School of Computer Science
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National University of Engineering

Midterm Exam - Solution

Topics: transformations; curves; linear programming - Simplex method
Subject: Computational Mathematics **Period: 2020.1**

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1. Let U be the unit cube with vertices (a, b, c), where each component is 0 or 1.

   (a) (2 pts.) Find the rotation of U of $2\pi/3$ clockwise around the line from $P_0 := (0, 0, 0)$ to $(1, 1, 1)$.

   (b) (1 pt.) Print the transformed cube of the item above.

Solution.

(a) As it was calculated in the solution of Question 3 of Test 1, the matrix transformation of this rotation is

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Moreover, let

$$\begin{aligned} v_0 &= (0, 0, 0) \\ v_1 &= (0, 0, 1) \\ v_2 &= (0, 1, 0) \\ v_3 &= (0, 1, 1) \\ v_4 &= (1, 0, 0) \\ v_5 &= (1, 0, 1) \\ v_6 &= (1, 1, 0) \\ v_7 &= (1, 1, 1) \end{aligned}$$

be the vertices of U. Then, as

$$\begin{aligned} M(v_0) &= v_0 \\ M(v_1) &= v_4 \\ M(v_2) &= v_1 \\ M(v_3) &= v_5 \\ M(v_4) &= v_2 \\ M(v_5) &= v_6 \\ M(v_6) &= v_3 \\ M(v_7) &= v_7, \end{aligned}$$

the rotated cube is 'the same' U; however, the vertices have been permuted.

(b)

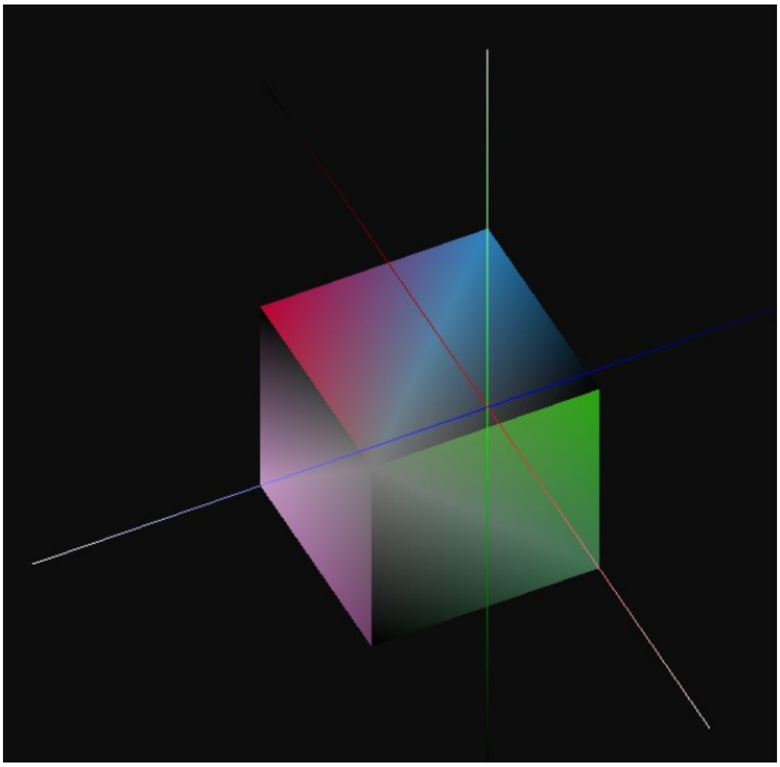


Figure 1: Transformed cube, taken from Jesús Torrejón's solution.

$\square$

2. Let $P_0 := (-2, 1)$, $P_1 := (0, -4)$, $P_2 := (3, 2)$ and $P_3 := (5, 0)$.

   (a) (2 pts.) Find the parametric description $P(t)$ of the cubic Bézier curve with control points: P_0, P_1, P_2 and P_3.

   (b) (1 pt.) Print the Bézier curve above with its control points.

(c) (2 pts.) Find the parametric description $P(t)$ of the uniform quadratic B-spline using control points: P_0, P_1, P_2 and P_3.

(d) (1 pt.) Print the B-spline above with its control points.

Solution.

(a) By virtue of Equation (7.9) in page 224 of [1],

$$P(t) = \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$
$$= \begin{bmatrix} -2 + 6t + 3t^2 - 2t^3 \\ 1 - 15t + 33t^2 - 19t^3 \end{bmatrix}$$

(b) From the parametric equation above: $P(0) = (-2, 1)$ and $P(1) = (5, 0)$.

