UNIVERSIDAD NACIONAL DE INGENIERÍA

ANÁLISIS Y MODELAMIENTO NUMÉRICO I
Práctica Dirigida I - Solucionario

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Problema 1

La función $f : [a, b] \to \mathbb{R}$ se dice que satisface una *condición de Lipschitz* con constante de Lipschitz L en $[a, b]$ si, para cada $x, y \in [a, b]$, se tiene

$$|f(x) - f(y)| \leq L\,|x - y|.$$

a. Demuestre que si f satisface la condición de Lipschitz con constante de Lipschitz L en el intervalo $[a, b]$, entonces $f \in C[a, b]$.
b. Demuestre que si f tiene una derivada que es acotada en $[a, b]$ por L, entonces f satisface la condición de Lipschitz con constante de Lipschitz L en el intervalo $[a, b]$.
c. Dé un ejemplo de una función que sea continua en un intervalo cerrado pero que no satisfaga la condición de Lipschitz en el intervalo.

Solución - Problema 1

Our function $f(x) : [a, b] \to \mathbb{R}$ satisfies **Lipschitz condition** on $f(x)$ i.e. there exists a **Lipschitz constant** $L > 0$ such that for every $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq L|x - y|$$

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a. A function $f(x)$ is said to be continuous at x_0, if for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta$$

If $f(x)$ is continuous at each point of its domain, it is called continuous function.
Assuming that $f(x)$ follows Lipschitz condition, for any $\epsilon > 0$, take $\delta = \dfrac{\epsilon}{L}$
As $|x - y| < \delta$, we get

$$|f(x) - f(y)| \leq L|x - y| < L \times \delta = \epsilon$$

Thus, $f(x)$ is continuous at all points $y \in [a, b]$.

Solución - Problema 1 (cont.)

b. Given that $|f'(x)| \leq L$, for all $x \in [a,b]$
If $f(x)$ is differentiable on $[a,b]$, then according to Mean Value Theorem:
For $x, y \in [a,b]$,

$$\frac{f(x)-f(y)}{x-y} = f'(\gamma) \quad \text{for some } \gamma \in [x,y]$$

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(\gamma)| \leq L \quad \text{for all } x, y \in [a,b]$$

$$|f(x)-f(y)| \leq L|x-y| \quad \text{for all } x, y \in [a,b]$$

Solución - Problema 1 (cont.)

c. $f(x) = \sqrt{x}$ defined on $x \in [0,1]$ will serve our purpose.
We can claim that this function doesn't satisfy Lipschitz condition, since its derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$ is unbounded at $x = 0$.
But, it can be shown that one-sided derivatives of Lipschitz functions are always bounded, as for $h > 0$:

$$f'_+(x) = \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \leq L$$

$$f'_-(x) = \lim_{h\to 0}\left|\frac{f(x)-f(x-h)}{h}\right| \leq L$$

Even, if both left hand and right hand derivatives at some point x_0 not become equal, i.e. derivative doesn't exist at point x_0, one-sided derivatives always remain bounded.

Problema 3

Sea $f \in C[a,b]$, y sea p en el intervalo (a,b).

1. Suponga que $f(p) \neq 0$. Demuestre que existe un $\delta > 0$ con $f(x) \neq 0$, para todo x en $[p-\delta, p+\delta]$, con $[p-\delta, p+\delta]$ subconjunto de $[a,b]$.
2. Suponga que $f(p) = 0$ y $k > 0$ es dado. Demuestre que existe $\delta > 0$ con $|f(x)| \leq k$, para todo x en $[p-\delta, p+\delta]$, con $[p-\delta, p+\delta]$ subconjunto de $[a,b]$.

Solución - Problema 3

$f \in C[a,b] \qquad p \in (a,b).$

(a) suppose $f(p) \neq 0$. then $|f(p)| > 0$.

Let $\epsilon = \frac{|f(p)|}{2} > 0$. Then by the definition of continuity,

$\exists\ \delta' > 0$ such that

$$f(p) - \epsilon \leq f(x) \leq f(p) + \epsilon \quad \forall\ x \in [p - \delta', p + \delta'].$$

$$\Rightarrow f(p) - \frac{|f(p)|}{2} \leq f(x) \leq f(p) + \frac{|f(p)|}{2}$$

If $f(p) > 0$, $\quad \frac{f(p)}{2} \leq f(x) \leq \frac{3f(p)}{2} \Rightarrow f(x) > 0$

If $f(p) < 0$, $\quad \frac{3f(p)}{2} \leq f(x) \leq \frac{f(p)}{2} \Rightarrow f(x) < 0.$

This will be true for any $\delta < \delta'$.

