{x+t}}{D_x}$.

6. An annuity-certain providing a payment of 1 at the end of each year for n years is bought by a person aged x. Prove that its value immediately after the t-th payment may be expressed in the form

$$P\cdot a_{x+t} + Q\cdot A_{x+t} + R\cdot \frac{1}{v^{x+t}}$$

and determine the values of P, Q, and R, independent of t and the mortality table.

7. Derive the relations

(a) $A^{1}_{x:\overline{n|}} = 1 - {}_nE_x - d\ddot{a}_{x:\overline{n|}}$

(b) $A_{x:\overline{n|}} = 1 - d\ddot{a}_{x:\overline{n|}}$

4. Insurances payable at the moment of death

8. Show that $A_x^{(m)} = m(v^{\frac{1}{m}}\ddot{a}_x^{(m)} - a_x^{(m)}) = 1 - d^{(m)}\ddot{a}_x^{(m)}$.

What is the effect of letting m become infinite?

9. Show that $\bar{C}_x + \delta\bar{D}_x = D_x - D_{x+1}$.

10. Show that

(a) $\dfrac{d\bar{M}_x}{dx} = -\mu_x D_x$

(b) $\dfrac{d\bar{A}_x}{dx} = \bar{A}_x(\mu_x + \delta) - \mu_x$

(c) $\dfrac{d\bar{A}^1_{x:\overline{n}|}}{dx} = \bar{A}^1_{x:\overline{n}|}(\mu_x + \delta) - \mu_x + \dfrac{D_{x+n}}{D_x}\mu_{x+n}$

(d) $\dfrac{d\bar{a}_{x:\overline{n-x}|}}{dx} = \mu_x\bar{a}_{x:\overline{n-x}|} - \bar{A}_{x:\overline{n-x}|}$

11. If a mortality table follows Makeham's Law, show that

$$\bar{A}_x = A\bar{a}_x + (\mu_x - A)\bar{a}_x',$$

where A is the usual Makeham constant and where $\bar{a}_x'$ is calculated at a special rate of interest.

5. Varying insurances

12. Express in commutation functions the net single premium at age 40 for a Double Protection to Age 65 policy which provides insurance of 2 in the event of death prior to age 65 and insurance of 1 after age 65.

13. A policy is issued at age 0 with the following graded scale of death benefits:

Age	*Death Benefit*
0	100
1	200
2	400
3	600
4	800
5–20	1,000
21 and over	5,000

Write the net single premium in terms of commutation functions.

14. Express in commutation symbols the net single premium at age 40

for an insurance providing for a payment of 1,000 the first year, 4,000 the second year, 7,000 the third year, and so on, increasing by 3,000 per year until 16,000 is reached, after which the payment decreases by 4,000 per year until it reaches 4,000 and then remains constant.

15. (a) Show that $\dfrac{dA_x}{di} = -v(IA)_x$. Hence suggest a method for approximating $(IA)_x$ when A_x is known at several rates of interest.

(b) What is $\dfrac{d\bar{A}_x}{di}$?

16. Show that

(a) $(I\bar{A})_x = \ddot{a}_x - \delta(I\ddot{a})_x$

(b) $(\bar{I}\bar{A})_x = \bar{a}_x - \delta(\bar{I}\bar{a})_x$

17. Show that

(a) $(\bar{I}\bar{a})_x = \dfrac{1}{D_x}\displaystyle\int_0^\infty \bar{N}_{x+t}\,dt$

(b) $(\bar{I}\bar{A})_x = \dfrac{1}{D_x}\displaystyle\int_0^\infty \bar{M}_{x+t}\,dt$

6. Summary of notation

18. Write the definitions of C and M for a mortality table with a 5-year select period.

19. Describe the functions denoted by the following symbols and give formulas for their evaluation in terms of commutation functions, using standard approximations where needed:

(a) $\bar{A}_{30:\overline{20}|}$

(b) $(DA)^{1}_{30:\overline{10}|}$

(c) $(I^{(4)}A)_{40}$

(d) $(\bar{I}\bar{A})_{25}$

20. The following symbols have not been specifically defined in the text. Assign meanings to these symbols in accordance with the principles of the International Notation, and derive practical formulas for their evaluation:

(a) ${}_{10|5}A_{25}$

(b) $(I_{\overline{5}|}A)^{1}_{40:\overline{15}|}$

(c) $(\bar{I}\bar{A})^{1}_{x:\overline{n}|}$

(d) $(\bar{I}A)_{45}$

Miscellaneous problems

21. (a) Determine whether or not a constant increase in the force of mortality has the same effect on A_x as the same increase in the force of interest.
    (b) Show that, if the single rate of mortality q_{x+n} is increased to $q_{x+n} + c$, then A_x will be increased by

$$cv^{n+1}{}_np_x(1 - A_{x+n+1}).$$

22. Give an expression in terms of commutation symbols and interest functions for the net single premium for an endowment for t years at age x which provides that the death benefit shall be an annuity of $\frac{1}{n}$ for n years certain, but that if the life survives to age $x + t$ the annuity shall be paid for n years certain and as long thereafter as (x) survives.

23. The net single premium for a pure endowment of $1,000 issued at a certain age and for a certain period is $700 with return of net premium in event of death during the period, or $650 with no return at death. Find the exact net single premium for a pure endowment of $1,000 issued at the same age and for the same period, if one-half of the net premium is to be returned at death during the period. (Ans. $674.07)

24. Assuming a uniform distribution of deaths, find a formula for $(\bar{I}\bar{A})^{1}_{x:\overline{1}|}$ in terms of interest and commutation symbols.

CHAPTER 4

NET ANNUAL PREMIUMS

1. Annual premiums for insurances and deferred annuities

Insurance is more commonly purchased by means of a series of periodic premium payments than by a single premium payment. These periodic payments form a life annuity-due payable by the insured to the insurer. The first premium is payable at the inception of the agreement and subsequent premiums are contingent on the insured's survival. The amount of the premium required for a given insurance may be determined from the principle that the present value of the sequence of net premiums must be equal to the present value of the insurance. In symbols,

$$P\cdot\ddot{a} = A$$

or

$$P = \frac{A}{\ddot{a}}, \tag{4.1}$$

where P is the net premium on an annual basis, $\ddot{a}$ the present value of the appropriate life annuity-due, and A the net single premium for the insurance, all functions being computed for the age at which the agreement becomes effective.

In the case of whole life, term, and endowment insurance where level premiums are paid once a year, principle (4.1) leads directly to the following expressions:

Whole life $$P_x = \frac{A_x}{\ddot{a}_x} = \frac{M_x}{N_x} \tag{4.2}$$

Term $$P^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}} \tag{4.3}$$

Endowment $$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} \tag{4.4}$$

The symbol $P_{x:\overline{n}|}^{\;\;1}$ is used for the pure endowment, and

$$P_{x:\overline{n}|}^{\;\;1} = \frac{A_{x:\overline{n}|}^{\;\;1}}{\ddot{a}_{x:\overline{n}|}} = \frac{D_{x+n}}{N_x - N_{x+n}}. \tag{4.5}$$

A relation connecting P_x and $\ddot{a}_x$ may be derived by dividing the expression $A_x = 1 - d\ddot{a}_x$ by $\ddot{a}_x$, yielding $\dfrac{A_x}{\ddot{a}_x} = \dfrac{1}{\ddot{a}_x} - d$,

or
$$P_x = \frac{1}{\ddot{a}_x} - d. \tag{4.6}$$

A similar relation for endowment premiums is

$$P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d. \tag{4.7}$$

Each of the above formulas assumes that premiums are to be paid over the entire duration of the contract; that is, for the whole of life in the case of a whole life insurance,[1] for n years in the case of an n-year endowment, and so on. Principle (4.1) also applies where the premium-paying period is less than the insurance period, as for example in the case of a whole life insurance where premiums are to be paid for a maximum period of t years. For "limited-payment" insurances of this type, with premiums payable for t years at most, the annual premiums are determined by dividing the net single premium by a t-year temporary annuity-due, as follows:

t-payment life

$$_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t}|}} = \frac{M_x}{N_x - N_{x+t}} \tag{4.8}$$

t-payment n-year term $(t < n)$

$$_tP^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}} = \frac{M_x - M_{x+n}}{N_x - N_{x+t}} \tag{4.9}$$

t-payment n-year endowment $(t < n)$

$$_tP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+t}} \tag{4.10}$$

It will be noted in the formulas above that the P symbol, with the associated prefixes and suffixes, defines not only the type of insurance but also the premium-paying period. All of these formulas apply to insurances payable at the end of the year of death. When

[1] Whole life insurance with premiums payable throughout the duration of the contract is commonly known as *ordinary life* insurance.

the insurance is payable at the moment of death, a different notational device is used to represent the net annual premium. In this case, the P symbol is used in conjunction with the net single premium symbol indicating immediate payment of claims. For example, the net annual premium for a whole life insurance payable at the moment of death is denoted by the symbol $P(\bar{A}_x)$. If premium payments are limited to t years, the symbol is ${}_tP(\bar{A}_x)$.

It is not uncommon for an insurer to issue a policy for which the annual premium is not level. Suppose, for example, that a whole life policy is issued with each premium payable during the first five years equal to one-half of each premium payable thereafter. If the initial annual premium for such a policy is denoted by P, the value of P is determined from the formula

$$P \cdot \ddot{a}_{x:\overline{5|}} + 2P \cdot {}_{5|}\ddot{a}_x = A_x ,$$

or

$$P = \frac{A_x}{\ddot{a}_{x:\overline{5|}} + 2\, {}_{5|}\ddot{a}_x} = \frac{M_x}{N_x + N_{x+5}} .$$

This is again an application of principle (4.1).

Annual premiums are also encountered in connection with deferred annuities. The annual premium, payable for t years, for an n-year deferred annuity-due of 1 to (x) is denoted by ${}_tP({}_{n|}\ddot{a}_x)$:

$$ {}_tP({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{t|}}} = \frac{N_{x+n}}{N_x - N_{x+t}} . \tag{4.11}$$

In practice, t is usually equal to n. The notational principle here is the same as that used to express the net annual premium for insurances payable at the moment of death: the P symbol is used in conjunction with the net single premium symbol.

2. Fractional premiums

An insured often elects to make his premium payments more frequently than once a year. In this case the amount of the net annual premium payable m times a year, $P^{(m)}$, is determined in accordance with principle (4.1) with the annuity factor adjusted for payments m times a year, thus:

$$P^{(m)} = \frac{A}{\ddot{a}^{(m)}} . \tag{4.12}$$

The adjustment in the annuity factor is usually made on the basis of the practical approximation illustrated in formula (2.22), $\ddot{a}_x^{(m)} \doteq \ddot{a}_x - \frac{m-1}{2m}$. This will be seen in the following applications of (4.12):

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}} \doteq \frac{A_x}{\ddot{a}_x - \frac{m-1}{2m}} = \frac{M_x}{N_x - \frac{m-1}{2m} D_x}$$

$$P_{x:\overline{n}|}^{1\,(m)} = \frac{A_{x:\overline{n}|}^{1}}{\ddot{a}_{x:\overline{n}|}^{(m)}} \doteq \frac{M_x - M_{x+n}}{N_x - N_{x+n} - \frac{m-1}{2m}(D_x - D_{x+n})}$$

$${}_tP^{(m)}({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{t}|}^{(m)}} \doteq \frac{N_{x+n}}{N_x - N_{x+t} - \frac{m-1}{2m}(D_x - D_{x+t})}$$

In using these formulas, it must be kept in mind that $P^{(m)}$ represents the *annual* amount of premium payable m times a year, and that the actual amount of each premium payment is $\frac{1}{m} P^{(m)}$.

When premium payments are made with great frequency so that m is large, as for example in weekly premium industrial insurance, it is convenient to assume that the payments are made continuously. The symbol $\bar{P}$ is used to denote a continuous premium. For whole life insurance with premiums payable for the duration of the contract, the continuous premium $\bar{P}_x$ is given by

$$\bar{P}_x = \frac{A_x}{\bar{a}_x} = \frac{M_x}{\bar{N}_x},$$

and corresponding expressions may be written for other forms of insurance. If the insurance is payable at the moment of death and premiums are payable continuously, we have

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = \frac{\bar{M}_x}{\bar{N}_x}.$$

The relation

$$\bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta$$

may also be obtained, using (3.14).

It is important to distinguish symbols like $\bar{P}_x$, $P(\bar{A}_x)$, and $\bar{P}(\bar{A}_x)$. The bar over the P symbol refers only to the continuous mode of premium payment, and the bar over the A symbol refers only to immediate payment of claims. To indicate an insurance payable at the moment of death with premiums payable continuously, a bar must appear over both P and A.

It is convenient to be able to calculate $P^{(m)}$ from the value of the corresponding annual premium P. For whole life insurance,

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}} \doteq \frac{A_x}{\ddot{a}_x - \dfrac{m-1}{2m}}.$$

Dividing numerator and denominator by $\ddot{a}_x$ and replacing $\dfrac{A_x}{\ddot{a}_x}$ by P_x, we have

$$P_x^{(m)} \doteq \frac{P_x}{1 - \dfrac{m-1}{2m}\cdot\dfrac{1}{\ddot{a}_x}}.$$

Since, from formula (4.6), $\dfrac{1}{\ddot{a}_x} = P_x + d$, we have finally

$$P_x^{(m)} \doteq \frac{P_x}{1 - \dfrac{m-1}{2m}(P_x + d)}. \tag{4.13}$$

The formula enables us to approximate $P_x^{(m)}$ when P_x and d are known.

Formula (4.13) lends itself readily to a verbal interpretation. When premiums are paid on the annual basis, the insurer receives a full year's premium in advance each year including the year in which death occurs. When fractional premiums of the $P^{(m)}$ type are payable, there are two significant differences. In the first place, the insurer earns less interest on the fractional premiums, since he receives them on the average at a later date. Secondly, he receives less premium on the average in the year of death, the fractional premiums for the balance of the year not being collectible after death has occurred. To compensate for these two differences, the amount of premium must be greater when paid on the fractional basis than when paid once a year.

Bearing these facts in mind, let us return to formula (4.13), which we now write in the following form:

$$P_x^{(m)} \doteq P_x + \frac{m-1}{2m} P_x^{(m)} d + \frac{m-1}{2m} P_x^{(m)} P_x. \qquad (4.14)$$

Here it is clear that $P_x^{(m)}$ exceeds P_x by the value of two adjustment terms. The first adjustment, $\frac{m-1}{2m} P_x^{(m)} d$, is equal to $\frac{1}{m} P_x^{(m)} \sum_{t=0}^{m-1} t \cdot \frac{d}{m}$ and represents the approximate annual loss of interest on the deferred portions of the fractional premium. The second adjustment, $\frac{m-1}{2m} P_x^{(m)} P_x$, may be interpreted as the annual premium required to provide insurance in the amount of $\frac{m-1}{2m} P_x^{(m)}$. Now $\frac{m-1}{2m} P_x^{(m)}$ is equal to $\frac{1}{m} P_x^{(m)} \sum_{t=0}^{m-1} \frac{t}{m}$ and represents the average loss in fractional premium incurred by the insurer in the year of death, on the assumption of a uniform distribution of deaths. The term $\frac{m-1}{2m} P_x^{(m)} \cdot P_x$ thus represents the adjustment required to compensate the insurer for this loss.

The above reasoning leads readily to formulas for fractional premiums for other forms of insurance. In the case of term insurance, the fractional premium $P^{(m)}_{\overset{1}{x}:\overline{n}|}$ will exceed the annual premium $P_{\overset{1}{x}:\overline{n}|}$ by (1) the approximate loss of interest adjustment $\frac{m-1}{2m} P^{(m)}_{\overset{1}{x}:\overline{n}|} d$ and (2) the approximate loss of premium adjustment $\frac{m-1}{2m} P^{(m)}_{\overset{1}{x}:\overline{n}|} P_{\overset{1}{x}:\overline{n}|}$. Hence,

$$P^{(m)}_{\overset{1}{x}:\overline{n}|} \doteq P_{\overset{1}{x}:\overline{n}|} + \frac{m-1}{2m} P^{(m)}_{\overset{1}{x}:\overline{n}|} \left(d + P_{\overset{1}{x}:\overline{n}|}\right), \qquad (4.15)$$

and

$$P^{(m)}_{\overset{1}{x}:\overline{n}|} \doteq \frac{P_{\overset{1}{x}:\overline{n}|}}{1 - \frac{m-1}{2m}\left(P_{\overset{1}{x}:\overline{n}|} + d\right)}. \qquad (4.16)$$

In the case of endowment insurance, the relations are

$$P^{(m)}_{x:\overline{n}|} \doteq P_{x:\overline{n}|} + \frac{m-1}{2m} P^{(m)}_{x:\overline{n}|} \left(d + P^{1}_{x:\overline{n}|}\right) \tag{4.17}$$

and

$$P^{(m)}_{x:\overline{n}|} \doteq \frac{P_{x:\overline{n}|}}{1 - \dfrac{m-1}{2m}\left(P^{1}_{x:\overline{n}|} + d\right)}. \tag{4.18}$$

It is important to note that in this case the loss of premium is covered by a term insurance and not by an endowment insurance. This is because the loss of premium is incurred only in the event of death, and not in the event that the insured survives to maturity.

In developing similar formulas for limited-payment insurances, it must be remembered that the loss of premium in the year of death can only occur while the policy is within the premium-paying period. Consequently, the loss of premium adjustment is covered by a term insurance for the premium-paying period. For example,

$${}_tP^{(m)}_x \doteq {}_tP_x + \frac{m-1}{2m}\, {}_tP^{(m)}_x \left(d + P^{1}_{x:\overline{t}|}\right) \tag{4.19}$$

and

$${}_tP^{(m)}_x \doteq \frac{{}_tP_x}{1 - \dfrac{m-1}{2m}\left(P^{1}_{x:\overline{t}|} + d\right)}. \tag{4.20}$$

If formulas (4.16), (4.18), and (4.20) are compared, it will be noted that they are all of the same form, with the premium in the denominator being determined by the premium-paying period of the policy. These formulas can also be derived by analytic methods similar to those used in deriving formula (4.13).

Corresponding formulas for continuous premiums can be obtained by letting m become infinite. Then $\dfrac{m-1}{2m}$ becomes $\frac{1}{2}$, and formula (4.13), for example, becomes

$$\bar{P}_x \doteq \frac{P_x}{1 - \frac{1}{2}(P_x + d)}. \tag{4.21}$$

Insurers sometimes accept premium payments m times a year with the condition that any unpaid instalments for the balance of the year of death will be deducted from the claim payment. For example, if the insured dies after paying one quarterly premium for the current year, the three unpaid instalments for the balance of the year are deducted from the policy proceeds. Premiums calculated on this basis are called *instalment* premiums and are denoted by the symbol $P^{[m]}$ to distinguish them from the $P^{(m)}$ type, which are often called *true* fractional premiums.

For an ordinary life policy with instalment premiums, the average amount of premium to be deducted from the proceeds is $\frac{m-1}{2m} P_x^{[m]}$. The net payment by the insurer in the year of death is thus $1 - \frac{m-1}{2m} P_x^{[m]}$ on the average. Hence,

$$P_x^{[m]} \doteq \frac{A_x\left(1 - \frac{m-1}{2m} P_x^{[m]}\right)}{\ddot{a}_x^{(m)}}$$

$$= P_x^{(m)}\left(1 - \frac{m-1}{2m} P_x^{[m]}\right).$$

Substituting for $P_x^{(m)}$ from (4.13) and solving, we have

$$P_x^{[m]} \doteq \frac{P_x}{1 - \frac{m-1}{2m} d}. \tag{4.22}$$

Since there is no loss of premium in the year of death in this case, it is only necessary to adjust the annual premium for loss of interest. This may be seen when (4.22) is written in the form

$$P_x^{[m]} \doteq P_x + \frac{m-1}{2m} P_x^{[m]} d. \tag{4.23}$$

Formulas analogous to (4.22) are valid for other plans of insurance.

A third type of fractional premium is the *apportionable* premium. It is the practice of some insurers to provide in their contracts that when death occurs a pro-rata premium refund will be made covering the period from the date of death to the date to which premiums are paid. For example, if death occurs one month after a quarterly premium has been paid, two-thirds of this quarterly premium is refunded with the sum insured. Premiums payable on this basis are called *apportionable*, and are denoted by the symbol $P^{\{m\}}$. If the apportionable premium is payable annually, it is denoted by $P^{\{1\}}$.

The effect of the apportionable feature is that the insurer refunds an average amount of $\frac{1}{2m} P^{\{m\}}$ as part of the policy proceeds. Thus, for ordinary life insurance, we have

$$P_x^{\{m\}} \doteq \frac{A_x\left(1 + \frac{1}{2m} P_x^{\{m\}}\right)}{\ddot{a}_x^{(m)}}$$

$$= P_x^{(m)}\left(1 + \frac{1}{2m} P_x^{\{m\}}\right).$$

Substituting from (4.13) and solving, we have

$$P_x^{\{m\}} \doteq \frac{P_x}{1 - \frac{m-1}{2m} d - \frac{1}{2} P_x}. \tag{4.24}$$

For the special case of the apportionable premium payable annually, this becomes

$$P_x^{\{1\}} \doteq \frac{P_x}{1 - \frac{1}{2}P_x}.$$

When (4.24) is written in the form

$$P_x^{\{m\}} \doteq P_x + \frac{m-1}{2m} P_x^{\{m\}} d + \tfrac{1}{2}P_x^{\{m\}}P_x, \tag{4.25}$$

we see that the adjustment for loss of premium in the year of death provides for an amount of $\frac{1}{2}P_x^{\{m\}}$, reflecting the fact that the insurer receives only a half year's premium on the average in the year of death.

In writing formulas for apportionable premiums for other forms of insurance, it must be remembered that the approximate loss of premium of $\frac{1}{2}P^{\{m\}}$ in the year of death is covered by a term insurance for the premium-paying period. This is illustrated in the following formulas:

$$P_{x:\overline{n}|}^{(m)} \doteq \frac{P_{x:\overline{n}|}}{1 - \frac{m-1}{2m}d - \frac{1}{2}P_{x:\overline{n}|}^{1}} \tag{4.26}$$

$$_tP_x^{(m)} \doteq \frac{_tP_x}{1 - \frac{m-1}{2m}d - \frac{1}{2}P_{x:\overline{t}|}^{1}} \tag{4.27}$$

3. Analysis of the endowment premium

We have seen that endowment insurance may be regarded as a combination of level term insurance and pure endowment, and the formula

$$P_{x:\overline{n}|} = P_{x:\overline{n}|}^{1} + P_{x:\overline{n}|}^{\ \ 1} \tag{4.28}$$

illustrates the separation of the net annual endowment premium into these two elements. There is an alternative analysis by which the endowment premium is seen to be mathematically equivalent to a combination of a savings fund premium and a premium for decreasing term insurance.

The endowment policy is a promise to pay a certain sum on a given future date, or prior to that date in the event of death. The same result can be achieved by using a savings fund, supported by equal periodic contributions of the amount necessary to accumulate the full endowment sum by the maturity date, together with a supplementary decreasing term insurance for the difference between the full sum and the amount accumulated in the savings fund at any time. Thus, if the insured survives the endowment period, the full sum is available in the savings fund and the term insurance expires, or, if death intervenes, the savings fund will have accumulated to a smaller amount and the term insurance provides the balance of the face amount.

Let us assume that the total sum insured is 1, payable at the end of n years at the latest, and that a savings fund is being accumulated by means of annual deposits of $\frac{1}{s_{\overline{n}|}}$. At the end of the first year, the savings fund will amount to $\frac{1+i}{s_{\overline{n}|}}$ or $\frac{s_{\overline{1}|}}{s_{\overline{n}|}}$, and the decreasing term insurance must therefore provide a first year death benefit of $1 - \frac{s_{\overline{1}|}}{s_{\overline{n}|}}$. Similarly, at the end of the second year, the insurance required is $1 - \dfrac{\ddot{s}_{\overline{2}|}}{\ddot{s}_{\overline{n}|}}$, and, generally, for the m-th year, the decreasing insurance must provide an amount of $1 - \dfrac{\ddot{s}_{\overline{m}|}}{\ddot{s}_{\overline{n}|}}$. For an insured aged x, the net annual premium for this decreasing term insurance is given by

$$P = \frac{\left\{\left(1 - \dfrac{\ddot{s}_{\overline{1}|}}{\ddot{s}_{\overline{n}|}}\right)C_x + \left(1 - \dfrac{\ddot{s}_{\overline{2}|}}{\ddot{s}_{\overline{n}|}}\right)C_{x+1} + \cdots + \left(1 - \dfrac{\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}}\right)C_{x+n-1}\right\}}{N_x - N_{x+n}}.$$

The total endowment premium is then

$$P_{x:\overline{n}|} = \frac{1}{\ddot{s}_{\overline{n}|}} + P. \tag{4.29}$$

It may now be shown that the savings fund-decreasing insurance combination of (4.29) is indeed mathematically equivalent to the term insurance-pure endowment combination of (4.28).

We first write the expression for P as

$$P = \frac{1}{N_x - N_{x+n}} \sum_{t=1}^{n} \left(1 - \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{\overline{n}|}}\right) C_{x+t-1}$$

$$= \frac{1}{N_x - N_{x+n}} \left[\sum_{t=1}^{n} C_{x+t-1} - \frac{1}{d\ddot{s}_{\overline{n}|}} \sum_{t=1}^{n} [(1+i)^t - 1]C_{x+t-1}\right].$$

Now, performing the summations and noting that

$$\sum_{t=1}^{n} (1+i)^t C_{x+t-1} = \sum_{t=1}^{n} v^x d_{x+t-1} = v^x(l_x - l_{x+n})$$

$$= D_x - (1+i)^n D_{x+n},$$

we obtain

$$P = \frac{M_x - M_{x+n}}{N_x - N_{x+n}} - \frac{D_x - (1+i)^n D_{x+n} - (M_x - M_{x+n})}{d\ddot{s}_{\overline{n}|}(N_x - N_{x+n})}.$$

Then, writing $D_x - dN_x$ for M_x, and $D_{x+n} - dN_{x+n}$ for M_{x+n} in the last term (from exercise 2(a) in Chapter 3) and rearranging, we have

$$P = P^{1}_{x:\overline{n}|} - \frac{d(N_x - N_{x+n}) - D_{x+n}[(1+i)^n - 1]}{d\ddot{s}_{\overline{n}|}(N_x - N_{x+n})}$$

$$= P^{1}_{x:\overline{n}|} - \frac{1}{\ddot{s}_{\overline{n}|}} + P_{x:\overline{n}|}^{\ \ 1}.$$

When the savings fund premium of $\frac{1}{\ddot{s}_{\overline{n}|}}$ is added to this decreasing insurance premium, we obtain the desired equivalence

$$P_{x:\overline{n}|} = P + \frac{1}{\ddot{s}_{\overline{n}|}} = P^{1}_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ 1}.$$

The student will gain further insight into the significance of this analysis in Section 3 of Chapter 5.

The same type of analysis may be applied equally well to ordinary life insurance. Under an ordinary life policy, the sum insured is payable at age ω at the latest and the contract may be regarded as an endowment to age ω, or an $(\omega - x)$-year endowment. The ordinary life premium, P_x, may be resolved into an investment element, $\frac{1}{\ddot{s}_{\overline{\omega-x}|}}$, which builds up to the amount of the policy at the limiting age of the mortality table, and a decreasing insurance element which provides insurance equal at any time to the difference between the accumulated savings fund and the amount of the policy. The endowment and the ordinary life policy are identical in principle, the apparent differences in the two forms of insurance arising solely from the fact that the savings fund accumulation is completed at an earlier date under the endowment.

4. Summary of notation

1. The principal symbol P is used to denote a net annual premium.

2. Except for simple insurance benefits payable at the end of the year of death, P is used in conjunction with the net single premium symbol; e.g., ${}_tP({}_{n|}\ddot{a}_x)$, $P(\bar{A}_x)$.

3. A prefixed subscript with P indicates the number of years in the premium-paying period (Principle C). If no subscript is used, the premium-paying period coincides with the term of the policy.

4. A suffixed superscript with P indicates the periodicity of premium payments (Principle F). If no superscript is used, annual payments are implied.

5. The horizontal bar above the symbol P indicates that premium payments are made continuously (Principle G).

Reference

The analysis of the endowment premium in Section 3 is taken from Linton (1919).

EXERCISES

1. Annual premiums for insurances and deferred annuities

1. Given $P_{35:\overline{20}|} = .0420$, ${}_{20}P_{35} = .0299$, and $A_{55} = .6099$, find the value of $P^{1}_{35:\overline{20}|}$. (Ans. .0110)

2. (a) Obtain a formula in commutation symbols for the net annual premium, payable for 5 years, for a 10-year deferred 20-year temporary immediate annuity of 1 to (30).
   (b) Find the net annual premium if the annuity in (a) has the additional provision that the net premiums paid will be returned in the event of death within the 10-year deferred period.

3. Express in commutation symbols the net annual premium at age 40 for a double protection to age 65 policy that provides level insurance for the initial amount of 2 to age 65 and insurance for the ultimate (one-half the initial) amount thereafter, if the net annual premium reduces at age 65 to the ordinary life net premium at the original age for the ultimate amount.

4. Show that $P[(IA)_x] = 1 - \dfrac{d(I\ddot{a})_x}{\ddot{a}_x}$.

2. Fractional premiums

5. The net annual premium at age 27 for an ordinary life policy of \$1000 on the 1958 CSO table with 3% interest is \$12.09. Using the standard approximation, compute the corresponding premium payable quarterly (a) on the true basis, (b) on the instalment basis, (c) on the apportionable basis. (Ans. \$12.28, \$12.22, \$12.30)

6. Obtain an approximate formula for ${}_{t}P^{(m)}_{x:\overline{n}|}$ in terms of net premiums payable annually and the rate of discount.

7. Show that $\lim\limits_{m\to\infty} P_x^{\{m\}} = \lim\limits_{m\to\infty} P_x^{(m)}$. What is the common limit? Why does $\lim\limits_{m\to\infty} P_x^{[m]}$ have a different value?

8. In practice, apportionable premiums are refunded only for complete months, with no refund for any part of the month of death. Show that on this assumption the formula for $P_x^{\{m\}}$ is

$$P_x^{\{m\}} \doteq \frac{P_x}{1 - \dfrac{m-1}{2m}\, d - {}^{11}\!/_{24}\, P_x}$$

9. Show that $\left(1 + \dfrac{d\bar{a}_x}{dx}\right)\cdot \bar{P}(\bar{A}_x) - \dfrac{d\bar{A}_x}{dx} = \mu_x$.

3. Analysis of the endowment premium

10. The following criticism is sometimes heard: "In buying endowment insurance, a man pays for two separate benefits on one only of which he can realize." Explain the implication, and state how you would reply to it.

11. (a) Show that

$$\frac{1}{\ddot{s}_{\overline{n}|}D_x}(\ddot{s}_{\overline{1}|}C_x + \ddot{s}_{\overline{2}|}C_{x+1} + \cdots + \ddot{s}_{\overline{n}|}C_{x+n-1}) = \frac{\ddot{a}_{x:\overline{n}|}}{\ddot{s}_{\overline{n}|}} - \frac{D_{x+n}}{D_x}.$$

(b) Find the net annual premium payable for n years for a deferred life annuity of 1 per annum, issued to a life aged x, with the first annuity payment n years after date of issue and with a provision that, if (x) dies within the n-year period, the net premiums paid are to be returned with compound interest to the end of the year of death. Assume that the interest allowed in the return of premium benefit is at the rate used in the calculation of the premium.

12. A man must meet an obligation of 1 at the end of n years. He purchases an insurance policy which provides an increasing death benefit, payable at the end of the year of death, equal at any time to the present value of this obligation at rate of interest i. Find the net level annual premium payable for n years, assuming interest at rate i.

13. An n-year decreasing term insurance policy issued to (x) provides a death benefit of $1 - \dfrac{\ddot{s}_{\overline{m}|}}{\ddot{s}_{\overline{n}|}}$ for the m-th policy year ($m = 1, 2, \ldots, n$). All calculations involving interest are made at the same rate. Express the net annual premium in terms of $P^{1}_{x:\overline{n}|}$, $P_{x:\overline{n}|}^{\;\;1}$, and interest functions.

14. An m-year endowment policy issued at age x provides for the payment of $\ddot{a}_{\overline{t}|}$ if death occurs during the first policy year, $\ddot{a}_{\overline{2t}|}$ if death occurs during the second policy year, etc., and $\ddot{a}_{\overline{mt}|}$ if death occurs during the m-th policy year or in event that (x) survives the m-year period. Show that the net annual premium P payable for r years is given by

$$P = \frac{A_{x:\overline{m}|} - A'_{x:\overline{m}|}}{1 - A_{x:\overline{r}|}},$$

where $A'_{x:\overline{m}|}$ is at rate of interest $j = (1 + i)^{t+1} - 1$, i being the rate assumed in the premium calculation.

4. Summary of notation

15. Express each of the following by a single symbol, and give formulas for evaluating each in terms of commutation functions (using standard approximations where necessary):

(a) the net annual premium payable continuously for 20 years to provide an insurance to age 60 on a life aged 25;

(b) the net annual premium payable quarterly for 15 years to provide a deferred annuity of 10 per month to (45) with first payment at age 60;

(c) the net annual premium payable monthly on the apportionable basis for 10 years to provide a 20 year endowment of 1 on (35) with death benefit payable at the moment of death.

16. Give formulas in commutation symbols for the evaluation of each of the following (using standard approximations where necessary):

(a) $_{10}P^{\{2\}}_{30:\overline{20}|}$

(b) $P\{(I\bar{A})^{\,1}_{25:\overline{15}|}\}$

(c) $_{10}P^{\{1\}}_{x}$

Miscellaneous problems

17. A new mortality table is constructed by increasing the p_x values of a standard table by a constant percentage so that

$$p_x' = (1 + r)p_x \qquad \text{for all } x.$$

Prove that ordinary life net annual premiums may be calculated on the new basis at rate of interest i from the formula

$$P_x' = \frac{1}{\ddot{a}_x} - d,$$

where $d = \dfrac{i}{1+i}$ and $\ddot{a}_x$ is from the standard table at rate of interest $\dfrac{i-r}{1+r}$.

18. A life aged 50 is subject to 1958 CSO mortality except at age 54. During that year of age, the life will be exposed to an extra hazard which is equivalent to an addition of 400% to the CSO rate of mortality. Calculate the net annual premium for a 5-year term insurance of \$1000 on this life, assuming interest at 3%. (Ans. \$18.20)

19. Compute the value of $P^{\{4\}}_{30:\overline{20}|}$ using the standard approximation, given $P_{30:\overline{20}|} = 1/40$, $P_{30:\overline{20}|}^{\;\;1} = 3/200$, $_{20}P_{30} = 1/50$, and $d = 4/100$.

(Ans. $5/196$)

20. Show that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{P}(\bar{A}_{x+t})\,dt = \log \frac{v\bar{N}_{x-\frac{1}{2}}}{\bar{N}_{x+\frac{1}{2}}}$.

21. A policy is issued at age 10 with level premiums payable for life. If death occurs prior to the attainment of age 15, the death benefit is the return of the net premiums paid with interest to the end of the year. If death occurs after age 15, the death benefit is 1. Assuming that the rate of interest in the return of premium provision is the same as that used in calculating the premium, show that the net annual premium is independent of the mortality prior to age 15.

22. Find the net annual premium payable for n years for a deferred annuity-due of 1 to (x) with first payment at age $x + n$ and with the net premiums returned in the event of death before that age. If (x) survives the n years, the annuity is to be payable for t years certain and for the subsequent lifetime of (x).

Chapter 5

NET LEVEL PREMIUM RESERVES

1. The nature of the reserve

When an annuity or an insurance policy is issued, the insurer assumes the obligation of providing certain sums in the future (the benefit payments). In return, the insured undertakes to make certain premium payments. At the outset, the present value of the payments to be made by the insurer is exactly equal to the present value of the net premiums which he expects to receive. As time goes on, the present value of the payments to be made by the insurer changes; it generally decreases for annuities and generally increases for life insurance policies. Similarly, the present value of the premium payments still to be received generally decreases during the premium-paying period. It is therefore important for the insurer to have a measure of the amount which he should have on hand at any time to assure the payment of the benefits, assuming of course that all future premium payments will be made by the insured during his lifetime as they fall due. The concept of a *reserve* arises from this necessity of measuring the insurer's net liability with respect to a group of policies at times subsequent to the date of issue.

For illustration, consider an ordinary life policy of unit amount issued at age x. At the end of t years, the present value of the benefit payment is A_{x+t}, and the present value of the future net premiums, including the one due at the beginning of the $(t+1)$-th year, is $P_x \cdot \ddot{a}_{x+t}$. The difference between these amounts,

$$A_{x+t} - P_x \cdot \ddot{a}_{x+t},$$

represents the insurer's net obligation at that time and is called the policy reserve. In this form, the reserve is defined as the excess of the present value of the future benefits over the present value of the future net premiums.

When the reserve calculation involves net premiums of uniform amount and is based on the mortality and interest assumptions used in computing these net premiums, the resulting reserve is known as the *net level premium reserve*. This is the classical type of reserve, and in the subsequent sections of this chapter the word "reserve" will be understood to imply "net level premium reserve."

2. Prospective formulas

The reserve definition given above may be called a *prospective* definition since it is in terms of *future* benefits and premiums, and the formulas which follow from this definition are known as prospective reserve formulas. The expression "future premiums" in this definition must be understood to include any premium currently due.

For an ordinary life insurance of 1 issued at age x, the symbol ${}_tV_x$ is used to denote the reserve at the end of t years. As shown above, the prospective formula is

$$\begin{aligned} {}_tV_x &= A_{x+t} - P_x \cdot \ddot{a}_{x+t} \\ &= \frac{M_{x+t} - P_x \cdot N_{x+t}}{D_{x+t}} . \end{aligned} \tag{5.1}$$

For an n-payment whole life insurance, the reserve at the end of t years is obtained from the following formulas:

$${}^n_tV_x = A_{x+t} - {}_nP_x \cdot \ddot{a}_{x+t:\overline{n-t|}} = \frac{M_{x+t} - {}_nP_x(N_{x+t} - N_{x+n})}{D_{x+t}} \text{ for } t < n,$$

$${}^n_tV_x = A_{x+t} = \frac{M_{x+t}}{D_{x+t}} \text{ for } t \geqslant n.$$

Note that in the second case, for $t \geqslant n$, there being no further premium payments, the reserve is simply the net single premium at the attained age. A new notational principle is illustrated here, the superscript n being used before the V symbol to denote the number of years in the limited-payment period.

As another example of a prospective formula, consider an n-year endowment:

$$\begin{aligned} {}_tV_{x:\overline{n|}} &= A_{x+t:\overline{n-t|}} - P_{x:\overline{n|}} \cdot \ddot{a}_{x+t:\overline{n-t|}} \\ &= \frac{M_{x+t} - M_{x+n} + D_{x+n} - P_{x:\overline{n|}}(N_{x+t} - N_{x+n})}{D_{x+t}} . \end{aligned}$$

For n-year term insurance, the reserve symbol is ${}_tV^1_{x:\overline{n|}}$. The student should be able to write the formula.

A different type of symbol is used to denote the reserve for insurances payable at the moment of death, and also for annuities. In these cases, the ${}_tV$ symbol is used in conjunction with the net single premium for the benefit. For an ordinary life insurance payable at the moment of death, for example, we write

$$ {}_tV(\bar{A}_x) = \bar{A}_{x+t} - P(\bar{A}_x)\cdot \ddot{a}_{x+t}. \tag{5.2}$$

For an n-year deferred annuity-due of 1 per annum with annual premiums payable for n years, the reserve at the end of t years is given by

$$ {}^n_tV({}_{n|}\ddot{a}_x) = {}_{n-t|}\ddot{a}_{x+t} - P\cdot \ddot{a}_{x+t:\overline{n-t|}} $$

$$ = \frac{N_{x+n} - P(N_{x+t} - N_{x+n})}{D_{x+t}} \text{ for } t < n $$

$$ \text{where} \quad P = {}_nP({}_{n|}\ddot{a}_x), $$

$$ {}^n_tV({}_{n|}\ddot{a}_x) = \ddot{a}_{x+t} = \frac{N_{x+t}}{D_{x+t}} \quad \text{for} \quad t \geqslant n. $$

The same notational device is used in any case where the simpler type of symbol does not adequately specify the function.

It has been assumed in the formulas above that the duration t is an *integral* number of years. Under this assumption, the function ${}_tV$ relates to the *end* of a policy year and is called the t-th year *terminal*, or *final*, reserve.

3. Retrospective formulas

The formula for the prospective reserve may be transformed in a significant way.

$$ {}_tV_x = A_{x+t} - P_x\cdot \ddot{a}_{x+t} \quad \text{where} \quad P_x = \frac{M_x}{N_x} $$

$$ = \frac{M_{x+t} - P_x\cdot N_{x+t}}{D_{x+t}} $$

$$ = \frac{M_{x+t} - P_x\cdot N_{x+t} + (P_x\cdot N_x - M_x)}{D_{x+t}} $$

$$ \text{since} \quad P_x\cdot N_x - M_x = 0 $$

$$ = \frac{P_x(N_x - N_{x+t})}{D_{x+t}} - \frac{M_x - M_{x+t}}{D_{x+t}} \tag{5.3}$$

In this result, the first term will be recognized as the accumulated value of the premiums paid during the first t years and may be expressed as $P \cdot \ddot{s}_{x:\overline{t|}}$. The second term is seen to be equal to $\dfrac{A^{1}_{x:\overline{t|}}}{{}_tE_x}$, and thus represents the net single premium for the first t years' insurance accumulated to the end of t years. This term is known as the *accumulated cost of insurance*, and is denoted by ${}_tk_x$, where

$$ {}_tk_x = \frac{M_x - M_{x+t}}{D_{x+t}}. \tag{5.4a} $$

When the insurance is payable at the moment of death, the corresponding function is

$$ {}_t\bar{k}_x = \frac{\bar{M}_x - \bar{M}_{x+t}}{D_{x+t}}. \tag{5.4b} $$

It is now clear that formula (5.3) expresses the reserve as the excess of the accumulated value of the premiums paid over the accumulated value of the benefits provided. This expression for the reserve, being in terms of *past* premiums and *past* benefits, is known as the *retrospective* definition, and reserves written in this form are called retrospective reserves.

Although the retrospective definition has been discussed with reference to a particular form of policy, its generality should be apparent. A few additional examples of retrospective formulas follow:

$$ \begin{aligned} {}_tV^{1}_{x:\overline{n|}} &= P^{1}_{x:\overline{n|}} \cdot \ddot{s}_{x:\overline{t|}} - {}_tk_x \\ &= \frac{P^{1}_{x:\overline{n|}}(N_x - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}} \end{aligned} $$

$$ {}^{n}_{t}V({}_{n|}\ddot{a}_x) = P \cdot \ddot{s}_{x:\overline{t|}} = \frac{P(N_x - N_{x+t})}{D_{x+t}} \qquad \text{for } t < n $$

$$ \text{where } P = {}_nP({}_{n|}\ddot{a}_x) $$

$$ {}_tV(\bar{A}_{x:\overline{n|}}) = P \cdot \ddot{s}_{x:\overline{t|}} - {}_t\bar{k}_x = \frac{P(N_x - N_{x+t}) - (\bar{M}_x - \bar{M}_{x+t})}{D_{x+t}} $$

$$ \text{where } P = P(\bar{A}_{x:\overline{n|}}). $$

In the case of an n-year limited-payment life insurance, the reserve at time $t > n$ is expressed in retrospective form as

$$
\begin{aligned}
{}^n_tV_x &= {}_nP_x \cdot \ddot{s}_{x:\overline{n}|} \cdot \frac{1}{{}_{t-n}E_{x+n}} - {}_tk_x \\
&= \frac{{}_nP_x(N_x - N_{x+n}) - (M_x - M_{x+t})}{D_{x+t}}.
\end{aligned}
$$

Notice how the formula is modified to recognize that premium payments cease after n years.

It is instructive to consider a specific policy and examine the way in which the financial details work themselves out as applied to a large group of lives insured at the same time and at the same age. Let us choose a 10-payment 15-year endowment of $1 issued at age 35, and let us adopt the 1958 CSO mortality basis with interest at 3% per annum. We note that in the CSO table l_{35} equals 9,373,807 and it is therefore convenient to commence with 9,373,807 policies issued at age 35. The net level premium for each policy is

$$
{}_{10}P_{35:\overline{15}|} = \frac{M_{35} - M_{50} + D_{50}}{N_{35} - N_{45}} = .074905,
$$

and the total amount of premium paid in the first year by the 9,373,807 lives is $702,145. Let us trace the history of this initial fund of $702,145 as it is augmented by interest earnings, reduced by claim payments, increased by additional premium payments, and so on, until at the end of 15 years all the remaining policies mature. The details are shown in Table 4.

At the end of the first year, the fund has grown to $723,209 by the addition of interest at 3%. The 1958 CSO table indicates 23,528 deaths at age 35 out of the original group and, after the claim payments have been deducted, the fund stands at $699,681. The second year's premium is now payable from each of the 9,350,279 survivors at age 36, a total of $700,383, which, added to the existing fund of $699,681, brings the fund to $1,400,064 at the beginning of the second year. The subsequent operation of the fund should now be clear from Table 4. After the first 10 years, no further premiums are credited, and the only increment is interest. At the end of the 15 year period, the fund has built up to the amount which is exactly sufficient to provide the required endowment of $1 to each of the lives that are surviving at age 50.

It is evident from the table that the net premiums received are more than sufficient to meet the death claim payments that become due, and that the accumulated excess gives rise to the total fund whose growth is shown in column (6). This excess of accumulated premiums over accumulated benefits is precisely the aggregate reserve for the group of policies according to the retro-

TABLE 4

ACCUMULATION OF FUNDS UNDER 9,373,807 ENDOWMENT POLICIES OF $1 ISSUED AT AGE 35

1958 CSO 3% Basis

$P = {}_{10}P_{35:\overline{15}|} = .074905$

(1) Year t	(2) Total Premiums Received $l_{35+t-1}\cdot P$	(3) Total Fund at Beginning of Year	(4) Fund with Interest $(1.03)\cdot(3)$	(5) Death Claims d_{35+t-1}	(6) Fund at End of Year t $(4)-(5)$	(7) Number of Survivors l_{35+t}	(8) Fund per Survivor ${}^{10}_{t}V_{35:\overline{15}|} = \frac{(6)}{(7)}$
1	702,145	702,145	723,209	23,528	699,681	9,350,279	.07483
2	700,383	1,400,064	1,442,066	24,685	1,417,381	9,325,594	.15199
3	698,534	2,115,915	2,179,392	26,112	2,153,280	9,299,482	.23155
4	696,578	2,849,858	2,935,354	27,991	2,907,363	9,271,491	.31358
5	694,481	3,601,844	3,709,899	30,132	3,679,767	9,241,359	.39818
6	692,224	4,371,991	4,503,151	32,622	4,470,529	9,208,737	.48547
7	689,780	5,160,309	5,315,118	35,362	5,279,756	9,173,375	.57555
8	687,132	5,966,888	6,145,895	38,253	6,107,642	9,135,122	.66859
9	684,266	6,791,908	6,995,665	41,382	6,954,283	9,093,740	.76473
10	681,167	7,635,450	7,864,514	44,741	7,819,773	9,048,999	.86416
11		7,819,773	8,054,366	48,412	8,005,954	9,000,587	.88949
12		8,005,954	8,246,133	52,473	8,193,660	8,948,114	.91569
13		8,193,660	8,439,470	56,910	8,382,560	8,891,204	.94279
14		8,382,560	8,634,037	61,794	8,572,243	8,829,410	.97087
15		8,572,243	8,829,410	67,104	8,762,306	8,762,306	1.00000

spective definition, and the average fund per policy as shown in column (8) is the reserve function ${}^{10}_{t}V_{35:\overline{15}|}$.

That the retrospective reserve is always equivalent to the prospective reserve in the case of net level premium valuation may readily be established. At any moment of time during the premium-paying period of an insurance policy, the value of all the premiums past and future under the contract must be equal to the value of the policy benefits already provided and promised for the future. Denoting the current value of the premiums, past and future, at time t by

$$P \cdot \ddot{s}_t + P \cdot \ddot{a}_t ,$$

and the value of the past and future benefits by

$${}_tk + A_t ,$$

we must have

$$P \cdot \ddot{s}_t + P \cdot \ddot{a}_t = {}_tk + A_t .$$

When this is written as

$$P \cdot \ddot{s}_t - {}_tk = A_t - P \cdot \ddot{a}_t ,$$

the left member will be recognized as the retrospective reserve and the right member as the prospective reserve. Similar reasoning shows that this equivalence holds also for durations beyond the premium-paying period.

An important principle is easily derived from the retrospective expression for the reserve. Consider two different insurance policies which provide the same death benefit. Their reserves may be written formally as

$${}_tV = P \cdot \ddot{s}_t - {}_tk$$

and

$${}_tV' = P' \cdot \ddot{s}_t - {}_tk.$$

Subtracting, we find

$${}_tV - {}_tV' = (P - P')\, \ddot{s}_t .$$

Thus, if two insurance policies provide the same death benefit and are valued on the same basis, the difference of the corresponding terminal reserves is equal to the difference of the net premiums accumulated with interest and survivorship.

In writing reserve formulas, the student should attempt to anticipate whether the prospective or retrospective method will yield the simpler expression. There are two obvious principles:

(1) The prospective method is more convenient for durations beyond the premium-paying period. The reserve is then simply the net single premium for the future benefits at the attained age. For example, for $t \geqslant n$,

$${}^n_tV_x = A_{x+t} \quad \text{and} \quad {}^n_tV({}_{n|}\ddot{a}_x) = \ddot{a}_{x+t} .$$

(2) The retrospective method is more convenient during a deferred period when no benefits have as yet been provided. The reserve is then simply the accumulation of the past premiums. For example, ${}^{n}_{t}V({}_{n|}\ddot{a}_x) = {}_{n}P({}_{n|}\ddot{a}_x)\cdot\ddot{s}_{x:\overline{t}|}$ for $t < n$.

The existence of the reserve is a natural consequence of the practice of financing insurance involving an increasing risk by means of level annual premiums. Most of the common forms of insurance are of this type, the premiums in the earlier years being more than sufficient to cover the current risk. This is not always the case, however. There are certain forms of insurance which provide for a decreasing scale of benefits, and under these circumstances the level premium may be inadequate in the early years of duration with the result that negative reserves are produced for a number of years. When this happens, the premiums in the later years are more than sufficient to cover the current death claims, and the earlier deficit is ultimately extinguished.

The possibility of negative reserves may be illustrated in the case of the hypothetical decreasing term insurance policy which was described in Chapter 4 as one of the elements into which the endowment policy may be analyzed. It will be recalled that the endowment policy may be envisaged as a combination of a savings fund which accumulates to the maturity value, together with a decreasing term insurance for the difference between the sum insured and the amount accumulated in the savings fund at any time. Let us apply this analysis to the 10-payment 15-year endowment policy of Table 4.

The savings fund premium applicable in this case is payable for 10 years and must accumulate with interest at 3% to the sum of 1 at the end of 15 years. This premium is

$$\frac{1}{(1.03)^5\ddot{s}_{\overline{10}|}} = .073054.$$

Since the total endowment premium is known from Table 4 to be .074905, it is apparent that the premium for the decreasing term insurance portion will be $.074905 - .073054 = .001851$.

Let us now examine the reserve at the end of 1 year on the decreasing insurance. The amount of insurance during the first year is equal to the excess of 1 over the accumulated savings fund. At the end of the year the savings fund amounts to

$$.073054(1.03) = .075246,$$

and the amount of insurance is accordingly $1 - .075246 = .924754$. Then, using the retrospective definition, the reserve at the end of 1 year will be

$$\frac{.001851\, D_{35} - .924754\, C_{35}}{D_{36}} = -.000416,$$

the negative result reflecting the initial inadequacy of the premium. In issuing a policy of this type, an insurer takes the risk that the insured may discontinue premium payments without repaying the deficit against his policy. In the above illustration, this contingency is avoided since the insurance described is not issued as an individual policy, and exists only as a component of the endowment policy. It will be seen that the negative reserve of $-.000416$ combined with the savings fund accumulation of .075246 produces the reserve of .074830 shown in Table 4 for the endowment policy.

The insurance that we have used for illustration is on a limited payment basis, and this feature helps to keep the magnitude of the negative reserve small. If premiums were payable throughout the full term of the policy, the decreasing insurance premium would be smaller and the first-year benefit larger and both of these effects would increase the deficit in the reserve. Negative reserves can be avoided entirely by suitably limiting the premium-paying period and this is one of the standard devices adopted by insurers when policies with decreasing benefits are issued.

4. Other formulas for terminal reserves

There are many useful transformations of the basic reserve formulas. From the prospective formula for ${}_tV_x$, for example, the following series of formulas may be derived:

$$\begin{aligned} {}_tV_x &= A_{x+t} - P_x \cdot \ddot{a}_{x+t} \\ &= 1 - (P_x + d)\ddot{a}_{x+t} \qquad (5.5) \\ &\qquad \text{since } A_{x+t} = 1 - d\ddot{a}_{x+t} \\ &= 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x} \qquad (5.6) \\ &\qquad \text{since } P_x + d = \frac{1}{\ddot{a}_x} \end{aligned}$$

$$= \frac{\ddot{a}_x - \ddot{a}_{x+t}}{\ddot{a}_x} \tag{5.7}$$

$$= \frac{A_{x+t} - A_x}{1 - A_x} \tag{5.8}$$

since $\ddot{a}_x = \dfrac{1 - A_x}{d}$ and $\ddot{a}_{x+t} = \dfrac{1 - A_{x+t}}{d}$.

The following development yields other relations:

$$_tV_x = A_{x+t} - P_x \cdot \ddot{a}_{x+t} = A_{x+t}\left(1 - \frac{P_x \cdot \ddot{a}_{x+t}}{A_{x+t}}\right)$$

$$= A_{x+t}\left(1 - \frac{P_x}{P_{x+t}}\right) \tag{5.9}$$

$$= (P_{x+t} - P_x)\ddot{a}_{x+t} \tag{5.10}$$

$$= \frac{P_{x+t} - P_x}{P_{x+t} + d}. \tag{5.11}$$

With suitable modifications, corresponding formulas also hold for endowment insurance (see Exercise 8).

Verbal interpretations may be given for all these formulas. Formula (5.5) is analogous to the formula $A_x = 1 - d\ddot{a}_x$, for example, and may be explained by a similar argument. If the death benefit were payable now, its value would be 1. Since this payment is deferred to the end of the year of death and is contingent upon the payment of premiums for each year that the insured enters upon, the present value of the interest and of the future premiums must be deducted. Thus,

$$_tV_x = 1 - (P_x + d)\ddot{a}_{x+t}.$$

As another example, formula (5.10) demonstrates that the reserve is equivalent to the present value of the difference between the premium that the insured should pay at his attained age for the future benefits and the premium that he is actually paying.

5. Formulas connecting successive terminal reserves

When the quantity P_x is added to both sides of the basic prospective equation (5.1), we have

$$_tV_x + P_x = A_{x+t} - P_x \cdot a_{x+t}.$$

With the substitutions

$$A_{x+t} = vq_{x+t} + vp_{x+t}A_{x+t+1}$$

and

$$a_{x+t} = vp_{x+t}\ddot{a}_{x+t+1}\,,$$

this becomes

$$\begin{aligned} {}_tV_x + P_x &= (vq_{x+t} + vp_{x+t}A_{x+t+1}) - P_x(vp_{x+t}\ddot{a}_{x+t+1}) \\ &= vq_{x+t} + vp_{x+t}(A_{x+t+1} - P_x\ddot{a}_{x+t+1}) \end{aligned}$$

or, $${}_tV_x + P_x = v(q_{x+t} + p_{x+t}\cdot {}_{t+1}V_x). \tag{5.12}$$

Formula (5.12) shows that the reserve at the end of t years, ${}_tV_x$, increased by the premium then due, P_x, provides insurance for the next year, vq_{x+t}, plus a pure endowment of the next year's reserve, $vp_{x+t}\cdot {}_{t+1}V_x$. Or, if the equation is multiplied by $(1 + i)l_{x+t}$,

$$l_{x+t}({}_tV_x + P_x)(1 + i) = d_{x+t} + l_{x+t+1}\cdot {}_{t+1}V_x\,, \tag{5.13}$$

we see that the aggregate reserve for a group of l_{x+t} lives, increased by the premiums due and a year's interest, is just sufficient to provide benefit payments of 1 for each of the d_{x+t} lives that fail during the year plus the new terminal reserve of ${}_{t+1}V_x$ for each of the l_{x+t+1} lives that survive to the end of the year. It will be recalled that this was the principle used in the construction of Table 4.

If p_{x+t} in formula (5.12) is replaced by $1 - q_{x+t}$, there follows

$$({}_tV_x + P_x)(1 + i) = {}_{t+1}V_x + q_{x+t}(1 - {}_{t+1}V_x), \tag{5.14}$$

$$l_{x+t}({}_tV_x + P_x)(1 + i) = l_{x+t}\cdot {}_{t+1}V_x + d_{x+t}(1 - {}_{t+1}V_x). \tag{5.15}$$

In this form, the equation shows that for a group of l_{x+t} policies the t-th year terminal reserve increased by one year's premiums and interest is sufficient to provide the $(t + 1)$-th year reserve for all the l_{x+t} policies plus the difference between the insured amount of 1 and the reserve for each of the d_{x+t} policies that becomes a claim during the year. The amount by which the reserve falls short of the face amount, $1 - {}_{t+1}V_x$, is called the *net amount at risk* for the $(t + 1)$-th year; and the expression, $q_{x+t}(1 - {}_{t+1}V_x)$, representing the insurer's expectation of having to pay the amount at risk, is called the *cost of insurance based upon the net amount at risk* for the $(t + 1)$-th year.

This discussion need not be confined to ordinary life insurance, and the results may be generally summarized in the equations:

$$({}_tV + P)(1 + i) = q_{x+t} + p_{x+t} \cdot {}_{t+1}V, \tag{5.16}$$

$$({}_tV + P)(1 + i) = {}_{t+1}V + q_{x+t}(1 - {}_{t+1}V). \tag{5.17}$$

With $P = 0$, these equations apply also to single premium insurance and to limited-payment insurance when t exceeds the premium-paying period.

Solving equation (5.17) for P, we obtain

$$P = vq_{x+t}(1 - {}_{t+1}V) + (v \cdot {}_{t+1}V - {}_tV).$$

This equation demonstrates that the net annual premium is sufficient in any year to provide (1) an amount which will accumulate with interest to the cost of insurance based upon the net amount at risk and (2) an amount which together with the previous year's reserve will accumulate with interest to the terminal reserve for the current year.

This analysis suggests that the reserve can be expressed as the accumulation of the net premiums with interest minus the accumulation at interest of the cost of insurance based upon the net amount at risk. We can obtain an algebraic justification by writing (5.17) as

$${}_{t+1}V - (1 + i){}_tV = P\ (1 + i) - K_{x+t}$$

where $K_{x+t} = q_{x+t}\ (1 - {}_{t+1}V)$, the cost of insurance based upon the net amount at risk for the $(t + 1)$-th year. Multiplying by $(1 + i)^{n-t-1}$ and summing, we have

$$\sum_{t=0}^{n-1} [(1 + i)^{n-t-1}\ {}_{t+1}V - (1 + i)^{n-t}\ {}_tV]$$

$$= \sum_{t=0}^{n-1} P\ (1 + i)^{n-t} - \sum_{t=0}^{n-1} (1 + i)^{n-t-1}\ K_{x+t}\,.$$

Since ${}_0V = 0$, this expression reduces to

$${}_nV = P \cdot \ddot{s}_{\overline{n}|} - \sum_{t=0}^{n-1} (1 + i)^{n-t-1}\ K_{x+t}\,. \tag{5.18}$$

Although P and K are functions which involve mortality, the accumulations in this formula are at interest only. The result is not restricted to policies with level amounts of insurance; it has useful applications to policies with varying insurance benefits as well.

6. Reserves on policies with fractional premiums

The symbol ${}_tV^{(m)}$ is used to denote the reserve at the end of t years on a policy with true fractional premiums $P^{(m)}$. If premiums are assumed to be payable momently, the reserve symbol becomes ${}_t\bar{V}$. For ordinary life insurance,

$$ {}_tV_x^{(m)} = A_{x+t} - P_x^{(m)} \cdot \ddot{a}_{x+t}^{(m)} \tag{5.19} $$

and

$$ {}_t\bar{V}_x = A_{x+t} - \bar{P}_x \cdot \bar{a}_{x+t} \,. \tag{5.20} $$

We can estimate how much ${}_tV_x^{(m)}$ differs from ${}_tV_x$ by using the following expressions for $P_x^{(m)}$ and $\ddot{a}_{x+t}^{(m)}$:

$$ P_x^{(m)} \doteq P_x + \frac{m-1}{2m} P_x^{(m)}(P_x + d) $$

$$ \ddot{a}_{x+t}^{(m)} \doteq \ddot{a}_{x+t} - \frac{m-1}{2m} \,. $$

Making these substitutions in (5.19),

$$ {}_tV_x^{(m)} \doteq A_{x+t} - P_x^{(m)} \left(\ddot{a}_{x+t} - \frac{m-1}{2m} \right) $$

$$ \doteq A_{x+t} - \left[P_x + \frac{m-1}{2m} P_x^{(m)}(P_x + d) \right] \ddot{a}_{x+t} + \frac{m-1}{2m} \cdot P_x^{(m)} $$

$$ = (A_{x+t} - P_x \ddot{a}_{x+t}) + \frac{m-1}{2m} P_x^{(m)} [1 - (P_x + d)\ddot{a}_{x+t}] $$

$$ = {}_tV_x \left(1 + \frac{m-1}{2m} P_x^{(m)} \right), \text{ using formulas (5.1) and (5.5).} $$

From this result,

$$ {}_tV_x^{(m)} \doteq {}_tV_x \left(1 + \frac{m-1}{2m} P_x^{(m)} \right), \tag{5.21} $$

we see that the reserve is larger when true fractional premiums are paid. This is because there will be a loss of premium in the year of death if death occurs before all the fractional premiums are due. The average loss is approximately $\frac{m-1}{2m} P_x^{(m)}$, and the reserve on the fractional premium basis is approximately equal to the reserve on an annual premium basis for an insurance with a benefit of 1 plus the average loss of premium.

When the instalment type of fractional premium is payable, the amount of the policy proceeds can be approximated as $1 - \frac{m-1}{2m} P_x^{[m]}$. We then have

$$
\begin{aligned}
{}_tV_x^{[m]} &\doteq A_{x+t}\left(1 - \frac{m-1}{2m} P_x^{[m]}\right) - P_x^{[m]}\ddot{a}_{x+t}^{(m)} \\
&\doteq A_{x+t} - \frac{m-1}{2m} P_x^{[m]}(1 - d\ddot{a}_{x+t}) \\
&\qquad - P_x^{[m]}\left(\ddot{a}_{x+t} - \frac{m-1}{2m}\right) \\
&= A_{x+t} - \ddot{a}_{x+t}\, P_x^{[m]}\left(1 - \frac{m-1}{2m}\, d\right) \\
&\doteq A_{x+t} - P_x\, \ddot{a}_{x+t} \qquad \text{by } (4.22) \\
&= {}_tV_x\,. \qquad\qquad (5.22)
\end{aligned}
$$

Hence, no reserve adjustment is necessary in this case. The result is not surprising, since there is no loss of premium in the year of death.

With apportionable premiums, there is an average loss of premium of $\frac{1}{2} P_x^{\{m\}}$ in the year of death, and general reasoning suggests that

$$
{}_tV_x^{\{m\}} \doteq {}_tV_x\,(1 + \tfrac{1}{2} P_x^{\{m\}}). \qquad (5.23)
$$

We can verify this result algebraically. Recalling that the policy proceeds can be estimated as $1 + \frac{1}{2m} P_x^{\{m\}}$, we have

$$
\begin{aligned}
{}_tV_x^{\{m\}} &\doteq A_{x+t}\left[1 + \frac{1}{2m} P_x^{\{m\}}\right] - P_x^{\{m\}}\ddot{a}_{x+t}^{(m)} \\
&= A_{x+t}\,[1 + \tfrac{1}{2}P_x^{\{m\}}] - A_{x+t}\,\frac{m-1}{2m}\, P_x^{\{m\}} - P_x^{\{m\}}\ddot{a}_{x+t}^{(m)}
\end{aligned}
$$

$$\doteq A_{x+t}\,[1 + \tfrac{1}{2}P_x^{\{m\}}] - \frac{m-1}{2m}\,P_x^{\{m\}}(1 - d\ddot{a}_{x+t})$$

$$- P_x^{\{m\}}\left(\ddot{a}_{x+t} - \frac{m-1}{2m}\right)$$

$$= A_{x+t}\,[1 + \tfrac{1}{2}P_x^{\{m\}}] - P_x^{\{m\}}\ddot{a}_{x+t}\left[1 - \frac{m-1}{2m}\,d\right].$$

But by (4.24), $P_x^{\{m\}}\left(1 - \dfrac{m-1}{2m}\,d\right) \doteq P_x(1 + \tfrac{1}{2}P_x^{\{m\}})$.

$$\therefore\ {}_tV_x^{\{m\}} \doteq A_{x+t}\,[1 + \tfrac{1}{2}P_x^{\{m\}}] - P_x(1 + \tfrac{1}{2}P_x^{\{m\}})\ddot{a}_{x+t}$$

$$= {}_tV_x\,[1 + \tfrac{1}{2}P_x^{\{m\}}].$$

For other forms of insurance, the added reserve which is required in the case of *true fractional* and *apportionable* premiums is on a term insurance basis, since the loss of premium will only be sustained during the premium-paying period. This is illustrated in the following formulas:

$${}_tV_{x:\overline{n}|}^{(m)} \doteq {}_tV_{x:\overline{n}|} + \frac{m-1}{2m}\,P_{x:\overline{n}|}^{(m)}\cdot{}_tV_{x:\overline{n}|}^{1}$$

$${}_t^nV_x^{\{m\}} \doteq {}_t^nV_x + \tfrac{1}{2}\cdot{}_nP_x^{\{m\}}\cdot{}_tV_{x:\overline{n}|}^{1} \qquad \text{for} \qquad t < n$$

and $\quad {}_t^nV_x^{\{m\}} = {}_t^nV_x \qquad \text{for} \qquad t \geqslant n.$

7. Reserves at fractional durations

In the formulas derived for ${}_tV$ thus far, an integral value of t has been assumed, and the resulting functions have been defined as terminal reserves. For accounting purposes, it is often necessary to compute reserves at fractional durations. The *mean* reserve ${}_{t+\frac{1}{2}}V$ is especially important since, in making an over-all valuation of an insurer's policy liabilities, it is customary to assume that on any particular day the average duration of all policies issued between t and $t + 1$ years earlier is $t + \frac{1}{2}$ years.

Consider an ordinary life policy of 1 with premiums payable annually. An exact expression for the prospective reserve at duration $t + h$ (t integral and $0 < h < 1$) is

$${}_{t+h}V_x = v^{1-h}({}_{1-h}q_{x+t+h} + {}_{1-h}p_{x+t+h}A_{x+t+1}) - P_x\cdot{}_{1-h|}\ddot{a}_{x+t+h}\,. \tag{5.24a}$$

It should be noted that it is not correct to use A_{x+t+h} in place of the first term of (5.24a). A_{x+t+h} would provide for death payments at the ends of years measured from time $t + h$ instead of at the ends of *policy* years.

Another exact expression for the reserve may be obtained by analogy from (5.14):

$$_{t+h}V_x\,(1+i)^{1-h} = {}_{t+1}V_x + {}_{1-h}q_{x+t+h}\,(1 - {}_{t+1}V_x). \quad (5.24b)$$

It can readily be shown that (5.24a) and (5.24b) are equivalent. Neither formula, however, is amenable to exact calculation, and neither suggests a simple approximation. Consequently it is customary to obtain a value for $_{t+h}V_x$ by means of linear interpolation based on reserve values for integral durations.

Since the reserve $_tV_x$ is based on the payment of exactly t annual premiums, and the reserves $_{t+h}V_x$ and $_{t+1}V_x$ are based on the payment of exactly $t + 1$ annual premiums, the interpolation must be between $_tV_x + P_x$, defined as the *initial* reserve for the $(t + 1)$-th year, and $_{t+1}V_x$, the terminal reserve for that year, rather than between $_tV_x$ and $_{t+1}V_x$, the successive terminal reserves. We thus have

$$_{t+h}V_x \doteq (1 - h)(_tV_x + P_x) + h\cdot{}_{t+1}V_x. \quad (5.24c)$$

In the general case, considering a policy with insurance of 1 and premium P payable annually, the formula is

$$\begin{aligned} _{t+h}V &\doteq (1 - h)\ (_tV + P) + h\cdot{}_{t+1}V \\ &= (1 - h)_tV + h\cdot{}_{t+1}V + (1 - h)P \quad (5.25) \end{aligned}$$

for integral t and $0 < h \leqslant 1$.

Since the terminal reserves are normally tabulated for all integral durations, formula (5.25) affords a convenient approximation.

It is interesting to consider the analogue of (5.24a) when the insurance is payable at the moment of death:

$$_{t+h}V(\bar{A}_x) = \bar{A}_{x+t+h} - P(\bar{A}_x)\cdot{}_{1-h|}\ddot{a}_{x+t+h}.$$

By using the linear interpolation

$$\bar{A}_{x+t+h} \doteq (1 - h)\ \bar{A}_{x+t} + h\bar{A}_{x+t+1}$$

and the standard approximation

$$_{1-h|}\ddot{a}_{x+t+h} \doteq \ddot{a}_{x+t+h} - (1 - h) \doteq (1 - h)\ \ddot{a}_{x+t} + h\ddot{a}_{x+t+1} - (1 - h),$$

the formula reduces to

$$_{t+h}V(\bar{A}_x) \doteq (1 - h)_tV(\bar{A}_x) + h \cdot {}_{t+1}V(\bar{A}_x) + (1 - h)P(\bar{A}_x),$$

which is the exact analogue of (5.25).

When (5.25) is applied to limited-payment plans, there is of course no need to adjust for the net premium at durations beyond the premium-paying period. In such cases, as for single premium insurance, the interim reserve is obtained by interpolation between the terminal reserves.

When fractional premiums are paid m times a year, two cases arise in evaluating the reserve at duration $t + h$.

Case 1: h is a multiple of $\frac{1}{m}$. Assuming a policy of 1 subject to true fractional premiums, we let $h = \frac{k}{m}$ where k is an integer, and consider the computation of $_{t+\frac{k}{m}}V^{(m)}$. Now the terminal reserve $_tV^{(m)}$ is based on the payment of premiums for exactly t years, and $_{t+1}V^{(m)}$ assumes premiums paid for exactly $t + 1$ years. If we interpolate directly between $_tV^{(m)}$ and $_{t+1}V^{(m)}$, we obtain a value consistent with the assumption that premiums have been paid for $t + \frac{k}{m}$ years, and this is precisely what we require in this particular case. Hence we can write

$$_{t+\frac{k}{m}}V^{(m)} \doteq \left(1 - \frac{k}{m}\right) {}_tV^{(m)} + \frac{k}{m} \cdot {}_{t+1}V^{(m)}, \tag{5.26}$$

without any premium adjustment.

Case 2: h is not a multiple of $\frac{1}{m}$. Let $h = \frac{k}{m} + r$ with $r < \frac{1}{m}$, and consider the value of $_{t+\frac{k}{m}+r}V^{(m)}$. Interpolating directly between $_tV^{(m)}$ and $_{t+1}V^{(m)}$ produces a value which is consistent with the assumption that premiums are paid for $t + \frac{k}{m} + r$ years. Since premiums are in fact paid for $t + \frac{k}{m} + \frac{1}{m}$ years, it is necessary to adjust the interpolated value by adding the difference of $\left(\frac{1}{m} - r\right) P^{(m)}$. We then have

$$
\begin{aligned}
{}_{t+\frac{k}{m}+r}V^{(m)} \doteq \left(1 - \frac{k}{m} - r\right) {}_tV^{(m)} \\
+ \left(\frac{k}{m} + r\right) {}_{t+1}V^{(m)} + \left(\frac{1}{m} - r\right) P^{(m)}.
\end{aligned} \tag{5.27}
$$

Formulas (5.25), (5.26), and (5.27) are all contained in the following general rule: Interpolate between successive terminal reserves and add the amount of net premium paid for any period beyond the date of valuation.

In practice, it is frequently desirable to express formulas of this kind in terms of existing tabulated values of annual-premium terminal reserves. This can always be done by substituting the expressions derived in Section 6 for the fractional-premium reserves in the formulas above.

8. Continuous reserves

When we consider a whole life insurance payable at the moment of death and purchased by continuous premiums $\bar{P}(\bar{A}_x)$, we encounter a reserve function, ${}_t\bar{V}(\bar{A}_x)$, which for a given age x is a continuous function of time t. Premiums are being paid continuously, interest may be considered to be credited momently, and the insurance costs are deducted at each instant. Although prospective and retrospective expressions for this reserve are readily formulated by analogy with the formulas already considered, a different approach is possible which gives valuable insights into the application of calculus methods to life contingency problems.