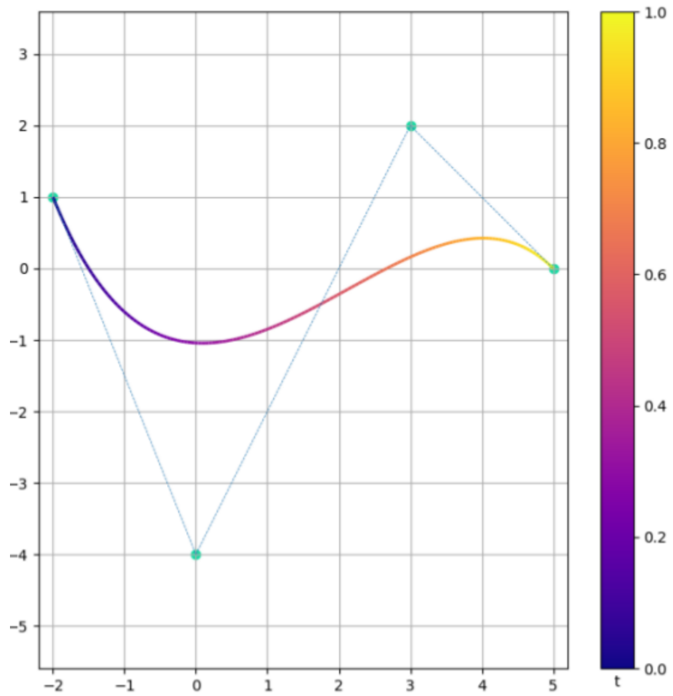


Figure 2: Bézier curve with control points P_0, P_1, P_2 and P_3, taken from Jesús Torrejón's solution.

(c) By virtue of Example 7.8 in page 238 of [1], we can build an uniform quadratic

B-spline with two segments: $[2,3]$ and $[3,4]$. Then, for $t \in [2,3]$ we have:

$$\begin{aligned}P(t) &= \left(\frac{1}{2} - (t-2) + \frac{1}{2}(t-2)^2\right)\begin{bmatrix}-2\\1\end{bmatrix} + \left(\frac{1}{2} + (t-2) - (t-2)^2\right)\begin{bmatrix}0\\-4\end{bmatrix}\\ &\quad + \frac{1}{2}(t-2)^2\begin{bmatrix}3\\2\end{bmatrix} + 0\cdot\begin{bmatrix}5\\0\end{bmatrix}\\ &= \begin{bmatrix}-3+\frac{1}{2}t^2\\ \frac{61}{2} - 27t + \frac{11}{2}t^2\end{bmatrix}\end{aligned}$$

And, for $t \in [3,4]$ we have:

$$\begin{aligned}P(t) &= 0\cdot\begin{bmatrix}-2\\1\end{bmatrix} + \left(\frac{1}{2} - (t-3) + \frac{1}{2}(t-3)^2\right)\begin{bmatrix}0\\-4\end{bmatrix}\\ &\quad + \left(\frac{1}{2} + (t-3) - (t-3)^2\right)\begin{bmatrix}3\\2\end{bmatrix} + \frac{1}{2}(t-3)^2\begin{bmatrix}5\\0\end{bmatrix}\\ &= \begin{bmatrix}-12+6t-\frac{1}{2}t^2\\ -55+30t-4t^2\end{bmatrix}.\end{aligned}$$

(d) From the parametric equations above: $P(2) = (-1, -1.5)$, $P(3) = (1.5, -1)$ and $P(4) = (4, 1)$.