Therefore we can choose $\delta > 0$ such that

$[p - \delta, p + \delta] \subseteq [a, b]$ and $f(x) \neq 0 \quad \forall\ x \in [p - \delta, p + \delta].$

Solución - Problema 3 (cont.)

(b) Suppose $f(p) = 0$ and $k > 0$. $p \in (a,b)$.

Take $\epsilon = k$ then $\exists\ \delta' > 0$ such that

$$|f(x) - f(p)| \leq \epsilon \quad \forall\ x \in [p-\delta', p+\delta'].$$

$$\Rightarrow |f(x)| \leq k \quad \forall\ x \in [p-\delta', p+\delta'].$$

This will be true for any $0 < \delta < \delta'$.

Therefore we can choose $\delta > 0$ such that $[p-\delta, p+\delta] \subseteq [a,b]$ and $|f(x)| \leq k$. $\forall\ x \in [p-\delta, p+\delta]$.

[Note: as $p \in (a,b)$ in both the problem (a) and (b). $p - a > 0$, $b - p > 0$ }.

We can take $\delta = \min\left\{\delta', \frac{p-a}{2}, \frac{b-p}{2}\right\}$.

Then $[p-\delta, p+\delta] \subseteq (a,b)$ and $\delta \leq \delta'$]

Problema 4

Sea $f(x) = \dfrac{x\cos x - \sin x}{x - \sin x}$

- (a) Encuentre $\lim_{x\to 0} f(x)$
- (b) Use aritmética de redondeo de cuatro dígitos para evaluar $f(0.1)$
- (c) Reemplace cada función trigonométrica con su tercer polinomio de Maclaurin, y repita la parte (b).
- (d) El valor real es $f(0.1) = -1.99899998$. Encuentre el error relativo para los valores obtenidos en las partes (b) y (c).

Solución - Problema 4

(a)

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x-\sin x}=(LHopital,0/0)=$$

$$=\lim_{x\to 0}\frac{-x\sin x}{1-\cos x}=(LH,0/0)$$

$$=\lim_{x\to 0}\frac{-\sin x-x\cos x}{\sin x}=\lim_{x\to 0}\frac{-2\cos x+x\sin x}{\cos x}=-2$$

Solución - Problema 4 (cont.)

(b) $f(0.1) \approx -1.941$

(c) see example 3 (or Common series section at the end of the book)

$$\frac{x(1-\frac{1}{2}x^2)-(x-\frac{1}{6}x^3)}{x-(x-\frac{1}{6}x^3)} = -2$$

(d)

The relative error $\dfrac{|p-p^*|}{|p|}$ in part (b) is 0.029.

The relative error in part (c) is 0.00050.

Richard L. Burden, J. Douglas Faires, *Numerical Analysis*, NINTH EDITION

Problema 6

Suppose that $fl(y)$ is a $k-$digit rounding approximation to y. Show that

$$\left|\frac{y - fl(y)}{y}\right| \leq 0.5 \times 10^{-k+1}$$

[*Hint:* If $d_{k+1} < 5$, then $fl(y) = 0.d_1.d_2...d_k \times 10^n$. If $d_{k+1} \geq 5$, then $fl(y) = 0.d_1.d_2...d_k \times 10^n + 10^{n-k}$.]

Solución - Problema 6

SOLUTION: We will consider the solution in two cases, first when $d_{k+1} \leq 5$, and then when $d_{k+1} > 5$.

When $d_{k+1} \leq 5$, we have

$$\left|\frac{y - fl(y)}{y}\right| = \frac{0.d_{k+1}\ldots \times 10^{n-k}}{0.d_1 \ldots \times 10^n} \leq \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$, we have

$$\left|\frac{y - fl(y)}{y}\right| = \frac{(1 - 0.d_{k+1}\ldots) \times 10^{n-k}}{0.d_1 \ldots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

Hence the inequality holds in all situations.