Consider a group of l_x whole life insurances of 1 issued at age x. The insurance is payable at the moment of death, and premiums are payable momently. Let us examine the fund which results from the continuous crediting of premiums and interest and the continuous debiting of death benefit payments. We write ${}_t\bar{V}$ for ${}_t\bar{V}(\bar{A}_x)$. Since ${}_t\bar{V}$ represents the share of each policy in this fund at duration t, the total fund at time t is given by $l_{x+t} \cdot {}_t\bar{V}$, and it is possible to find an expression for ${}_t\bar{V}$ from consideration of the rate of change of $l_{x+t} \cdot {}_t\bar{V}$ with respect to t. At any instant each existing policy is contributing premium to the fund at the rate of $\bar{P}$ per annum, and the total fund $l_{x+t} \cdot {}_t\bar{V}$ is increasing on this account at time t at the rate of $l_{x+t} \cdot \bar{P}$ per annum. Interest is being added to the fund at time t at the rate of $\delta \cdot l_{x+t} \cdot {}_t\bar{V}$ per annum. The fund is being decreased at each instant by the claims incurred, and the rate of decrement on this account at time t is $l_{x+t}\mu_{x+t}$. The net effect of these three factors is to produce a total rate of change at time t of $\bar{P}l_{x+t} + \delta l_{x+t}\cdot{}_t\bar{V} - l_{x+t}\mu_{x+t}$, and we may write the following differential equation for $l_{x+t}\cdot{}_t\bar{V}$:

$$\frac{d(l_{x+t}\cdot{}_t\bar{V})}{dt} = \bar{P}l_{x+t} + \delta l_{x+t}\cdot{}_t\bar{V} - l_{x+t}\mu_{x+t}.$$

Multiplying this equation by v^t and rearranging terms,

$$v^t\cdot\frac{d(l_{x+t}\cdot{}_t\bar{V})}{dt} - \delta v^t(l_{x+t}\cdot{}_t\bar{V}) = v^t(\bar{P}l_{x+t} - l_{x+t}\mu_{x+t}),$$

we note that the left-hand side is the derivative of $v^t l_{x+t}\cdot{}_t\bar{V}$, and integrating both sides between the limits 0 and t, we have

$$\begin{aligned} v^t l_{x+t}\cdot{}_t\bar{V} &= \int_0^t v^t\bar{P}l_{x+t}\,dt - \int_0^t v^t l_{x+t}\mu_{x+t}\,dt \\ &= \bar{P}l_x\int_0^t v^t{}_tp_x\,dt - l_x\int_0^t v^t{}_tp_x\mu_{x+t}\,dt. \end{aligned}$$

Then,

$${}_t\bar{V} = \frac{l_x}{v^t l_{x+t}}(\bar{P}\bar{a}_{x:\overline{t|}} - \bar{A}^1_{x:\overline{t|}}),$$

that is,

$${}_t\bar{V}(\bar{A}_x) = \bar{P}\bar{s}_{x:\overline{t|}} - {}_t\bar{k}_x. \tag{5.28}$$

The reserve is thus seen to be equal to the accumulation of the past premiums minus the accumulation of the insurance coverage provided—a direct expression of the usual retrospective principle. The prospective formula may also be derived. Expressing (5.28) in commutation symbols,

$${}_t\bar{V}(\bar{A}_x) = \frac{\bar{P}(\bar{N}_x - \bar{N}_{x+t}) - (\bar{M}_x - \bar{M}_{x+t})}{D_{x+t}},$$

and replacing $\bar{P}(\bar{N}_x - \bar{N}_{x+t})$ by $\bar{M}_x - \bar{P}\bar{N}_{x+t}$, since $\bar{P} = \dfrac{\bar{M}_x}{\bar{N}_x}$ we have

$${}_t\bar{V}(\bar{A}_x) = \frac{(\bar{M}_x - \bar{P}\cdot\bar{N}_{x+t}) - (\bar{M}_x - \bar{M}_{x+t})}{D_{x+t}} = \frac{\bar{M}_{x+t} - \bar{P}\cdot\bar{N}_{x+t}}{D_{x+t}}$$

or
$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - {}_t\bar{P}\cdot\bar{a}_{x+t}\,, \tag{5.29} $$
the corresponding formula in prospective terms.

The difference between ${}_t\bar{V}(\bar{A}_x)$ and ${}_tV_x$ can be estimated. We write
$$ {}_tV_x = \frac{\ddot{a}_x - \ddot{a}_{x+t}}{\ddot{a}_x} \qquad \text{from (5.7)} $$
and there is a similar expression for ${}_t\bar{V}(\bar{A}_x)$:
$$ {}_t\bar{V}(\bar{A}_x) = \frac{\bar{a}_x - \bar{a}_{x+t}}{\bar{a}_x}. $$
Since $\bar{a}_x - \bar{a}_{x+t} \doteq \ddot{a}_x - \ddot{a}_{x+t}$ from (2.27), it follows that
$$ {}_t\bar{V}(\bar{A}_x) \doteq {}_tV_x\cdot\frac{\ddot{a}_x}{\bar{a}_x}. $$
Since
$$ \frac{\ddot{a}_x}{\bar{a}_x} \doteq \frac{\bar{a}_x + \frac{1}{2}}{\bar{a}_x} = 1 + \frac{1}{2\bar{a}_x} = 1 + \tfrac{1}{2}(\bar{P} + \delta), $$
we have
$$ {}_t\bar{V}(\bar{A}_x) \doteq {}_tV_x[1 + \tfrac{1}{2}(\bar{P} + \delta)]. \tag{5.30} $$
In this result, the adjustment of $\frac{1}{2}\bar{P}$ covers the average loss of a half-year's premium in the year of death, and the factor of $\frac{1}{2}\delta$ covers the average loss of interest on the sum insured due to the immediate payment of the death benefit.

9. The Fackler reserve accumulation formula

Solving for ${}_{t+1}V$ in equation (5.16), we obtain
$$ {}_{t+1}V = \frac{({}_tV + P)(1 + i) - q_{x+t}}{p_{x+t}}. $$
With the substitutions
$$ \frac{1 + i}{p_{x+t}} = u_{x+t} $$
and
$$ \frac{q_{x+t}}{p_{x+t}} = k_{x+t}\,, $$
this becomes

$$_{t+1}V = ({}_tV + P)u_{x+t} - k_{x+t}\,. \tag{5.31}$$

The u_x and k_x functions are called *Fackler valuation functions*, after the American actuary, David Parks Fackler.

$$u_x = \frac{1+i}{p_x} = \frac{(1+i)l_x}{l_{x+1}} = \frac{D_x}{D_{x+1}} \tag{5.32}$$

$$k_x = \frac{q_x}{p_x} = \frac{d_x}{l_{x+1}} = \frac{C_x}{D_{x+1}} \tag{5.33}$$

Note that k_x is a function of mortality alone, being independent of the rate of interest. Both u_x and k_x are special cases of more general functions defined earlier. From (2.15), we see that u_x is the same as $\ddot{s}_{x:\overline{1}|}$ (or ${}_1u_x$), and from (5.4a), k_x is the same as ${}_1k_x$. The continuous forms $\bar{u}_x$ and $\bar{k}_x$ are also used.

The Fackler formula (5.31) forms the basis of a convenient method for preparing reserve tables. When the values of u_x and k_x are available at all ages, we require only the value of the net premium on a policy to produce terminal reserves sequentially for all durations, starting with the initial value ${}_0V = 0$. Since the formula produces these values by a continued accumulation process, the work is self-checking, and the calculations can be assumed to be accurate if the reserves accumulate to the correct amount at maturity.

10. Methods of valuation

The process of computing the aggregate reserve for a group of policies is called *valuation*. We consider here certain practical aspects of the valuation process.

One of the standard valuation methods is based on the retrospective formula

$$_tV = \frac{P(N_x - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}}\,, \tag{5.34}$$

valid for any life, term, or endowment insurance of level amount 1 for which premiums have not ceased. By this *retrospective method*, all policies of the above type having the same year of issue and age at issue can be grouped together for valuation. For each of the resulting groups, the total net premiums (ΣP) and the total sums insured (ΣS) are obtained. Then the formula

$$\Sigma P\left(\frac{N_x - N_{x+t}}{D_{x+t}}\right) - \Sigma S\left(\frac{M_x - M_{x+t}}{D_{x+t}}\right) = \Sigma_t V \qquad (5.35)$$

produces the total t-th year terminal reserve for the group.

The retrospective method can be further improved by a device which permits the replacement of the two-fold grouping by age and year of issue by a single grouping by attained age at the valuation date. This *attained age* formula is derived from formula (5.34) by adding M_x to the numerator and subtracting the equivalent expression $P_x \cdot N_x$. Then,

$$_tV = \frac{M_{x+t} - P \cdot N_{x+t} + (P - P_x)N_x}{D_{x+t}}$$

or

$$_tV = A_{x+t} - P\ddot{a}_{x+t} + \frac{(P - P_x)N_x}{D_{x+t}}. \qquad (5.36)$$

It must be remembered that P stands for the net premium on the plan being valued and P_x is the ordinary life net premium at the same age. The expression $(P - P_x)N_x$ is constant throughout the duration of the policy and its use is the characteristic feature of the attained age method. It is denoted by θ_x, and its value is recorded for each policy issued. The actual valuation then requires a single sorting of policies into attained age groups. For each group, the total net premiums (ΣP), the total sums insured (ΣS), and the total θ-factors $(\Sigma\theta)$ are obtained. Then a single application of the formula

$$\Sigma S \cdot A_{x+t} - \Sigma P \cdot \ddot{a}_{x+t} + \frac{\Sigma\theta}{D_{x+t}}$$

produces the total terminal reserve $(\Sigma_t V)$ for the group of policies.

The retrospective method and the attained age method are conveniently adapted to punched-card systems using mechanical tabulating equipment, and these methods have been widely applied in this way for many years. Through the use of electronic computers, a high degree of efficiency can be obtained without introducing special methods which require a considerable amount of prior data processing. It is found, for example, that a simple seriatim method of valuing each individual policy, or a grouping method based on plans of insurance, can be used advantageously with high-speed computers.

11. Effects of variation in interest and mortality

It is often useful to be able to determine in advance the effect on reserves when changes are to be made in the interest and mortality assumptions. Even though the effect of such changes on net premiums may be readily determined, it is not always easy to analyze their effect on reserves, and even the most careful mathematical investigations of the problem have been found to lead only to a partial solution. The most significant contribution to the subject was made in a paper presented to the Institute of Actuaries by G. J. Lidstone (1905). His principal result may be summarized in the following theorem.

Lidstone's Theorem. For an insurance of uniform amount 1 with level annual premiums payable throughout the n-year duration of the policy, let P and ${}_tV$ denote the net level premium and the terminal reserve at the end of t years on a standard basis with interest rate i and mortality rates q, and let P' and ${}_tV'$ denote the corresponding net premium and reserve on a special basis with interest rate i' and mortality rates q'.

Consider the "critical function"

$$c_t = ({}_tV + P)(i' - i) - (q'_t - q_t)(1 - {}_{t+1}V), \quad 0 \leqslant t < n.$$

If the critical function always decreases as t increases, the special reserves ${}_tV'$ exceed the standard reserves ${}_tV$ for $0 < t < n$; if the critical function always increases with t, the special reserves are less than the standard reserves for $0 < t < n$; and, if the critical function is constant, the special reserves are equal to the standard reserves for $0 \leqslant t \leqslant n$.

It should be understood that the theorem refers only to terminal reserves, with t assuming only integral values from 0 to n. The proof of the theorem involves a closely reasoned argument, the steps of which are set forth for conciseness in the accompanying deductive scheme (page 119).

Two important corollaries to Lidstone's theorem may be derived. Suppose that the special basis involves only a change in the rate of interest. Then $q'_t = q_t$, and the critical function becomes

$$c_t = ({}_tV + P)(i' - i).$$

Statements	Reasons
1. $({}_tV + P)(1 + i) = {}_{t+1}V + q_t(1 - {}_{t+1}V)$ and $({}_tV' + P')(1 + i') = {}_{t+1}V' + q'_t(1 - {}_{t+1}V')$	Formula (5.17)
2. $({}_tV + P)(i' - i) + ({}_tV' - {}_tV + P' - P)(1 + i') = (q'_t - q_t)(1 - {}_{t+1}V) + (1 - q'_t)({}_{t+1}V' - {}_{t+1}V)$	Subtracting the equations in Statement 1 and rearranging terms
3. Define $R_t = {}_tV' - {}_tV$ and $S_t = ({}_tV + P)(i' - i) + (P' - P)(1 + i') - (q'_t - q_t)(1 - {}_{t+1}V)$	Definitions
4. $R_t(1 + i') + S_t = p'_t\, R_{t+1}$	Substituting for R_t and S in Statement 2
5. $R_0 = R_n = 0$	${}_0V = {}_0V'$ and ${}_nV = {}_nV'$
6. S_t can neither be positive for every t nor be negative for every t.	Statements 4 and 5
7. If S_t is constant, it must be equal to zero.	Statement 6
8. If S_t is constant, then $R_t = 0$ for every t, and ${}_tV'$ is always equal to ${}_tV$.	Statements 4, 5, and 7
9. If S_t is first positive and then becomes *and remains* negative, then $R_t > 0$ and ${}_tV' > {}_tV$ for $0 < t < n$; if S_t is first negative and then becomes *and remains* positive, then $R_t < 0$ and ${}_tV' < {}_tV$ for $0 < t < n$.	Statements 4 and 5
10. If $c_t = ({}_tV + P)(i' - i) - (q'_t - q_t)(1 - {}_{t+1}V)$ always decreases, then S_t is first positive and then becomes and remains negative; if c_t always increases, then S_t is first negative and then positive; if c_t is constant, S_t is constant.	$S_t = c_t + (P' - P)(1 + i')$, and the term $(P' - P)(1 + i')$ is constant. Also Statement 6
11. If c_t always decreases, then ${}_tV' > {}_tV$ for $0 < t < n$; if c_t always increases, then ${}_tV' < {}_tV$ for $0 < t < n$; if c_t is constant, then ${}_tV' = {}_tV$ for $0 < t < n$.	Statements 8, 9, and 10 q.e.d.

If i' is greater than i and ${}_tV$ increases with t, it follows that c_t is an increasing function, and the theorem indicates that the special reserves are less than the standard reserves. If i' is less than i and ${}_tV$ increases, c_t is a decreasing function, and the special reserves exceed the standard reserves. These results may be summarized as follows:

> *Corollary 1.* An increase in the rate of interest produces a decrease in reserves, and a decrease in the rate of interest produces an increase in reserves, provided that the reserves increase with duration.

The second corollary relates to the effect of a constant change in the rate of mortality, with no change in the rate of interest. Let $q_t' = q_t + k$, and, since $i' = i$, we then have

$$\begin{aligned} c_t &= -(q_t' - q_t)(1 - {}_{t+1}V) \\ &= -k(1 - {}_{t+1}V). \end{aligned}$$

If the reserves increase with duration, the net amount at risk, $1 - {}_{t+1}V$, is a decreasing function. Hence, if k is positive, c_t increases, and the special reserves are decreased; if k is negative, c_t decreases, and the special reserves are increased.

> *Corollary 2.* A constant increase in the rate of mortality produces a decrease in reserves, and a constant decrease in the rate of mortality produces an increase in reserves, provided that the reserves increase with duration.

The requirement that reserves increase with duration is normally satisfied in the case of ordinary life and endowment insurance, but not in the case of term insurance. The two corollaries thus do not hold for term insurance.

It will be seen that Lidstone's theorem is of limited utility, applying only to certain forms of insurance and being restricted to three special types of critical function. Furthermore, the behavior of the critical function is not usually evident unless the standard and special bases are connected by a simple mathematical relationship.

When Lidstone's theorem cannot be applied, it is sometimes possible to use other criteria. For example, with ordinary life insurance, it follows from formula (5.6) that

$$_tV'_x \gtreqless {}_tV_x$$

according as
$$1 - \frac{\ddot{a}'_{x+t}}{\ddot{a}'_x} \gtreqless 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x},$$

or
$$\frac{\ddot{a}'_{x+t}}{\ddot{a}'_x} \lesseqgtr \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

or
$$\frac{\ddot{a}'_{x+t}}{\ddot{a}_{x+t}} \lesseqgtr \frac{\ddot{a}'_x}{\ddot{a}_x}.$$

Here it is only necessary to calculate the values of $\frac{\ddot{a}'_x}{\ddot{a}_x}$ for all values of x, and a simple inspection of these values then shows the ages and durations at which ordinary life reserves on the special basis exceed those on the standard basis. The criterion is

$$_tV'_x \gtreqless {}_tV_x \qquad \text{according as} \qquad \frac{\ddot{a}'_{x+t}}{\ddot{a}_{x+t}} \lesseqgtr \frac{\ddot{a}'_x}{\ddot{a}_x}.$$

There is a similar criterion for endowment insurance. Thus,

$$_tV'_{x:\overline{n|}} \gtreqless {}_tV_{x:\overline{n|}} \qquad \text{according as} \qquad \frac{\ddot{a}'_{x+t:\overline{n-t|}}}{\ddot{a}_{x+t:\overline{n-t|}}} \lesseqgtr \frac{\ddot{a}'_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}.$$

These criteria apply equally to changes in interest, or mortality, or both.

A problem of some interest is to determine the condition under which two different mortality tables produce identical reserves over a given interval. Suppose that an ordinary life policy issued at age x has identical terminal reserves for the first m years on two different mortality bases, the rate of interest being the same on both bases. Then,

$$_tV'_x = {}_tV_x \qquad 0 \leqslant t \leqslant m$$

and
$$\frac{\ddot{a}'_{x+t}}{\ddot{a}_{x+t}} = \frac{\ddot{a}'_x}{\ddot{a}_x} \qquad 0 \leqslant t \leqslant m.$$

This implies that the ratio $\frac{\ddot{a}'}{\ddot{a}}$ is constant over the interval, and we may write

$$\ddot{a}_y = (1 + k)\,\ddot{a}'_y$$

where k is constant and $x \leqslant y \leqslant x + m$.

Then $$1 + vp_y\ddot{a}_{y+1} = (1 + k) + (1 + k)vp'_y\ddot{a}'_{y+1}\,.$$

Replacing $(1 + k)\,\ddot{a}'_{y+1}$ in the right member by $\ddot{a}_{y+1}$, we have

$$vp_y\ddot{a}_{y+1} = k + vp'_y\ddot{a}_{y+1}\,.$$

Then $$p_y = \frac{k}{v\ddot{a}_{y+1}} + p'_y$$

or $$q'_y = q_y + \frac{k}{v\ddot{a}_{y+1}} \qquad \text{for} \qquad x \leqslant y < x + m. \qquad (5.37)$$

The quantity $\ddot{a}_{y+1}$ normally decreases as y increases, and hence we see from (5.37) that when the reserves are equal, the mortality rates differ by an amount which increases with age.

A sufficient condition for the equality of these reserves can be given in the following form:

If $$q'_y = q_y + \frac{k}{v\ddot{a}_{y+1}} \qquad \text{for} \qquad x \leqslant y < x + m,$$

where $$k = \frac{\ddot{a}_{x+m}}{\ddot{a}'_{x+m}} - 1, \qquad \text{then} \qquad {}_tV'_x = {}_tV_x$$

for $0 \leqslant t \leqslant m$.

The proof is by induction. Suppose that, for some age u on the interval $x < u \leqslant x + m$,

$$\ddot{a}_u = (1 + k)\ddot{a}'_u\,.$$

Then,

$$\begin{aligned}
\ddot{a}_{u-1} &= 1 + vp_{u-1}\ddot{a}_u \\
&= 1 + v\left(p'_{u-1} + \frac{k}{v\ddot{a}_u}\right)(1 + k)\ddot{a}'_u \\
&= (1 + k)(1 + vp'_{u-1}\ddot{a}'_u) \\
&= (1 + k)\ddot{a}'_{u-1}\,.
\end{aligned}$$

Since, by hypothesis,

$$\ddot{a}_{x+m} = (1 + k)\ddot{a}'_{x+m},$$

it follows by induction that

$$\ddot{a}_y = (1 + k)\ddot{a}'_y$$

for all y such that $x \leqslant y \leqslant x + m$,

and hence that

$${}_tV'_x = {}_tV_x \text{ for all } t \text{ such that } 0 \leqslant t \leqslant m.$$

12. Payment of reserve in addition to face amount

The nature of the reserve is not always clearly understood by those who purchase life insurance, and there have been many demands upon insurance companies for the payment of the reserve in addition to the face amount of insurance upon the death of the insured. In the case of the usual insurance contract, such a demand must be met with an explanation that the premiums are calculated to provide only the face amount of insurance at death, and that they are not sufficient to cover the increasing amounts of insurance that would result from the additional payment of the amount of the reserve. The insured must be shown that the reserve at any time while the policy is in existence is a measure of the value of the policy at that time, not a surplus accumulated in excess of that value, and that it exists only as a technical or accounting consequence of the payment of a level premium for a risk which increases from year to year.

It is possible, however, to prepare a special type of policy which will provide for the payment of the amount of its reserve in addition to the face amount of insurance upon death. Such a policy requires a net premium correspondingly higher than when only the face amount is payable. The method of determining the necessary special premium will be illustrated for the case of an ordinary life insurance.

We will assume that an ordinary life insurance of 1 is issued at age x with the provision that the reserve will be paid in addition to the face amount in the event of death during the first n years, the extra premium for the additional benefit to be payable for n

years. The death benefit in the $(t + 1)$-th year is $1 + {}_{t+1}V$ and the corresponding cost of insurance based upon the net amount at risk is

$$K_{x+t} = q_{x+t}[(1 + {}_{t+1}V) - {}_{t+1}V] = q_{x+t} .$$

We can now determine the premium P payable during the first n years by using formula (5.18), where ${}_nV$ is simply the ordinary life reserve ${}_nV_x$, the usual ordinary life death benefit being payable after n years. We thus have

$${}_nV_x = P\ddot{s}_{\overline{n}|} - \sum_{t=0}^{n-1} (1 + i)^{n-t-1}q_{x+t} .$$

The summation term can be simplified as follows:

$$\sum_{t=0}^{n-1} (1 + i)^{n-t-1}q_{x+t} = (1 + i)^{x+n} \sum_{t=0}^{n-1} v^{x+t+1}q_{x+t}$$

$$= (1 + i)^{x+n}(M'_x - M'_{x+n})$$

where $$M'_x = \sum_{t=0}^{\infty} v^{x+t+1}q_{x+t} .$$

Solving for P, we have

$$P = \frac{{}_nV_x + (1 + i)^{x+n}(M'_x - M'_{x+n})}{\ddot{s}_{\overline{n}|}} . \tag{5.38}$$

After n years, when the extra death benefit is no longer in effect, the premium reduces to P_x.

It should be understood that the special symbol M'_x has been introduced here as a convenient simplifying device for this particular context only. It does not define a standard function, and will not necessarily be used in the same sense elsewhere.

13. Summary of notation for reserves

1. The principal symbol V is used to denote a reserve value.

2. Except for simple insurance benefits payable at the end of the year of death, the V symbol is used in conjunction with the net single premium for the benefit; e.g., ${}_tV(\bar{A}_x)$, ${}^n_tV({}_{n|}a_x)$.

3. A prefixed subscript with V indicates the duration in years at which the reserve value is taken.

4. A prefixed superscript is used with V when there is a limited premium-payment period, and indicates its duration in years.

5. A suffixed superscript and a horizontal bar over the V symbol relate to the frequency of premium payment and have the same significance as when used with P; e.g., ${}_tV_x^{(m)}$, ${}_t\bar{V}_{x:\overline{n}|}$, ${}^n_tV_x^{(m)}$.

References

7. Mereu (1961) has investigated the error involved in the assumption of linearity of reserves. Procedures for obtaining interim reserves by adapting these formulas to digital computers are described by Davidson and Birkenshaw (1958).

8. A generalization to other forms of insurance is given by Mereu (1963).

11. The classical paper is that of Lidstone (1905). Extensions of Lidstone's results are given by Gershenson (1951) and Baillie (1951). The inductive proof of the sufficiency of the condition for equal ordinary life reserves is taken from T. N. E. Greville's discussion of Gershenson (1951), TSA 3, pp. 533–4.

12. The original paper is that of Marshall (1933). Using a different approach, Nowlin and Greville (1956) obtained a result that is slightly more general than our formula (5.38). Further generalizations are cited in Chapter 6.

EXERCISES

1, 2. Prospective formulas

1. Write prospective reserve formulas in terms of commutation functions for the following:
   (a) ${}^{20}_{10}V_{30:\overline{25}|}$
   (b) ${}_{15}V(\bar{A}^{\,1}_{20:\overline{25}|})$

2. Show that ${}_{\omega-x-1}V_x = v - P_x$.

3. Show that

$${}_tV(\bar{A}_x) \doteq \left(1 + \frac{i}{2}\right){}_tV_x.$$

Write the corresponding approximation for ${}_tV(\bar{A}_{x:\overline{n}|})$.

3. Retrospective formulas

4. Write retrospective formulas for ${}^n_tV_x$ in commutation functions, considering the cases where $t < n$ and $t > n$. In each case, show that the retrospective expression is equivalent to the prospective formula.

5. Suppose that 90,000 lives aged 30 each purchase an ordinary life insurance of 1. The net annual premium is .02. If interest is at 3% and if the mortality table used provides for 700 deaths at age 30, compute ${}_1V_{30}$.

(Ans. .013)

6. Show that the product of the t-th year terminal reserve for an annual premium n-year pure endowment issued at age x and the annual premium for a pure endowment of like amount issued at the same age but maturing in t years is constant for all values of t.

4. Other formulas for terminal reserves

7. Show that (a) ${}_tV_x = A_{x+t}\left(1 + \dfrac{1+i}{i} P_x\right) - \dfrac{1+i}{i} P_x$.

(b) ${}_tV_x = 1 - (1 - {}_1V_x)(1 - {}_1V_{x+1}) \cdots (1 - {}_1V_{x+t-1})$.

8. Show that

(a) $${}_tV_{x:\overline{n}|} = 1 - (P_{x:\overline{n}|} + d)\ddot{a}_{x+t:\overline{n-t}|}$$
$$= 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$$
$$= \frac{\ddot{a}_{x:\overline{n}|} - \ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{A_{x+t:\overline{n-t}|} - A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}$$

(b) $${}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|}\left(1 - \frac{P_{x:\overline{n}|}}{P_{x+t:\overline{n-t}|}}\right)$$
$$= (P_{x+t:\overline{n-t}|} - P_{x:\overline{n}|})\ddot{a}_{x+t:\overline{n-t}|}$$
$$= \frac{P_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}}{P_{x+t:\overline{n-t}|} + d}.$$

9. (a) Given tables of P_x for $x = 10$ to 70 and tables of ${}_nV_x$ for the same range of x and for $n = 1$ to 20, show how you would find the rate of interest and the values of q_x for ages from 10 to 89.

(b) Find the fifteenth-year terminal reserve for an endowment at age 65 policy issued at age 47 with annual premiums, given only a complete table of terminal reserves for annual premium twenty-year endowment policies issued at all ages.

10. Find the value of $P^1_{x:\overline{n}|}$ if ${}_nV_x = .080$, $P_x = .024$, and $P_{x:\overline{n}|}^{\;\;\,1} = .200$.

(Ans. .008)

5. Formulas connecting successive terminal reserves

11. Show that

$$P + d\cdot {}_tV = vq_{x+t}(1 - {}_tV) + vp_{x+t}({}_{t+1}V - {}_tV),$$

and give a verbal interpretation of this equation.

12. A life aged 35, whose health is impaired, purchases a $1000 policy on the 10-payment 15-year endowment plan. The substandard net premium which he pays is $5 greater than the normal net premium at that age. If the insurer maintains the usual 1958 CSO 3% reserves, for what rate of mortality does this total premium provide in the first policy year? (Use the net premium and reserves given in Table 4.) (Ans. .0081)

13. An insured under an ordinary life policy for $1000, issued at age 20, is to be subject during the eleventh policy year to an extra mortality not covered by the terms of the policy. This extra risk may be expressed as an addition of .01 to the normal rate of mortality during that year. If the eleventh year terminal reserve is $81.54 and the tabular rate of mortality at age 30 is .008427, calculate on a net basis the theoretical reduction in the amount of insurance the company should require during the eleventh year if the policy is to be amended to include the extra risk. (Ans. $498)

14. A special retirement plan provides an annual payment of 1 at the end of each year of survival and a death benefit of one-half of the reserve at the end of the year of death. If retirement begins at age x and if ${}_tR_x$ denotes the reserve immediately before the t-th retirement payment, find ${}_{t+1}R_x$ in terms of ${}_tR_x$, p, q, and i.

6. Reserves on policies with fractional premiums

15. Show that ${}_t\bar{V}_x \doteq (1 + \frac{1}{2}\bar{P}_x)\,{}_tV_x$.

16. Given $P_x = .0240$, ${}_4V_x = .0600$, and $d = .026$, estimate the values of ${}_4V_x^{(2)}$, ${}_4V_x^{\{12\}}$, ${}_4V_x^{\{1\}}$, and ${}_4\bar{V}_x$, using the standard approximation.

17. Write formulas which express the following in terms of reserves subject to annual premiums (using the standard approximation):

(a) ${}_t\overset{(m)}{V}{}^{1}_{x:\overline{n}|}$,

(b) ${}^{k}_{t}V_x^{\{m\}}$ for $t < k$,

(c) ${}_t\overset{\{m\}}{V}_{x:\overline{n}|}$,

(d) ${}_t\bar{V}_{x:\overline{n}|}$.

7. Reserves at fractional durations

18. Write the expressions you would use in obtaining approximations to the following:

(a) ${}_{5\,1/4}^{\;10}V_{[40]}$

(b) ${}_{5\,1/4}^{\;10}V_{[40]}^{(4)}$

(c) ${}_{5\,1/4}^{\;10}V_{[40]}^{\{2\}}$

19. On February 1, 1922, a $1000 ordinary life policy was issued subject to true net monthly premiums of $1.80 each. If the reserve on the corre-

sponding annual premium policy would have been \$240 on January 31, 1936, and \$260 on January 31, 1937, compute an approximate value for the reserve on the monthly premium policy on January 31, 1936; January 31, 1937; September 30, 1936; and October 20, 1936.

20. Verify the following formula for computing mean reserves:

$$ {}_rV_x \cdot P_{x+r} + 1000\, Q_{x+r}\,, $$

where ${}_rV_x$ is the r-th year terminal reserve for insurance of \$1000,

$$ P_{x+r} = \tfrac{1}{2}(1 + vp_{x+r-1}) $$

and

$$ Q_{x+r} = \tfrac{1}{2}vq_{x+r-1}\,. $$

8. Continuous reserves

21. Show that the basic differential equation for ${}_t\bar{V}$ can be written as

$$ \frac{d}{dt}\,{}_t\bar{V} = \bar{P} + \delta\cdot{}_t\bar{V} - \mu_{x+t}\,(1 - {}_t\bar{V}). $$

Give a verbal interpretation.

22. Show that, if premiums are paid continuously and claims are payable at the moment of death, formula (5.12) becomes

$$ {}_t\bar{V} + \bar{P}\bar{a}_{x+t:\overline{1}|} = \bar{A}^{\,1}_{x+t:\overline{1}|} + vp_{x+t}\cdot{}_{t+1}\bar{V}. $$

23. Show that

(a)
$$ \frac{d}{dt}\,[{}_t\bar{V}(\bar{A}_x)] = \frac{\bar{A}_{x+t} - \mu_{x+t}\bar{a}_{x+t}}{\bar{a}_x}\,, $$

(b)
$$ \frac{d}{dx}\,[{}_t\bar{V}(\bar{A}_x)] = -\frac{\bar{a}_{x+t}}{\bar{a}_x}\,(\mu_{x+t} - \mu_x) + \frac{{}_t\bar{V}(\bar{A}_x)}{\bar{a}_x}. $$

9. The Fackler reserve accumulation formula

24. Find ${}_1V_x$, given $P_x = .025$, $u_x = 1.032$, and $k_x = .015$.

25. Show that $\dfrac{k_x}{u_x} = A^{\,1}_{x:\overline{1}|}$.

26. Wright's reserve accumulation formula is

$$ {}_{t+1}V = ({}_tV + P - c_{x+t})u_{x+t}\,. $$

How must c_{x+t} be defined in this formula?

10. Methods of valuation

27. How should θ_x be defined so that the attained age formula can be applied to a single premium n-year term insurance?

28. Show that the reserve at the end of t years for an annual premium n-year deferred immediate annuity issued at age x can be expressed as

$$\frac{\theta_x}{D_{x+t}} - P\ddot{a}_{x+t} \qquad \text{where} \qquad t < n$$

and P is the net premium. What is θ_x ?

29. Show that the n-th terminal reserve under any net level annual premium policy for an insurance of unity issued at age x, premiums not having ceased, equals

$$(1 + \theta_x D_{x+n}^{-1}) - (P + d)\ddot{a}_{x+n}$$

where P is the net annual premium and θ_x is a function independent of duration.

11. Effects of variation in interest and mortality

30. Determine the effect on ordinary life reserves if each p_x in the mortality table is multiplied by $(1 + k)$, where k is a positive constant.

31. If the critical function in Lidstone's theorem has the constant value c, show that $P = P' + cv'$.

32. A certain substandard class is subject to extra mortality which results in ordinary life annual net premiums P'_x which are related to standard ordinary life net premiums P_x by the formula

$$P'_x = (1 + k)P_x + c.$$

Show that the substandard class reserves will be greater than, less than, or equal to the standard ordinary life reserves according as c is less than, greater than, or equal to kd.

33. Given the following data based upon two mortality tables T and T':

x	$\dfrac{\ddot{a}_x}{\ddot{a}'_x}$	$\dfrac{\ddot{a}'_{x+10}}{\ddot{a}'_x}$
30	.981	.870
40	.988	.815
50	.979	.757

Rank in order of magnitude the values of ${}_{10}V_{30}$, ${}_{10}V_{40}$, ${}_{10}V'_{30}$, and ${}_{10}V'_{40}$.

12. Payment of reserve in addition to face amount

34. Draft a letter of reply to a beneficiary under one of your company's ordinary life policies who claims that the policy proceeds should include the amount of the reserve on the policy in addition to the face amount.

35. An ordinary life policy of 1 is issued to (x) with the provision that the reserve will be paid in addition to the face amount in the event of death during the first n years. Show that the net annual premium for this policy payable for the lifetime of (x) is given by

$$\frac{A_{x+n} + (1+i)^{x+n}(M'_x - M'_{x+n})}{\ddot{s}_{\overline{n}|} + \ddot{a}_{x+n}},$$

where $M'_x = \sum_{k=x}^{\omega-1} v^{k+1} q_k$.

36. Derive a formula for calculating net annual premiums for a special 20-year endowment policy which provides for the payment of the terminal reserve on the entire policy in addition to the face amount in the event of death within 20 years but which matures for the face amount only. Give a formula for obtaining a reasonably close approximation to the true premium for this policy, indicating the nature of the error involved.

Miscellaneous problems

37. Given $\sum_{x=31}^{60} \log(1 - {}_1V_x) = -\log 7/5$, find ${}_{20}V_{31}$ correct to two decimal places. (Ans. .29)

38. An applicant aged 25 purchases an ordinary life policy. Each year he invests elsewhere the difference between the net premium for his policy and the corresponding net premium for a twenty-year endowment policy. Approximately what interest rate must he earn on this investment in order that it may accumulate at the end of twenty years to the difference between the face amount and the reserve for his policy? Given:

$$\ddot{a}_{25} = 23 \qquad \ddot{a}_{\overline{20}|} \text{ at } 3\tfrac{1}{2} \text{ per cent} = 14.7$$

$$\ddot{a}_{45} = 16 \qquad \ddot{a}_{\overline{20}|} \text{ at 4 per cent} = 14.1$$

$$\ddot{a}_{25:\overline{20}|} = 15 \qquad d = .0338 \text{ for } i = .035$$

$$d = .0385 \text{ for } i = .04$$

Explain briefly why the rate he must earn on his investment differs from that used in the calculation of the annual premium.

39. An n-year term policy providing decreasing insurance calls for level annual premiums payable for k years $(k < n)$. The amount of insurance in the t-th policy year is B_t, and

$$B_1 \cdot q_x > B_2 \cdot q_{x+1} > \cdots > B_n \cdot q_{x+n-1},$$

where x is the age at issue.

(a) Show that all terminal reserves will be positive if the first year terminal reserve is positive.

(b) Prove that the largest value of k permissible if terminal reserves are never to be negative is the largest value of k for which

$$N_{x+k} > N_x - \frac{D_x}{B_1 C_x} \sum_{t=1}^{n} B_t C_{x+t-1}.$$

40. An ordinary life policy of unit amount is issued on a life aged x. The annual premium is payable for five years, at the end of which time the insured has the option of either continuing the policy or converting it, by

payment of an increased annual premium, to an endowment for the same face amount and maturing at age $x + n$.

(a) If the conversion option is elected, find an expression for the net annual premium payable for the remaining term of the endowment.

(b) Show that the reserve by the prospective method at the end of five years after such conversion, immediately before payment of the 6th increased premium, is equal to the reserve which would have been held had the policy not been converted, together with the difference in the net premiums paid, accumulated with interest and survivorship. Assume that the reserve is calculated on the same mortality and interest basis as the net premiums.

41. Two mortality tables T and T' are such that $q'_y = q_y - \dfrac{.52}{\ddot{a}_{y+1}}$ at ages 90 through 99, where the interest rate used for calculating $\ddot{a}_{y+1}$ is 4%. At the same interest rate, ${}_tV_{90} = {}_tV'_{90}$ for $0 < t < 10$. The limiting age ω for table T is 101. Find the value of $\ddot{a}'_{100}$ at 4% interest.

CHAPTER 6

THE EXPENSE FACTOR

1. The evaluation of expenses

The determination of life insurance and annuity premiums on a commercial basis involves certain elements not yet considered. The rate of premium charged for a given contract must not merely cover the net value of the benefit; it must also be sufficient to provide for the expenses which the insurer will incur in carrying out the contract. It is important to understand the character of these expenses, since they have a significant effect upon the actuarial basis of the policy of insurance.

A considerable portion of the total expense on a policy is incurred when the policy is issued. This initial expense includes the cost of underwriting the risk, which may involve medical fees, and the clerical expense of issuing the policy and listing the details of the transaction on the insurer's records. After the policy is established on the insurer's books, there are continuing administrative expenses, and finally, when the policy is terminated by death or maturity, there is the cost of settlement. In addition to these types of expense, there are commission payments and premium taxes to be taken into account. Under the usual American system of agents' compensation, the rate of commission is higher in the first year than in the renewal years, and thus the initial expenses are further weighted on this account.

In calculating premiums which make allowance for expenses, insurers first make detailed cost studies in which each expense item is expressed as a percentage of the premium, or as a flat amount per $1000 of insurance, or as a flat amount per policy. The expenses which appear as a flat amount per policy can be translated into a flat amount per $1000 of insurance by the use of an assumption as to the average amount of insurance per policy. The expense data can thus be reduced to a form which is easily assimilated in the premium formula. Once the expense rates are available in this form, the premium formula is established from the principle that the present value of the premiums to be received must equal the present value of the benefits to be provided plus the present value of the expenses to be paid.

For example, suppose that the expenses on an ordinary life policy are assumed to be as follows: 75% of the first premium, 20% of the second premium, 10% of the third through sixth premiums, and 5% of each premium thereafter, plus \$10 at the beginning of the first year and \$2 at the beginning of each subsequent year per \$1000 of insurance, with \$5 per \$1000 as the cost of settlement. If G is the annual premium per \$1000 of insurance with provision for expenses and the immediate payment of claims, we have

$$G\cdot\ddot{a}_x \doteq 1005\left(1+\frac{i}{2}\right)A_x + .75G + .2G(\ddot{a}_{x:\overline{2}|} - \ddot{a}_{x:\overline{1}|})$$
$$+ .1G(\ddot{a}_{x:\overline{6}|} - \ddot{a}_{x:\overline{2}|}) + .05G(\ddot{a}_x - \ddot{a}_{x:\overline{6}|}) + 10 + 2a_x$$

or

$$G \doteq \frac{1005\left(1+\frac{i}{2}\right)A_x + 8 + 2\ddot{a}_x}{.95\ddot{a}_x - .05\ddot{a}_{x:\overline{6}|} - .1\ddot{a}_{x:\overline{2}|} - .55}.$$

The premium actually charged, including provision for expenses and any provision for safety margins and other factors which the insurer may feel to be necessary, is called the *gross premium*. The amount by which the gross premium exceeds the assumed net premium is called *loading*.

Gross premium formulas may become quite complex. It is often found, however, that the result of a complicated gross premium formula can be approximated with sufficient accuracy by adding some simple loading to the net premium. For example, the net premium may be loaded by a percentage of itself, or by a constant, or by a percentage plus a constant. We shall often use a simple loading of this type when gross premiums are required in our subsequent work.

Fractional gross premiums are usually established as a constant percentage of the corresponding annual premiums. Thus, a particular insurer might calculate all semi-annual premiums as one-half of the annual premium increased by 2%. The actual percentages used are determined from an analysis of the net fractional premiums of the appropriate type—true, instalment, or apportionable—and by a study of the extra expense involved in making the more frequent premium collections.

2. Modified reserve systems

When the net level premium reserve is used as the basis of an insurer's valuation, the loading available for expenses is a level amount from year to year, although the actual expenses are higher in the first year than in subsequent years. There is accordingly a deficiency in the first year which must be made up by drawing temporarily on the insurer's surplus, with the amount drawn in this way being returned in later years as the loading becomes more than enough to meet current expenses. For a small or a newly-formed company, with a relatively small working capital, this need to draw on surplus for the financing of new business could be a real hardship. Such a company might find itself in a position where it could not afford to accept any additional new business for a temporary period.

This situation can be alleviated by the use of a reserve system which gives recognition to the decreasing incidence of expense and provides a larger loading in the first policy year than in the renewal years. This of course implies that the reserve must be built from a first year net premium which is less than the renewal net premiums.

In any modified reserve system of this kind, the sequence of net level premiums P is replaced during a specified number of years k by a reduced first year net premium α followed by a series of increased renewal premiums β. If the modification period of k years is less than the premium-paying period, net level premiums and full net level premium reserves are used after k years. The initial value of the sequence of modified net premiums must be equal to the initial value of the net level premiums P during the k-year period. If x is the age at issue, we have

$$\alpha + \beta \cdot a_{x:\overline{k-1|}} = P\ddot{a}_{x:\overline{k|}} .$$

From this equation, the renewal net premium β may be obtained when the value of α is known:

$$\beta = \frac{P\ddot{a}_{x:\overline{k|}} - \alpha}{a_{x:\overline{k-1|}}} . \tag{6.1}$$

If we substitute $a_{x:\overline{k-1|}} = \ddot{a}_{x:\overline{k|}} - 1$, the formula may be transformed to

$$\beta = P + \frac{\beta - \alpha}{\ddot{a}_{x:\overline{k|}}} , \tag{6.2}$$

which is useful since some modifications are defined in terms of the excess of the renewal net premium β over the first year net premium α.

The following formulas give the reserves for the general case of an n-payment m-year endowment insurance under a modified valuation method with a modification period of k years. For $t \leqslant k \leqslant n$, the prospective formula is

$$
\begin{aligned}
&{}^{n}_{t}V^{\text{Mod}}_{x:\overline{m}|} \\
&\quad = A_{x+t:\overline{m-t}|} - {}_nP_{x:\overline{m}|}\cdot \ddot{a}_{x+t:\overline{n-t}|} - (\beta - {}_nP_{x:\overline{m}|})\ddot{a}_{x+t:\overline{k-t}|}
\end{aligned}
\tag{6.3a}
$$

and the retrospective formula is

$$
{}^{n}_{t}V^{\text{Mod}}_{x:\overline{m}|} = \alpha\cdot\frac{1}{{}_tE_x} + \beta\cdot\ddot{s}_{x+1:\overline{t-1}|} - {}_tk_x\,. \tag{6.3b}
$$

For $t \geqslant k$, the reserves are net level premium reserves.

The reserves can also be computed sequentially using the Fackler accumulation formula (5.31) with P replaced by the appropriate modified net premium.

In the sections that follow, we shall examine the details of several specific modification methods. In the notation, we shall continue to use certain special conventions which have been introduced above. The symbol P refers to the net level premium for an unspecified plan and issue age. The symbols α and β refer to modified net premiums with the plan, issue age, and modification method all unspecified. The symbols ${}_tV$ and ${}_tV^{\text{Mod}}$ refer to net level reserves and modified reserves for an unspecified plan and issue age. When subscripts are used, as in P_x and ${}_tV^{\text{Mod}}_{x:\overline{n}|}$, they will identify the plan and age in accordance with the standard rules. Although we shall sometimes use these symbols without subscripts even when they refer to a specific plan or age, the specification should always be clear from the context. Particular modification methods will always be indicated by a superscript, as in α^F, β^{Can}, ${}_tV^{\text{Com}}$.

3. The full preliminary term plan

In order to avoid a negative reserve at the end of the first year, the first year modified net premium must be not less than the cost of insurance for that year. In other words, we should have for *any* modified reserve system $\alpha \geqslant A^{\,1}_{x:\overline{1}|}$. The reserve method for which

$\alpha = A^{\;1}_{x:\overline{1}|}$, and under which the largest amount of first year loading is therefore available, is known as *full preliminary term* valuation. In this context, $A^{\;1}_{x:\overline{1}|}$ is usually represented by the symbol c_x :

$$c_x = A^{\;1}_{x:\overline{1}|} = \frac{C_x}{D_x}.$$

The modification extends over the entire premium-paying period, and, with $\alpha = c_x$, the renewal net premium β may be determined from formula (6.1). We let α^F and β^F denote the first year and the renewal net premiums respectively on the full preliminary term basis, and then, considering again an n-payment m-year endowment insurance issued at age x,

$$\alpha^F = c_x \tag{6.4a}$$

and from (6.1),

$$\beta^F = \frac{{}_nP_{x:\overline{m}|}\cdot \ddot{a}_{x:\overline{n}|} - c_x}{a_{x:\overline{n-1}|}} = \frac{M_x - M_{x+m} + D_{x+m} - C_x}{N_{x+1} - N_{x+n}}$$

$$= \frac{M_{x+1} - M_{x+m} + D_{x+m}}{N_{x+1} - N_{x+n}} = \frac{A_{x+1:\overline{m-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}} = {}_{n-1}P_{x+1:\overline{m-1}|}\,. \tag{6.4b}$$

Thus the renewal net premium is the net level premium for a similar insurance issued at an age one year older, with premiums payable for one year less, and maturing at the same age as the original insurance.

It is apparent that this method assumes that the entire first year gross premium is needed for expenses and death claim payments. Accordingly, the reserve at the end of the first year is zero. In the renewal years the premium required is the same as the net level premium for a policy issued at the end of the first year at an age one year higher. It follows that the reserves on the full preliminary term basis, ${}_tV^F$, can be expressed in terms of the net level premium reserves for an insurance issued at an age one year older, thus:

$${}^{n}_{t}V^F_{x:\overline{m}|} = {}^{n-1}_{t-1}V_{x+1:\overline{m-1}|}\,. \tag{6.5}$$

The full preliminary term reserves can also be calculated from formulas (6.3) with α taken as c_x and β as ${}_{n-1}P_{x+1:\overline{m-1}|}$.

4. Modified preliminary term plans

In the full preliminary term method, the first year net premium at a given age is the same for all plans of insurance, and the extra first year expense allowance is large or small according as the insurance is on a high or low premium form. For example, the first year net premium c_x is a much smaller part of the 20-year endowment gross premium than of the ordinary life gross premium at the same age, so that a relatively larger portion of the endowment premium is available as first year loading. If the full preliminary term method is regarded as providing equitable first year expense allowances on low premium forms such as ordinary life insurance, it must be concluded that it provides unnecessarily large allowances for high premium forms such as 20-year endowment insurance.

Because of this fact, most state laws which specify a minimum valuation basis restrict the use of the full preliminary term method to certain low premium forms of insurance. For the higher premium forms a *modified preliminary term* basis is prescribed. A number of different modified preliminary term systems have been used. The various modifications differ in the length of the modification period and in the amount of the extra first year expense allowance. In each case, the modified net premiums can be determined from (6.1) or (6.2). We shall consider two methods which are of current interest; others will be described in the exercises.

(a) *The Commissioners reserve valuation method*

Most states which specify a minimum valuation basis prescribe the Commissioners reserve valuation method for policies currently being issued. This method provides a special modification for policies for which the net renewal premium on the full preliminary term basis exceeds the full preliminary term renewal premium for 20-payment life insurance at the same age. Thus, this modification, which we shall call the Commissioners modification, is used for policies for which $\beta^F > {}_{19}P_{x+1}$, where x is the age at issue. For all other policies, those for which $\beta^F \leqslant {}_{19}P_{x+1}$, the method requires full preliminary term valuation.

The Commissioners modification extends over the full premium-paying period of the policy and prescribes the following relationship between α and β:

$$\beta^{\text{Com}} - \alpha^{\text{Com}} = {}_{19}P_{x+1} - c_x\,. \tag{6.6}$$

This is equivalent to a requirement that the first year loading exceed the renewal loading by the same amount as the full preliminary term method allows for 20-payment life insurance.

Combining (6.6) with (6.2), we have

$$\beta^{\text{Com}} = P + \frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{n}|}}\,, \tag{6.7a}$$

where P is the net level premium, x is the age at issue, and n is the premium-paying period. From (6.6),

$$\alpha^{\text{Com}} = \beta^{\text{Com}} - ({}_{19}P_{x+1} - c_x). \tag{6.7b}$$

The Commissioners reserves may now be obtained by using formulas similar to (6.3a) or (6.3b), or by using the Fackler accumulation method.

(*b*) *The Canadian reserve valuation method*

The Canadian method prescribes a special modification for all policies with a net level premium greater than the ordinary life net level premium at the same age, and full preliminary term valuation for all other policies. The modification, which extends over the entire premium-paying period, is based upon the excess of the net level premium P over the first year modified net premium α, and this excess, $P - \alpha$, is set equal to the corresponding difference, $P_x - c_x$, produced by the full preliminary term method applied to ordinary life insurance. We thus have

$$P - \alpha^{\text{Can}} = P_x - c_x \qquad \text{for } P > P_x\,,$$

whence

$$\alpha^{\text{Can}} = P - (P_x - c_x). \tag{6.8a}$$

Then, from (6.1)

$$\beta^{\text{Can}} = \frac{P\ddot{a}_{x:\overline{n}|} - \alpha^{\text{Can}}}{a_{x:\overline{n-1}|}}$$

$$= \frac{P\ddot{a}_{x:\overline{n}|} - P + (P_x - c_x)}{a_{x:\overline{n-1}|}}\,,$$

which reduces to

$$\beta^{\text{Can}} = P + \frac{P_x - c_x}{a_{x:\overline{n-1}|}}\,. \tag{6.8b}$$

5. Cash values

An important feature of the modern life insurance policy is the provision for nonforfeiture in the event of default in premium payment. The usual nonforfeiture clause provides for a cash surrender value, together with a choice of other options involving insurance benefits in place of the cash value. The determination of the equitable amount of cash value to be paid to a discontinuing policyholder is a problem requiring careful analysis. The surrendering policyholder is entitled to a value based on the amount which he has contributed to the insurer's funds, after deductions have been made for the cost of insurance and expenses. In policies of term insurance, the resulting value is often so small that no nonforfeiture clause is required. With life and endowment plans, however, the nonforfeiture clause is a standard provision.

For many years, it was the practice to compute cash values by applying a scale of deductions to the terminal reserves. These deductions varied by duration and by plan of insurance, and were designed so that the resulting cash values would have a proper relation to the actual funds accumulated by the insurer on the given policies. The method was somewhat artificial, and since the deductions were called *surrender charges*, they were often misunderstood by the policyholders.

Modern practice involves the computation of cash values by a more realistic approach, the *adjusted premium method.* The laws of most states now prescribe this method as the basis for determining minimum cash values.

With this method, the expense on a policy is assumed to be of two kinds: (1) a level amount E incurred each year throughout the premium-paying period, and (2) an additional first year amount E^1. The total first year expense is thus $E + E^1$.

The gross premium G is assumed to be made up of an *adjusted premium* P^A plus a loading equal to the level annual expense E:

$$G = P^A + E.$$

Letting A denote the net single premium at issue for the insurance benefit, and $\ddot{a}$ the present value at issue of a temporary life annuity for the premium-paying period, we have the gross premium equation

$$G \cdot \ddot{a} = (P^A + E)\ddot{a} = A + E \cdot \ddot{a} + E^1,$$

whence
$$P^A = \frac{A + E^1}{\ddot{a}}. \tag{6.9}$$

This formula can be rewritten as follows (replacing $\ddot{a}$ by $a + 1$):

$$(P^A - E^1) + P^A \cdot a = A.$$

This shows that the first year net premium can be taken as $P^A - E^1$ and the renewal net premium as P^A. The cash values are now defined as the modified preliminary term reserves with $\alpha = P^A - E^1$ and $\beta = P^A$. Using the prospective form, and assuming for illustration an ordinary life policy issued at age x, the cash value at the end of t years is given by

$$_tCV_x = A_{x+t} - P_x^A \cdot \ddot{a}_{x+t}, \qquad t \geqslant 1. \tag{6.10}$$

When appropriate assumptions are made for interest and mortality and for the extra initial expense E^1, the adjusted premium P^A is fully defined by (6.9) and the values computed as shown in (6.10) provide a scale of cash values which makes allowance for the extra initial expense.

The *minimum* cash values which an insurer may offer are defined by law to be adjusted premium surrender values computed on a prescribed mortality and interest basis with the extra initial expense E^1 derived from the following formula (based on a level insurance amount of \$1000):

(a) 40% of the adjusted premium for the policy, but the amount not to exceed \$16; plus

(b) 25% of the adjusted premium for the policy or of the adjusted premium for an otherwise similar ordinary life policy, whichever is less, but the amount not to exceed \$10; plus

(c) \$20.

The formula allows the extra initial expense to vary with the plan of insurance—but only within a reasonable range, from about \$25 per \$1000 of insurance on the lowest premium plan to a maximum of \$46.

The simple formula for P^A given by (6.9) cannot usually be directly applied to find the adjusted premium, since E^1 is defined partly in terms of certain percentages of P^A. Furthermore, care must be taken that the maximum elements, defined in (a) and (b), are not exceeded. It will be noted that the maximum amount of

\$16 in (a) must be used whenever P^A is greater than \$40, and the \$10 maximum in (b) must be used if the lesser of P^A and the ordinary life adjusted premium at the same age exceeds \$40. The formula for P^A will thus vary according as

(1) P^A is greater or less than \$40,

(2) P^A is greater or less than the ordinary life adjusted premium P_x^A at the same age, and

(3) P_x^A is greater or less than \$40.

These limitations on E^1 can be reduced to the following formula, assuming a unit amount of insurance issued at age x:

$$E^1 = .4\begin{bmatrix} P^A \\ \\ .04 \end{bmatrix} + .25\begin{bmatrix} P^A \\ P_x^A \\ .04 \end{bmatrix} + .02\,, \tag{6.11}$$

where the smallest of the quantities in each bracket is to be used. P^A is the required adjusted premium, payable for n years, and P_x^A is the ordinary life adjusted premium.

In using the formula, one first computes the ordinary life adjusted premium. We have

$$P_x^A = \frac{A_x + E^1}{\ddot{a}_x}$$

where $E^1 = .40\,P_x^A + .25\,P_x^A + .02 = .65\,P_x^A + .02 \qquad \text{for } P_x^A \leqslant .04$

and $\qquad E^1 = .016 + .010 + .020 = .046 \qquad \text{for } P_x^A > .04.$

These lead to the following formulas:

$$P_x^A = \frac{A_x + .02}{\ddot{a}_x - .65} \qquad \text{for} \qquad P_x^A \leqslant .04 \tag{6.12a}$$

and

$$P_x^A = \frac{A_x + .046}{\ddot{a}_x} \qquad \text{for} \qquad P_x^A > .04. \tag{6.12b}$$

Suppose now that we wish to compute the adjusted premium for a 10-payment 15-year endowment issued at age 35. Let us assume, as will usually be the case, that the value of P_{35}^A is less than .04 and the value of ${}_{10}P_{35:\overline{15}|}^A$ is greater than .04. We can compute P_{35}^A from (6.12a), and then

$${}_{10}P_{35:\overline{15}|}^A = \frac{A_{35:\overline{15}|} + E^1}{\ddot{a}_{35:\overline{10}|}}\,,$$

where $$E^1 = .016 + .25\, P^A_{35} + .020.$$

The minimum cash values for the policy may be expressed prospectively as

$${}^{10}_{\;t}CV_{35:\overline{15}|} = A_{35+t:\overline{15-t}|} - {}_{10}P^A_{35:\overline{15}|} \cdot \ddot{a}_{35+t:\overline{10-t}|}\,, \quad 1 \leqslant t \leqslant 10.$$

Although the statutes define minimum cash values in terms of the adjusted premium method, insurers are free to calculate their cash values on any basis that produces values which are not less than the prescribed minimum. Many policies provide the full reserve as a cash value after a certain number of years.

The cash value forms the basis of another important policy provision—the loan clause. This provides that the insurer will grant, on the security of the policy, a loan not greater than the cash value. If, as a result of the operation of this clause, there is any indebtedness outstanding at the time of settlement of the policy, the amount of the debt is deducted from the policy proceeds.

6. Paid-up insurance and extended insurance

One of the options available in place of a cash value is *reduced paid-up insurance*. Under this option, the insurance is reduced to such an amount as the cash value will purchase on a net single premium basis, the calculation being made on the mortality and interest assumptions specified in the policy. With ordinary life insurance, for example, the amount of paid-up insurance provided at the end of t years by the cash value of ${}_tCV_x$ will be

$$\frac{{}_tCV_x}{A_{x+t}}.$$

If there is a loan against the policy when the paid-up option is elected, the amount of the cash value is reduced by the amount of the indebtedness before being applied to purchase the paid-up insurance. The paid-up policy is then free of indebtedness.

The amount of paid-up insurance that can be provided on an ordinary life policy at duration t by the full net level premium reserve ${}_tV_x$ is denoted by ${}_tW_x$, and

$${}_tW_x = \frac{{}_tV_x}{A_{x+t}}. \tag{6.13}$$

This can be expressed as

$$
\begin{aligned}
{}_tW_x &= \frac{A_{x+t} - P_x \cdot \ddot{a}_{x+t}}{A_{x+t}} \\
&= 1 - \frac{P_x}{P_{x+t}}. \qquad (6.14)
\end{aligned}
$$

Formula (6.14) can be obtained by general reasoning. After t years the net annual premium for an insurance of 1 effected at the attained age is P_{x+t}. An annual premium of P_x payable at that age is sufficient to provide insurance of only $\dfrac{P_x}{P_{x+t}}$. Since P_x is the amount of premium actually being paid for a policy providing insurance of 1, the amount of paid-up insurance provided by the reserve must be the balance, or $1 - \dfrac{P_x}{P_{x+t}}$.

Formulas (6.13) and (6.14) take the following forms for endowment insurance:

$$
{}_tW_{x:\overline{n}|} = \frac{{}_tV_{x:\overline{n}|}}{A_{x+t:\overline{n-t}|}} = 1 - \frac{P_{x:\overline{n}|}}{P_{x+t:\overline{n-t}|}}.
$$

For an n-payment life policy, we have

$$
{}^n_tW_x = \frac{{}^n_tV_x}{A_{x+t}} = 1 - \frac{{}_nP_x}{{}_{n-t}P_{x+t}}.
$$

A second insurance option, known as *extended insurance*, provides for the use of the cash value to purchase term insurance for the full amount of the policy for a limited period. If the face amount is 1, the extended insurance period n is determined from the relation

$$
A^{\,1}_{x+t:\overline{n}|} = {}_tCV
$$

where the value of n is customarily interpolated to days.

In the case of endowment policies, the cash value at the later durations may be more than sufficient to purchase term insurance for the full amount to the end of the endowment period. In that case, the excess of the cash value over the amount required to pay for the term insurance to the end of the endowment period is applied to purchase a pure endowment payable at maturity. The amount of the pure endowment is given by the expression

$$\frac{{}_tCV - A^{\ 1}_{x+t:\overline{n}|}}{A_{x+t:\overline{n}|}^{\ \ \ \ \ 1}},$$

where n is the number of years remaining in the endowment period.

If the policy is subject to indebtedness of amount L, the policy usually provides that the extended insurance will be for the amount of $1 - L$. Otherwise the policyholder would be able to increase the insurer's risk by his own action in electing this option. The term of the extended insurance is then the value of n for which

$$(1 - L)A^{\ 1}_{x+t:\overline{n}|} = {}_tCV - L.$$

This method provides a smaller amount of extended insurance for a longer period, and helps to protect the insurer against possible anti-selection in the use of the option.

7. Summary of notation

In this chapter, a few special conventions have been adopted which are not part of the standard notation. These are described at the end of Section 2.

References

1. Expense analysis and the other topics in this chapter are covered in greater detail in the course of study for the Fellowship examinations given by the Society of Actuaries. Students who wish to pursue these subjects in more detail at this time will find suitable references in the course of reading prescribed for these examinations.

2, 3, 4. The chapter on Modified Reserve Systems in Menge and Fischer (1965) provides a good elementary introduction to this subject. A more detailed treatment may be seen in the papers by Menge (1936), (1946).

An early version of the text of the Standard Valuation Law, which defines the Commissioners reserve valuation method, is quoted by Menge (1946). The language of the statute does not define the method in the same terms that we use here, but it may be seen in Menge's paper that the two definitions are equivalent.

This paper also discusses the application of the method to policies with varying benefits or premiums.

Commissioners reserves using continuous functions are discussed by Smith (1961).

5. An early version of the text of the Standard Nonforfeiture Law, which defines the adjusted premium method, may be seen in *Reports and Statements on Nonforfeiture Benefits and Related Matters*, National Association of Insurance Commissioners, 1942, pp. 266–270. The Standard Valuation Law is also included here, pp. 271–274.

A policy which provides for the payment of the cash value in addition to the face amount is discussed by Gold and Wilson (1963). A more general treatment of this type of policy is given by Smith (1964).

6. Hummel and Stedman (1958) give an interesting mathematical analysis of the rate of increase in the paid-up insurance values as compared with the rate of increase in the cash values.

EXERCISES

1. The evaluation of expenses

1. Derive a formula in terms of annuity values for the gross annual premium at age x payable for n years for a whole life insurance of 1 if the expenses consist of a flat amount of h per year plus an additional amount of k in the first year plus j per cent of each premium paid.

2. Gross premiums for single premium endowment insurances are to be calculated with the following expense assumptions:

    Tax: $2\frac{1}{2}\%$ of the gross premium;
    Commission: 4% of the gross premium;
    Other expenses (per \$1000 of insurance): \$5 in first year and \$2.50 in renewal years.

Derive a formula in commutation functions for the gross single premium per \$1000 of insurance including provision for immediate payment of claims.

3. Obtain a formula for the gross single premium at age x for an immediate life annuity of 1 per annum payable m times a year if the expenses are k per cent of the gross premium plus a flat amount of h per year.

4. Derive a formula in commutation symbols for the net annual premium for a whole life insurance providing for the return of the gross premiums paid along with the sum insured of 1. Assume that the gross annual premium G is related to the net annual premium P as follows: $G = (P + c)(1 + k)$.

2. Modified reserve systems

5. For a preliminary term valuation method where the modification covers the entire premium-paying period, show that the modified reserve

${}_tV^{\text{Mod}}$ differs from the net level premium reserve ${}_tV$ on a policy with premiums payable for n years by the amount

$${}_tV - {}_tV^{\text{Mod}} = (\beta - P)\ddot{a}_{x+t:\overline{n-t}|} = (\beta - \alpha)\,\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}.$$

6. Show that when $k = n$

$${}_tV^{\text{Mod}}_{x:\overline{n}|} = 1 - (\beta + d)\ddot{a}_{x+t:\overline{n-t}|}.$$

7. The attained age method is to be used in valuing a group of policies on a modified reserve basis with first year net premium α and renewal premium β. The formula is

$${}_tV^{\text{Mod}} = A_{x+t} - \beta\cdot\ddot{a}_{x+t} + \frac{\theta_x}{D_{x+t}}.$$

Find the expression for θ_x in this formula.

8. In a modified reserve system with an n-year modification period, the reduced first-year net premium is equal to one-fourth of the net level annual premium. Express $\dfrac{\beta - P}{P}$ in terms of $a_{x:\overline{n-1}|}$, where β is the increased renewal net premium and P is the net level annual premium.

3. The full preliminary term plan

9. Give a mathematical demonstration of the following equivalence:

$${}^{n}_{t}V^{F}_{x:\overline{m}|} = {}^{n-1}_{t-1}V_{x+1:\overline{m-1}|}.$$

10. Show that

$${}_tV^{F}_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x+1:\overline{n-1}|}} \qquad \text{for } 1 < t < n.$$

11. A twenty-five payment whole life policy issued at age 40 provides insurance of 5 during the first twenty-five years and 1 thereafter. Give expressions for α^F and β^F in terms of commutation functions.

4. Modified preliminary term plans

12. The n-th year terminal reserve by the Commissioners reserve valuation method for a premium-paying policy issued at age x, where $\beta^F > {}_{19}P_{x+1}$, may be written as

$${}_nV^{\text{Com}} = {}^{20}_{n}V^{\text{Com}}_{x} + \frac{T(N_x - N_{x+n})}{D_{x+n}}, \qquad n < 20.$$

Find the expression for T.

13. Give formulas for the net premiums and the first, seventh, and fifteenth year terminal reserves for a 10-payment 20-year endowment policy issued

at age 25 and valued according to the Commissioners reserve valuation method.

14. The New Jersey modified preliminary term reserve method provides for a first year net premium of c_x, renewal premiums of β^J from the second to the twentieth year inclusive, with net level premiums payable thereafter. For an n-payment m-year endowment, show that

$$\beta^J = P + \frac{P - c_x}{a_{x:\overline{19}|}}$$

where $P = {}_nP_{x:\overline{m}|}$, $n > 20$.

15. The Illinois modified preliminary term reserve method defines a modification for policies issued at age x, with $\beta^F > {}_{19}P_{x+1}$, such that $\beta^I - \alpha^I = {}_{19}P_{x+1} - c_x$, where the modification period is the lesser of 20 years and the premium-payment period.

For an n-payment m-year endowment with $\beta^F > {}_{19}P_{x+1}$ and $n > 20$, show that

(a) $\beta^I = {}_nP_{x:\overline{m}|} + \dfrac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{20}|}}$,

(b) ${}^n_tV^I_{x:\overline{m}|} > {}^n_tV^{\text{Com}}_{x:\overline{m}|}$ for $t < n$.

5. Cash values

16. Derive an expression, corresponding to formula (6.9), for the adjusted premium P^A on a policy issued at age x when it is known that $P^A_x < P^A < .04$.

17. Compute ${}_{10}P^A_{35:\overline{15}|}$ given the following values on the 1958 CSO 3% basis: ${}_{10}P_{35:\overline{15}|} = .07490$, $A_{35} = .35866$, $\ddot{a}_{35} = 22.01926$, $\ddot{a}_{35:\overline{10}|} = 8.67455$.

(Ans. .07957)

18. Assuming that the value of the adjusted premium is known, write formulas for computing adjusted premium surrender values for the fifth and the twelfth years for a 10-payment 15-year endowment issued at age 35.

6. Paid-up insurance and extended insurance

19. For a 30-year endowment policy with face amount 1 issued to (x), the amount of reduced paid-up insurance at the end of 10 years is .55. For a 20-year endowment policy with face amount 1 issued to $(x + 10)$, the amount of reduced paid-up insurance at the end of 5 years is .30. For the 30-year endowment policy issued to (x), find the amount of reduced paid-up insurance at the end of 15 years. Assume that cash values are equal to net level reserves. (Ans. .685)

20. At the end of the sixteenth policy year, as a result of the non-payment of the premium then due, a \$2,000 endowment at age 65 policy issued at age 43 is changed to an extended insurance basis. Loan indebtedness of \$100 is outstanding. The cash surrender value is equal to the reserve according to the Commissioners reserve valuation method. Derive an expression for the

cash surrender value and, assuming that the net value is used to purchase extended term insurance at net rates for the face amount of the policy less the loan indebtedness, show how to calculate the amount of pure endowment payable to the insured at age 65.

21. A 20-year endowment policy of 1, issued at age 25, is lapsed at the end of 15 years when there is indebtedness of L outstanding against the cash surrender value of C. Express in terms of commutation symbols:

(i) The amount of reduced paid-up insurance; and the amount of pure endowment at the regular maturity date if extended insurance for the face amount of the policy less indebtedness continues to the regular maturity date.

(ii) The retrospective and prospective expressions for the reserve on the extended insurance and pure endowment three years after the date of lapse. Show that the two expressions are equivalent.

Miscellaneous problems

22. Find an expression in commutation functions for the gross annual premium for a whole life policy for \$1000 issued to (x) and subject to the condition that, in the event of a claim before age $x + n$, the only death benefit will be the return of gross premiums paid without interest. Allow for a loading of 20% of the net premium plus \$3.

23. Write expressions for the difference between the Commissioners reserve and the net level premium reserve at the end of t years for

(a) 30-year endowment insurance issued at age x, where $\beta^F > {}_{19}P_{x+1}$;

(b) whole life insurance with premiums to age 65 issued at age $x > 45$.

24. Give the retrospective formula for the net level premium reserve at attained age y for a 20-payment life policy issued at age x, for $y < x + 20$. Derive and give a verbal interpretation of the difference between this reserve and the reserve at the same attained age for paid-up term insurance for the face amount of the policy, purchased by the net level premium reserve on the 20-payment life policy at age $x + t$, for $t < y - x$.

25. A 20-payment life policy issued 10 years ago at age x is to be changed to a fully-paid life policy for the same amount. Each of the following expressions has been suggested as the net cost to the insured for this change:

(1) ${}_{20}P_x \cdot \ddot{a}_{x+10:\overline{10|}}$,

(2) $(1 - {}^{20}_{10}W_x)A_{x+10}$,

(3) $A_{x+10} - {}^{20}_{10}V_x$.

Explain the reasoning by which each expression has been obtained. Demonstrate the mathematical equivalence of the three expressions.

26. A 20-payment life policy issued at age 30 has outstanding against it at the end of the fifteenth year a loan equal to the full net level premium terminal reserve. If the rate at which the loan draws interest and the interest rate at which the insured may expect to invest are each equal to the valuation interest rate of 4 per cent, and if all limited payment life premiums

are loaded 10 per cent of the net, plus C dollars for each dollar of insurance, what is the maximum value of C in order that it be to the insured's advantage to continue his present policy instead of taking out a 5-payment life policy for the present net amount at risk?

Given that $\ddot{a}_{45} = 16.09$ and $\ddot{a}_{45:\overline{5}|} = 4.53$. (Ans. .01366)

27. A policy is issued at age 20 with a maturity value of \$1,500 at age 50. Obtain an expression involving commutation symbols for the net annual premium where the death benefit prior to age 50 is return of gross premiums paid, with interest at the rate used in the premium calculation. Assume a loading of 10% of the gross premium.

Chapter 7

SPECIAL TOPICS

1. Introduction

The practice of combining more than one insurance or annuity benefit in a single policy has resulted in many unusual problems of premium determination. The solution of these problems always depends upon the application of principles established in the earlier chapters, but special methods are often required in order to obtain results which are suitable for computation. We shall discuss a few specific cases which will illustrate some important techniques not yet described.

2. Cash refund annuities

A *cash refund annuity* is a combination of the usual life annuity benefit with a death benefit equal at any time to the excess of the total premium paid over the sum of the annuity payments received. There is no death benefit when the sum of the annuity payments exceeds the amount of premium paid.

Let G denote here the gross single premium for an immediate cash refund annuity of 1 per annum issued at age x, with a loading of r per unit of the gross premium. Then the net single premium, $(1 - r)G$, is equal to the combined present value of the annuity and the death benefit. The annuity portion alone has the value a_x. The decreasing death benefit is G during the first year and reduces by 1 each year. Now if n is the greatest integer in G, the death benefit will extend over $n + 1$ years and its present value, assuming payment at the end of the year of death, may be expressed as

$$\frac{G(M_x - M_{x+n+1}) - (R_{x+1} - R_{x+n+1} - nM_{x+n+1})}{D_x}.$$

Hence,

$$(1 - r)G$$
$$= \frac{N_{x+1} + G(M_x - M_{x+n+1}) - (R_{x+1} - R_{x+n+1} - nM_{x+n+1})}{D_x},$$

and solving for G,

$$G = \frac{N_{x+1} - (R_{x+1} - R_{x+n+1} - nM_{x+n+1})}{(1 - r)D_x - (M_x - M_{x+n+1})}. \tag{7.1}$$

Since n in this formula depends upon the magnitude of G, trial values of n must be tested until a consistent value of G is obtained.

Let us now consider the general case in which the annuity payments are made m times a year. The death benefit now decreases by $1/m$ at the end of each m-th part of a year. If n is the greatest integer in G, the death benefit will extend over n complete years, those between ages x and $x + n$, and over a portion of the $(n + 1)$-th year which starts at age $x + n$.

An expression for the value of the death benefit during the year starting at age $x + t$, $0 \leqslant t < n$, can readily be derived by assuming a uniform distribution of deaths during the year of age. During the age interval $x + t$ to $x + t + 1/m$, the death benefit is $G - t$, and, with a uniform distribution of deaths, the present value at age x of the insurance for this period is $\dfrac{(G - t)C_{x+t}}{mD_x}$. During the age interval $x + t + 1/m$ to $x + t + 2/m$, the death benefit is $G - t - 1/m$, and has a present value at age x of $\dfrac{(G - t - 1/m)C_{x+t}}{mD_x}$. Proceeding throughout the year of age in this way and summing the results, we obtain

$$\begin{aligned} \frac{C_{x+t}}{mD_x} \sum_{i=0}^{m-1} \left(G - t - \frac{i}{m}\right) &= \frac{C_{x+t}}{mD_x}\left[mG - mt - \frac{m(m-1)}{2m}\right] \\ &= \frac{C_{x+t}}{D_x}\left(G - t - \frac{m-1}{2m}\right) \end{aligned}$$

as the present value at age x of the insurance benefit during the year of age $x + t$. Now summing for all ages from x through $x + n - 1$, we have

$$\sum_{t=0}^{n-1} \frac{C_{x+t}}{D_x}\left(G - t - \frac{m-1}{2m}\right) =$$

$$\frac{1}{D_x}\left\{\left(G - \frac{m-1}{2m}\right)(M_x - M_{x+n}) - [R_{x+1} - R_{x+n} - (n-1)M_{x+n}]\right\}.$$

To this must be added the value of the death benefit for a fraction of the year of age $x + n$ to $x + n + 1$. The death benefit at the beginning of this year is $G - n$, and if we let k denote the greatest integer in the ratio $\dfrac{G-n}{\frac{1}{m}}$, we can express the value of this partial year's insurance as

$$\frac{C_{x+n}}{mD_x}\sum_{i=0}^{k}\left(G - n - \frac{i}{m}\right) = \frac{C_{x+n}}{mD_x}\left[(k+1)(G-n) - \frac{k(k+1)}{2m}\right]$$
$$= \frac{(k+1)C_{x+n}}{mD_x}\left(G - n - \frac{k}{2m}\right).$$

To the above two expressions must be added the present value of the annuity payments, $a_x^{(m)}$, and the total can then be equated to the net single premium $(1 - r)G$:

$$(1 - r)G \doteq \frac{1}{D_x}\left\{N_{x+1} + \frac{m-1}{2m}D_x + \left(G - \frac{m-1}{2m}\right)(M_x - M_{x+n})\right.$$
$$\left. - [R_{x+1} - R_{x+n} - (n-1)M_{x+n}] + \frac{(k+1)C_{x+n}}{m}\left(G - n - \frac{k}{2m}\right)\right\}.$$

Solving for G, we have

$$G \doteq \frac{A}{B}, \tag{7.2}$$

where

$$A = N_{x+1} + \frac{m-1}{2m}(D_x - M_x + M_{x+n})$$
$$- [R_{x+1} - R_{x+n} - (n-1)M_{x+n}] - \frac{k+1}{m}\left(n + \frac{k}{2m}\right)C_{x+n}$$

and

$$B = (1 - r)D_x - (M_x - M_{x+n}) - \frac{k+1}{m}C_{x+n}.$$

In this formula, both n and k depend upon the magnitude of G, and trial values of n and k must be substituted until a consistent value of G is obtained.

