AMN

SOLUCIÓN DE LA PC1

1a) Supongamos que si A es estrictamente diagonalmente dominante y singular $|A|=0$, entonces existe un vector $X\neq 0$ con $AX=0$, esto es X tiene alguna entrada $x_i>0$ tal que $|x_i| = \max_{1\leq j\leq n}\{|x_j|\}$.

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \quad i=1,2,\cdots,n$$

$$a_{ii}x_i = -\sum_{j\neq i=1}^{n} a_{ij}x_j$$

$$a_{ii} = -\sum_{j\neq i=1}^{n}\frac{x_j}{x_i}a_{ij}$$

$$|a_{ii}| = \left|-\sum_{j\neq i=1}^{n}\frac{x_j}{x_i}a_{ij}\right| \leq \sum_{j\neq i=1}^{n}\left|\frac{x_j}{x_i}a_{ij}\right|$$

$$|a_{ii}| \leq \sum_{j\neq i=1}^{n}|a_{ij}|$$

lo cual es una contradicción a la definición de matriz diagonalmente estrictamente dominante.

- $A=(a_{ij})_{n\times n}$ es estrictamente dominante por filas, cuando
$$|a_{ii}| > \sum_{j\neq i=1}^{n}|a_{ij}|, \quad \forall i=1,2,\cdots,n$$
- A es diagonalmente estrictamente dominante si lo es por filas o por columnas

1b) Por teorema, toda matriz A diagonalmente estrictamente dominante tiene todos los menores principales no singulares. Por lo tanto tiene una factorización LU Doolitle única (teorema).

Dada una matriz cuadrada A, un menor principal es aquel determinante de una submatriz cuadrada de A, once que los elementos de su diagonal principal pertenecen a la diagonal principal de la matriz A

2) Factorización LU (Doolitle)

$$A=\begin{pmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} \quad \begin{array}{l} R_2 \to \boxed{-\frac{1}{3}} R_1 + R_2 \\ R_3 \to \boxed{-\frac{1}{6}} R_1 + R_3 \\ R_4 \to \boxed{\frac{1}{6}} R_1 + R_4 \end{array} \begin{pmatrix} 6 & 2 & 1 & -1 \\ 0 & 10/3 & 2/3 & 1/3 \\ 0 & 2/3 & 23/6 & -5/6 \\ 0 & 1/3 & -5/6 & 17/6 \end{pmatrix}$$

$$\begin{array}{l} R_3 \to \boxed{-1/5} R_2 + R_3 \\ R_4 \to \boxed{-\frac{1}{10}} R_2 + R_4 \end{array} \begin{pmatrix} 6 & 2 & 1 & -1 \\ 0 & 10/3 & 2/3 & 1/3 \\ 0 & 0 & 37/10 & -9/10 \\ 0 & 0 & -9/10 & 14/5 \end{pmatrix} \quad R_4 \to \boxed{\frac{9}{37}} R_3 + R_4 \quad \underbrace{\begin{pmatrix} 6 & 2 & 1 & -1 \\ 0 & 10/3 & 2/3 & 1/3 \\ 0 & 0 & 37/10 & -9/10 \\ 0 & 0 & 0 & 191/74 \end{pmatrix}}_{U}$$

$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{1/3} & 1 & 0 & 0 \\ \boxed{1/6} & \boxed{1/5} & 1 & 0 \\ \boxed{-1/6} & \boxed{1/10} & -9/37 & 1 \end{pmatrix}$$

3) Factorización Cholesky (Matriz Simétrica)

$$A=\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad AU=\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}; \quad A = BB^T = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} & l_{n1} \\ 0 & l_{22} & l_{32} & l_{n2} \\ 0 & 0 & l_{33} & l_{n3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & l_{nn} \end{pmatrix}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{r=1}^{k-1} l_{kr}^2}, \quad l_{kj} = \frac{1}{l_{jj}}\left(a_{kj} - \sum_{r=1}^{j-1} l_{kr} l_{jr}\right)$$