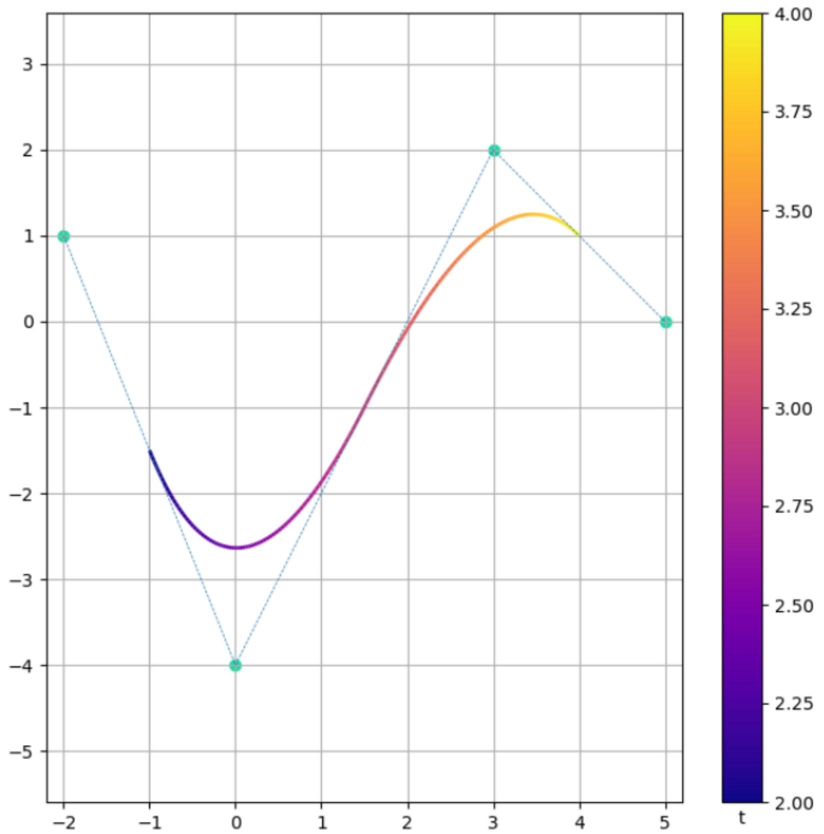


Figure 3: B-spline with control points P_0, P_1, P_2 and P_3, taken from Jesús Torrejón's solution.

□

3. In a Python program implement:

   (a) (2 pts.) A function that receives a 'tableau', a basic `b` and a non-basic element `n`, and returns the resulting 'tableau' of conducting the process: `pivot(b,n)`.

   (b) (2 pts.) A function called `iterate` that receives a 'tableau' with $\tilde{b} \geq 0$, and returns the final 'tableau' of conducting the iteration process of the Simplex method.

   (c) (1 pt.) Pass Table 1 to the `iterate` function above and print the final 'tableau'.

	1	x_1	x_3
z	4.30	-0.2	6
x_2	1.05	-1.2	1
x_4	0.05	-0.2	1

Table 1: 'Tableau' for iteration process of Simplex method.

Solution. see `code_for_q3.py` file, taken from Jesús Torrejón's solution. □

4. Considering the following linear optimization problem:

"Jose builds electrical cable using two types of metallic alloys. Alloy 1 is 55% aluminum and 45% copper, while alloy 2 is 75% aluminum and 25% copper. Market prices for alloys 1 and 2 are \$5 and \$4 per ton, respectively. Formulate a linear optimization problem to determine the cost-minimizing quantities of the two alloys that Jose should use to produce 1 ton of cable that is at least 30% copper."

do the following:

   (a) (2 pts.) Formulate in the general form of a LPP.

   (b) (2 pts.) Conduct a geometric analysis: find (sketch) the feasible region, find the level sets, find the optimal solution and optimal value.

   (c) (2 pts.) Conduct the Simplex method with $K = 6$ for the regularization step.

Solution.

   (a) First, let

$$x_1 : \text{quantity in ton of alloy 1;}$$
$$x_2 : \text{quantity in ton of alloy 2.}$$

Second, because we are asked to "determine the cost-minimizing quantities of the two alloys", the problem is to minimize the objective function:

$$z = 5x_1 + 4x_2.$$

Third, as Jose should "produce 1 ton of cable" one restriction is

$$x_1 + x_2 = 1.$$

Fourth, since that ton of cable must be "at least 30% copper", another restriction is