In practice, formula (7.2) is simplified by arbitrarily introducing one of several possible substitutions which will eliminate k. For example, the substitution $k = m - 1$ leads to the approximate formula

$$G \doteqdot \qquad (7.3)$$

$$\frac{N_{x+1} - (R_{x+1} - R_{x+n+1} - nM_{x+n+1}) + \frac{m-1}{2m}(D_x - M_x + M_{x+n+1})}{(1 - r)D_x - (M_x - M_{x+n+1})}.$$

The value of G given by formula (7.3) never exceeds the value given by (7.2), since when the true value of k is less than $m - 1$ the effect of the approximation employed is to include in the premium the value of a negative death benefit in the latter part of the year under consideration. This introduces a credit element into G which would not be present in the true value. The magnitude of the error is negligible for practical purposes.

It can be shown that the average degree of error introduced by substituting $k = 0$ is the same as the average degree of error in the approximation represented by (7.3), and is again in the direction of understating the premium. However, the resulting formula is more complex than (7.3).

Formulas for single premium *deferred* cash refund annuities may be similarly developed. Let us assume that the annuity is issued at age x and provides an annual income of 1 payable m times a year commencing at age $x + t$. The benefit will then be a t-year deferred cash refund annuity-due. During the deferred period, the death benefit is equal to the gross single premium, and this death benefit is reduced by the payments made at age $x + t$ and thereafter. The student should undertake to verify the following expressions for the present values at age x of the various elements in this benefit. The method and assumptions are similar to those used in the derivation of formula (7.2).

	Present value at age x
Net premium	$(1 - r)G$
Annuity benefit	$\frac{1}{D_x}\left(N_{x+t} - \frac{m-1}{2m} D_{x+t}\right)$
Death benefit between ages x and $x + t$	$\frac{G}{D_x}(M_x - M_{x+t})$

Death benefit between ages $x + t$ and $x + t + n$: $\frac{1}{D_x}\Big[G(M_{x+t} - M_{x+t+n}) - (R_{x+t} - R_{x+t+n} - nM_{x+t+n}) + \frac{m-1}{2m}(M_{x+t} - M_{x+t+n})\Big]$

where n is the greatest integer in G

Death benefit between ages $x + t + n$ and $x + t + n + \frac{k}{m}$: $\frac{kC_{x+t+n}}{mD_x}\left(G - n - \frac{k+1}{2m}\right)$

where k is the greatest integer in $m(G - n)$.

Equating the net single premium to the sum of the present values of the benefit elements, and solving for G, we find

$$G \doteq \frac{E}{F}, \tag{7.4}$$

where

$$E = N_{x+t} - (R_{x+t} - R_{x+t+n} - nM_{x+t+n}) - \frac{m-1}{2m}(D_{x+t} - M_{x+t} + M_{x+t+n}) - \frac{k}{m}\left(n + \frac{k+1}{2m}\right)C_{x+t+n}$$

and

$$F = (1 - r)D_x - (M_x - M_{x+t+n}) - \frac{k}{m}C_{x+t+n}.$$

The formula may be expressed more concisely if we modify the standard notation here and write

$$N_y^{(m)} = N_y - \frac{m-1}{2m}D_y$$

and

$$R_y^{(m)} = R_y - \frac{m-1}{2m}M_y.$$

Then the numerator E in (7.4) can be written:

$$E = N_{x+t}^{(m)} - (R_{x+t}^{(m)} - R_{x+t+n}^{(m)} - nM_{x+t+n}) - \frac{k}{m}\left(n + \frac{k+1}{2m}\right)C_{x+t+n}.$$

In this case, a more convenient practical formula may be obtained by substituting $k = 0$:

$$G \doteq \frac{N_{x+t}^{(m)} - (R_{x+t}^{(m)} - R_{x+t+n}^{(m)} - nM_{x+t+n})}{(1 - r)D_x - (M_x - M_{x+t+n})}. \tag{7.5}$$

As in the case of formula (7.3) for the immediate annuity, the approximation understates the premium slightly (unless the true value of k is 0), although the understatement is here due to the elimination of positive benefits in the final year rather than the introduction of negative benefits. The approximation resulting from $k = m - 1$ gives results which are, on the average, equally accurate, but produces a less convenient formula.

Another practical formula independent of k, similar to (7.5), may be derived by assuming a uniformly diminishing death benefit over the *entire* year of age $x + t + n$ to $x + t + n + 1$, in place of the true benefit which diminishes uniformly over the fractional age period $x + t + n$ to $x + t + n + \frac{k}{m}$. The student may show that with this assumption the present value at age x of the final year's death benefit is

$$\frac{C_{x+t+n}}{D_x}\left(\frac{m - 1}{2m}\right)(G - n),$$

leading to the following approximation, which yields a result never less than the true value:

$$G \doteq \frac{N_{x+t}^{(m)} - (R_{x+t}^{(m)} - R_{x+t+n}^{(m)} - nM_{x+t+n}^{(m)})}{(1 - r)D_x - (M_x - M_{x+t+n}^{(m)})}, \tag{7.6}$$

where $N^{(m)}$ and $R^{(m)}$ are defined as above and

$$M_y^{(m)} = M_y - \frac{m - 1}{2m} C_y.$$

Although the formulas appear complicated, the work can be arranged in a systematic way so that the computation, including the testing of trial values of n, is not unduly burdensome. With a digital computer, it is possible to produce complete tables of premiums by age very quickly.

3. Annuities with a term-certain

Another type of refund annuity provides for the continuation of payments after the death of the annuitant in the event that the annuitant has failed to receive a given minimum number of payments. This minimum may be either a specified number of years, such as 5, 10, or 20, or the number of years required for the total payments to equal the amount of premium.

In the first case, in which the minimum number of annuity payments is independent of the gross premium, the benefit may be expressed as a combination of an annuity-certain and a deferred life annuity. Thus, the gross single premium, loaded r per unit, for an immediate life annuity of 1 payable m times a year issued to (x) with payments guaranteed for n years may be obtained from the formula,

$$G(1 - r) = a_{\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)}. \tag{7.7}$$

In the second case, in which the annuity payments are continued until a sum equal to the gross premium G is returned, the length of the certain period n depends upon the magnitude of G. If the gross premium G is an exact multiple of the annuity payment, we can let $n = G$ in (7.7) and write

$$G(1 - r) = a_{\overline{G}|}^{(m)} + {}_{G|}a_x^{(m)}. \tag{7.8}$$

Since the gross premium G is normally not an exact multiple of the annuity payment, the annuity functions in (7.8) will in general not be defined. As a result, formula (7.8) cannot be directly applied, and it is customary to find G by means of an interpolation device. If we write

$$f(G) = a_{\overline{G}|}^{(m)} + {}_{G|}a_x^{(m)} - G(1 - r), \tag{7.9}$$

then the value of G is required for which $f(G) = 0$. The procedure is to determine by trial an integer n for which $f(n) > 0$ and $f(n + 1) < 0$. Then by a linear interpolation,

$$G \doteq n + \frac{f(n)}{f(n) - f(n + 1)}. \tag{7.10}$$

This type of annuity is called a *refund* or *instalment refund annuity.*

As an example, consider an instalment refund annuity providing a monthly income of \$10 issued to a life aged 65 with a loading of 6% of the gross premium. Using formula (7.9), we write

$$f(n) = a_{\overline{n}|}^{(12)} + {}_{n|}a_{65}^{(12)} - .94\,n.$$

Suppose that after a few trials we find $f(15) = .4739$ and $f(16) = -.1094$. Then from (7.10),

$$G \doteq 15 + \frac{.4739}{.4739 - (-.1094)} = 15.8124.$$

For \$10 per month, we have \$120 $\times$ 15.8124 = \$1,897.49.

4. Complete annuities

An annuity which provides for a final payment at death proportional to the time elapsed since the last payment is called a *complete*, or *apportionable*, annuity. The usual annuity, which does not provide a final fractional payment of this type, is sometimes called a *curtate* annuity, and the value of the complete annuity differs from the value of the corresponding curtate annuity by the value of the proportionate payment at death.

The complete annuity payable m times a year, denoted by $\mathring{a}_x^{(m)}$, is equivalent to a curtate annuity, $a_x^{(m)}$, plus a continuously increasing insurance payable at the moment of death, $(\bar{I}\bar{A})_x$, reduced at the end of each interval by $\frac{1}{m}$. The reduction is equivalent to an increasing insurance which provides insurance of $0, \frac{1}{m}, \frac{2}{m}, \cdots$, and its value is accordingly given by $(I^{(m)}\bar{A})_x - \frac{1}{m}\bar{A}_x$. We then have

$$\mathring{a}_x^{(m)} = a_x^{(m)} + (\bar{I}\bar{A})_x - (I^{(m)}\bar{A})_x + \frac{1}{m}\bar{A}_x. \qquad (7.11)$$

If we now substitute in this formula for $(\bar{I}\bar{A})_x$ from formula (3.30) and for $(I^{(m)}\bar{A})_x$ from formula (3.28), we obtain

$$\mathring{a}_x^{(m)} \doteq a_x^{(m)} + \frac{1}{2m}\bar{A}_x. \qquad (7.12)$$

As the formula shows, the value of the apportionable feature is roughly equal to $\frac{1}{2m}\bar{A}_x$, an insurance of half the periodical payment, death occurring on the average halfway between two payment dates.

Another approximation,

$$\ddot{a}_x^{(m)} \doteq \left(1 - \frac{\delta}{2m}\right) \bar{a}_x \,, \tag{7.13}$$

is derived from (7.12) by means of the two substitutions

$$a_x^{(m)} \doteq \bar{a}_x - \frac{1}{2m}$$

and

$$\bar{A}_x = 1 - \delta \bar{a}_x \,.$$

5. The retirement income policy

Many insurers offer policies which provide a combination of insurance coverage up to a fixed retirement age and an annuity income after retirement. Premiums are normally payable until the retirement age is reached. The retirement income is often at the rate of \$10 per month for each \$1000 of face amount and is customarily payable for a fixed number of years certain and the remainder of life.

Under normal circumstances the present value at the retirement age of the annuity income of \$10 per month exceeds the face amount of \$1000. This means that the policy must mature at the retirement age for an amount greater than the face amount, and that the policy reserves and cash values, in building up to the required maturity value, will also exceed the face amount for a period prior to retirement. Since it is impractical to have the face amount of insurance less than the cash value, these policies provide that the death benefit will be equal to the cash value during the period when the cash value exceeds the face amount.

Let $1 + k$ denote the present value at the retirement age, $x + n$, of the annuity to be provided at that age for each unit of face amount issued at age x. Then the policy can be regarded as a combination of a pure endowment of $1 + k$ due at the retirement age and a term insurance up to that age providing a death benefit equal to 1 or the total cash value on the policy whichever is greater. The net premium, P, for the policy can therefore be written

$$P = \frac{M_x - M_{x+a} + \sum_{t=a+1}^{n} ({}_tCV \cdot C_{x+t-1}) + (1 + k)D_{x+n}}{N_x - N_{x+n}} \,, \tag{7.14}$$

where ${}_tCV$ is the cash value at the end of t years and ${}_{a+1}CV$ is the first cash value that exceeds the face amount of 1.

Since the cash values and the premium P are usually interrelated, formula (7.14) cannot generally be solved for P directly. Suppose, however, that the cash values at durations a and greater are equal to the net level premium reserves. With this assumption we are able to derive an explicit formula for P.

We may assume that the terminal reserves ${}_tV$ increase from ${}_0V = 0$ to ${}_nV = 1 + k$. Hence, there is exactly one integer a such that ${}_aV \leqslant 1$ and ${}_{a+1}V > 1$. During the first a years, the death benefit is the face amount, and we have

$$({}_tV + P)(1 + i) = q_{x+t} + p_{x+t}\cdot{}_{t+1}V.$$

After a years, the death benefit is the reserve, and

$$({}_tV + P)(1 + i) = q_{x+t}\cdot{}_{t+1}V + p_{x+t}\cdot{}_{t+1}V,$$

or

$$({}_tV + P)(1 + i) = {}_{t+1}V. \tag{7.15}$$

This relation shows that the reserve accumulation is independent of mortality after a years, there being no amount at risk and hence no insurance element. The prospective formula for ${}_aV$ will thus be

$${}_aV = (1 + k)v^{n-a} - P\cdot\ddot{a}_{\overline{n-a}|}\,. \tag{7.16}$$

The retrospective formula is

$${}_aV = P\cdot\ddot{s}_{x:\overline{a}|} - {}_ak_x\,. \tag{7.17}$$

Equating the two expressions for ${}_aV$ and solving for P, we find

$$\begin{aligned} P &= \frac{{}_ak_x + (1 + k)v^{n-a}}{\ddot{s}_{x:\overline{a}|} + \ddot{a}_{\overline{n-a}|}} \\ &= \frac{M_x - M_{x+a} + (1 + k)v^{n-a}D_{x+a}}{N_x - N_{x+a} + \ddot{a}_{\overline{n-a}|}\,D_{x+a}}\,. \end{aligned} \tag{7.18}$$

The value of a used in this formula depends upon the magnitude of P, and may be determined by a trial-and-error process. It is possible, however, to develop a simple criterion which permits the independent determination of a.

We shall use the relations

$$P \leqslant P_{x:\overline{a}|} \qquad \text{and} \qquad P > P_{x:\overline{a+1}|}$$

which follow from ${}_aV \leqslant 1$ and ${}_{a+1}V > 1$.

We first transform (7.16) as follows:

$${}_aV \cdot (1+i)^{n-a} + P \cdot \ddot{s}_{\overline{n-a}|} = 1 + k.$$

In this expression, if we replace $(1+i)^{n-a}$ by $1 + d\,\ddot{s}_{\overline{n-a}|}$ and use the inequalities ${}_aV \leqslant 1$ and $P \leqslant P_{x:\overline{a}|}$, we obtain

$$1 + (P_{x:\overline{a}|} + d)\ddot{s}_{\overline{n-a}|} \geqslant 1 + k,$$

which reduces to

$$\frac{\ddot{s}_{\overline{n-a}|}}{\ddot{a}_{x:\overline{a}|}} \geqslant k. \tag{7.19}$$

Now if we replace a by $a + 1$ in formula (7.16) and proceed similarly, we obtain

$$\frac{\ddot{s}_{\overline{n-a-1}|}}{\ddot{a}_{x:\overline{a+1}|}} < k. \tag{7.20}$$

Since $\dfrac{\ddot{s}_{\overline{n-t}|}}{\ddot{a}_{x:\overline{t}|}}$ is a decreasing function of t, we can conclude from (7.19) and (7.20) that a is the greatest integer for which

$$\frac{\ddot{s}_{\overline{n-a}|}}{\ddot{a}_{x:\overline{a}|}} \geqslant k. \tag{7.21}$$

6. Family income benefits

Family income benefit is the name given to a form of decreasing term insurance which provides that in the event of death within n years a monthly income will be payable to the beneficiary for the balance of the n-year period. The income commences on the date of death and is payable at monthly intervals thereafter until the last regular payment date falling within the original period of n years. This n-year period is called the family income period.

The symbol ${}_nF_x$ will be used to denote the net single premium for an n-year family income benefit issued at age x and providing an annual income of 1 payable monthly commencing at the date of death and continuing for the balance of the n-year period. If it is assumed that the income is payable at monthly intervals measured from the date of issue, rather than from the date of death, with the first payment falling due on the first day of the policy month following the date of death, an exact expression for the present value of the benefit would be

$$a_{\overline{n}|}^{(12)} - a_{x:\overline{n}|}^{(12)}. \tag{7.22}$$

The difference between the present values of the annuity-certain and the temporary life annuity clearly gives the value of the series of income payments for the balance of the n years after the death of (x).

We may now consider how expression (7.22) should be adjusted if payments are to commence upon the date of death instead of at the beginning of a policy month. It is clear that the same number of payments will be made in either case. The series of payments commencing at the date of death, however, will be made on the average one-half a month earlier. We thus obtain the following approximate formula for ${}_nF_x$:

$${}_nF_x \doteq (1 + i)^{\frac{1}{24}}(a_{\overline{n}|}^{(12)} - a_{x:\overline{n}|}^{(12)}). \tag{7.23}$$

It is sometimes necessary to determine family income premiums on the assumption of one interest rate during the insurance period and another rate during the period when the annuity is payable. Formula (7.23) obviously cannot be adapted to this situation, and a new formula must be constructed in which the annuity portion is evaluated separately from the insurance portion.

In the following development, the rate of interest before death is denoted by i and the rate after death by i', with all unprimed symbols referring to rate i and all primed symbols to rate i'.

Since the death benefit varies for each month of age, we will express the net single premium for the total contract as the sum of the present values of the insurance for each month. If death occurs in the month of age from $x + t + \frac{s}{12}$ to $x + t + \frac{s+1}{12}$, t and s being integers, the death benefit, evaluated at the date of death, has the present value $\ddot{a}'^{(12)}_{\overline{n-t-\frac{s}{12}}|}$, and the present value at age x of this month's insurance may be expressed as

$$I_{x+t+\frac{s}{12}} = v^t {}_tp_x \int_0^{\frac{1}{12}} v^{\frac{s}{12}+r} {}_{\frac{s}{12}+r}p_{x+t}\mu_{x+t+\frac{s}{12}+r}\, \ddot{a}'^{(12)}_{\overline{n-t-\frac{s}{12}}|}\, dr.$$

The value of the insurance for the year of age $x + t$ to $x + t + 1$

is then obtained by summing $I_{x+t+\frac{s}{12}}$ from $s = 0$ to $s = 11$, and finally the value of the insurance for the period from age x to $x + n$ is obtained by a second summation from $t = 0$ to $n - 1$:

$$_nF_x^{i\&i'} = \sum_{t=0}^{n-1} \sum_{s=0}^{11} I_{x+t+\frac{s}{12}}.$$

This exact formula is not in suitable form for calculation, but it can be reduced to a usable form by introducing the assumption of a uniform distribution of deaths in each year of age.

We will first simplify the definite integral for $I_{x+t+\frac{s}{12}}$ on this assumption. From (1.24b) we can write

$$_{\frac{s}{12}+r}p_{x+t}\mu_{x+t+\frac{s}{12}+r} \doteq q_{x+t},$$

so that

$$I_{x+t+\frac{s}{12}} \doteq v^t\,{}_tp_xq_{x+t}\ddot{a}'^{(12)}_{\overline{n-t-\frac{s}{12}}|}\int_0^{\frac{1}{12}} v^{\frac{s}{12}+r}\,dr.$$

Now

$$\int_0^{\frac{1}{12}} v^{\frac{s}{12}+r}\,dr = \left[\frac{v^{\frac{s}{12}+r}}{\log v}\right]_0^{\frac{1}{12}} = \frac{v^{\frac{s}{12}}}{\delta}\left(1 - v^{\frac{1}{12}}\right) = \frac{v^{\frac{s}{12}}d^{(12)}}{12\delta},$$

where $d^{(12)} = 12(1 - v^{\frac{1}{12}})$.

Making this substitution, and with

$$\ddot{a}'^{(12)}_{\overline{n-t-\frac{s}{12}}|} = \frac{1 - v'^{n-t-\frac{s}{12}}}{d'^{(12)}},$$

we have

$$I_{x+t+\frac{s}{12}} \doteq v^t\,{}_tp_xq_{x+t}\cdot\frac{d^{(12)}}{12\delta d'^{(12)}}\left[v^{\frac{s}{12}} - v'^{n-t}\left(\frac{v}{v'}\right)^{\frac{s}{12}}\right].$$

At this point, the introduction of a new rate of interest i'' such that $v'' = \frac{v}{v'}$ will simplify the subsequent derivation. With this substitution, we perform the summation of $I_{x+t+\frac{s}{12}}$ from $s = 0$ to 11, and noting that

$$\sum_{s=0}^{11} v^{\frac{s}{12}} = 12\ddot{a}_{\overline{1}|}^{(12)} = \frac{12(1-v)}{d^{(12)}} = \frac{12\,iv}{d^{(12)}},$$

we obtain

$$\sum_{s=0}^{11} I_{x+t+\frac{s}{12}} \doteq v'^{t}\,{}_tp_x q_{x+t}\cdot\frac{d^{(12)}}{\delta d'^{(12)}}\left(\frac{iv}{d^{(12)}} - \frac{v'^{n-t}i''v''}{d''^{(12)}}\right).$$

Finally, summing from $t = 0$ to $n - 1$, we have

$$_nF_x^{i\&i'} \doteq \frac{i}{\delta d'^{(12)}}A^{1}_{x:\overline{n}|} - \frac{v'^{n}i''d^{(12)}}{\delta d'^{(12)}d''^{(12)}}A''^{1}_{x:\overline{n}|}. \qquad (7.24)$$

The special rate of interest i'' is equal to $\dfrac{i - i'}{1 + i'}$.

The use of the special rate of interest i'' is not merely an algebraic device for simplifying a complex expression. It also provides a convenient practical method for performing an otherwise tedious calculation. The values of $A''^{1}_{x:\overline{n}|}$ are easily obtained from the special commutation functions $D''_x = (1 + i')^x D_x$ and $C''_x = (1 + i')^{x+1}C_x$.

When $i' = i$, the above derivation leads to the formula

$$_nF_x \doteq \frac{i}{\delta d^{(12)}}A^{1}_{x:\overline{n}|} - \frac{v^n}{\delta}\,{}_nq_x \doteq \frac{1}{d^{(12)}}\bar{A}^{1}_{x:\overline{n}|} - \frac{v^n}{\delta}\,{}_nq_x. \qquad (7.25)$$

The approximations produced by this formula are very close to those given by (7.23), which is derived from a similar assumption.

Family income benefits are commonly issued in combination with permanent plans of insurance. When this is done, the permanent plan is usually modified so that the face amount is not payable until the end of the n-year family income period in the event of death during those n years. When the face amount is retained by the insurer in this way, it may be assumed that a portion of the family income benefit is provided by the interest earned on the face amount. If, for example, the permanent plan has a face amount of \$1000 and the family income benefit is \$10 per month, the face amount will provide a monthly interest income of $1000\,\dfrac{d^{(12)}}{12}$ payable in advance, leaving a balance of $10 - 1000\,\dfrac{d^{(12)}}{12}$ to be provided by the family income premium. The net single premium for the combination benefit thus consists of the premium for the permanent plan plus the family income premium of

$$(120 - 1000d^{(12)})_nF_x \,. \tag{7.26}$$

It should be noted that an additional approximation has been introduced in the reasoning by which (7.26) has been derived. If death occurs during the family income period, the face amount becomes payable at the end of n years from the date of issue. The monthly payment dates, however, do not usually correspond with the beginning of policy months, and the final income payment normally becomes due less than a month before the face amount is payable. Consequently, it should not be assumed that the face amount provides a full interest payment of $\frac{1000d^{(12)}}{12}$ for this final fractional period. The error is obviously very small, and is often ignored in practice. The derivation of a formula which is free from this error will be suggested as an exercise for the student.[1]

Annual premiums for family income benefits may be derived in accordance with the usual principles by dividing the net single premium by the appropriate life annuity factor. These policies are sometimes issued with premiums on a limited-payment basis in order to eliminate the negative reserves which may occur when a decreasing benefit of this kind is subject to level annual premiums.

7. Notation

One new notational principle has been introduced in this chapter:

J. A small circle placed over the principal symbol shows that the function is complete. This is illustrated by the complete annuity $\mathring{a}_x$.

It should be understood that symbols like the following are not standard notation: $N_x^{(m)}$, ${}_nF_x$, D_x''. They will not be used outside of the special context for which they were defined in this chapter.

References

2. Employee pension plans are often provided on a group annuity basis, and these plans usually include some variation of the cash refund benefit. A specific example may be seen in Exercise 2.

4. Rasor and Greville (1952) have explored the consequences of making a somewhat different assumption as to the amount of

[1] See Exercise 17.

the fractional payment in the final year under a complete annuity. Their alternative assumption follows naturally from certain compound interest considerations, and their results are in many ways more consistent than those which follow from the classical definition. This paper will be of interest to students who wish to see additional relationships between the theories of compound interest and life contingencies.

5. The original paper on the retirement income policy was that of Fassel (1930). Our treatment here is in a form suggested to the author by Harry Gershenson, F.S.A. An approach using continuous functions is given by Smith (1961).

Adjusted premiums and minimum cash values for this policy are discussed by Hahn (1946). Rosser's discussion of the paper by Walker and Lewis (1949) gives references to formulas for most of the major reserve modification methods. Nesbitt and Van Eenam (1952) have investigated a related type of policy which provides insurance for face amount or paid-up insurance amount if greater.

6. The treatment of family income benefits is adapted from the paper by Cody (1948).

EXERCISES

2. Cash refund annuities

1. Obtain an expression in a form involving an integral for a continuous annuity to a person aged y which provides 1 per annum payable for life, and the refund in cash immediately upon the death of (y) of the excess of an amount k over the total annuity payments previously made. From this expression derive a formula which will enable you to evaluate it assuming you are given k, i, and a table of values of D_x, N_x, M_x, and R_x. State any approximations used.

2. An employee pension plan is established under which the employee contributes \$3 toward the single premium for each \$1 of annual income provided at retirement. In the event of death prior to retirement, the employee's contribution is returned together with compound interest to the end of the year of death. After retirement, the death benefit at any time is equal to the excess of the death benefit at retirement over the sum of the annuity payments made.

For an annuity issued at age x to provide an annual income of 1 commencing at age $x + t$, show that the gross single premium loaded 8% of itself is given by

$$\frac{N_{x+t} - (R_{x+t} - R_{x+t+n} - nM_{x+t+n}) + 3D_x - 3(1+i)^t(D_{x+t} - M_{x+t} + M_{x+t+n})}{.92D_x}$$

where n is the greatest integer in $3(1 + i)^t$. It is assumed that the interest in the death benefit is at the same rate as that used in the premium calculation.

3. The gross annual premium payable for t years is required for an annuity of 1 payable m times a year with first payment at the end of t years with a return at death of the excess of premiums paid over the annuity payments made. The loading is r per unit of the gross premium. Obtain the premium formula, corresponding to formula (7.5) for the single premium case.

3. Annuities with a term-certain

4. Given the table of values below, compute the approximate gross single premium for an instalment refund annuity providing a monthly income of \$10 to a life aged 65. The loading is 5% of the gross premium.

n	$a^{(12)}_{\overline{n}\vert}$	$_{n\vert}a^{(12)}_{65}$
13	11.13	2.28
14	11.83	1.89
15	12.52	1.55

(Ans. \$1,764)

5. Show that the present value of an immediate life annuity of 1 per annum, payable for n years certain and the lifetime of (x), given by $a_{\overline{n}|} + {}_{n|}a_x$, may be alternatively expressed as

$$a_x + \frac{\ddot{a}_{\overline{n}|}M_x - v^n N_{x+1} + N_{x+n+1}}{D_x}.$$

By what reasoning may this latter formula be derived?

6. Determine in terms of an expression involving commutation symbols the net annual premium payable to age 65 for an annual annuity deferred to age 65 with payments guaranteed for 10 years under which the sum of all gross premiums paid is payable in the event of death before age 65. Net premiums are loaded by a percentage plus a constant.

4. Complete annuities

7. Show that

$$\mathring{a}_x = a_x + \sum_{n=0}^{\infty} v^n {}_np_x \int_0^1 t v^t {}_tp_{x+n}\mu_{x+n+t}\, dt.$$

Assuming a uniform distribution of deaths, derive

$$\mathring{a}_x \doteq a_x + A_x\left(\frac{i - \delta}{\delta^2}\right).$$

8. Show that $\mathring{a}_x = a_x - A^{\;1}_{x:\overline{1}|}\cdot\bar{A}_{x+1} - A^{\;1}_{x:\overline{2}|}\cdot\bar{A}_{x+2} - \cdots + (\bar{I}\bar{A})_x$ and explain the result by general reasoning.

9. Explain by general reasoning the relation

$$\bar{A}_x \doteq 1 - i\bar{a}_x .$$

Why is the equivalence only approximate?

10. Compute the exact value of $\bar{a}_x$ on the assumption of a uniform distribution of deaths throughout each year of age, given $q_x = \frac{3}{4}$, $q_{x+1} = 1$, and $i = 0$.

5. The retirement income policy

11. A retirement income policy issued at age 40 has a maturity value of 1.5 at age 65. The death benefit prior to age 65 is the terminal reserve or 1, whichever is greater. The terminal reserves are less than 1 to the end of the 17th year, but greater thereafter. Give an expression for the net annual premium.

12. Derive a formula in terms of commutation functions for the net single premium at age x for a retirement income policy providing a death benefit equal to 1 or the reserve if greater and maturing at the end of n years for $1 + k$. Give a criterion for determining c, the number of years during which the death benefit is constant at 1.

13. How will formula (7.18) be modified if premium payments are limited to r years, $r < a + 1$, where a is the greatest integer for which ${}_aV \leqslant 1$?

14. An annual premium retirement annuity provides a life income of \$1 a month, first payment at the retirement age, and the return of the gross premiums paid, or the reserve if greater, if death occurs before that age. Derive the net level premium for such a policy issued at age x with retirement age $x + n$. Assume that the gross premium is loaded a percentage of the net premium plus a constant.

6. Family income benefits

15. (a) Show that the net single premium at age x for an n-year family income benefit providing a yearly income of 1 payable continuously may be expressed in the following four ways:

$$\begin{aligned}
{}_nF_x &= \bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|} \\
&= \int_0^n v^t(1 - {}_tp_x)\, dt \\
&= \int_0^n v^t\, {}_tp_x \mu_{x+t} \bar{a}_{\overline{n-t}|}\, dt \\
&= \frac{1}{\delta} \bar{A}^{\,1}_{x:\overline{n}|} - \frac{v^n}{\delta}\, {}_nq_x .
\end{aligned}$$

(b) If the rate of interest before death is i and the rate after death is i', show that, without approximation, the net premium in (a) is

$$\frac{1}{\delta'}\,(\bar{A}^1_{x:\overline{n}|} - v'^n \overset{\prime\prime}{\bar{A}}{}^1_{x:\overline{n}|})$$

where
$$i'' = \frac{i - i'}{1 + i'}.$$

16. A policy provides a continuous annuity-certain of 1 per annum beginning at the date of death of (x). If death occurs within 15 years of issue, the annuity is payable to the end of 20 years from issue. If death occurs between 15 and 20 years from issue, the annuity is payable for 5 years certain. Coverage ceases 20 years from issue. Find an exact expression for the net single premium.

17. (a) An m-year endowment contract for \$1000 is modified so that if death occurs within the first n years the death benefit is retained by the insurer until the end of the n-year period and is paid at that time. If death occurs after n years, the death benefit is paid immediately.

Show that the net single premium at age x is given by

$$1000(v^n{}_nq_x + \bar{A}_{x:\overline{m}|} - \bar{A}^1_{x:\overline{n}|}).$$

(b) An n-year family income benefit providing \$10 per month is issued in combination with the m-year endowment described above. Using (7.25) to evaluate the family income portion, show that the total net single premium for the combination policy exceeds the single premium of 1000 $\bar{A}_{x:\overline{m}|}$ for the normal endowment insurance by the amount of

$$\left(\frac{120}{d^{(12)}} - 1000\right)\bar{A}^1_{x:\overline{n}|} - \left(\frac{120}{\delta} - 1000\right)v^n{}_nq_x.$$

This formula corrects the error noted in the text in connection with formula (7.26).

18. A contract provides for the payment of \$1000 at the end of 20 years if the insured is then living, or an income of \$10 a month in the event of death before the twentieth anniversary of the policy. The first income payment is due at the end of the policy month of death but no payments are made after 20 years from date of issue. Give the formula for the net annual premium at age x.

Miscellaneous problems

19. A life insurance company sells a single premium complete annuity of $\$K$ per year payable quarterly, with the provision that, if death occurs in the first policy year, five-sixths of the gross premium will be refunded, this death benefit decreasing by one-sixth of the gross premium each policy year thereafter until it becomes zero.

Find, in terms of commutation symbols, the net single premium at age x

for the annuity, if the gross premium is equal to the net premium loaded c per cent. Use standard approximations.

20. A 20-year term policy provides that in the event of the death of the insured during the 20-year period the company will pay a yearly income of \$100, payable continuously, until 20 years from issue or for 10 years from the date of death if the latter period is longer. Determine the net single premium at age x for this benefit in terms of an expression reduced to a form which does not contain integrals.

21. A retirement income policy is issued at age x and matures at age $x + n$ for $1 + k$. The death benefit is payable at the end of the policy year in which death occurs and is equal to 1 or the terminal reserve if greater. Annual premiums are payable for n years. Reserves are computed under a modification, extending over the entire premium-paying period, such that $\beta - \alpha = g_x$. Numerical values of g_x are available. Derive the criterion and the premium formulas.

CHAPTER 8
POPULATION THEORY

1. Some additional mortality functions

The analysis of population statistics forms a specialized branch of actuarial science known as demography. While this subject is in general outside the scope of this text, there are certain basic mortality functions which are common to demography and to insurance mathematics. An introductory explanation of these functions in demographic terms will assist the student in grasping their significance.

We first define several new functions:

$$L_x = \int_x^{x+1} l_y\,dy = \int_0^1 l_{x+t}\,dt \tag{8.1}$$

$$T_x = \int_x^{\infty} l_y\,dy = \int_0^{\infty} l_{x+t}\,dt = \sum_{y=x}^{\infty} L_y = \sum_{t=0}^{\infty} L_{x+t} \tag{8.2}$$

$$Y_x = \int_x^{\infty} T_y\,dy = \int_0^{\infty} T_{x+t}\,dt \tag{8.3}$$

As consequences of the definitions, we note that

$$\frac{dL_x}{dx} = l_{x+1} - l_x = -d_x$$

$$\frac{dT_x}{dx} = -l_x$$

$$\frac{dY_x}{dx} = -T_x$$

Throughout this chapter, we shall be using integral expressions and the student will find it helpful to keep in mind the following short table of indefinite integrals (arbitrary constants omitted):

$$\int l_y\mu_y\,dy = -l_y$$

$$\int l_y\,dy = -T_y$$

$$\int T_y \, dy = -Y_y$$

$$\int y l_y \mu_y \, dy = -y l_y - T_y$$

$$\int y l_y \, dy = -y T_y - Y_y$$

The last two formulas can be derived using integration by parts and may be verified by differentiation.

The new functions can be given concrete interpretations in terms of the mortality table model. Consider the l_x lives which survive to age x. We shall show that T_x represents the total number of years lived by the members of this group from age x until death. The number of these lives which die at the moment of attaining age y is given by $l_y \mu_y \, dy$. Since each of these has lived $y - x$ years since age x, the integral

$$\int_x^\infty (y - x) l_y \mu_y \, dy$$

represents the number of years lived by the whole group subsequent to attaining age x. Using the formulas just listed, we find

$$\int_x^\infty (y - x) l_y \mu_y \, dy = \int_x^\infty (y l_y \mu_y - x l_y \mu_y) \, dy$$

$$= [-y l_y - T_y + x l_y]_{y=x}^{y=\infty} = x l_x + T_x - x l_x = T_x .$$

The definition of L_x shows that it represents the mean value of the function l_y between x and $x + 1$. By changing the upper limit of integration to $x + 1$ in the above integrals, it can be seen that L_x also represents the number of years lived between ages x and $x + 1$ by the l_x lives which survive to age x.

T_x may also be described as the total future lifetime of the l_x lives which survive to age x. Then Y_x, as defined by (8.3), represents the total future lifetime of those aged x *and over*.

Numerical values of these functions are usually obtained by using the trapezoidal rule to compute the integrals:

$$L_x \doteq \tfrac{1}{2}(l_x + l_{x+1}) = l_x - \tfrac{1}{2} d_x \tag{8.4}$$

$$T_x \doteq \tfrac{1}{2}l_x + \sum_{y=x+1}^{\infty} l_y \tag{8.5}$$

$$Y_x \doteq \tfrac{1}{2}T_x + \sum_{y=x+1}^{\infty} T_y \tag{8.6}$$

The trapezoidal rule may not be sufficiently accurate when dealing with the values at the extreme ends of the table, and more refined methods are sometimes used for the very low and the very high ages.

2. The central death rate

The ratio of d_x to L_x is called the *central death rate at age x* and is denoted by m_x :

$$m_x = \frac{d_x}{L_x}. \tag{8.7}$$

It is known as a central rate since it relates the number of deaths between ages x and $x + 1$ to the mean value of l_x over that age interval.

Evaluating L_x by (8.4), we have the following formulas connecting m_x with q_x and p_x :

$$m_x \doteq \frac{d_x}{l_x - \frac{1}{2}d_x} = \frac{q_x}{1 - \frac{1}{2}q_x} \tag{8.8}$$

$$q_x \doteq \frac{m_x}{1 + \frac{1}{2}m_x} \tag{8.9}$$

$$p_x \doteq \frac{1 - \frac{1}{2}m_x}{1 + \frac{1}{2}m_x} \tag{8.10}$$

The relationship between the central death rate and the force of mortality can be seen when m_x is written in the following form:

$$m_x = \frac{\int_0^1 l_{x+t} \cdot \mu_{x+t}\, dt}{\int_0^1 l_{x+t}\, dt}.$$

Here it is apparent that m_x is the weighted mean value of the force of mortality over the year of age x to $x + 1$ where the weights are the number of lives attaining each age $x + t$ in that interval. We may use $\mu_{x+\frac{1}{2}}$ as a reasonable estimate for this average value.

This can be seen in another way. We use the fact that $d_x = -\frac{dL_x}{dx}$ and introduce the assumption of a uniform distribution of deaths:

$$m_x = \frac{d_x}{L_x} = -\frac{1}{L_x}\cdot\frac{dL_x}{dx} \doteqdot \frac{-1}{l_{x+\frac{1}{2}}}\cdot\frac{dl_{x+\frac{1}{2}}}{dx} = \mu_{x+\frac{1}{2}}\,.$$

In formulas (1.18) and (1.19a) we noted that $-\log p_x$ represents an unweighted mean value of μ_x which can also be approximated as $\mu_{x+\frac{1}{2}}$. Hence,

$$m_x \doteqdot \mu_{x+\frac{1}{2}} \doteqdot -\log p_x\,. \tag{8.11}$$

3. The expectation of life

A function which is frequently encountered in connection with mortality tables based on population statistics is the expectation of life. This appears in two forms, the *curtate* expectation e_x and the *complete* expectation $\mathring{e}_x$, defined as follows:

$$e_x = \frac{1}{l_x}\sum_{t=1}^{\infty} l_{x+t} = \sum_{t=1}^{\infty} {}_tp_x \tag{8.12a}$$

$$\mathring{e}_x = \frac{1}{l_x}\int_0^{\infty} l_{x+t}\,dt = \int_0^{\infty} {}_tp_x\,dt. \tag{8.12b}$$

In either form, the function may be interpreted as representing the average future lifetime at age x. The form e_x includes only full years of future lifetime and hence is curtate in the sense that it ignores final fractional years of life, while $\mathring{e}_x$ includes the complete future lifetime. On the average, the complete future lifetime exceeds by half a year the number of integral years in the future lifetime, and hence

$$\mathring{e}_x \doteqdot e_x + \tfrac{1}{2}. \tag{8.13}$$

This formula is analogous to the relationship between the two forms of life annuity, $\bar{a}_x \doteqdot a_x + \frac{1}{2}$, and the expectation of life is in fact a special case of the life annuity when the rate of interest is zero.

There are expectation functions corresponding to other types of life annuity. For example, temporary expectations of life, $e_{x:\overline{n|}}$ and

$\mathring{e}_{x:\overline{n}|}$, may be defined:

$$e_{x:\overline{n}|} = \sum_{t=1}^{n} {}_tp_x, \qquad \mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\, dt.$$

Computing the integral by the trapezoidal rule, we obtain the relationship

$$\mathring{e}_{x:\overline{n}|} \doteq e_{x:\overline{n}|} + \tfrac{1}{2}(1 - {}_np_x).$$

When exact values of T_x are available, the expectations may be computed from the formulas:

$$\mathring{e}_x = \frac{T_x}{l_x} \tag{8.14a}$$

$$e_x \doteq \frac{T_x}{l_x} - \tfrac{1}{2}. \tag{8.14b}$$

It should be noted, however, that tabulated values of T_x are usually approximate. When values of T_x computed from (8.5) are used, formula (8.14a) is approximate and formula (8.14b) is exact.

It is popularly believed that the expectation of life is widely used in actuarial calculations. In reality, it is of interest to actuaries only because it affords an index for comparing different mortality tables. One of the persistent misconceptions is that the present value of a life annuity at age x is equal to the value of an annuity certain for a term equal to the life expectancy at age x; that is, that a_x and $a_{\overline{e_x}|}$ have the same value. We shall show that in fact $a_{\overline{e_x}|}$ exceeds a_x, assuming $i > 0$.

Since e_x is in general not an integer, we write

$$e_x = \sum_{t=1}^{\infty} {}_tp_x = n + f,$$

where n is integral and $0 \leqslant f < 1$. The annuity-certain with fractional term $n + f$ provides in theory a final payment of $\dfrac{(1+i)^f - 1}{i}$ at the end of $n + f$ years.[1] At the usual rates of interest, the value of $\dfrac{(1+i)^f - 1}{i}$ is very close to f, and in practice the final payment is taken to be f. It can be shown that both $v^{n+f} \cdot \dfrac{(1+i)^f - 1}{i}$ and

[1] See Donald (1956), pp. 51f.

$v^{n+f}\cdot f$ are never less than $v^{n+1}\cdot f$, so that in either case we have the following results for $n \neq 0$:

$$\begin{aligned}
a_{\overline{\mathring{e}_x}|} = a_{\overline{n+f}|} &\geqslant a_{\overline{n}|} + v^{n+1}\cdot f \\
&= a_{x:\overline{n+1}|} + \sum_{t=1}^{n} v^t(1 - {}_tp_x) + v^{n+1}(f - {}_{n+1}p_x) \\
&> a_{x:\overline{n+1}|} + v^{n+1}\left[\sum_{t=1}^{n}(1 - {}_tp_x) + (f - {}_{n+1}p_x)\right] \\
&= a_{x:\overline{n+1}|} + v^{n+1}\left(n + f - \sum_{t=1}^{n+1} {}_tp_x\right) \\
&= a_{x:\overline{n+1}|} + v^{n+1}\left(\sum_{t=1}^{\infty} {}_tp_x - \sum_{t=1}^{n+1} {}_tp_x\right) \\
&= a_{x:\overline{n+1}|} + v^{n+1}\sum_{t=n+2}^{\infty} {}_tp_x \\
&\geqslant a_{x:\overline{n+1}|} + \sum_{t=n+2}^{\infty} v^t\,{}_tp_x \\
&= a_x
\end{aligned}$$

Since the same result can be obtained when $n = 0$ (the derivation is left for the student in Exercise 11), we see that $a_{\overline{\mathring{e}_x}|}$ exceeds a_x in all cases.

4. Analysis of the survivorship group

The mortality table, as we have discussed it thus far, can be interpreted either as a model for the expected survivorship of a newly-born group of l_0 lives or as the actual historical record of the survival of l_0 lives each of whom was observed from birth until death. Some applications may require a combination of these two points of view. In each case, we refer to the mortality table as a survivorship group model. We consider now certain properties of the closed population of lives represented by the mortality table when interpreted in this way.

For those members of the population who survive to age x, the average age at death will be $x + \mathring{e}_x$, since $\mathring{e}_x$ represents the average future lifetime at age x. This is the result that would be obtained if it were possible to observe each of the lives from age x until death, recording the exact age at death for each one, and finding the average of these ages.

It will be useful to derive this result in another way. Since each of the l_x lives has already survived for x years, we know that their *total past lifetime* is xl_x. Their *total future lifetime* is T_x, as we noted in Section 1. We thus see that their *total lifetime* is $xl_x + T_x$, which is the same as the total of their ages at death. Dividing by l_x, the number of lives, we obtain the average age at death as

$$\frac{xl_x + T_x}{l_x} = x + \frac{T_x}{l_x} = x + \mathring{e}_x .$$

For $x = 0$, we have $\dfrac{T_0}{l_0} = \mathring{e}_0$, the average age at death for the entire original group of l_0 lives.

Let us now determine the average age at death for those who survive to age x but die before age $x + n$. The number of such persons (i.e., the number of deaths) is

$$\int_x^{x+n} l_y \mu_y \, dy = l_x - l_{x+n} .$$

The total of the ages at death is

$$\int_x^{x+n} y l_y \mu_y \, dy = [-y l_y - T_y]_x^{x+n}$$

$$= -(x + n) l_{x+n} - T_{x+n} + x l_x + T_x .$$

Hence the average age at death is

$$\frac{x(l_x - l_{x+n}) + (T_x - T_{x+n} - n l_{x+n})}{l_x - l_{x+n}} = x + \frac{T_x - T_{x+n} - n l_{x+n}}{l_x - l_{x+n}} .$$

Since the total past lifetime prior to age x for those who survive to age x but die before age $x + n$ is $x(l_x - l_{x+n})$, this result shows that the total future lifetime subsequent to age x for the same persons is $T_x - T_{x+n} - n l_{x+n}$. The total future lifetime can also be obtained directly by evaluating the expression

$$\int_x^{x+n} (y - x) l_y \mu_y \, dy.$$

5. The stationary population

Problems in demography are sometimes concerned with communities with a stable age distribution for which the population is roughly stationary. It is possible to use the mortality table as a model for a stationary population if we interpret the familiar mortality functions in a somewhat different way.

Suppose that a population is supported by l_0 annual births, l_0 being the radix of a given mortality table, and suppose also that these births are uniformly distributed over each calendar year. Let the deaths among the population occur in accordance with the given mortality table, and let there be no migration into or out of the population. Then it may be shown that when the birth and death process has continued for a period of years at least equal to the terminal age of the mortality table, the total population remains stationary and its age distribution is constant.

This may be seen by considering the consequences of the assumption that the l_0 annual births are uniformly distributed over each calendar year. This clearly means that there will be l_0 births uniformly distributed over *any* year of time, and that in *any fraction* of a year h, however small, there will be hl_0 births. It follows that there will be l_x lives attaining age x in any year and hl_x lives attaining age x in any time interval h, as survivors of the births which occurred in the corresponding periods of time x years ago.

Now consider the incidence of deaths. Each of the hl_y lives attaining age y in any interval h is subject to the force of mortality μ_y, and hence the differential expression $hl_y\mu_y\,dy$ represents the number of lives dying at exact age y in that interval. Then the number dying between ages x and $x + 1$ in any interval h will be given by $\int_x^{x+1} hl_y\mu_y\,dy = hd_x$. There are two conclusions from this result: (1) letting h equal 1, we find that the number dying between ages x and $x + 1$ in any year of time is d_x; (2) since the number dying between ages x and $x + 1$ in any fraction of a year h is proportional to h, it is clear that the deaths between ages x and $x + 1$, and hence all deaths, occurring in any period of time are uniformly distributed over that period.

It may now be seen that such a population is indeed stationary. For, the total of the deaths at all ages in any interval h is $\int_0^\omega hl_y\mu_y\,dy = hl_0$, which is the same as the number of births occurring in the interval. Since the interval h is an arbitrarily small period of time, we may conclude that each life which leaves the population by death is simultaneously replaced by a new birth.

Furthermore, the distribution of the total population by ages is stationary. For, consider the lives which are aged x last birthday at any time. These are the lives which have attained integral age x but not age $x + 1$. In any interval h, the number of lives which leave this group by attaining age $x + 1$ is hl_{x+1}, and the number which leave the group by death is hd_x, making a total decrement of $hl_{x+1} + hd_x = hl_x$. Thus, the total decrement is exactly equal to the number entering the group during the interval by attaining age x. In other words, when a life leaves this age group, either by death or by attaining age $x + 1$, its place is simultaneously taken by a life entering from the next lower age group. Under these circumstances, the number living at a given age last birthday is always constant.

Now $\frac{1}{r} l_{x+\frac{m}{r}}$ represents the number of lives attaining exact age $x + \frac{m}{r}$ in any *interval* of time $\frac{1}{r}$ and therefore approximates the number of lives between exact ages $x + \frac{m}{r}$ and $x + \frac{m+1}{r}$ at any *moment* of time. Hence the number of lives between exact ages x and $x + 1$ at any moment of time is

$$\lim_{r\to\infty} \sum_{m=0}^{r-1} \frac{1}{r} l_{x+\frac{m}{r}} = \int_0^1 l_{x+t}\, dt = L_x .$$

Thus L_x is the constant number of lives between exact ages x and $x + 1$ at any moment of time. Similarly, T_x is the number of lives aged x and over at any moment of time. The student should note carefully how this conception of the mortality table contrasts with the survivorship group interpretation. In the stationary population concept, the function l_x represents the number of lives attaining age x in any year of time, and d_x represents the number of deaths between ages x and $x + 1$ in any year of time. The radix l_0 is the number of births (and of deaths) in any year of time. The functions L_x and T_x represent numbers living in the population at any moment of time. If a census enumeration of the population were made at any moment, the number of lives reporting age x last birthday would be L_x, and the number reporting an age of x or over would be T_x. The central death rate m_x can be described as the ratio of the number of deaths between ages x and $x + 1$ in any year of time to the number living between those ages at any moment of time.

Although the stationary population concept is admittedly artificial, the theory finds application in the solution of problems where the stationary condition is approximately realized. A problem such as the following illustrates a practical situation in which the "population" is substantially stationary.

> A peacetime army is maintained by 500,000 annual entrants inducted on their nineteenth birthday. Sixty per cent of the survivors of each year's entrants are discharged on their twenty-first birthday and the remainder on their twenty-second birthday. Assuming that the mortality follows the table for U.S. White Males 1959–61, determine the size of the army when it reaches a stationary condition.

If the number of annual entrants were l_{19}, the ultimate size of the army would be

$$L_{19} + L_{20} + .4L_{21} = (T_{19} - T_{21}) + .4L_{21} .$$

Since the number of annual entrants is 500,000 instead of l_{19}, the actual size will be

$$\frac{500,000}{l_{19}} (T_{19} - T_{21} + .4L_{21}),$$

and evaluating the functions from Table B of Appendix I, we find the result to be 1,197,695 men.

A number of other questions may be easily answered. For instance, the number of men who die in service each year will be

$$\frac{500,000}{l_{19}} (d_{19} + d_{20} + .4\, d_{21}) = 1,878.$$

The number of men discharged each year will be

$$\frac{500,000}{l_{19}} (.6l_{21} + .4l_{22}) = 498,122.$$

As a check, it may be noted that the number of men dying and discharged each year is equal to 500,000, the number of annual entrants.

6. Average ages in the stationary population

Some of the problems that were discussed in terms of the survivorship group have exact mathematical counterparts in the stationary population. Although the interpretation is different, the solution involves the same mortality functions and the same mathematical processes. Thus, the average age at death of all those in the stationary population who die after attaining age x is $x + \mathring{e}_x$, and the average age at death of all those in the stationary population is $\mathring{e}_0$. Similarly, the average age at death of all those in the stationary population whose deaths occur between ages x and $x + n$ is

$$x + \frac{T_x - T_{x+n} - nl_{x+n}}{l_x - l_{x+n}}.$$

In considering this result for the stationary population, the expression $l_x - l_{x+n}$ may be interpreted as the number of deaths between ages x and $x + n$ *in any year of time*, and thus the result corresponds to what would be obtained by observing the deaths at those ages for a period of a year. The population being stationary, the particular yearly period used is immaterial.

Problems which involve segments of the population *now living* at certain ages are peculiar to the stationary population and have no analogy in the survivorship group. Such problems are typically concerned with some subset of the present population, and we shall discuss them under two headings: (1) those involving all the members of the subset, and (2) those involving only the members who die under certain specified conditions.

(1) As an example of problems involving all the members of a subset, we consider the subset consisting of those persons now living at ages x and over. We note first that the number of members in this subset is $T_x = \int_x^\infty l_y\, dy$. Their total lifetime can be computed in three parts. First, their total past lifetime before age x is clearly xT_x. Second, we require their total past lifetime from age x to their present ages. For the $l_y\, dy$ lives attaining age y at any moment, this total past lifetime after age x is $(y - x)l_y\, dy$. Hence, for the entire subset, we have

$$\int_x^\infty (y - x)l_y\, dy = [-yT_y - Y_y + xT_y]_x^\infty = Y_x.$$

Third, since the total future lifetime for those aged y is T_y, the total future lifetime for the entire subset is

$$\int_x^\infty T_y\, dy = Y_x.$$

Summarizing, for the T_x persons now living at ages x and over, we have:

Total past lifetime before age x	$= xT_x$
Total past lifetime after age x	$= Y_x$
Total future lifetime	$= Y_x$
Total lifetime	$= xT_x + 2Y_x$

It is of interest to note that Y_x can be interpreted here in two quite different ways.

From this analysis, it is easy to express the average attained age of the members of the present population. For the T_x lives aged x and over, the average attained age is equal to x plus the average past lifetime after age x; that is, $x + \frac{Y_x}{T_x}$. The average attained age of the entire present population is $\frac{Y_0}{T_0}$.

Since the total lifetime represents the total of the ages at death, we can now find the average age at death for the T_x lives aged x and over:

$$\frac{xT_x + 2Y_x}{T_x} = x + \frac{2Y_x}{T_x}.$$

When $x = 0$, the average age at death of the entire present population is found to be $\frac{2Y_0}{T_0}$.

It is important to note the distinction between $\frac{T_0}{l_0}$, which represents the average age at death for newly born lives and also the average age at death in any period of time for the entire population, and $\frac{2Y_0}{T_0}$, which represents the average age at death of the present members of the population. The present members form a closed group of lives, and if their average age at death were to be obtained from observation of published death notices, it would be necessary to ignore the deaths of those who were born after the group is closed. The closed group constitutes an aging population in which the average age of the deaths increases in successive intervals of time. It follows that the average age at death of the closed group, being a composite of the averages in each successive interval of time, will be greater than the average age at death in any interval of time for the entire stationary population (for which all deaths are observed); that is,

$$\frac{2Y_0}{T_0} > \frac{T_0}{l_0}.$$

This kind of analysis can be extended to other segments of the population. Consider the members now living between ages y_1 and y_2. If all the members of the subset are included, then

the number of members is $\int_{y_1}^{y_2} l_y \, dy$;

their total past lifetime is $\int_{y_1}^{y_2} y l_y \, dy$;

their total future lifetime is $\int_{y_1}^{y_2} T_y \, dy$;

their total lifetime is $\int_{y_1}^{y_2} (y l_y + T_y) \, dy$.

The following averages can then be obtained:

$$\text{average attained age} = \frac{\text{total past lifetime}}{\text{number of members}}$$

$$\text{average age at death} = \frac{\text{total lifetime}}{\text{number of members}}.$$

Other properties of the subset can often be obtained by making simple modifications in the integrands specified above. Suppose we wish to find the total past lifetime *since age 20* for the persons now living between ages 30 and 65. The proper integrand is $(y - 20)\,l_y$, and we thus find

$$\int_{30}^{65} (y - 20) l_y \, dy = 30T_{30} + Y_{30} - 65T_{65} - Y_{65} - 20(T_{30} - T_{65}).$$

For the total future lifetime *prior to attaining age 80* we have

$$\int_{30}^{65} (T_y - T_{80}) \, dy = Y_{30} - Y_{65} - 35T_{80}.$$

(2) When we are concerned with the members whose deaths

occur under specified conditions, we have a two-variable problem requiring double integration. Given those who are now living at some range of ages, $y_1 \leqslant y \leqslant y_2$, we wish to consider those members who die within some interval of years in the future, $t_1 \leqslant t \leqslant t_2$, where t_1 and t_2 may be functions of y. The integral

$$\int_{y_1}^{y_2} \int_{t_1}^{t_2} f(y,t) l_{y+t}\mu_{y+t}\, dt\, dy \tag{8.15}$$

represents different properties of the group of members who die, depending on the form of $f(y,t)$ as follows:

Number of members who die	$f(y,t) = 1$
Their total past lifetime	$f(y,t) = y$
Their total future lifetime	$f(y,t) = t$
Their total lifetime	$f(y,t) = y + t$

In (8.15), the first integration with respect to t produces a summation over the deaths among those now aged y, and the second integration sums these values for all y in the interval $y_1 \leqslant y \leqslant y_2$.

Example 1. Let the members of the subset be those persons now living between ages 20 and 40 who will die before age 70. Find the number of members and their total lifetime.

With $f(y,t) = 1$, the number of members is

$$\int_{20}^{40} \int_{0}^{70-y} l_{y+t}\mu_{y+t}\, dt\, dy = \int_{20}^{40} [-l_{y+t}]_{t=0}^{t=70-y}\, dy$$
$$= \int_{20}^{40} (l_y - l_{70})\, dy = [-T_y - yl_{70}]_{20}^{40}$$
$$= T_{20} - T_{40} - 20l_{70}.$$

With $f(y,t) = y + t$, the total lifetime is

$$\int_{20}^{40} \int_{0}^{70-y} (y+t) l_{y+t}\mu_{y+t}\, dt\, dy = \int_{20}^{40} [-(y+t)l_{y+t} - T_{y+t}]_{t=0}^{t=70-y}\, dy$$
$$= \int_{20}^{40} (yl_y + T_y - 70l_{70} - T_{70})\, dy$$
$$= [-yT_y - 2Y_y - 70yl_{70} - yT_{70}]_{20}^{40}$$
$$= 20T_{20} + 2Y_{20} - 40T_{40} - 2Y_{40} - 20(70l_{70} + T_{70}).$$

Example 2. Find the average age at death of those persons now living between ages 20 and 70 who die between ages 60 and 80.

Since the two age intervals intersect, the integrals must be expressed as follows:

$$\int_{20}^{60}\int_{60-y}^{80-y} f(y,t)l_{y+t}\mu_{y+t}\,dt\,dy + \int_{60}^{70}\int_{0}^{80-y} f(y,t)l_{y+t}\mu_{y+t}\,dt\,dy,$$

where $f(y,t) = 1$ for the number of deaths and $f(y,t) = y + t$ for the total lifetime. The student may verify that the average age at death is

$$\frac{2400l_{60} - 4000l_{80} + 100T_{60} - 70T_{70} - 50T_{80} + 2(Y_{60} - Y_{70})}{40l_{60} - 50l_{80} + T_{60} - T_{70}}.$$

It is interesting to note that if $t_1 = 0$ and $t_2 = \infty$, the four formulas obtained from (8.15) by using the four different forms of $f(y,t)$ reduce to the four formulas given on page 182. It is clear that this should be so, since setting $t_1 = 0$ and $t_2 = \infty$ merely eliminates the requirement of any specified conditions on the time of the deaths.

If in addition $y_1 = x$ and $y_2 = \infty$, then $f(y,t) = 1$ leads to T_x and $f(y,t) = y + t$ leads to $xT_x + 2Y_x$. This result was previously obtained on page 181 in considering the average age at death for all lives aged x and over.

Finally, if we omit the integration with respect to y, we obtain formulas applicable to the survivorship group and to similar problems in the stationary population. For example, if $t_1 = 0$ and $t_2 = n$, then $f(y,t) = 1$ leads to $l_y - l_{y+n}$ and $f(y,t) = y + t$ leads to $y(l_y - l_{y+n}) + T_y - T_{y+n} - nl_{y+n}$. This result was previously obtained on page 176 and was discussed again on page 180.

It is clear from this discussion that (8.15) is a very general expression and can be used, in conjunction with the table of indefinite integrals on pages 170 and 171, as a basis for solving a wide variety of problems.

Problems relating to the stationary population have given rise to an extensive literature in the actuarial journals. Much ingenuity has been expended on the development of heuristic devices for simplifying the solution of this type of problem. One of the most useful methods, suggested by Grace and Nesbitt (1950), follows from the definition of two special functions:

$$F_x = xl_x + T_x$$

$$G_x = xT_x + 2Y_x\,.$$

F_x is the total lifetime of the l_x lives that attain age x, and G_x is the total lifetime of the T_x lives now living at ages x and over. Suppose that we have determined an expression for the number of members in some segment of the stationary population as a linear combination of the functions l_x and T_x. The Grace-Nesbitt principle states that we can then obtain the total lifetime of these members by substituting F_x for l_x and G_x for T_x throughout the expression.

In example 1 above, the number of members is $T_{20} - T_{40} - 20l_{70}$. Making the prescribed substitutions, we immediately find the total lifetime to be $G_{20} - G_{40} - 20F_{70} = (20T_{20} + 2Y_{20}) - (40T_{40} + 2Y_{40}) - 20(70l_{70} + T_{70})$. This affords a simple check on the accuracy of the original integrations. A similar check can be made for example 2 as well as for the average age at death problem discussed on page 176, and (trivially) for the average age at death expressions $x + \dfrac{T_x}{l_x}$ and $x + \dfrac{2Y_x}{T_x}$. In cases where the expression for the number of deaths is known to be correct so that no checking is necessary, this provides a method for finding the total lifetime without additional integration.

References

3. The proof that the expectancy annuity exceeds the life annuity is taken from King (1902), p. 112. An interesting geometric demonstration of this fact is given by Sarason (1960).

5, 6. Students who feel that they need additional help in analyzing stationary population problems will find useful material in the discussion of the paper by Grace and Nesbitt (1950) and in the paper by Veit (1964) and its discussion.

EXERCISES

1. Some additional mortality functions

1. Show that $\dfrac{d}{dx}\left(\dfrac{T_x}{l_x}\right) = \dfrac{\mu_x T_x}{l_x} - 1.$

2. What mortality function is represented by each of the following when $i = 0$?

(a) $l_x\bar{A}_x$ (c) $\bar{N}_x$ (e) $l_x(\bar{I}\bar{A})_x$

(b) $\bar{D}_x$ (d) $l_x\bar{a}_x$ (f) $l_x(\bar{I}\bar{a})_x$

3. Evaluate (a) $\displaystyle\int_0^n t l_{x+t}\mu_{x+t}\,dt$

(b) $\displaystyle\int_0^\infty t l_{x+t}\,dt$

(c) $\displaystyle\int_n^\infty l_{x+t}\,dt.$

2. The central death rate

4. Show that m_x is constant for all values of x if $l_x = ke^{-x}$.
5. Estimate m_{25} if $l_{25} = 10{,}075$ and $l_{26} = 9925$.
6. Show that $L_{x+1} = L_x e^{-\int_0^1 m_{x+t}\,dt}$.
7. Calculate the exact value of the central death rate at age x if mortality follows the de Moivre law.

3. The expectation of life

8. From the following data, compute the curtate expectation of life at age 90 and estimate the complete expectation:

x	l_x	x	x
90	21	95	5
91	15	96	3
92	12	97	1
93	9	98	0
94	7		

9. Find $\mathring{e}_x$ assuming de Moivre's law.

10. (a) Show that $\dfrac{d \log T_x}{dx} = -\dfrac{1}{\mathring{e}_x}$.

(b) Show that if $\mu'_x = \mu_x + \dfrac{.05}{\mathring{e}_x}$, then $p'_x = p_x\left(\dfrac{T_{x+1}}{T_x}\right)^{.05}$.

11. Prove that $a_{\overline{e_x|}} > a_x$ for the case where $0 < e_x < 1$.

4. Analysis of the survivorship group

12. For the l_x lives who survive to age x, write expressions for
   (a) the total past lifetime subsequent to age y $(y < x)$;
   (b) the total future lifetime prior to age z $(z > x)$;
   (c) the average lifetime between ages y and z $(y < x < z)$.

13. For those in a survivorship group who survive to age 30 but die before age 50, find

(a) the total past lifetime between ages 10 and 30;
(b) the average future lifetime subsequent to age 30.

5. The stationary population

14. A staff is maintained in a stationary condition by 300 annual entrants at exact age 20. Ten per cent leave at the end of five years, 5% of those remaining are promoted after ten years into jobs outside the staff, and at age 60 the balance retire on a pension. Express in terms of mortality table functions
(a) the size of the staff,
(b) the number promoted each year,
(c) the number of pensioners on the books.

15. An organization of 1000 active members is kept in a stationary state by admission of a uniform number of entrants at exact age 30. There are no withdrawals other than by death except that ¼ of those who reach age 55 retire at that age, ⅓ of those who reach age 60 in service retire at that age, and all those remaining in service are retired at 65. Express in terms of tabular functions
(a) the number of annual entrants at age 30,
(b) the total number of deaths in service each year.

16. In a stationary community supported by 5000 annual births, each person contributes $200 on attaining the age of 25 and $10 on each succeeding birthday up to, and including, his 65th birthday. On each birthday thereafter, he receives an annuity payment of $150. A payment of $50 is made at the death of each contributor who dies before receiving the first annuity payment. Find expressions for
(a) the constant number of persons who have made their first contribution but not their last,
(b) the receipts in any year,
(c) the total annuity payments in any year,
(d) the death claims in any year.

17.(a) Show that $\frac{d}{dx}(l_x\bar{a}_x) = -l_x\bar{A}_x$.

(b) In a stationary community, all persons now aged x and over agree to contribute a single sum equally to a fund from which a unit will be paid at the death of each of them. Show that the payment to be made by each is $\frac{\bar{a}_x}{\mathring{e}_x}$.

6. Average ages in the stationary population

18. An organization of 10,000 members who enter at age 20 and retire at age 60 has reached a stationary condition. Each year there are 100 deaths, the average age at death being 40. What is the number entering and retiring each year? (Ans. 300, 200)

19. (a) Find the total number of future years that those in the stationary population now between ages x and $x + n$ will live before attaining age $x + n$.

(b) Find the total number of future years that those now between ages x and $x + n$ will live in the next m years.

20. (a) Find the average age at death of those in the stationary population who are now living between ages x and $x + n$.

(b) Find the average age at death of those in the stationary population who are now living between ages x and $x + n$ and who will die before attaining age $x + n$.

21. Find the average attained age of those in the stationary population now living at ages under 30.

22. Find the average age at death of those in the stationary population who are now living between ages 20 and 30 and who will die between ages 30 and 50.

Miscellaneous problems

23. (a) Given $e_{25:\overline{n|}} = n\left(1 - \dfrac{n+1}{150}\right)$ for all n, derive an expression for ${}_np_{25}$.

(b) Find the derivative of $\mathring{e}_x$ with respect to l_x.

(c) Show that $\displaystyle\int_0^\infty v^t\,\mathring{e}_{x:\overline{t|}}\,dt = \frac{\bar{a}_x}{\delta}$.

24. In a stationary community of 600,000 lives, the number of deaths is 10,000 annually. The complete expectation of life on attaining majority at age 21 is 50 years. If one-third of the population is under age 21, how many lives attain majority each year, and what is the average age at death of those who die under age 21?

25. Find the total lifetime of those in the stationary population now living between ages 20 and 70 who will die between ages 60 and 80 within 50 years from now.

26. (a) Show that $\dfrac{d}{dt}\,({}_tp_x\mathring{e}_{x+t}) = -{}_tp_x$.

(b) Show that $\displaystyle\int_0^\infty v^t\,{}_tp_x\mathring{e}_{x+t}\,dt = \frac{1}{\delta}\,(\mathring{e}_x - \bar{a}_x)$.

Part II

MULTI-LIFE FUNCTIONS

Chapter 9

THE JOINT-LIFE STATUS

1. Introduction

The mathematical theory of life contingencies has been developed in Part I in terms of functions of a single life only. This chapter and the succeeding chapters in Part II are devoted to an expansion of the basic theory to cover the realm of multi-life functions which provide the means of solving problems involving two or more lives in combination.

The theory is developed from a concept of group survival. Consider a body of m lives with ages $x_1, x_2, \cdots, x_m$. Looked upon as a single collective entity, this body of lives can be said to "survive" at least as long as all of its component members survive; it becomes a matter of choice whether we say that it ceases to exist upon the first death among its members, or when the last survivor dies, or at some intermediate point. In practical applications, the component lives which are combined in an annuity or an insurance are often associated in some common undertaking, or are perhaps related by blood or marriage, and through their mutual activities may be simultaneously exposed to many of the same mortality hazards. For development of the general theory, however, the survival probabilities are assumed to be independent. *Joint-life* functions, as discussed in this chapter, are defined on the assumption that the group terminates its existence when the first death occurs. Thus the joint-life status $(x_1x_2 \cdots x_m)$ continues in existence as long as *all* the lives (x_1), (x_2), $\cdots$, (x_m) survive, and fails upon the occurrence of the *first* death.

It will be seen that this joint-life concept is precisely the type of status involved in the practical situation of a joint-life insurance payable upon the first death among the insureds, or a joint-life annuity payable as long as all the annuitants survive.

2. Joint-life probabilities

Let us consider how the probabilities of failure and survival for a joint-life status may be obtained. Although single-life mortality is measured by means of statistical studies, it is obviously impractical to attempt to obtain joint-life mortality rates by similar observation of groups of lives. This recourse is in fact unnecessary, since the probabilities of joint survival can be easily calculated from single-life mortality tables.

The probability that the joint-life status $(x_1x_2 \cdots x_m)$ will survive for n years is denoted by ${}_np_{x_1x_2\cdots x_m}$, and, since joint survival requires the individual survival of all the component lives, we have

$$ {}_np_{x_1x_2\cdots x_m} = {}_np_{x_1} \cdot {}_np_{x_2} \cdots {}_np_{x_m} . \tag{9.1} $$

The other basic probabilities then follow in terms of ${}_np_{x_1x_2\cdots x_m}$. The probability that the joint-life status will fail within n years is

$$ {}_nq_{x_1x_2\cdots x_m} = 1 - {}_np_{x_1x_2\cdots x_m} . \tag{9.2} $$

The probability that the status will fail in the $(n + 1)$-th year is

$$ {}_{n|}q_{x_1x_2\cdots x_m} = {}_np_{x_1x_2\cdots x_m} - {}_{n+1}p_{x_1x_2\cdots x_m} . \tag{9.3} $$

The usual annuity and insurance functions may now be defined for the joint-life status. For example, consider an immediate life annuity under which a payment of 1 is made at the end of each year as long as (x_1), (x_2), $\cdots$, (x_m) are all surviving, that is, as long as the status $(x_1x_2 \cdots x_m)$ survives. The present value of this benefit is denoted by $a_{x_1x_2\cdots x_m}$:

$$ a_{x_1x_2\cdots x_m} = \sum_{t=1}^{\infty} v^t \, {}_tp_{x_1x_2\cdots x_m} . $$

The joint-life insurance, with present value $A_{x_1x_2\cdots x_m}$, provides a payment of 1 at the end of the year in which the status $(x_1x_2 \cdots x_m)$ fails:

$$ A_{x_1x_2\cdots x_m} = \sum_{t=0}^{\infty} v^{t+1} \, {}_{t|}q_{x_1x_2\cdots x_m} . $$

Other joint-life functions may be formulated by replacing the single-life probabilities in the corresponding single-life functions by the appropriate joint-life probabilities. For example,

$$ e_{xy} = \sum_{t=1}^{\infty} {}_tp_{xy} $$

$$ \bar{a}_{xyz} = \int_0^{\infty} v^t \, {}_tp_{xyz} \, dt $$

$$(IA)_{wxyz} = \sum_{t=0}^{\infty} (t+1)v^{t+1} \cdot {}_{t|}q_{wxyz} \,.$$

3. Joint-life mortality and commutation functions

The formula ${}_np_x = \dfrac{l_{x+n}}{l_x}$ has the formal joint-life counterpart

$$ {}_np_{x_1x_2\cdots x_m} = \frac{l_{x_1+n:x_2+n:\cdots:x_m+n}}{l_{x_1x_2\cdots x_m}}, \tag{9.4}$$

where $l_{x_1x_2\cdots x_m}$ remains to be defined.
Also

$$ {}_np_{x_1x_2\cdots x_m} = {}_np_{x_1} \cdot {}_np_{x_2} \cdots {}_np_{x_m} = \frac{l_{x_1+n}l_{x_2+n}\cdots l_{x_m+n}}{l_{x_1}l_{x_2}\cdots l_{x_m}}.$$

From these two expressions for ${}_np_{x_1x_2\cdots x_m}$, we find that

$$\frac{l_{x_1+n:x_2+n:\cdots:x_m+n}}{l_{x_1x_2\cdots x_m}} = \frac{l_{x_1+n}l_{x_2+n}\cdots l_{x_m+n}}{l_{x_1}l_{x_2}\cdots l_{x_m}},$$

and this suggests that we define the joint-life function $l_{x_1x_2\cdots x_m}$ as proportional to the product of the single-life l's; i.e.,

$$l_{x_1x_2\cdots x_m} = kl_{x_1}l_{x_2} \cdots l_{x_m} \,. \tag{9.5}$$

In practice, the constant of proportionality k is generally taken to be some power of 10^{-1}, so chosen as to reduce the product values to a desired order of magnitude.

Then

$$\begin{aligned} q_{x_1x_2\cdots x_m} &= 1 - p_{x_1x_2\cdots x_m} \\ &= \frac{l_{x_1x_2\cdots x_m} - l_{x_1+1:x_2+1:\cdots:x_m+1}}{l_{x_1x_2\cdots x_m}} = \frac{d_{x_1x_2\cdots x_m}}{l_{x_1x_2\cdots x_m}} \end{aligned}$$

where, by definition,

$$d_{x_1x_2\cdots x_m} = l_{x_1x_2\cdots x_m} - l_{x_1+1:x_2+1:\cdots:x_m+1} \,. \tag{9.6}$$

It is important to note that $d_{x_1x_2\cdots x_m}$ is *not* the same as $kd_{x_1}d_{x_2}\cdots d_{x_m}$.

