Lagrangian Mechanics

$L = KE - PE$

$= T - V$

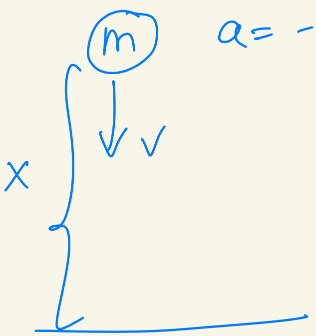
Equation of motion can be determined by.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

$\dot{x} \to$ velocity.

$x \to$ position.

Freely falling body – Example



$a = -9.8\, m/s^2$

$KE = \frac{1}{2} m\dot{x}^2$

$PE = mgx$

$L = KE - PE$

$L = \frac{1}{2} m\dot{x}^2 - mgx$

$$\frac{\partial L}{\partial x} = -mg$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

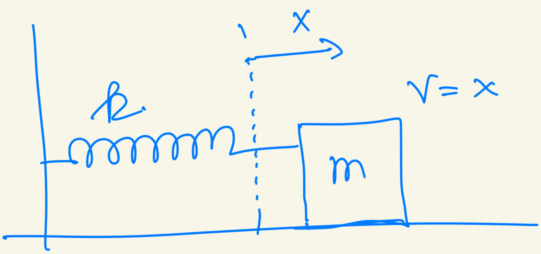
Equation of motion

$$m\ddot{x} - (-mg) = 0$$

$$m\ddot{x} = -mg$$

$$\boxed{\ddot{x} = -g}$$

Simple Harmonic motion with Spring:



$v = \dot{x}$

$$KE = \frac{1}{2} m\dot{x}^2$$

$$PE = \frac{1}{2} k x^2$$

k → spring constant

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{d}{dx}(L) = -kx$$

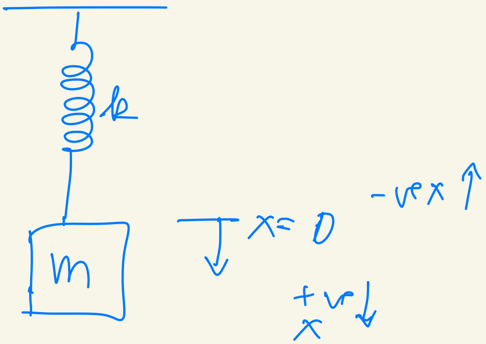
$$\frac{d}{d\dot{x}}(L) = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{d}{d\dot{x}}(L)\right) = m\ddot{x}$$

Equation of motion ⇒ $m\ddot{x} - (-kx) = 0$

$$\boxed{F = -kx}$$

Modeling Spring Mass - Logan Dihel Video
(Vertical)



$$KE = \frac{1}{2} m \dot{x}^2$$

$$PE = PE_{mass} + PE_{spring}$$

$$= -mgx + \frac{1}{2} k x^2$$

PE of mass occurs for a vertical system.

$$L = KE - PE$$

$$L = \frac{1}{2} m \dot{x}^2 + mgx - \frac{1}{2} k x^2$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0.$$

$$\frac{d}{dt}\left(m\dot{x}\right) - (mg - kx) = 0$$

$$m\ddot{x} = mg - kx$$

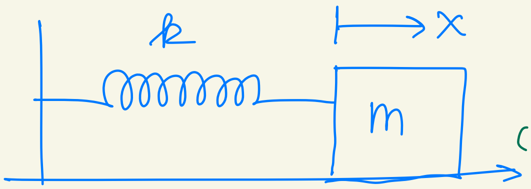
$$\boxed{\ddot{x} = g - \frac{k}{m} x}$$

ODE

$$\Rightarrow \frac{d(\dot{x})}{dt} = g - \frac{kx}{m}.$$

Solving this will $\dot{x}$ as function of time

Spring mass system: Horizontal



(no elevation)

$$KE = \frac{1}{2} m \dot{x}^2$$

$$PE = \frac{1}{2} k x^2$$

$$L = KE - PE$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

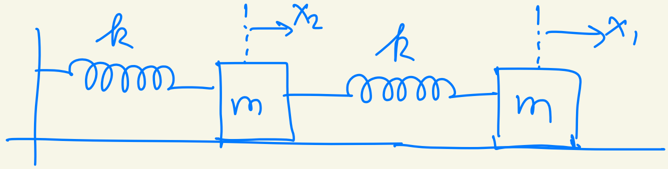
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(m\dot{x}\right) - \left(-kx\right) = 0.$$

$$\frac{d}{dt}(m\dot{x}) = -kx \rightarrow \text{ODE}$$

solve this for finding $\dot{x}$ as function of time.

