Finite-Dimensional Variational Inequalities and Complementarity Problems

復習

1.1.1 Definition. Given a subset K of the Euclidean n-dimensional space $\mathbb{R}^n$ and a mapping $F : K \to \mathbb{R}^n$, the *variational inequality*, denoted VI (K, F), is to find a vector $x \in K$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall\, y \in K. \tag{1.1.1}$$

The set of solutions to this problem is denoted SOL(K, F). □

$\mathrm{SOL}(K, F) = \{x \mid (y - x)^\top F(x) \geq 0 \ \forall y \in K\}$

1.1.2 Definition. Given a cone K and a mapping $F : K \to \mathbb{R}^n$, the *complementarity problem*, denoted CP (K, F), is to find a vector $x \in \mathbb{R}^n$ satisfying the following conditions:

$$K \ni x \perp F(x) \in K^*,$$

where the notation $\perp$ means "perpendicular" and K^* is the *dual cone* of K defined as:

$$K^* \equiv \{\, d \in \mathbb{R}^n : v^T d \geq 0 \ \forall\, v \in K \,\};$$

that is, K^* consists of all vectors that make a non-obtuse angle with every vector in K. □

1.1.1 Affine problems

The CP is a special case of the VI (K, F) where the set K is a cone. In what follows, we introduce several other special cases of the VI (K, F) where either K or F has some other interesting structures. To begin, let F be the affine function given by:

$$F(x) \equiv q + Mx, \qquad \forall\, x \in \mathbb{R}^n, \tag{1.1.6}$$

for some vector $q \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{n\times n}$; in this case, we write VI (K, q, M) to mean VI (K, F). The solution set of the VI (K, q, M) is denoted $\mathrm{SOL}(K, q, M)$. If in addition K is a polyhedral set, we attach the adjective "affine" and use the notation AVI (K, q, M) to describe the all affine VI. Finally, if K is a polyhedral set but F is not necessarily affine, we use the adjective "linearly constrained" to describe the VI (K, F). Unlike the AVI where both the defining set and function are affinely structured, the VI (K, q, M) and the linearly constrained VI have only one affine member in the pair (K, F)..

key recession properties of a set in $\mathbb{R}^n$. By definition, a *recession direction* of a set X (not necessarily convex) is a vector d such that for some vector $x \in X$, the ray $\{x + \tau\, d : \tau \geq 0\}$ is contained in X. The set of all recession directions of X is denoted X_∞ and called the *recession cone* of X. Clearly, if X_∞ contains a nonzero vector, then X is unbounded. If X is a closed and convex set, then $x + \tau\, d \in X$ for all $x \in X$, all $d \in X_\infty$, and $\tau \geq 0$; moreover, in this case, X is bounded if and only if $X_\infty = \{0\}$. If X is a closed cone, then $X = X_\infty$. If X is a polyhedral set, say,

$$X = \{\, x \in \mathbb{R}^n \,:\, Ax \leq b \,\},$$

for some matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$, then

$$X_\infty = \{\, d \in \mathbb{R}^n \,:\, Ad \leq 0 \,\}.$$

For clarity of notation, we caution the reader that we write $\mathrm{int}(X_\infty)^*$ to mean $\mathrm{int}((X_\infty)^*)$; i.e., the interior of $(X_\infty)^*$.

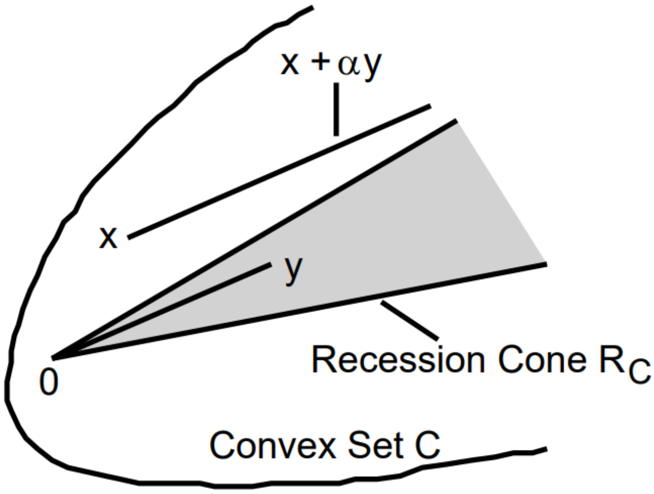


Figure 1.2.8. Illustration of the recession cone R_C of a convex set C. A direction of recession y has the property that $x + \alpha y \in C$ for all $x \in C$ and $\alpha \geq 0$.