Just as

$$\mu_{x+t} = -\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt} = -\frac{d \log l_{x+t}}{dt},$$

so for the joint-life status we have, by definition,

$$\mu_{x_1+t:x_2+t:\cdots:x_m+t} = -\frac{1}{l_{x_1+t:x_2+t:\cdots:x_m+t}} \frac{dl_{x_1+t:x_2+t:\cdots:x_m+t}}{dt}$$
$$= -\frac{d \log l_{x_1+t:x_2+t:\cdots:x_m+t}}{dt}. \tag{9.7}$$

But this may be expressed in terms of single-life l's as

$$-\frac{d \log kl_{x_1+t}l_{x_2+t} \cdots l_{x_m+t}}{dt}$$

$$= -\frac{d \log l_{x_1+t}}{dt} - \frac{d \log l_{x_2+t}}{dt} - \cdots - \frac{d \log l_{x_m+t}}{dt}$$

$$= \mu_{x_1+t} + \mu_{x_2+t} + \cdots + \mu_{x_m+t}.$$

Thus the joint-life force of mortality is equal to the sum of the single-life forces of mortality:

$$\mu_{x_1x_2\cdots x_m} = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \tag{9.8}$$

Commutation functions for the joint-life status are defined by analogy with those for single lives; thus,

$$D_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}} l_{x_1x_2\cdots x_m}. \tag{9.9}$$

By using the exponent $\dfrac{x_1 + x_2 + \cdots + x_m}{m}$, the various lives in the status are made to enter the function symmetrically, and an increase of one year in the age of each life produces an increase of unity in the exponent, so that the function behaves mathematically like its single-life counterpart.

Similarly, by definition,

$$C_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1} d_{x_1x_2\cdots x_m}. \tag{9.10}$$

The functions N, S, M, and R are defined by summation in the standard way.

Since these functions bear the same relation to the joint-life mortality functions as the corresponding single-life commutation functions bear to the basic single-life mortality functions, it follows that each joint-life benefit will have a formula in commutation symbols which exactly parallels the single-life formula for the same benefit. As illustrations, we may write

$$a_{x_1 x_2 \cdots x_m} = \frac{N_{x_1+1:x_2+1:\cdots:x_m+1}}{D_{x_1 x_2 \cdots x_m}}$$

$$A_{x_1 x_2 \cdots x_m} = \frac{M_{x_1 x_2 \cdots x_m}}{D_{x_1 x_2 \cdots x_m}}.$$

Many cases arise in practice in which it is not reasonable to assume that all the individual lives of a joint-life status are subject to the same single-life mortality table. Suppose, for example, that the joint two-life status (xy) is composed of the life (x) subject to mortality table T and the life (y) subject to mortality table T'. The joint-life functions of this status will reflect the dual mortality basis if the values of l_{xy} are computed as $k \cdot l_x \cdot l'_y$, using l_x from table T and l'_y from table T'.

It is important to observe that the functions $l_{x_1 x_2 \cdots x_m}$ and $d_{x_1 x_2 \cdots x_m}$ are mathematical abstractions which are even more highly idealized than their single-life counterparts l_x and d_x. Whereas the latter functions lend themselves readily to an interpretation in terms of numbers of lives surviving and dying, the functions $l_{x_1 x_2 \cdots x_m}$ and $d_{x_1 x_2 \cdots x_m}$, if they are to be given a physical meaning, must be regarded as numbers of *groups* of lives existing and failing, where existence and failure are construed in the special sense applicable to the joint-life status. Since, however, their usefulness in joint-life theory requires only that they satisfy certain mathematical definitions, any physical interpretation is irrelevant, and the student should regard them as having the nature of commutation functions, that is, purely as mathematical tools for simplifying the formulation of a particular class of problem.

4. Joint-life functions under Makeham's law

The problem of tabulating numerical values for joint-life functions presents certain obvious difficulties. A single function, even in the simplest case of two lives, requires a very extensive table if provision is made for all combinations of ages. In the case of tables following Makeham's or Gompertz's law, a remarkable simplification is possible and the volume of tabulated data can be considerably reduced. We shall examine first the effect of a Makeham assumption.

Under Makeham's law, the probability of survival for a single life is given by

$$_np_x = s^n g^{c^x(c^n-1)}.$$

Then for a joint-life status $(x_1x_2 \cdots x_m)$, the probability of joint survival will be

$$_np_{x_1x_2\cdots x_m} = s^{mn} g^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^n-1)}, \tag{9.11}$$

and, by substituting

$$c^{x_1} + c^{x_2} + \cdots + c^{x_m} = mc^w,$$

we obtain

$$_np_{x_1x_2\cdots x_m} = s^{mn} g^{mc^w(c^n-1)} = {}_np_{ww\cdots w}\,.$$

Thus in calculating probabilities of survival, the status $(x_1x_2 \cdots x_m)$ involving m lives of unequal age can be replaced by the simpler m-life status $(ww \cdots w)$ where all the components are of the same age. Furthermore, there is a simple relation for the determination of the "equivalent equal age" w in the condition

$$mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \tag{9.12}$$

This condition may also be expressed in terms of the force of mortality, for, if we multiply through by B and add mA to both sides, A and B being the usual Makeham constants, we obtain

$$m(A + Bc^w) = (A + Bc^{x_1}) + (A + Bc^{x_2}) + \cdots + (A + Bc^{x_m}),$$

whence

$$m\mu_w = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}\,. \tag{9.13}$$

Thus, the equivalent equal age w may be described as the age for which the single-life force of mortality is equal to the arithmetic mean of the single-life forces of mortality at the individual ages $x_1, x_2, \cdots, x_m$.

When joint-life functions are tabulated on a Makehamized basis, the values of μ_x or of c^x are customarily shown for all ages. From these, values of w can be obtained by interpolation using formula (9.12) or (9.13).

This property of Makeham's law has the important consequence that any joint-life function, regardless of the ages involved, may be evaluated as the same function of joint lives of equal age. For, if a group of m lives of unequal ages may be replaced by m lives of equal age in calculating probabilities of survival, the same replacement may be made in any joint-life function depending on those probabilities. Thus, since ${}_np_{x_1x_2\cdots x_m} = {}_np_{ww\cdots w}$, it follows that $a_{x_1x_2\cdots x_m} = a_{ww\cdots w}$, $A_{x_1x_2\cdots x_m} = A_{ww\cdots w}$, $P_{x_1x_2\cdots x_m} = P_{ww\cdots w}$, and so on. Under these circumstances, it is only necessary to tabulate joint-life functions for equal ages, the extensive tables required for $l_{x_1x_2\cdots x_m}$, $d_{x_1x_2\cdots x_m}$, $D_{x_1x_2\cdots x_m}$, and the other functions being now replaced by *single* columns of $l_{xx\cdots x}$, $d_{xx\cdots x}$, $D_{xx\cdots x}$, and so on.

In the past many important mortality tables were constructed on a Makeham basis in order to gain this advantage. The 1941 CSO table follows Makeham's law between ages 15 and 95. Table 5 shows certain annuity values on this basis, together with values of c^x.

As an example, suppose that the value of $\ddot{a}_{30:35:45}$ is required on the 1941 CSO $2\frac{1}{2}$% basis. Using Table 5, we first compute

$$c^w = \frac{c^{30} + c^{35} + c^{45}}{3} = 26.477.$$

Then, by linear interpolation in the table, we find $w = 38.363$, and hence

$$\ddot{a}_{30:35:45} = \ddot{a}_{38.363:38.363:38.363}\,.$$

Finally, a linear interpolation between $x = 38$ and $x = 39$ in the table of values of $\ddot{a}_{xxx}$ produces the required annuity value of 15.27735. In this result, we cannot expect all five decimal places to be correct in view of the interpolation procedure, but the loss of accuracy is slight enough to be unimportant in most applications.

There is a minor theoretical error which should be noted, although it has very little practical importance. The function which represents l_x in Makeham's law is positive for all values of x, implying an infinite limiting age in the mortality table. The 1941 CSO table used above, although Makehamized over most of its range, has l_{100} equal to 0. If we recognize this limiting age in computing the life annuity value above, we have, since ${}_tp_{45} = 0$ for $t > 54$,

TABLE 5

Values from a Makeham Table
1941 CSO 2½%

x	$\ddot{a}_{xx}$	$\ddot{a}_{xxx}$	c^x	x
30	20.63897	18.48172	12.954	30
31	20.27405	18.11153	14.100	31
32	19.90413	17.73708	15.366	32
33	19.52954	17.35885	16.736	33
34	19.15026	16.97680	18.228	34
35	18.76695	16.59181	19.852	35
36	18.37945	16.20371	21.622	36
37	17.98838	15.81326	23.549	37
38	17.59385	15.42063	25.648	38
39	17.19596	15.02593	27.934	39
40	16.79545	14.63009	30.424	40
41	16.39232	14.23310	33.136	41
42	15.98714	13.83565	36.089	42
43	15.58011	13.43796	39.306	43
44	15.17162	13.04049	42.809	44
45	14.76233	12.64403	46.625	45
46	14.35245	12.24878	50.781	46
47	13.94244	11.85524	55.307	47
48	13.53291	11.46409	60.236	48
49	13.12403	11.07548	65.605	49
50	12.71666	10.69038	71.453	50
51	12.31105	10.30898	77.822	51
52	11.90776	9.93186	84.758	52
53	11.50720	9.55942	92.313	53
54	11.11010	9.19240	100.541	54
55	10.71676	8.83102	109.502	55
56	10.32771	8.47580	119.262	56
57	9.94354	8.12727	129.892	57
58	9.56463	7.78572	141.470	58
59	9.19149	7.45161	154.079	59
60	8.82458	7.12530	167.812	60
61	8.46435	6.80713	182.770	61
62	8.11113	6.49731	199.060	62
63	7.76562	6.19648	216.803	63
64	7.42783	5.90444	236.127	64

$$\ddot{a}_{30:35:45} = \sum_{t=0}^{54} v^t {}_tp_{30:35:45} = \sum_{t=0}^{54} v^t {}_tp_{www} = \ddot{a}_{www:\overline{55}|}$$

where $w = 38.363$. For greater accuracy then, $\ddot{a}_{30:35:45}$ should be evaluated as $\ddot{a}_{www:\overline{55}|}$ instead of as $\ddot{a}_{www}$. The difference between these two values, however, involves a small number of annuity payments discounted over a long period and contingent on the joint survival of three lives to advanced ages. It can be shown in this particular case that the error is too small to affect the fifth decimal place. Although this error can safely be ignored in practice, the student should be aware of its existence.

5. The law of uniform seniority

In the case of a joint *two*-life status, the equivalent equal age under Makeham's law may be determined in a particularly simple way. We know that the status $(x:x + n)$ can be replaced by the equal-age status $(x + t:x + t)$ provided that

$$2c^{x+t} = c^x + c^{x+n}.$$

Removal of the factor c^x in this relation yields

$$2c^t = 1 + c^n,$$

whence
$$t = \frac{\log (1 + c^n) - \log 2}{\log c}.$$

The result indicates that the seniority t of the substituted lives over the age of the younger of the original lives depends only on the difference n in the original ages. This makes it possible to construct a simple table showing the value t of this uniform seniority for each difference in ages n. When such a table is available, it is unnecessary to refer to the values of μ_x or of c^x in order to determine the age w.

The table of uniform seniority for the 1941 CSO basis is shown in Table 6. Because of the special methods used in the construction of the 1941 CSO table at the youngest and the oldest ages, the table of uniform seniority does not apply accurately if either age is under 15 or over 90.

Suppose that we require the equivalent equal age on this basis for the status (30:40). Since the difference in ages is 10, we enter Table 6 with $n = 10$ and find $t = 6.036$. Adding this value of t to the younger age 30, we find that the equivalent equal age is 36.036.

TABLE 6

UNIFORM SENIORITY

1941 CSO TABLE

Difference of Age n	Addition to Younger Age t
1	0.511
2	1.043
3	1.596
4	2.170
5	2.765
6	3.380
7	4.015
8	4.670
9	5.344
10	6.036
11	6.747
12	7.474
13	8.218
14	8.978
15	9.753
16	10.543
17	11.346
18	12.163
19	12.992
20	13.833
21	14.685
22	15.547
23	16.420
24	17.301
25	18.191

Does not apply if either age is under 15 or over 90.

The principle may be extended to the case of three joint lives, but the process is more complicated, and it is simpler to interpolate for w in a table of μ_x or c^x.

The case of four joint lives can be treated, however, by means of the table of uniform seniority for two lives. Consider first the special case of the status $(xxyy)$, involving four lives but only two distinct ages. The equivalent equal age w must satisfy the condition

$$4c^w = 2c^x + 2c^y,$$

but this is the same as the condition

$$2c^w = c^x + c^y$$

for which (xy) is equivalent to (ww). Hence the equivalent equal age w for $(xxyy)$ is the same as for (xy).

The case of four lives of unequal age $(uvxy)$ may be reduced to the preceding case. First the equivalent equal age for each of the pairs (uv) and (xy) is found from the table of uniform seniority. Suppose that $(uv) = (rr)$ and $(xy) = (ss)$. The joint four-life status now becomes $(rrss)$, and, as above, the equivalent equal age for (rs) will be the equivalent equal age for $(rrss)$, and hence for $(uvxy)$. The ages r and s are normally fractional, so that the last step involves interpolation in the table of uniform seniority. In this case, the method is not necessarily more convenient than the process of averaging the values of μ_x or c^x.

In the case of the two-life status the table of uniform seniority makes possible a precise determination of w, since it gives values based on an exact mathematical formula. In addition, it has the practical advantage of eliminating the calculation involved in interpolating w from the table of values of c^x or μ_x.

Even when a mortality table is not Makehamized, it is sometimes possible to adopt a table of uniform seniority that will produce joint-life values for unequal ages with sufficient accuracy for many practical purposes. Although the 1958 CSO table is not a Makeham table, extensive test calculations have shown that a table of uniform seniority with $\log_{10} c = .04$ can be used to produce interpolated values for annuities, net annual premiums, and reserves which are reasonably close to the true values. The table of uniform seniority is given in Appendix I. The degree of accuracy in the resulting joint-life values depends upon the ages involved and the particular function being computed. This method should not be used with other tables without adequate testing, and if a high degree of precision is necessary, the method cannot be used.

6. Joint-life functions under Gompertz's law

With Gompertz's law, the evaluation of joint-life functions can be simplified still further. Since

$$ {}_np_{x_1x_2\cdots x_m} = g^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^n-1)}, \tag{9.14} $$

by substituting $\quad c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m},$

we obtain

$$ {}_np_{x_1x_2\cdots x_m} = g^{c^w(c^n-1)} = {}_np_w . $$

Thus, the joint-life status $(x_1x_2 \cdots x_m)$ can be replaced by the single life (w), and joint-life functions can be evaluated directly from single-life tables, it being only necessary to determine first the age of the equivalent single life from the relation

$$ c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m} \tag{9.15} $$

or
$$ \mu_w = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m} = \mu_{x_1x_2\cdots x_m} . \tag{9.16} $$

In this case, w is the age at which the single-life force of mortality is equal to the force of mortality for the original joint-life status. It should be noted that the substituted single life (w) is always older than any of the original lives (x_i), whereas with Makeham's law the equivalent equal age is intermediate in value to the original ages.

A law of uniform seniority holds also for Gompertz's law. The joint two-life status $(x:x+n)$ may be replaced by the single-life status $(x+t)$ provided that

$$ c^{x+t} = c^x + c^{x+n} $$

or
$$ c^t = 1 + c^n. $$

Hence,
$$ t = \frac{\log(1+c^n)}{\log c} . $$

The seniority of the substituted life can be measured equally well from the older age, and, where this is done, values of $t - n$ are tabulated for each value of n.

In the Gompertz case, the same table of uniform seniority may be used with any number of lives by simply replacing pairs of joint lives by single lives until the original age status is reduced to the age of a single life. The result is independent of the order in which the lives are combined in pairs. This extension to more than two lives is not of great importance since the age of the equivalent single life is normally fractional, and it becomes necessary to interpolate in the table. It is usually easier to interpolate in the table of μ_x or of c^x.

Table 7 shows annuity values at 3% for a Gompertz table together with the table of uniform seniority. Suppose that the value of $a_{50:55}$ is required. We enter the table of uniform seniority with a difference in ages of 5 and obtain a seniority of 6.858, and, noting that the table measures seniority from the older life, we have $(50:55) = (61.858)$. Then, by interpolation in the column of annuity values,

$$a_{50:55} = a_{61.858} = 11.70476.$$

If the value of $a_{50:55:65}$ is also required, we interpolate in the table of uniform seniority for a difference in ages of $65 - 61.858 = 3.142$ to find $(61.858:65) = (72.645)$. Hence,

$$a_{50:55:65} = a_{72.645} = 7.88446.$$

7. Summary of notation

1. The special symbol $(x_1x_2 \cdots x_m)$ denotes a joint-life status in which the individual lives are aged $x_1, x_2, \cdots, x_m$. When, for the sake of distinctness, it is desired to separate the ages in such a status, a colon is placed between them; e.g., $(20:25)$, $(x+t:y+t)$.

2. When a suffixed subscript consists of two or more ages without any distinguishing mark, a joint-life status is intended. Thus: l_{xy}, $a_{20:30:40}$.

3. Brackets are used to denote select mortality; e.g., $A_{[40][50]}$, $l_{[x]+t:[y]+t}$.

4. When a joint-life status is involved and the absence of any distinguishing mark would lead to ambiguity, the symbol $\overline{}$ is placed above the lives included—as in $A^{\;1}_{\overline{xy}:\overline{n}|}$.

5. *Commutation functions*

$$D_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}} l_{x_1x_2\cdots x_m}$$

$$\bar{D}_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\, dt$$

$$C_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1} d_{x_1x_2\cdots x_m}$$

$$\bar{C}_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\,\mu_{x_1+t:x_2+t:\cdots:x_m+t}\, dt$$

The functions N, $\bar{N}$, S, $\bar{S}$, M, $\bar{M}$, R, $\bar{R}$ are obtained by summation in the usual way.

TABLE 7

UNIFORM SENIORITY IN A GOMPERTZ TABLE

Annuity Values at 3%		Table of Uniform Seniority	
x	a_x	Difference of Ages	Addition to Older Age
40	19.52384	0	9.122
41	19.19752	1	8.632
42	18.86680	2	8.160
43	18.53179	3	7.707
44	18.19268	4	7.273
45	17.84964	5	6.858
46	17.50283	6	6.461
47	17.15249	7	6.082
48	16.79884	8	5.721
49	16.44204	9	5.377
50	16.08243	10	5.050
51	15.72020	11	4.739
52	15.35566	12	4.445
53	14.98909	13	4.166
54	14.62080	14	3.902
55	14.25107	15	3.653
56	13.88028	16	3.417
57	13.50872	17	3.195
58	13.13677	18	2.986
59	12.76477	19	2.789
60	12.39309	20	2.604
61	12.02211	21	2.430
62	11.65224	22	2.266
63	11.28381	23	2.113
64	10.91726	24	1.970
65	10.55296	25	1.835
66	10.19131	26	1.709
67	9.83271	27	1.591
68	9.47754	28	1.481
69	9.12620	29	1.378
70	8.77904	30	1.282
71	8.43646	31	1.192
72	8.09883	32	1.109
73	7.76647	33	1.031
74	7.43975	34	.958

TABLE 7—*Continued*

Annuity Values at 3%		Table of Uniform Seniority	
x	a_x	Difference of Ages	Addition to Older Age
75	7.11896	35	.890
76	6.80441	36	.827
77	6.49642	37	.768
78	6.19521	38	.714
79	5.90105	39	.663
80	5.61416	40	.615

References

4. The values of c^x shown in Table 5 are taken from Heckman (1946). This paper also gives joint-life mortality and commutation functions on the 1941 CSO basis with interest at $2\frac{1}{2}$ % and 3 %.

When Makeham's law holds, joint-life annuities can be computed from a single-life table with a change in the rate of interest. The method is described by Jones (1930) and forms the basis of Exercises 11 and 12.

The theoretical error in replacing a_{xy} by a_{ww} is pointed out by Gershenson (1957).

5. The data of Table 6 are taken from Thompson (1941). The extension of this method to three joint lives requires two tables of uniform seniority. It is illustrated by Spurgeon (1932), p. 259.

For the 1958 CSO table, the complete table of uniform seniority and the tests of accuracy for various functions may be seen in the Report of the General Committee on Publication of Monetary Tables, TSA 11, pp. 1049 ff.

6. The data of Table 7 are based on the 1937 Standard Annuity Table, described by Kineke (1938).

A general treatment of laws of mortality which satisfy a uniform seniority principle is given by Greville (1956).

EXERCISES

1, 2. Joint-life probabilities

1. Express each of the following in terms of the single-life probabilities ${}_np_x$ and ${}_np_y$:

(a) the probability that (xy) will survive n years;
(b) the probability that exactly one of the lives (x) and (y) will survive n years;
(c) the probability that at least one of the lives (x) and (y) will survive n years;
(d) the probability that (xy) will fail within n years;
(e) the probability that at least one of the lives will die within n years;
(f) the probability that both lives will die within n years.

2. Express in terms of ${}_{n|}q_x$, ${}_{n|}q_y$, and ${}_{n|}q_z$:
(a) the probability that all three lives (x), (y), and (z) will die in the $(n + 1)$-th year;
(b) the probability that none of the lives will die in the $(n + 1)$-th year;
(c) the probability that at least one of the lives will die in the $(n + 1)$-th year.

3. Show that the probability that (x) survives n years and (y) survives $n - 1$ years may be expressed either as

$$\frac{{}_{n}p_{x:y-1}}{p_{y-1}} \qquad \text{or as} \qquad p_x \cdot {}_{n-1}p_{x+1:y} .$$

4. Show that the probability that two lives (30) and (40) will die at the same age last birthday can be expressed as

$${}_{10}p_{30}(1 + e_{40:40}) - 2\, {}_{11}p_{30}(1 + e_{40:41}) + p_{40} \cdot {}_{11}p_{30}(1 + e_{41:41}).$$

3. Joint-life mortality and commutation functions

5. Since $l_{xy} = kl_x l_y$, a student assumes that the following relations are correct:
(a) $d_{xy} = kd_x d_y$
(b) $\mu_{xy} = k\mu_x \mu_y$
(c) $p_{xy} = p_x p_y$
(d) $q_{xy} = q_x q_y$
Which of these are incorrect? Give the correct expressions.

6. If a special mortality table is derived from a standard table by the relationship $\mu_x' = k\mu_x$ (k integral), show that ${}_tp_x' = {}_tp_{xx\cdots(k)}$.

7. Show that
(a) $D_{xy} = k(1 + i)^{\frac{x+y}{2}} D_x \cdot D_y$
(b) $C_{xy} = vD_{xy} - D_{x+1:y+1}$
(c) $D_{xy} = \sqrt{D_{xx} \cdot D_{yy}}$

8. Find the derivatives with respect to x of:
(a) $\log {}_tp_{xx}$ (b) $\mathring{e}_{xx}$.

4. Joint-life functions under Makeham's law

9. Use the fact that the 1941 CSO table is Makehamized to evaluate the following functions on the 1941 CSO $2\frac{1}{2}\%$ basis:

(a) $a_{40:50}$ (Ans. 13.33810)
(b) $A_{40:50}$ (Ans. .65029)
(c) $P_{40:50}$ (Ans. .04535)
(d) $\ddot{a}_{30:40:50}$ (Ans. 13.56522)

Explain how you would proceed if the 1941 CSO table were not Makehamized.

10. Two single-life tables, one for male and one for female lives, both follow Makeham's law and have a common value of the constant c. From these two tables, a joint two-life table is constructed applying to pairs of lives one of which is male and the other female.

Prove that by this joint-life table for a male life (x) and a female life (y)

$$a_{xy} = a_{ww}$$

where
$$c^w = \frac{c^x + kc^y}{1+k},$$

k being equal to
$$\frac{\log g'}{\log g} = \frac{B'}{B},$$

the primed symbols applying to the original female table and unprimed symbols to the original male table.

11. Assuming Makeham's law, show that

$${}_np_{xy} = {}_nE_w ,$$

where $c^w = c^x + c^y$ and ${}_nE_w$ is calculated at rate of interest $i = \frac{1}{s} - 1$, s being the usual Makeham constant.

12. Prove that if a mortality table is constructed according to Makeham's law the value of an annuity on m joint lives aged $x, y, z, \cdots$ calculated at a rate of interest i is equal to the value of a single-life annuity at age w calculated at a rate of interest i' where

$$i' = \frac{1+i}{s^{m-1}} - 1.$$

Show that the age w may be determined from the relation

$$\mu_{w+h} = \frac{\mu_x + \mu_y + \mu_z + \cdots}{m}$$

where
$$h = -\frac{\log m}{\log c}.$$

5. The law of uniform seniority

13. (a) Use the table of uniform seniority to evaluate $a_{40:50}$ on the 1941 CSO $2\frac{1}{2}\%$ basis. Compare your answer with that obtained in exer-

cise 9(a) and explain why there is not an exact agreement in the two results. (Ans. 13.33769)

(b) Use the tables in Appendix I to compute an approximate value for $a_{40:50}$ on the 1958 CSO 3% basis. (Ans. 14.08285)

14. Show that the law of uniform seniority holds for two lives when the law of mortality is

$$\mu_x = a + bx.$$

15. What is the formula for calculating the table of uniform seniority for the joint-life table of Exercise 10?

6. Joint-life functions under Gompertz's law

16. (a) Calculate $a_{60:70}$ from Table 7. (Ans. 7.10323)

(b) For the calculation of $a_{50:55:65}$ shown in the text, verify that the same result is obtained if the lives (55) and (65) are combined first.

17. With Gompertz's law, if (z) is the single life which is equivalent to the joint status (xy), show that there is a theoretical inaccuracy in writing $a_z = a_{xy}$ for a table which has a finite limiting age ω. What is the correct relationship? Why can the inaccuracy generally be ignored in practice?

18. (a) A mortality table for male lives follows Makeham's law, and another table for female lives follows Gompertz's law with the constants B and c the same as for the male table. Show that in the evaluation of a joint-life annuity on m male lives and n female lives, all of unequal ages, m male lives of equal age can be substituted for the $m + n$ lives.

(b) Show how the table of uniform seniority may be constructed for the case where $m = n = 1$.

Miscellaneous problems

19. If the Fackler valuation functions $u_{xx} = \dfrac{D_{xx}}{D_{x+1:x+1}}$ and $k_{xx} = \dfrac{C_{xx}}{D_{x+1:x+1}}$ are computed at a certain age y on a certain mortality basis with $i = .03$, it is found that $u_{yy} = 4\, k_{yy}$. Find p_y. (Ans. .86)

20. Compute $\mathring{e}_{25:25}$ if $\mu_x = \dfrac{1}{100 - x}$, $0 < x < 100$. (Ans. 25)

21. You are given the following probabilities:

(i) The probability that three persons aged 20, 30 and 40 will live 10 years is .758.

(ii) The probability that a person aged 45 will die within 5 years, while another person aged 40 is alive at the end of 5 years, is .063.

(iii) The probability that four persons aged 20, 25, 30, and 35 will live for 5 years, while a fifth person aged 40 will not be alive at the end of 5 years, is .045.

Find the probability that a person aged 20 will be alive at the end of 25 years. (Ans. .812)

22. A mortality table is graduated according to Makeham's law: $\mu_x = A + Bc^x$. A new mortality table is constructed such that

$$\mu'_x = A' + B'c^x,$$

where

$$A' = -\frac{\delta}{2} \qquad \text{and} \qquad B' = Bc^{\frac{1}{2}}.$$

Express a_{xy}, based on the new mortality table, as a single-life function derived from the Gompertz table which is a specialization of the original Makeham table.

CHAPTER 10

LAST-SURVIVOR AND GENERAL MULTI-LIFE STATUSES

1. The last-survivor status

In the last chapter we noted that a body of m lives, considered as a single collective entity, can be said to fail upon the first death among its component lives, or upon the death of its last surviving member, or arbitrarily upon some intermediate death. The concept of group failure upon the first death leads to the joint-life functions; the major alternative, that of defining the failure of the group as occurring when its last survivor dies, leads to the *last-survivor* status and the various functions of that status. The expressions "longest-life" and "joint-and-survivor" are also used in insurance practice as synonymous with "last-survivor."

The symbol $(\overline{x_1x_2 \cdots x_m})$ denotes a group of m lives aged x_1, x_2, $\cdots$, x_m, comprising a last-survivor status. The group continues to exist as long as *at least one* of (x_1), (x_2), $\cdots$, (x_m) is alive, and fails upon the occurrence of the *last* death.

The symbol ${}_np_{\overline{x_1x_2\cdots x_m}}$ denotes the probability that the last-survivor status $(\overline{x_1x_2 \cdots x_m})$ will survive for n years. This function may also be described as the probability that at least one of the lives (x_1), (x_2), $\cdots$, (x_m) will survive for n years, or as the probability that the last survivor will be alive at the end of n years. There is, of course, no way of knowing at the outset which one of the lives will be the last survivor. It should be understood that the language, "the last survivor will be alive," does not imply that none of the other lives will also be surviving; it is merely an alternative way of saying that at least one of the lives will be in existence.

It is evident that ${}_np_{\overline{x_1x_2\cdots x_m}}$ may be expressed as the complement of the probability that all the lives will die within n years. Thus,

$$\begin{aligned}{}_np_{\overline{x_1x_2\cdots x_m}} &= 1 - (1 - {}_np_{x_1})(1 - {}_np_{x_2}) \cdots (1 - {}_np_{x_m}) \\ &= ({}_np_{x_1} + {}_np_{x_2} + \cdots + {}_np_{x_m}) \\ &\quad - ({}_np_{x_1x_2} + {}_np_{x_1x_3} + \cdots + {}_np_{x_{m-1}x_m})\end{aligned}$$

$$+ \ ({}_np_{x_1x_2x_3} + {}_np_{x_1x_2x_4} + \cdots + {}_np_{x_{m-2}x_{m-1}x_m})$$
$$- \ ({}_np_{x_1x_2x_3x_4} + \cdots + {}_np_{x_{m-3}x_{m-2}x_{m-1}x_m})$$
$$+ \cdots$$
$$+ \ (-1)^{m+1}{}_np_{x_1x_2\ldots x_m}.$$

This expression may be written more concisely as follows:

$$\begin{aligned}{}_np_{\overline{x_1x_2\cdots x_m}} = \sum {}_np_{x_1} - \sum {}_np_{x_1x_2} + \sum {}_np_{x_1x_2x_3} \\ - \sum {}_np_{x_1x_2x_3x_4} + \cdots + (-1)^{m+1}{}_np_{x_1x_2\ldots x_m}\end{aligned} \tag{10.1}$$

where the summations are understood to cover all possible combinations of the lives taken one at a time, two at a time, three at a time, and so on. It is thus evident that the probabilities of survival for the last-survivor status may be obtained in terms of single-life and joint-life probabilities.

The probabilities of failure for the status $(\overline{x_1x_2 \cdots x_m})$ may now be defined in terms of ${}_np_{\overline{x_1x_2\cdots x_m}}$. The probability that the status will fail within n years is

$${}_nq_{\overline{x_1x_2\cdots x_m}} = 1 - {}_np_{\overline{x_1x_2\cdots x_m}}. \tag{10.2}$$

The probability that $(\overline{x_1x_2 \cdots x_m})$ will fail in the $(n + 1)$-th year is given by

$${}_{n|}q_{\overline{x_1x_2\cdots x_m}} = {}_np_{\overline{x_1x_2\cdots x_m}} - {}_{n+1}p_{\overline{x_1x_2\cdots x_m}}. \tag{10.3}$$

The most common last-survivor functions are those which involve only two lives or three lives. In these cases, assuming that the lives are (x), (y), and (z), we have from (10.1)

$${}_np_{\overline{xy}} = \sum {}_np_x - {}_np_{xy} = {}_np_x + {}_np_y - {}_np_{xy} \tag{10.4}$$

$$\begin{aligned}{}_np_{\overline{xyz}} &= \sum {}_np_x - \sum {}_np_{xy} + {}_np_{xyz} \\ &= {}_np_x + {}_np_y + {}_np_z - {}_np_{xy} - {}_np_{xz} - {}_np_{yz} + {}_np_{xyz}.\end{aligned} \tag{10.5}$$

Other functions can be expressed in a similar way; for example,

$$\begin{aligned}{}_nq_{\overline{xy}} &= 1 - {}_np_{\overline{xy}} = 1 - ({}_np_x + {}_np_y - {}_np_{xy}) \\ &= {}_nq_x + {}_nq_y - {}_nq_{xy} \\ &= \sum {}_nq_x - {}_nq_{xy}\end{aligned}$$

$$
\begin{aligned}
{}_{n|}q_{\overline{xyz}} &= {}_{n}p_{\overline{xyz}} - {}_{n+1}p_{\overline{xyz}} \\
&= ({}_{n}p_x + {}_{n}p_y + {}_{n}p_z - {}_{n}p_{xy} - {}_{n}p_{xz} - {}_{n}p_{yz} + {}_{n}p_{xyz}) - \\
&\quad ({}_{n+1}p_x + {}_{n+1}p_y + {}_{n+1}p_z - {}_{n+1}p_{xy} - {}_{n+1}p_{xz} - {}_{n+1}p_{yz} + {}_{n+1}p_{xyz}) \\
&= {}_{n|}q_x + {}_{n|}q_y + {}_{n|}q_z - {}_{n|}q_{xy} - {}_{n|}q_{xz} - {}_{n|}q_{yz} + {}_{n|}q_{xyz} \\
&= \sum {}_{n|}q_x - \sum {}_{n|}q_{xy} + {}_{n|}q_{xyz} \\
a_{\overline{xyz}} &= \sum_{t=1}^{\infty} v^t\, {}_{t}p_{\overline{xyz}} \\
&= \sum_{t=1}^{\infty} v^t ({}_{t}p_x + {}_{t}p_y + {}_{t}p_z - {}_{t}p_{xy} - {}_{t}p_{xz} - {}_{t}p_{yz} + {}_{t}p_{xyz}) \\
&= a_x + a_y + a_z - a_{xy} - a_{xz} - a_{yz} + a_{xyz} \\
&= \sum a_x - \sum a_{xy} + a_{xyz}
\end{aligned}
$$

In each of the special cases considered above for two lives and for three lives, it will be observed that the form of the result is the same as the form of (10.4) or (10.5). It will be apparent upon reflection that this is not merely coincidental, but that the similarity follows necessarily from the manner in which each of the functions depends upon the probabilities of survival. The functions, ${}_{n}q$, ${}_{n|}q$, and a, can all be expressed[1] as linear combinations of the values of ${}_{t}p$. If we let $F({}_{t}p)$ denote such a linear function of ${}_{t}p$, it will follow from the property of linearity that

$$
\begin{aligned}
F({}_{t}p_{\overline{x_1x_2\cdots x_m}}) &= F\left(\sum {}_{t}p_{x_1} - \sum {}_{t}p_{x_1x_2} + \cdots + (-1)^{m+1}\,{}_{t}p_{x_1x_2\cdots x_m}\right) \\
&= F\left(\sum {}_{t}p_{x_1}\right) - F\left(\sum {}_{t}p_{x_1x_2}\right) + \cdots + (-1)^{m+1}F({}_{t}p_{x_1x_2\cdots x_m}) \\
&= \sum F({}_{t}p_{x_1}) - \sum F({}_{t}p_{x_1x_2}) + \cdots + (-1)^{m+1}F({}_{t}p_{x_1x_2\cdots x_m}).
\end{aligned}
$$

Hence, any last-survivor function, $F_{\overline{x_1x_2\cdots x_m}}$, which can be written as a linear combination of values of ${}_{t}p$, may be expressed in terms of single-life functions F_{x_1} and joint-life functions $F_{x_1x_2}$, $F_{x_1x_2x_3}$, $\cdots$, $F_{x_1x_2\cdots x_m}$, in accordance with the relation

$$
F_{\overline{x_1x_2\cdots x_m}} = \qquad (10.6)
$$

$$
\sum F_{x_1} - \sum F_{x_1x_2} + \sum F_{x_1x_2x_3} - \cdots + (-1)^{m+1}F_{x_1x_2\cdots x_m}.
$$

[1] The function ${}_{n}q$ becomes linear in ${}_{t}p$ when written as ${}_{0}p - {}_{n}p$, where ${}_{0}p = 1$.

The net single premiums for the common annuity and insurance benefits can normally be expressed as linear combinations of the probabilities of survival, and relation (10.6) therefore holds in these cases. Examples of functions which are *not* linear in ${}_tp$ are net annual premiums and reserves. For example,

$$P_{\overline{xy}} = \frac{A_{\overline{xy}}}{\ddot{a}_{\overline{xy}}} = \frac{A_x + A_y - A_{xy}}{\ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}},$$

which is clearly different from $P_x + P_y - P_{xy}$. For the reserve value, the expression to be used depends upon which lives are still in existence at the time of valuation. Thus, in the case of the insurance $A_{\overline{xy}}$, the t-th year terminal reserve will be

$$A_{\overline{x+t:y+t}} - P_{\overline{xy}} \cdot \ddot{a}_{\overline{x+t:y+t}}$$

if (x) and (y) are both alive,

$$A_{x+t} - P_{\overline{xy}} \cdot \ddot{a}_{x+t}$$

if only (x) is alive,

or

$$A_{y+t} - P_{\overline{xy}} \cdot \ddot{a}_{y+t}$$

if only (y) is alive.

2. Compound statuses

Both the joint-life status and the last-survivor status have been defined only for groups of single lives. It is possible, however, and frequently useful, to relate these concepts to groups of individual statuses which are not necessarily single lives. For example, the annuity $a_{\overline{wx}:\overline{yz}}$ is payable during the *joint* existence of the *last survivor* of (w) and (x) and the *last survivor* of (y) and (z); the compound status $(\overline{wx}:\overline{yz})$ is a *joint* status of which the individual components are themselves *last-survivor* statuses. As another example, the insurance benefit represented by $A_{\overline{wx:\overline{yz}}}$ is payable when the second failure of (wx) and $(\overline{yz})$ has occurred, so that $(\overline{wx:\overline{yz}})$ is a compound last-survivor status of which the components are a joint-life status and a last-survivor status.

Functions of this kind can always be expressed in terms of single-life and simple joint-life functions. The process will be illustrated for the two cases cited above.

$$a_{\overline{wx}:\overline{yz}} = \sum_{t=1}^{\infty} v^t \, {}_tp_{\overline{wx}:\overline{yz}}$$

$$= \sum_{t=1}^{\infty} v^t \, {}_tp_{\overline{wx}} \cdot {}_tp_{\overline{yz}} \text{ (since both } (\overline{wx}) \text{ and } (\overline{yz}) \text{ must survive)}$$

$$= \sum_{t=1}^{\infty} v^t ({}_tp_w + {}_tp_x - {}_tp_{wx})({}_tp_y + {}_tp_z - {}_tp_{yz})$$

$$= \sum_{t=1}^{\infty} v^t ({}_tp_{wy} + {}_tp_{wz} + {}_tp_{xy} + {}_tp_{xz} - {}_tp_{wyz} - {}_tp_{xyz} - {}_tp_{wxy}$$
$$- {}_tp_{wxz} + {}_tp_{wxyz})$$

$$= a_{wy} + a_{wz} + a_{xy} + a_{xz} - a_{wyz} - a_{xyz} - a_{wxy} - a_{wxz} + a_{wxyz}$$

$$A_{\overline{wx:\overline{yz}}} = \sum_{t=0}^{\infty} v^{t+1} \, {}_{t|}q_{\overline{wx:\overline{yz}}} = \sum_{t=0}^{\infty} v^{t+1} ({}_tp_{\overline{wx:\overline{yz}}} - {}_{t+1}p_{\overline{wx:\overline{yz}}})$$

$$= \sum_{t=0}^{\infty} v^{t+1} \{[1 - (1 - {}_tp_{wx})(1 - {}_tp_{\overline{yz}})]$$
$$- [1 - (1 - {}_{t+1}p_{wx})(1 - {}_{t+1}p_{\overline{yz}})]\}$$

(since ${}_tp_{\overline{wx:\overline{yz}}}$ is the complement of the probability that both (wx) and $(\overline{yz})$ will fail in t years)

$$= \sum_{t=0}^{\infty} v^{t+1} \{[1 - (1 - {}_tp_{wx})(1 - {}_tp_y - {}_tp_z + {}_tp_{yz})]$$
$$- [1 - (1 - {}_{t+1}p_{wx})(1 - {}_{t+1}p_y - {}_{t+1}p_z + {}_{t+1}p_{yz})]\}$$

$$= \sum_{t=0}^{\infty} v^{t+1} [({}_tp_y + {}_tp_z - {}_tp_{yz} + {}_tp_{wx} - {}_tp_{wxy} - {}_tp_{wxz} + {}_tp_{wxyz})$$
$$- ({}_{t+1}p_y + {}_{t+1}p_z - {}_{t+1}p_{yz} + {}_{t+1}p_{wx} - {}_{t+1}p_{wxy}$$
$$- {}_{t+1}p_{wxz} + {}_{t+1}p_{wxyz})]$$

$$= \sum_{t=0}^{\infty} v^{t+1} ({}_{t|}q_y + {}_{t|}q_z - {}_{t|}q_{yz} + {}_{t|}q_{wx} - {}_{t|}q_{wxy} - {}_{t|}q_{wxz}$$
$$+ {}_{t|}q_{wxyz})$$

$$= A_y + A_z - A_{yz} + A_{wx} - A_{wxy} - A_{wxz} + A_{wxyz} \, .$$

While it is of fundamental importance for the understanding of the subject that the student be able to resolve compound expressions of this type into their single- and joint-life elements, it will be noted that in practice the expansions required can become tedious. Certain general relationships between last-survivor and joint-life statuses make it possible to reduce very markedly the steps needed for these expansions.

Consider first the function $a_{\overline{uv}}$, where (u) and (v) represent any type of status thus far defined—single-life, joint-life, last-survivor, or term-certain. Whatever the individual statuses (u) and (v) may be, we know that the last-survivor annuity $a_{\overline{uv}}$ will provide annual payments of 1 for a period determined by the *longer* surviving status. On the other hand, the joint-life annuity a_{uv} provides payments only during the existence of the *shorter* surviving status. Hence, $a_{\overline{uv}}$ provides an annuity payable during the continuance of one of the statuses (u) and (v), and a_{uv} provides an annuity on the other. It follows that

$$a_{\overline{uv}} + a_{uv} = a_u + a_v \,,$$

whence

$$a_{\overline{uv}} = a_u + a_v - a_{uv} \,. \tag{10.7a}$$

It will be observed that the two-life case of relation (10.6) applied to annuities is merely a special case of the above result. Similar reasoning produces an analogous relation for the present values of insurances:

$$A_{\overline{uv}} = A_u + A_v - A_{uv} \,. \tag{10.7b}$$

Next consider the function $a_{u:\overline{vw}}$, where again the statuses (u), (v), (w) are not necessarily single lives. This function provides annuity payments for as long as (u) survives jointly with the *longer* surviving of the two statuses (v) and (w). The function a_{uvw}, on the other hand, provides payments for as long as (u) survives jointly with the *shorter* surviving of the two statuses (v) and (w). Hence, $a_{u:\overline{vw}}$ is payable during the continuance of one of the statuses (uv) and (uw), and a_{uvw} provides an annuity on the other. It follows that

$$a_{u:\overline{vw}} + a_{uvw} = a_{uv} + a_{uw} \,,$$

whence

$$a_{u:\overline{vw}} = a_{uv} + a_{uw} - a_{uvw} \,. \tag{10.8a}$$

The corresponding result for insurances is

$$A_{u:\overline{vw}} = A_{uv} + A_{uw} - A_{uvw}\,. \tag{10.8b}$$

The two examples discussed earlier may now be used to illustrate how these general relationships reduce the number of steps needed to obtain the required expansion. Thus, for the function $a_{\overline{\overline{wx}:\overline{yz}}}$, we use (10.8a):

$$a_{\overline{\overline{wx}:\overline{yz}}} = a_{\overline{wx}:y} + a_{\overline{wx}:z} - a_{\overline{wx}:yz}\,.$$

Applying (10.8a) a second time,

$$a_{\overline{\overline{wx}:\overline{yz}}} =$$

$$(a_{wy} + a_{xy} - a_{wxy}) + (a_{wz} + a_{xz} - a_{wxz}) - (a_{wyz} + a_{xyz} - a_{wxyz}),$$

agreeing with the previous result.

Similarly, $A_{\overline{wx:\overline{yz}}}$ is first expanded by (10.7b) to

$$A_{wx} + A_{\overline{yz}} - A_{wx:\overline{yz}}\,,$$

and then expanding $A_{\overline{yz}}$ and applying (10.8b) to the term $A_{wx:\overline{yz}}$, we find

$$A_{\overline{wx:\overline{yz}}} = A_{wx} + (A_y + A_z - A_{yz}) - (A_{wxy} + A_{wxz} - A_{wxyz}),$$

as before.

It should be noted that the general reasoning by which formulas (10.7) and (10.8) have been derived is equally valid when the compound status involves a term-certain. Consider an annuity payable until the death of (x) or until (y) attains age $y + n$, whichever period is longer. The two statuses are (x) and $(y:\overline{n}|)$, and the annuity is payable during the continuance of the last-surviving status. We thus have, from (10.7a),

$$a_{\overline{(x)(y:\overline{n}|)}} = a_x + a_{y:\overline{n}|} - a_{xy:\overline{n}|}\,.$$

If the annuity is payable until (x) attains age $x + n$ or for m years certain, whichever is longer, the statuses are $(x:\overline{n}|)$ and $(\overline{m}|)$, and we have $a_{\overline{(x:\overline{n}|)(\overline{m}|)}} = a_{x:\overline{n}|} + a_{\overline{m}|} - a_{x:\overline{n}|:\overline{m}|}$. For $m < n$, the status $(\overline{m}|)$ fails before the status $(\overline{n}|)$, and hence the joint status $(x:\overline{n}|:\overline{m}|)$ is the same as $(x:\overline{m}|)$. The function $a_{x:\overline{n}|:\overline{m}|}$ can be written in this case as $a_{x:\overline{m}|}$. We thus have

$$a_{\overline{(x:\overline{n}|)(\overline{m}|)}} = a_{x:\overline{n}|} + a_{\overline{m}|} - a_{x:\overline{m}|}\,, \qquad m < n.$$

An annuity payable until either (x) attains age $x + n$ or (y) attains age $y + m$ $(m < n)$, whichever period is longer, has the present value

$$
\begin{aligned}
a_{\overline{(x:\overline{n|})(y:\overline{m|})}} &= a_{x:\overline{n|}} + a_{y:\overline{m|}} - a_{xy:\overline{n|}:\overline{m|}} \\
&= a_{x:\overline{n|}} + a_{y:\overline{m|}} - a_{xy:\overline{m|}}, \qquad m < n.
\end{aligned}
$$

3. The general multi-life status

The joint-life status and the last-survivor status may be regarded as special cases of a general multi-life status. Let the m lives, (x_1), (x_2), $\cdots$, (x_m), comprise a status which is defined as surviving as long as at least r of the individual lives survive and which therefore fails upon the occurrence of the $(m - r + 1)$-th death. This general status is denoted by the symbol $(\overset{r}{\overline{x_1x_2 \cdots x_m}})$. When r equals m, this reduces to the joint-life status; when r equals 1, a last-survivor status is defined.

The symbol ${}_np_{\overset{r}{\overline{x_1x_2\cdots x_m}}}$ denotes the probability that the status $(\overset{r}{\overline{x_1x_2 \cdots x_m}})$ will survive for n years, that is, that at least r out of the original m lives will still be in existence n years hence. In order to evaluate this function, it is first necessary to find an expression for the probability that *exactly* r lives will be surviving at the end of n years, denoted by ${}_np_{\overset{[r]}{\overline{x_1x_2\cdots x_m}}}$.

If the m lives are all of the same age x, the probability that exactly r *designated* lives will be surviving at the end of n years, the other $m - r$ being dead, is clearly

$$({}_np_x)^r(1 - {}_np_x)^{m-r}.$$

Now there are $\binom{m}{r}$ ways of selecting a group of r lives out of m, and the probability that *any* group of exactly r lives will be surviving, and the others dead, is

$${}_np_{\overset{[r]}{\overline{xx\cdots x}}} = \binom{m}{r}({}_np_x)^r(1 - {}_np_x)^{m-r}. \tag{10.9}$$

Note that this expression is equivalent to the coefficient of t^r in the binomial expansion of $({}_np_x \cdot t + {}_nq_x)^m$.

Suppose now that the m lives are not all of the same age. The probability that exactly r specified lives, (x_1), (x_2), $\cdots$, (x_r), will be surviving at the end of n years and the others dead is

$$ {}_np_{x_1}\cdot{}_np_{x_2}\cdots{}_np_{x_r}(1-{}_np_{x_{r+1}})(1-{}_np_{x_{r+2}})\cdots(1-{}_np_{x_m}). $$

The probability ${}_np_{\overline{x_1x_2\cdots x_m}}^{[r]}$ is the sum of all such expressions containing exactly r factors ${}_np_{x_i}$ and exactly $m - r$ factors $1 - {}_np_{x_j}$. To find this sum, we write the following product:

$$ ({}_np_{x_1}\cdot t + {}_nq_{x_1})({}_np_{x_2}\cdot t + {}_nq_{x_2})\cdots({}_np_{x_m}\cdot t + {}_nq_{x_m}). $$

In the expansion of this product, the coefficient of t^r is the sum of all the terms which can be formed by multiplying r factors of the form ${}_np_{x_i}$ and $m - r$ factors of the form ${}_nq_{x_j}$, and is thus equal to the required sum for ${}_np_{\overline{x_1x_2\cdots x_m}}^{[r]}$. We rewrite the product as

$$ [1 + (t-1)\,{}_np_{x_1}][1 + (t-1)\,{}_np_{x_2}]\cdots[1 + (t-1)\,{}_np_{x_m}] $$
$$ = 1 + (t-1)\,Z_1 + (t-1)^2\,Z_2 + \cdots + (t-1)^m\,Z_m, $$

where Z_s denotes the summation $\sum {}_np_{x_1x_2\ldots x_s}$ taken over all the possible combinations of s lives out of m. Now t^r appears only in the terms involving Z_s where $s \geqslant r$. The coefficient of t^r in the term involving Z_s, that is, the term $(t-1)^s\,Z_s$ $(s \geqslant r)$, is

$$ (-1)^{s-r}\binom{s}{s-r}Z_s. $$

Hence the total coefficient of t^r is

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{[r]} = \sum_{s=r}^{m}(-1)^{s-r}\binom{s}{s-r}Z_s = Z_r - \binom{r+1}{1}Z_{r+1} $$
$$ + \binom{r+2}{2}Z_{r+2} - \cdots + (-1)^{m-r}\binom{m}{m-r}Z_m. \quad (10.10) $$

The presence of the binomial coefficients in this formula suggests the following formal device. If we regard the indices of Z as exponents and assume that $Z^t = 0$ for $t > m$, the formula may be written in the following symbolic form:

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{[r]} = \frac{Z^r}{(1+Z)^{r+1}}. \quad (10.11) $$

As an example of the use of formula (10.11), we obtain the probability that exactly two of the four lives, (w), (x), (y), (z), will survive for n years. Since $m = 4$ and $r = 2$, we first write

$$ \frac{Z^2}{(1+Z)^3} = Z^2 - 3Z^3 + 6Z^4. $$

Then,

$$
{}_np_{\overline{wxyz}}^{[2]} = {}_np_{wx} + {}_np_{wy} + {}_np_{wz} + {}_np_{xy} + {}_np_{xz} + {}_np_{yz}
$$

$$
- 3\left({}_np_{wxy} + {}_np_{wxz} + {}_np_{wyz} + {}_np_{xyz}\right) + 6\,{}_np_{wxyz}\,.
$$

A partial check can be obtained by letting ${}_np = 1$ for each single life. We then find

$$
{}_np_{\overline{wxyz}}^{[2]} = 6 - 3\cdot(4) + 6 = 0.
$$

This is clearly correct since if each life is certain to survive, the probability that exactly two will survive is 0.

We now return to the probability ${}_np_{\overline{x_1x_2\cdots x_m}}^{\,r}$ that *at least* r lives will survive for n years. We shall show that this probability can be expressed as follows:

$$
{}_np_{\overline{x_1x_2\cdots x_m}}^{\,r} = \sum_{s=r}^{m}(-1)^{s-r}\binom{s-1}{s-r}Z_s = Z_r - \binom{r}{1}Z_{r+1}
$$

$$
+ \binom{r+1}{2}Z_{r+2} - \cdots + (-1)^{m-r}\binom{m-1}{m-r}Z_m\,. \qquad (10.12)
$$

The proof is by induction on r. When $r = 1$, the formula is

$$
{}_np_{\overline{x_1x_2\cdots x_m}}^{\,1} = Z_1 - Z_2 + Z_3 - \cdots + (-1)^{m-1}Z_m\,.
$$

Since this is equivalent to (10.1), we see that the formula is valid in this case. We now show that if the formula holds for $r = k$, it also holds for $r = k + 1$.

$$
{}_np_{\overline{x_1x_2\cdots x_m}}^{\,k+1} = {}_np_{\overline{x_1x_2\cdots x_m}}^{\,k} - {}_np_{\overline{x_1x_2\cdots x_m}}^{[k]}
$$

$$
= \sum_{s=k}^{m}(-1)^{s-k}\binom{s-1}{s-k}Z_s - \sum_{s=k}^{m}(-1)^{s-k}\binom{s}{s-k}Z_s
$$

$$
= \sum_{s=k}^{m}(-1)^{s-k}\left[\binom{s-1}{s-k} - \binom{s}{s-k}\right]Z_s
$$

$$
= \sum_{s=k+1}^{m}(-1)^{s-k-1}\binom{s-1}{s-k-1}Z_s\,,
$$

where the expression $\binom{s-1}{s-k} - \binom{s}{s-k}$ has been replaced by $-\binom{s-1}{s-k-1}$ for $s \geqslant k + 1$ and by 0 for $s = k$. This relation is familiar from Pascal's triangle of the binomial coefficients; it can be easily derived as an algebraic identity. Since the result is the same as (10.12) for $r = k + 1$, the induction is complete and we conclude that (10.12) holds for all positive integers r. The formula may be written in symbolic form as

$$ {}_np_{\overline{x_1x_2\cdots x_m}}^{\;r} = \frac{Z^r}{(1+Z)^r}. \tag{10.13}$$

As an example of the use of formula (10.13), we obtain the probability that at least two of the three lives, (x), (y), (z), will be alive at the end of n years. Since $m = 3$ and $r = 2$, we first write

$$\frac{Z^2}{(1+Z)^2} = Z^2 - 2Z^3.$$

Then,

$$ {}_np_{\overline{xyz}}^{\;2} = {}_np_{xy} + {}_np_{xz} + {}_np_{yz} - 2\,{}_np_{xyz}\,.$$

As a check in this case, if we let ${}_np = 1$ for each single life, we have

$$ {}_np_{\overline{xyz}}^{\;2} = 1 + 1 + 1 - 2 = 1.$$

If each life is certain to survive, the probability that at least two will survive is 1.

The expression $\frac{Z^r}{(1+Z)^r}$ of formula (10.13) may also be used to expand any linear function of the probability ${}_np_{\overline{x_1x_2\cdots x_m}}^{\;r}$ in terms of the corresponding joint-life functions. For example,

$$a_{\overline{xyz}}^{\;2} = a_{xy} + a_{xz} + a_{yz} - 2a_{xyz}$$

$$A_{\overline{wxyz}}^{\;3} = A_{wxy} + A_{wxz} + A_{wyz} + A_{xyz} - 3A_{wxyz}\,.$$

When the insurance symbol is used in this way, it must be kept in mind that $A_{\overline{x_1x_2\cdots x_m}}^{\;r}$ represents an insurance payable upon the failure of the status $\overset{r}{\overline{(x_1x_2\cdots x_m)}}$. The insurance, therefore, is payable not at the r-th death but at the $(m - r + 1)$-th death.

Similarly, the expression $\frac{Z^r}{(1+Z)^{r+1}}$ of formula (10.11) may be used to expand linear functions of ${}_np_{\overline{x_1x_2\cdots x_m}}^{\;[r]}$. As an example, the present value of an annuity of 1 payable at the end of each year if exactly one of the three lives, (x), (y), and (z), is then surviving is given by

$$a_{\overline{xyz}}^{\;[1]} = a_x + a_y + a_z - 2(a_{xy} + a_{xz} + a_{yz}) + 3a_{xyz}\,.$$

It should be noted that this benefit is a form of deferred annuity, no payments being made until two deaths have occurred. The expression can be checked by noting the annuity payments that are provided while the different numbers of lives are surviving. When all three are alive, each of the annuity functions contributes a payment of 1, and the total payment is $1 + 1 + 1 - 2 - 2 - 2 + 3 = 0$. When exactly two are alive, there is a payment for *two* of the single-life functions and for *one* of the two-life functions, and the total payment is $1 + 1 - 2 = 0$. When exactly one is alive, there is a payment of 1, as required, from one of the single-life functions. This is also a useful device for checking the corresponding probability expansion.

4. Summary of notation

If a horizontal bar appears above a suffixed subscript, then *survivors* of the lives, and not *joint* lives, are intended. The number of survivors is denoted by a symbol over the right end of the bar. If that symbol, say r, is enclosed in brackets, the meaning is *exactly* r survivors; without brackets, the meaning is *at least* r survivors. If no symbol appears over the bar, then unity is assumed and the meaning is *at least one* survivor.

References

3. The formula for ${}_np_{\overline{x_1x_2\cdots x_m}}^{\,[r]}$ was derived by Waring in 1792. In its general form, Waring's Theorem gives the probability that exactly r out of m events will occur, and thus has wide applications. The theorem is discussed in standard texts on probability, e.g., Bizley (1957). White and Greville (1959) have discussed computational devices which simplify the application of the theorem. A related method of computing the values of the individual shares in a last-survivor annuity has been given by Rasor and Myers (1952).

The proof by induction for formula (10.12) was suggested by Harry Gershenson, F.S.A.

EXERCISES

1. The last-survivor status

1. Show that $q_{\overline{xy}} = q_x \cdot q_y$, and give an alternative expression in terms of q_x, q_y, and q_{xy}.

2. Express each of the following in terms of single- and joint-life functions:

(a) $\bar{a}_{\overline{wxyz}}$

(b) $P^{\ 1}_{\overline{xy}:\overline{n|}}$

3. Find the probability that at least one of the two lives (x) and (y) will die in the $(n + 1)$-th year. Is this the same as ${}_{n|}q_{\overline{xy}}$? Explain.

4. Describe the benefit whose net single premium is ${}_{n|}a_{\overline{xy}}$. Show that ${}_{n|}a_{\overline{xy}}$ is *not* equivalent to $v^n{}_np_{xy}a_{\overline{x+n:y+n}}$ nor to $v^n{}_np_{\overline{xy}}a_{\overline{x+n:y+n}}$. Give a correct expression for ${}_{n|}a_{\overline{xy}}$.

5. Given a mortality table following Makeham's law and two ages x and y for which (ww) is the equivalent equal-age status.

(a) Show that ${}_tp_w$ is the geometric mean of ${}_tp_x$ and ${}_tp_y$.

(b) If $x \neq y$, show that ${}_tp_x + {}_tp_y > 2{}_tp_w$, and hence that $a_{\overline{xy}} > a_{\overline{ww}}$.

6. Given $a_{xy} = a_{ww}$ on a Makehamized mortality table, and $y < x$. Rank the following in order of magnitude: a_y , a_w , a_{xx} , a_{ww} , $a_{\overline{xy}}$, $a_{\overline{yy}}$.

2. Compound statuses

7. Describe the benefit whose net single premium is denoted by $A_{\overline{x:\overline{n|}}}$. Show that $A_{\overline{x:\overline{n|}}} = A_x - A_{x:\overline{n|}} + v^n$.

8. Show that

$$a_{\overline{wx}:\overline{yz}} = a_{\overline{wx}} + a_{\overline{yz}} - a_{\overline{wxyz}} ,$$

and explain by general reasoning.

9. Show that

$$a_{\overline{\overline{xy}:\overline{n|}}} = a_{\overline{n|}} + {}_{n|}a_{xy} .$$

Describe this benefit.

10. Express $a_{abc:\overline{xyz}}$ in terms of joint-life annuities.

11. Express in annuity symbols the value of an annuity of 1 payable at the end of each year until the survivor of (25) or (30) reaches age 50, or until his death if both lives die before attaining that age.

12. Express in annuity symbols the value of a deferred annuity of 1 payable as long as either (25) or (30) is living after age 50.

13. Give an expression for evaluating $a_{\overline{\overline{x:y}:\overline{n|}}}$ in terms of annuities-certain and single- and joint-life annuities.

3. The general multi-life status

14. Show that the symbolic formula

$${}_np_{\overline{xy\ldots z}}^{\ \ r} = \frac{Z^r}{(1 + Z)^r}$$

may be obtained formally by an algebraic summation over s of

$$_np_{\overline{xy\ldots s}}^{[s]} = \frac{Z^s}{(1+Z)^{s+1}}.$$

15. Express in terms of single- and joint-life probabilities:

(a) $_np_{\overline{xyz}}^{2}$

(b) $_np_{\overline{xyz}}^{[1]}$

16. Describe in words the probability indicated by the following expression:

$$_tp_{wx} + {_tp_{wy}} + {_tp_{wz}} + {_tp_{xy}} + {_tp_{xz}} + {_tp_{yz}}$$
$$- 3({_tp_{wxy}} + {_tp_{wxz}} + {_tp_{wyz}} + {_tp_{xyz}}) + 6{_tp_{wxyz}}.$$

17. Give an expression in terms of joint-life annuity values for the net single premium for an annuity of 1 payable at the end of each year if exactly 3 of the 4 lives (w), (x), (y), (z) are then alive.

18. Find the value of $a_{\overline{xxxxxx}}^{[4]}$, given that

$a_{xxxx} = 11.900$, $a_{xxxxx} = 10.750$ and $a_{xxxxxx} = 9.675$. (Ans. 1.125)

19. Prove mathematically and by general reasoning:

$$a_{\overline{xyz}} - a_{\overline{xyz}}^{[1} = a_{\overline{xyz}}^{2}.$$

20. Express in terms of annuity values and the rate of discount the net single premium for an insurance payable at the end of the year in which the third death occurs among the four lives (w), (x), (y), (z).

21. Obtain an expression for evaluating $\ddot{a}_{w:\overline{xyz}}^{2}$, an annuity-due payable for as long as (w) survives jointly with at least two of the lives, (x), (y), and (z).

Miscellaneous problems

22. A partnership is formed by five men, all of whom are aged x exactly. Find:

(a) The probability that not more than three of the original partners will be alive at the end of n years.

(b) The probability that not more than one of the original partners will die in the n-th year.

23. An extract from a table of joint-life annuities valued at $3\frac{1}{2}\%$ reads as follows:

Age of Lives	Net Single Premium Joint-Life Immediate Annuity
26: 20: 28	14.40
29: 26: 20	14.30
28: 29: 26	13.80
20: 28: 29	14.00
29: 26: 28: 20	12.50

Find the net single premium for an insurance of \$1000 payable at the end of the year of death of the second life to fail out of four lives aged 20, 26, 28, and 29. (Ans. \$324)

24. An annuity of 1 is payable to (x) and (y) on a last-survivor basis commencing at the end of $m + k$ years, provided that both lives survive m years. Show that its present value is

$$_mp_y \cdot {}_{k+m-1|}a_x + {}_mp_x \cdot {}_{k+m-1|}a_y - {}_{k+m-1|}a_{xy} \, .$$

25. An annuity of 1 is payable yearly to (x) as long as he lives jointly with (y) and for n years after the death of (y) except that in no event will payments be made after m years from the present time, $m > n$. Show that its present value is

$$a_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\, a_{x+n:y:\overline{m-n}|} \, .$$

26. An insurance of 1 is to be payable in the event that (x) dies before age $x + n$ and (y) dies before age $y + m$, where $m < n$, the face amount becoming payable at the end of the year in which the second death occurs. Show that the present value may be expressed as

$$A^{\;\;1}_{\overline{xy}:\overline{m}|} + v^m {}_mp_x(1 - {}_mp_y)A^{\;\;1}_{x+m:\overline{n-m}|} \, ,$$

and that the net annual premium would be determined by dividing by

$$\ddot{a}_{\overline{xy}:\overline{m}|} + v^m {}_mp_x(1 - {}_mp_y)\ddot{a}_{x+m:\overline{n-m}|} \, .$$

27. Last-survivor annuities often provide for a reduction in the amount of the annuity payment after the first death. An n-year temporary annuity-due payable to $(\overline{xy})$ provides annual payments of 1 while both lives survive, reducing to $\frac{1}{2}$ on the death of (x) or to $\frac{1}{3}$ on the death of (y). Express the present value in terms of annuity values.

28. An annual annuity is to be payable so long as any of the lives (w), (x), (y), and (z) survive. At each death the rate of annual payment is reduced 50% from the rate in effect prior to such death. Find a simplified expression involving only single- and joint-life annuities for the cost of the contract per unit of initial annual payment.

29. Obtain an expression in annuity symbols for the net single premium for a continuous annuity of 1 per annum payable while at least one of the two lives (40) and (50) is living and is over age 60 but not while either of them is living under age 55.

CHAPTER 11

CONTINGENT FUNCTIONS

1. Introduction

In the various types of multi-life status discussed in the preceding chapters, the component lives are not distinguished according to the order in which they die. The status (xyz), for example, terminates upon the first death, whether it is the death of (x), of (y), or of (z); and the status $(\overline{xyz})$ terminates upon the last death regardless of the particular order in which the individual deaths occur.

A different group of functions is required in situations where the order of deaths is significant, as, for example, in the case of an insurance payable upon the death of (x) provided that he dies before (y). Functions of this type, depending upon a specified order of deaths, are known as *contingent functions*.

The present chapter deals only with the simplest class of contingent function—that in which the order of failure is prescribed with reference to only one of the individual components involved. A subsequent chapter will describe functions in which the order of failure is prescribed for more than one of the components—the *compound* contingent functions.

2. Contingent probabilities

When the lives of a group are all of the same age and subject to the same mortality, each has the same chance of dying first, or second, and contingent probabilities are easily evaluated by reference to the possible orders of death. Thus, in the case of two lives each aged x, the probability that a specified one of them will die first is simply equal to $\frac{1}{2}$. Similarly, the probability that the specified life will die first and within n years is

$$\tfrac{1}{2} \cdot {}_nq_{xx} ,$$

${}_nq_{xx}$ representing the chance that the joint lifetime ends within n years, and $\frac{1}{2}$ being the chance that it is terminated by the failure of the specified life.

In general, the lives involved are not of equal age. Consider first the case of two lives aged x and y, and let $q_{\overset{1}{x}y}$ denote the probability that (x) will die before (y) and within one year— in other words, that the joint lifetime will fail within one year because of the death of (x). This probability may be expressed in definite integral form as

$$q_{\overset{1}{x}y} = \int_0^1 {}_tp_{xy}\mu_{x+t}\,dt = \frac{1}{l_{xy}}\int_0^1 l_{x+t:y+t}\mu_{x+t}\,dt, \tag{11.1}$$

since the differential ${}_tp_{xy}\mu_{x+t}\,dt$ represents the probability that (x) will die at the moment of attaining age $x + t$, (y) being still alive.

The other basic contingent probabilities for two lives may be written as follows:

$${}_nq_{\overset{1}{x}y} = \int_0^n {}_tp_{xy}\mu_{x+t}\,dt \tag{11.2}$$

$${}_\infty q_{\overset{1}{x}y} = \int_0^\infty {}_tp_{xy}\mu_{x+t}\,dt \tag{11.3}$$

$${}_{n|}q_{\overset{1}{x}y} = \int_n^{n+1} {}_tp_{xy}\mu_{x+t}\,dt. \tag{11.4}$$

Note the symbol used in (11.3) for the probability that (x) will die before (y) with no limit on the time of death.

If the specified life (x) is to die *second* within n years, the probability is

$$\begin{aligned}
{}_nq_{\overset{2}{x}y} &= \int_0^n {}_tp_x(1 - {}_tp_y)\mu_{x+t}\,dt \\
&= \int_0^n {}_tp_x\mu_{x+t}\,dt - \int_0^n {}_tp_{xy}\mu_{x+t}\,dt \\
&= {}_nq_x - {}_nq_{\overset{1}{x}y}.
\end{aligned} \tag{11.5}$$

The result may be obtained by general reasoning, since if (x) dies within n years he will die either before or after (y); i.e., ${}_nq_x = {}_nq_{\overset{1}{x}y} + {}_nq_{\overset{2}{x}y}$.

The probability that (x) will die second and during the $(n + 1)$-th year may similarly be seen to be

$${}_{n|}q_{\overset{2}{x}y} = {}_{n|}q_x - {}_{n|}q_{\overset{1}{x}y}. \tag{11.6}$$

Note that (y) need not die in the $(n + 1)$-th year, but may die at any time preceding the death of (x).

The probabilities involving three lives, contingent upon the death of a specified life (x), are as follows:

$$ {}_nq_{\overset{1}{x}yz} = \int_0^n {}_tp_{xyz}\mu_{x+t}\,dt \tag{11.7}$$

$$\begin{aligned} {}_nq_{\overset{2}{x}yz} &= \int_0^n {}_tp_x \cdot {}_tp_{\overline{yz}}^{[1]}\mu_{x+t}\,dt \\ &= \int_0^n {}_tp_x({}_tp_y + {}_tp_z - 2\,{}_tp_{yz})\mu_{x+t}\,dt \\ &= {}_nq_{\overset{1}{x}y} + {}_nq_{\overset{1}{x}z} - 2\,{}_nq_{\overset{1}{x}yz} \end{aligned} \tag{11.8}$$

$$\begin{aligned} {}_nq_{\overset{3}{x}yz} &= \int_0^n (1 - {}_tp_y)(1 - {}_tp_z)\,{}_tp_x\mu_{x+t}\,dt \\ &= {}_nq_x - {}_nq_{\overset{1}{x}y} - {}_nq_{\overset{1}{x}z} + {}_nq_{\overset{1}{x}yz}\,. \end{aligned} \tag{11.9}$$

The probability ${}_nq_{\overset{1}{\overline{xy}}:z}$, that either (x) or (y) will be the first to die within n years, is an instance of a function in which the specified status is not a single life:

$$ {}_nq_{\overset{1}{\overline{xy}}:z} = {}_nq_{\overset{1}{x}yz} + {}_nq_{x\overset{1}{y}z}\,. \tag{11.10}$$

The probability that (x) will die within n years and before the survivor of (y) and (z) is

$$\begin{aligned} {}_nq_{\overset{1}{x}:\overline{yz}} &= \int_0^n {}_tp_{x:\overline{yz}}\mu_{x+t}\,dt \\ &= \int_0^n {}_tp_x({}_tp_y + {}_tp_z - {}_tp_{yz})\mu_{x+t}\,dt \\ &= {}_nq_{\overset{1}{x}y} + {}_nq_{\overset{1}{x}z} - {}_nq_{\overset{1}{x}yz}\,. \end{aligned} \tag{11.11}$$

This function may also be described as the probability that (x) will die either first or second and within n years, and an alternative formulation is

$$ {}_nq^{1:2}_{x\ yz} = {}_nq_{\overset{1}{x}yz} + {}_nq_{\overset{2}{x}yz}\,. \tag{11.12}$$

In view of (11.8), this latter expression may easily be seen to be equivalent to that obtained above. Another expression for this probability is

$$ {}_nq_z - {}_nq_{xyz}^{\;\;\;\;3} . $$

Consider the function ${}_nq^{\;\;2}_{\overline{xy}:z}$, representing the probability that the survivor of (x) and (y) will die within n years, predeceased by (z). In formulating this probability as a definite integral, it is necessary to express the differential probability that the survivor of (x) and (y) will die at moment of time t. This expression is

$$ [{}_tp_x(1 - {}_tp_y)\mu_{x+t} + {}_tp_y(1 - {}_tp_x)\mu_{y+t}]\, dt $$

or

$$ ({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\, dt. $$

Hence

$$ {}_nq^{\;\;2}_{\overline{xy}:z} = \int_0^n (1 - {}_tp_z)({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\, dt $$

$$ = ({}_nq_x + {}_nq_y - {}_nq_{xy}) - ({}_nq^{1}_{xz} + {}_nq^{1}_{yz} - {}_nq^{\;\;1}_{\overline{xy}:z}). \quad (11.13) $$

3. Evaluation of contingent probabilities

In the elementary case where the lives are of equal age, contingent probabilities can be evaluated directly from general probability principles. It is sometimes helpful to write the integral expression and carry out the integration as a check on the direct reasoning.

For three lives all aged x, we know that the probability that a specified life will die first within n years is

$$ {}_nq^{1}_{xxx} = \tfrac{1}{3}\cdot {}_nq_{xxx} = \tfrac{1}{3}\,[1 - ({}_np_x)^3]. $$

Using the integral form, we have

$$ {}_nq^{1}_{xxx} = \int_0^n {}_tp_{xxx}\mu_{x+t}\, dt $$

$$ = \int_0^n ({}_tp_x)^2\,{}_tp_x\mu_{x+t}\, dt $$

$$ = -\int_{t=0}^{t=n} ({}_tp_x)^2 d({}_tp_x) $$

$$ = [-\tfrac{1}{3}\,({}_tp_x)^3]_0^n = \tfrac{1}{3}\,[1 - ({}_np_x)^3]. $$

It is important to distinguish this probability from two others that are similar.

(1) If the failing life is not specified, the probability that the first death occurs within n years is simply the probability that the joint-life status fails within n years:

$$\begin{aligned} {}_nq_{xxx} &= \int_0^n {}_tp_{xxx}\mu_{x+t:x+t:x+t}\,dt \\ &= \int_0^n {}_tp_{xxx}\cdot 3\mu_{x+t}\,dt \\ &= 3\,{}_nq^{\,1}_{xxx} = 1 - ({}_np_x)^3. \end{aligned}$$

(2) The probability that a specified life dies within n years and the other two survive the n-year period is

$$\int_0^n {}_tp_x\mu_{x+t}\cdot{}_np_{xx}\,dt = {}_nq_x\cdot{}_np_{xx} = (1 - {}_np_x)({}_np_x)^2.$$

If the failing life is not specified, the probability that exactly one will die within n years is $3(1 - {}_np_x)\ ({}_np_x)^2$.