Problema 8

Las ecuaciones (1.2) y (1.3) en la Sección 1.2 dan fórmulas alternativas para las raíces x_1 y x_2 de $ax^2 + bx + c = 0$. Construya un algoritmo con entrada a, b, c y salida x_1, x_2 que calcule las raíces x_1 y x_2 (que puede ser igual o ser conjugados complejos) usando la mejor fórmula para cada raíz.

$$x_1 = \frac{-2c}{b + \sqrt{b^2 - 4ac}} \tag{1.2}$$

$$x_1 = \frac{-2c}{b - \sqrt{b^2 - 4ac}} \tag{1.3}$$

Richard L. Burden, J. Douglas Faires, *Numerical Analysis*, NINTH EDITION

Solución - Problema 8

SOLUTION: The following algorithm uses the most effective formula for computing the roots of a quadratic equation.

INPUT A, B, C.
OUTPUT x_1, x_2.

Step 1 If $A = 0$ then
if $B = 0$ then OUTPUT ('NO SOLUTIONS');
STOP.
else set $x_1 = -C/B$;
OUTPUT ('ONE SOLUTION',x_1);
STOP.

Step 2 Set $D = B^2 - 4AC$.

Step 3 If $D = 0$ then set $x_1 = -B/(2A)$;
OUTPUT ('MULTIPLE ROOTS', x_1);
STOP.

Solución - Problema 8 (cont.)

Step 4 If $D < 0$ then set

$$b = \sqrt{-D}/(2A);$$
$$a = -B/(2A);$$

OUTPUT ('COMPLEX CONJUGATE ROOTS');

$$x_1 = a + bi;$$
$$x_2 = a - bi;$$

OUTPUT (x_1, x_2);

STOP.

Step 5 If $B \geq 0$ then set

$$d = B + \sqrt{D};$$
$$x_1 = -2C/d;$$
$$x_2 = -d/(2A)$$

else set

$$d = -B + \sqrt{D};$$
$$x_1 = d/(2A);$$
$$x_2 = 2C/d.$$

Step 6 OUTPUT (x_1, x_2);

STOP.

Problema 10

Suppose that as x approaches zero,
$F_1(x) = L_1 + O(x^{\alpha})$ and $F_2(x) = L_2 + O(x^{\beta})$.
Let c_1 and c_2 be nonzero constants, and define
$F(x) = c_1 F_1(x) + c_2 F_2(x)$ and $G(x) = F_1(c_1 x) + F_2(c_2 x)$.
Show that if $\gamma = \text{minimum}\,\{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1 L_1 + c_2 L_2 + O(x^{\gamma})$
b. $G(x) = L_1 + L_2 + O(x^{\gamma})$

Solución - Problema 10

Suppose for sufficiently small $|x|$ we have positive constants k_1 and k_2 independent of x, for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

a. We have

$$\begin{aligned}
|F(x) - c_1L_1 - c_2L_2| =& |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\
\leq& |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \\
\leq& cK\left(|x|^\alpha + |x|^\beta\right) \\
\leq& cK|x|^\gamma\left(1 + |x|^{\delta-\gamma}\right) \leq K|x|^\gamma,
\end{aligned}$$

for sufficiently small $|x|$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

b. We have

$$\begin{aligned}
|G(x) - L_1 - L_2| =& |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\
\leq& K_1|c_1x|^\alpha + K_2|c_2x|^\beta \\
\leq& Kc^\delta\left(|x|^\alpha + |x|^\beta\right) \\
\leq& Kc^\delta|x|^\gamma\left(1 + |x|^{\delta-\gamma}\right) \leq K''|x|^\gamma,
\end{aligned}$$

for sufficiently small $|x|$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

Problema 12

Dada la función $f : \mathbb{R} \to \mathbb{R}$, definida como $f(x) = \sqrt{1+x} - \sqrt{x}$ para $|x| \gg 1$.
Determine el número de condicionamiento de f.

Solución - Problema 12

Se define el número de condición o condicionamiento de la función f como
$\kappa\{f(x)\} = \left|\dfrac{xf'(x)}{f(x)}\right|$.

La derivada de la función f, es $f'(x) = \dfrac{1}{2\sqrt{1+x}} - \dfrac{1}{2\sqrt{x}}$

Luego

$$\kappa\{f(x)\} = \left|\left(\frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{x}}\right)\frac{x}{\sqrt{1+x}-\sqrt{x}}\right| = \left|\frac{1}{2}\sqrt{\frac{x}{1+x}}\right|$$

Como se observa, el número de condicionamiento para $x \gg 1$ es $\kappa\{f(x)\} \approx \dfrac{1}{2}$