Two Mass - Two spring System (no elevation):



$KE = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$ $\quad PE = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_2)^2$

$PE = \frac{1}{2} k (x_1^2 - 2x_1 x_2 + x_2^2) + \frac{1}{2} k x_2^2$

$L = KE - PE = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k (x_1 - x_2)^2 - \frac{1}{2} k x_2^2$

$\frac{dL}{d\dot{x}_1} = m\dot{x}_1 \qquad \frac{dL}{d\dot{x}_2} = m\dot{x}_2$

$\frac{dL}{dx_1} = kx_1 - kx_2 \qquad \frac{dL}{dx_2} = -kx_1 + kx_2 + kx_2$

$= -kx_1 + 2kx_2$

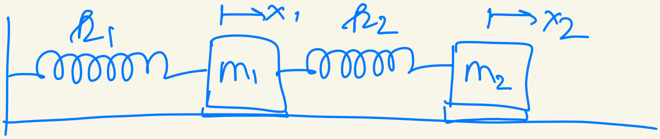
For right side m

$$\frac{d}{dt}(m\dot{x}_1) = k(x_1 - x_2)$$

For left side m

$$\frac{d}{dt}(m\dot{x}_2) = k(2x_2 - x_1)$$

Two Mass Two spring (different values) (no elevation):



$$KE = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2^2 - 2x_1 x_2 + x_1^2)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{\partial L}{\partial x_1} = -k_1 x_1 + k_2 x_2 - k_2 x_1$$

$$\frac{\partial L}{\partial x_2} = -k_2 x_2 + k_2 x_1$$

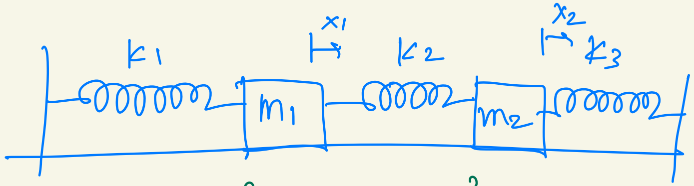
ODE x_1

$$\boxed{\frac{d}{dt}(m_1 \dot{x}_1) = -k_1 x_1 + k_2 (x_2 - x_1)}$$

ODE x_2

$$\boxed{\frac{d}{dt}(m_2 \dot{x}_2) = k_2 (x_1 - x_2)}$$

Two Mass, Two spring with walls (no elevation):



k_2 — force constant

$$KE = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_3 x_2^2 - \frac{1}{2} k_2 (x_2^2 + x_1^2 - 2x_1 x_2)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{\partial L}{\partial x_1} = -k_1 x_1 - k_2 x_1 + k_2 x_2 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\frac{\partial L}{\partial x_2} = -k_3 x_2 - k_2 x_2 + k_2 x_1 = -k_3 x_2 - k_2 (x_2 - x_1)$$

ODE

$$\frac{d}{dt}(m_1 \dot{x}_1) = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\frac{d}{dt}(m_2 \dot{x}_2) = -k_3 x_2 - k_2 (x_2 - x_1)$$

$$\frac{d^2}{dt^2} \begin{bmatrix} m_1 x_1 \\ m_2 x_2 \end{bmatrix} = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$m \ddot{x} = A x$$

$$\frac{d^2}{dt^2}\begin{bmatrix} m_1 x_1 \\ m_2 x_2 \end{bmatrix} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{--- (1)}$$

M = $N \times N$ diagonal matrix of masses

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

F = $N \times N$ matrix $\rightarrow$ discrete Laplacian of a weighted graph.

diagonal $\rightarrow f_{jj} = \sum_k k_{jk}$

off diagonal $\rightarrow f_{jk} = -k_{jk}$.

$$F = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y(t) = \sqrt{M}\, x(t)$$

$$(1) \Rightarrow M\ddot{X} = -FX$$

$$\Rightarrow \sqrt{M}\,\boxed{\sqrt{M}\,\ddot{X}} = -FX \Rightarrow \ddot{Y} = -\frac{FX}{\sqrt{M}}$$

$$A = \sqrt{M}^{-1} F \sqrt{M}^{-1} \geq 0$$

$$\ddot{Y} = -\sqrt{M}^{-1} F X \Rightarrow \frac{A}{\sqrt{M}^{-1}} = \sqrt{M}^{-1} F$$

$$\Rightarrow \ddot{Y} = -A\sqrt{M}X$$

$$\Rightarrow \boxed{\ddot{Y} = -AY} \quad \text{--- (2)}$$

$\ddot{y} = -Ay$

Using Schodinger equation caused problems because of $-\sqrt{A}$. So,

$B = N \times M$ matrix $\quad BB^{\dagger} = A$. and

Hamiltonian $\mathcal{H} = -\begin{bmatrix} 0 & B \\ B^{\dagger} & 0 \end{bmatrix}$