Suppose that the three lives aged x are A, B, and C. What is the probability that either A or B will be the first to die within n years? The joint lifetime of A and B must fail before C, and hence the solution is given by ${}_nq^{\,1}_{\overline{xx}:x}$. Using (11.10), we find ${}_nq^{\,1}_{\overline{xx}:x} = 2\,{}_nq^{\,1}_{xxx} = \frac{2}{3}[1 - ({}_np_x)^3]$. The solution by integration is

$$\begin{aligned} {}_nq^{\,1}_{\overline{xx}:x} &= \int_0^n {}_tp_{xx}\mu_{x+t:x+t}\cdot{}_tp_x\,dt \\ &= \int_0^n {}_tp_{xxx}\cdot 2\,\mu_{x+t}\,dt \\ &= \tfrac{2}{3}[1 - ({}_np_x)^3]. \end{aligned}$$

When these functions involve lives of unequal age, the integrations cannot be directly carried out and special methods must be used. Standard methods of numerical integration are available which can be programmed for a digital computer if a large volume of calculations must be made.

Another method of evaluation is based upon the assumption of a uniform distribution of deaths. In the formula

$$q^{\,1}_{xy} = \int_0^1 {}_tp_{xy}\mu_{x+t}\,dt,$$

we substitute from formula (1.24b)

$$ {}_tp_x\mu_{x+t} \doteq q_x \,, $$

and obtain

$$ q_{xy}^{\,1} \doteq q_x \int_0^1 {}_tp_y \, dt \doteq \frac{d_x l_{y+\frac{1}{2}}}{l_x l_y} \tag{11.14} $$

where $l_{y+\frac{1}{2}}$ is taken as $\frac{1}{2}\ (l_y + l_{y+1})$. Then

$$ {}_nq_{xy}^{\,1} \doteq \frac{1}{l_x l_y} \sum_{t=0}^{n-1} d_{x+t} \cdot l_{y+t+\frac{1}{2}} \,. \tag{11.15} $$

A similar approximation can be used when more than two lives are involved:

$$ {}_nq_{xyz}^{\,1} \doteq \frac{1}{l_x l_y l_z} \sum_{t=0}^{n-1} d_{x+t} l_{y+t+\frac{1}{2}} l_{z+t+\frac{1}{2}} \,. \tag{11.16} $$

When either Gompertz's or Makeham's law applies, special formulas are available which make possible an exact evaluation. For instance, with Gompertz's law,

$$
\begin{aligned}
{}_nq_{x_1x_2\cdots x_m}^{\,1} &= \int_0^n {}_tp_{x_1x_2\cdots x_m} Bc^{x_1+t}\, dt \\
&= \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} \int_0^n {}_tp_{x_1x_2\cdots x_m} Bc^t(c^{x_1} + c^{x_2} + \cdots + c^{x_m})\, dt \\
&= \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} \cdot {}_nq_{x_1x_2\cdots x_m} \\
&= \frac{c^{x_1}}{c^w} \cdot {}_nq_w = \frac{\mu_{x_1}}{\mu_w} \cdot {}_nq_w \,,
\end{aligned} \tag{11.17}
$$

where w is the age of the equivalent single life. A special case of the above result is

$$ {}_\infty q_{x_1x_2\cdots x_m}^{\,1} = \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}} \,. \tag{11.18} $$

We may then write

$$ {}_nq_{x_1x_2\cdots x_m}^{\,1} = {}_\infty q_{x_1x_2\cdots x_m}^{\,1} \cdot {}_nq_{x_1x_2\cdots x_m} \,. \tag{11.19} $$

Thus, with Gompertz's law, the probability that (x_1) will die first within n years is equal to the probability that (x_1) will be the first to die multiplied by the probability that the joint lives will fail within n years.

A precise formula may also be obtained when Makeham's law applies. With this law, we have

$$
{}_nq^{\,1}_{x_1x_2\cdots x_m} = \int_0^n {}_tp_{x_1x_2\cdots x_m}(A + Bc^{x_1+t})\,dt
$$

$$
= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt
$$

$$
+ \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}}\int_0^n {}_tp_{x_1x_2\cdots x_m}Bc^t(c^{x_1} + c^{x_2} + \cdots + c^{x_m})\,dt
$$

$$
= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt + \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}}
$$

$$
\cdot \int_0^n {}_tp_{x_1x_2\cdots x_m}(\mu_{x_1+t} + \mu_{x_2+t} + \cdots + \mu_{x_m+t} - mA)\,dt
$$

$$
= A\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt
$$

$$
+ \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}}\left({}_nq_{x_1x_2\cdots x_m} - mA\cdot\int_0^n {}_tp_{x_1x_2\cdots x_m}\,dt\right)
$$

$$
= A\cdot\mathring{e}_{x_1x_2\cdots x_m:\overline{n|}} + \frac{c^{x_1}}{c^{x_1} + c^{x_2} + \cdots + c^{x_m}}
$$

$$
\cdot({}_nq_{x_1x_2\cdots x_m} - mA\cdot\mathring{e}_{x_1x_2\cdots x_m:\overline{n|}})
$$

$$
= A\mathring{e}_{ww\cdots w:\overline{n|}} + \frac{c^{x_1}}{mc^w}({}_nq_{ww\cdots w} - mA\cdot\mathring{e}_{ww\cdots w:\overline{n|}})
$$

$$
\text{where } mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}.
$$

Replacing A by $-\log s$ and rearranging terms,

$$
{}_nq^{\,1}_{x_1x_2\cdots x_m}
$$

$$
= \frac{c^{x_1}}{c^w}\left(\frac{1}{m}\cdot{}_nq_{ww\cdots w} + \log s\cdot\mathring{e}_{ww\cdots w:\overline{n|}}\right) - \log s\cdot\mathring{e}_{ww\cdots w:\overline{n|}}\,. \qquad (11.20)
$$

Then as a special case

$$_{\infty}q^{\,1}_{x_1x_2\cdots x_m} = \frac{c^{x_1}}{c^{w}}\left(\frac{1}{m} + \log s \cdot \mathring{e}_{ww\cdots w}\right) - \log s \cdot \mathring{e}_{ww\cdots w}. \quad (11.21)$$

No special techniques are required for the evaluation of probabilities contingent upon a second or subsequent death, since these can always be expressed in terms of probabilities contingent upon the first death.

4. Contingent insurance functions

Let $A^{\,1}_{xy}$ denote the net single premium for a contingent insurance of 1 payable at the end of the year of death of (x) provided that (x) dies before (y). Then

$$A^{\,1}_{xy} = \sum_{t=0}^{\infty} v^{t+1}\ {}_{t|}q^{\,1}_{xy}. \quad (11.22)$$

When the value of $A^{\,1}_{xy}$ is known, the complementary function $A_{x\overset{1}{y}}$ is easily obtained from the relation

$$A^{\,1}_{xy} + A_{x\overset{1}{y}} = A_{xy}.$$

For the contingent term insurance, we have

$$A^{\,1}_{xy:\overline{n}|} = \sum_{t=0}^{n-1} v^{t+1}\ {}_{t|}q^{\,1}_{xy}. \quad (11.23)$$

The payment is made at the end of the year of death of (x), provided that (x) dies before (y) and within n years—or, in other words, if the status (x) is the first to fail of the three statuses, (x), (y), and $(\overline{n}|)$. It may be noted that the case where (x) is the first to fail of the *two* statuses, (x) and $(\overline{n}|)$, is merely the familiar single-life function $A^{\,1}_{x:\overline{n}|}$.

When the insurance is payable at the moment of death, instead of at the end of the year of death, the formulas appear in definite integral form.

$$\bar{A}^{\,1}_{xy} = \int_0^{\infty} v^t\ {}_tp_{xy}\mu_{x+t}\, dt \quad (11.24)$$

$$\bar{A}^{\,1}_{xy:\overline{n}|} = \int_0^{n} v^t\ {}_tp_{xy}\mu_{x+t}\, dt \quad (11.25)$$

A few examples involving three lives follow:

$$\bar{A}^{\,1}_{xyz} = \int_0^\infty v^t\,{}_tp_{xyz}\mu_{z+t}\,dt \tag{11.26}$$

$$\bar{A}_{\overset{1}{\overline{xy}}:z} = \int_0^\infty v^t\,{}_tp_{xyz}\mu_{x+t:y+t}\,dt = \int_0^\infty v^t\,{}_tp_{xyz}(\mu_{x+t} + \mu_{y+t})\,dt$$

$$= \bar{A}^{\,1}_{xyz} + \bar{A}_{x\overset{1}{y}z} \tag{11.27}$$

$$\bar{A}_{\overset{1}{\overline{xy}}:z} = \int_0^\infty v^t\,{}_tp_z({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t})\,dt$$

$$= \bar{A}^{\,1}_{xz} + \bar{A}^{\,1}_{yz} - \bar{A}_{\overset{1}{\overline{xy}}:z}$$

$$= \bar{A}^{\,1}_{xz} + \bar{A}^{\,1}_{yz} - \bar{A}^{\,1}_{xyz} - \bar{A}_{x\overset{1}{y}z}\,. \tag{11.28}$$

Note that the contingent functions in this last result can all be made to depend upon (z) dying first:

$$\bar{A}_{\overline{xy}:\overset{1}{z}} = (\bar{A}_{xz} - \bar{A}_{x\overset{1}{z}}) + (\bar{A}_{yz} - \bar{A}_{y\overset{1}{z}}) - (\bar{A}_{xyz} - \bar{A}_{xy\overset{1}{z}}). \tag{11.29}$$

Contingent insurances can also be defined when the specified status is not the *first* to fail. For example, the symbol $A^{\,2}_{xy}$ denotes the net single premium for an insurance of 1 payable at the end of the year of death of (x) provided that (y) has died before (x):

$$A^{\,2}_{xy} = \sum_{t=0}^{\infty} v^{t+1}\,{}_{t|}q^{\,2}_{xy} = \sum_{t=0}^{\infty} v^{t+1}({}_{t|}q_x - {}_{t|}q^{\,1}_{xy}) = A_x - A^{\,1}_{xy}\,. \tag{11.30}$$

The result is evident from general reasoning, since $A^{\,1}_{xy} + A^{\,2}_{xy} = A_x$. Similarly,

$$\bar{A}^{\,2}_{xy} = \int_0^\infty v^t(1 - {}_tp_y)\,{}_tp_x\mu_{x+t}\,dt = \bar{A}_x - \bar{A}^{\,1}_{xy}\,. \tag{11.31}$$

The definite integral which appears in (11.31) can be transformed in an interesting way. We first note that the expression $1 - {}_tp_y$ can itself be written as the definite integral

${}_tq_y = \int_0^t {}_sp_y\mu_{y+s}\,ds$, and this leads to a double integral for $\bar{A}^{\,2}_{xy}$:

$$\bar{A}^{\,2}_{xy} = \int_0^\infty \int_0^t v^t\,{}_tp_x\mu_{x+t}\,{}_sp_y\mu_{y+s}\,ds\,dt.$$

By inverting the order of integration, we obtain the following equivalent expression:

$$\bar{A}^{2}_{xy} = \int_0^\infty \int_s^\infty v^t {}_tp_x\mu_{x+t}\, {}_sp_y\mu_{y+s}\, dt\, ds.$$

Performing the integration with respect to t, we have

$$\bar{A}^{2}_{xy} = \int_0^\infty {}_sp_y\mu_{y+s}\, {}_{s|}\bar{A}_x\, ds$$

$$= \int_0^\infty v^s {}_sp_{xy}\mu_{y+s}\, \bar{A}_{x+s}\, ds. \qquad (11.32)$$

In this form, the function appears as a varying insurance on the life of (y), contingent on the survival of (x). The amount of the varying insurance at any time is the net single premium for insurance of 1 on (x) at his then attained age.

It is not necessary in general to derive separate formulas for contingent functions of both the A and $\bar{A}$ type as we have done above for A^{2}_{xy} and $\bar{A}^{2}_{xy}$ in (11.30) and (11.31). The relations among the net single premiums are the same whether the insurance is payable at the end of the year of death or at the moment of death.

It will be found that the net single premium for any contingent insurance, when written as a definite integral, can readily be transformed to an expression which is solely in terms of single- and joint-life functions and contingent functions dependent on the *first* death. This is illustrated in the examples that follow:

$$\begin{aligned} \bar{A}^{2}_{xyz} &= \int_0^\infty v^t {}_tp_x \cdot {}_tp^{[1]}_{\overline{yz}}\mu_{x+t}\, dt \\ &= \bar{A}^{1}_{xy} + \bar{A}^{1}_{xz} - 2\bar{A}^{1}_{xyz} \end{aligned} \qquad (11.33)$$

$$\begin{aligned} \bar{A}^{\;2}_{\overline{xy}:z} &= \int_0^\infty v^t(1 - {}_tp_z)\, {}_tp_{xy}\mu_{x+t:y+t}\, dt \\ &= \bar{A}_{xy} - \bar{A}^{\;1}_{\overline{xy}:z} = \bar{A}_{xy} - \bar{A}^{1}_{xyz} - \bar{A}_{\overset{\;1}{xy}z} \end{aligned} \qquad (11.34)$$

$$\begin{aligned} \bar{A}^{3}_{xyz} &= \int_0^\infty v^t(1 - {}_tp_y)(1 - {}_tp_z)\, {}_tp_x\mu_{x+t}\, dt \\ &= \bar{A}_x - \bar{A}^{1}_{xy} - \bar{A}^{1}_{xz} + \bar{A}^{1}_{xyz}\,. \end{aligned} \qquad (11.35)$$

A special notation is used when these insurances are on a temporary basis, in that the term period is indicated by a prefixed subscript. For example, an insurance on the life of (x) payable if he dies after (y) and within n years is represented by ${}_nA^{2}_{xy}$.

It should be noted that the symbol $A^{\;2}_{xy:\overline{n}|}$ denotes an insurance payable upon the death of (x) provided that (x) is the second of the three statuses, (x), (y), $(\overline{n}|)$, to fail, and this benefit is different from the term insurance described above. It is thus important that symbols like ${}_nA^{\;2}_{xy}$ and $A^{\;2}_{xy:\overline{n}|}$ be correctly distinguished.

5. Evaluation of contingent insurances

When it is necessary to evaluate a large number of contingent insurance functions on the same mortality and interest basis, the construction of special commutation columns will simplify the calculations. These are usually based on the approximation afforded by the assumption of a uniform distribution of deaths,

$$ {}_{t|}q^{\;1}_{xy} \doteq \frac{d_{x+t}\cdot l_{y+t+\frac{1}{2}}}{l_x \cdot l_y}, $$

so that

$$ \begin{aligned} A^{\;1}_{xy} &= \sum_{t=0}^{\infty} v^{t+1}\,{}_{t|}q^{\;1}_{xy} \doteq \sum_{t=0}^{\infty} \frac{v^{\frac{x+y}{2}+t+1} d_{x+t}\cdot l_{y+t+\frac{1}{2}}}{v^{\frac{x+y}{2}} l_x \cdot l_y} \\ &= \frac{\sum_{t=0}^{\infty} C_{\overset{1}{x+t}:y+t}}{D_{xy}} = \frac{M^{\;1}_{xy}}{D_{xy}}, \end{aligned} $$

where $$ C^{\;1}_{xy} = v^{\frac{x+y}{2}+1} k d_x l_{y+\frac{1}{2}} \qquad \text{with } k = \frac{l_{xy}}{l_x \cdot l_y} \tag{11.36} $$

and $$ M^{\;1}_{xy} = \sum_{t=0}^{\infty} C_{\overset{1}{x+t}:y+t}. \tag{11.37} $$

For the contingent term insurance, we have

$$ A^{\;1}_{xy:\overline{n}|} = \frac{M^{\;1}_{xy} - M_{\overset{1}{x+n}:y+n}}{D_{xy}}. $$

The exact definition of $C^{\;1}_{xy}$ is

$$ C^{\;1}_{xy} = v^{\frac{x+y}{2}+1} \int_0^1 l_{x+t:y+t}\mu_{x+t}\, dt \tag{11.38} $$

the form given by (11.36) being merely a convenient approximation to the true value.

These commutation functions apply to insurances payable at the end of the year of death. When the insurance is payable at the moment of death, the payment is made on the average one-half year earlier, and one of the usual approximate relationships may be assumed, for example:

$$\bar{A} \doteq (1+i)^{\frac{1}{2}} A.$$

In most instances in practice, only a few isolated contingent insurance values are required, and the simplest method of evaluation is approximate integration. This device is particularly convenient when the mortality assumption must be changed at some point in the contract. Suppose, for example, that $\bar{A}^{\;2}_{xy}$ is to be evaluated on the basis of mortality from Table A while both lives survive and from Table B after the death of (y). This is easily handled by first using (11.32) to write

$$\bar{A}^{\;2}_{xy} = \frac{1}{l_x l_y} \int_0^\infty v^t l_{x+t} l_{y+t} \mu_{y+t} \bar{A}_{x+t}\, dt.$$

Then, after an appropriate formula of approximate integration has been chosen, the required values of l_{x+t}, l_{y+t}, and μ_{y+t} are taken from Table A and those of $\bar{A}_{x+t}$ from Table B.

As in the case of contingent probabilities, an exact evaluation is possible when either Gompertz's or Makeham's law applies. With Gompertz's law, we have

$$\begin{aligned} \bar{A}^{\;1}_{x_1x_2\cdots x_m} &= {}_\infty q^{\;1}_{x_1x_2\cdots x_m} \cdot \bar{A}_{x_1x_2\cdots x_m} \\ &= \frac{c^{x_1}}{c^w} \bar{A}_w, \text{ where } c^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \end{aligned} \tag{11.39}$$

In this case, the value of the contingent insurance is the same as the value of the joint-life insurance multiplied by the probability that (x) will die first.

With Makeham's law, we have

$$\bar{A}^{\;1}_{x_1x_2\cdots x_m} = \frac{c^{x_1}}{c^w}\left(\frac{1}{m}\bar{A}_{ww\cdots w} + \log s \cdot \bar{a}_{ww\cdots w}\right) - \log s \cdot \bar{a}_{ww\cdots w},$$

$$\text{where } mc^w = c^{x_1} + c^{x_2} + \cdots + c^{x_m}. \tag{11.40}$$

The derivations of these formulas are not given here, since they parallel closely those of the corresponding contingent probabilities which were set forth in detail in Section 3.

6. Annual premiums and reserves

A contract of contingent insurance may terminate in either of two ways: (1) by the occurrence of the event insured against, in which case a claim is incurred; or (2) by the occurrence of a death contrary to the order of deaths insured against, in which case the contract expires. When contracts of this type are subject to annual premiums, both of the above possibilities must be considered in determining the premium-paying period, since premiums should not be made payable beyond the period during which the insurance is in force.

In the case of an insurance contingent upon a specified status being the *first* to fail, the contract will either become a claim or expire upon the occurrence of the first failure, and the annual premiums are accordingly made payable during the joint existence of all the statuses. This is illustrated by

$$P^{1}_{xy} = \frac{A^{1}_{xy}}{\ddot{a}_{xy}}$$

and

$$P^{1}_{x:\overline{yz}} = \frac{A^{1}_{x:\overline{yz}}}{\ddot{a}_{x:\overline{yz}}}.$$

When an insurance is contingent upon a specified status being the *last* to fail, the insurance continues in force until the failure of the specified status, and the annual premiums may be made payable during the existence of the failing status alone. In the general case involving m statuses where the insurance is contingent upon a specified status being the n-th to fail, the annual premiums may be made payable during the joint existence of the failing status with at least m-n of the other statuses. In practice, however, there are often underwriting considerations which require that premium payments be limited to a period shorter than the maximum indicated above, and consequently it is not possible to give unique definitions for functions like P^{2}_{xyz}.

Terminal reserves for contingent insurances are computed by the prospective method, and the amount of the reserve at any time depends upon the then surviving combination of lives. As an illustration, consider the terminal reserve at the end of t years for an insurance on (x) if he dies second of the three lives (x), (y), and (z), with annual premiums payable during the joint lifetime of all three. If the lives are all surviving at the end of t years, the reserve is

$$A_{\overset{2}{x+t}:y+t:z+t} - P^{2}_{xyz} \cdot \ddot{a}_{x+t:y+t:z+t} \,.$$

If (y) has died, the reserve is $A_{\overset{1}{x+t}:z+t}$, and if (z) has died, the reserve is $A_{\overset{1}{x+t}:y+t}$.

7. Some special contingent functions

The probability that (x) will be alive n years after the death of (y) is

$$\int_0^\infty {}_{t+n}p_x \cdot {}_tp_y \mu_{y+t}\, dt = {}_np_x \int_0^\infty {}_tp_{x+n:y}\mu_{y+t}\, dt = {}_np_x \cdot {}_\infty q_{x+n:\overset{1}{y}} . \quad (11.41)$$

Similarly the probability that (y) will be alive n years after the death of (x) is ${}_np_y \cdot {}_\infty q_{\overset{1}{x}:y+n}$, and the probability that at least n years will elapse between the two deaths is therefore

$${}_np_x \cdot {}_\infty q_{x+n:\overset{1}{y}} + {}_np_y \cdot {}_\infty q_{\overset{1}{x}:y+n} \,. \quad (11.42)$$

It follows that the probability that (x) and (y) will die within n years of each other is

$$1 - ({}_np_x \cdot {}_\infty q_{x+n:\overset{1}{y}} + {}_np_y \cdot {}_\infty q_{\overset{1}{x}:y+n}) . \quad (11.43)$$

The probability that (x) will die before (y) or within n years after the death of (y) is the complement of the probability that (x) will be alive n years after the death of (y), or

$$1 - {}_np_x \cdot {}_\infty q_{x+n:\overset{1}{y}} \,. \quad (11.44)$$

The probability that (x) will die during the n years following the death of (y) is

$$\int_0^\infty ({}_tp_x - {}_{t+n}p_x)\, {}_tp_y \mu_{y+t}\, dt = {}_\infty q_{x\overset{1}{y}} - {}_np_x \cdot {}_\infty q_{x+n:\overset{1}{y}} , \quad (11.45)$$

which is evidently correct since the required probability is the probability that (x) will survive (y) but not be alive n years after the death of (y).

We may now consider the values of insurances which depend on contingent probabilities of the above type.

The net single premium for a pure endowment of 1 payable if (x) is alive n years after the death of (y) is

$$\int_0^\infty v^{t+n}{}_{t+n}p_x\ {}_tp_y\mu_{y+t}\,dt = \frac{D_{x+n}}{D_x}\bar{A}_{x+n:\overset{1}{y}}. \tag{11.46}$$

The annual premium is obtained by dividing the single premium by the present value of an annuity-due payable until the death of (x) or until n years after the death of (y), whichever period is shorter. The present value is

$$\ddot{a}_{x:\overline{n}|} + \sum_{t=0}^{\infty} v^{n+t}{}_{n+t}p_x\ {}_tp_y = \ddot{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\sum_{t=0}^{\infty} v^t\ {}_tp_{x+n:y}$$

$$= \ddot{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\ddot{a}_{x+n:y}. \tag{11.47}$$

In this expression, the first term represents the present value of the first n payments, which are independent of the survival of (y), and the second term takes account of the payments after n years, which are made only if (y) was alive n years before.

The present value of an insurance payable upon the death of (x) provided it occurs at least n years after the death of (y) is

$$\int_0^\infty v^{n+t}(1 - {}_tp_y)_{n+t}p_x\cdot\mu_{x+n+t}\,dt = \frac{D_{x+n}}{D_x}(\bar{A}_{x+n} - \bar{A}_{\overset{1}{x+n}:y}). \tag{11.48}$$

The annual premium is determined by dividing by $\ddot{a}_x$.

The present value of an insurance payable if (x) dies before (y) or within n years after (y) is equal to $\bar{A}_x$ minus the value of an insurance payable if (x) dies at least n years after (y):

$$\bar{A}_x - \frac{D_{x+n}}{D_x}(\bar{A}_{x+n} - \bar{A}_{\overset{1}{x+n}:y}). \tag{11.49a}$$

Alternatively, this value may be expressed as

$$\bar{A}^1_{x:\overline{n}|} + \int_0^\infty v^{n+t}{}_{n+t}p_x\cdot{}_tp_y\cdot\mu_{x+n+t}\,dt$$

since the insurance must be payable if (x) dies during the first n years, and is payable after that time if (y) is alive n years before the death of (x). This alternative expression reduces to

$$\bar{A}^1_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\bar{A}_{\overset{1}{x+n}:y}, \tag{11.49b}$$

which may be seen to be equivalent to the formula first obtained.

The annual premium is determined by dividing by

$$\ddot{a}_{x:\overline{n}|} + \frac{D_{x+n}}{D_x}\,\ddot{a}_{x+n:y}\,.$$

8. Summary of notation

1. The notational principle used with simple contingent functions is an extension of Principle I (Chapter 3):

When a suffixed subscript consists of several statuses, a numeral, say r, written above one of the statuses indicates that the function is determined by that status being the r-th to fail. For example, the suffix $x\overset{2}{y}z$ refers to the contingency that (y) will be the second to die of the three lives; the suffix $\overline{xy}:\overset{2}{z}w$ to the contingency that (z) will be the second to fail of the three statuses $(\overline{xy})$, (z), (w).

2. With contingent term insurances, the temporary period is denoted by a prefixed subscript; e.g., ${}_nA^{\;2}_{xy}$. (When the insurance is contingent upon the first death, however, either ${}_nA^{1}_{xy}$ or $A^{\;1}_{xy:\overline{n}|}$ is correct.) This device is sometimes used as an alternative notation under other circumstances. For example, the symbol $a_{x:\overline{n}|}$ is sometimes written ${}_na_x$, and $A^{1}_{x:\overline{n}|}$ is sometimes written ${}_nA_x$.

3. Contingent commutation functions:

$$C^{1}_{x_1x_2\cdots x_m} = v^{\frac{x_1+x_2+\cdots+x_m}{m}+1}\int_0^1 l_{x_1+t:x_2+t:\cdots:x_m+t}\,\mu_{x_1+t}\,dt$$

$$\bar{C}^{1}_{x_1x_2\cdots x_m} = \int_0^1 D_{x_1+t:x_2+t:\cdots:x_m+t}\,\mu_{x_1+t}\,dt$$

The functions M, $\bar{M}$, R, $\bar{R}$ are obtained by summation in the usual way.

References

1, 2, 3, 4. This material is also covered by Spurgeon (1932) and by Hooker and Longley-Cook (1957).

5. The method of formula (11.40) is due to Hume and Stott (1905). Another method of evaluating $\bar{A}^{1}_{x_1x_2\cdots x_m}$ under Makeham's law was given by Evans (1925) and may be seen in Hooker and Longley-Cook (1957), page 147. A paper by Groeschell and Snell (1951) shows how Evans' method is applied to the valuation of reversionary interests for taxation purposes. The numerical illustrations in this paper, comparing the results of Evans' method with others described in the present chapter, will be of interest to students.

Spurgeon (1932) and Hooker and Longley-Cook (1957) describe several other approximate methods of evaluation. Some of these form the bases for Exercises 5, 6, 19, 20, and 21.

6. The family-type policy currently issued by many insurance companies affords an interesting application of contingent insurance principles. A paper by Sarnoff (1958) discusses this policy, with special reference to the problem of valuation.

EXERCISES

1, 2. Contingent probabilities

1. Show that ${}_\infty q^{\;2}_{\overline{xx}:x} = 2/3$.

2. Given the three lives A, B, C, all aged x. Write an expression in terms of ${}_np_x$ for the probability that A and B will die within n years, A predeceasing B, and that C will survive for n years.

3. Given ${}_\infty q_{x\overset{2}{y}} = .4$, ${}_\infty q_{x\overset{2}{z}} = .5$, and ${}_\infty q_{\overset{2}{x}yz} = .3$, find the numerical value of ${}_\infty q_{\overset{3}{x}yz}$. (Ans. .3)

4. Express ${}_\infty q_{w\overset{2}{x}yz}$ in terms of probabilities contingent upon the first death.

3. Evaluation of contingent probabilities

5. Using the fact that $\dfrac{\partial \mathring{e}_{xy}}{\partial x} = \mu_x \mathring{e}_{xy} - {}_\infty q^{\,1}_{xy}$ (see Appendix II), derive the approximate formula

$$ {}_\infty q^{\,1}_{xy} \doteq \mu_x \mathring{e}_{xy} + \tfrac{1}{2}(\mathring{e}_{x-1:y} - \mathring{e}_{x+1:y}). $$

6. (a) Assuming that $\mu_{x+t} \doteq \dfrac{1}{2l_{x+t}}(l_{x+t-1} - l_{x+t+1})$ for all values of t, show that

$$ {}_\infty q^{\,1}_{xy} \doteq \tfrac{1}{2}\left\{\frac{\mathring{e}_{x-1:y}}{p_{x-1}} - p_x \cdot \mathring{e}_{x+1:y}\right\}. $$

(b) Determine whether (x) is older than (y), given

$$ \begin{aligned} \mathring{e}_{x-1:y} &= 10.682 & l_{x-1:y} &= 10{,}250{,}000 \\ \mathring{e}_{x+1:y} &= 10.000 & l_{x:y} &= 10{,}000{,}000 \\ & & l_{x+1:y} &= 9{,}740{,}000. \end{aligned} $$

7. In a mortality table known to follow Makeham's law, you are given:

$$ A = .003,\ c^{10} = 3, \text{ and } \mathring{e}_{40:50} = 17. $$

Find ${}_\infty q^{\,1}_{40:50}$. (Ans. .2755)

8. Assuming that Makeham's law holds and given the following data find as closely as possible the value of ${}_{\infty}q^{\;\;1}_{\overline{40:40:40}:42}$.

$$\log_{10} 4.0 = .60206$$
$$\log_{10} 4.1 = .61278 \qquad c = 1.1$$
$$\log_{10} 4.2 = .62325 \qquad \log_e s = -.00619$$
$$\log_{10} 4.3 = .63347 \qquad \mathring{e}_{40:40:40:40} = 13.880$$
$$\log_{10} 4.4 = .64345 \qquad \mathring{e}_{41:41:41:41} = 13.429$$

(Ans. .725)

4. Contingent insurance functions

9. Show that the net single premium for an insurance of 1 payable at the end of the year of death of (x) provided that (y) is then alive may be expressed as

$$vp_y\ddot{a}_{x:y+1} - a_{xy}\,.$$

10. Show by general reasoning that

$$A^{1}_{xy} - A^{2}_{xy} > 0.$$

Is this true if $i = 0$? Explain.

11. Show that

$$A^{\;2}_{xy} = A_y - A_{xy} + A^{1}_{xy}\,.$$

12. Show that

$$\text{(a)} \int_0^{\infty} v^t{}_tp_{xy}\mu_{y+t}\;{}_nE_{x+t}\,dt = \frac{D_{x+n}}{D_x}\,\bar{A}_{x+n:\overset{1}{y}}$$

$$\text{(b)} \int_0^{\infty} v^t{}_tp_{xy}\mu_{y+t}\cdot{}_{n|}\bar{A}_{x+t}\,dt = \frac{D_{x+n}}{D_x}\left(\bar{A}_{x+n} - \bar{A}^{\;\;1}_{\overline{x+n}:y}\right)$$

In each case, describe the benefit.

13. Express $\bar{A}^{\;3}_{wxyz}$ in terms of insurances contingent on the first death.

5. Evaluation of contingent insurances

14. Describe the benefit denoted by the symbol $A^{\;\;1}_{x:\overline{y:\overline{n}|}}$ and give a formula for it in commutation symbols.

15. Given $\mu_x = 0.002 + 10^{-5}\,c^x$ and $c^{10} = 3$, express $\bar{A}^{1}_{40:50}$ in terms of $\bar{A}_{40:50}$ and $\bar{a}_{40:50}$.

16. Give the full derivation of formulas (11.39) and (11.40).

17. Show that, if a mortality table follows Makeham's law,

$$\bar{A}^1_{zy} = \frac{1}{1+c^{y-z}}\bar{A}_{zy} + \frac{1-c^{y-z}}{1+c^{y-z}}\log s\cdot\bar{a}_{zy}.$$

18. Given the following data from a table known to follow Makeham's law, find the value of $\bar{A}^{\,1}_{30:40:60}$.

$$\mu_{30} = .0106$$

$$\mu_{40} = .0136$$

$$\mu_{60} = .0316$$

x	c^x	$\bar{a}_{xxx}$	$\bar{A}_{xxx}$
49	80.6	9.0	.73
50	88.2	8.7	.74

(Ans. .099)

19. Show that $\dfrac{\partial \bar{a}_{xy}}{\partial x} = \mu_x\bar{a}_{xy} - \bar{A}^1_{xy}$. Hence derive the approximate formula

$$\bar{A}^1_{xy} \doteq \mu_x\bar{a}_{xy} + \tfrac{1}{2}(\bar{a}_{x-1:y} - \bar{a}_{x+1:y}).$$

20. Assuming that $\mu_{x+t} \doteq \dfrac{l_{x+t-1} - l_{x+t+1}}{2l_{x+t}}$

for all values of t, derive the formula

$$\bar{A}^1_{xy} \doteq \tfrac{1}{2}\left\{\frac{\bar{a}_{x-1:y}}{p_{x-1}} - p_x\cdot\bar{a}_{x+1:y}\right\}.$$

What is the corresponding formula for $\bar{A}^1_{x:\overline{yz}}$?

21. Assuming that $\mu_{x+t} \doteq q_{x+t}$ for all values of t, derive the formula

$$\bar{A}^1_{xy} \doteq \bar{a}_{xy} - p_x\cdot\bar{a}_{x+1:y}.$$

6. Annual premiums and reserves

22. Give formulas for ${}_tV^2_{xyz}$, assuming that the annual premium is payable during the joint lifetime of (x) and the survivor of (y) and (z).

23. A policy insures a husband aged 40 and his wife aged 35. If (35) dies first, the payment at her death is \$1000. If (40) dies first, the payment at his death is \$10,000 during the first policy year; thereafter it decreases by \$100 at the beginning of each policy year until it reaches \$8000 and then remains constant. Give a formula for the net annual premium in terms of commutation functions.

7. Some special contingent functions

24. In the case of 3 lives aged 70, 75, and 79, the probability that exactly 2 lives survive 5 years is .45 and that at least one dies within 5 years is

.775. Given that $\frac{1}{{}_5p_{75}}$, $\frac{1}{{}_5p_{75}}$, and $\frac{1}{{}_5p_{70}}$ are in arithmetic progression and that ${}_\infty q^{1}_{70:80} = .3$, find the probability that 2 lives aged 70 and 75 will die within 5 years of each other. (Ans. .445)

25. Show that the net annual premium for a pure endowment of 1 payable if (x) is alive n years after the death of (y) is given by

$$\frac{\bar{A}_{x+n:\overset{1}{y}}}{\ddot{s}_{x:\overline{n}|} + \ddot{a}_{x+n:y}}.$$

26. A deferred annuity of 1 is payable annually, beginning m years from now. Annuity payments are made if at the time of payment (x) is alive and (y) either is alive or has been dead less than n years $(m > n)$. What is the present value of this annuity?

27. Find the present value of 1 payable at the moment of death of (50) provided that (20) has then either died or attained age 40.

28. Derive and simplify an expression, in terms of single premium symbols, for an insurance payable on the death of (x) after (y), provided (y) dies during the n years preceding the death of (x).

Miscellaneous problems

29. Find the single premium with a loading of 7½ per cent of the net premium for a contingent survivorship policy payable if a life aged 30 dies before a life aged 60, or within 5 years after the death of the latter, with return of the single premium without interest 5 years after the death of the life aged 60 if no claim under the policy arises by the death of the life aged 30.

30. Find the value of $\int_0^\infty {}_tq_{xz}^{\;2} \; {}_tp_{xz} \; \mu_{x+t} \, dt.$

31. An insurance is to be paid at the moment of death of (x) if he survives (y). The payment is equal to the number of years (including fractional parts of a year) elapsed from the date of issue to the moment of death of (y). Write the net single premium as a definite integral.

32. Develop a formula in terms of an expression which does not involve integrals for the net single premium for an insurance of 1 payable 10 years after the death of (x), if both or either of (y) and (z) survive (x) and both are dead before the end of the 10-year period.

33. A corporation has a president, aged 50, and two vice-presidents aged 48 and 45 respectively. They wish to purchase the following insurance: If the president dies first, \$1000 is to be paid immediately and \$2000 is to be paid upon the first death of the two vice-presidents. If one of the vice-presidents dies first, nothing is to be paid until the second death among the three officers at which time \$3000 is to be paid.

Compute the net single premium for this insurance, assuming immediate payment of claims, given the following values at $2\frac{1}{2}\%$ from the mortality table to be employed:

$$\ddot{a}_{50:48} = 10.95 \qquad \ddot{a}_{48:45} = 11.46$$

$$\ddot{a}_{50:45} = 11.26 \qquad \ddot{a}_{50:48:45} = 9.30$$

(Ans. \$1974.38)

Chapter 12

COMPOUND CONTINGENT FUNCTIONS

1. Introduction

The contingent functions considered in the preceding chapter were of a type in which the sequence of deaths was restricted by a condition on a single one of the lives (or statuses) involved. Thus a typical problem arose from the condition that (y) die second of the lives (x), (y), and (z), and nothing was said about the multiple condition exemplified by, say, the requirement that (y) die second predeceased by (x) among (x), (y), and (z). The present chapter deals with functions of this latter type, known as *compound contingent functions.*

It is obvious that no compound contingent function can arise from a two-life status, since an ordering restriction on one life will automatically determine a complementary condition on the other. A compound contingent function thus requires three or more lives with two or more conditions on the sequence of deaths.

2. Probabilities

Given the three lives (x), (y), and (z), what is the probability that their deaths will occur in the order named? This probability is denoted by ${}_\infty q_{\underset{1}{x}\overset{2}{y}z}$, and it may be expressed in several different ways:

$$ {}_\infty q_{\underset{1}{x}\overset{2}{y}z} = \int_0^\infty {}_tp_{xyz}\mu_{x+t}\cdot{}_\infty q_{\overset{1}{\overline{y+t}}:z+t}\,dt \qquad (12.1a)$$

or

$$\int_0^\infty {}_tq_x\cdot{}_tp_{yz}\mu_{y+t}\,dt \qquad (12.1b)$$

or

$$\int_0^\infty {}_tq_{x\overset{2}{y}}\cdot{}_tp_z\mu_{z+t}\,dt. \qquad (12.1c)$$

These three expressions are derived by letting t refer to the moment of death of (x), of (y), and of (z) respectively. The integrand in the first case expresses the condition that at the moment of death of (x), the lives (y) and (z) are both surviving and that (y) will later predecease (z). The second integral reflects the fact that when (y)

dies, (x) is already dead and (z) is surviving. The third expression is based on the fact that (x) and (y) have already died in the order named when the death of (z) occurs.

The three integrals may easily be shown to be equivalent mathematically. If, in (12.1b), the substitution of $\int_0^t {}_sp_x\mu_{x+s}\,ds$ is made for ${}_tq_x$, a double integral results. By inverting the order of integration and simplifying, formula (12.1a) is obtained. If, on the other hand, the substitution of $\int_t^\infty {}_sp_z\mu_{z+s}\,ds$ is made for ${}_tp_z$ in formula (12.1b), then when the order of integration is inverted formula (12.1c) will be obtained.

All three expressions will normally require approximate integration methods, and formulas (12.1a) and (12.1c) are less convenient to evaluate because of the presence of contingent functions under the integral sign.

By using (12.1b), it is possible to express ${}_\infty q_{\underset{1}{x}\overset{2}{y}z}$ in terms of simple contingent probabilities. Writing $1 - {}_tp_x$ for ${}_tq_x$, we have

$$\begin{aligned} {}_\infty q_{\underset{1}{x}\overset{2}{y}z} &= \int_0^\infty (1 - {}_tp_x)\,{}_tp_{yz}\mu_{y+t}\,dt \\ &= \int_0^\infty {}_tp_{yz}\mu_{y+t}\,dt - \int_0^\infty {}_tp_{xyz}\mu_{y+t}\,dt \\ &= {}_\infty q_{\overset{1}{y}z} - {}_\infty q_{x\overset{1}{y}z}\,. \end{aligned} \tag{12.2}$$

In the symbols used for compound contingent functions, it is important to note the significance of the numerals which appear above and below the suffix. The numeral above the suffix distinguishes the life upon whose failure the function is finally determined, and numerals below the suffix indicate the order in which the other lives are to fail. In the case of ${}_\infty q_{\underset{1}{x}\overset{2}{y}z}$, for example, the probability is determined upon (y)'s dying second, predeceased by (x).

When more than three lives are involved, the functions become more complex and it is not always possible to set up integrals which are themselves free of contingent functions. For example, to take a four-life case, the probability that (w), (x), (y), and (z) will die in the order named is

$$ {}_{\infty}q_{\underset{1\;2}{wxy}\overset{3}{z}} = \int_0^\infty {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t} \cdot {}_{\infty}q_{\overline{y+t}:\overset{1}{z+t}}\, dt \qquad (12.2a) $$

or

$$ \int_0^\infty {}_tq_{w\overset{2}{x}} \cdot {}_tp_{yz}\mu_{y+t}\, dt. \qquad (12.2b) $$

Other equivalent expressions can be formulated by basing the integrals on the moment of death of (w) or of (z). It will be found, however, that the formulas are simpler when the variable of integration t refers to the moment of death of one of the intermediate lives, as (x) in (12.2a) or (y) in (12.2b).

3. Insurances

A single order of deaths among a group of lives may give rise to several different insurance functions. If there are three lives which are to die in the order (x), (y), (z), there are three insurance functions corresponding to payment of the benefit upon the death of (x), of (y), or of (z). Actually, the first case presents nothing new since if the insurance is payable upon (x)'s dying first, the order of the subsequent deaths is immaterial. There remain the two cases where the insurance is payable (1) upon the death of (y) predeceased by (x) and survived by (z), and (2) upon the death of (z) predeceased by (x) and (y) in that order.

For case (1), where the payment is made upon the death of (y), the net single premium for an insurance of 1 is given by

$$ \begin{aligned} \bar{A}_{\underset{1}{x}\overset{2}{y}z} &= \int_0^\infty v^t \cdot {}_tq_x \cdot {}_tp_{yz}\mu_{y+t}\, dt \\ &= \int_0^\infty v^t(1 - {}_tp_x)\,{}_tp_{yz}\mu_{y+t}\, dt \\ &= \bar{A}_{\overset{1}{y}z} - \bar{A}_{x\overset{1}{y}z}\,. \end{aligned} \qquad (12.3) $$

For case (2), where the payment is made upon the death of (z), we have

$$ \bar{A}_{\underset{1}{x}y\overset{3}{z}} = \int_0^\infty v^t \cdot {}_tq_x \cdot {}_tp_{yz}\mu_{y+t}\bar{A}_{z+t}\, dt. \qquad (12.4) $$

Analogous functions may be written for four lives. The student should describe the insurance represented by each of the following:

$$\bar{A}_{\underset{1}{w}\overset{2}{x}yz} = \int_0^\infty v^t\, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\, dt \tag{12.5}$$

$$\bar{A}_{\underset{12}{wx}\overset{3}{y}z} = \int_0^\infty v^t\, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}_{\overset{1}{\overline{y+t}}:z+t}\, dt \tag{12.6}$$

$$\bar{A}_{\underset{12}{wx}y\overset{4}{z}} = \int_0^\infty v^t\, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}_{y+t:\overset{2}{\overline{z+t}}}\, dt \tag{12.7}$$

$$\bar{A}_{\underset{12}{wx}\overset{3:4}{y\ z}} = \int_0^\infty v^t\, {}_tq_w \cdot {}_tp_{xyz}\mu_{x+t}\bar{A}_{y+t}\, dt \tag{12.8}$$

Each of the above may also be written as a double integral. For example,

$$\bar{A}_{\underset{12}{wx}\overset{3}{y}z} = \int_0^\infty {}_tq_w \cdot {}_tp_x\mu_{x+t}\left(\int_t^\infty v^s\, {}_sp_{yz}\mu_{y+s}\, ds\right) dt.$$

In more complex cases, multiple integration is sometimes the only means of writing a precise expression for a given function.

4. Compound contingencies with terms-certain

A compound contingent function may involve conditions other than the mere order of deaths, as for example where a condition as to the time interval between deaths is introduced. No new theory is needed for the formulation of such functions, and in fact they are always best handled from first principles without any attempt at classification. Since these functions are of such varying types, it is not feasible to assign specific symbols for them. The examples which follow will illustrate the techniques used in formulating expressions for such functions.

Consider the probability that (x), (y), and (z) will die in the order named within twenty years, at least five years separating each death. These conditions require the death of (y) within the period from five to fifteen years from the present, and the probability may be expressed as

$$\int_5^{15} (1 - {}_{t-5}p_x)\, {}_tp_y\mu_{y+t}({}_{t+5}p_z - {}_{20}p_z)\, dt.$$

This may be reduced to a form involving simple contingent probabilities. Making the change of variable $s = t - 5$, we have

$$\int_0^{10} (1 - {}_sp_x)_{s+5}p_y\mu_{y+s+5}({}_{s+10}p_z - {}_{20}p_z)\, ds$$

$$= {}_5p_y \cdot {}_{10}p_z \int_0^{10} (1 - {}_sp_x)_sp_{y+5}\mu_{y+5+s} \cdot {}_sp_{z+10}\, ds$$

$$- {}_5p_y \cdot {}_{20}p_z \int_0^{10} (1 - {}_sp_x)_sp_{y+5}\mu_{y+5+s}\, ds$$

$$= {}_5p_y \cdot {}_{10}p_z({}_{10}q^{\;1}_{\overline{y+5}:z+10} - {}_{10}q_{x:\overset{1}{\overline{y+5}}:z+10}) - {}_5p_y \cdot {}_{20}p_z({}_{10}q_{y+5} - {}_{10}q_{x:\overset{1}{\overline{y+5}}}).$$

Consider next the probability that (x), (y), and (z) will die in the order named, (y) within five years of (x), and (z) within ten years of (y). This probability may be expressed as

$$\int_0^5 (1 - {}_tp_x)_tp_y\mu_{y+t}({}_tp_z - {}_{t+10}p_z)\, dt$$

$$+ \int_5^\infty ({}_{t-5}p_x - {}_tp_x)_tp_y\mu_{y+t}({}_tp_z - {}_{t+10}p_z)\, dt,$$

since, if (y) dies within the first five years, he must be predeceased by (x), and if (y) dies after five years, the prior death of (x) must have occurred within five years, the death of (z) occurring within ten years after that of (y) in either case. The student may verify that the integrals reduce to the following expression involving simple contingent functions:

$${}_5q^{\;1}_{yz} - {}_\infty q_{x\overset{1}{y}z} - {}_{10}p_z({}_5q^{\;1}_{y:z+10} - {}_\infty q_{x\overset{1}{y}:z+10})$$

$$+ {}_5p_{yz}({}_\infty q_{x:\overset{1}{\overline{y+5}}:z+5} - {}_{10}p_{z+5} \cdot {}_\infty q_{x:\overset{1}{\overline{y+5}}:z+15}).$$

Suppose that we require the probability that the three lives (10), (20), (30) will all die between ages 40 and 80 and in the order named, the first and second deaths occurring within five years of each other, with the second and third deaths separated by at least five years. It may be noted that, in order to fulfill these conditions, the life (20) must die within the period from thirty to forty-five years from the present, and that two separate integrals must be written to take account of (10)'s predeceasing (20) within five years. We thus have

$$\int_{20}^{35} ({}_{30}p_{10} - {}_tp_{10})_tp_{20}\mu_{20+t}({}_{t+5}p_{30} - {}_{50}p_{30})\, dt$$

$$+ \int_{35}^{45} ({}_{t-5}p_{10} - {}_tp_{10})_tp_{20}\mu_{20+t}({}_{t+5}p_{30} - {}_{50}p_{30})\, dt.$$

As an example involving four lives, consider the probability that (w), (x), (y), and (z) will die in the order named, the first two deaths occurring within the first five years with not more than two years separating each of the last three deaths. This probability requires a double integral:

$$\int_0^5 (1 - {}_tp_w)\,{}_tp_x\mu_{x+t}\left(\int_t^{t+2} {}_sp_y\mu_{y+s}({}_sp_z - {}_{s+2}p_z)\,ds\right)dt.$$

Insurances which involve contingencies of this nature may be similarly formulated. Suppose that an insurance of 1 is payable at the death of (x) if he dies before (y) or within n years after (y), leaving a third life (z) surviving. In expressing the net single premium, we note that during the first n years the insurance is independent of the survival of (y), and write

$$\int_0^n v^t\,{}_tp_{xz}\mu_{x+t}\,dt + v^n\,{}_np_{xz}\int_0^\infty v^t\,{}_tp_{x+n:y:z+n}\mu_{x+n+t}\,dt$$

$$= \bar{A}^{\,1}_{xz:\overline{n}|} + \frac{D_{x+n:z+n}}{D_{xz}}\cdot\bar{A}_{\overset{1}{x+n}:y:z+n}\,.$$

As a second example, consider an insurance payable at the death of (x) if he is predeceased by (y) and (z) in the order named with the death of (z) occurring at least n years earlier. This may be regarded as a contingent insurance on the life of (z) in which the amount of insurance at any time is equal to the present value at that time of an n-year deferred insurance on (x). The value may thus be written as

$$\int_0^\infty v^t\,{}_tq_y\,{}_tp_{xz}\mu_{z+t}\cdot{}_{n|}\bar{A}_{x+t}\,dt$$

$$= \frac{D_{x+n}}{D_x}\int_0^\infty v^t\,{}_tq_y\,{}_tp_{x+n:z}\mu_{z+t}\cdot\bar{A}_{x+n+t}\,dt = \frac{D_{x+n}}{D_x}\cdot\bar{A}_{\overset{3}{x+n}:y:\underset{1}{z}}\,.$$

5. Summary of notation

Numerals are used with a suffixed subscript to indicate the order of failure of the statuses involved. The numeral written above the suffix distinguishes the status upon whose failure the event is finally determined, and the numerals written below the suffix indicate the order in which the other statuses are to fail. In cases where a numeral below the suffix is clearly implied, it is unnecessary to write it, as, for example, the numeral 2 below the y in $\bar{A}_{\underset{1}{x}y\overset{3}{z}}$, or the numeral 3 below the y in $\bar{A}_{\overset{2}{\overline{wx}}:\underset{1}{y}z}$.

References

This material is treated by Spurgeon (1932), Chapter 19, and more briefly by Hooker and Longley-Cook (1957), Chapter 24.

EXERCISES

1, 2. Probabilities

1. Express ${}_{\infty}q_{\underset{1}{w}\overset{2}{x}yz}$

   (a) as a definite integral;
   (b) in terms of simple contingent probabilities.

2. If the mortality table follows Gompertz's law, show that

   (a) ${}_{\infty}q_{\underset{1}{x}\overset{2}{y}z} = {}_{\infty}q_{\overset{1}{y}z} \cdot {}_{\infty}q_{\overset{1}{x}yz}$

   (b) $${}_{\infty}q_{\underset{1}{25}:\underset{2}{30}:\overset{3}{35}:40} = \frac{1}{1+c^5} \cdot {}_{\infty}q_{\underset{1}{25}:\overset{2}{30}:35:40}.$$

3. Show that

$$\begin{aligned}{}_{\infty}q_{\overset{1}{w}:\overline{xyz}} = {}_{\infty}q_{w\overset{2}{x}yz} + {}_{\infty}q_{w\overset{1}{x}y} - {}_{\infty}q_{wx\overset{3}{y}} \\ - {}_{\infty}q_{\overset{2}{w}z} - {}_{\infty}q_{\overset{1}{\overline{wx}}:y} - 2({}_{\infty}q_{\overset{2}{w}\underset{1}{x}yz} + {}_{\infty}q_{\overset{2}{w}x\underset{1}{y}z} - 1).\end{aligned}$$

4. Given

$${}_{\infty}q_{\overset{1}{x}y} = .5537,\ {}_{\infty}q_{\overset{1}{x}z} = .6484,\ {}_{\infty}q_{\overset{1}{x}yz} = .5325,\ {}_{\infty}q_{xy\overset{2}{z}} = {}_{\infty}q_{xy\overset{3}{z}}.\ \text{Find } {}_{\infty}q_{x\underset{1}{y}\overset{2}{z}}.$$

(Ans. .2145)

3. Insurances

5. Show that

   (a) $\bar{A}_{\underset{1}{x}y\overset{3}{z}} = \bar{A}_{y\overset{2}{z}} - \bar{A}_{x\underset{1}{y}\overset{2:3}{z}}$

   (b) $\bar{A}_{\underset{1}{w}\overset{2}{x}yz} = \bar{A}_{\overset{1}{x}yz} - \bar{A}_{w\overset{1}{x}yz}$.

6. The expressions below have been given for the value of $\bar{A}_{\overset{4}{w}xy\underset{1}{z}}$. Show clearly why each expression is correct or incorrect. Also give another expression, which is correct.

(a) $\int_0^\infty v^t \, {}_tp_w(1 - {}_tp_z)[{}_tp_y(1 - {}_tp_x)\mu_{y+t} + {}_tp_x(1 - {}_tp_y)\mu_{x+t}]\bar{A}_{w+t}\, dt$

(b) $\int_0^\infty v^t \, {}_tp_w \left({}_tp_x \cdot {}_tq^{\;2}_{yz}\bar{A}_{\overline{w+t}:x+t}^{\;2} + {}_tp_y \cdot {}_tq^{\;2}_{xz}\bar{A}_{\overline{w+t}:y+t}^{\;2}\right) dt.$

7. Assume that the mortality of (y) and (z) is given in terms of a standard table and that (x) is subject to the mortality of a special table in which the force of mortality is twice that of the standard table. On the basis of this assumption, derive an expression in terms of the standard table for the value of a unit payable to (z) upon the death of (x), if (z) is then living, provided (x) dies after (y).

8. What annuity value should be used in obtaining the net annual premium corresponding to each of the following net single premiums?

(a) $A_{\overset{2}{x}\underset{1}{y}z}$

(b) $A_{w\overset{2}{x}y\underset{1}{z}}$

4. Compound contingencies with terms-certain

9. Express in the form of integrals the probabilities that of the four lives (w), (x), (y), and (z),

(a) the deaths will occur in the order (w), (x), (y), (z), and that (x) and (y) will die within four years of each other;

(b) (w) only will die within 10 years of the present time and (x), (y), and (z) will die after the 10 years in that order, (y) within 5 years of (x) and (z) within 5 years of (y).

10. Express in integral form:

(a) The probability that three lives aged 10, 20, and 30 all die before attaining age 60, with (20) being the second to die;

(b) the probability that (w), (x), (y), and (z) will die in the order named, not more than two years separating any two consecutive deaths.

11. Derive an expression in terms of contingent insurances and commutation functions for the net single premium for an insurance payable at the death of a person now aged 35 if he dies before the survivor of two persons now aged 40 and 50 or within ten years after the death of such survivor, leaving another person now aged 30 surviving.

Miscellaneous problems

12. According to a certain mortality table, the probability that three lives aged 70, 55, and 40 will die in that order at intervals of not less than fifteen years is .048, and the probability that at least one of two lives now aged 70 will be alive fifteen years before the death of a life now aged 55 is .8. Find the probability that neither of two lives now aged 40 will survive to age 70. (Ans. .2704)

13. If it is assumed that $\mu_{x+t} = q_{x+t}$ for all values of t, show that

$$\bar{A}^{2}_{\underset{1}{xyz}} = \frac{1}{\delta}\left[\bar{A}_{xyz} - \bar{A}_{xz} - (1 - \mu_x)(\bar{A}_{x+1:yz} - \bar{A}_{x+1:z})\right].$$

14. Write an expression in the form of integrals for the gross single premium, including a loading of 10 per cent of the net, for an insurance payable on the death of a life aged 30, provided that he dies either before a life aged 35 or within three years after his death and that both are predeceased by two lives each aged 40 who must die in a specified order. Should the life aged 30 die within the three years after the death of the life aged 35, the sum insured is to be reduced by 25 per cent, but should he survive the life aged 35 by three years, the other lives dying as before, the premium is to be returned.

15. The will of H, now dead, leaves to his wife (w) the income for life from a \$200,000 trust fund, the principal to be divided at (w)'s death in accordance with the following conditions. If H's sons (x) and (y) are both alive, they alone share the principal. If either son predeceases (w), his children living at (w)'s death share the principal. (x) has three children, (a), (b), and (c); and (y) two, (d) and (e). All beneficiaries at (w)'s death are to share the principal per capita, i.e., equally without regard to generation. Ignore the possibility of the birth of any further children to (x) or (y).

Find an expression for the present value of (x)'s expectation (not including that of his children) in terms of continuous contingent insurances payable on the first death.

Chapter 13

REVERSIONARY ANNUITIES

1. Introduction

A reversionary annuity is an annuity which commences upon the failure of a given status (A) if a second status (B) is then in existence and which continues thereafter during the existence of (B). A simple example is the familiar deferred annuity ${}_{n|}a_x$, in which $(\overline{n}|)$ is the failing status (A) and (x) the surviving status (B). In the general case where the failing status involves life contingencies, reversionary annuities are a type of contingent function analogous to the contingent insurances discussed in the preceding chapters.

2. The basic function

The simplest form of reversionary annuity where the failing status involves a life contingency is an annuity of 1 payable to a life now aged x commencing at the end of the year of death of (y)—or, more briefly, an annuity to (x) after (y). The present value of this annuity is denoted by $a_{y|x}$. Since the annuity consists of a series of payments made at the end of each year provided that (x) is alive and (y) is not alive, its present value may be written

$$a_{y|x} = \sum_{t=1}^{\infty} v^t \, {}_tp_x(1 - {}_tp_y) = a_x - a_{xy} . \qquad (13.1)$$

This result is obviously correct since this reversionary annuity is the same as a life annuity to (x) with payments omitted during the joint lifetime of (x) and (y). The same reasoning applies to the generalized function $a_{v|u}$, whatever age status or combination of statuses may be represented by (u) and (v), and we may write

$$a_{v|u} = a_u - a_{uv} . \qquad (13.2)$$

For example, if (u) represents the joint-life status (xy) and (v) the single life (z), we have

$$a_{z|xy} = a_{xy} - a_{xyz} .$$

Reversionary annuities of this type may thus be expressed directly in terms of single-life and joint-life annuity values. Two further examples follow:

$$a_{\overline{yz}|x} = a_x - a_{x:\overline{yz}} = a_x - a_{xy} - a_{xz} + a_{xyz}$$

$$a_{yz|\overline{wx}} = a_{\overline{wx}} - a_{\overline{wx}:yz}$$

$$= a_w + a_x - a_{wx} - a_{wyz} - a_{xyz} + a_{wxyz} \,.$$

In the first of these, the annuity is payable during the lifetime of (x) after the death of the survivor of (y) and (z). In the second, the annuity commences upon the first death of (y) and (z) and is payable until the second death of (w) and (x).

Although known as reversionary *annuities*, these functions are essentially insurances since the payments commence upon the failure of a specified status.

3. Modifications of the basic function

The basic function $a_{y|x}$ makes provision for annual payments, and the payment periods are measured from the end of the year of death of (y). In practice, the payments are often made more frequently than once a year, and the payment periods are measured from different points of time.

If the annuity is payable to (x) m times a year and commences at the end of the m-th part of the year in which (y) dies, the present value is given by

$$a_{y|x}^{(m)} = a_x^{(m)} - a_{xy}^{(m)}$$

$$\doteq a_x - a_{xy} + \frac{m^2 - 1}{12m^2}\,\mu_y\,, \tag{13.3}$$

using (2.18) and recalling that

$$\mu_{xy} = \mu_x + \mu_y \,.$$

If m becomes infinite in the formula above, we have

$$\bar{a}_{y|x} = \bar{a}_x - \bar{a}_{xy} \doteq a_x - a_{xy} + 1/12\mu_y \,. \tag{13.4}$$

This represents the present value of an annuity of 1 commencing at the moment of death of (y) and payable continuously throughout the remaining lifetime of (x). An integral expression for this function is

$$\bar{a}_{y|x} = \int_0^\infty v^t \, {}_tp_x(1 - {}_tp_y) \, dt. \tag{13.5a}$$

When the substitution of $\int_0^t {}_sp_y\mu_{y+s} \, ds$ is made for $1 - {}_tp_y$ in (13.5a), we obtain

$$\bar{a}_{y|x} = \int_0^\infty \int_0^t v^t \, {}_tp_x \, {}_sp_y\mu_{y+s} \, ds \, dt.$$

Then, inverting the order of integration,

$$\begin{aligned} \bar{a}_{y|x} &= \int_0^\infty \int_s^\infty v^t \, {}_tp_x \, {}_sp_y\mu_{y+s} \, dt \, ds \\ &= \int_0^\infty {}_sp_y\mu_{y+s} \cdot {}_{s|}\bar{a}_x \, ds \\ &= \int_0^\infty v^s \, {}_sp_{xy}\mu_{y+s}\bar{a}_{x+s} \, ds. \end{aligned} \tag{13.5b}$$

In this form, the contingent insurance character of the function is clearly apparent.

The symbol $\hat{a}$ is used to distinguish a type of reversionary annuity in which the payments are made periodically with the payment periods measured from the date of death. The annuity which provides payments m times a year to (x) with the first payment of $1/m$ due at the end of $1/m$ years from the moment of death of (y) has the present value:

$$\begin{aligned} \hat{a}_{y|x}^{(m)} &= \int_0^\infty v^t \, {}_tp_{xy}\mu_{y+t}a_{x+t}^{(m)} \, dt \\ &\doteq \int_0^\infty v^t \, {}_tp_{xy}\mu_{y+t}\left(\bar{a}_{x+t} - \frac{1}{2m}\right) dt \end{aligned}$$

using (2.19) and (2.27),

$$\begin{aligned} &= \bar{a}_{y|x} - \frac{1}{2m}\bar{A}_{xy}^{\;\,1} \\ &\doteq a_x - a_{xy} + \tfrac{1}{12}\mu_y - \frac{1}{2m}\bar{A}_{xy}^{\;\,1}. \end{aligned} \tag{13.6}$$

It will be observed in each of the three cases considered above that the formula for the modified function consists of the expression

$a_x - a_{xy}$ plus certain correction terms. In practice, the correction terms are often ignored, and the net single premium for all three types is computed as $a_x - a_{xy}$.

This convenient approximation may also be extended to the general case of an annuity to (u) after (v):

$$a_{v|u}^{(m)} \doteq \bar{a}_{v|u} \doteq \mathring{a}_{v|u}^{(m)} \doteq a_{v|u} = a_u - a_{uv} .$$

When necessary, the approximation can be improved by adding the proper correction terms. In particular, when either of the statuses (u) and (v) involves a term-certain, the correction terms are important and should not be neglected. This case will be examined in the next section.

From the relation $a_{v|u} \doteq \bar{a}_{v|u}$, it follows that annuities with periodic payments may be evaluated as continuous annuities and that either type may therefore be directly computed by methods of numerical integration in cases where the necessary joint-life functions are not already tabulated. As long as (u) and (v) involve only lives, the definite integrals always assume the following forms:

$$\bar{a}_{v|u} = \int_0^\infty v^t \, {}_tp_u(1 - {}_tp_v) \, dt \tag{13.7a}$$

or

$$\int_0^\infty v^t \, {}_tp_{uv}\mu_{v+t}\bar{a}_{u+t} \, dt. \tag{13.7b}$$

The second form is useful when there is a change in the interest or mortality assumptions after the failure of (v).

4. Functions involving terms-certain

We have already noted that the basic formula

$$a_{v|u} = a_u - a_{uv}$$

is valid regardless of the type of status represented by (u) and (v). Let us consider a few cases where these statuses include a term-certain.

If (v) is the term-certain $(\overline{n|})$ and (u) the single life (x), we have

$$a_{\overline{n|}|x} = a_x - a_{x:\overline{n|}} ,$$

and this will be recognized as the single-life deferred annuity, ${}_{n|}a_x$, cited earlier as a simple example of a reversionary function.

An annuity to (x) after (y) with no payments to be made after n years from the present time has the present value

$$a_{y|x:\overline{n|}} = a_{x:\overline{n|}} - a_{xy:\overline{n|}}\,.$$

This function may be described as an n-year temporary reversionary annuity. An alternative notation is ${}_n a_{y|x}$.

An annuity to (x) after (y) with no payments to be made within the first n years from the present time may be interpreted as an annuity to (x) after $(\overline{y:\overline{n|}})$, and

$$a_{\overline{y:\overline{n|}}|x} = a_x - a_{x:\overline{y:\overline{n|}}} = a_x - a_{xy} - a_{x:\overline{n|}} + a_{xy:\overline{n|}}$$
$$= {}_{n|}a_x - {}_{n|}a_{xy}\,.$$

The function is an n-year deferred reversionary annuity. An alternative symbol for the present value is ${}_{n|}a_{y|x}$.

It should be noted that the term-certain in the above functions is measured from the date of issue of the contract. It is important to distinguish clearly another type of function in which the temporary or deferred period is measured from the *date of death* instead of the date of issue. In the case where the deferred period is measured from the end of the year of death of (y), the benefit is an annuity to (x) commencing n years after the death of (y), and the present value is

$$\sum_{t=1}^{\infty} v^{n+t}\ {}_{n+t}p_x(1 - {}_tp_y) = \frac{D_{x+n}}{D_x}\,(a_{x+n} - a_{x+n:y}) = \frac{D_{x+n}}{D_x}\,a_{y|x+n}\,.$$

For the case where the annuity is payable to (x) for n years after the death of (y), the present value is obtained by deducting the value of the above deferred benefit from the normal reversionary annuity:

$$a_{y|x} - \frac{D_{x+n}}{D_x}\,a_{y|x+n}\,.$$

Although the approximation $a^{(m)}_{v|u} \doteq a_{v|u}$ is often used when (u) and (v) involve only lives, it should not be used when terms-certain are involved. This may be seen in the case of the simple function $a^{(m)}_{\overline{n|}|x}$:

$$\begin{aligned} a^{(m)}_{\overline{n|}|x} &= a^{(m)}_x - a^{(m)}_{x:\overline{n|}} \\ &\doteq \left(a_x + \frac{m-1}{2m}\right) - \left[a_{x:\overline{n|}} + \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right)\right] \\ &= a_x - a_{x:\overline{n|}} + \frac{m-1}{2m}\cdot\frac{D_{x+n}}{D_x}\,. \end{aligned} \tag{13.8}$$

The term $\frac{m-1}{2m}\cdot\frac{D_{x+n}}{D_x}$ may make an appreciable contribution to the total value of the function and should not be neglected.

5. Annual premiums and reserves

When the reversionary annuity $a_{v|u}$ is subject to annual premiums, the premium-paying period coincides with the joint lifetime of the statuses (u) and (v). If (v) is the first to fail, the annuity becomes payable, whereas if (u) is the first to fail, the contract expires, and in either case no further premium payments are made. The following annual premium formulas illustrate this principle:

$$P(a_{y|x}) = \frac{a_{y|x}}{\ddot{a}_{xy}} = \frac{a_x - a_{xy}}{\ddot{a}_{xy}} = \frac{\ddot{a}_x - \ddot{a}_{xy}}{\ddot{a}_{xy}} = \frac{\ddot{a}_x}{\ddot{a}_{xy}} - 1 \tag{13.9}$$

$$P(a_{yz|x}) = \frac{a_{yz|x}}{\ddot{a}_{xyz}}$$

$$P(a_{\overline{yz}|x}) = \frac{a_{\overline{yz}|x}}{\ddot{a}_{x:\overline{yz}}}$$

Reserve formulas are easily written in prospective form. For example,

$$_tV(a_{y|x}) = a_{y+t|x+t} - P\cdot\ddot{a}_{x+t:y+t} \tag{13.10}$$

if both (x) and (y) are surviving, or $_tV(a_{y|x}) = a_{x+t}$ if (x) alone is surviving. Reversionary annuities often give rise to negative reserves—depending on the ages involved, the duration, and the mortality table used. This feature may be avoided by suitably limiting the premium-paying period.

6. Continuous instalments

Reversionary annuities are sometimes issued in combination with other insurance benefits. One such combination policy is known as *insurance with continuous instalments.*

Suppose that a policy provides that at the death of (x) a monthly annuity of 1 per annum will be payable for n years certain and for as long thereafter as a beneficiary, aged y at date of issue, survives. At the date of issue, the certain payments have a present value of $\ddot{a}^{(12)}_{\overline{n}|}\cdot\bar{A}_x$, and the deferred annuity to (y), the continuous instalment portion, has the following value:

$$\int_0^\infty v^t\,{}_tp_{xy}\mu_{x+t}\cdot{}_{n|}\ddot{a}^{(12)}_{y+t}\,dt = \frac{D_{y+n}}{D_y}\int_0^\infty v^t\,{}_tp_{x:y+n}\mu_{x+t}\ddot{a}^{(12)}_{y+n+t}\,dt. \tag{13.11}$$

The expressions assume that the first monthly income payment is made on the date of death of (x). Earlier in the chapter the symbol $\hat{a}$ was used to indicate that an immediate annuity commences at the moment of death, and if we now use $\hat{\ddot{a}}$ to indicate that an annuity-due becomes payable at death, the definite integral in (13.11) may be written as $\hat{\ddot{a}}^{(12)}_{x|y+n}$. The total net single premium, including the certain payments, will then be

$$\ddot{a}^{(12)}_{\overline{n|}}\bar{A}_x + \frac{D_{y+n}}{D_y}\hat{\ddot{a}}^{(12)}_{x|y+n}.$$

This particular combination of benefits is known as *whole life insurance with continuous instalments.*

The continuous instalment portion of this policy expires without value if (y) predeceases (x), and when the policy is subject to annual premiums, the payment of the extra premium for the continuous instalment portion is made to depend upon the survival of both (x) and (y). The total annual premium is thus given by

$$\frac{\ddot{a}^{(12)}_{\overline{n|}}\bar{A}_x}{\ddot{a}_x} + \frac{\dfrac{D_{y+n}}{D_y}\hat{\ddot{a}}^{(12)}_{x|y+n}}{\ddot{a}_{xy}}, \tag{13.12}$$

reducing to $\dfrac{\ddot{a}^{(12)}_{\overline{n|}}\bar{A}_x}{\ddot{a}_x}$ upon the death of (y).

Continuous instalments may also be issued in combination with other forms of insurance. In the case of *term insurance with continuous instalments,* the benefit is payable only if the insured life (x) dies within m years. The net single premium for the n-year annuity-certain is $\ddot{a}^{(12)}_{\overline{n|}}\bar{A}^{1}_{x:\overline{m|}}$, and the extra premium required to continue the instalments to (y) is

$$\int_0^m v^t\,{}_tp_{xy}\mu_{x+t}\cdot{}_{n|}\ddot{a}^{(12)}_{y+t}\,dt$$

$$= \int_0^\infty v^t\,{}_tp_{xy}\mu_{x+t}\cdot{}_{n|}\ddot{a}^{(12)}_{y+t}\,dt - v^m\,{}_mp_{xy}\int_0^\infty v^t\,{}_tp_{x+m:y+m}\mu_{x+m+t}\cdot{}_{n|}\ddot{a}^{(12)}_{y+m+t}\,dt$$

$$= \frac{D_{y+n}}{D_y}\hat{\ddot{a}}^{(12)}_{x|y+n} - v^m\,{}_mp_{xy}\frac{D_{y+m+n}}{D_{y+m}}\hat{\ddot{a}}^{(12)}_{x+m|y+m+n}$$

$$= \frac{D_{y+n}}{D_y}\left[\hat{\ddot{a}}^{(12)}_{x|y+n} - \frac{D_{x+m:y+n+m}}{D_{x:y+n}}\hat{\ddot{a}}^{(12)}_{x+m|y+n+m}\right].$$

The total annual premium is

$$\frac{\ddot{a}_{\overline{n}|}^{(12)}\bar{A}^{1}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} + \frac{\dfrac{D_{y+n}}{D_y}\left(\mathring{a}^{(12)}_{x|y+n} - \dfrac{D_{x+m:y+n+m}}{D_{x:y+n}}\mathring{a}^{(12)}_{x+m|y+n+m}\right)}{\ddot{a}_{xy:\overline{m}|}},$$

reducing to $\dfrac{\ddot{a}_{\overline{n}|}^{(12)}\bar{A}^{1}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}}$ upon the death of (y).

In practice, the $\mathring{a}^{(12)}$ functions appearing in these formulas are most conveniently evaluated as simple reversionary annuities, thus:

$$\mathring{a}^{(12)}_{v|u} \doteq a_u - a_{uv}\,.$$

This approximation understates the true value, a fact which is sometimes reflected in the loading formula.

Reserves for continuous instalment policies may be obtained from the usual valuation principles. As an example, consider an ordinary life policy issued at age x with continuous instalments to a beneficiary aged y. The reserve at the end of t years is given by the following expression:

$$\ddot{a}_{\overline{n}|}^{(12)}\,{}_tV(\bar{A}_x) + \frac{D_{y+n+t}}{D_{y+t}}\mathring{a}^{(12)}_{x+t|y+n+t} - P\cdot\ddot{a}_{x+t:y+t}\,,$$

where $P = \dfrac{\dfrac{D_{y+n}}{D_y}\mathring{a}^{(12)}_{x|y+n}}{\ddot{a}_{xy}}$, the extra annual premium for the continuous instalments. This expression involves the assumption that both (x) and (y) are alive on the valuation date. If only (x) is alive, the reserve is simply $\ddot{a}_{\overline{n}|}^{(12)}\cdot{}_tV(\bar{A}_x)$.

7. Compound reversionary annuities

Although rarely encountered in practice, compound reversionary annuities are of theoretical interest since they represent the annuity counterpart of the compound contingent insurances discussed in the preceding chapter. Under such an annuity, payments are made only in the event that the antecedent deaths have occurred in a prescribed order. Many functions of this type may be defined, and, since little classification is possible, each particular case should be approached from first principles.

