METCS 248 HW#7

1) Fill in the following table so that ({*x*, *y*, *z*}, *) forms a group.

*	*x*	*y*	*z*
x	x	y	z
y	y	z	x
z	z	x	y

Identify the identity element, and the inverses of x, y, and z.

Identity element = x
Inverse of x = x
y = z
z = y

2) Does [{0000, 0010, 1101, 1111}, $+_2$] form a subgroup of [B^4 , $+_2$] ? If yes, find all the cosets. ($+_2$: represents addition modulo 2, $B^4 = B \times B \times B \times B$ where $\times$ is the Cartesian product, and B = {0, 1}).

$H = [\{0000, 0010, 1101, 1111\}, +_2]$
$G = [B^4, +_2]$

	0000	0010	1101	1111
0000	0000	0010	1101	1111
0010	0010	0000	1111	1101
1101	1101	1111	0000	0010
1111	1111	1101	0010	0000

H is a subgroup of G.

$\frac{|G|}{|H|} = 4$ There exists 4 cosets.

$0000 +_2 H = \{0000, 0010, 1101, 1111\}$
$0001 +_2 H = \{0001, 0011, 1100, 1110\}$
$0100 +_2 H = \{0100, 0110, 1001, 1011\}$
$0101 +_2 H = \{$ ~~0100,~~ 0101, 0111, 1000, 1010$\}$

3) Check if the following is a homomorphism or an isomorphism: f: $[B^3, +_2] \rightarrow [Z_4, +_4]$
And $f(x) = w(x)$, *where* $w(x)$ is the weight of *x* i.e. the number of ones (1's) in the binary number x. (e.g. if *x* = 101, then *w(x)* = 2), Z_4 = {0, 1, 2, 3}, $+_4$: addition modulo 4. (1 $+_4$ 2 = 3, 3 $+_4$ 3 = 2 ...etc.)

Cardinality of B^3 is 8 and Z_4 is 4.
Therefore, no possibility of isomorphism.
Verification of homomorphism.
1) $f(e_1) = e_2$
$f(000) = w(000) = 0$
2) $f(x * y) = f(x) \# f(y)$
~~f(000~~

$f(001 +_2 011) = f(010) = 1$
$f(001) +_4 f(011) = 1 +_4 2 = 3$
$f(001 +_2 011) \neq f(001) +_4 f(011)$
Therefore, the given function is not homomorphism.

4) Let G be the set of all-nonzero real numbers and let $a * b = \frac{ab}{2}$. Show that (G,*) is an abelian group.

1) Structure?
Yes, closure and unique.

2) Commutative?
Yes. $a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$

3) Associative?
Yes. $(a*b)*c = \frac{\frac{ab}{2} \cdot c}{2} = \frac{a \cdot \frac{bc}{2}}{2}$
$= a * (b * c)$

4) Identity element?
$a * e = a$
$\frac{ae}{2} = a \rightarrow e = 2$

5) Inverse?
$a * a^{-1} = e$
$\frac{a \cdot a^{-1}}{2} = e \rightarrow a^{-1} = \frac{4}{a}$

6) Abelian Group?
Yes, since it is a commutative group.

5) Let $S = \{a, b, c\}$ and $T = \{x, y, z\}$, and let $[S, *]$ and $[T, \#]$ be defined as:

*	*a*	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*c*	*a*
c	*c*	*a*	*b*

#	*x*	*y*	*z*
x	*z*	*x*	*y*
y	*x*	*y*	*z*
z	*y*	*z*	*x*

Can you find an isomorphism between [S,*] and [T, #]?

$e_1 = a \quad e_2 = z$

$f_1: a \to z$
$\quad b \to x$
$\quad c \to z$

$f_2: a \to z$
$\quad b \to z$
$\quad c \to x$

Isomorphism? Yes.
Surjective? Yes. Every element in T has a preimage.
Injective? Yes. No elements in S share images.
Bijective? Yes.

6) Check if the following mappings form a homomorphism or an isomorphism from [Z, +] to [Z, +].

a) $f(x) = 5x$

Homomorphism? Yes.
$f(x * z) = f(x) \# f(z)$
$f(x * z) = f(x + z) = 5(x + z)$
$f(x) \# f(z) = 5x + 5z = 5(x + z)$

Isomorphism? No.
1 in the codomain has no preimage.
Hence, it's not surjective nor bijective.

b) $g(x) = x + 1$

Homomorphism? No.
$f(x * y) \neq f(x) \# f(z)$
$f(x * z) = f(x + z) = \underline{x + z + 1}$
$f(x) \# f(z) = \underline{x + 1 + z + 1}$

Isomorphism? Yes.
Surjective? Yes. Every element in the codomain has preimage.
Injective? Yes. No elements in the domain share the same image.
Bijective? Yes.

7) Can you find an isomorphism from [5Z, +] to [12Z, +]?

$f(x) = \frac{12}{5}x$

Isomorphism? Yes.
Surjective? Yes
Injective? Yes
Bijective? Yes.