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$l_{22} = \sqrt{a_{22} - \sum_{r=1}^{1} l_{2r}^2} = \sqrt{4 - l_{21}^2} = \sqrt{4-(-1/2)^2} = \frac{\sqrt{15}}{2}$$

$$l_{33} = \sqrt{a_{33} - \sum_{r=1}^{2} l_{3r}^2} = \sqrt{4-(l_{31}^2 + l_{32}^2)} = \sqrt{4-(0+\frac{4}{15})} = \frac{2\sqrt{210}}{15}$$

$$l_{21} = \frac{1}{l_{11}}(a_{21}) = \frac{1}{2}(-1) = -\frac{1}{2}$$

$$l_{31} = \frac{1}{l_{11}}(a_{31}) = \frac{1}{2}(0) = 0$$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - \sum_{r=1}^{1} l_{3r}\right) = \frac{1}{\sqrt{15}/2}(-1-0) = -\frac{2\sqrt{15}}{15}$$

luego

$$A=\begin{pmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2\sqrt{15}/15 & 2\sqrt{210}/15 \end{pmatrix}\begin{pmatrix} 2 & -1/2 & 0 \\ 0 & \sqrt{15}/2 & -2\sqrt{15}/15 \\ 0 & 0 & 2\sqrt{210}/15 \end{pmatrix} \blacksquare$$

- $(x_1, x_2, x_3)\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4x_2^2 - 2x_1x_2 - 2x_2x_3 + 4x_1^2 + x_3^2 = (x_2-x_1)^2 + 2x_2^2 + (x_2-x_3)^2 > 0$

  A es definida positiva

- Autovalores de A, $\lambda_1 = 4$, $\lambda_2 = 4+\sqrt{2}$, $\lambda_3 = 4-\sqrt{2}$ >0

3b). $Ax=b$, $B(\underbrace{B^T x}_{Y})=b$, $BY=b$

$$\begin{pmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2\sqrt{15}/15 & 2\sqrt{210}/15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix};$$

$2y_1 = 2$, $y_1 = 1$

$-\frac{1}{2}y_1 + \frac{\sqrt{15}}{2}y_2 = 6$, $y_2 = \frac{13\sqrt{15}}{15}$

$-\frac{2\sqrt{15}}{15}y_2 + \frac{2\sqrt{210}}{15}y_3 = 2$, $y_3 = \frac{2\sqrt{210}}{15}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 13\sqrt{15}/15 \\ 2\sqrt{210}/15 \end{pmatrix}$$

$B^T x = Y$

$$\begin{pmatrix} 2 & -1/2 & 0 \\ 0 & \sqrt{15}/2 & -2\sqrt{15}/15 \\ 0 & 0 & 2\sqrt{210}/15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 13\sqrt{15}/15 \\ 2\sqrt{210}/15 \end{pmatrix}$$

$2x_1 - 1/2\, x_2 = 1$, $x_1 = 1$

$\frac{\sqrt{15}}{2}x_2 - \frac{2\sqrt{15}}{15}x_3 = \frac{13\sqrt{15}}{15}$, $x_2 = 2$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\frac{2\sqrt{210}}{15}x_3 = \frac{2\sqrt{210}}{15}$, $x_3 = 1$

4) Sistema tridiagonal (número de operaciones, eliminación gaussiana).

Algoritmo

$p_1 := \frac{c_1}{a_1}$, $q_1 := \frac{d_1}{a_1}$

For $i=2$ to n

$r_i = a_i - b_i p_{i-1}$, $p_i = \frac{c_i}{r_i}$, $q_i = \frac{d_i - b_i q_{i-1}}{r_i}$

end

$x_n := q_n$

$i := n-1$

repeat $x_i := q_i - p_i x_{i+1}$, $i := i-1$

until

$i = 0$

La solución es $x_1, x_2, \ldots, x_n$

Terminar.