The symbol $\bar{a}_{\overset{1}{y}z|x}$ denotes the present value of a continuous annuity of 1 per annum payable to (x) commencing upon the death of (y) provided that (y) dies before a third life (z). This benefit may be regarded as an insurance contingent on (y) being the first of the three lives to die, with the amount of insurance at any time being equal to the present value of a continuous annuity at (x)'s attained age. We thus have

$$\bar{a}_{\overset{1}{y}z|x} = \int_0^\infty v^t\,{}_tp_{xyz}\mu_{y+t}\bar{a}_{x+t}\,dt. \tag{13.13a}$$

An equivalent definite integral may be written by considering the function from another point of view. Annuity payments will be made at time t provided that (x) is then alive and that (y) has previously died before (z). We may thus write

$$\bar{a}_{\overset{1}{y}z|x} = \int_0^\infty v^t\,{}_tp_x\cdot{}_tq_{\overset{1}{y}z}\,dt. \tag{13.13b}$$

When $\bar{a}_{\overset{1}{y}z|x}$ is known, the value of $\bar{a}_{y\overset{1}{z}|x}$ may be obtained from the relation

$$\bar{a}_{\overset{1}{y}z|x} + \bar{a}_{y\overset{1}{z}|x} = \bar{a}_{yz|x}\,. \tag{13.14}$$

The symbol $\bar{a}_{\overset{2}{y}z|x}$ denotes the present value of an annuity to (x) commencing upon the death of (y) provided that (y) dies after (z). This value is given by

$$\begin{aligned}\bar{a}_{\overset{2}{y}z|x} &= \int_0^\infty v^t\,{}_tp_{xy}(1 - {}_tp_z)\mu_{y+t}\bar{a}_{x+t}\,dt \\ &= \bar{a}_{y|x} - \bar{a}_{\overset{1}{y}z|x}\,. \end{aligned} \tag{13.15}$$

The relation

$$\bar{a}_{\overset{1}{y}z|x} + \bar{a}_{\overset{2}{y}z|x} = \bar{a}_{y|x}$$

is clearly correct.

Another way of expressing $\bar{a}_{\overset{2}{y}z|x}$ is by regarding the benefit as an insurance contingent upon (z) dying first with the amount of insurance at any time equal to the current present value of a reversionary annuity to (x) after (y). This point of view leads to the following result:

$$\bar{a}_{\overset{2}{y}z|x} = \int_0^\infty v^t\,{}_tp_{xyz}\mu_{z+t}\bar{a}_{y+t|x+t}\,dt = \bar{a}_{y\overset{1}{z}|x} - \bar{a}_{z|xy}\,. \tag{13.16}$$

The student should give a verbal interpretation of this result and show that it is equivalent to the one obtained above.

In the case of more complex functions, a notation similar to that used with compound contingent insurances can be adopted. A numeral is written above the life upon whose failure the benefit becomes payable, and numerals written below indicate the relative order of death of the other lives. For example, $\bar{a}_{\overset{2}{x}\underset{1}{y}z|w}$ represents an annuity to (w) commencing upon the death of (x) provided that (x) dies before (z) and after (y):

$$\bar{a}_{\overset{2}{x}\underset{1}{y}z|w} = \int_0^\infty v^t\,{}_tp_{wxz}(1 - {}_tp_y)\mu_{x+t}\bar{a}_{w+t}\,dt = \bar{a}_{\overset{1}{x}z|w} - \bar{a}_{\overset{1}{x}yz|w} \tag{13.17}$$

In the two examples that follow, the student should describe the benefit and verify the expressions given:

$$\bar{a}_{\overset{2:3}{x}\,\underset{1}{y}z|w} = \int_0^\infty v^t\,{}_tp_{wxyz}\mu_{y+t}\bar{a}_{x+t|w+t}\,dt = \bar{a}_{x\overset{1}{y}z|w} - \bar{a}_{\overset{1}{y}z|wx} \tag{13.18}$$

$$\bar{a}_{\overset{4}{w}\underset{2}{x}\underset{1}{y}z|v} = \int_0^\infty v^t\,{}_tp_{vwxy}(1 - {}_tp_z)\mu_{y+t}\bar{a}_{\overset{2}{\overline{w+t}}:x+t|v+t}\,dt \tag{13.19}$$

It is not necessary to give special attention to compound reversionary annuities in non-continuous form since, just as with simple reversionary annuities, the values of the discrete annuities are approximately the same as the values of the corresponding continuous annuities.

8. Summary of notation

A vertical bar separating the elements of a suffixed subscript indicates a reversionary function, the status after the bar being understood to follow the status before the bar. Thus $a_{y|x}^{(m)}$ is an annuity on the life of (x) after the death of (y) with the first annuity payment of $1/m$ due at the end of the $1/m$-th part of a year in which (y) dies, the payment periods being measured from the date of issue of the contract. When the payment periods are measured from the date of death, the distinguishing symbol $\hat{a}^{(m)}$ is used.

References

1–5. Similar material is covered by Spurgeon (1932), Chapter 18, and by Hooker and Longley-Cook (1957), Chapter 25.

6. Policies with continuous instalments are discussed by Bergstresser (1941).

7. Compound reversionary annuities are treated by Spurgeon (1932), Chapter 19.

EXERCISES

1, 2. The basic function

1. Explain by general reasoning and show algebraically:

$$a_{y|x} = \sum_{t=1}^{\infty} v^t {}_tp_x \cdot {}_{t-1|}q_y \ddot{a}_{x+t} \, .$$

2. Express in terms of single- and joint-life annuities:
   (a) $a_{z|\overline{xy}}$
   (b) $a_{\overline{yz}|wx}$.

3. The last-survivor annuity $a_{\overline{xyz}}$ is payable in the following way. While all three lives are surviving, (x) and (y) each receive ¼ of the annual payment and (z) receives ½. Upon the death of (x), his share passes to (y), if surviving, otherwise to (z). Upon the death of either (y) or (z), their shares are divided equally among the survivors, the last survivor receiving the total annuity payment of 1. Find expressions for the present values of the individual shares of (x), (y), and (z) in terms of single- and joint-life annuities.

3. Modifications of the basic function

4. Give two independent integral expressions for $\bar{a}_{\overline{xy}|z}$. Demonstrate their equivalence by reducing both to the same expression in terms of simple continuous annuities.

5. Show that

(a) $\dfrac{\partial}{\partial x} \bar{a}_{y|x} = \mu_x \bar{a}_{y|x} - \bar{A}^{\,2}_{xy}$

(b) $\bar{A}^{\,2}_{xy} = \bar{A}_{x\overset{1}{y}} - \delta \bar{a}_{y|x}$.

6. Show that

$$\mathring{a}^{(m)}_{y|x} \doteq \bar{a}_{y|x} + \frac{1}{2m} \bar{A}_{x\overset{1}{y}} \, .$$

4. Functions involving terms-certain

7. Express in terms of commutation functions and simple annuity values:
   (a) the present value of a temporary annuity of 1 payable to (x) with the first payment at the end of n years from the end of the year of death of (y), but with no payments to be made after m years from the present time $(m > n)$;
   (b) the present value of a temporary annuity of 1 payable to (x) during the lifetime of (y) and for n years after the death of (y), but with no payments to be made after m years from the present time $(m > n)$.

8. An annuity of 1 becomes payable at the end of the year of death of (x) and continues until n years after the death of (y). Express the present value in terms of net single premium symbols on each of the following assumptions:

(a) the annuity is entered upon only if (y) is alive at the death of (x);

(b) the annuity is payable whether or not (y) is alive at the death of (x).

9. Find an expression in a form free from integrals or summations for the value of an annuity of \$100 per annum payable to (y), the first payment to be made at the end of the t-th year succeeding the year in which (x) dies, provided (x) dies within n years, the annuity to be void if (x) lives beyond n years.

10. For each of the following, describe the benefit and give a simple formula (using standard approximations if necessary):

(a) $a_{y|\overline{x:\overline{n}|}}$, (b) $a^{(4)}_{x:\overline{n}||y}$.

5. Annual premiums and reserves

11. Find as closely as possible the net annual premium for a reversionary annuity of 1 per annum to a life aged 40 after the death of another life aged 43, the first annuity payment to be a full payment of 1 and to be made on the anniversary of the date of issue, given the following data from a Gompertz table:

x	a_x	μ_x	x	a_x	μ_x
40	15.136	.00990	54	11.331	.01950
43	14.413	.01106	55	11.024	.02077
51	12.230	.01631	56	10.714	.02216
52	11.935	.01727	62	8.808	.03381
53	11.635	.01833	63	8.488	.03645

(Ans. .347)

12. Show that ${}_tV(a_{y|x})$ is negative if

$$\frac{\ddot{a}_{x+t}}{\ddot{a}_x} < \frac{\ddot{a}_{x+t:y+t}}{\ddot{a}_{xy}}.$$

13. Express in annuity symbols the reserve t years after issue for a single premium reversionary annuity payable to a life (z) after the death of the survivor of two lives (x) and (y).

6. Continuous instalments

14. An m-year endowment policy with continuous instalments provides that if the insured, aged x at issue, dies before the maturity of the policy as an endowment, a monthly income will be paid for n years certain and for as long thereafter as the beneficiary, aged y at issue, survives. At maturity as an endowment the income is payable for n years certain and as long thereafter as either the insured or beneficiary survives.

Show that the net single premium for the policy consists of the following four parts:

(a) for the annuity certain: $\ddot{a}_{\overline{n}|}^{(12)}\bar{A}_{x:\overline{m}|}$

(b) for the continuous instalments to the beneficiary in the event of the insured's death prior to maturity:

$$\frac{D_{y+n}}{D_y}\left(\mathring{\ddot{a}}_{x|y+n}^{(12)} - \frac{D_{x+m:y+n+m}}{D_{x:y+n}}\mathring{\ddot{a}}_{x+m|y+n+m}^{(12)}\right)$$

(c) for the continuous instalments to the insured after maturity:

$${}_{m+n|}\ddot{a}_x^{(12)}$$

(d) for the continuous instalments to the beneficiary in the event of the insured's death after maturity:

$$({}_mp_x - {}_{m+n}p_x)\frac{D_{y+m+n}}{D_y}\ddot{a}_{y+m+n}^{(12)} + \frac{D_{x+m+n:y+m+n}}{D_{xy}}\ddot{a}_{x+m+n|y+m+n}^{(12)}$$

15. In Exercise 14, if the $\mathring{\ddot{a}}^{(12)}$ functions are replaced by $\ddot{a}^{(12)}$, show that the sum of parts (b) and (d) of the net single premium reduces to

$${}_{n|}\ddot{a}_y^{(12)} - {}_{m+n|}\ddot{a}_{xy}^{(12)} - \frac{D_{y+n}}{D_y}\ddot{a}_{x:y+n:\overline{m}|}^{(12)}.$$

16. In the policy described in Exercise 14, the annual premiums are arranged so that if the beneficiary dies during the premium-paying period, the premium will reduce to that for a policy on the insured's life only. Using the approximation indicated in Exercise 15, write the formula for the net annual premium.

7. Compound reversionary annuities

17. When Gompertz's law holds, show that

$$\bar{a}_{\overset{1}{y}z|x} = {}_\infty q_{\overset{1}{y}z}\cdot\bar{a}_{yz|x}.$$

18. Give two different expressions in the form of integrals for the value of a continuous annuity of 1 per annum, payable so long as at least one of two lives aged x and y shall survive after the death of z, provided z predeceases a fourth life w, and show that your expressions are equal.

19. Show that $\bar{a}_{x\overset{1}{y}|z} = \int_0^\infty v^t\,{}_tp_z\bar{A}_{x\overset{1}{y}:z+t}\,dt.$

Miscellaneous applications

20. Develop a formula in terms of annuity and insurance symbols for the gross single premium to provide the following benefits with a loading of 6½% of the gross premium: A last-survivor annuity on the lives of a husband and wife, deferred n years and reducing by one-third on the first death. If the husband's death occurs prior to that of the wife and during the de-

ferred period, the annuity on the reduced basis is to commence immediately to the wife. If the husband's death occurs after that of the wife and during the deferred period, the single premium is to be refunded.

21. A man aged 35 has a wife aged 30 and two children aged 5 and 10, respectively. Under an insurance program, upon the man's death while either of his two children is under age 18 a monthly annuity will become payable subject to the following conditions:

(i) \$30 monthly payable to his wife as long as she survives and either child survives and has not attained age 18,

(ii) \$20 monthly payable to each child as long as the child survives and has not attained age 18, payable whether or not the man's wife is alive at the time the payments come due.

Assuming that the monthly functions may be replaced by continuous functions without appreciable loss of accuracy, express the value of the insurance program in terms of continuous life annuities.

22. An insured holds a \$5000 policy, whole life fully paid up at age 65, issued at age x five years ago. An option in the policy provides for settlement in equal monthly instalments for a period of twenty years, the first payment to be made at the date of death, and such that the present value of the instalments at the date of death is \$5000. The insured now desires to provide for an additional benefit, whereby the monthly instalments will continue during the lifetime of the beneficiary now aged y. Find an expression for the additional net annual premium, assuming such premiums are not to be payable after the death of the beneficiary. Give an expression for the reserve on the additional benefit after five years, assuming both the insured and the beneficiary are then alive.

Part III

MULTIPLE-DECREMENT FUNCTIONS

Chapter 14

MULTIPLE-DECREMENT TABLES

1. Introduction

The analytical methods applied so far to problems involving mortality alone can be extended to form a more general theory involving the simultaneous operation of several causes of decrement to a particular body of lives. For example, one may be concerned with an insurance coverage in which disability and mortality are distinct causes of claim and the interacting effects of exposure to both causes of decrement must be analyzed. Similarly, one may wish to study a mortality experience in terms of its component causes of death, treating each cause of death as a separate decrement. Or, one may have the problem of valuation of a pension plan where death, disability, withdrawal, and retirement all operate as decremental forces against a staff of employees. The mathematical model from which analyses of this kind can be made is known as the *multiple-decrement table.*

2. Probabilities of decrement

A multiple-decrement table is a mathematical model which assumes a large body of lives subject to several independent causes of decrement which are operating continuously. The body of lives forms a closed group, there being no new entrants and no re-entrants after the operation of the various decrements.

The following notation is adopted:

$l_x^{(T)}$ is the number of lives attaining age x in a body of lives subject to the operation of m causes of decrement (1), (2), $\cdots$, (m);

$d_x^{(k)}$ is the number of decrements from cause (k) between ages x and $x+1$;

$d_x^{(T)}$ is the total number of decrements from all causes between ages x and $x+1$, so that

$$d_x^{(T)} = \sum_{k=1}^{m} d_x^{(k)} \tag{14.1}$$

and

$$l_x^{(T)} - d_x^{(T)} = l_{x+1}^{(T)} \,. \tag{14.2}$$

$q_x^{(k)}$ is the probability that (x) will leave the body of lives within one year as a result of cause (k):

$$q_x^{(k)} = \frac{d_x^{(k)}}{l_x^{(T)}}. \tag{14.3}$$

$q_x^{(T)}$ is the probability that (x) will leave the body of lives within one year regardless of cause:

$$q_x^{(T)} = \frac{d_x^{(T)}}{l_x^{(T)}} = \sum_{k=1}^{m} q_x^{(k)}. \tag{14.4}$$

TABLE 8

SECTION OF DOUBLE-DECREMENT TABLE

Age x	$l_x^{(T)}$	$d_x^{(1)}$	$d_x^{(2)}$
24	901,020	299	92,762
25	807,959	314	86,632
26	721,013	324	80,385
27	640,304	329	74,117
28	565,858	329	67,909
29	497,620	324	61,839

$p_x^{(T)}$ is the probability that (x) will remain in the body of lives for at least one year:

$$p_x^{(T)} = 1 - q_x^{(T)} = \frac{l_{x+1}^{(T)}}{l_x^{(T)}}. \tag{14.5}$$

Similarly,

$$_np_x^{(T)} = \frac{l_{x+n}^{(T)}}{l_x^{(T)}} \tag{14.6}$$

and

$$_nq_x^{(T)} = 1 - {_np_x^{(T)}}. \tag{14.7}$$

If the values of $q_x^{(k)}$ are known for all k, the complete multiple-decrement table is easily built up. An arbitrary radix is assumed, and the values of $d_x^{(k)}$, $d_x^{(T)}$, and $l_x^{(T)}$ are obtained for each successive age by using formulas (14.3), (14.1), and (14.2) respectively.

Table 8 shows a section of a multiple-decrement table with two causes of decrement. When such a table is available, it affords an instrument for calculating various probabilities for the lives in question. The following examples may be verified from Table 8:

$$q_{24}^{(1)} = \frac{299}{901{,}020}$$

$$q_{26}^{(T)} = \frac{314 + 86{,}632}{807{,}959}$$

$$_3p_{26}^{(T)} = \frac{497{,}620}{721{,}013}$$

$$_2q_{26}^{(2)} = \frac{80{,}385 + 74{,}117}{721{,}013}$$

$$_{2|}q_{27}^{(1)} = \frac{324}{640{,}304}$$

3. Central rates of decrement

The central rate of decrement from all causes at age x is defined as

$$m_x^{(T)} = \frac{d_x^{(T)}}{L_x^{(T)}} \tag{14.8}$$

where

$$L_x^{(T)} = \int_0^1 l_{x+t}^{(T)}\, dt. \tag{14.9}$$

This function is analogous to the central death rate m_x in the mortality table.

The central rate of decrement from cause (k) is

$$m_x^{(k)} = \frac{d_x^{(k)}}{L_x^{(T)}}. \tag{14.10}$$

It is evident that

$$\sum_{k=1}^{m} m_x^{(k)} = m_x^{(T)}. \tag{14.11}$$

In order to evaluate $m_x^{(k)}$, it is convenient to assume that the total decrement at each age is uniformly distributed over the year of age. This assumption is equivalent to the approximation

$$l_{x+t}^{(T)} \doteq l_x^{(T)} - td_x^{(T)}, \qquad 0 < t < 1 \tag{14.12}$$

whence

$$L_x^{(T)} = \int_0^1 l_{x+t}^{(T)}\,dt \doteq \int_0^1 (l_x^{(T)} - t d_x^{(T)})\,dt$$

$$= l_x^{(T)} - \tfrac{1}{2}d_x^{(T)}. \tag{14.13}$$

We then have

$$m_x^{(k)} \doteq \frac{d_x^{(k)}}{l_x^{(T)} - \frac{1}{2}d_x^{(T)}}. \tag{14.14}$$

From this it follows that $m_x^{(k)}$ can be expressed in terms of probabilities of decrement as

$$m_x^{(k)} \doteq \frac{q_x^{(k)}}{1 - \frac{1}{2}q_x^{(T)}}. \tag{14.15}$$

The experience obtained from a body of lives subject to several causes of decrement is often summarized in the form of central rates, and it becomes important to have a method for constructing the multiple-decrement table when the central rates of decrement are given for each cause.

From (14.13), we have $l_x^{(T)} \doteq L_x^{(T)} + \frac{1}{2}d_x^{(T)}$. Then

$$q_x^{(k)} \doteq \frac{d_x^{(k)}}{L_x^{(T)} + \frac{1}{2}d_x^{(T)}} = \frac{m_x^{(k)}}{1 + \frac{1}{2}m_x^{(T)}}, \tag{14.16}$$

and similarly

$$p_x^{(T)} \doteq \frac{1 - \frac{1}{2}m_x^{(T)}}{1 + \frac{1}{2}m_x^{(T)}}. \tag{14.17}$$

Formula (14.17) can be used to build up the $l_x^{(T)}$ column of the multiple-decrement table, and the total decrement $d_x^{(T)}$ at each age can then be distributed in proportion to the various central rates $m_x^{(k)}$:

$$d_x^{(k)} = d_x^{(T)} \cdot \frac{m_x^{(k)}}{m_x^{(T)}}.$$

4. Forces of decrement

In a multiple-decrement table, the total force of decrement at age x is defined as

$$\mu_x^{(T)} = \lim_{h \to 0} \frac{{}_hq_x^{(T)}}{h} \tag{14.18a}$$

$$= -\frac{1}{l_x^{(T)}} \cdot \frac{dl_x^{(T)}}{dx} \tag{14.18b}$$

$$= -\frac{d \log l_x^{(T)}}{dx}. \tag{14.18c}$$

This function represents the force of decrement from all causes combined, and is mathematically the same kind of function as the force of mortality in a single-decrement mortality table. The following relations may be derived in the same way as in Chapter 1:

$$l_x^{(T)} = l_0^{(T)} e^{-\int_0^x \mu_y^{(T)}\, dy} \tag{14.19}$$

$$_np_x^{(T)} = e^{-\int_x^{x+n} \mu_y^{(T)}\, dy} = e^{-\int_0^n \mu_{x+t}^{(T)}\, dt}. \tag{14.20}$$

In order to define the forces of decrement for each of the individual causes, we first introduce the new functions

$$l_x^{(k)} = \sum_{y=x}^{\infty} d_y^{(k)},\ k = 1, 2, \cdots, m. \tag{14.21a}$$

It will be seen that $l_x^{(k)}$ represents the number of lives at age x that will eventually be removed by cause (k). We now seek to define a force of decrement $\mu^{(k)}$ for each cause (k) such that

$$\int_x^{\infty} l_y^{(T)} \mu_y^{(k)}\, dy = l_x^{(k)}, \qquad k = 1, 2, \cdots, m. \tag{14.21b}$$

Differentiating with respect to x, we find

$$-l_x^{(T)} \mu_x^{(k)} = \frac{dl_x^{(k)}}{dx},$$

whence

$$\mu_x^{(k)} = -\frac{1}{l_x^{(T)}} \cdot \frac{dl_x^{(k)}}{dx}. \tag{14.22}$$

This is taken as the definition of $\mu_x^{(k)}$. It is important to observe that the denominator in (14.22) is *not* $l_x^{(k)}$ but $l_x^{(T)}$. We now have

$$d_x^{(k)} = \int_0^1 l_{x+t}^{(T)} \mu_{x+t}^{(k)}\, dt \tag{14.23}$$

and

$$_nq_x^{(k)} = \int_0^n {}_tp_x^{(T)} \mu_{x+t}^{(k)}\, dt. \tag{14.24}$$

A simple relationship exists between the total force $\mu_x^{(T)}$ and the individual forces $\mu_x^{(k)}$. Since

$$l_x^{(T)} = \sum_{k=1}^{m} l_x^{(k)},$$

it follows from (14.18b) and (14.22) that

$$\mu_x^{(T)} = \sum_{k=1}^{m} \mu_x^{(k)}. \tag{14.25}$$

Thus, the total force of decrement is equal to the sum of the several partial forces.

An important distinction between forces of decrement and probabilities of decrement should be noted. The probability functions involve a certain interval of time during which the body of lives is being depleted by all the causes of decrement. Hence, the number of lives which drop out in the interval on account of cause (k) is not independent of the magnitude of the decrements due to the competing causes. The greater the effect of these other forces, the fewer will be the decrements on account of cause (k), and the smaller will be the value of the probability of decrement from this cause. The probability values for the various causes thus depend on one another, and probabilities of decrement must be regarded as *dependent* probabilities in any situation in which several causes are combined.

This may be seen mathematically by writing the probability of decrement from cause (k) as

$$q_x^{(k)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(k)}\, dt$$

and substituting for ${}_tp_x^{(T)}$ from (14.20):

$$q_x^{(k)} = \int_0^1 e^{-\int_0^t \mu_{x+s}^{(T)}\, ds}\, \mu_{x+t}^{(k)}\, dt.$$

In this form, it is clear that the probability of decrement for any individual cause (k) depends upon the forces of decrement for *all* the causes.

On the other hand, the function $\mu_x^{(k)}$, being an instantaneous rate of decrement, is not based upon any time *interval*, and is not affected by the operation of competing causes. The forces of decrement for the various causes are therefore independent functions, in contrast to the probabilities of decrement, which are dependent upon one another.

When numerical values of $\mu_x^{(k)}$ are needed, they can be estimated from the multiple-decrement table by means of an approximate formula. The following three formulas may be derived under the same assumptions as formulas (1.20), (1.21), and (1.22) for the force of mortality μ_x :

$$\mu_x^{(k)} \doteq \frac{d_{x-1}^{(k)} + d_x^{(k)}}{2l_x^{(T)}} \tag{14.26}$$

$$\mu_x^{(k)} \doteq \frac{7(d_{x-1}^{(k)} + d_x^{(k)}) - (d_{x-2}^{(k)} + d_{x+1}^{(k)})}{12l_x^{(T)}} \tag{14.27}$$

$$\mu_x^{(k)} \doteq \frac{1}{l_x^{(T)}} (d_x^{(k)} - \tfrac{1}{2}\Delta d_x^{(k)} + \tfrac{1}{3}\Delta^2 d_x^{(k)} - \cdots) \tag{14.28}$$

The formulas are given in terms of $d_x^{(k)}$ instead of $l_x^{(k)}$ since the values of $l_x^{(k)}$ are not usually shown in the multiple-decrement table.

5. The associated single-decrement tables

For each of the causes in a multiple-decrement table, it is possible to define a single-decrement table which exhibits the properties of the cause in an independent form. If we have, for example, a double-decrement table for mortality and disability, we can obtain a single-decrement table for mortality and one for disability. Each table will be based upon its own force of decrement, which is independent of the operation of the other cause. In general, for each cause (k), we take $l_0'^{(k)}$ as the radix of an associated single-decrement table and define the functions

$$l_x'^{(k)} = l_0'^{(k)} e^{-\int_0^x \mu_y^{(k)} dy} \tag{14.29}$$

and

$$q_x'^{(k)} = 1 - e^{-\int_0^1 \mu_{x+t}^{(k)} dt} \tag{14.30}$$

These expressions will be recognized as the standard relations which hold in any single-decrement table. When the single cause of decrement is death, for example, the functions $l_x'^{(k)}$ and $q_x'^{(k)}$ are merely the familiar l_x and q_x of the mortality table.

In the single-decrement table, the function $q_x'^{(k)}$ is both an annual rate and a probability of decrement. In the context of the multiple-decrement table, however, $q_x'^{(k)}$ is solely a rate of decrement and must be distinguished from the probability $q_x^{(k)}$. When several decrements are operating simultaneously, we shall see that the rate of decrement for a given cause is greater than the corresponding probability of decrement. In this book, the expression *rate of decrement* will always refer to the function $q_x'^{(k)}$ and will not be used as an abbreviation for *central rate of decrement*. The function $q_x'^{(k)}$ has often been called the *absolute* rate of decrement in other actuarial literature.

Since formulas (14.29) and (14.30) cannot normally be applied directly, the values of the rate of decrement $q_x'^{(k)}$ are usually determined by approximation methods from the data given in the multiple-decrement table.

The traditional formula may be obtained from the following general argument. In a single-decrement table, when the rate of decrement for a given age is multiplied by the number of lives attaining that age, the result is the number of lives which fail during the year as shown in the table; e.g., $q_x \cdot l_x = d_x$. In a multiple-decrement table, however, if the rate of decrement $q_x'^{(k)}$ is multiplied by $l_x^{(T)}$, the resulting number of lives exceeds the value of $d_x^{(k)}$ shown in the table, since some of the $l_x^{(T)}$ lives are not exposed to cause (k) for a full year due to their removal by one of the other causes. The number of lives which are removed during the year from other causes is $d_x^{(T)} - d_x^{(k)}$, which we denote by $d_x^{(-k)}$. If it is assumed that these lives are exposed to cause (k) for one-half year on the average before exit, the loss in exposure is the equivalent of $\frac{1}{2}d_x^{(-k)}$ lives. Hence, in order to obtain the decrement $d_x^{(k)}$ shown in the multiple-decrement table, the rate $q_x'^{(k)}$ should be multiplied by $l_x^{(T)} - \frac{1}{2}d_x^{(-k)}$, which is the average exposure. We thus have

$$q_x'^{(k)}\,[l_x^{(T)} - \tfrac{1}{2}d_x^{(-k)}] \doteq d_x^{(k)},$$

whence

$$q_x'^{(k)} \doteq \frac{d_x^{(k)}}{l_x^{(T)} - \frac{1}{2}d_x^{(-k)}} \tag{14.31a}$$

$$= \frac{q_x^{(k)}}{1 - \frac{1}{2}q_x^{(-k)}}, \tag{14.31b}$$

where we define $q_x^{(-k)} = q_x^{(T)} - q_x^{(k)}$.

For the special case of two decrements, the formulas are

$$q_x^{\prime(1)} \doteq \frac{q_x^{(1)}}{1 - \frac{1}{2}q_x^{(2)}} \quad \text{and} \quad q_x^{\prime(2)} \doteq \frac{q_x^{(2)}}{1 - \frac{1}{2}q_x^{(1)}}. \tag{14.32}$$

Other formulas can be obtained on the basis of more formal mathematical assumptions. If we assume that each decrement $d_x^{(k)}$ is uniformly distributed over the year of age, we have

$$l_{x+t}^{(k)} \doteq l_x^{(k)} - t \cdot d_x^{(k)}$$

for integral x and $0 < t < 1$.

It follows that

$$l_{x+t}^{(T)} \doteq l_x^{(T)} - t \cdot d_x^{(T)} \tag{14.33}$$

and

$$\frac{dl_{x+t}^{(k)}}{dt} \doteq -d_x^{(k)},$$

and therefore

$$\mu_{x+t}^{(k)} = -\frac{1}{l_{x+t}^{(T)}} \frac{dl_{x+t}^{(k)}}{dt} \doteq \frac{d_x^{(k)}}{l_{x+t}^{(T)}}. \tag{14.34}$$

An approximate formula for $q_x^{\prime(k)}$ can now be derived by using this assumption in formula (14.30). From (14.33) and (14.34) we have

$$\int_0^1 \mu_{x+t}^{(k)}\,dt \doteq \int_0^1 \frac{d_x^{(k)}}{l_x^{(T)} - t \cdot d_x^{(T)}}\,dt$$

$$= \left[-\frac{d_x^{(k)}}{d_x^{(T)}} \cdot \log\left(l_x^{(T)} - t \cdot d_x^{(T)}\right) \right]_0^1$$

$$= -\frac{d_x^{(k)}}{d_x^{(T)}} \log p_x^{(T)}.$$

Substituting this result in (14.30), we obtain

$$q_x^{\prime(k)} \doteq 1 - \left(p_x^{(T)}\right)^{\frac{d_x^{(k)}}{d_x^{(T)}}}. \tag{14.35}$$

To obtain a formula for practical use, we write the right-hand side as

$$1 - \left(1 - q_x^{(T)}\right)^{\frac{d_x^{(k)}}{d_x^{(T)}}},$$

and expand the binomial:

$$1 - \left\{1 - \frac{d_x^{(k)}}{d_x^{(T)}} \cdot q_x^{(T)} + \tfrac{1}{2} \cdot \frac{d_x^{(k)}}{d_x^{(T)}} \left(\frac{d_x^{(k)}}{d_x^{(T)}} - 1\right) (q_x^{(T)})^2 - \cdots \right\}.$$

Retaining only the terms indicated and simplifying, we have

$$\begin{aligned} q_x^{\prime(k)} &\doteq q_x^{(k)}[1 + \tfrac{1}{2}(q_x^{(T)} - q_x^{(k)})] \\ &= q_x^{(k)}(1 + \tfrac{1}{2} q_x^{(-k)}). \end{aligned} \tag{14.36}$$

Formulas (14.31b) and (14.36) are easily seen to be substantially equivalent. Writing (14.31b) as

$$\begin{aligned} q_x^{\prime(k)} &\doteq q_x^{(k)} \ (1 - \tfrac{1}{2} q_x^{(-k)})^{-1} \\ &= q_x^{(k)} \ (1 + \tfrac{1}{2} q_x^{(-k)} + \cdots), \end{aligned}$$

we obtain (14.36) by truncating the binomial expansion.

6. Construction of a multiple-decrement table from the associated single-decrement tables

We have seen that a given multiple-decrement table may be resolved into a family of associated single-decrement tables. We now consider the inverse problem of constructing a multiple-decrement table based on a given family of single-decrement tables.

Let us assume that m single-decrement tables are given, each representing a different cause of decrement, and that we wish to obtain a multiple-decrement table embodying these m causes of decrement. The following relation holds between the required parent table and the given family of tables:

$$\mu_x^{(T)} = \mu_x^{(1)} + \mu_x^{(2)} + \cdots + \mu_x^{(m)}.$$

Since, from (14.20), $p_x^{(T)} = e^{-\int_0^1 \mu_{x+t}^{(T)} dt}$, it follows that

$$\begin{aligned} p_x^{(T)} &= e^{-\int_0^1 (\mu_{x+t}^{(1)} + \mu_{x+t}^{(2)} + \cdots + \mu_{x+t}^{(m)}) \, dt} \\ &= e^{-\int_0^1 \mu_{x+t}^{(1)} dt} \cdot e^{-\int_0^1 \mu_{x+t}^{(2)} dt} \cdot \ \cdots \ \cdot e^{-\int_0^1 \mu_{x+t}^{(m)} dt} \\ &= p_x^{\prime(1)} \cdot p_x^{\prime(2)} \cdot \ \cdots \ \cdot p_x^{\prime(m)}, \end{aligned} \tag{14.37a}$$

where $p_x^{\prime(k)} = 1 - q_x^{\prime(k)}$.

Hence,

$$l_x^{(T)} = k l_x^{\prime(1)} \cdot l_x^{\prime(2)} \cdot \ \cdots \ \cdot l_x^{\prime(m)}, \tag{14.37b}$$

where k is a constant of proportionality.

Although (14.37b) provides an exact basis for constructing $l_x^{(T)}$, the individual decrements at each age,

$$d_x^{(k)} = \int_0^1 l_{x+t}^{(T)}\mu_{x+t}^{(k)}\,dt,$$

must normally be computed by a method of approximation.

A device which is particularly convenient for the special case of two decrements is based on (14.32). In the present instance, the values of the rates $q_x'^{(1)}$ and $q_x'^{(2)}$ are known from the given single-decrement tables, and we desire to find the values of the corresponding probabilities $q_x^{(1)}$ and $q_x^{(2)}$. The relations in (14.32) lead to the following system of simultaneous linear equations in $q_x^{(1)}$ and $q_x^{(2)}$:

$$\begin{aligned} q_x^{(1)} + \tfrac{1}{2}q_x'^{(1)}\cdot q_x^{(2)} &\doteq q_x'^{(1)} \\ \tfrac{1}{2}q_x'^{(2)}\cdot q_x^{(1)} + q_x^{(2)} &\doteq q_x'^{(2)}. \end{aligned}$$

Solving, we find

$$\begin{aligned} q_x^{(1)} &\doteq \frac{q_x'^{(1)}(1 - \frac{1}{2}q_x'^{(2)})}{1 - \frac{1}{4}q_x'^{(1)}q_x'^{(2)}} \\ q_x^{(2)} &\doteq \frac{q_x'^{(2)}(1 - \frac{1}{2}q_x'^{(1)})}{1 - \frac{1}{4}q_x'^{(1)}q_x'^{(2)}}. \end{aligned} \tag{14.38}$$

If $d_x^{(1)}$ and $d_x^{(2)}$ are now computed as $q_x^{(1)}\cdot l_x^{(T)}$ and $q_x^{(2)}\cdot l_x^{(T)}$, the total $d_x^{(1)} + d_x^{(2)}$ will not always agree with the value $d_x^{(T)}$ given by $l_x^{(T)} - l_{x+1}^{(T)}$, where $l_x^{(T)}$ has been found from the exact formula (14.37b). Hence it is better to compute $d_x^{(1)}$ and $d_x^{(2)}$ by distributing the exact value of $d_x^{(T)}$ in proportion to the approximate values of $q_x^{(1)}$ and $q_x^{(2)}$ from (14.38); that is, by using

$$d_x^{(k)} = \frac{q_x^{(k)}}{q_x^{(1)} + q_x^{(2)}}\cdot d_x^{(T)}, \qquad k = 1, 2.$$

The extension of this method to the cases of three or more decrements is inconvenient since it requires the solution of larger systems of linear equations.

Another method, which is easily applied for any number of decrements, is to translate the rates of decrement into central rates. The central rates can be readily estimated. Since

$$l_x^{(T)} - \tfrac{1}{2}d_x^{(T)} \doteq L_x^{(T)},$$

formula (14.31a) may be written as

$$q_x'^{(k)} \doteq \frac{d_x^{(k)}}{L_x^{(T)} + \frac{1}{2}d_x^{(k)}} = \frac{m_x^{(k)}}{1 + \frac{1}{2}m_x^{(k)}} \tag{14.39}$$

and solving for $m_x^{(k)}$,

$$m_x^{(k)} \doteq \frac{q_x'^{(k)}}{1 - \frac{1}{2}q_x'^{(k)}} . \tag{14.40}$$

The exact value of $d_x^{(T)}$ can then be distributed in proportion to the central rates.

It will be noted that some type of approximation is almost always necessary in carrying out the methods described in this chapter. For convenient reference, the basic formulas are collected in Table 9.

7. The analogy with joint-life functions

The student should realize that he has already encountered a mathematical model of the same type as a multiple-decrement table in his study of joint-life functions. Just as $l_x^{(T)}$ represents a number of lives subject to m causes of decrement, so the joint-life function $l_{x_1x_2\cdots x_m}$ may be interpreted as a number of life-groups, each containing m lives, and, since the death of any one of the m lives in a group is sufficient to cause the failure of the group, we may say that the function is subject to m causes of decrement, namely the death of (x_1), the death of (x_2), $\cdots$, the death of (x_m). The number of failures during the year, denoted by $d_{x_1x_2\cdots x_m}$, corresponds to the total decrement function $d_x^{(T)}$.

In constructing a joint-life mortality table, we begin with a single-life table and use the relation

$$l_{x_1x_2\cdots x_m} = kl_{x_1} \cdot l_{x_2} \cdots l_{x_m} .$$

Similarly, the multiple-decrement table may be constructed from the associated single-decrement tables by using the relation

$$l_x^{(T)} = kl_x'^{(1)} \cdot l_x'^{(2)} \cdots l_x'^{(m)}.$$

Also, corresponding to the joint-life formula

$$\mu_{x_1x_2\cdots x_m} = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m} ,$$

TABLE 9

THE STANDARD APPROXIMATION FORMULAS

Formula for	In terms of $q_x^{(k)}$	In terms of $m_x^{(k)}$	In terms of $q_x'^{(k)}$
Probability $q_x^{(k)}$	$q_x^{(k)}$	$q_x^{(k)} \doteq \dfrac{m_x^{(k)}}{1 + \frac{1}{2}m_x^{(T)}}$ (14.16)	For two decrements only: $q_x^{(k)} \doteq \dfrac{q_x'^{(k)}(1 - \frac{1}{2}q_x'^{(-k)})}{1 - \frac{1}{4}q_x'^{(k)}\, q_x'^{(-k)}}$ (14.38)
Central rate $m_x^{(k)}$	$m_x^{(k)} \doteq \dfrac{q_x^{(k)}}{1 - \frac{1}{2}q_x^{(T)}}$ (14.15)	$m_x^{(k)}$	$m_x^{(k)} \doteq \dfrac{q_x'^{(k)}}{1 - \frac{1}{2}q_x'^{(k)}}$ (14.40)
Rate $q_x'^{(k)}$	$q_x'^{(k)} \doteq \dfrac{q_x^{(k)}}{1 - \frac{1}{2}q_x^{(-k)}}$ (14.31b)	$q_x'^{(k)} \doteq \dfrac{m_x^{(k)}}{1 + \frac{1}{2}m_x^{(k)}}$ (14.39)	$q_x'^{(k)}$

we have the multiple-decrement relation

$$\mu_x^{(T)} = \mu_x^{(1)} + \mu_x^{(2)} + \cdots + \mu_x^{(m)}.$$

In the joint-life mortality table, it is not customary to tabulate the number of failures due to the death of a specified life. However, if it were desired to express the number of failures in a year due to the death of (x_1), the formula would be

$$d^{1}_{x_1 x_2 \cdots x_m} = \int_0^1 l_{x_1+t:x_2+t:\cdots:x_m+t}\mu_{x_1+t}\, dt,$$

just as the number of decrements due to cause (1) is given by

$$d_x^{(1)} = \int_0^1 l_{x+t}^{(T)}\mu_{x+t}^{(1)}\, dt.$$

Also, the contingent probability

$$q^{1}_{x_1 x_2 \cdots x_m} = \int_0^1 {}_tp_{x_1 x_2 \cdots x_m}\mu_{x_1+t}\, dt,$$

is analogous to the multiple-decrement probability

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(T)}\mu_{x+t}^{(1)}\, dt.$$

8. Monetary applications

The probabilities of the multiple-decrement table may be combined with compound interest functions to produce monetary values in accordance with the usual principles. The process is facilitated by the definition of special commutation functions.

$$\text{Let } v^x l_x^{(T)} = D_x^{(T)} \quad \text{and} \quad \sum_{t=0}^{\infty} D_{x+t}^{(T)} = N_x^{(T)}.$$

Then the present value at age x of a series of annual payments of 1 continuing as long as (x) remains in the body of lives defined by the multiple-decrement table may be expressed as

$$\ddot{a}_x^{(T)} = \frac{N_x^{(T)}}{D_x^{(T)}}.$$

Let $v^{x+1}d_x^{(k)} = C_x^{(k)}$ and $\sum_{t=0}^{\infty} C_{x+t}^{(k)} = M_x^{(k)}$. Then the present value at age x of a payment of 1 to be made at the end of the year in which (x) leaves the body of lives as a result of cause (k) is given by

$$A_x^{(k)} = \frac{M_x^{(k)}}{D_x^{(T)}}.$$

If this benefit is payable immediately instead of at the end of the year, we have

$$\bar{A}_x^{(k)} = \frac{\bar{M}_x^{(k)}}{D_x^{(T)}}$$

with the usual approximation $\bar{M}_x^{(k)} \doteq \frac{i}{\delta} M_x^{(k)}$.

Life insurance policies often include a provision for an additional payment in the event of accidental death. The theoretical model for evaluating this supplementary benefit is a double-decrement table showing the number of accidental deaths d_x^{ad} and the number of deaths from all other causes $d_x^{\overline{ad}}$. We define

$$C_x^{ad} = v^{x+1} d_x^{ad}$$

and

$$M_x^{ad} = \sum_{t=0}^{\infty} C_{x+t}^{ad}.$$

Then the present value at age x of a payment of 1 at the end of the year in which accidental death occurs is

$$A_x^{ad} = \frac{M_x^{ad}}{D_x^{(T)}}.$$

The net annual premium payable for m years for an accidental death benefit providing coverage for n years is

$${}_mP^{1\ ad}_{x:\overline{n}|} = \frac{M_x^{ad} - M_{x+n}^{ad}}{N_x^{(T)} - N_{x+m}^{(T)}}.$$

In practice, a special kind of approximation is often used in obtaining net premiums for the accidental death benefit. If the values of the probability of accidental death, q_x^{ad}, are available, the construction of the double-decrement table can be avoided. In place of $D_x^{(T)}$ and $N_x^{(T)}$, the functions D_x and N_x from a standard mortality table are used. The values of C_x^{ad} are then obtained from the formula

$$C_x^{ad} \doteq vq_x^{ad} D_x .$$

Functions computed in this way are said to be based upon the accidental death table combined with the standard mortality table.

Special methods have been developed in two other important fields of application, pension plans and disability benefits. They will be taken up in Chapter 16.

9. Notation

At the present time there is no generally accepted standard notation for multiple-decrement functions. The usual practice is to assign symbols according to the nature of the problem under consideration. If the cause of decrement is withdrawal, for example, the symbol w_x may be used in place of $d_x^{(w)}$. Mortality is normally one of the decrements considered, and the symbol d_x is often used for $d_x^{(d)}$. When no confusion will arise, the superscribed parentheses are often omitted, and symbols like l_x^T and d_x^k are commonly encountered. In order to accustom the student to this notational flexibility, we shall frequently use alternative symbols in the following chapters. A few notations involving different symbols have already appeared, and others will be encountered in the exercises that follow.

References

A general treatment of multiple-decrement theory is given by Bailey and Haycocks (1946) and by Hooker and Longley-Cook (1957). Papers which discuss the subject from a special point of view are Nesbitt and Van Eenam (1948), Bicknell and Nesbitt (1956), and Hickman (1964).

Greville (1948, 1954) has discussed a special type of multiple-decrement table in which the decrements are deaths arising from different causes.

4. Methods of approximating the forces of decrement are given by Menge (1932).

6. The problem of this section is discussed by Gershenson (1957).

8. Values of the accidental death functions q_x^{ad} and C_x^{ad}, based upon recent insurance experience, have been tabulated by Brodie and November (1959).

EXERCISES

1. Introduction; 2. Probabilities of decrement

1. Compute from Table 8
   (a) the probability that (26) will leave the body of lives as a result of cause (1) at age 29 last birthday;
   (b) the probability that (24) will attain age 26 in the body of lives and will then leave as a result of cause (2) within the succeeding 2 years.

2. A two-year junior college expects that its students will leave in accordance with the following probabilities:

Class	Probability of	
	Academic Failure	Withdrawal for Other Causes
First year	.10	.30
Second year	.05	.20

   (a) How many new students should be admitted at the beginning of each school year if 180 graduates are desired at the end of each year?
   (b) How many new students should be admitted at the beginning of each school year if the total enrollment then is to be 800 students?

3. Central rates of decrement

3. In Table 8, estimate the value of the central rate of decrement from cause (2) at age 25.

4. Extend the data of Table 8 to age 30 being given that $m_{30}^{(1)} = .0008$ and $m_{30}^{(2)} = .1374$. Use standard approximations.

5. Suppose that for the first year following the training period of pilots the central death and withdrawal rates are a and b respectively. What is the probability that a pilot who has just finished training will still be active as a pilot at the end of the first year? Use standard approximations.

4. Forces of decrement

6. In a double-decrement table, it is known that $\mu_{x+t}^{(1)} = j$ and $\mu_{x+t}^{(2)} = k$, $0 \leqslant t \leqslant 1$, where j and k are constants. Compute the exact value of $q_x^{(1)}$.

7. In a multiple-decrement table associated with a certain body of lives there are three forces of decrement, $\mu_x^{(1)}$, $\mu_x^{(2)}$, and $\mu_x^{(3)}$, where $\mu_x^{(k)}$ is equal to $\dfrac{3}{11\,k(100 - x)}$. What is the probability that a life aged 10 will remain in the body of lives until age 60? (Ans. $\frac{2}{3}$)

8. Given $l_x^{(T)} = (a - x^2)e^{-x}$ and $d_x^{(1)} = 2[e^{-x}(x + 1) - e^{-x-1}(x + 2)]$, derive an exact algebraic expression for $\mu_x^{(1)}$.

9. Given $\mu_x^{(1)} = \dfrac{1}{a - x}$ and $\mu_x^{(2)} = 1$, derive exact algebraic expressions for $l_x^{(T)}$, $d_x^{(1)}$, and $d_x^{(2)}$, assuming $l_0^{(T)} = a$.

5. The associated single-decrement tables

10. Consider the associated single-decrement tables corresponding to Table 8. Assuming $l_{24}'^{(1)} = l_{24}'^{(2)} = l_{24}^{(T)}$, compute values for $d_{24}'^{(1)}$ and $d_{24}'^{(2)}$. Use standard approximations.

11. A body of lives is subject to two causes of decrement. Using standard approximations, express $q_x^{(1)}$ in terms of $q_x'^{(1)}$ and $q_x^{(T)}$.

12. In a certain double-decrement table involving deaths d_x and withdrawals w_x, the decrements are distributed throughout the year of age in such a way as to produce the following formulas for the rates of decrement:

$$q_x'^{(d)} = \frac{d_x}{l_x^{(T)} - .8w_x}, \qquad q_x'^{(w)} = \frac{w_x}{l_x^{(T)} - .5d_x}.$$

For a certain age x, $l_x^{(T)} = 1000$, $l_{x+1}^{(T)} = 900$, and $q_x'^{(w)} = .080$. Find the value of $q_x'^{(d)}$. (Ans. .022)

6. Construction of a multiple-decrement table from the associated single-decrement tables

13. Two single-decrement tables are defined by $l_x'^{(1)} = a - x$ and $l_x'^{(2)} = ce^{-x}$. Find expressions for $l_x^{(T)}$, $d_x^{(1)}$, and $d_x^{(2)}$ in the corresponding double-decrement table, assuming $l_0^{(T)} = a$.

14. Given $q_{25}'^{(1)} = 2\%61$, $q_{25}'^{(2)} = 2/201$, $q_{25}'^{(3)} = 6/403$, and $q_{25}'^{(4)} = 14/407$, compute values for $d_{25}^{(1)}$, $d_{25}^{(2)}$, $d_{25}^{(3)}$, and $d_{25}^{(4)}$, assuming $l_{25}^{(T)} = 10{,}000$. Use standard approximations.

15. A company which employs a large number of female office workers has just constructed a service table showing the number of females who attain age x in employment. The workers must resign when they marry. They also withdraw for other reasons. The following values are given: $l_{20}^{(T)} = 10{,}000$, $l_{21}^{(T)} = 8600$, $l_{22}^{(T)} = 7400$. If the rate of marriage for each of these ages is 10% and if the central rate of mortality for each age is 1%, calculate (using standard approximations) the number of deaths, marriages, and other withdrawals for ages 20 and 21.

16. If $q_x'^{(1)} = .080$ and $m_x^{(2)} = 3\,m_x^{(1)}$ in a double-decrement table, compute the value of $q_x^{(T)}$ using standard approximations. (Ans. .286)

7. The analogy with joint-life functions

17. (a) What is the multiple-decrement analogue of the joint-life relation $p_{x_1 x_2 \ldots x_m} = p_{x_1} \cdot p_{x_2} \cdots p_{x_m}$?
(b) What is the joint-life analogue of the multiple-decrement formula $m_x^{(1)} = \dfrac{d_x^{(1)}}{L_x^{(T)}}$?

8. Monetary applications

18. Given a double-decrement table showing decrements by death $d_x^{(d)}$ and by disability $d_x^{(i)}$. The net annual premium at age 52 is required for a disability benefit of 1 which is payable at the end of the year of disability provided that disability occurs prior to age 55.
(a) Express this premium in terms of the given double-decrement functions and interest functions.
(b) Express the premium in terms of appropriate commutation functions.

19. An analysis of the mortality among civilians in a war combat area shows that at a certain age the rate of death resulting from war is .05 and that the central rate of death resulting from war is ten times that of death not resulting from war. Compute the net premium for a \$1000 one-year term policy which would give full coverage at this age irrespective of whether death resulted from war or not. Interest is at 3 per cent.

20. A mutual benefit society is being formed to offer the following benefits:
(i) On withdrawal, return of premiums without interest.
(ii) On death, \$1000 if before retirement, none if after retirement.
(iii) On retirement at age 65, a lump sum of \$500 times the number of years the member has paid premiums.

Assuming that the necessary service table is available, show what commutation columns you would set up to facilitate arriving at the annual premiums for the above benefits. Give the formula for the total annual net premium P at age x assuming P' (gross premium) $= (P + c)(1 + k)$.

21. When an accidental death table is combined with a standard mortality table, show that the terminal reserve ${}^{m}_{t}V^{1\,ad}_{x:\overline{n}|}$ $(t < m)$ can be expressed as

$$\left({}_{m-t}P^{1\,ad}_{x+t:\overline{n-t}|} - {}_{m}P^{1\,ad}_{x:\overline{n}|}\right) \ddot{a}_{x+t:\overline{m-t}|}.$$

Miscellaneous problems

22. Show that $\mu_{x+\frac{1}{2}}^{(k)}$ is equal to $m_x^{(k)}$ on the assumption of a uniform distribution of decrements.

23. Given the probability of death, the rate of retirement, and the central rate of withdrawal, find approximate formulas for the probability of withdrawal and the probability of retirement in terms of the given functions.

24. The following is an incomplete extract from an employees' service table involving two causes of decrement—death and withdrawal:

x	$l_x^{(T)}$	$d_x^{(w)}$	$d_x^{(d)}$	$q_x'^{(d)}$
20	10,000	—	41	—
21	9,100	—	—	.0046
22	8,328	—	—	—

Using standard approximations (where necessary):

(a) find the values for $d_{20}^{(w)}$, $d_{21}^{(w)}$, and $d_{21}^{(d)}$;

(b) find the probability that an individual aged 20 will withdraw within 2 years;

(c) find the central death rate at age 21;

(d) find the rate of withdrawal at age 21;

(e) find the complete two-year work-life expectancy at age 20, $\mathring{e}_{20:\overline{2}|}^{(T)}$.

CHAPTER 15

TABLES WITH SECONDARY DECREMENTS

1. Primary and secondary decrements

In the multiple-decrement tables discussed thus far, each life is observed only until it is removed by the operation of one of the causes of decrement. The model does not include the subsequent history of those lives that are removed by causes other than death. In calculating the values of certain benefits, the actuary must sometimes make assumptions about the subsequent survival of these lives. An example is the calculation of the value of a disability benefit which provides an income to the insured in the event of disability. A possible model for this situation is a double-decrement table for mortality and disability with additional columns showing the effects of mortality and recovery on the lives that become disabled. In this particular table, the original body of lives is said to be subject to two *primary* decrements, mortality and disability, and the disabled lives are subject to two *secondary* decrements, mortality and recovery. The student will readily visualize other situations which require a model of this type.

The simplest form for such a model comprises two primary decrements and a single secondary decrement, as illustrated in Table 10. The primary decrements are death (d) and an unspecified cause (h), and the first three columns thus constitute the usual double-decrement table. Columns (4) and (5) show the effects of mortality after the operation of cause (h), the function $(hl)_x$ being defined as the number of lives attaining age x after leaving the original body of lives as a result of cause (h), and the secondary decrement $(hd)_x$ being defined as the number of lives dying between ages x and $x + 1$ after the operation of cause (h). It should not be thought that these two columns form a single-decrement table, for the function $(hl)_x$ is subject to the increment $d_x^{(h)}$ as well as to the decrement $(hd)_x$. Thus the two sections of the combined table are linked by the function $d_x^{(h)}$, which forms a decrement from $l_x^{(T)}$ and an increment to $(hl)_x$. The relations among these functions are

$$l_x^{(T)} - d_x^{(d)} - d_x^{(h)} = l_{x+1}^{(T)} \qquad (15.1)$$

$$(hl)_x + d_x^{(h)} - (hd)_x = (hl)_{x+1}\,. \tag{15.2}$$

It is instructive to verify these relations for the illustrative numerical values that have been entered in Table 10.

In discussing this model, it is convenient to adopt a special terminology. We shall refer to the lives denoted by $l_x^{(T)}$ as being in state (T) and to the $(hl)_x$ lives as being in state (h). The lives in state (T) are subject to the risk of death and to the risk of transfer to state (h), and the lives in state (h) are subject only to the risk of death. It is assumed in this particular model that re-entry to state (T) from state (h) is impossible.

TABLE 10

Age x	(1)	(2)	(3)	(4)	(5)
	$l_x^{(T)}$	$d_x^{(d)}$	$d_x^{(h)}$	$(hl)_x$	$(hd)_x$
30	32,176	506	2,203	52,066	489
31	29,467	494	2,014	53,780	527

The probabilities for the primary decrements take the usual form:

$$q_x^{(d)} = \frac{d_x^{(d)}}{l_x^{(T)}} \tag{15.3a}$$

$$q_x^{(h)} = \frac{d_x^{(h)}}{l_x^{(T)}}\,. \tag{15.3b}$$

The rates of decrement, by (14.31a), are

$$q_x^{\prime(d)} \doteq \frac{d_x^{(d)}}{l_x^{(T)} - \frac{1}{2}d_x^{(h)}} \tag{15.4a}$$

$$q_x^{\prime(h)} \doteq \frac{d_x^{(h)}}{l_x^{(T)} - \frac{1}{2}d_x^{(d)}}\,. \tag{15.4b}$$

In obtaining the rate of mortality for lives in state (h), which we denote by $(hq)_x$, it must be remembered that $(hl)_x$ is subject both to the decrement $(hd)_x$ and to the increment $d_x^{(h)}$. A multiple-decrement formula must therefore be used with $d_x^{(h)}$ taken as a negative decrement. With this adjustment, formula (14.31a) gives

$$(hq)_x \doteq \frac{(hd)_x}{(hl)_x + \frac{1}{2}d_x^{(h)}}. \tag{15.5}$$

This result is clearly a reasonable one, for the $(hl)_x$ lives are exposed to the risk of death for a full year and the $d_x^{(h)}$ lives are exposed in state (h) for about one-half year on the average.

Consider the probability that a life aged x in state (T) will transfer to state (h) and die before attaining age $x + 1$. We will denote this function by $q_x^{(Th)}$. Here we wish to know how many of the lives shown in the table as dying in state (h) between ages x and $x + 1$ were in state (T) at the beginning of that year of age. The total number of lives dying in state (h) at that age is $(hd)_x$, and this is made up of deaths among the $(hl)_x$ lives attaining age x together with the deaths occurring among the $d_x^{(h)}$ lives which transferred to state (h) during the year. It is these latter deaths to which the probability $q_x^{(Th)}$ refers. The rate of mortality for lives in state (h) is $(hq)_x$, and the number of deaths during the year among the $(hl)_x$ lives is given by $(hl)_x \cdot (hq)_x$. It follows that the number of deaths among the $d_x^{(h)}$ lives is

$$(hd)_x - (hl)_x \cdot (hq)_x ,$$

and the required probability is therefore

$$q_x^{(Th)} = \frac{(hd)_x - (hl)_x \cdot (hq)_x}{l_x^{(T)}}. \tag{15.6}$$

There is an alternative approach to the value of $q_x^{(Th)}$. The probability that a life aged (x) in state (T) will transfer to state (h) within one year and then die before the end of that year may be expressed as

$$q_x^{(Th)} = \int_0^1 {}_tp_x^{(T)} \mu_{x+t}^{(h)} \cdot {}_{1-t}(hq)_{x+t}\, dt,$$

where $\mu_x^{(h)}$ is the force of decrement for cause (h). Using the Balducci hypothesis,

$${}_{1-t}(hq)_{x+t} \doteq (1 - t) \cdot (hq)_x ,$$

and assuming a uniform distribution of primary decrements so that

$${}_tp_x^{(T)} \mu_{x+t}^{(h)} \doteq \frac{d_x^{(h)}}{l_x^{(T)}},$$

we have

$$q_x^{(Th)} \doteq \int_0^1 \frac{d_x^{(h)}}{l_x^{(T)}} \cdot (1 - t) \cdot (hq)_x \, dt$$

$$= \tfrac{1}{2} \cdot \frac{d_x^{(h)}}{l_x^{(T)}} \cdot (hq)_x$$

$$= \tfrac{1}{2} \cdot q_x^{(h)} \cdot (hq)_x . \tag{15.7}$$

The result appears reasonable, since $q_x^{(h)}$ is the probability of transfer to state (h), and the lives entering state (h) are then subject to the risk of death for approximately half a year on the average before attaining age $x + 1$.

Although (15.6) is an exact formula and (15.7) an approximation if $(hq)_x$ is known, in practice $(hq)_x$ must be estimated from (15.5), so that both formulas produce an approximate result. It can easily be demonstrated that the substitution of (15.5) for $(hq)_x$ leads to the same algebraic result in both (15.6) and (15.7).

Consider next the probability $p_x^{(Th)}$ that a life aged x in state (T) will be in state (h) upon attaining age $x + 1$. Of the $(hl)_{x+1}$ lives in state (h) at age $x + 1$, $(hl)_x \cdot (hp)_x$ have been in that state for the entire past year, where $(hp)_x = 1 - (hq)_x$. The remaining lives, $(hl)_{x+1} - (hl)_x \cdot (hp)_x$, must have attained age $x + 1$ in state (h) after having been in state (T) at age x, and it follows that the required probability is

$$p_x^{(Th)} = \frac{(hl)_{x+1} - (hl)_x \cdot (hp)_x}{l_x^{(T)}} . \tag{15.8}$$

The formula may also be derived by noting that

$$p_x^{(Th)} = q_x^{(h)} - q_x^{(Th)} \tag{15.9}$$

and substituting for $q_x^{(Th)}$ from (15.6). If we substitute for $q_x^{(Th)}$ from (15.7), the alternative formula

$$p_x^{(Th)} \doteq q_x^{(h)}[1 - \tfrac{1}{2}(hq)_x] \tag{15.10}$$

is obtained.

The probability that a life (x) in state (T) will attain age $x + n$ in state (h) may be expressed as

$$ {}_np_x^{(Th)} = \frac{(hl)_{x+n} - (hl)_x \cdot {}_n(hp)_x}{l_x^{(T)}} \tag{15.11} $$

by an obvious extension of formula (15.8). Here the function ${}_n(hp)_x$ is given by

$$ \begin{aligned} {}_n(hp)_x &= (hp)_x \cdot (hp)_{x+1} \cdot \cdots \cdot (hp)_{x+n-1} \\ &= [1 - (hq)_x][1 - (hq)_{x+1}] \cdots \\ &\qquad [1 - (hq)_{x+n-1}]. \end{aligned} \tag{15.12} $$

A number of other functions can be similarly expressed. For example, the symbol ${}_{n|}q_x^{(Th)}$ denotes the probability that a life aged x in state (T) will die in state (h) between ages $x + n$ and $x + n + 1$, and

$$ {}_{n|}q_x^{(Th)} = \frac{(hd)_{x+n} - (hl)_x \cdot {}_{n|}(hq)_x}{l_x^{(T)}}, \tag{15.13} $$

where ${}_{n|}(hq)_x = {}_n(hp)_x - {}_{n+1}(hp)_x$. (15.14)

In the sections that follow, we shall consider two specific realizations of the model of Table 10. Each of these will be discussed in terms of its own characteristic notation. It is therefore important for the student to understand the sense of each of the basic formulas derived above without regard to the particular symbols in which it happens to be expressed.

2. Disability and mortality

In Table 11, the two primary decrements are disability and mortality, and the secondary decrement is mortality among the disabled lives. This table, known as a combined disability and mortality table, is the model that was used in the calculations for the older forms of disability benefits included in life insurance policies. Payments were made only for disability which was *permanent* as well as total, so that no recoveries were assumed to occur. More recent practice involves a different model, which we shall discuss in the next chapter.

In the combined disability and mortality table, the symbol l_x^{aa} is used for the original body of lives, the active lives. The decrements from death and from disability are denoted by d_x^{aa} and by i_x respectively. The disabled lives attaining age x are denoted by l_x^{ii} and the deaths among the disabled lives by d_x^{ii}. We thus have

$$l_x^{aa} - d_x^{aa} - i_x = l_{x+1}^{aa} \tag{15.15}$$

$$l_x^{ii} + i_x - d_x^{ii} = l_{x+1}^{ii} \,. \tag{15.16}$$

The letters a and i used as superscripts in these symbols are suggested by the terms *active* and *invalid.* When used to distinguish the l_x and d_x functions of the combined disability and mortality table, as above, these letters are always repeated—aa and ii. When they are used as a superscript with probability, annuity, and insurance functions, a special convention is ob-

TABLE 11

SECTION OF COMBINED DISABILITY AND MORTALITY TABLE

Age x	(1)	(2)	(3)	(4)	(5)
	l_x^{aa}	d_x^{aa}	i_x	l_x^{ii}	d_x^{ii}
20	92,483	686	53	154	37
21	91,744	684	52	170	38
22	91,008	683	52	184	38
23	90,273	682	51	198	38
24	89,540	681	51	211	38

served. In such cases, when there is a double superscript, as in p_x^{aa} and a_x^{ai}, the first letter indicates the present state and the second letter specifies a future state. For example, p_x^{aa} represents the probability that a life now active will be alive and active one year later; and a_x^{ai} is the present value of an annuity which is payable when a life now active becomes disabled. When there is a single superscript, the future state is not specified. Thus, q_x^a represents the probability that a life now active will die within one year, whether active or disabled.

In the case of q_x^i, there is only one decrement operating, and the symbol represents both the probability and the rate of mortality among the disabled lives. The function q_x^i is the counterpart of the rate $(hq)_x$ in the general model of Section 1.

Formulas for the basic probabilities may be written in terms of the combined disability and mortality functions. Each one listed below corresponds to a formula derived in the preceding section. Instead of repeating the derivation in each case, we shall indicate at the left the number of the analogous general formula.

The probability that an active life aged x will die while active within a year is

(15.3a) $$q_x^{aa} = \frac{d_x^{aa}}{l_x^{aa}}. \tag{15.17}$$

The probability that an active life aged x will become disabled within a year is

(15.3b) $$r_x = \frac{i_x}{l_x^{aa}}. \tag{15.18}$$

Note the use of the special symbol "r" for the probability of disability.

The rate of disability for age x is denoted by r'_x and

(15.4b) $$r'_x \doteq \frac{i_x}{l_x^{aa} - \frac{1}{2}d_x^{aa}}. \tag{15.19}$$

For the rate of mortality among disabled lives, we have

(15.5) $$q_x^i \doteq \frac{d_x^{ii}}{l_x^{ii} + \frac{1}{2}i_x}. \tag{15.20}$$

The probability that an active life aged x will die disabled within one year is

(15.6) $$q_x^{ai} = \frac{d_x^{ii} - l_x^{ii}\cdot q_x^i}{l_x^{aa}} \tag{15.21}$$

or

$$q_x^{ai} \doteq \frac{1}{2}\cdot\frac{i_x}{l_x^{aa}}\cdot q_x^i$$

(15.7) $$= \tfrac{1}{2}r_x\cdot q_x^i. \tag{15.22}$$

The probability that an active life aged x will be alive and disabled at age $x + 1$ is

(15.8) $$p_x^{ai} = \frac{l_{x+1}^{ii} - l_x^{ii}\cdot p_x^i}{l_x^{aa}} \tag{15.23}$$

or

(15.10) $$p_x^{ai} \doteq \frac{i_x}{l_x^{aa}}\left(1 - \tfrac{1}{2}q_x^{i}\right). \tag{15.24}$$

The probability that an active life aged x will be alive and disabled at age $x + n$ is

(15.11) $$_np_x^{ai} = \frac{l_{x+n}^{ii} - l_x^{ii}\cdot {}_np_x^{i}}{l_x^{aa}}. \tag{15.25}$$

The probability that an active life now aged x will die disabled in the $(n + 1)$-th year from now is

(15.13) $$_{n|}q_x^{ai} = \frac{d_{x+n}^{ii} - l_x^{ii}\cdot {}_{n|}q_x^{i}}{l_x^{aa}}. \tag{15.26}$$

Many other probabilities may be formulated in terms of those listed above. For example, let us obtain the probability that an active life aged x will become disabled before attaining age $x + 1$ and will die before attaining age $x + 2$. These conditions are satisfied if the life becomes disabled and dies before age $x + 1$, the chance of which is q_x^{ai}, or if the life becomes disabled and survives to age $x + 1$ and then dies before age $x + 2$, the chance of which is $p_x^{ai}\cdot q_{x+1}^{i}$. The required probability is thus given by

$$q_x^{ai} + p_x^{ai}\cdot q_{x+1}^{i}.$$

This probability may also be expressed as the probability of becoming disabled before age $x + 1$, which is r_x, minus the probability of becoming disabled before age $x + 1$ and then surviving to age $x + 2$, which is $p_x^{ai}\cdot p_{x+1}^{i}$. Thus an alternative formulation is

$$r_x - p_x^{ai}\cdot p_{x+1}^{i}.$$

The two expressions are clearly equivalent since $p_x^{ai} + q_x^{ai} = r_x$.

3. Marriage and mortality

As a second application, we shall consider a few probabilities from a combined marriage and mortality table, a typical section of which is shown in Table 12. The definitions of the symbols used will be obvious from the column headings of the table.

For a married man, the rate of mortality, $(mq)_x$, is obtained from

$$(mq)_x \doteq \frac{(md)_x}{(ml)_x + \frac{1}{2}(bm)_x}.$$

Let us use Table 12 to evaluate the probability that a bachelor aged 33 will attain age 34 as a bachelor and then die a married man before attaining age 35. The probability of attaining age 34 as a bachelor is given by $\frac{(bl)_{34}}{(bl)_{33}}$, and the probability that a bachelor aged 34 will die a married man before age 35 is approximately

TABLE 12

EXTRACT FROM COMBINED MARRIAGE AND MORTALITY TABLE FOR MALE LIVES

Age x	Bachelors living $(bl)_x$	Bachelors dying $(bd)_x$	Bachelors marrying $(bm)_x$	Married men living $(ml)_x$	Married men dying $(md)_x$
33	51,002	376	4,335	29,711	207
34	46,291	345	4,027	33,839	240
35	41,919	320	3,773	37,626	273
36	37,826	—	—	41,126	—

$\frac{\frac{1}{2}(bm)_{34} \cdot (mq)_{34}}{(bl)_{34}}$. The required probability is then the product of these two expressions, or

$$\frac{\frac{1}{2}(bm)_{34} \cdot (mq)_{34}}{(bl)_{33}}.$$

In evaluating this expression, we first compute

$$(mq)_{34} \doteq \frac{240}{33{,}839 + \frac{1}{2}(4{,}027)} = .0067.$$

Then the required probability is

$$\frac{\frac{1}{2}(4{,}027)(.0067)}{51{,}002} = .0003.$$

The student may show that this probability can also be computed from the expression

$$\frac{(md)_{34} - (ml)_{34} \cdot (mq)_{34}}{(bl)_{33}}.$$

As a second example, let us find the probability that a bachelor aged 34 will be alive and married at age 36. This will be recognized as analogous to the probability ${}_2p_x^{(r\bar{h})}$ of Section 1, and, recalling the reasoning which underlies formula (15.11), we may write

$$\frac{(ml)_{36} - (ml)_{34}\cdot{}_2(mp)_{34}}{(bl)_{34}},$$

where ${}_2(mp)_{34} = [1 - (mq)_{34}][1 - (mq)_{35}]$.

4. Monetary applications

The combined disability and mortality table can be used to evaluate benefits payable in the event of total and permanent disability. As an example, suppose that we require the present value of an annuity benefit which is to become payable to an active life now aged x in the event of disability. Let the benefit consist of a payment of 1 at the end of each year if the insured is then surviving and disabled. The present value may be expressed as

$$\begin{aligned} a_x^{ai} &= \sum_{t=0}^{\infty} v^t\,{}_tp_x^{ai} \\ &= \sum_{t=0}^{\infty} v^t\cdot\frac{l_{x+t}^{ii} - l_x^{ii}\cdot{}_tp_x^{i}}{l_x^{aa}} && \text{from (15.25)} \\ &= \sum_{t=0}^{\infty} \frac{v^{x+t}l_{x+t}^{ii} - v^x l_x^{ii}\cdot v^t\,{}_tp_x^{i}}{v^x l_x^{aa}} \\ &= \frac{N_x^{ii} - D_x^{ii}\cdot\ddot{a}_x^{i}}{D_x^{aa}} && (15.27) \end{aligned}$$

where $\quad D_x^{ii} = v^x l_x^{ii}$ and $D_x^{aa} = v^x l_x^{aa}$,

$$N_x^{ii} = \sum_{t=0}^{\infty} D_{x+t}^{ii},$$

and $\quad \ddot{a}_x^{i} = \sum_{t=0}^{\infty} v^t\,{}_tp_x^{i}.$

It will be seen that $\ddot{a}_x^i$ represents the present value of an annuity-due payable to a disabled life aged x.

If a net *annual* premium is required for the above benefit, it will be payable only while the insured remains an active life.

The present value of an annuity of 1 payable to (x) while active is

$$\begin{aligned}\ddot{a}_x^{aa} &= \sum_{t=0}^{\infty} v^t\ {}_tp_x^{aa} \\ &= \sum_{t=0}^{\infty} \frac{v^{x+t} l_{x+t}^{aa}}{v^x l_x^{aa}} \\ &= \frac{N_x^{aa}}{D_x^{aa}}, \qquad (15.28)\end{aligned}$$

$$\text{where } D_x^{aa} = v^x l_x^{aa}$$

$$\text{and } N_x^{aa} = \sum_{t=0}^{\infty} D_{x+t}^{aa}\,.$$

Then the net annual premium corresponding to the net single premium a_x^{ai} is

$$\frac{a_x^{ai}}{\ddot{a}_x^{aa}} = \frac{N_x^{ii} - D_x^{ii}\cdot \ddot{a}_x^i}{N_x^{aa}}\,.$$

We consider next an example using the combined marriage and mortality table. Let us determine the net single premium for an insurance of 1 for a bachelor aged x, the insurance to be payable at the end of the year of death provided that the insured dies a married man. We first require an expression for the probability that a bachelor aged x will die a married man in the $(t + 1)$-th year from now. This is analogous to the probability ${}_{n|}q_x^{(Th)}$ of Section 1, and recalling formula (15.13), we write

$${}_{t|}q_x^{bm} = \frac{(md)_{x+t} - (ml)_x \cdot {}_{t|}(mq)_x}{(bl)_x}\,.$$

Denoting the required net single premium by A_x^{bm}, we now have

$$\begin{aligned}A_x^{bm} &= \sum_{t=0}^{\infty} v^{t+1}\ {}_{t|}q_x^{bm} \\ &= \sum_{t=0}^{\infty} v^{t+1}\, \frac{(md)_{x+t} - (ml)_x \cdot {}_{t|}(mq)_x}{(bl)_x} \\ &= \sum_{t=0}^{\infty} \frac{v^{x+t+1}(md)_{x+t} - v^x (ml)_x \cdot v^{t+1}\ {}_{t|}(mq)_x}{v^x (bl)_x} \\ &= \frac{(mM)_x - (mD)_x \cdot (mA)_x}{(bD)_x}\end{aligned}$$

where $$(mM)_x = \sum_{t=0}^{\infty} v^{x+t+1}(md)_{x+t},$$

$$(mD)_x = v^x(ml)_x \qquad \text{and} \qquad (bD)_x = v^x(bl)_x,$$

and $$(mA)_x = \sum_{t=0}^{\infty} v^{t+1}{}_{t|}(mq)_x.$$

The function $(mA)_x$ will be recognized as the net single premium for a whole life insurance for a married man aged x. As before, the formulation of the benefit has been simplified by the introduction of special commutation functions.

It can be seen from the simple examples above that a table with secondary decrements is not a particularly convenient model for computation. The construction of such a table requires the careful compilation of extensive data, and the evaluation of insurance and annuity functions depends upon the availability of various special commutation functions. In the case of benefits which are based on more than one secondary decrement, the mathematical analysis becomes even more complex. In the next chapter, we shall examine an alternative model which has proved more convenient in certain applications.