Approach 2

Total energy $\Rightarrow E = PE + KE$

$$E = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

We want to map $x_1, x_2, \dot{x}_1, \dot{x}_2$ at any time t to quantum state $|\psi(t)\rangle$

$\dot{x}$ amp. embedding - take square root & normalized map

$$|\psi(t)\rangle = \frac{1}{\sqrt{2E}} \begin{bmatrix} \sqrt{m_1}\,\dot{x}_1 \\ \sqrt{m_2}\,\dot{x}_2 \\ \sqrt{k_2}\,(x_2 - x_1) \\ \sqrt{k_1}\,x_1 \\ \sqrt{k_3}\,x_2 \end{bmatrix} \quad \because y = \sqrt{m}\,x. \quad = \frac{1}{\sqrt{2E}} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \sqrt{k_2}\,(x_2 - x_1) \\ \sqrt{k_1}\,x_1 \\ \sqrt{k_3}\,x_2 \end{bmatrix}$$

$\div$ by E to get normalized value.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2E}} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \sqrt{k_2}(x_2 - x_1) \\ \sqrt{k_1}x_1 \\ \sqrt{k_3}x_2 \end{bmatrix}$$

incidence matrix $\underline{B}$ and so $A = BB^{+}$

$B = N \times m$ matrix

Let $H = -\begin{bmatrix} 0 & B \\ B^{+} & 0 \end{bmatrix}$

Schrodinger equation induced by H is

$$|\dot{\psi}(t)\rangle = -iH|\psi(t)\rangle \quad \text{--- } (3)$$

$$= +i \begin{bmatrix} 0 & B \\ B^{+} & 0 \end{bmatrix} \frac{1}{\sqrt{2E}} \begin{bmatrix} \dot{y}_1 \\ y_2 \\ \sqrt{k_2}(x_2 - x_1) \\ \sqrt{k_1}x_1 \\ \sqrt{k_3}x_2 \end{bmatrix}$$

Laplacian F

$$F = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Incidence matrix

$N = 2$ masses

$B = N \times M$.

$M = 3$ springs (edges)

$\hookrightarrow 1$ iff vertex v is incident upon edge e

$$B = \begin{array}{c|c} & \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} +1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix} \end{array}$$

directed graph

$-1 \to$ start node of edge

$1 \to$ end node of edge.

$$B = \begin{bmatrix} k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} k_1 & 0 \\ -k_2 & k_2 \\ 0 & -k_3 \end{bmatrix}$$

$2 \times 3 \; 3 \times 2$

$$BB^T = \begin{bmatrix} k_1^2 + k_2^2 & -k_2^2 \\ -k_2^2 & k_2^2 + k_3^2 \end{bmatrix} := F$$

$$B = \begin{bmatrix} k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix}$$

$$B^{\dagger} y(t) = \begin{bmatrix} k_1 & 0 \\ -k_2 & k_2 \\ 0 & -k_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_1 y_1 \\ -k_2 y_1 + k_2 y_2 \\ -k_3 y_2 \end{bmatrix}$$

$$3 \times \boxed{2 \quad 2} \times 1$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2E}} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \sqrt{k_2}(x_2 - x_1) \\ \sqrt{k_1}\, x_1 \\ \sqrt{k_3}\, x_2 \end{bmatrix}$$

$\because y = \sqrt{m}\, x$

$x = \sqrt{m}^{-1} y$.

$$|\psi(t)\rangle = \alpha \begin{bmatrix} \dot{y}(t) \\ i B^{\dagger} y(t) \end{bmatrix} \longrightarrow B_f$$

Valid sol. for 3 $|\dot{\psi}(t)\rangle = -iH|\psi(t)\rangle$

$$\begin{pmatrix} \dot{y}(t) \\ iB^{+}y(t) \end{pmatrix} = e^{-itH} \begin{pmatrix} \dot{y}(0) \\ iB^{+}y(0) \end{pmatrix} \rightarrow \text{Solution}$$

$\rightarrow$ S

Choose B such that $B^{+}y(t) = B^{+}\sqrt{M}\,x(t)$

$$\begin{bmatrix} k_1 & 0 \\ -k_2 & k_2 \\ 0 & -k_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ -k_2 & k_2 \\ 0 & -k_3 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}^{1/2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

B from incidence matrix of F.

$$\sqrt{M}\,B = \begin{bmatrix} k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix}$$

$$H = -\begin{bmatrix} 0 & B \\ B^{\dagger} & 0 \end{bmatrix}$$

$$B^{\dagger} y(t) = B^{\dagger} \sqrt{M}\, x(t)$$

$$H = -\begin{bmatrix} 0 & 0 & k_1 & -k_2 & 0 \\ 0 & 0 & 0 & k_2 & -k_3 \\ k_1 & 0 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & 0 & 0 \end{bmatrix}$$