4) sistema tridiagonal (número de operaciones: eliminación gaussiana)

$$\left(\begin{array}{cccccc|c} a_1 & c_1 & & & & & d_1 \\ b_2 & a_2 & c_2 & & 0 & & d_2 \\ & b_3 & a_3 & c_3 & & & d_3 \\ & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right) \sim \quad R_1 \to \frac{1}{a_1} R_1 \quad \left(\begin{array}{cccccc|c} 1 & p_1 & & & & & q_1 \\ b_2 & a_2 & c_2 & & 0 & & d_2 \\ & b_3 & a_3 & c_3 & & & d_3 \\ & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right)$$

$\boxed{p_1 = \frac{c_1}{a_1}}$, $\boxed{q_1 = \frac{d_1}{a_1}}$

$R_2 \to -b_2 R_1 + R_2$

$$\left(\begin{array}{cccccc|c} 1 & p_1 & & & & & q_1 \\ 0 & r_2 & c_2 & & 0 & & d_2 - b_2 q_1 \\ & b_3 & a_3 & c_3 & & & d_3 \\ & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right) R_2 \to \frac{1}{r_2} R_2 \left(\begin{array}{cccccc|c} 1 & p_1 & 0 & & & 0 & q_1 \\ 0 & 1 & p_2 & & & & q_2 \\ & b_3 & a_3 & c_3 & & & d_3 \\ & 0 & \ddots & \ddots & \ddots & & \vdots \\ & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right)$$

$r_2 = -b_2 p_1 + a_2$

$p_2 = \frac{c_2}{r_2}$, $q_2 = \frac{d_2 - b_2 q_1}{r_2}$

$R_3 \to -b_3 R_2 + R_3$

$$\left(\begin{array}{cccccc|c} 1 & p_1 & 0 & & & & q_1 \\ 0 & 1 & p_2 & & 0 & & q_2 \\ & 0 & r_3 & c_3 & & & d_3 - b_3 q_2 \\ 0 & & \ddots & \ddots & \ddots & & \vdots \\ & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right) R_3 \to \frac{1}{r_3} R_3 \left(\begin{array}{cccccc|c} 1 & p_1 & 0 & & & 0 & q_1 \\ 0 & 1 & p_2 & & & & q_2 \\ & 0 & 1 & p_3 & & & q_3 \\ 0 & & \ddots & \ddots & \ddots & & \vdots \\ & & & b_{n-1} & a_{n-1} & c_{n-1} & d_{n-1} \\ & & & & b_n & a_n & d_n \end{array}\right)$$

$r_3 = a_3 - b_3 p_2$

$p_3 = \frac{c_3}{r_3}$, $q_3 = \frac{d_3 - b_3 q_2}{r_3}$

Así sucesivamente

$$\left(\begin{array}{cccccc|c} 1 & p_1 & 0 & & & & q_1 \\ 0 & 1 & p_2 & & 0 & & q_2 \\ & & 1 & p_3 & & & q_3 \\ & & & \ddots & \ddots & & \vdots \\ 0 & & & & 1 & p_{n-1} & q_{n-1} \\ & & & & 0 & 1 & q_n \end{array}\right)$$

$p_1 = \frac{c_1}{a_1}$, $q_1 = \frac{d_1}{a_1}$

$i = 1, 2, \dots, n$

$r_i = a_i - b_i p_{i-1}$

$p_i = \frac{c_i}{r_i}$, $q_i = \frac{d_i - b_i q_{i-1}}{r_i}$

En general para $A = [a_{ij}]_{n \times n}$ tridiagonal. La cantidad de operaciones para resolver un sistema tridiagonal de n-ecuaciones lineales, el método de Gauss requiere <u>8n operaciones</u>

$$\begin{aligned} x_1 + p_1 x_2 &= q_1 \\ x_2 + p_2 x_3 &= q_2 \\ x_3 + p_3 x_4 &= q_3 \\ &\vdots \\ x_{n-1} + p_{n-1} x_n &= q_{n-1} \\ x_n &= q_n \end{aligned} \quad \left\{ \begin{aligned} x_n &= q_n \\ x_{n-1} &= q_{n-1} - p_{n-1} x_n \\ x_{n-2} &= q_{n-2} - p_{n-2} x_{n-1} \\ &\vdots \\ x_2 &= q_2 - p_2 x_3 \\ x_1 &= q_1 - p_1 x_2 \end{aligned} \right.$$

b.1) $\|x\|_F = \sqrt{tr(A^TA)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{i=1}^{\min\{m,n\}}\sigma_i^2}$