$$0.45x_1 + 0.25x_2 \geq 0.3.$$

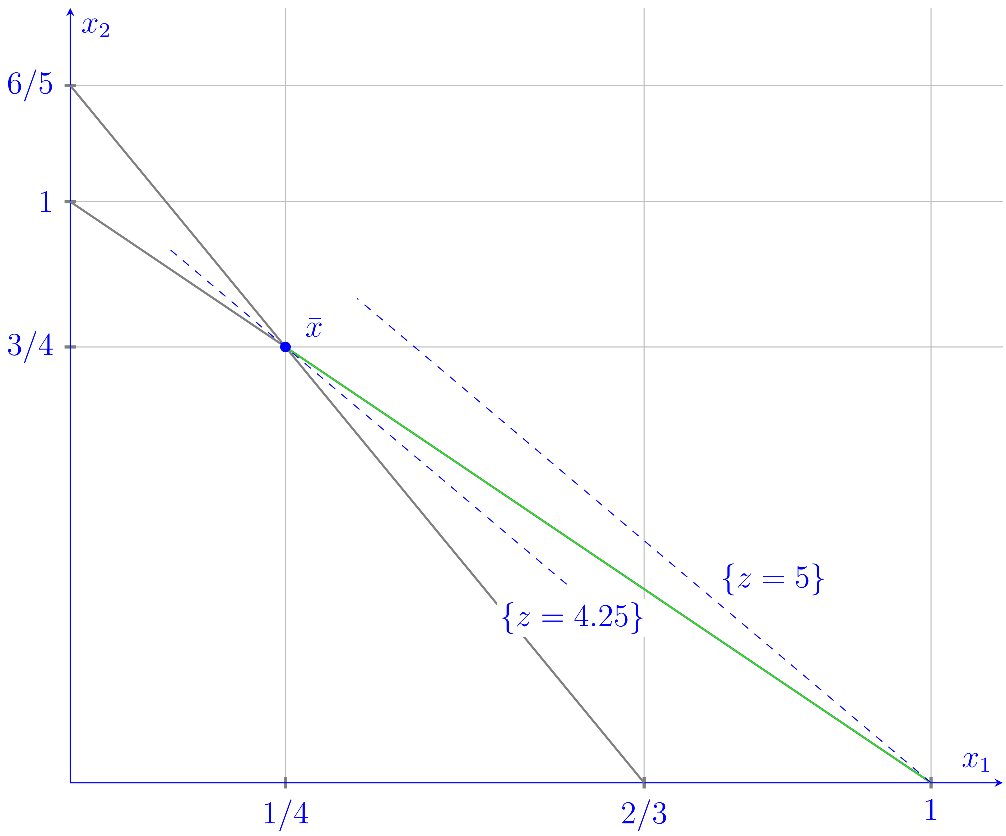
Moreover, it is an implicit restriction that

$$x_1, x_2 \geq 0.$$

Finally, the general form of our LPP is

$$\begin{array}{lrcrcl} \text{minimize} & 5x_1 & + & 4x_2 & & \\ \text{subject to} & x_1 & + & x_2 & = & 1 \\ & 0.45x_1 & + & 0.25x_2 & \geq & 0.3 \\ & & & x_1, x_2 & \geq & 0. \end{array}$$

(b) Geometric analysis:



The feasible region is the green segment line; as our optimal problem is a minimization one, the optimal solution is $\bar{x} = \{1/4, 3/4\}$ and the optimal value is $\bar{z} = 4.25$.

(c) First, the standard form of our original problem is

$$\begin{array}{rrcl} \text{minimize} & c^\top x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0.45 & 0.25 & -1 \end{bmatrix}$$
$$b = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}$$
$$c = (5, 4, 0)$$
$$x = (x_1, x_2, x_3).$$

Second, setting $x_B := (x_2, x_3)$ and $x_N = (x_1)$, and by virtue of section 2.4.2 of [2], we have Table 2.

	1	x_1
z	4	1
x_2	1	-1
x_3	-0.05	0.2

Table 2: First ‘tableau’ of Simplex method.

Third, as $\widetilde{b}$ is not greater than zero, a regularization process is necessary, we work with $K = 6$ and get Table 3.

	1	x_1	x_4
z	4	1	6
x_2	1	-1	1
x_3	-0.05	0.2	1

Table 3: ‘Tableau’ for regularization process.

Then, after the regularization process we get Table 1 above. Finally, as it was shown in the solution of item (c) of Question 3 above, we got Table 4

	1	x_4	x_3
z	4.25	1	5
x_2	0.75	6	-5
x_1	0.25	-5	5

Table 4: Final 'tableau'.

and as it was shown in the previous item, the optimal solution and optimal value are the same. □

Bibliography

[1] Janke, S. J. *Mathematical Structures for Computer Graphics.* Wiley, 2015.

[2] Sioshansi, R., and Conejo, A. J. *Optimization in Engineering: Models and Algorithms.* Springer Optimization and Its Applications. Springer, 2017.

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