METCS248 - HW#3

1. Test the following binary relations on the given sets S for reflexivity, symmetry, anti-symmetry, and transitivity. Check whether the binary relations are equivalence, and/or partial/total ordering relations or neither, describe the equivalence classes if applicable.

a) $S = \{0, 1, 2, 3, 4\}$, $R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (2, 4), (0, 1), (1, 2), (4, 3)\}$

Ref : Yes. Trival.
Sym : No. $(2,4) \in R_1 \rightarrow (4,2) \in R_1$ is false.
Tran : No. $(0,1) \in R_1$ & $(1,2) \in R_1 \rightarrow (0,2) \in R_1$ is false.
Anti : Yes. By inspection.
Eq : No. It's $R, \overline{S}, \overline{T}$.
Ord : No. It's $R, A, \overline{T}$.
Eq class : Not applicable.

b) $S = P(\{a, b, c, d, e, f, g, h, i\})$ [P: Power set], (A,B) belongs to R_2 if and only if $|A| = |B|$.

Ref : Yes. $\forall A \in S \ |A| = |A| \rightarrow (A,A) \in R_2$
Sym : Yes. $\forall A, B \in S \ |A| = |B| \rightarrow |B| = |A| \rightarrow (A,B) \in R_2 \rightarrow (B,A) \in R_2$
Tran : Yes. $\forall A, B, C \in S \ |A| = |B|$ and $|B| = |C| \rightarrow |A| = |C| \rightarrow (A,C) \in R_2$
Anti : No. $\forall A, B \in S \ |A| = |B|$ and $|B| = |A| \rightarrow (A,B) \in R_2$ & $(B,A) \in R_2 \rightarrow \exists A, B \ A \neq B$
Eq : Yes. It's R, S, T.
Ord : No. It's $R, \overline{A}, T$.
Eq class : $[A] = \{B \mid |A| = |B|\}$
10 equivalence classes from $|A| = 0$ to $|A| = 9$.

c) $S = N$ [N: set of positive integers]. (x, y) belongs to R_3 if and only if $x^2 - y^2$ is even. [x^2 : stands for x squared].

Ref : Yes. $x^2 - x^2 = 0$ is even $\rightarrow (x,x) \in R_3$
Sym : Yes. $(x,y) \in R_3 \rightarrow x^2 - y^2 =$ even $\rightarrow -(x^2 - y^2) =$ even $\rightarrow y^2 - x^2 =$ even $\rightarrow (y,x) \in R_3$
Tran : Yes. $(x,y) \in R_3$ & $(y,z) \in R_3 \rightarrow x^2 - y^2 =$ even & $y^2 - z^2 =$ even $\rightarrow (x^2 - y^2) + (y^2 - z^2) =$ even
$\rightarrow x^2 - z^2 =$ even $\rightarrow$ $\cancel{(x,y)}$ $(x,z) \in R_3$
Anti : No. $(2,4) \in R_3$ & $(4,2) \in R_3 \rightarrow 2^2 - 4^2 \neq 4^2 - 2^2$
Eq : Yes. It's R, S, T.
Ord : No. It's $R, \overline{A}, T$.
Eq classes : $x^2 - y^2 =$ even $\rightarrow x^2$ even y^2 even or x^2 odd y^2 odd $\rightarrow x$ even y even or $\overset{x}{\cancel{y}}$ odd y odd. 2 equivalence classes.

2. Let $S = \{0, 2, 4, 6\}$, and $T = \{1, 3, 5, 7\}$. Determine whether each of the following sets of ordered pairs is a function from S to T. If so, is it injective, surjective, and bijective?

a. $\{(0, 2), (2, 4), (4, 6), (6, 0)\}$

Not a function. (0,2) going from S to S.

b. $\{(6, 3), (2, 1), (0, 3), (4, 5)\}$

Function : Yes. By inspection.
Injective : No. $f(6) = f(0) = 3$.
Surjective : No. $f^{-1}(7)$ is not defined.
Bijective : No.

c. {(2, 3), (4, 7), (0, 1), (6, 5)}

Function: Yes. By inspection.
Injective: Yes. By inspection.
Surgective: Yes. By inspection.
Bijective: Yes.

3. For each case, think of set *S* and a binary relation *R* on *S* (different from any example in class or in any textbook) satisfying the given conditions.

a. *R* is reflexive and symmetric but not transitive

S = {あ、い、う}
R = {(あ, あ)、(い, い)、(う, う)、(あ, い)、(い, あ)、(い, う)、(う, い)}

b. *R* is reflexive and transitive but not symmetric.

S = {ア, イ, ウ}
R = {(ア, ア)、(イ, イ)、(ウ, ウ)、(ア, イ)、(イ, ウ)、(ア, ウ)}

c. *R* is reflexive but not symmetric and not transitive.

S = {壱, 弐, 参}
R = {(壱, 壱)、(弐, 弐)、(参, 参)、(壱, 弐)、(弐, 参)}

4. The following two functions map form R to R. Give an equation describing the composition of function $g \text{ o } f$ *and* $f \text{ o } g$.

a. $f(x) = 6x^3$, $g(x) = 2x$

$$g \circ f = g[f(x)] = 2 \cdot 6x^3 = 12x^3$$

$$f \circ g = f[g(x)] = 6 \cdot (2x)^3 = 48x^3$$

b. $f(x) = \frac{(x-1)}{2}$, $g(x) = \sqrt{(x+5)}$

$$g \circ f = g[f(x)] = \sqrt{(f(x)+5)} = \sqrt{\left(\frac{x-1}{2}\right)+5} = \sqrt{\frac{x+9}{2}}$$

$$f \circ g = f[g(x)] \cancel{= \frac{g(x-1)}{2}} = \frac{g(x)-1}{2} = \frac{\sqrt{x+5}-1}{2}$$

c. Find the inverse, if it exists, of the functions $f(x) = \frac{x}{2} - 7$ and $g(x) = x^2$.

$$f(x) = \frac{x}{2} - 7$$
$$f(x) + 7 = \frac{x}{2}$$
$$2(f(x)+7) = x$$
$$f^{-1}(x) = 2(x+7)$$

$$g(x) = x^2$$
$$\pm\sqrt{g(x)} = x$$
$$g^{-1}(x) = \pm\sqrt{x}$$