MET CS662 - Assignment #1

1. Show that

$$\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$$

2. Prove that $\forall n \ge 4$ the inequality $2^n < n!$ Holds.

3. Show that $2 - \sqrt{2}$ is irrational.

4. Prove that the set of all prime number is infinite.

---

1. BASE $\frac{1}{1^2} \le 1 - \frac{1}{1}$

$1 \le 1$ ✓

ASSUME $P(n)$

PROVE $P(n) \rightarrow P(n+1)$

$$\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

$$\rightarrow \sum_{i=1}^{n+1} \frac{1}{i^2} \le 2 - \frac{1}{n} + \frac{1}{n(n+1)}$$

$$\because \frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}$$

$$\rightarrow \sum_{i=1}^{n+1} \frac{1}{i^2} \le 2 - \left(\frac{1}{n} - \frac{1}{n(n+1)}\right)$$

$$\le 2 - \left(\frac{n+1-1}{n(n+1)}\right)$$

$$\le 2 - \frac{1}{n+1}$$

2. BASE $2^4 < 4!$

$16 < 24$ ✓

ASSUME $P(n)$

PROVE $P(n) \rightarrow P(n+1)$

$$2^n \cdot 2 < n! \cdot 2$$

$$< n! \cdot (n+1)$$

$$2^{n+1} < (n+1)!$$

3. If $n = 2-\sqrt{2}$, ~~is irrational,~~
then n is not a rational #.

Assume n is a rational #.

$n = 2-\sqrt{2} = \frac{a}{b}$

$\rightarrow \sqrt{2} = 2 - \frac{a}{b}$

$\rightarrow \sqrt{2}$ is rational since subtraction of rational #'s is rational.

$\rightarrow \sqrt{2} = \frac{a'}{b'}$

$\rightarrow 2 = \frac{a'^2}{b'^2}$

$\rightarrow a'^2 = 2b'^2 \rightarrow a'^2$ is even $\rightarrow a'$ is even

$\rightarrow a' = 2m$

$\rightarrow a'^2 = 4m^2 = 2b'^2$

$\rightarrow b'^2 = \frac{4m^2}{2} = 2m^2$

$\rightarrow b'^2$ is even $\rightarrow b'$ is even

$\rightarrow b = 2n$

$\rightarrow \sqrt{2} = \frac{2m}{2n} \rightarrow$ Reducable

Contradiction!

4. Assume the set of all prime number is finite.
Product of all the known prime numbers be denoted by P such that

$P = p_1 p_2 \cdots p_n$

Let Q be a number such that

$Q = P+1$

Clearly, Q is greater than any of prime numbers.
so Q can't be prime.
Then, it must be divisible by any of prime #'s.
But when we divide it with any of prime numbers, we get 1 as remainder.
Thus, the assumption that there are only finite prime number is wrong.