References

2. Hunter and Phillips (1932) discuss the methods of constructing a combined disability and mortality table.

3. The combined marriage and mortality table is treated briefly by Spurgeon (1932) and by Hooker and Longley-Cook (1957).

EXERCISES

(Use standard approximations where necessary.)

1. Primary and secondary decrements

1. Evaluate $q_{30}^{(Th)}$ from Table 10
   (a) using (15.6),
   (b) using (15.7).

2. Give an expression for the probability that a life aged 30 in state (T) will attain age 31 in state (h) and will then die before attaining age 32.

3. Compute the entries for age 32 in Table 10 being given $q_{32}^{(d)} = .020$ and $q_{32}^{(h)} = .060$, and being also given that the rate of mortality at age 32 for lives in state (h) is the same as for lives at that age in state (T).

4. Find an exact expression for $\int_0^1 {}_tp_x^{(T)}\mu_{x+t}^{(h)} \cdot {}_{1-t}(hp)_{x+t}\, dt$ in terms of functions based on the model of Table 10.

2. Disability and mortality

5. Show that $p_x^{ai} < r_x < r_x'$.

6. Obtain a formula for evaluating p_x^a from the combined disability and mortality table.

7. From the given data find d_{15}^{aa}, d_{15}^{ii}, and l_{17}^{ii}, assuming $l_{15}^{ii} = 0$.

x	$l_x^{aa} + l_x^{ii}$	i_x	q_x^i
15	96,285	57	.267
16	95,550	56	.254

8. Which of the following expressions are exactly equivalent to each other?

(a) $q_x^{ai} + p_x^{ai} \cdot q_{x+1}^i$

(b) $r_x - p_x^{ai} \cdot p_{x+1}^i$

(c) $\dfrac{d_x^{ii} + d_{x+1}^{ii}}{l_x^{aa}}$

(d) $\int_0^2 {}_tp_x^{ai} \cdot \mu_{x+t}^i\, dt,$

where μ_x^i is the force of mortality among disabled lives.

3. Marriage and mortality

9. Using Table 12, find numerical values for the following:
   (a) the rate of mortality for married men aged 33;
   (b) the probability that a man, who at age 33 is married, will be alive at age 34;
   (c) the probability that a man, who is a bachelor at age 33, will be alive at age 34.

10. For the model of Table 12, let the forces of mortality and marriage among bachelors be denoted by $\mu^{(bd)}$ and $\mu^{(bm)}$ respectively and the rate of death among married men by $q^{(m)}$. Write a definite integral for the probability that a man who is now a bachelor aged 34 will die within 2 years.

11. Given the following data for each of ages 57 to 59 inclusive:
the rate of marriage for spinsters,
the rate of death for married women,
the central rate of death for spinsters.

Taking $(ml)_{57}$ to be 0, describe how the combined marriage and mortality table may be completed for ages 57 to 59.

4. Monetary applications

12. Define the symbols and show that $A_x^{ai} = \dfrac{M_x^{ii} - D_x^{ii} \cdot A_x^{i}}{D_x^{aa}}$.

13. Find an expression for evaluating $a_x^{a(i:\overline{n}|)}$, the present value to an active life (x) of a disability income of 1 per annum for life if disability occurs within n years.

14. Assume that the combined marriage and mortality table of Exercise 11 is available. Given the value of $\ddot{a}_{60}$ and the rate of interest i, and assuming that no spinsters marry after age 60 and that after age 60 spinsters experience the same mortality as married women, develop an expression for the net annual premium, payable to age 60 or to prior marriage or death, for an insurance of 1 payable upon marriage and also upon death (either married or a spinster) of a spinster now aged 57, with the proviso that the insurance will be doubled if death occurs while a spinster.

Miscellaneous problems

15. Find the probability that an employee now aged 40 of the company whose experience is shown in the following table will die between the ages of 41 and 42, upon the assumption that the rate of mortality is the same among lives that have withdrawn as among those still in service. Indicate but do not perform the numerical work.

Age x	l_x^T	d_x	w_x
40	10,000	76	1,290
41	8,634	80	1,268

16. (a) Given the rates of death and disability among active lives and the rates of death among disabled lives, explain how you would construct a combined disability and mortality table, assuming no recoveries from disability.

(b) Develop a formula, based on the table in (a), for the net annual premium payable by a life aged x while active for a policy providing a payment of \$10 upon death while active, \$5 upon becoming disabled, \$1 at the end of each year after becoming disabled, and \$5 upon death while disabled. Assume all claim payments are made at the end of the policy year. Express your answer in terms of suitably defined commutation columns.

CHAPTER 16

A GENERALIZED MODEL

1. Description of the model

We consider further the problem of calculating the present value of benefits which become payable when an individual life, subject to several causes of decrement, is transferred, through the operation of one of these causes, to a specified state g. We shall often be concerned with benefits which consist either partly or wholly of annuity payments to be made while the insured is in state g. As we have seen in Chapter 15, the required value can be obtained if the multiple-decrement table is extended in a natural way to show the secondary decrements from state g. We now develop an alternative model, somewhat simpler in conception, which is often more convenient to use in practice.

The model requires the following basic elements:

1. A multiple-decrement table which measures the risk of transfer to state g. In this table, we shall use l_x^T to refer to the original body of lives and g_x to denote the transfers to state g.
2. A rate of interest i for use with the multiple-decrement table.
3. A table of present values of annuities for lives in state g, computed at an appropriate rate of interest, which is not necessarily the same as i. These values are based on the probabilities of continuance in state g, where continuance may depend on survival only, or on some additional condition as well, such as the continuation of disability. The form of annuity is determined by the particular application, and standard notational devices are used to distinguish the different types, as, for example, $\ddot{a}_x^g$ and $\bar{a}_x^g$.

Using these three elements, it is easy to express the net single premium for annuity payments to be received in state g. Suppose that a continuous life annuity of 1 per annum is payable while (x) is in state g, provided that entrance into state g occurs before age y. Assuming that the annuity becomes payable at the moment of entrance to state g, we can write the net single premium as

$$\frac{1}{v^x\, l_x^T}\int_x^y v^z\, l_z^T \mu_z^g\, \bar{a}_z^g\, dz \doteq \frac{1}{v^x\, l_x^T}\sum_{z=x}^{y-1} v^{z+\frac{1}{2}}\, l_z^T q_z^g\, \bar{a}_{z+\frac{1}{2}}^g$$

$$= \frac{1}{v^x\, l_x^T}\sum_{z=x}^{y-1} v^{z+\frac{1}{2}}\, g_z\, \bar{a}_{z+\frac{1}{2}}^g\,. \qquad (16.1)$$

The only functions required for the evaluation of (16.1) are those given by the three elements described above.

As usual, the calculation of the value of the benefits is easier if commutation functions are available. We shall use the following basic functions, derived from the given multiple-decrement table:

$$D_x^T = v^x\, l_x^T \quad \text{and} \quad \bar{D}_x^T = \tfrac{1}{2}\,(D_x^T + D_{x+1}^T) \qquad (16.2)$$

$$C_x^g = v^{x+1}\, g_x \quad \text{and} \quad \bar{C}_x^g = (1+i)^{\frac{1}{2}}\, C_x^g\,. \qquad (16.3)$$

Note that $\bar{D}_x^T$ and $\bar{C}_x^g$, as defined above, are approximations for

$$\int_x^{x+1} D_z^T\, dz \quad \text{and} \quad \int_x^{x+1} v^z\, l_z^T \mu_z^g\, dz$$

respectively.

A special feature of this model is the use of weighted values of C and D, distinguished by a prefixed superscript:

$$^F\!D_x^T = F_x \cdot D_x^T \quad \text{and} \quad ^F\!\bar{D}_x^T = F_x \cdot \bar{D}_x^T \qquad (16.4)$$

$$^F\!C_x^g = F_x \cdot C_x^g \quad \text{and} \quad ^F\!\bar{C}_x^g = F_x \cdot \bar{C}_x^g\,, \qquad (16.5)$$

where F_x is a function of x on which the required monetary values depend. For example, if $F_x = \ddot{a}_{x+1}^g$, the function $^F\!C_x^g = C_x^g \cdot \ddot{a}_{x+1}^g$ is a natural one to use in finding the present value of a life annuity with first annual payment at the end of the year of transfer to state g.

The weight function F_x takes on many different forms in the applications, and we shall usually replace the superscript F by a symbol which will suggest the nature of the particular function being used. For example, when F_x is some form of annuity function, the symbol $^a C_x^g$ may be used in place of $^F C_x^g$.

It should be noted that the exact form of annuity used in $^a C_x^g$ depends on the application involved, and this symbol, like all other symbols involving weight functions, must always be defined specifically.

Other commutation functions are defined in the usual way by summation. From D_x^T and $\bar{D}_x^T$, we obtain N_x^T, $\bar{N}_x^T$, S_x^T, and $\bar{S}_x^T$; and from C_x^g and $\bar{C}_x^g$, we obtain M_x^g, $\bar{M}_x^g$, R_x^g, and $\bar{R}_x^g$. The corresponding weighted functions are defined in the same way; for example,

$$^r\bar{M}_x^g = \sum_{z=x}^{\infty} {^r\bar{C}_z^g} \quad \text{and} \quad ^r\bar{R}_x^g = \sum_{z=x}^{\infty} {^r\bar{M}_z^g}.$$

Returning to the net premium expression in (16.1), we can introduce commutation functions as follows:

$$\frac{1}{v^x l_x^T} \sum_{z=x}^{y-1} v^{z+\frac{1}{2}} g_z \, \bar{a}_{z+\frac{1}{2}}^g$$

$$= \frac{\sum_{z=x}^{y-1} \bar{C}_z^g \bar{a}_{z+\frac{1}{2}}^g}{D_x^T}$$

$$= \frac{\sum_{z=x}^{y-1} {^a\bar{C}_z^g}}{D_x^T} \qquad \text{where} \qquad {^a\bar{C}_z^g} = \bar{C}_z^g \cdot \bar{a}_{z+\frac{1}{2}}^g$$

$$= \frac{^a\bar{M}_x^g - {^a\bar{M}_y^g}}{D_x^T}. \tag{16.6}$$

In certain applications, such as the one just described, the benefits are payable only if transfer to state g occurs prior to a specified age y. It is then convenient to tabulate the values of the summation of the C^g functions up to age y only,

$$\sum_{z=x}^{y-1} C_z^g .$$

When this is done, the resulting values are distinguished by a prefixed subscript with the M^g function:

$$_yM_x^g = \sum_{z=x}^{y-1} C_z^g = M_x^g - M_y^g. \tag{16.7}$$

This convention may be used with any of the forms of the M^g function. The premium in (16.6), for example, can be expressed as

$$\frac{{}^{s}_{y}\bar{M}^{e}_{x}}{D^{T}_{x}}.$$

2. Pension plans

As a first application of the general model described above, we consider the methods used in calculating the values of benefits and contributions for the participants in a pension plan. The participants in such a plan are usually employees who share a common status, either in the service of a particular employer or industry, or as members of a union organization. The plan may provide benefits upon disability, withdrawal, and death, in addition to the pension benefits at retirement. The cost of the plan may be shared by the participants and the employer (or other sponsor), or it may be borne entirely by the sponsor without contributions from the participants.

The methods to be discussed here can be used in making periodic estimates, in the aggregate, of the value of the future benefits and contributions for all participants in the plan. These figures are used in estimating the aggregate costs and for other purposes. Although the values of the contributions and the benefits of the plan as a whole are the sum of these values for each individual participant, the individual results are normally not of interest in themselves.

There are wide variations in the provisions of pension plans currently in force, and we shall develop formulas for only a few typical situations. In practice, these formulas will often be modified because of special features in a particular plan.

A common type of multiple-decrement table used in pension plan calculations includes four modes of decrement: mortality, disability, withdrawal, retirement. The following notation will be used:

l^T_x is the number of lives attaining age x in service;

d_x is the number dying in service in the year following attainment of age x;

i_x is the number becoming disabled in service in the year following attainment of age x;

w_x is the number withdrawing from service in the year following attainment of age x;

r_x is the number retiring from service in the year following attainment of age x.[1]

Then $$l^T_{x+1} = l^T_x - d_x - i_x - w_x - r_x .$$

Since some of the decrements, especially withdrawal, may change rapidly as length of service increases, it is sometimes advisable to use a select service table, but this is normally justifiable only when the number of participants is large. The development of formulas on a select basis presents no additional difficulties in the theory and will not be considered here.

In writing the formulas for retirement benefits, we shall use the general notation of Section 1 with the unspecified decrement g replaced by the retirement decrement r. For example, the present value at age x of a payment of 1 at the moment of retirement is evaluated as

$$\frac{\sum_{t=0}^{\infty} v^{t+\frac{1}{2}}\, r_{x+t}}{l^T_x} = \frac{\sum_{t=0}^{\infty} \bar{C}^r_{x+t}}{D^T_x} = \frac{\bar{M}^r_x}{D^T_x} . \tag{16.8}$$

The present value at age x of a payment at the moment of retirement of 1 for each completed year of service from age x to retirement is evaluated as

$$\frac{\sum_{t=0}^{\infty} t\bar{C}^r_{x+t}}{D^T_x} = \frac{\sum_{t=1}^{\infty} \bar{M}^r_{x+t}}{D^T_x} = \frac{\bar{R}^r_{x+1}}{D^T_x} . \tag{16.9}$$

It should be remembered that retirements will generally occur only within a certain range of ages. If the service table shows nonzero values of r_x only for $y \leqslant x \leqslant z$, then $\bar{C}^r_x = 0$ for all $x < y$ and all $x > z$. The summations in formulas like (16.8) and (16.9) will not necessarily include a very large number of terms.

In some plans, it may not be realistic to assume that the retirements are spread uniformly over the year. It may be provided, for example, that all retirements at age 65 occur at either the beginning or the end of the year of age. Then $v^{\frac{1}{2}}$ would not be a

[1] The symbol r_x has been used in Chapter 15 to denote the probability of disability. In the present chapter, it will always be used in the sense of the definition above. We shall use r'_x in Section 9 as the rate of disability, but this should cause no confusion.

suitable factor for age 65, and an appropriate modification should be introduced.

Retirement benefits are generally in the form of life annuities. In evaluating such benefits, it is convenient to use the annuity value for retired lives as a weight function. If we define

$$^{a}\bar{C}_x^r = \bar{C}_x^r \cdot \bar{a}_{x+\frac{1}{2}}^r , \tag{16.10}$$

the present value at age x of a continuous annuity of 1 commencing at retirement may be evaluated as

$$\frac{\sum_{t=0}^{\infty} \bar{C}_{x+t}^r \cdot \bar{a}_{x+t+\frac{1}{2}}^r}{D_x^T} = \frac{\sum_{t=0}^{\infty} {}^{a}\bar{C}_{x+t}^r}{D_x^T} = \frac{{}^{a}\bar{M}_x^r}{D_x^T} .$$

Other kinds of annuities may be used as weights in (16.10), depending on the nature of the benefit. For example, if the life annuity payments were guaranteed for five years, it would be appropriate to define $^{a}\bar{C}_x^r$ as

$$\bar{C}_x^r \cdot \bar{a}_{\overline{x+\frac{1}{2}:\overline{5|}}}^r .$$

Pension plans may also provide annuity payments in the event of disability. These benefits may be evaluated in the same way by defining

$$\bar{C}_x^i = v^{x+\frac{1}{2}}\, i_x \tag{16.11}$$

and

$$^{a}\bar{C}_x^i = \bar{C}_x^i \cdot \bar{a}_{x+\frac{1}{2}}^i . \tag{16.12}$$

3. Pension benefits based on salary

The amount of the pension benefit to be received by a participant often depends on his compensation, and it is important to have a means of estimating future changes in compensation. This is provided by a salary scale function, either continuous or discrete, depending on whether salary changes are recognized immediately or on some fixed date each year. For simplicity, the definitions below assume a discrete salary scale function.

We shall use the following salary functions, defined for integral values of x and y:

$(TPS)_x$ is total actual salary for service prior to attainment of age x;

$(AS)_x$ is actual salary for service during the year following attainment of age x;

$(ES)_y$ is estimated salary for service during the year following attainment of age y;

S_y is a salary scale function such that

$$(ES)_y = \frac{S_y}{S_x}(AS)_x, \qquad y > x;$$

$(ES)_{y:\overline{m}|}$ is estimated average annual salary for service during the m years preceding the occurrence, in the year following attainment of age y, of a decremental event (retirement, disability, etc.);

${}_mZ_y$ is a salary scale function such that

$$(ES)_{y:\overline{m}|} = \frac{{}_mZ_y}{S_x}(AS)_x, \qquad y \geqslant x + m.$$

Then

$${}_mZ_y \doteq \frac{1}{2m}[S_y + 2(S_{y-1} + S_{y-2} + \cdots + S_{y-m+1}) + S_{y-m}].$$

Since m is generally a constant for any one plan, the subscript m in ${}_mZ_y$ is often suppressed.

The magnitude of a particular value of S_y, or of Z_y, is not in itself significant, any more than is a value of l_x without reference to the radix of the table. It is the ratio of these values that is important, as

$$\frac{S_y}{S_x} \quad \text{for} \quad y > x, \quad \text{or} \quad \frac{Z_y}{S_x} \quad \text{for } y \geqslant x + m.$$

In practice, separate salary scale functions may be required for the male and the female participants, or for the salaried and the hourly-rated participants. The term "salary scale function" is a general one and is used for all types of compensation. In what follows, we shall confine ourselves to a discussion of salary only, but it is to be understood that the same methods can be applied to other forms of compensation.

In evaluating pension benefits based on salary, we shall use the salary scale functions as weights in computing commutation functions:

$$^{S}D_x^T = S_x\, D_x^T \tag{16.13a}$$

$$^{S}\bar{C}_x^r = S_x\, \bar{C}_x^r \tag{16.13b}$$

$$^{Sa}\bar{C}_x^r = S_x \cdot {}^{a}\bar{C}_x^r = \bar{C}_x^r \cdot S_x\, \bar{a}_{x+\frac{1}{2}}^r \tag{16.14}$$

$$^{Za}\bar{C}_x^r = Z_x \cdot {}^{a}\bar{C}_x^r = \bar{C}_x^r \cdot Z_x\, \bar{a}_{x+\frac{1}{2}}^r\,. \tag{16.15}$$

The weighting of the function $\bar{C}_x^r$ by the product of two functions of x is indicated here by a prefixed superscript consisting of two letters.

Consider a plan providing an annual retirement pension of $k\%$ of one year's salary at the rate of pay in the year of retirement The present value at age x is

$$.01\, k(AS)_x \frac{\sum_{t=0}^{\infty} S_{x+t}\, \bar{C}_{x+t}^r\, \bar{a}_{x+t+\frac{1}{2}}^r}{S_x\, D_x^T}$$

$$= .01\, k(AS)_x \frac{\sum_{t=0}^{\infty} {}^{Sa}\bar{C}_{x+t}^r}{^{S}D_x^T} = .01\, k(AS)_x \frac{^{Sa}\bar{M}_x^r}{^{S}D_x^T}\,. \tag{16.16}$$

If the pension is based on the m-year final average salary, the present value is

$$.01\, k(AS)_x \frac{\sum_{t=0}^{\infty} {}_{m}Z_{x+t} \cdot \bar{C}_{x+t}^r\, \bar{a}_{x+t+\frac{1}{2}}^r}{S_x\, D_x^T}$$

$$= .01\, k(AS)_x \frac{\sum_{t=0}^{\infty} {}^{Za}\bar{C}_{x+t}^r}{^{S}D_x^T} = .01\, k(AS)_x \frac{^{Za}\bar{M}_x^r}{^{S}D_x^T}\,. \tag{16.17}$$

Note that this formula cannot, as it stands, be applied to lives within m years of the earliest possible retirement age, since ${}_{m}Z_{x+t}$ has not been defined for $t < m$. An appropriate definition can be made in any specific case; this would take account not only of any actual salary paid before attainment of age x but also of the fact that for retirements before age $x + m$ there may be less than m years of salary to average.

Most pension plans give some recognition to length of service (or, alternatively, length of participation in the plan) in determining the pension benefit. Consider a plan providing, for each year of service at retirement, an annual retirement pension of $k\%$ of annual salary at the rate of pay applicable in the year of retirement. If an employee at age x has n years of past service (n is not necessarily an integer here), the present value of the portion of the benefit arising from past service is

$$.01\, k \cdot n(AS)_x \frac{{}^{Sa}\bar{M}_x^r}{{}^{S}D_x^T}, \tag{16.18}$$

and the present value of the portion arising from future service (assuming credit only for completed years of future service) is

$$\begin{aligned} .01\, k(AS)_x \sum_{t=1}^{\infty} \frac{t \cdot {}^{Sa}\bar{C}_{x+t}^r}{{}^{S}D_x^T} &= .01\, k(AS)_x \sum_{t=1}^{\infty} \frac{{}^{Sa}\bar{M}_{x+t}^r}{{}^{S}D_x^T} \\ &= .01\, k(AS)_x \frac{{}^{Sa}\bar{R}_{x+1}^r}{{}^{S}D_x^T}. \end{aligned} \tag{16.19}$$

If, instead of giving credit only for completed years of future service, the plan provides that fractional portions of a year are counted as well, there will be, on the average, an additional half-year's credit at retirement. This can be taken into account by modifying the formula for future service benefits to

$$.01\, k(AS)_x \frac{\frac{1}{2}\,{}^{Sa}\bar{M}_x^r + {}^{Sa}\bar{R}_{x+1}^r}{{}^{S}D_x^T}. \tag{16.20}$$

It should be noted that when past service is combined with future service the number of completed years of total service is not necessarily equal to the number of completed years of past service plus the number of completed years of future service. Moreover, the addition of a half-year's credit to both past service and future service does not necessarily give the best approximation to total service with fractional years of service considered. In any calculation in which past service and future service are combined, care must be taken to use appropriate methods.

If pension benefits are based on aggregate salary during service rather than on final salary, the nature of the problem of evaluating pension benefits is altered considerably. A benefit based on aggregate salary is sometimes called a *career average* pension, since the pension formula may be expressed either as $k\%$ of aggregate salary, or as $k\%$ of average annual salary during service multiplied by the number of years of service.

For the present value at age x of the portion of the benefit related to past service, we have simply

$$.01\, k(TPS)_x \frac{{}^a\bar{M}_x^r}{D_x^T}. \tag{16.21}$$

The present value at age x of the portion of the benefit related to future service will be developed in several steps.

First, we write an expression for the present value at age x of the benefit arising from salary during the year following attainment of age $x + t$, assuming for the moment that retirement occurs during the year following attainment of age $x + t + n$. We also assume temporarily that there is no credit for salary during the fraction of a year immediately preceding the date of retirement. Hence the present value is

$$.01\, k(AS)_x \cdot \frac{S_{x+t}}{S_x} \cdot \frac{{}^a\bar{C}_{x+t+n}^r}{D_x^T},$$

where the only values of n to be considered are those $\geqslant 1$. Summing this expression from $n = 1$ to the end of the table, we obtain the present value, without regard to date of retirement, of all benefits arising from salary during the year following attainment of age $x + t$:

$$.01\, k(AS)_x \cdot \frac{S_{x+t}}{S_x} \cdot \frac{\sum_{n=1}^{\infty} {}^a\bar{C}_{x+t+n}^r}{D_x^T}$$

$$= .01\, k(AS)_x \cdot \frac{S_{x+t}}{S_x} \cdot \frac{{}^a\bar{M}_{x+t+1}^r}{D_x^T}. \tag{16.22}$$

If we define

$${}^{S'a}\bar{M}_x^r = S_{x-1}\, {}^a\bar{M}_x^r, \tag{16.23}$$

the formula becomes

$$.01\ k(AS)_x\ \frac{{}^{S'a}\bar{M}^r_{x+t+1}}{{}^{S}D^T_x}. \qquad (16.24)$$

We now disregard the temporary assumption used in deriving (16.24) and make an adjustment to take account of the credit for salary during the fraction of a year immediately preceding the date of retirement. Assuming that retirement will occur, on the average, half-way through the year, the adjustment term is

$$.01\ k(AS)_x\cdot\frac{S_{x+t}}{S_x}\cdot\frac{\frac{1}{2}{}^{a}\bar{C}^r_{x+t}}{D^T_x} = .01\ k(AS)_x\cdot\frac{\frac{1}{2}{}^{Sa}\bar{C}^r_{x+t}}{{}^{S}D^T_x},$$

and adding this expression to (16.24) we have

$$.01\ k\ (AS)_x\ \frac{\frac{1}{2}{}^{Sa}\bar{C}^r_{x+t} + {}^{S'a}\bar{M}^r_{x+t+1}}{{}^{S}D^T_x}. \qquad (16.25)$$

This represents the value of only that portion of the benefit arising from salary during the year following attainment of age $x + t$.

We now obtain the present value of the entire benefit attributable to future service as

$$.01\ k(AS)_x\ \frac{\sum_{t=0}^{\infty}\left(\frac{1}{2}{}^{Sa}\bar{C}^r_{x+t} + {}^{S'a}\bar{M}^r_{x+t+1}\right)}{{}^{S}D^T_x}$$
$$= .01\ k(AS)_x\ \frac{\frac{1}{2}{}^{Sa}\bar{M}^r_x + {}^{S'a}\bar{R}^r_{x+1}}{{}^{S}D^T_x}. \qquad (16.26)$$

In defining commutation symbols for this benefit, we have introduced the device of a weight function applied to the values of M. When this is done, a prime is shown with the prefixed superscript. In the symbol ${}^{S'a}\bar{M}^r_x$, it is to be understood from the superscribed $S'a$ that $\bar{C}^r_x$ has been weighted with an annuity function and that the resulting ${}^{a}\bar{M}^r_x$ has been weighted with a salary function. The fact that the salary function has a subscript one age lower than that of ${}^{a}\bar{M}^r_x$ is important in the application, but this is a matter of definition, and the same notation could be used even if this were not so.

It should be kept in mind that the notation used in this chapter is far from standardized and is not always the same as the notation used elsewhere in the literature. For example, some authors use the symbol C_x^{ra} where we use ${}^a\bar{C}_x^r$ and reserve the use of the bar over a function for definitions like the following:

$$\begin{aligned} {}^{S'}\bar{M}_x^{ra} &= \tfrac{1}{2}\,{}^{S}C_x^{ra} + {}^{S'}M_{x+1}^{ra} \\ {}^{S'}\bar{R}_x^{ra} &= \sum_{t=0}^{\infty} {}^{S'}\bar{M}_{x+t}^{ra}\,. \end{aligned}$$

These special definitions are designed to simplify the writing of expressions like (16.25) and (16.26).

4. Participants' contributions

Participants' contributions to a pension plan are usually either a flat amount per year or a percentage of compensation. The contributions are usually made by payroll deduction. There are wide variations in payroll practices as to frequency of payment, and it is convenient to assume that the contributions are received continuously.

On this assumption, the present value at age x of a unit to be contributed during the year following attainment of age $x + t$ is $\dfrac{\bar{D}_{x+t}^T}{D_x^T}$. The present value of a unit to be contributed during each year of future service is

$$\sum_{t=0}^{\infty} \frac{\bar{D}_{x+t}^T}{D_x^T} = \frac{\bar{N}_x^T}{D_x^T}\,. \tag{16.27}$$

The present value at age x of a contribution of $C\%$ of salary during the year following attainment of age $x + t$ is

$$.01C(AS)_x \frac{S_{x+t}}{S_x} \cdot \frac{\bar{D}_{x+t}^T}{D_x^T}$$

or

$$.01C(AS)_x \frac{{}^S\bar{D}_{x+t}^T}{{}^SD_x^T}\,,$$

where ${}^S\bar{D}_x^T$ is defined as $S_x\bar{D}_x^T$. Then the present value of all such contributions over the period of future service is

$$.01C(AS)_x \sum_{t=0}^{\infty} \frac{{}^S\bar{D}_{x+t}^T}{{}^SD_x^T} = .01C(AS)_x \frac{{}^S\bar{N}_x^T}{{}^SD_x^T}\,. \tag{16.28}$$

Plans requiring contributions by participants frequently provide, on death or withdrawal before retirement, for a refund of the participant's contributions with interest. Consider a plan requiring contributions of $C\%$ of earnings, and providing on withdrawal a refund of contributions with interest at rate j. We discuss here the case where j is different from the basic rate i. The case where $j = i$ is included as an exercise for the student.

The present value at age x of the benefit arising from contributions during the year following attainment of age $x + t$, assuming for the moment that withdrawal occurs during the year following attainment of age $x + t + n$ $(n \geqslant 1)$ is

$$.01C(AS)_x \frac{S_{x+t}}{S_x} (1 + j)^n \frac{\bar{C}^w_{x+t+n}}{D^T_x}.$$

The factor $(1 + j)^n$ in this expression is appropriate under the further assumption that interest is credited from the moment of contribution to the moment of withdrawal. The present value can then be written

$$.01C(AS)_x \frac{S_{x+t}\,(1 + j)^{x+t+n}\,\bar{C}^w_{x+t+n}}{S_x\,(1 + j)^{x+t}\,D^T_x}$$

$$= .01C(AS)_x \frac{S_{x+t}\;{}^j\bar{C}^w_{x+t+n}}{(1 + j)^{x+t}\;{}^S D^T_x}, \qquad (16.29)$$

where we define

$${}^j\bar{C}^w_x = (1 + j)^x\,\bar{C}^w_x.$$

If we sum (16.29) over n for $n \geqslant 1$ and add one-half the corresponding amount for $n = 0$ (since on the average only one-half year's contributions are made in the year of withdrawal), we obtain the present value, without regard to the date of withdrawal, of all benefits arising from contributions during the year following attainment of age $x + t$:

$$.01C(AS)_x \cdot \frac{S_{x+t}}{(1 + j)^{x+t}} \cdot \frac{\frac{1}{2}\,{}^j\bar{C}^w_{x+t} + \sum_{n=1}^{\infty} {}^j\bar{C}^w_{x+t+n}}{{}^S D^T_x} \qquad (16.30)$$

In this expression, we note that

$$\frac{S_{x+t}}{(1 + j)^{x+t}} \cdot {}^j\bar{C}^w_{x+t} = S_{x+t}\,\bar{C}^w_{x+t} = {}^S\bar{C}^w_{x+t}$$

and

$$\frac{S_{x+t}}{(1+j)^{x+t}} \cdot \sum_{n=1}^{\infty} {}^{j}\bar{C}^{w}_{x+t+n} = \frac{S_{x+t}}{(1+j)^{x+t}} \cdot {}^{j}\bar{M}^{w}_{x+t+1} = {}^{S''j}\bar{M}^{w}_{x+t+1} ,$$

where we define[2]

$$^{S''j}\bar{M}^{w}_{x} = \frac{S_{x-1}}{(1+j)^{x-1}} \cdot {}^{j}\bar{M}^{w}_{x} . \tag{16.31}$$

With these substitutions, (16.30) becomes

$$.01C(AS)_x \frac{\frac{1}{2}{}^{S}\bar{C}^{w}_{x+t} + {}^{S''j}\bar{M}^{w}_{x+t+1}}{{}^{S}D^{T}_{x}} . \tag{16.32}$$

Finally, we obtain the present value of the entire benefit attributable to future contributions by summing (16.32) from $t = 0$ to the end of the table, obtaining

$$.01C(AS)_x \frac{\frac{1}{2}{}^{S}\bar{M}^{w}_{x} + {}^{S''j}\bar{R}^{w}_{x+1}}{{}^{S}D^{T}_{x}} . \tag{16.33}$$

Turning now to the present value of the benefit attributable to past contributions, we require the accumulation of such contributions with interest to age x at rate j. We denote this accumulation by ${}^{j}(TPC)_x$. Such an accumulation is normally maintained as a running total in the administrative records of the plan. The present value at age x of the benefit attributable to past contributions is then

$$\begin{aligned} & {}^{j}(TPC)_x \frac{\sum_{t=0}^{\infty} (1+j)^{t+\frac{1}{2}}\, \bar{C}^{w}_{x+t}}{D^{T}_{x}} \\ & \qquad = {}^{j}(TPC)_x (1+j)^{\frac{1}{2}} \frac{\sum_{t=0}^{\infty} (1+j)^{x+t}\, \bar{C}^{w}_{x+t}}{(1+j)^{x}\, D^{T}_{x}} \\ & \qquad = {}^{j}(TPC)_x (1+j)^{\frac{1}{2}} \frac{\sum_{t=0}^{\infty} {}^{j}\bar{C}^{w}_{x+t}}{(1+j)^{x}\, D^{T}_{x}} \end{aligned}$$

[2] Consistency would suggest a more complicated symbol to express the function defined in (16.31). For convenience, we are simply using a superscribed S'' to indicate that the function ${}^{j}\bar{M}^{w}_{x}$ is being weighted here in a special way.

$$= {}^j(TPC)_x(1+j)^{\frac{1}{2}} \frac{{}^j\bar{M}_x^w}{{}^jD_x^T}, \tag{16.34}$$

where
$${}^jD_x^T = (1+j)^x D_x^T. \tag{16.35}$$

5. Disability benefits

We consider now a second application of the general model of Section 1. This relates to the type of disability insurance which is offered as a supplementary benefit, often on an optional basis, in individual life insurance policies.

The disability clauses in current use provide a benefit for total and permanent disability. For the purpose of defining such disability, total disability which has been continuous for a period specified in the policy is presumed to be permanent. This period is known as the *waiting period* and is usually six months.

The most common benefit consists of waiver during the continuance of disability of the premiums for the life insurance policy which contains the disability clause. Premiums are usually waived retroactively to the beginning of the waiting period. No benefits are payable unless disability begins before some expiry age, usually 60 or 65. Some companies also offer a monthly income benefit, payable from the end of the waiting period. In the case of endowment policies, the disability income is usually payable only until the maturity date. In other cases, it is payable during the continuance of disability, either for life or up to some specified age.

Similar benefits are provided by individual health insurance policies, and the formulas discussed below can be adapted to this type of coverage as well.

We shall use the notation of Section 1 with l_x^T replaced by l_x^{aa} and the contingency g taken as the contingency i of total disability of at least m months' duration (where m months is the waiting period). We assume that i_x represents the number of lives that become disabled between ages x and $x+1$ and remain disabled for m months.

In order to evaluate annuity benefits commencing at the end of the waiting period, we shall use disabled life annuities of the forms $\bar{a}^i_{[x+\frac{1}{2}]+m/12}$ and $\bar{a}^i_{[x+\frac{1}{2}]+m/12:\overline{n}|}$. The first of these represents the present value at age $x + \frac{1}{2} + m/12$ of a continuous annuity, payable during the continuance of disability, to a life who entered state i at age

$x + \frac{1}{2}$ (used as the average age at disability of those who become disabled between ages x and $x + 1$). It is thus the present value at the end of the waiting period of an annuity with payments beginning at that time. The annuity terminates at recovery or prior death. Because of the high rates of recovery in the early months following the end of the m months waiting period, select annuity values are used.

It is of interest to note that the equation

$$_{m/12|}\bar{a}^i_{[x+\frac{1}{2}]} = v^{m/12}\ {}_{m/12}p^i_{[x+\frac{1}{2}]}\ \bar{a}^i_{[x+\frac{1}{2}]+m/12}\,,$$

where $_{m/12}p^i_{[x+\frac{1}{2}]}$ represents the probability that a life just entering state i at age $x + \frac{1}{2}$ will remain disabled for m months, reduces to

$$_{m/12|}\bar{a}^i_{[x+\frac{1}{2}]} = v^{m/12}\bar{a}^i_{[x+\frac{1}{2}]+m/12}$$

since, by the definition of the contingency i,

$$_{m/12}p^i_{[x+\frac{1}{2}]} = 1.$$

In the formulas that follow, we shall use y to denote the age at which the disability coverage expires. It is not required that the waiting period be completed before age y, but only that the initial date of disability occur before age y. The expiry age y should also be distinguished from the age at which any benefits cease, which will be denoted by u. A disability clause may provide, for example, that the coverage expires at age 60 ($y = 60$) and the benefits cease at age 65 ($u = 65$).

The simplest benefit to evaluate is a lump sum payment of 1 at the end of the m months waiting period. We approximate the net single premium at age x on the assumption that disability occurs in the middle of the policy year:

$$\frac{1}{l^{aa}_x}\sum_{z=x}^{y-1} v^{z-x+\frac{1}{2}+m/12}\,i_z = \frac{v^{m/12}\sum\limits_{z=x}^{y-1}\bar{C}^i_z}{D^{aa}_x} = \frac{v^{m/12}\,{}_y\bar{M}^i_x}{D^{aa}_x}\,, \tag{16.36}$$

where we define $_y\bar{M}^i_x$ as in (16.7).

If the benefit is a continuous life annuity of 1 per annum payable beginning at the end of the waiting period and thereafter during continuance of disability, the net single premium is

$$\frac{1}{l_x^{aa}}\sum_{z=x}^{y-1} v^{z-x+\frac{1}{2}}\, i_z \cdot {}_{m/12|}\bar{a}^i_{[z+\frac{1}{2}]} = \frac{\sum_{z=x}^{y-1} v^{m/12}\, \bar{C}^i_z\, \bar{a}^i_{[z+\frac{1}{2}]+m/12}}{D_x^{aa}}$$

$$= \frac{\sum_{z=x}^{y-1} {}^{w}\bar{C}^i_z}{D_x^{aa}} = \frac{{}^{w}_{y}\bar{M}^i_x}{D_x^{aa}} \qquad (16.37)$$

where

$${}^{w}\bar{C}^i_x = \bar{C}^i_x \cdot v^{m/12}\, \bar{a}^i_{[x+\frac{1}{2}]+m/12}\,. \qquad (16.38)$$

In (16.38) it is convenient to absorb the factor $v^{m/12}$ into the weight function. Hence $v^{m/12}$ does not appear explicitly in (16.37) as it does in (16.36), where no weight function is involved.

If the benefit is a continuous annuity of 1 per annum payable beginning at the end of the waiting period and thereafter during continuance of disability, but not beyond age u, $u > y$, the net single premium is

$$\frac{{}^{u}_{y}\bar{M}^i_x}{D_x^{aa}}, \qquad (16.39)$$

where we define

$${}^{u}\bar{C}^i_x = \bar{C}^i_x \cdot v^{m/12}\, \bar{a}^i_{[x+\frac{1}{2}]+m/12:\overline{u-x-\frac{6+m}{12}|}}\,. \qquad (16.40)$$

If $u \leqslant y$, (16.39) becomes

$$\frac{{}^{u}_{u}\bar{M}^i_x}{D_x^{aa}}.$$

The symbol in the numerator of this last expression is generally written as ${}^{u}\bar{M}^i_x$; that is,

$${}^{u}\bar{M}^i_x = \sum_{z=x}^{u-1} {}^{u}\bar{C}^i_z\,. \qquad (16.41)$$

The definitions of (16.38) and (16.40) depart somewhat from the general plan described in Section 1. Instead of using a superscribed a to indicate that the weight function involves an annuity, we use the superscript ω to indicate that it involves a *life* annuity and the superscript u to indicate that it involves a *temporary* annuity ceasing at age u. The reason for this departure is that disability commutation functions are required for many values of u, and it is desirable to have a convenient means of distinguishing them.

6. Income benefits

Life insurance policies which offer disability income benefits usually provide payments of \$5 or \$10 per month for each thousand dollars of face amount. The net single premium for a monthly income of \$10 commencing at the end of the waiting period in the event of disability prior to age y and payable for life during continuance of disability is

$$\frac{120}{l_x^{aa}} \sum_{z=x}^{y-1} v^{z-x+\frac{1}{2}}\, \dot{i}_z \cdot {}_{m/12|}\ddot{a}^{i(12)}_{[z+\frac{1}{2}]} = \frac{120 \sum_{z=x}^{y-1} v^{m/12}\, \bar{C}_z^i\, \ddot{a}^{i(12)}_{[z+\frac{1}{2}]+m/12}}{D_x^{aa}}. \tag{16.42}$$

We can rewrite this expression in terms of the commutation functions used in (16.36) and (16.37). Since the functions of (16.37) involve continuous annuities, we must make an adjustment for the monthly payments specified in (16.42). From the formulas

$$\ddot{a}_x^{(12)} \doteq \ddot{a}_x - {}^{11}\!/_{24}$$

and

$$\bar{a}_x \doteq \ddot{a}_x - \tfrac{1}{2},$$

we have

$$\ddot{a}_x^{(12)} \doteq \bar{a}_x + {}^{1}\!/_{24}, \tag{16.43}$$

and we can make a substitution in (16.42) based on this relation. We then have

$$\frac{120 \sum_{z=x}^{y-1} v^{m/12}\, \bar{C}_z^i (\bar{a}^i_{[z+\frac{1}{2}]+m/12} + {}^{1}\!/_{24})}{D_x^{aa}}$$

$$= \frac{120({}^{\omega}_{y}\bar{M}_x^i + {}^{1}\!/_{24}\, v^{m/12}\, {}_{y}\bar{M}_x^i)}{D_x^{aa}}. \tag{16.44}$$

The payment period for premiums for the disability income benefit is the same as that for premiums for the basic life insurance policy, except that disability income premiums cease upon the occurrence of a qualifying disability or upon the attainment of age y. When the policy is an ordinary life policy, the net annual premium for the disability income is

$$\frac{120({}^{\omega}_{y}\bar{M}^{i}_{x} + \tfrac{1}{24}\, v^{m/12}\, {}_{y}\bar{M}^{i}_{x})}{N^{aa}_{x} - N^{aa}_{y}}. \tag{16.45}$$

When the policy is an n-payment life policy, the net annual premium is the same as (16.45) if $x + n \geqslant y$, and the denominator is changed to $N^{aa}_{x} - N^{aa}_{x+n}$ if $x + n < y$.

Endowment policies usually provide that the disability income will cease upon maturity of the endowment. If the endowment matures at age $x + n$, the corresponding formulas are

$$\frac{120({}^{x+n}_{y}\bar{M}^{i}_{x} + \tfrac{1}{24}\, v^{m/12}\, {}_{y}\bar{M}^{i}_{x})}{N^{aa}_{x} - N^{aa}_{y}} \quad \text{for} \quad x + n > y \tag{16.46a}$$

and

$$\frac{120({}^{x+n}\bar{M}^{i}_{x} + \tfrac{1}{24}\, v^{m/12}\, {}_{x+n}\bar{M}^{i}_{x})}{N^{aa}_{x} - N^{aa}_{x+n}} \quad \text{for} \quad x + n \leq y. \tag{16.46b}$$

These formulas assume that the monthly *temporary* annuity can be evaluated as a continuous temporary annuity with an additive adjustment of $\frac{1}{24}$, exactly as the monthly *life* annuity is evaluated in (16.43). The student will recognize that a further small adjustment would be made if the standard approximation for a temporary annuity were used.

7. Waiver of premium benefits

When a waiver of premium provision is included in a life insurance policy, it provides for waiving premiums for the policy during the continuance of disability. The benefit is thus similar to a disability income benefit for the amount of premium being waived. In order to avoid consideration of the many different fractional premium arrangements which are encountered in practice, it is customary to assume that the premiums being waived are payable continuously. With this assumption, it is possible to use the symbol ${}^{u}\bar{C}_x^i$ without adjustment. Indeed, it is for this reason that ${}^{u}\bar{C}_x^i$ is defined in terms of a continuous disabled life annuity, the waiver benefit being a more common type of disability benefit than the income benefit.

For an n-payment policy, the net annual premium to provide the waiver of a unit of premium from the end of the waiting period is

$$\frac{{}^{x+n}{}_{y}\bar{M}_x^i}{N_x^{aa} - N_y^{aa}} \quad \text{for} \quad x + n > y \tag{16.47a}$$

and

$$\frac{{}^{x+n}\bar{M}_x^i}{N_x^{aa} - N_{x+n}^{aa}} \quad \text{for} \quad x + n \leq y, \tag{16.47b}$$

assuming that such net annual premium ceases upon the occurrence of a qualifying disability or upon the attainment of age y.

A further adjustment must be added when the waiver benefit is made retroactive to the beginning of the waiting period. In this case, any premiums paid during the waiting period are refunded at the end of the waiting period if the insured remains disabled. Under the assumption of continuous premium payments, the retroactive feature is thus equivalent to a lump sum payment of the total premium for m months. The net annual premium for this additional benefit (on a unit premium basis) is

$$\frac{\frac{m}{12}\, v^{m/12}\ {}_{y}\bar{M}_x^i}{N_x^{aa} - N_y^{aa}} \quad \text{for} \quad x + n > y \tag{16.48a}$$

and

$$\frac{\frac{m}{12}\, v^{m/12}\ {}_{x+n}\bar{M}_x^i}{N_x^{aa} - N_{x+n}^{aa}} \quad \text{for} \quad x + n \leq y. \tag{16.48b}$$

These expressions are added to the corresponding (16.47) formulas.

8. Disability reserves

Reserves for disability benefits are of two kinds, active life reserves and disabled life reserves. As the name implies, the active life reserve is the present value of future disability benefits less the present value of future disability premiums, both calculated on the assumption that the insured is not disabled. Similarly, the disabled life reserve is the present value of future disability benefits, calculated on the assumption that the insured has already incurred a qualifying disability.

To illustrate the active life reserve, we consider an n-payment life policy issued at age x with a disability clause providing waiver of premium and a disability income for life of \$10 per month in the event of disability prior to age y. It will be assumed that $x + n$ is greater than y.

Taking first the income portion of the benefit, we denote the net annual disability premium per dollar of annual income benefits by P_x^I. The active life reserve at the end of t years $(t < y - x)$ is

$$120\left[\frac{{}^{\omega}_{y}\bar{M}^{i}_{x+t} + \frac{1}{24}v^{m/12}\,{}_{y}\bar{M}^{i}_{x+t}}{D^{aa}_{x+t}} - P^I_x \cdot \ddot{a}^{aa}_{x+t:\overline{y-x-t}|}\right]. \quad (16.49a)$$

Turning now to the waiver benefit, we assume that the amount of annual premium to be waived is P. Denoting the net annual disability premium per unit of annual premium waived by P_x^W, we can write the active life reserve at the end of t years $(t < y - x)$ as

$$P\left[\frac{{}^{x+n}_{y}\bar{M}^{i}_{x+t} + \frac{m}{12}v^{m/12}\,{}_{y}\bar{M}^{i}_{x+t}}{D^{aa}_{x+t}} - P^W_x \cdot \ddot{a}^{aa}_{x+t:\overline{y-x-t}|}\right]. \quad (16.49b)$$

Active life terminal reserves can also be obtained from a formula analogous to the insurance formula

$${}_tV_x = (P_{x+t} - P_x)\,\ddot{a}_{x+t}\,.$$

In this form, the active life reserves shown above may be written

$$120\ (P^I_{x+t} - P^I_x)\ \ddot{a}^{aa}_{x+t:\overline{y-x-t}|} \quad (16.50a)$$

and

$$P(P^W_{x+t} - P^W_x)\ \ddot{a}^{aa}_{x+t:\overline{y-x-t}|}\,. \quad (16.50b)$$

There are no special problems involved in obtaining the second type of disability reserve, the disabled life reserve. The amount of premium waived or income paid is multiplied by the appropriate disabled life annuity value, taking account of age at initial date of disability, duration, and terminal age for the annuity.

9. Special methods used in current disability practice

The general model on which the preceding disability formulas are based requires a double-decrement table for the active lives. In current practice, it is customary to use values of D_x and N_x from an existing standard mortality table in place of D_x^{aa} and N_x^{aa} wherever they occur in the formulas above. Similarly, the active life annuity values in these formulas are replaced by the corresponding annuities based on the standard mortality table. Although this may appear to be a rather crude practical expedient, it has been found that only a slight distortion is produced in the resulting monetary values.

When the double-decrement table is eliminated in this way, the function i_x is unavailable and special methods must be used to obtain values of $\bar{C}_x^i$. Modern disability investigations produce values of the rate of disability r_x', and $\bar{C}_x^i$ can then be obtained from the following development, which makes liberal use of approximations:

$$r_x' \doteq \frac{i_x}{l_x^{aa} - \frac{1}{2}d_x^{aa}}$$

and hence

$$i_x \doteq r_x'(l_x^{aa} - \tfrac{1}{2}d_x^{aa}).$$

Consistent with the assumptions just described for D_x^{aa}, we can write

$$i_x \doteq r_x'(l_x - \tfrac{1}{2}d_x) \doteq r_x'\, l_{x+\frac{1}{2}}\,.$$

Hence

$$\begin{aligned} \bar{C}_x^i &= v^{x+\frac{1}{2}}\, i_x \\ &\doteq v^{x+\frac{1}{2}}\, l_{x+\frac{1}{2}}\, r_x' \\ &\doteq \bar{D}_x\, r_x'. \end{aligned}$$

Thus we can calculate values of $\bar{C}_x^i$ using the function $\bar{D}_x$ based on the standard mortality table.

These special methods are not fully standardized. In using published volumes of disability functions, the student should be careful to note the specifications for the functions. It is also important to be aware of the notation being used, which may differ from that developed in this chapter.

References

2. A basic reference for the pension plan material in this chapter is the paper by Noback (1950). Note, however, that Noback's notation differs in many respects from that used in this chapter.

Many pension plans provide benefits to the widow of a participant. The evaluation of widows' benefits has been treated by Niessen (1949, 1963).

3. A survey of the actuarial techniques connected with salary scale mathematics is given by Marples (1962).

5. Our discussion of disability benefits relates to the type of coverage offered as a supplementary benefit with life insurance policies. The notation and the formulas used when disability benefits are provided by individual health insurance policies are treated in the book by Bartleson (1963).

The details of the calculation of disabled life annuity values may be seen in Cueto (1954).

9. The nature of the approximation involved in replacing D_x^{aa} by D_x is discussed by Phillips (1929).

The methods used in the 1952 Disability Study are described in the Report of the Committee on Disability and Double Indemnity, published in the Society of Actuaries' 1952 Reports of Mortality and Morbidity Experience. They are also discussed by Cueto (1954).

EXERCISES

1. Description of the model

1. Consider the example in the text, with net single premium given by (16.6).
   - (a) What change should be made in the definition of ${}^a\bar{C}_x^g$ if the annuity benefit is to be payable monthly with first payment at the time of transfer to state g?
   - (b) Define a similar commutation function for the case where the annuity is to be payable quarterly with first payment at the end of the policy year of transfer to state g.
   - (c) Write the net single premium for the benefit in (b) using a function of the type defined in (16.7).

2. The function ${}_yM_x^g$, where y is fixed, is tabulated for values of x. It is desired to obtain values of ${}_zM_x^g$, where z is a fixed age less than y. Show that

$$ {}_zM_x^g = {}_yM_x^g - {}_yM_z^g\,, \; z < y. $$

3. Using ${}^{s}\bar{C}_x^s$ as defined in the text, obtain an approximating expression for

$$\int_{20}^{50} (x-20)v^x l_x^T \mu_x^s \bar{a}_x^s \, dx.$$

2. Pension plans

4. For a given age x (less than 65), the probability of retirement $q_{x+t}^r =$ .002 $(t+1)$ for all t. Show that

$${}_{65}\bar{M}_x^r = .002\, v^{\frac{1}{2}}[S_x^T - S_{65}^T - (65-x)\, N_{65}^T].$$

5. If an employee aged x retires or becomes permanently disabled within the next year, he will receive an annual life income of \$4000. Find the net single premium for this one year's coverage, assuming that the income is payable on a continuous basis and given the following:

$D_x^T = 1000, \quad {}^{a}\bar{C}_x^r = 160, \quad i_x = 1.5r_x\,, \quad \bar{a}_{x+\frac{1}{2}}^i = .08\bar{a}_{x+\frac{1}{2}}^r\,.$ (Ans. \$716.80)

6. How should (16.8) be modified if the plan provides that participants (entering the plan at age x) who are still in service at the end of $65 - x$ years must retire at that time?

3. Pension benefits based on salary

7. A salary scale function has the property that $S_y = (1+i)^{y-x} S_x$. Find an expression for ${}_5Z_{65}$ in terms of S_x and functions involving interest only.

8. A pension plan provides a retirement income of 20% of the five-year final average salary. For an employee now aged x (where x is not within five years of the earliest retirement age) who is earning \$10,000 per year, express the present value of this benefit in terms of commutation functions.

9. Express in terms of commutation functions the present value of the following benefits for an employee hired at age x:

(i) a retirement pension equal to 2% of annual salary at the rate of pay applicable in the year of retirement for each of the first 20 completed years of service, plus 1% of such salary for each of the following 10 completed years of service; and

(ii) a disability pension equal to \$100 a year for each completed year of service, such pension not to exceed \$3,000 a year.

4. Participants' contributions

10. To what does ${}^{s^w}{}_i\bar{R}_{x+1}^w$ reduce if $j = i$?

11. To what does (16.34) reduce if $j = i$?

12. A pension plan requires contributions by members of 3% of salary. It provides on withdrawal a return of contributions with interest at rate j. An employee now aged 40 has a monthly salary of \$500. Give an expression in terms of commutation functions for each of the following:

(a) the present value of the withdrawal benefit arising from contributions in the year of age 50 to 51;

(b) the total present value of the withdrawal benefit.

5. Disability benefits

13. For each of the following disability benefits, payable in the event of disability prior to age 60, find the net single premium at age 35 in terms of elementary functions and also in terms of commutation functions:
   (a) a payment of \$100 at the end of a four months waiting period;
   (b) a continuous disabled life annuity of \$1000 per year, commencing at the end of a 6 months waiting period;
   (c) a continuous disabled life annuity of \$2000 per year, commencing at the end of a 3 months waiting period and payable up to age 65.

6. Income benefits

14. In deriving formulas (16.46a) and (16.46b), the following approximation is used:

$$\ddot{a}^{i(12)}_{[z+\frac{1}{2}]+m/12:\overline{x+n-z-\frac{6+m}{12}}|} \doteq \bar{a}^{i}_{[z+\frac{1}{2}]+m/12:\overline{x+n-z-\frac{6+m}{12}}|} + k, \text{ where } k = \frac{1}{24}.$$

What would be the expression for k on the basis of the standard approximations?

15. Find, in terms of commutation functions, the net annual premium at age 30 for a disability income benefit of \$10 per month issued as a rider with a 20-year endowment policy. The waiting period is 6 months, and the disability income ceases upon maturity of the endowment.

16. For the benefit described in exercise 15, find the net annual premium at age 45 if disability must occur before age 60.

7. Waiver of premium benefits

17. A disability waiver of premium benefit included in an ordinary life policy issued at age 30 provides for waiver of premiums falling due after a six month waiting period and during continued disability, but only if disability occurs before age 55. Premiums falling due during the waiting period are waived retroactively. If the gross annual premium to be waived is \$100, find the net annual premium for the waiver of premium benefit.

18. For the benefit described in exercise 17, find the net annual premium if the benefit is included:
   (a) in a 15-payment 25-year endowment issued at age 35;
   (b) in a 30-payment life policy issued at age 45.

8. Disability reserves

19. Write formulas for the active life reserves corresponding to (16.49a) and (16.49b) for the case where $x + n < y$ and $t < n$.

20. If the reserve given by (16.49a) is denoted by 120 ${}_tV^I_x$, show that

$${}_{t+1}V^I_x = ({}_tV^I_x + P^I_x)u^{aa}_{x+t} - ({}^{\omega}\bar{k}^i_{x+t} + 1/24\, v^{m/12}\, \bar{k}^i_{x+t}),$$

where

$$u^{aa}_x = \frac{D^{aa}_x}{D^{aa}_{x+1}}, \quad \bar{k}^i_x = \frac{\bar{C}^i_x}{D^{aa}_{x+1}}, \quad \text{and} \quad {}^{\omega}\bar{k}^i_x = \frac{{}^{\omega}\bar{C}^i_x}{D^{aa}_{x+1}}.$$

Miscellaneous problems

21. A pension plan provides at retirement an annual pension of 20% of one year's salary at the rate of pay in the year of retirement plus, for each completed year of service at retirement, 1% of m-year final average salary. The pension is paid on a five years certain and life thereafter basis. The employee contributes ½% of each year's salary. Employee contributions cease after 30 years even if the employee has not retired. If the employee terminates or dies prior to retirement, the accumulation of his contributions at rate of interest j is returned. What is the present value of all future benefits for an employee hired at age x at an annual salary of \$5,000?

22. A disability rider, attached at issue to an ordinary life policy on (x), provides the following benefits if (x) becomes totally disabled before age y and remains disabled during a waiting period of 6 months:

(i) a monthly income commencing at the end of the waiting period and payable for life during continued disability;

(ii) waiver of the life insurance premiums falling due on or after the initial date of disability and during continued disability.

Find the amount of the monthly income and the amount of the annual premium to be waived (assumed payable continuously) if the net annual premium for the disability rider is

$$\frac{1720\ {}^{\omega}_{y}\bar{M}^{i}_{x} + 200\ v^{\frac{1}{2}}\ {}_{y}\bar{M}^{i}_{x}}{N^{aa}_{x} - N^{aa}_{y}}\ .$$

APPENDICES

APPENDIX I

TABLE A

The Commissioners 1958 Standard Ordinary (1958 CSO) Mortality Table
Male Lives

x	l_x	d_x	$1000q_x$
0	10,000,000	70,800	7.08
1	9,929,200	17,475	1.76
2	9,911,725	15,066	1.52
3	9,896,659	14,449	1.46
4	9,882,210	13,835	1.40
5	9,868,375	13,322	1.35
6	9,855,053	12,812	1.30
7	9,842,241	12,401	1.26
8	9,829,840	12,091	1.23
9	9,817,749	11,879	1.21
10	9,805,870	11,865	1.21
11	9,794,005	12,047	1.23
12	9,781,958	12,325	1.26
13	9,769,633	12,896	1.32
14	9,756,737	13,562	1.39
15	9,743,175	14,225	1.46
16	9,728,950	14,983	1.54
17	9,713,967	15,737	1.62
18	9,698,230	16,390	1.69
19	9,681,840	16,846	1.74
20	9,664,994	17,300	1.79
21	9,647,694	17,655	1.83
22	9,630,039	17,912	1.86
23	9,612,127	18,167	1.89
24	9,593,960	18,324	1.91
25	9,575,636	18,481	1.93
26	9,557,155	18,732	1.96
27	9,538,423	18,981	1.99
28	9,519,442	19,324	2.03
29	9,500,118	19,760	2.08
30	9,480,358	20,193	2.13
31	9,460,165	20,718	2.19
32	9,439,447	21,239	2.25
33	9,418,208	21,850	2.32
34	9,396,358	22,551	2.40

Male Lives

x	l_x	d_x	$1000q_x$
35	9,373,807	23,528	2.51
36	9,350,279	24,685	2.64
37	9,325,594	26,112	2.80
38	9,299,482	27,991	3.01
39	9,271,491	30,132	3.25
40	9,241,359	32,622	3.53
41	9,208,737	35,362	3.84
42	9,173,375	38,253	4.17
43	9,135,122	41,382	4.53
44	9,093,740	44,741	4.92
45	9,048,999	48,412	5.35
46	9,000,587	52,473	5.83
47	8,948,114	56,910	6.36
48	8,891,204	61,794	6.95
49	8,829,410	67,104	7.60
50	8,762,306	72,902	8.32
51	8,689,404	79,160	9.11
52	8,610,244	85,758	9.96
53	8,524,486	92,832	10.89
54	8,431,654	100,337	11.90
55	8,331,317	108,307	13.00
56	8,223,010	116,849	14.21
57	8,106,161	125,970	15.54
58	7,980,191	135,663	17.00
59	7,844,528	145,830	18.59
60	7,698,698	156,592	20.34
61	7,542,106	167,736	22.24
62	7,374,370	179,271	24.31
63	7,195,099	191,174	26.57
64	7,003,925	203,394	29.04
65	6,800,531	215,917	31.75
66	6,584,614	228,749	34.74
67	6,355,865	241,777	38.04
68	6,114,088	254,835	41.68
69	5,859,253	267,241	45.61
70	5,592,012	278,426	49.79
71	5,313,586	287,731	54.15
72	5,025,855	294,766	58.65
73	4,731,089	299,289	63.26
74	4,431,800	301,894	68.12

1958 CSO Mortality Table—Continued

Male Lives

x	l_x	d_x	$1000q_x$
75	4,129,906	303,011	73.37
76	3,826,895	303,014	79.18
77	3,523,881	301,997	85.70
78	3,221,884	299,829	93.06
79	2,922,055	295,683	101.19
80	2,626,372	288,848	109.98
81	2,337,524	278,983	119.35
82	2,058,541	265,902	129.17
83	1,792,639	249,858	139.38
84	1,542,781	231,433	150.01
85	1,311,348	211,311	161.14
86	1,100,037	190,108	172.82
87	909,929	168,455	185.13
88	741,474	146,997	198.25
89	594,477	126,303	212.46
90	468,174	106,809	228.14
91	361,365	88,813	245.77
92	272,552	72,480	265.93
93	200,072	57,881	289.30
94	142,191	45,026	316.66
95	97,165	34,128	351.24
96	63,037	25,250	400.56
97	37,787	18,456	488.42
98	19,331	12,916	668.15
99	6,415	6,415	1000.00

COMMUTATION COLUMNS
1958 CSO 3%
Male Lives

x	D_x	N_x	S_x	x
0	10,000,000.0	288,963,016.7	6,979,643,888.8	0
1	9,640,000.0	278,963,016.7	6,690,680,872.1	1
2	9,342,751.4	269,323,016.7	6,411,717,855.4	2
3	9,056,844.9	259,980,265.3	6,142,394,838.7	3
4	8,780,215.6	250,923,420.4	5,882,414,573.4	4
5	8,512,546.9	242,143,204.8	5,631,491,153.0	5
6	8,253,451.8	233,630,657.9	5,389,347,948.2	6
7	8,002,642.6	225,377,206.1	5,155,717,290.3	7
8	7,759,766.4	217,374,563.5	4,930,340,084.2	8
9	7,524,487.1	209,614,797.1	4,712,965,520.7	9
10	7,296,488.1	202,090,310.0	4,503,350,723.6	10
11	7,075,397.6	194,793,821.9	4,301,260,413.6	11
12	6,860,868.5	187,718,424.3	4,106,466,591.7	12
13	6,652,644.7	180,857,555.8	3,918,748,167.4	13
14	6,450,352.6	174,204,911.1	3,737,890,611.6	14
15	6,253,773.3	167,754,558.5	3,563,685,700.5	15
16	6,062,760.0	161,500,785.2	3,395,931,142.0	16
17	5,877,109.8	155,438,025.2	3,234,430,356.8	17
18	5,696,688.0	149,560,915.4	3,078,992,331.6	18
19	5,521,418.1	143,864,227.4	2,929,431,416.2	19
20	5,351,272.8	138,342,809.3	2,785,567,188.8	20
21	5,186,111.0	132,991,536.5	2,647,224,379.5	21
22	5,025,845.1	127,805,425.5	2,514,232,843.0	22
23	4,870,385.5	122,779,580.4	2,386,427,417.5	23
24	4,719,592.6	117,909,194.9	2,263,647,837.1	24
25	4,573,377.1	113,189,602.3	2,145,738,642.2	25
26	4,431,602.4	108,616,225.2	2,032,549,039.9	26
27	4,294,093.7	104,184,622.8	1,923,932,814.7	27
28	4,160,726.8	99,890,529.1	1,819,748,191.9	28
29	4,031,340.5	95,729,802.3	1,719,857,662.8	29
30	3,905,782.0	91,698,461.8	1,624,127,860.5	30
31	3,783,944.4	87,792,679.8	1,532,429,398.7	31
32	3,665,686.8	84,008,735.4	1,444,636,718.9	32
33	3,550,911.6	80,343,048.6	1,360,627,983.5	33
34	3,439,488.9	76,792,137.0	1,280,284,934.9	34

1958 CSO 3%
Male Lives

x	D_x	N_x	S_x	x
35	3,331,295.4	73,352,648.1	1,203,492,797.9	35
36	3,226,149.5	70,021,352.7	1,130,140,149.8	36
37	3,123,914.9	66,795,203.2	1,060,118,797.1	37
38	3,024,434.7	63,671,288.3	993,323,593.9	38
39	2,927,506.2	60,646,853.6	929,652,305.6	39
40	2,833,001.8	57,719,347.4	869,005,452.0	40
41	2,740,778.0	54,886,345.6	811,286,104.6	41
42	2,650,731.3	52,145,567.6	756,399,759.0	42
43	2,562,794.0	49,494,836.3	704,254,191.4	43
44	2,476,878.2	46,932,042.3	654,759,355.1	44
45	2,392,904.8	44,455,164.1	607,827,312.8	45
46	2,310,779.5	42,062,259.3	563,372,148.7	46
47	2,230,395.8	39,751,479.8	521,309,889.4	47
48	2,151,660.7	37,521,084.0	481,558,409.6	48
49	2,074,472.4	35,369,423.3	444,037,325.6	49
50	1,998,744.0	33,294,950.9	408,667,902.3	50
51	1,924,383.0	31,296,206.9	375,372,951.4	51
52	1,851,312.7	29,371,823.9	344,076,744.5	52
53	1,779,488.9	27,520,511.2	314,704,920.6	53
54	1,708,844.9	25,741,022.3	287,184,409.4	54
55	1,639,329.7	24,032,177.4	261,443,387.1	55
56	1,570,891.7	22,392,847.7	237,411,209.7	56
57	1,503,465.3	20,821,956.0	215,018,362.0	57
58	1,436,991.7	19,318,490.7	194,196,406.0	58
59	1,371,420.2	17,881,499.0	174,877,915.3	59
60	1,306,723.8	16,510,078.8	156,996,416.3	60
61	1,242,859.2	15,203,355.0	140,486,337.5	61
62	1,179,823.4	13,960,495.8	125,282,982.5	62
63	1,117,613.4	12,780,672.4	111,322,486.7	63
64	1,056,231.5	11,663,059.0	98,541,814.3	64
65	995,687.8	10,606,827.5	86,878,755.3	65
66	935,994.9	9,611,139.7	76,271,927.8	66
67	877,163.6	8,675,144.8	66,660,788.1	67
68	819,219.7	7,797,981.2	57,985,643.3	68
69	762,208.4	6,978,761.5	50,187,662.1	69

1958 CSO 3%
Male Lives

x	D_x	N_x	S_x	x
70	706,256.4	6,216,553.1	43,208,900.6	70
71	651,545.5	5,510,296.7	36,992,347.5	71
72	598,314.8	4,858,751.2	31,482,050.8	72
73	546,819.2	4,260,436.4	26,623,299.6	73
74	497,308.1	3,713,617.2	22,362,863.2	74
75	449,933.5	3,216,309.1	18,649,246.0	75
76	404,778.5	2,766,375.6	15,432,936.9	76
77	361,872.0	2,361,597.1	12,666,561.3	77
78	321,222.8	1,999,725.1	10,304,964.2	78
79	282,844.4	1,678,502.3	8,305,239.1	79
80	246,818.8	1,395,657.9	6,626,736.8	80
81	213,275.5	1,148,839.1	5,231,078.9	81
82	182,350.6	935,563.6	4,082,239.8	82
83	154,171.2	753,213.0	3,146,676.2	83
84	128,818.2	599,041.8	2,393,463.2	84
85	106,305.0	470,223.6	1,794,421.4	85
86	86,577.7	363,918.6	1,324,197.8	86
87	69,529.5	277,340.9	960,279.2	87
88	55,007.3	207,811.4	682,938.3	88
89	42,817.6	152,804.1	475,126.9	89
90	32,738.4	109,986.5	322,322.8	90
91	24,533.4	77,248.1	212,336.3	91
92	17,964.9	52,714.7	135,088.2	92
93	12,803.4	34,749.8	82,373.5	93
94	8,834.3	21,946.4	47,623.7	94
95	5,861.0	13,112.1	25,677.3	95
96	3,691.7	7,251.1	12,565.2	96
97	2,148.5	3,559.4	5,314.1	97
98	1,067.1	1,410.9	1,754.7	98
99	343.8	343.8	343.8	99

1958 CSO 3%
Male Lives

x	C_x	M_x	R_x	x
0	68,737.864	1,583,601.456	85,672,418.432	0
1	16,471.864	1,514,863.592	84,088,816.976	1
2	13,787.524	1,498,391.728	82,573,953.384	2
3	12,837.749	1,484,604.204	81,075,561.656	3
4	11,934.192	1,471,766.455	79,590,957.452	4
5	11,156.965	1,459,832.263	78,119,190.997	5
6	10,417.328	1,448,675.298	76,659,358.734	6
7	9,789.464	1,438,257.970	75,210,683.436	7
8	9,266.745	1,428,468.506	73,772,425.466	8
9	8,839.092	1,419,201.761	72,343,956.960	9
10	8,571.528	1,410,362.669	70,924,755.199	10
11	8,449.523	1,401,791.141	69,514,392.530	11
12	8,392.725	1,393,341.618	68,112,601.389	12
13	8,525.775	1,384,948.893	66,719,259.771	13
14	8,704.932	1,376,423.118	65,334,310.878	14
15	8,864.550	1,367,718.186	63,957,887.760	15
16	9,064.961	1,358,853.636	62,590,169.574	16
17	9,243.829	1,349,788.675	61,231,315.938	17
18	9,346.988	1,340,544.846	59,881,527.263	18
19	9,327.222	1,331,197.858	58,540,982.417	19
20	9,299.603	1,321,870.636	57,209,784.559	20
21	9,214.012	1,312,571.033	55,887,913.923	21
22	9,075.863	1,303,357.021	54,575,342.890	22
23	8,936.960	1,294,281.158	53,271,985.869	23
24	8,751.644	1,285,344.198	51,977,704.711	24
25	8,569.542	1,276,592.554	50,692,360.513	25
26	8,432.941	1,268,023.012	49,415,767.959	26
27	8,296.154	1,259,590.071	48,147,744.947	27
28	8,200.069	1,251,293.917	46,888,154.876	28
29	8,140.858	1,243.093.848	45,636,860.959	29
30	8,076.941	1,234,952.990	44,393,767.111	30
31	8,045.567	1,226,876.049	43,158,814.121	31
32	8,007.661	1,218,830.482	41,931,938.072	32
33	7,998.081	1,210,822.821	40,713,107.590	33
34	8,014.251	1,202,824.740	39,502,284.769	34

1958 CSO 3%

Male Lives

x	C_x	M_x	R_x	x
35	8,117.923	1,194,810.489	38,299,460.029	35
36	8,269.054	1,186,692.566	37,104,649.540	36
37	8,492.305	1,178,423.512	35,917,956.974	37
38	8,838.258	1,169,931.207	34,739,533.462	38
39	9,237.171	1,161,092.949	33,569,602.255	39
40	9,709.221	1,151,855.778	32,408,509.306	40
41	10,218.176	1,142,146.557	31,256,653.528	41
42	10,731.609	1,131,928.381	30,114,506.971	42
43	11,271.289	1,121,196.772	28,982,578.590	43
44	11,831.248	1,109,925.483	27,861,381.818	44
45	12,429.129	1,098,094.235	26,751,456.335	45
46	13,079.355	1,085,665.106	25,653,362.100	46
47	13,772.152	1,072,585.751	24,567,696.994	47
48	14,518.518	1,058,813.599	23,495,111.243	48
49	15,306.897	1,044,295.081	22,436,297.644	49
50	16,145.109	1,028,988.184	21,392,002.563	50
51	17,020.413	1,012,843.075	20,363,014.379	51
52	17,902.007	995,822.662	19,350,171.304	52
53	18,814.279	977,920.655	18,354,348.642	53
54	19,743.028	959,106.376	17,376,427.987	54
55	20,690.546	939,363.348	16,417,321.611	55
56	21,672.210	918,672.802	15,477,958.263	56
57	22,683.398	897,000.592	14,559,285.461	57
58	23,717.295	874,317.194	13,662,284.869	58
59	24,752.177	850,599.899	12,787,967.675	59
60	25,804.703	825,847.722	11,937,367.776	60
61	26,836.036	800,043.019	11,111,520.054	61
62	27,846.132	773,206.983	10,311,477.035	62
63	28,830.119	745,360.851	9,538,270.052	63
64	29,779.577	716,530.732	8,792,909.201	64
65	30,692.340	686,751.155	8,076,378.469	65
66	31,569.313	656,058.815	7,389,627.314	66
67	32,395.427	624,489.502	6,733,568.499	67
68	33,150.537	592,094.075	6,109,078.997	68
69	33,751.833	558,943.538	5,516,984.922	69

1958 CSO 3%
Male Lives

x	C_x	M_x	R_x	x
70	34,140.262	525,191.705	4,958,041.384	70
71	34,253.619	491,051.443	4,432,849.679	71
72	34,069.048	456,797.824	3,941,798.236	72
73	33,584.287	422,728.776	3,485,000.412	73
74	32,889.905	389,144.489	3,062,271.636	74
75	32,050.095	356,254.584	2,673,127.147	75
76	31,116.905	324,204.489	2,316,872.563	76
77	30,109.191	293,087.584	1,992,668.074	77
78	29,022.371	262,978.393	1,699,580.490	78
79	27,787.431	233,956.022	1,436,602.097	79
80	26,354.463	206,168.591	1,202,646.075	80
81	24,712.992	179,814.128	996,477.484	81
82	22,868.200	155,101.136	816,663.356	82
83	20,862.501	132,232.936	661,562.220	83
84	18,761.225	111,370.435	529,329.284	84
85	16,631.093	92,609.210	417,958.849	85
86	14,526.529	75,978.117	325,349.639	86
87	12,497.068	61,451.588	249,371.522	87
88	10,587.550	48,954.520	187,919.934	88
89	8,832.090	38,366.970	138,965.414	89
90	7,251.375	29,534.880	100,598.444	90
91	5,853.988	22,283.505	71,063.564	91
92	4,638.273	16,429.517	48,780.059	92
93	3,596.142	11,791.244	32,350.542	93
94	2,715.983	8,195.102	20,559.298	94
95	1,998.652	5,479.119	12,364.196	95
96	1,435.657	3,480.467	6,885.077	96
97	1,018.801	2,044.810	3,404.610	97
98	692.218	1,026.009	1,359.800	98
99	333.791	333.791	333.791	99

Net Single Premiums
1958 CSO 3%
Male Lives

x	$1000A_x$	$\ddot{a}_x$	$\ddot{a}_{xx}$
0	158.360 15	28.896 30	27.046 89
1	157.143 53	28.938 07	27.212 26
2	160.380 13	28.826 95	27.094 00
3	163.920 68	28.705 39	26.958 78
4	167.623 04	28.578 28	26.815 84
5	171.491 83	28.445 45	26.664 98
6	175.523 57	28.307 02	26.506 50
7	179.722 88	28.162 85	26.340 17
8	184.086 53	28.013 03	26.166 32
9	188.611 10	27.857 69	25.985 23
10	193.293 36	27.696 93	25.797 22
11	198.121 89	27.531 15	25.603 10
12	203.085 31	27.360 74	25.403 68
13	208.180 20	27.185 81	25.199 30
14	213.387 27	27.007 04	24.991 25
15	218.702 87	26.824 53	24.779 88
16	224.131 19	26.638 16	24.565 00
17	229.668 79	26.448 04	24.346 94
18	235.320 04	26.254 01	24.125 52
19	241.097 09	26.055 67	23.900 06
20	247.019 86	25.852 32	23.669 44
21	253.093 51	25.643 79	23.433 41
22	259.330 92	25.429 64	23.191 29
23	265.745 12	25.209 42	22.942 38
24	272.342 19	24.982 92	22.686 40
25	279.135 64	24.749 68	22.422 65
26	286.131 94	24.509 47	22.150 83
27	293.330 83	24.262 31	21.871 09
28	300.739 26	24.007 95	21.583 13
29	308.357 44	23.746 39	21.287 04
30	316.185 85	23.477 62	20.982 94
31	324.232 05	23.201 37	20.670 49
32	332.497 17	22.917 60	20.349 73
33	340.989 29	22.626 03	20.020 12
34	349.710 31	22.326 61	19.681 85

1958 CSO 3%
Male Lives

x	$1000A_x$	$\ddot{a}_x$	$\ddot{a}_{xx}$
35	358.662 43	22.019 26	19.334 92
36	367.835 58	21.704 31	18.980 05
37	377.226 51	21.381 89	18.617 57
38	386.826 41	21.052 29	18.248 10
39	396.615 03	20.716 22	17.872 96
40	406.584 91	20.373 92	17.492 68
41	416.723 48	20.025 83	17.108 07
42	427.024 94	19.672 14	16.719 55
43	437.490 01	19.312 84	16.326 98
44	448.114 68	18.948 06	15.930 80
45	458.895 91	18.577 91	15.531 24
46	469.826 35	18.202 63	15.128 60
47	480.894 80	17.822 61	14.723 70
48	492.091 34	17.438 20	14.316 95
49	503.402 74	17.049 84	13.909 10
50	514.817 40	16.657 94	13.500 77
51	526.320 94	16.262 98	13.092 74
52	537.900 84	15.865 40	12.685 64
53	549.551 42	15.465 40	12.279 59
54	561.260 05	15.063 40	11.875 20
55	573.016 73	14.659 76	11.472 90
56	584.809 76	14.254 86	11.073 11
57	596.622 08	13.849 31	10.676 59
58	608.435 80	13.443 70	10.284 05
59	620.232 88	13.038 67	9.896 17
60	631.998 68	12.634 71	9.513 44
61	643.711 71	12.232 56	9.136 78
62	655.358 24	11.832 70	8.766 52
63	666.921 90	11.435 68	8.403 12
64	678.384 17	11.042 14	8.047 12
65	689.725 39	10.652 76	7.699 24
66	700.921 36	10.268 37	7.360 15
67	711.941 88	9.890 00	7.030 96
68	722.753 71	9.518 79	6.712 87
69	733.321 15	9.155 98	6.407 22

1958 CSO 3%
Male Lives

x	$1000A_x$	$\ddot{a}_x$	$\ddot{a}_{xx}$
70	743.627 53	8.802 12	6.114 48
71	753.671 76	8.457 27	5.834 44
72	763.474 05	8.120 73	5.565 96
73	773.068 64	7.791 31	5.307 21
74	782.501 81	7.467 44	5.055 88
75	791.793 86	7.148 41	4.810 63
76	800.942 96	6.834 29	4.571 12
77	809.920 59	6.526 06	4.338 03
78	818.679 10	6.225 35	4.112 94
79	827.154 51	5.934 37	3.898 08
80	835.303 43	5.654 59	3.694 98
81	843.107 29	5.386 64	3.504 23
82	850.565 54	5.130 58	3.325 87
83	857.701 93	4.885 56	3.159 04
84	864.555 12	4.650 29	3.002 44
85	871.165 14	4.423 34	2.854 76
86	877.571 44	4.203 38	2.714 85
87	883.820 36	3.988 82	2.581 45
88	889.964 06	3.777 89	2.453 11
89	896.056 06	3.568 72	2.328 40
90	902.147 94	3.359 56	2.206 08
91	908.292 57	3.148 69	2.085 14
92	914.534 29	2.934 32	1.964 79
93	920.946 31	2.714 11	1.844 15
94	927.645 88	2.484 23	1.721 41
95	934.843 71	2.237 18	1.591 28
96	942.781 65	1.964 16	1.446 98
97	951.738 42	1.656 69	1.281 26
98	961.492 83	1.322 18	1.106 92
99	970.887 14	1.000 00	1.000 00

1958 CSO Table
Table of Uniform Seniority[1]

Difference of Age	Addition to Younger Age	Difference of Age	Addition to Younger Age
1	.512	41	33.720
2	1.046	42	34.699
3	1.603	43	35.679
4	2.183	44	36.661
5	2.785	45	37.645
6	3.409	46	38.630
7	4.055	47	39.617
8	4.721	48	40.604
9	5.407	49	41.593
10	6.113	50	42.582
11	6.837	51	43.573
12	7.580	52	44.564
13	8.340	53	45.556
14	9.116	54	46.549
15	9.907	55	47.543
16	10.714	56	48.537
17	11.534	57	49.531
18	12.368	58	50.526
19	13.214	59	51.522
20	14.072	60	52.517
21	14.940	61	53.514
22	15.819	62	54.510
23	16.707	63	55.507
24	17.604	64	56.504
25	18.509	65	57.502
26	19.422	66	58.499
27	20.342	67	59.497
28	21.268	68	60.495
29	22.201	69	61.493
30	23.139	70	62.491
31	24.082	71	63.490
32	25.030	72	64.489
33	25.982	73	65.487
34	26.938	74	66.486
35	27.898	75	67.485
36	28.862		
37	29.828		
38	30.797		
39	31.769		
40	32.744		

[1] Based on $\log_{10} c = .04$. Produces approximate values for joint two-life functions.