$\sigma_i = \sqrt{\lambda_i}$: Valores singulares de A
λ_i: valor propio de A.

$$A^TA = \begin{pmatrix} 89 & 34 & -2 \\ 34 & 49 & 14 \\ -2 & 14 & 69 \end{pmatrix}$$

$tr(A^TA) = 89+49+69 = 207$

$\|A\|_F = \sqrt{207} \approx 14.387494..$ ∎

$$A^TA = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

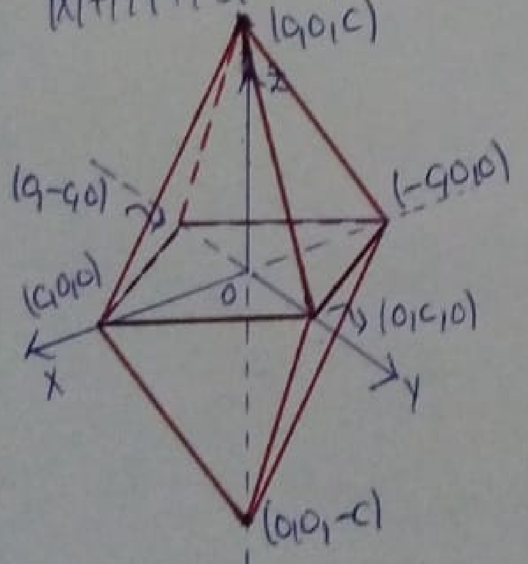
$$= \begin{pmatrix} \underbrace{a_{11}a_{11}+a_{21}a_{21}+\cdots+a_{m1}a_{m1}}_{\sum_{i=1}^{m} a_{i1}^2} & * & * \\ * & \underbrace{a_{12}a_{12}+a_{22}a_{22}+\cdots+a_{m2}a_{m2}}_{\sum_{i=1}^{m} a_{i2}^2} & * \\ \vdots & & \ddots \\ * & * & \underbrace{a_{1n}a_{1n}+a_{2n}a_{2n}+\cdots+a_{mn}a_{mn}}_{\sum_{i=1}^{m} a_{in}^2} \end{pmatrix}$$

$tr(A^TA) = \sum_{i=1}^{m} a_{i1}^2 + \sum_{i=1}^{m} a_{i2}^2 + \cdots + \sum_{i=1}^{m} a_{in}^2$

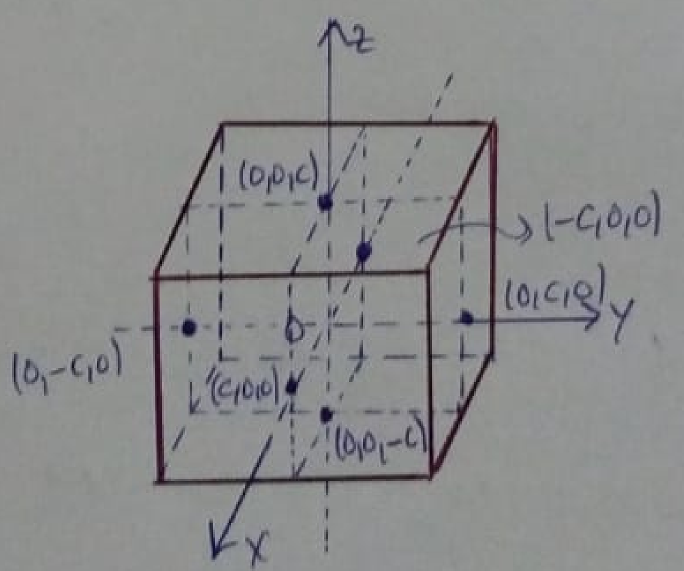
$= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^2\right)$ □

5a) $\vec{x} = (x,y,z)$, $\|\vec{x}\|_1 = C$, $\|\vec{x}\|_\infty = C$, C = constante > 0: significado geométrico