TABLE B

Life Table for White Males: United States, 1959–61

x	q_x	l_x	d_x	L_x	T_x	$\mathring{e}_x$	x
0	.02592	100,000	2,592	97,764	6,754,846	67.55	0
1	.00153	97,408	149	97,334	6,657,082	68.34	1
2	.00101	97,259	99	97,210	6,559,748	67.45	2
3	.00081	97,160	78	97,121	6,462,538	66.51	3
4	.00069	97,082	67	97,048	6,365,417	65.57	4
5	.00062	97,015	60	96,985	6,268,369	64.61	5
6	.00057	96,955	55	96,927	6,171,384	63.65	6
7	.00053	96,900	52	96,874	6,074,457	62.69	7
8	.00049	96,848	47	96,825	5,977,583	61.72	8
9	.00045	96,801	43	96,779	5,880,758	60.75	9
10	.00042	96,758	40	96,738	5,783,979	59.78	10
11	.00042	96,718	40	96,698	5,687,241	58.80	11
12	.00047	96,678	46	96,655	5,590,543	57.83	12
13	.00059	96,632	56	96,604	5,493,888	56.85	13
14	.00075	96,576	73	96,539	5,397,284	55.89	14
15	.00093	96,503	90	96,458	5,300,745	54.93	15
16	.00111	96,413	107	96,359	5,204,287	53.98	16
17	.00126	96,306	121	96,246	5,107,928	53.04	17
18	.00139	96,185	134	96,118	5,011,682	52.10	18
19	.00149	96,051	143	95,979	4,915,564	51.18	19
20	.00159	95,908	153	95,831	4,819,585	50.25	20
21	.00169	95,755	162	95,674	4,723,754	49.33	21
22	.00174	95,593	167	95,509	4,628,080	48.41	22
23	.00172	95,426	163	95,345	4,532,571	47.50	23
24	.00165	95,263	157	95,184	4,437,226	46.58	24
25	.00156	95,106	149	95,032	4,342,042	45.65	25
26	.00149	94,957	141	94,887	4,247,010	44.73	26
27	.00145	94,816	137	94,747	4,152,123	43.79	27
28	.00145	94,679	137	94,611	4,057,376	42.85	28
29	.00149	94,542	141	94,471	3,962,765	41.92	29
30	.00156	94,401	147	94,327	3,868,294	40.98	30
31	.00163	94,254	154	94,177	3,773,967	40.04	31
32	.00171	94,100	161	94,020	3,679,790	39.10	32
33	.00181	93,939	170	93,855	3,585,770	38.17	33
34	.00193	93,769	180	93,679	3,491,915	37.24	34

x	q_x	l_x	d_x	L_x	T_x	$\mathring{e}_x$	x
35	.00207	93,589	194	93,491	3,398,236	36.31	35
36	.00225	93,395	210	93,290	3,304,745	35.38	36
37	.00246	93,185	229	93,070	3,211,455	34.46	37
38	.00270	92,956	251	92,830	3,118,385	33.55	38
39	.00299	92,705	278	92,566	3,025,555	32.64	39
40	.00332	92,427	306	92,274	2,932,989	31.73	40
41	.00368	92,121	339	91,952	2,840,715	30.84	41
42	.00409	91,782	376	91,594	2,748,763	29.95	42
43	.00454	91,406	415	91,198	2,657,169	29.07	43
44	.00504	90,991	458	90,762	2,565,971	28.20	44
45	.00558	90,533	505	90,280	2,475,209	27.34	45
46	.00617	90,028	556	89,751	2,384,929	26.49	46
47	.00686	89,472	613	89,165	2,295,178	25.65	47
48	.00766	88,859	681	88,519	2,206,013	24.83	48
49	.00856	88,178	754	87,801	2,117,494	24.01	49
50	.00955	87,424	835	87,007	2,029,693	23.22	50
51	.01058	86,589	916	86,131	1,942,686	22.44	51
52	.01162	85,673	995	85,176	1,856,555	21.67	52
53	.01264	84,678	1,071	84,142	1,771,379	20.92	53
54	.01368	83,607	1,144	83,035	1,687,237	20.18	54
55	.01475	82,463	1,216	81,855	1,604,202	19.45	55
56	.01593	81,247	1,295	80,599	1,522,347	18.74	56
57	.01730	79,952	1,383	79,261	1,441,748	18.03	57
58	.01891	78,569	1,486	77,826	1,362,487	17.34	58
59	.02074	77,083	1,598	76,284	1,284,661	16.67	59
60	.02271	75,485	1,714	74,628	1,208,377	16.01	60
61	.02476	73,771	1,827	72,858	1,133,749	15.37	61
62	.02690	71,944	1,935	70,976	1,060,891	14.75	62
63	.02912	70,009	2,039	68,990	989,915	14.14	63
64	.03143	67,970	2,136	66,902	920,925	13.55	64
65	.03389	65,834	2,231	64,718	854,023	12.97	65
66	.03652	63,603	2,323	62,441	789,305	12.41	66
67	.03930	61,280	2,409	60,076	726,864	11.86	67
68	.04225	58,871	2,487	57,627	666,788	11.33	68
69	.04538	56,384	2,559	55,105	609,161	10.80	69
70	.04871	53,825	2,621	52,514	554,056	10.29	70
71	.05230	51,204	2,678	49,865	501,542	9.80	71
72	.05623	48,526	2,729	47,161	451,677	9.31	72
73	.06060	45,797	2,775	44,410	404,516	8.83	73
74	.06542	43,022	2,815	41,615	360,106	8.37	74

x	q_x	l_x	d_x	L_x	T_x	$\mathring{e}_x$	x
75	.07066	40,207	2,841	38,786	318,491	7.92	75
76	.07636	37,366	2,853	35,940	279,705	7.49	76
77	.08271	34,513	2,855	33,086	243,765	7.06	77
78	.08986	31,658	2,844	30,236	210,679	6.65	78
79	.09788	28,814	2,821	27,403	180,443	6.26	79
80	.10732	25,993	2,789	24,599	153,040	5.89	80
81	.11799	23,204	2,738	21,835	128,441	5.54	81
82	.12895	20,466	2,639	19,146	106,606	5.21	82
83	.13920	17,827	2,482	16,586	87,460	4.91	83
84	.14861	15,345	2,280	14,205	70,874	4.62	84
85	.16039	13,065	2,096	12,017	56,669	4.34	85
86	.17303	10,969	1,898	10,020	44,652	4.07	86
87	.18665	9,071	1,693	8,225	34,632	3.82	87
88	.20194	7,378	1,490	6,633	26,407	3.58	88
89	.21877	5,888	1,288	5,244	19,774	3.36	89
90	.23601	4,600	1,086	4,058	14,530	3.16	90
91	.25289	3,514	888	3,070	10,472	2.98	91
92	.26973	2,626	709	2,271	7,402	2.82	92
93	.28612	1,917	548	1,643	5,131	2.68	93
94	.30128	1,369	413	1,163	3,488	2.55	94
95	.31416	956	300	806	2,325	2.43	95
96	.32915	656	216	548	1,519	2.32	96
97	.34450	440	152	364	971	2.21	97
98	.36018	288	103	237	607	2.10	98
99	.37616	185	70	150	370	2.01	99
100	.39242	115	45	92	220	1.91	100
101	.40891	70	29	56	128	1.83	101
102	.42562	41	17	32	72	1.75	102
103	.44250	24	11	19	40	1.67	103
104	.45951	13	6	10	21	1.60	104
105	.47662	7	3	6	11	1.53	105
106	.49378	4	2	2	5	1.46	106
107	.51095	2	1	2	3	1.40	107
108	.52810	1	1	0	1	1.35	108
109	.54519	0	0	1	1	1.29	109

At the oldest ages, where the available data are scanty and unreliable, the life table functions have been adjusted in a special way. The student will note that the usual mathematical relationships do not hold consistently at the end of the table.

APPENDIX II

DERIVATIVES OF ACTUARIAL FUNCTIONS

1. $\dfrac{dl_x}{dx} = -l_x\mu_x$ See (1.9).

2. $\dfrac{d \log l_x}{dx} = -\mu_x$ See (1.10).

3. $\dfrac{\partial\, {}_tp_x}{\partial x} = {}_tp_x(\mu_x - \mu_{x+t})$

   See Exercise 11 (d) in Chapter 1.

4. $\dfrac{\partial\, {}_tp_x}{\partial t} = -{}_tp_x\mu_{x+t}$

   See Exercise 11 (e) in Chapter 1.

5. $\dfrac{dD_x}{dx} = -D_x(\mu_x + \delta)$ See (2.17).

6. $\dfrac{d\bar{N}_x}{dx} = -D_x$ See (2.36).

7. $\dfrac{d\bar{a}_x}{dx} = \bar{a}_x(\mu_x + \delta) - 1 = \mu_x\bar{a}_x - \bar{A}_x.$ See (2.37).

8. $\dfrac{da_x}{di} = -v(Ia)_x$

   See (2.48).

9. $\dfrac{d\bar{M}_x}{dx} = -\mu_xD_x$

   See Exercise 10 (a) in Chapter 3.

10. $\dfrac{d\bar{A}_x}{dx} = \bar{A}_x(\mu_x + \delta) - \mu_x$

    See Exercise 10 (b) in Chapter 3.

11. $\dfrac{dA_x}{di} = -v(IA)_x$

    See Exercise 15 (a) in Chapter 3.

12. $\dfrac{dL_x}{dx} = -d_x$ See page 170.

13. $\dfrac{dT_x}{dx} = -l_x$ See page 170.

14. $\dfrac{dY_x}{dx} = -T_x$ See page 170.

15. $\dfrac{d\mathring{e}_x}{dx} = \mu_x\mathring{e}_x - 1$

See Exercise 1 in Chapter 8.

16. $\dfrac{\partial\mathring{e}_{xy}}{\partial x} = \mu_x\mathring{e}_{xy} - {}_\infty q^{1}_{xy}$

Write $\mathring{e}_{xy} = \displaystyle\int_0^\infty {}_tp_{xy}\,dt.$

$$\text{Then } \frac{\partial\mathring{e}_{xy}}{\partial x} = \int_0^\infty \frac{\partial_t p_{xy}}{\partial x}\,dt$$

$$= \int_0^\infty {}_tp_y\,\frac{\partial_t p_x}{\partial x}\,dt$$

$$= \int_0^\infty {}_tp_y\cdot{}_tp_x(\mu_x - \mu_{x+t})\,dt$$

$$= \mu_x\mathring{e}_{xy} - {}_\infty q^{1}_{xy}\,.$$

17. $\dfrac{\partial\bar{a}_{xy}}{\partial x} = \mu_x\bar{a}_{xy} - \bar{A}^{1}_{xy}$

See Exercise 19 in Chapter 11.

18. $\dfrac{\partial\bar{a}_{y|x}}{\partial x} = \mu_x\bar{a}_{y|x} - \bar{A}^{2}_{xy}$

See Exercise 5(a) in Chapter 13.

ANSWERS TO THE EXERCISES

Chapter 1

1. $\omega = 100$

3. (a) $s(x) = \frac{1}{98}(98 - x)$

4. (d) $s(x) = 1 - \dfrac{x^{2/3}}{25}, \qquad \omega = 125$

5. $l_0 = 100{,}000 \qquad d_0 = 505$

$l_1 = 99{,}495 \qquad d_1 = 515$

$l_2 = 98{,}980 \qquad d_2 = 525$

6. $l_0 = 10{,}000 \qquad d_0 = 110$

$l_1 = 9{,}890 \qquad d_1 = 49$

$l_2 = 9{,}841 \qquad d_2 = 30$

7. (a) $\dfrac{65{,}834}{93{,}589}$

(b) 1 − (a)

(c) $\dfrac{2{,}136}{93{,}589}$

(d) $\dfrac{82{,}463 - 65{,}834}{93{,}589}$

(e) 1 − (d)

9. $\mu_x = -\log s - 2x \log w - c^x \log g \log c$

10. $l_x = k(100 - x)$

12. $1 - .992\, e^{-.03}$

17. (a) $\dfrac{1}{4}\left(\dfrac{149}{95{,}106}\right)$

(b) $\dfrac{\frac{1}{2}(90{,}533 + 90{,}028)}{92{,}427}$

(c) $\dfrac{835}{87{,}424 - \frac{1}{3}(835)}$

18. $\dfrac{k}{1 - 2k}$

21. $\log c$

22. $A = -\log s \qquad H = -2 \log w \qquad B = -\log c \cdot \log g$

23. $k(1 + Bc^x)^{-\frac{A}{B \log c}}$

24. $\dfrac{l_{[30]+4} - l_{40}}{l_{[30]+2}}$

25. (a) $\dfrac{941{,}143 - 934{,}572}{941{,}143}$

(b) $\dfrac{942{,}001 - 934{,}572}{942{,}001}$

26. (a) ${}_np^A_{[x]} > {}_np^B_x$

(b) $q^B_x > q^C_x > q^A_{[x]}$

(c) $\mu^A_x = \mu^B_x > \mu^C_x$

27. (a) ${}_{8|}q_{[47]+3}$

(b) ${}_{67|5}q_0$

(c) $d_{[27]+2}$

28. $q_x = 1 - e^{-\int_0^1 \mu_{x+t}\,dt} = 1 - e^{-\int_x^{x+1} \mu_y\,dy}$

30. $q'_x < 2q_x$

Chapter 2

10. $\dfrac{1}{D_{30}} [120(N_{40} - N_{60}) - 55(D_{40} - D_{60})]$

15. (a) $S_{x+1} - S_{x+n+1} - nN_{x+n+1}$

(b) $\dfrac{1}{D_x} (S_x - S_{x+n} - nN_{x+n})$

(c) $\dfrac{1}{D_x}\Big\{nN_x - (S_{x+1} - S_{x+n+1})$

$$- \frac{m-1}{2m}[nD_x - (N_{x+1} - N_{x+n+1})]\Big\}$$

16. $\dfrac{1}{D_x}[N_x + .1(S_{x+1} - S_{x+6}) - 1.5N_{x+25}]$

17. $\dfrac{1}{D_x}[100N_{x+1} + 300S_{x+2} - 700S_{x+7} + 400S_{x+10}]$

18. $\dfrac{1}{D_x}[hN_y - k(S_{y+1} - S_{y+n+1}) - (h - nk)N_{y+n}]$

$\dfrac{1}{D_x}[hN_y + k(S_{y+1} - S_{y+n})]$

20. $-(\bar{I}\bar{a})_x$

21. (b) $(Ia)_{40}^{2\%} \doteq -\dfrac{1.02}{.005}[(a_{40}^{2\frac{1}{4}\%} - a_{40}^{2\%}) - \frac{1}{2}(a_{40}^{3\%} - 2a_{40}^{2\frac{1}{2}\%} + a_{40}^{2\%})]$

26. (a) ${}_{6|}a_{35}$

(b) $120\,\ddot{a}^{(12)}_{35:\overline{15|}}$

(c) $2\cdot{}_{1/4|}\ddot{a}^{(2)}_{40}$

(d) $144(I^{(12)}a)^{(12)}_{60}$

(e) ${}_{15|10}\bar{a}_{20}$

27. (a) $\dfrac{N_{35}}{D_{[30]}}$

(b) $\dfrac{N_{42} - N_{54} + \frac{3}{8}(D_{41} - D_{53})}{D_{34}}$

(c) $\dfrac{N_{35} - \frac{1}{2}D_{35}}{D_{20}}$

(d) $\dfrac{N_{25} - \frac{1}{3}D_{25}}{D_{35}}$

(e) $\dfrac{S_{25} - \frac{1}{4}N_{25}}{D_{25}}$

(f) $\dfrac{S_{51} + \frac{1}{12}D_{50}}{D_{50}}$

29. $\dfrac{N^{2\%}_{[30]} - N^{2\%}_{35}}{D^{2\%}_{[30]}} + \dfrac{D^{2\%}_{35}}{D^{2\%}_{[30]}} \cdot \dfrac{N^{3\%}_{35}}{D^{3\%}_{35}}$

30. $\dfrac{\log (1 + k)}{\log c}$

Chapter 3

3. $v^{20}\left(\dfrac{l_x - l_{x+20}}{l_x}\right) + \dfrac{M_{x+20}}{D_x}$

6. $P = 1, \qquad Q = \dfrac{1 + i}{i}, \qquad R = -\dfrac{v^{x+n}}{i}$

11. $i' = \dfrac{1 + i}{c} - 1$

12. $\dfrac{1}{D_{40}}(2M_{40} - M_{65})$

13. $\dfrac{100}{D_0}[C_0 + 2(R_1 - R_6) + 40M_{21}]$

14. $\dfrac{1000}{D_{40}}(M_{40} + 3R_{41} - 7R_{46} + 4R_{49})$

15. (b) $-v(\bar{I}\bar{A})_x$

18. For $0 \leqslant t \leqslant 4$, $C_{[x]+t} = v^{x+t+1}\, d_{[x]+t}$

and $M_{[x]+t} = \sum_{r=t}^{4} C_{[x]+r} + \sum_{r=5}^{\omega-x-1} C_{x+r}$.

For $t \geqslant 5$, $C_{x+t} = v^{x+t+1}\, d_{x+t}$

and $M_{x+t} = \sum_{r=t}^{\omega-x-1} C_{x+r}$.

19. (a) $\dfrac{\left(1 + \dfrac{i}{2}\right)(M_{30} - M_{60}) + D_{60}}{D_{30}}$

(b) $\dfrac{10M_{30} - (R_{31} - R_{41})}{D_{30}}$

(c) $\dfrac{R_{40} - \frac{3}{8}M_{40}}{D_{40}}$

(d) $\dfrac{\left(1+\dfrac{i}{2}\right)(R_{35} - \frac{1}{2}M_{35})}{D_{35}}$

20. (a) $\dfrac{M_{35} - M_{40}}{D_{25}}$

(b) $\dfrac{R_{40} - R_{45} - 5M_{65}}{D_{40}}$

(c) $\dfrac{\bar{R}_x - \frac{1}{2}\bar{M}_x - \bar{R}_{x+n} - (n - \frac{1}{2})\bar{M}_{x+n}}{D_x}$

(d) $\dfrac{R_{45} - \frac{1}{2}M_{45}}{D_{45}}$

21. (a) No.

22. $\dfrac{1}{nD_x}[\ddot{a}_{\overline{n}|}(M_x - M_{x+t}) + \ddot{a}_{\overline{n}|}D_{x+t} + N_{x+t+n}]$

24. $\dfrac{i - \delta}{\delta^2}\cdot\dfrac{C_x}{D_x}$

Chapter 4

2. (a) $\dfrac{N_{41} - N_{61}}{N_{30} - N_{35}}$

(b) $\dfrac{N_{41} - N_{61}}{N_{30} - N_{35} - R_{30} + R_{35} + 5M_{40}}$

3. $\dfrac{M_{40} - M_{65}}{N_{40} - N_{65}} + \dfrac{M_{40}}{N_{40}}$

6. ${}_tP^{(m)}_{x:\overline{n}|} \doteq \dfrac{{}_tP_{x:\overline{n}|}}{1 - \dfrac{m-1}{2m}(P^{1}_{x:\overline{t}|} + d)}$

7. Limit is $\bar{P}_x$.

11. (b) $P = \dfrac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}$

12. $\dfrac{v^n}{\ddot{a}_{x:\overline{n}|}}\left(1 - \dfrac{l_{x+n}}{l_x}\right)$

13. $P^{1}_{x:\overline{n}|} + \frac{\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{2n}|}} \cdot P_{x:\overline{n}|}^{\;\;1} - \frac{1}{\ddot{s}_{\overline{2n}|}}$

15. (a) ${}_{20}\bar{P}^{\;1}_{25:\overline{35}|} = \frac{M_{25} - M_{60}}{\bar{N}_{25} - \bar{N}_{45}}$

(b) $120P^{(4)}({}_{15|}\ddot{a}^{(12)}_{45}) \doteq \frac{120(N_{60} - 11/24D_{60})}{N_{45} - N_{60} - 3/8(D_{45} - D_{60})}$

(c) ${}_{10}P^{\{12\}}(\bar{A}_{35:\overline{20}|}) \doteq \frac{\bar{M}_{35} - \bar{M}_{55} + D_{55}}{N_{35} - N_{45} - 11/24(D_{35} - D_{45}) - 1/24(\bar{M}_{35} - \bar{M}_{45})}$

or

${}_{10}P^{\{12\}}(\bar{A}_{35:\overline{20}|}) \doteq \frac{\bar{M}_{35} - \bar{M}_{55} + D_{55}}{N_{35} - N_{45} - 11/24\delta(\bar{N}_{35} - \bar{N}_{45}) - \frac{1}{2}(\bar{M}_{35} - \bar{M}_{45})}$

16. (a) $\frac{M_{30} - M_{50} + D_{50}}{(1 - 1/4d)(N_{30} - N_{40})}$

(b) $\frac{\bar{R}_{25} - \bar{R}_{40} - 15\bar{M}_{40}}{N_{25} - N_{40}}$

(c) $\frac{M_x}{(N_x - N_{x+10}) - \frac{1}{2}(M_x - M_{x+10})}$

21. $P = \frac{M_{15}}{\ddot{s}_{\overline{5}|}D_{15} + N_{15}}$

22. $\frac{\ddot{a}_{\overline{t}|}D_{x+n} + N_{x+n+t}}{N_x - N_{x+n} - (R_x - R_{x+n} - nM_{x+n})}$

Chapter 5

1. (a) $\frac{1}{D_{40}}\{M_{40} - M_{55} + D_{55} - P(N_{40} - N_{50})\}$

where $P = \frac{M_{30} - M_{55} + D_{55}}{N_{30} - N_{50}}$

(b) $\frac{1}{D_{35}}\{\bar{M}_{35} - \bar{M}_{45} - P(N_{35} - N_{45})\}$

where $P = \frac{\bar{M}_{20} - \bar{M}_{45}}{N_{20} - N_{45}}$

3. $\left(1 + \frac{i}{2}\right) {}_tV^{1}_{x:\overline{n}|} + {}_tV_{x:\overline{n}|}^{\;\;1}$

4. $\frac{1}{D_{x+t}}\{{}_nP_x(N_x - N_{x+t}) - (M_x - M_{x+t})\}$ for $t \leqslant n$

$$\frac{1}{D_{x+t}}\{{}_nP_x(N_x - N_{x+n}) - (M_x - M_{x+t})\} \qquad \text{for} \qquad t > n$$

9. (b) $\dfrac{{}_{17}V_{45:\overline{20}|} - {}_2V_{45:\overline{20}|}}{1 - {}_2V_{45:\overline{20}|}}$

14. $\dfrac{({}_tR_x - 1)(1 + i)}{p_{x+t} + \frac{1}{2}q_{x+t}}$

16. .0604, .0600, .0607, .0607.

17. (a) ${}_tV^{1}_{x:\overline{n}|}\left(1 + \dfrac{m-1}{2m}\,P^{1(m)}_{x:\overline{n}|}\right)$

(b) ${}^{k}_{t}V_x + \frac{1}{2}\cdot{}_kP^{(m)}_x\cdot{}_tV^{1}_{x:\overline{k}|}$

(c) ${}_tV_{x:\overline{n}|}$

(d) ${}_tV_{x:\overline{n}|} + \frac{1}{2}\cdot\bar{P}_{x:\overline{n}|}\cdot{}_tV^{1}_{x:\overline{n}|}$

18. (a) $\frac{3}{4}\cdot{}^{10}_{5}V_{[40]} + \frac{1}{4}\cdot{}^{10}_{6}V_{[40]} + \frac{3}{4}\cdot{}_{10}P_{[40]}$

(b) $\frac{3}{4}\cdot{}^{10}_{5}V_{[40]} + \frac{1}{4}\cdot{}^{10}_{6}V_{[40]}$

$$+ \tfrac{3}{8}\cdot{}_{10}P^{(4)}_{[40]}\left(\tfrac{3}{4}\cdot{}_5V^{\,1}_{[40]:\overline{10}|} + \tfrac{1}{4}\cdot{}_6V^{\,1}_{[40]:\overline{10}|}\right)$$

(c) $\frac{3}{4}\cdot{}^{10}_{5}V_{[40]} + \frac{1}{4}\cdot{}^{10}_{6}V_{[40]}$

$$+ \tfrac{1}{2}\cdot{}_{10}P^{\{2\}}_{[40]}\left(\tfrac{3}{4}\cdot{}_5V^{\,1}_{[40]:\overline{10}|} + \tfrac{1}{4}\cdot{}_6V^{\,1}_{[40]:\overline{10}|}\right) + \tfrac{1}{4}\cdot{}_{10}P^{\{2\}}_{[40]}$$

19. \$242.38, \$262.57, \$255.84, \$257.56.

24. .0108.

26. $c_{x+t} = vq_{x+t}$

27. $-M_{x+n}$

28. PN_x

33. ${}_{10}V_{30} < {}_{10}V'_{30} < {}_{10}V'_{40} < {}_{10}V_{40}$

36. $P = \dfrac{1 + (1+i)^{x+20}(M'_x - M'_{x+20})}{s_{\overline{20}|}}$

$$\text{where} \qquad M'_x = \sum_{k=x}^{\omega-1} v^{k+1}q_k$$

$$P \doteq \frac{1}{s_{\overline{20}|}} + P^{1}_{x:\overline{20}|}$$

38. $3\frac{3}{4}\,\%$.

40. (a) $\dfrac{P_x N_{x+5} - M_{x+n} + D_{x+n}}{N_{x+5} - N_{x+n}}$

41. 2

Chapter 6

1. $\dfrac{1 - (d - h)\ddot{a}_x + k}{\left(1 - \dfrac{j}{100}\right)\ddot{a}_{x:\overline{n}|}}$

2. $$\frac{1}{.935D_x}\left[1000\left(1 + \frac{i}{2}\right)(M_x - M_{x+n}) + 1000D_{x+n} + 5D_x + 2.5(N_{x+1} - N_{x+n})\right]$$

3. $\dfrac{(1 + h)\ddot{a}_x - \dfrac{m + 1}{2m}}{1 - \dfrac{k}{100}}$

4. $P = \dfrac{M_x + c(1 + k)R_x}{N_x - (1 + k)R_x}$

7. $\alpha D_x + \beta N_{x+1} - M_x$

8. $\dfrac{3}{4a_{x:\overline{n-1}|}}$

11. $\alpha^F = \dfrac{5C_{40}}{D_{40}}$

$\beta^F = \dfrac{5M_{41} - 4M_{65}}{N_{41} - N_{65}}$

12. $\alpha^{\text{Com}} - c_x = \beta^{\text{Com}} - {}_{19}P_{x+1}$

13. $\beta^{\text{Com}} = {}_{10}P_{25:\overline{20}|} + \dfrac{{}_{19}P_{26} - c_{25}}{\ddot{a}_{25:\overline{10}|}}$

$\alpha^{\text{Com}} = \beta^{\text{Com}} - ({}_{19}P_{26} - c_{25})$

${}_1V = \dfrac{1}{D_{26}}(\alpha^{\text{Com}}D_{25} - C_{25})$

$${}_7V = \frac{1}{D_{32}}[\alpha^{\text{Com}}D_{25} + \beta^{\text{Com}}(N_{26} - N_{32}) - (M_{25} - M_{32})]$$

$${}_{15}V = A_{40:\overline{5}|}$$

16. $$P^A = \frac{A + .25P_x^A + .02}{\ddot{a} - .4} \quad \text{where} \quad P_x^A = \frac{A_x + .02}{\ddot{a}_x - .65}$$

18. $${}_5CV = A_{40:\overline{10}|} - {}_{10}P^A_{35:\overline{15}|}\cdot\ddot{a}_{40:\overline{5}|}$$

$${}_{12}CV = A_{47:\overline{3}|}$$

20. $$\text{Net } CV = 2000[A_{59:\overline{6}|} - \beta^{\text{Com}}\,\ddot{a}_{59:\overline{6}|}] - 100$$

$$\text{where} \qquad \beta^{\text{Com}} = P_{43:\overline{22}|} + \frac{{}_{19}P_{44} - c_{43}}{\ddot{a}_{43:\overline{22}|}}$$

$$\text{Pure endowment} = \frac{(\text{net } CV) - 1900A^{1}_{59:\overline{6}|}}{A_{59:\overline{6}|}^{\;\;1}}$$

21. (i) $$\text{Paid-up amount} = \frac{(C - L)D_{40}}{M_{40} - M_{45} + D_{45}}$$

Pure endowment

$$E = \frac{(C - L)D_{40} - (1 - L)(M_{40} - M_{45})}{D_{45}}$$

(ii) Retrospective reserve

$$= (C - L)\frac{D_{40}}{D_{43}} - (1 - L)\frac{M_{40} - M_{43}}{D_{43}}$$

$$\text{Prospective reserve} = E\frac{D_{45}}{D_{43}} + (1 - L)\frac{M_{43} - M_{45}}{D_{43}}$$

where E is the pure endowment given in (i).

22. $$\frac{1200M_{x+n} + 3N_x}{N_x - 1.2\,(R_x - R_{x+n} - nM_{x+n})}$$

23. (a) $${}_tV - {}_tV^{\text{Com}} = \left(\frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{30}|}}\right)\ddot{a}_{x+t:\overline{30-t}|}$$

(b) $${}_tV - {}_tV^{\text{Com}} = \left(\frac{{}_{19}P_{x+1} - c_x}{\ddot{a}_{x:\overline{65-x}|}}\right)\ddot{a}_{x+t:\overline{65-x-t}|}$$

24. ${}^{20}_{y-x}V_x = {}_{20}P_x \dfrac{N_x - N_y}{D_y} - \dfrac{M_x - M_y}{D_y}$

Paid-up reserve $= {}_{20}P_x \dfrac{N_x - N_{x+t}}{D_y} - \dfrac{M_x - M_y}{D_y}$

Difference is ${}_{20}P_x \dfrac{N_{x+t} - N_y}{D_y}$

27. $\dfrac{13{,}500D_{60}}{10\ddot{s}_{\overline{10}|}D_{60} - (N_{20} - N_{60})}$

Chapter 7

1. $\bar{a}_y + \displaystyle\int_0^k (k - t)\, v^t{}_tp_y\mu_{y+t}\, dt$

$$\doteq \frac{N_{y+1}}{D_y} + \tfrac{1}{2} + \frac{(1 + i)^{\frac{1}{2}}}{D_y} [(k + \tfrac{1}{2})(M_y - M_{y+k})$$
$$- (R_y - R_{y+k} - kM_{y+k})]$$

3. $\dfrac{N_{x+t}^{(m)} - (R_{x+t}^{(m)} - R_{x+t+n}^{(m)} - nM_{x+t+n})}{(1 - r)(N_x - N_{x+t}) - (R_x - R_{x+t} - tM_{x+t+n})}$

6. $\dfrac{\ddot{a}_{\overline{10}|}D_{65} + N_{75} + c[R_x - R_{65} - (65 - x)M_{65}]}{N_x - N_{65} - (1 + k)[R_x - R_{65} - (65 - x)M_{65}]}$

where the gross premium equals $(1 + k)$ times the net premium plus c.

10. $3/4$

11. $\dfrac{M_{40} - M_{67} + 1.5\, v^8\, D_{67}}{N_{40} - N_{67} + \ddot{a}_{\overline{8}|}D_{67}}$

12. $\dfrac{1}{D_x} [M_x - M_{x+c} + (1 + k)v^{n-c}D_{x+c}]$

where c is the greatest integer for which $1 + k \leqslant (1 + i)^{n-c}$

13. $\dfrac{1}{N_x - N_{x+r}} [M_x - M_{x+a} + (1 + k)v^{n-a}D_{x+a}]$

14. $\dfrac{12\, v^{n-a}D_{x+a}\cdot\ddot{a}_{x+n}^{(12)} + c(R_x - R_{x+a} - aM_{x+a})}{N_x - N_{x+a} + D_{x+a}\cdot\ddot{a}_{\overline{n-a}|} - (1 + k)(R_x - R_{x+a} - aM_{x+a})}$

where $P' = (1+k)P + c$
and a is the greatest integer for which

$$12v^{n-a}\ddot{a}^{(12)}_{x+n} - P\ddot{a}_{\overline{n-a}|} \leqslant a[P(1+k)+c]$$

16. $\frac{1}{\delta}(\bar{A}^{\;1}_{x:\overline{15}|} - v^{20}\cdot{}_{15}q_x) + \bar{a}_{\overline{5}|}(\bar{A}^{\;1}_{x:\overline{20}|} - \bar{A}^{\;1}_{x:\overline{15}|})$

18. $\frac{1}{\ddot{a}_{x:\overline{20}|}}[1000A_{x:\overline{20}|}^{\;\;\;1} + 120(a^{(12)}_{\overline{20}|} - a^{(12)}_{x:\overline{20}|})]$

19. $$A \doteq \frac{K(N_{x+1} + \frac{3}{8}D_x + \frac{1}{8}\bar{M}_x)}{D_x - \left(1 + \frac{c}{100}\right)[M_x - \frac{1}{6}(R_x - R_{x+6})]}$$

20. $\frac{100}{\delta}(\bar{A}^{\;1}_{x:\overline{10}|} - v^{20}\,{}_{10}q_x) + 100\bar{a}_{\overline{10}|}\frac{D_{x+10}}{D_x}\bar{A}_{x+10:\overline{10}|}^{\;\;\;1}$

21. $$\beta = \frac{{}_ak_x + (1+k)\,v^{n-a} + g_x\frac{D_x}{D_{x+a}}}{\ddot{s}_{x:\overline{a}|} + \ddot{a}_{\overline{n-a}|}}$$

where a is the greatest integer for which

$$\frac{\ddot{s}_{\overline{n-a}|}}{\ddot{a}_{x:\overline{a}|}} \geqslant \frac{k}{1+g_x}$$

CHAPTER 8

2. (a) l_x (b) L_x (c) T_x (d) T_x (e) T_x (f) Y_x
3. (a) $T_x - T_{x+n} - nl_{x+n}$

   (b) Y_x

   (c) T_{x+n}
4. $m_x = 1$ for all x
5. .015
7. $\frac{2}{2\omega - 2x - 1}$
8. 2.48, 2.98
9. $\frac{\omega - x}{2}$

12. (a) $(x - y)l_x$

(b) $T_x - T_z = l_x\, \mathring{e}_{x:\overline{z-x}|}$

(c) $\dfrac{(x - y)\, l_x + T_x - T_z}{l_x} = x - y + \mathring{e}_{x:\overline{z-x}|}$

13. (a) $20(l_{30} - l_{50})$

(b) $\dfrac{T_{30} - T_{50} - 20l_{50}}{l_{30} - l_{50}}$

14. (a) $\dfrac{300}{l_{20}}(T_{20} - .1T_{25} - .045T_{30} - .855T_{60})$

(b) $\dfrac{300}{l_{20}}(.045l_{30})$

(c) $\dfrac{300}{l_{20}}(.855T_{60})$

15. (a) $\dfrac{1000l_{30}}{T_{30} - \frac{1}{4}T_{55} - \frac{1}{4}T_{60} - \frac{1}{2}T_{65}}$

(b) $\dfrac{1000(l_{30} - \frac{1}{4}l_{55} - \frac{1}{4}l_{60} - \frac{1}{2}l_{65})}{T_{30} - \frac{1}{4}T_{55} - \frac{1}{4}T_{60} - \frac{1}{2}T_{65}}$

16. (a) $\dfrac{5000}{l_0}(T_{25} - T_{65})$

(b) $\dfrac{50{,}000\, l_{25}}{l_0}(20 + e_{25:\overline{40}|})$

(c) $\dfrac{750{,}000\, l_{65}\, e_{65}}{l_0}$

(d) $\dfrac{250{,}000}{l_0}(l_{25} - l_{65})$

19. (a) $Y_x - Y_{x+n} - nT_{x+n}$

(b) $Y_x - Y_{x+n} - Y_{x+m} + Y_{x+m+n}$

20. (a) $x + \dfrac{2(Y_x - Y_{x+n}) - nT_{x+n}}{T_x - T_{x+n}}$

(b) $x + \dfrac{2(Y_x - Y_{x+n} - nT_{x+n}) - n^2 l_{x+n}}{T_x - T_{x+n} - nl_{x+n}}$

21. $\dfrac{Y_0 - Y_{30} - 30T_{30}}{T_0 - T_{30}}$

22. $30 + \dfrac{T_{30} - T_{50} - 20l_{50}}{l_{30} - l_{50}}$

23. (a) $1 - \dfrac{n}{75}$

(b) $\dfrac{1 - \mu_x \mathring{e}_x}{\mu_x l_x}$

24. 8000, 16

25. $2400\, l_{60} - 3200\, l_{80} + 100\, T_{60} - 140\, T_{70}$
$+ 40\, T_{80} + 2\, Y_{60} - 4Y_{70} + 2Y_{80}$

Chapter 9

1. (a) ${}_np_x \cdot {}_np_y$

(b) ${}_np_x + {}_np_y - 2{}_np_x \cdot {}_np_y$

(c) ${}_np_x + {}_np_y - {}_np_x \cdot {}_np_y$

(d) $1 - {}_np_x \cdot {}_np_y$

(e) $1 - {}_np_x \cdot {}_np_y$

(f) $(1 - {}_np_x)(1 - {}_np_y)$

2. (a) ${}_{n|}q_x \cdot {}_{n|}q_y \cdot {}_{n|}q_z$

(b) $(1 - {}_{n|}q_x)(1 - {}_{n|}q_y)(1 - {}_{n|}q_z)$

(c) $1 -$ (b)

5. (a) $d_{xy} = l_{xy} - l_{x+1:y+1}$

(b) $\mu_{xy} = \mu_x + \mu_y$

(c) Correct.

(d) $q_{xy} = q_x + q_y - q_x \cdot q_y$

8. (a) $2(\mu_x - \mu_{x+t})$

(b) $2\mu_x\, \mathring{e}_{xx} - 1$

15. $t = \dfrac{\log(1 + kc^n) - \log(1 + k)}{\log c}$

where $t = w - x$ and $n = y - x$

17. $a_z = a_{xy:\overline{w-z}|}$

22. e_{w+2}, where $c^w = c^x + c^y$

Chapter 10

1. $q_x + q_y - q_{xy}$

2. (a) $\bar{a}_w + \bar{a}_x + \bar{a}_y + \bar{a}_z - \bar{a}_{wx} - \bar{a}_{wy} - \bar{a}_{wz}$

$- \bar{a}_{xy} - \bar{a}_{xz} - \bar{a}_{yz} + \bar{a}_{wxy} + \bar{a}_{wxz}$

$+ \bar{a}_{wyz} + \bar{a}_{xyz} - \bar{a}_{wxyz}$

(b) $\dfrac{A^{1}_{x:\overline{n}|} + A^{1}_{y:\overline{n}|} - A^{1}_{\overline{xy}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|} + \ddot{a}_{y:\overline{n}|} - \ddot{a}_{xy:\overline{n}|}}$

3. ${}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_x \cdot {}_{n|}q_y$

No

4. ${}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}$

6. $a_{xz} < a_{ww} < a_w < a_y < a_{\overline{xy}} < a_{\overline{yy}}$

10. $a_{abcx} + a_{abcy} + a_{abcz} - a_{abcxy} - a_{abcxz} - a_{abcyz} + a_{abcxyz}$

11. $a_{25:\overline{25}|} + a_{30:\overline{20}|} - a_{25:30:\overline{20}|}$

12. ${}_{25|}a_{25} + {}_{20|}a_{30} - {}_{25|}a_{25:30}$

13. $a_{\overline{n}|} + a_x + a_y - a_{xy} - a_{x:\overline{n}|} - a_{y:\overline{n}|} + a_{xy:\overline{n}|}$

15. (a) ${}_np_{xy} + {}_np_{xz} + {}_np_{yz} - 2\,{}_np_{xyz}$

(b) ${}_np_x + {}_np_y + {}_np_z - 2({}_np_{xy} + {}_np_{xz} + {}_np_{yz}) + 3\,{}_np_{xyz}$

16. Exactly 2 of the 4 lives will survive for t years.

17. $a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz} - 4a_{wxyz}$

20. $1 - d(\ddot{a}_{wx} + \ddot{a}_{wy} + \ddot{a}_{wz} + \ddot{a}_{xy} + \ddot{a}_{xz} + \ddot{a}_{yz} - 2\ddot{a}_{wxy} - 2\ddot{a}_{wxz}$

$- 2\ddot{a}_{wyz} - 2\ddot{a}_{xyz} + 3\ddot{a}_{wxyz})$

21. $\ddot{a}_{wxy} + \ddot{a}_{wxz} + \ddot{a}_{wyz} - 2\ddot{a}_{wxyz}$

22. (a) $1 - 5({}_np_x)^4 + 4({}_np_x)^5$

(b) $[1 + 4({}_{n-1}p_x - {}_np_x)][1 - ({}_{n-1}p_x - {}_np_x)]^4$

27. $\frac{1}{3}\ddot{a}_{x:\overline{n}|} + \frac{1}{2}\ddot{a}_{y:\overline{n}|} + \frac{1}{6}\ddot{a}_{xy:\overline{n}|}$

28. $\frac{1}{8}(a_w + a_x + a_y + a_z + a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz})$

29. ${}_{20|}\bar{a}_{40} + {}_{10|}\bar{a}_{50} - {}_{10|5}\bar{a}_{40:50} - {}_{20|}\bar{a}_{40:50}$

Chapter 11

2. $\frac{1}{2}(1 - {}_np_x)^2{}_np_x$

4. ${}_\infty q_{wxy}^{\;1} + {}_\infty q_{wxz}^{\;1} + {}_\infty q_{xyz}^{1} - 3{}_\infty q_{wxyz}^{\;\;1}$

6. (b) (x) is older than (y)

13. $\bar{A}_{wx}^{\;1} + \bar{A}_{xy}^{1} + \bar{A}_{xz}^{1} - 2(\bar{A}_{wxy}^{\;1} + \bar{A}_{wxz}^{\;1} + \bar{A}_{xyz}^{1}) + 3\bar{A}_{wxyz}^{\;\;1}$

14. Insurance payable upon death of (y) before (x) within n years, or at end of n years if both are surviving.

$$A_{\overline{x:\overset{1}{y}:\overline{n}|}} = \frac{M_{x\overset{1}{y}} - M_{x+n:\overset{1}{\overline{y+n}}} + D_{x+n:y+n}}{D_{xy}}$$

15. $.25\bar{A}_{40:50} + .001\bar{a}_{40:50}$

20. $\bar{A}_{x:\overline{yz}}^{1} \doteq \frac{1}{2}\left(\frac{\bar{a}_{x-1:\overline{yz}}}{p_{x-1}} - p_x \cdot \bar{a}_{x+1:\overline{yz}}\right)$

22. If all 3 lives are surviving:

$$A_{\overline{x+t}:y+t:z+t}^{\;2} - P\cdot\ddot{a}_{x+t:\overline{y+t:z+t}}$$

If only (x) and (y) are surviving:

$$A_{\overline{x+t}:y+t}^{\;1} - P\cdot\ddot{a}_{x+t:y+t}$$

If only (x) and (z) are surviving:

$$A_{\overline{x+t}:z+t}^{\;1} - P\cdot\ddot{a}_{x+t:z+t}$$

23. $\dfrac{1{,}000\bar{M}_{35:40}^{\;1} + 10{,}000\bar{M}_{35:\overset{1}{40}} - 100(\bar{R}_{36:\overset{1}{41}} - \bar{R}_{56:\overset{1}{61}})}{N_{35:40}}$

26. $\frac{D_{x+m}}{D_x} \, {}_{m-n}p_y \, \ddot{a}_{x+m:y+m-n}$

27. $\bar{A}_{50} - \bar{A}^{\;1}_{50:20:\overline{20|}}$

28. $\bar{A}^{\;2}_{xy} - \frac{D_{x+n}}{D_x} (\bar{A}_{x+n} - \bar{A}^{\;1}_{\overline{x+n}:y})$

29. $\dfrac{\bar{A}^{\;1}_{30:\overline{5|}} + \frac{D_{35}}{D_{30}} \bar{A}^{\;1}_{35:60}}{\frac{1}{1.075} - \frac{D_{35}}{D_{30}} \bar{A}_{35:\overset{1}{60}}}$

30. 1/24

31. $\int_0^\infty tv^t \, {}_tp_{xy} \, \mu_{y+t} \, \bar{A}_{x+t} \, dt$

32. $v^{10}(\bar{A}^{\;1}_{x:\overline{yz}} - {}_{10}p_y \, \bar{A}^{\;1}_{x:y+10} - {}_{10}p_z \, \bar{A}^{\;1}_{x:z+10} + {}_{10}p_{yz} \, \bar{A}^{\;1}_{x:y+10:z+10})$

Chapter 12

1. (a) $\int_0^\infty (1 - {}_tp_w) {}_tp_{xyz} \, \mu_{x+t} \, dt$

   (b) ${}_\infty q^{\;1}_{xyz} - {}_\infty q^{\;\;1}_{wxyz}$

6. Both are incorrect. Should be

$$\int_0^\infty v^t \, {}_tp_{wxyz} \, \mu_{z+t} \, \bar{A}^{\;\;3}_{\overline{w+t}:x+t:y+t} \, dt$$

$$\text{or } \int_0^\infty v^t \, {}_tq_z \, {}_tp_{wxy} \, (\mu_{x+t} \, \bar{A}^{\;\;2}_{\overline{w+t}:y+t} + \mu_{y+t} \, \bar{A}^{\;\;2}_{\overline{w+t}:x+t}) \, dt$$

7. $2\bar{A}^{\;2}_{\underset{1}{zxyz}}$ or $\bar{A}^{\;2}_{\underset{1}{\overline{zxyz}}}$

8. (a) $\ddot{a}_{xz}$

   (b) $\ddot{a}_{wxy}$

9. (a) $\int_0^\infty (1 - {}_tp_w) {}_tp_{xyz} \, \mu_{x+t} \cdot {}_5q^{\;1}_{\overline{y+t}:z+t} \, dt$

   (b) $(1 - {}_{10}p_w) {}_{10}p_{xyz} \left[\int_0^5 (1 - {}_tp_{z+10}) {}_tp_{y+10} \, \mu_{y+10+t}\right.$

$$\left. ({}_tp_{z+10} - {}_{t+5}p_{z+10}) \, dt\right]$$

$$+\int_5^\infty ({}_{t-5}p_{x+10} - {}_tp_{x+10})\,{}_tp_{y+10}\,\mu_{y+10+t}({}_tp_{z+10} - {}_{t+5}p_{z+10})\,dt\Big]$$

10. (a) $\displaystyle\int_0^{30} (1 - {}_tp_{10})\,{}_tp_{20}\,\mu_{20+t}({}_tp_{30} - {}_{30}p_{30})\,dt$

$$+\int_0^{30} (1 - {}_tp_{30})\,{}_tp_{20}\,\mu_{20+t}({}_tp_{10} - {}_{50}p_{10})\,dt$$

$$+\int_{30}^{40} (1 - {}_{30}p_{30})\,{}_tp_{20}\,\mu_{20+t}({}_tp_{10} - {}_{50}p_{10})\,dt$$

(b) $\displaystyle\int_0^2 (1 - {}_tp_w)\,{}_tp_x\,\mu_{x+t}\int_t^{t+2} {}_sp_y\,\mu_{y+s}({}_sp_z - {}_{s+2}p_z)\,ds\,dt$

$$+\int_2^\infty ({}_{t-2}p_w - {}_tp_w)\,{}_tp_x\,\mu_{x+t}\int_t^{t+2} {}_sp_y\,\mu_{y+s}({}_sp_z - {}_{s+2}p_z)\,ds\,dt$$

11. $\bar{A}^{1}_{35:30:\overline{10|}} + \dfrac{D_{40:45}}{D_{30:35}}(\bar{A}^{1}_{45:40:40} + \bar{A}^{\;1}_{45:40:50} - \bar{A}^{\;1}_{45:40:40:50})$

14. $\dfrac{1.1(a + b)}{1 - 1.1c}$

where $\displaystyle a = \int_0^\infty \tfrac{1}{2}(1 - {}_tp_{40})^2\, v^t\, {}_tp_{30:35}\,\mu_{30+t}\,dt$

$$b = .75\int_0^\infty \tfrac{1}{2}(1 - {}_tp_{40})^2\, v^t\, {}_tp_{30:35}\,\mu_{35+t}\bar{A}_{\overset{1}{30+t}:\overline{3|}}\,dt$$

$$c = \int_0^\infty \tfrac{1}{2}(1 - {}_tp_{40})^2\, v^{t+3}\, {}_tp_{35}\,\mu_{35+t}\cdot {}_{t+3}p_{30}\,dt$$

15. $200{,}000\,[\bar{A}^{1}_{wx} - \tfrac{1}{2}(\bar{A}^{1}_{wxy} + \bar{A}^{1}_{wxd} + \bar{A}^{1}_{wxe})$
$\qquad + \tfrac{1}{2}(\bar{A}^{1}_{wxyd} + \bar{A}^{1}_{wxye}) + \tfrac{1}{3}(\bar{A}^{1}_{wxde} - \bar{A}^{1}_{wxyde})]$

CHAPTER 13

2. (a) $a_x + a_y - a_{xy} - a_{xz} - a_{yz} + a_{xyz}$

(b) $a_{wx} - a_{wxy} - a_{wxz} + a_{wxyz}$

3. Share of (x): $a_x - \tfrac{1}{2}a_{xy} - \tfrac{5}{8}a_{xz} + \tfrac{3}{8}a_{xyz}$

Share of (y): $a_y - \frac{1}{2}a_{yz} - \frac{1}{2}a_{xy} + \frac{1}{4}a_{xyz}$

Share of (z): $a_z - \frac{3}{8}a_{xz} - \frac{1}{2}a_{yz} + \frac{3}{8}a_{xyz}$

4. $\int_0^\infty v^t {}_tp_z(1 - {}_tp_{\overline{xy}})\, dt$

$$= \int_0^\infty v^t {}_tp_z({}_tp_x\,\mu_{x+t} + {}_tp_y\,\mu_{y+t} - {}_tp_{xy}\,\mu_{x+t:y+t})\bar{a}_{z+t}\, dt$$

$$= \bar{a}_z - \bar{a}_{xz} - \bar{a}_{yz} + \bar{a}_{xyz}$$

7. (a) $\frac{D_{x+n}}{D_x}(a_{x+n:\overline{m-n}|} - a_{x+n:y:\overline{m-n}|})$

(b) $a_{x:\overline{n}|} + \frac{D_{x+n}}{D_x} a_{x+n:y:\overline{m-n}|}$

8. (a) $a_{x|y} + \ddot{a}_{\overline{n}|}\cdot A_{xy}^{\;2}$

(b) $a_{\overline{n}|} - a_{x:\overline{n}|} + v^n a_y - \frac{D_{x+n}}{D_x} a_{x+n:y}$

9. $100\,\frac{D_{y+t}}{D_y}(a_{y+t} - a_{x:y+t:\overline{n}|} - {}_np_x\cdot{}_{n|}a_{y+t})$

10. (a) $a_x + a_{\overline{n}|} - a_{x:\overline{n}|} - a_{xy} - a_{y:\overline{n}|} + a_{xy:\overline{n}|}$

(b) $a_y - a_{xy:\overline{n}|} + \frac{3\,D_{x+n:y+n}}{8\,D_{xy}}$

13. All alive: $a_{z+t} - a_{x+t:z+t} - a_{y+t:z+t} + a_{x+t:y+t:z+t}$

(x) dead: $a_{z+t} - a_{y+t:z+t}$

(y) dead: $a_{z+t} - a_{x+t:z+t}$

(x) and (y) dead: a_{z+t}

16. $$\frac{\ddot{a}_{\overline{n}|}^{(12)}\cdot\bar{A}_{x:\overline{m}|} + {}_{m+n|}\ddot{a}_x^{(12)}}{\ddot{a}_{x:\overline{m}|}} + \frac{{}_{n|}\ddot{a}_y^{(12)} - {}_{m+n|}\ddot{a}_{xy}^{(12)} - \frac{D_{y+n}}{D_y}\ddot{a}_{x:y+n:\overline{m}|}^{(12)}}{\ddot{a}_{xy:\overline{m}|}},$$

reducing upon the death of the beneficiary to

$$\frac{\ddot{a}_{n}^{(12)}\cdot\bar{A}_{x:\overline{m}|} + {}_{m+n|}\ddot{a}_x^{(12)}}{\ddot{a}_{x:\overline{m}|}}$$

18. $\int_0^\infty v^t\,{}_tp_{\overline{xy}}\cdot{}_tq^{\,1}_{zw}\,dt$

$\int_0^\infty v^t\,{}_tp_{zw}\,\mu_{z+t}({}_tp_x\bar{a}_{x+t} + {}_tp_y\bar{a}_{y+t} - {}_tp_{xy}\bar{a}_{x+t:y+t})\,dt$

20. $\dfrac{\frac{2}{3}(\ddot{a}_y - \ddot{a}_{xy} + {}_{n|}\ddot{a}_x) + \frac{1}{3}\,{}_{n|}\ddot{a}_{xy}}{.935 - {}_nA^{\,2}_{xy}}$ where (x) is husband and (y) is wife

21. $360(\bar{a}_{10:20:\overline{8|}} - \bar{a}_{5:10:20:\overline{8|}} - \bar{a}_{10:20:35:\overline{8|}} + \bar{a}_{5:10:20:35:\overline{8|}}$

$+ \bar{a}_{5:20:\overline{13|}} - \bar{a}_{5:20:35:\overline{13|}}) + 240(\bar{a}_{5:\overline{13|}} - \bar{a}_{5:35:\overline{13|}}$

$+ \bar{a}_{10:\overline{8|}} - \bar{a}_{10:35:\overline{8|}})$

22. $P = \dfrac{1}{\ddot{a}_{x+5:y:\overline{60-x|}}}\left(\dfrac{5000}{\ddot{a}^{(12)}_{\overline{20|}}}\right)\left(\dfrac{D_{y+20}}{D_y}\,\mathring{a}^{(12)}_{x+5|y+20}\right)$

${}_5V = \dfrac{5000}{\ddot{a}^{(12)}_{\overline{20|}}}\left(\dfrac{D_{y+25}}{D_{y+5}}\,\mathring{a}^{(12)}_{x+10|y+25}\right) - P\cdot\ddot{a}_{x+10:y+5:\overline{55-x|}}$

CHAPTER 14

1. (a) $\dfrac{324}{721{,}013}$

   (b) $\dfrac{80{,}385 + 74{,}117}{901{,}020}$

2. (a) 400

   (b) 500

3. $\dfrac{86{,}632}{764{,}486}$

4. $l^{(T)}_{30} = 435{,}457$

   $d^{(1)}_{30} = 326$

   $d^{(2)}_{30} = 55{,}964$

5. $\dfrac{1 - \frac{1}{2}(a + b)}{1 + \frac{1}{2}(a + b)}$

6. $\dfrac{j}{j + k}\,[1 - e^{-(j+k)}]$

8. $\mu_x^{(1)} = \dfrac{2x}{a - x^2}$

9. $l_x^{(T)} = (a - x)e^{-x}$

$d_x^{(1)} = e^{-x} - e^{-x-1}$

$d_x^{(2)} = e^{-x}(a - x - 1) - e^{-x-1}(a - x - 2)$

10. 315; 92,777

11. $\dfrac{2 - q_x^{(T)}}{2 - q_x'^{(1)}} \cdot q_x'^{(1)}$ if (14.31b) is used.

Answer is different if (14.36) is used.

13. Same as Exercise 9.

14. 3,520; 80; 120; 280.

15.

x	$d_x^{(d)}$	$d_x^{(m)}$	$d_x^{(w)}$
20	93	979	328
21	80	842	278

17. (a) $p_x^{(T)} = p_x'^{(1)} \cdot p_x'^{(2)} \cdots p_x'^{(m)}$

(b) $m^1_{x_1x_2\cdots x_m} = \dfrac{d^1_{x_1x_2\cdots x_m}}{L_{x_1x_2\cdots x_m}}$

where $L_{x_1x_2\cdots x_m} = \displaystyle\int_0^1 l_{x_1+t:x_2+t:\cdots:x_m+t}\,dt$

18. (a) $\dfrac{vd_{62}^{(i)} + v^2d_{63}^{(i)} + v^3d_{64}^{(i)}}{l_{62}^{(T)} + vl_{63}^{(T)} + v^2l_{64}^{(T)}}$

(b) $\dfrac{M_{62}^{(i)} - M_{65}^{(i)}}{N_{62}^{(T)} - N_{65}^{(T)}}$

19. \$53.26.

20. $P = \dfrac{c(1 + k)R_x^{(w)} + 1000M_x^{(d)} + 500(65 - x)D_{65}^{(T)}}{N_x^{(T)} - N_{65}^{(T)} - (1 + k)R_x^{(w)}}$

23. $q_x^{(w)} \doteq m_x^{(w)} \dfrac{[1 - \frac{1}{2}q_x^{(d)}][1 - \frac{1}{2}q_x'^{(r)}]}{1 + \frac{1}{2}m_x^{(w)} - \frac{1}{4}q_x'^{(r)}m_x^{(w)}}$

$$q_x^{(r)} \doteq q_x'^{(r)} \frac{1 - \frac{1}{2}q_x^{(d)}}{1 + \frac{1}{2}m_x^{(w)} - \frac{1}{4}q_x'^{(r)}m_x^{(w)}}$$

24. (a) 859, 732, 40

(b) $\frac{1,591}{10,000}$

(c) $\frac{40}{8,714}$ or $\frac{46}{9,977}$ depending on formula used

(d) $\frac{732}{9,080}$

(e) 1.83

Chapter 15

1. .0003.

2. $p_{30}^{(Th)} \cdot (hq)_{31} \doteq \frac{1}{l_{30}^{(T)}} \{d_{30}^{(h)}[1 - \frac{1}{2}(hq)_{30}](hq)_{31}\}$

3. $l_{32}^{(T)} = 26,959$; $d_{32}^{(d)} = 539$; $d_{32}^{(h)} = 1,618$; $(hl)_{32} = 55,267$; $(hd)_{32} = 1,156$

4. $p_x^{(Th)}$, formula (15.8)

6. $p_x^a \doteq \frac{1}{l_x^{aa}} [l_{x+1}^{aa} + i_x(1 - \frac{1}{2}q_x^i)]$ or

$$p_x^a = \frac{1}{l_x^{aa}} [l_{x+1}^{aa} + l_{x+1}^{ii} - l_x^{ii}p_x^i]$$

7. 727, 20, 85

8. (a) and (b)

9. (a) $\frac{207}{29,711 + \frac{1}{2}\cdot 4,335}$

(b) $1 - $ (a)

(c) $1 - \frac{376 + 4,335\cdot\frac{(a)}{2}}{51,002}$ or

$$\frac{1}{51{,}002}\{46{,}291 + 33{,}839 - 29{,}711[1 - (a)]\}$$

10. $\displaystyle\int_0^2 \frac{(bl)_{34+t}}{(bl)_{34}} \left(\mu_{34+t}^{(bd)} + \mu_{34+t}^{(bm)} \cdot {}_{2-t}q_{34+t}^{(m)}\right) dt$

13. $\dfrac{N_x^{ii} - D_x^{ii} \cdot \ddot{a}_x^i - (N_{x+n}^{ii} - D_{x+n}^{ii} \cdot \ddot{a}_{x+n}^i)}{D_x^{aa}}$

14. $\dfrac{\displaystyle\sum_{t=0}^{2} v^{t+1}[2(sd)_{57+t} + (sm)_{57+t} + (md)_{57+t}] + v^3[2(sl)_{60} + (ml)_{60}](1 - d\ddot{a}_{60})}{\displaystyle\sum_{t=0}^{2} v^t (sl)_{57+t}}$

15. $\left[1 - \dfrac{76}{10{,}000 - \frac{1}{2}(1{,}290)}\right]\left[\dfrac{80}{8{,}634 - \frac{1}{2}(1{,}268)}\right]$

16. (b) $\dfrac{10M_x^{aa} + 5M_x^{r} + (N_x^{ii} - D_x^{ii} \cdot \ddot{a}_x^i) + 5(M_x^{ii} - D_x^{ii} \cdot A_x^i)}{N_x^{aa}}$

Chapter 16

1. (a) ${}^a\bar{C}_x^g = v^{x+\frac{1}{2}} g_x \cdot \ddot{a}_{x+\frac{1}{2}}^{g(12)} = \bar{C}_x^g \cdot \ddot{a}_{x+\frac{1}{2}}^{g(12)}$

   (b) ${}^a\bar{C}_x^g = v^{x+\frac{1}{2}} g_x \cdot {}_{\frac{1}{4}|}\ddot{a}_{x+\frac{1}{2}}^{g(4)} = \bar{C}_x^g \cdot {}_{\frac{1}{4}|}\ddot{a}_{x+\frac{1}{2}}^{g(4)}$

   (c) $\dfrac{{}^a_v\bar{M}_x^g}{D_x^T}$, where ${}^a\bar{C}_x^g$ is defined as in (b).

3. $\displaystyle\sum_{x=20}^{49} (x - 19 - \tfrac{1}{2}) v^{x+\frac{1}{2}} g_x \cdot \ddot{a}_{x+\frac{1}{2}}^g$

   $= {}^a\bar{R}_{20}^g - {}^a\bar{R}_{50}^g - 30\,{}^a\bar{M}_{50}^g - \frac{1}{2}({}^a\bar{M}_{20}^g - {}^a\bar{M}_{50}^g)$

6. $\dfrac{\bar{M}_x^r - \bar{M}_{65}^r + D_{65}^T}{D_x^T}$ or $\dfrac{\bar{M}_x^r - \bar{M}_{65}^r + (1 + i)C_{65}^r}{D_x^T}$

7. $\frac{1}{10}(1 + i)^{65-x}(\ddot{a}_{\overline{6}|} + a_{\overline{4}|})S_x$

8. $\dfrac{2000\,{}^{Za}\bar{M}_x^r}{{}^S D_x^T}$

9. $$\frac{.01(AS)_x(2\,{}^{Sa}\bar{R}^r_{x+1} - {}^{Sa}\bar{R}^r_{x+21} - {}^{Sa}\bar{R}^r_{x+31})}{{}^{S}D^T_x} + \frac{100({}^{a}\bar{R}^i_{x+1} - {}^{a}\bar{R}^i_{x+31})}{D^T_x}$$

10. $$\sum_{t=0}^{\infty}\left(S_{x+t}\, v^{x+t+\frac{1}{2}} \sum_{n=1}^{\infty} w_{x+t+n}\right)$$

11. $${}^{i}(TPC)_x \frac{\sum_{t=0}^{\infty} w_{x+t}}{l^T_x}$$

12. (a) $$\frac{180(\frac{1}{2}{}^{S}\bar{C}^w_{60} + {}^{S''j}\bar{M}^w_{51})}{{}^{S}D^T_{40}}$$

(b) $$\frac{180(\frac{1}{2}{}^{S}\bar{M}^w_{40} + {}^{S''j}\bar{R}^w_{41})}{{}^{S}D^T_{40}}$$

13. (a) $$\frac{100}{l^{aa}_{35}} \sum_{z=35}^{59} v^{z-34\frac{1}{2}}\, i_z = \frac{100\, v^{\frac{1}{2}}\,{}_{60}\bar{M}^i_{35}}{D^{aa}_{35}}$$

(b) $$\frac{1000}{l^{aa}_{35}} \sum_{z=35}^{59} v^{z-34}\, i_z \cdot \bar{a}^i_{[z+\frac{1}{2}]+\frac{1}{2}} = \frac{1000\,{}^{\omega}_{60}\bar{M}^i_{35}}{D^{aa}_{35}}$$

(c) $$\frac{2000}{l^{aa}_{35}} \sum_{z=35}^{59} v^{z-34\frac{1}{2}}\, i_z \cdot \bar{a}^i_{[z+\frac{1}{2}]+\frac{1}{2}:\overline{64\frac{1}{2}-z}|} = \frac{2000\,{}^{65}_{60}\bar{M}^i_{35}}{D^{aa}_{35}}$$

14. $$\frac{1}{24}\left(1 - v^t\,{}_tp^i_{[z+\frac{1}{2}]+m/12}\right)$$

where $t = x + n - z - \dfrac{6+m}{12}$

15. $$120\,\frac{\left({}^{50}\bar{M}^i_{30} + \frac{1}{24}\, v^{\frac{1}{2}}\,{}_{60}\bar{M}^i_{30}\right)}{N^{aa}_{30} - N^{aa}_{50}}$$

16. $$\frac{120({}^{65}_{60}\bar{M}^i_{45} + \frac{1}{24}\, v^{\frac{1}{2}}\,{}_{60}\bar{M}^i_{45})}{N^{aa}_{45} - N^{aa}_{60}}$$

17. $$\frac{50(2\,{}^{\omega}_{55}\bar{M}^i_{30} + v^{\frac{1}{2}}\,{}_{55}\bar{M}^i_{30})}{N^{aa}_{30} - N^{aa}_{55}}$$

18. (a) $\dfrac{50(2\,{}^{50}\bar{M}^{i}_{35} + v^{\frac{1}{2}}\,{}_{50}\bar{M}^{i}_{35})}{N^{aa}_{35} - N^{aa}_{50}}$

(b) $\dfrac{50(2\,{}^{75}_{55}\bar{M}^{i}_{45} + v^{\frac{1}{2}}\,{}_{55}\bar{M}^{i}_{45})}{N^{aa}_{45} - N^{aa}_{55}}$

19. (a) $120\left[\dfrac{{}^{w}_{y}\bar{M}^{i}_{x+t} + \dfrac{1}{24}\,v^{m/12}\,{}_{y}\bar{M}^{i}_{x+t}}{D^{aa}_{x+t}} - P^{I}_{x}\cdot\ddot{a}^{aa}_{x+t:\overline{n-t}|}\right]$

(b) $P\left[\dfrac{{}^{x+n}\bar{M}^{i}_{x+t} + \dfrac{m}{12}\,v^{m/12}\,{}_{x+n}\bar{M}^{i}_{x+t}}{D^{aa}_{x+t}} - P^{W}_{x}\cdot\ddot{a}^{aa}_{x+t:\overline{n-t}|}\right]$

21. The present value is equal to the sum of the four expressions below:

(1) $$1000\,\frac{\sum_{t=0}^{\infty} S_{x+t}\,\bar{C}^{r}_{x+t}\,(\bar{a}_{\overline{5}|} + {}_{5|}\bar{a}^{r}_{x+t+\frac{1}{2}})}{S_x D^{T}_x}$$

$$= \frac{1000\,{}^{Sa}\bar{M}^{r}_{x}}{{}^{S}D^{T}_{x}}$$

where $${}^{Sa}\bar{C}^{r}_{x} = \bar{C}^{r}_{x}\cdot S_x(\bar{a}_{\overline{5}|} + {}_{5|}\bar{a}^{r}_{x+\frac{1}{2}}).$$

(2) $$\frac{50\sum_{t=1}^{\infty} t\cdot{}_{m}Z_{x+t}\,\bar{C}^{r}_{x+t}(\bar{a}_{\overline{5}|} + {}_{5|}\bar{a}^{r}_{x+t+\frac{1}{2}})}{S_x D^{T}_x}$$

$$= \frac{50\,{}^{Za}\bar{R}^{r}_{x+1}}{{}^{S}D^{T}_{x}},$$

where ${}^{Za}\bar{C}^{r}_{x}$ is defined analogously to ${}^{Sa}\bar{C}^{r}_{x}$ in (1).

(3) $$\frac{25\sum_{t=0}^{29}(\frac{1}{2}{}^{S}\bar{C}^{w}_{x+t} + {}^{S''j}\bar{M}^{w}_{x+t+1})}{{}^{S}D^{T}_{x}}$$

$$= 25\,\frac{\frac{1}{2}({}^{S}\bar{M}^{w}_{x} - {}^{S}\bar{M}^{w}_{x+30}) + ({}^{S''j}\bar{R}^{w}_{x+1} - {}^{S''j}\bar{R}^{w}_{x+31})}{{}^{S}D_{x}{}^{T}}$$

(4) Same as (3) with the superscribed w replaced by d in the commutation symbols.

22. \$120, \$280.

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