1) $\|\vec{x}\|_1 = C$ (norma de la suma).
$|x|+|y|+|z| = C$



2) $\|\vec{x}\|_\infty = C$ (norma del máximo)
$\max\{|x|,|y|,|z|\}$



Los profesores

5b). $A=\begin{pmatrix} 8 & 3 & 4 \\ 3 & 6 & -2 \\ 4 & -2 & -7 \end{pmatrix}$, $\|*\|_1$, $\|*\|_\infty$, $\|*\|_2$, $\|*\|_F$?

$\begin{array}{ccc} \tilde{15} & \tilde{11} & \tilde{13} \end{array}$

$$A=\begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{array}{c} \tilde{F_1} \\ \tilde{F_2} \\ \\ \tilde{F_m} \end{array} \\ \begin{array}{ccc} \tilde{C_1} & \tilde{C_2} & \tilde{C_n} \end{array} \end{array}$$

b1) $\|*\|_1 = \max_{1\le j\le n} \{\tilde{C_1}, \tilde{C_2}, \cdots, \tilde{C_n}\}$

donde $\tilde{C_j} = |a_{1j}| + |a_{2j}| + \cdots + |a_{mj}| = \sum_{i=1}^{m} |a_{ij}|$

$\|*\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$: máxima suma absoluta de las columnas de la matriz.

$A=\begin{pmatrix} 8 & 3 & 4 \\ 3 & 6 & -2 \\ 4 & -2 & -7 \end{pmatrix}$, $\|A\|_1 = \max\{15, 11, 13\} = 15$

$\begin{array}{ccc} \tilde{15} & \tilde{11} & \tilde{13} \end{array}$

b2) $\|*\|_\infty = \max_{1\le i\le m} \{\tilde{F_1}, \tilde{F_2}, \cdots, \tilde{F_m}\}$

donde $\tilde{F_i} = |a_{i1}| + |a_{i2}| + \cdots + |a_{in}| = \sum_{j=1}^{n} |a_{ij}|$

$\|*\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$: máxima suma absoluta de las filas de la matriz.

$A=\begin{pmatrix} 8 & 3 & 4 \\ 3 & 6 & -2 \\ 4 & -2 & -7 \end{pmatrix} \begin{array}{c} 15 \\ 11 \\ 13 \end{array}$

$\|A\|_\infty = \max\{15, 11, 13\} = 15$

b3) $\|*\|_2 = \sqrt{\underbrace{\lambda_{max}}_{\text{valor propio}}(A^*A)} = \max_{1\le i\le n} \sqrt{\lambda_i(A^TA)}$

$$A^TA = \begin{pmatrix} 8 & 3 & 4 \\ 3 & 6 & -2 \\ 4 & -2 & -7 \end{pmatrix}\begin{pmatrix} 8 & 3 & 4 \\ 3 & 6 & -2 \\ 4 & -2 & -7 \end{pmatrix} = \begin{pmatrix} 89 & 34 & -2 \\ 34 & 49 & 14 \\ -2 & 14 & 69 \end{pmatrix}$$

$\underbrace{|A-\lambda I|}_{P(\lambda)\text{ polinomio característico}} = 0$

$$\begin{vmatrix} 89-\lambda & 34 & -2 \\ 34 & 49-\lambda & 14 \\ -2 & 14 & 69-\lambda \end{vmatrix} = 0, \quad -\lambda^3 + 207\lambda^2 - 12527\lambda + 201601 = 0$$

$$\lambda^3 - 207\lambda^2 + 12527\lambda - 201601 = 0$$

de donde

$\lambda_1 \approx 25.5467\ldots$, $\lambda_2 \approx 72.29137\ldots$, $\lambda_3 \approx 109.16190\ldots$: valores propios

$\|A\|_2 = \max_{1\le i\le 3}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}\} = \max\{5.0543753\ldots, 8.502433\ldots, 10.448057\ldots\}$

$\approx 10.448057\ldots$ Norma espectral de la matriz A