(出典: https://www.researchgate.net/publication/2486516_Convexity_Duality_and_Lagrange_Multipliers)

K_∞は、Kが無限に伸びている方向の集合。

As a tool for studying the VI (K, q, M), we introduce several basic point sets associated with this problem. These sets have their origin from LCP theory where they form the basis for the definitions of various matrix classes. The development in this section is largely motivated by this theory. The sets defined below are particularly relevant when we deal with the VI (K, q, M) with a fixed pair (K, M) but with q being arbitrary. Specifically,

given the pair (K, M), where K is an arbitrary subset of $\mathbb{R}^n$ and M is an arbitrary $n \times n$ matrix, we define

$$\begin{aligned} \mathcal{R}(K, M) &\equiv \{ q \in \mathbb{R}^n : \mathrm{SOL}(K, q, M) \neq \emptyset \} \\ \mathcal{D}(K, M) &\equiv (K_\infty)^* - MK \\ \mathcal{K}(K, M) &\equiv \mathrm{SOL}(K_\infty, 0, M). \end{aligned}$$

The first set $\mathcal{R}(K, M)$ consists of all vectors q for which the VI (K, q, M) has a solution. We call $\mathcal{R}(K, M)$ the *VI range* of the pair (K, M). It is easy to see that

$$-MK \subseteq \mathcal{R}(K, M). \tag{2.5.1}$$

Sec. 2.5.2

AVI range of the pair (K, M): $\mathcal{R}(K, M)$

$$\begin{aligned} \mathcal{R}(K, M) &= \{ q \in \mathbb{R}^n \mid \mathrm{SOL}(K, q, M) \neq \emptyset \} \\ &= \{ q \in \mathbb{R}^n \mid \exists x \in K \text{ s.t. } (y - x)^\top (q + Mx) \geq 0 \ \forall y \in K \} \end{aligned}$$

Thm. 2.5.15

2.5.15 Theorem. Let K be a polyhedron in $\mathbb{R}^n$ and M be an $n \times n$ matrix. The following two statements are valid.

(a) For every vector $q \in \mathbb{R}^n$, the set $\mathrm{SOL}(K, q, M)$ is the union of finitely many polyhedra in $\mathbb{R}^n$.

(b) The AVI range $\mathcal{R}(K, M)$ is the union of finitely many polyhedra in $\mathbb{R}^n$; thus it is closed.

(a): 本質はThm2.4.13を使うだけ (b): (a)を変形して、qの方の存在範囲に変えただけ

以下は、Thm. 2.4.13の復習

Proof. Let K be given by

$$K \equiv \{ x \in \mathbb{R}^n : Cx = d, \ Ax \leq b \},$$

for some given matrices $C \in \mathbb{R}^{\ell \times n}$ and $A \in \mathbb{R}^{m \times n}$ and vectors $d \in \mathbb{R}^{\ell}$ and $b \in \mathbb{R}^m$. By Proposition **1.2.1**, a vector x is a solution of the AVI (K, q, M) if and only if there exists a pair of multipliers $(\mu, \lambda) \in \mathbb{R}^{\ell+m}$ such that the triple (x, μ, λ) is a solution to the augmented MLCP (1.2.3) whose defining matrix $\mathbf{Q}$, given by (1.2.7), is positive semidefinite if M is so. For ease of reference, the latter MLCP is repeated below:

$$\begin{aligned} 0 &= q + Mx + C^T \mu + A^T \lambda \\ 0 &= d - Cx \\ 0 &\leq \lambda \perp b - Ax \geq 0. \end{aligned} \tag{2.4.6}$$

By the aforementioned observation, the solution set of the latter augmented MLCP (being a special monotone affine CP) is polyhedral. Since the solution set of the AVI (K, q, M) is the image of the latter polyhedral set under

the canonical projection:

$$\begin{pmatrix} x \\ \mu \\ \lambda \end{pmatrix} \in \mathbb{R}^{n+\ell+m} \mapsto x \in \mathbb{R}^n, \tag{2.4.7}$$

it follows that the solution set of the AVI (K, q, M) is also polyhedral. $\square$

Cor. 2.5.16

2.5.16 Corollary. The solution set of an AVI (K, q, M) is bounded if and only if there exists no nonzero solution ray.

Proof. It suffices to show that if there exists no nonzero solution ray, then $\mathrm{SOL}(K, q, M)$ is bounded. Indeed, if $\mathrm{SOL}(K, q, M)$ is unbounded, then one of its polyhedral pieces is unbounded. Being an unbounded polyhedral set, this piece must contain a ray emanating from a certain vector in the piece. The direction of the ray furnishes a nonzero solution ray of the AVI (K, q, M). $\square$

直感的にも明らかだが、

$$(y-x)^\top(q+Mx) \geq 0 \ \forall y \in K$$

であるとして、Prop. 2.5.4を用いれば、計算しても導かれる:

$$\begin{aligned}
&(y-x-\tau d)^\top(q+M(x+\tau d)) \\
&\geq (y-x)^\top(M\tau d) - \tau d^\top(q+M(x+\tau d)) \\
&\geq 0 - 0 - \tau^2 d^\top M d \\
&= 0 \quad (\because d \perp Md)
\end{aligned}$$

2.5.4 Proposition. A vector $d \in \mathbb{R}^n$ is a solution ray of the VI (K, q, M) if and only if there exists $x \in \mathrm{SOL}(K, q, M)$ such that

(a) $d \in \mathcal{K}(K, M)$,

(b) $d^T(q + Mx) = 0$, and

(c) $x^T M d \leq y^T M d$ for all $y \in K$.

Thm. 2.5.17

これまでに見てきた結果の精緻化にすぎない。証明もかなり基礎的。飛ばします。

大事そうな結論としては、$\mathrm{SOL}(K, q, M)$の凸性の、同値な特徴づけが出来て、計算するときに便利ということ? (何が嬉しいのかまでは詳しく分からず)

Sec. 2.5.3

(2.3.4)の復習

In general, for a convex set K, we have

$$F(x) \in (K_\infty)^*, \quad \forall x \in \mathrm{SOL}(K,F);$$

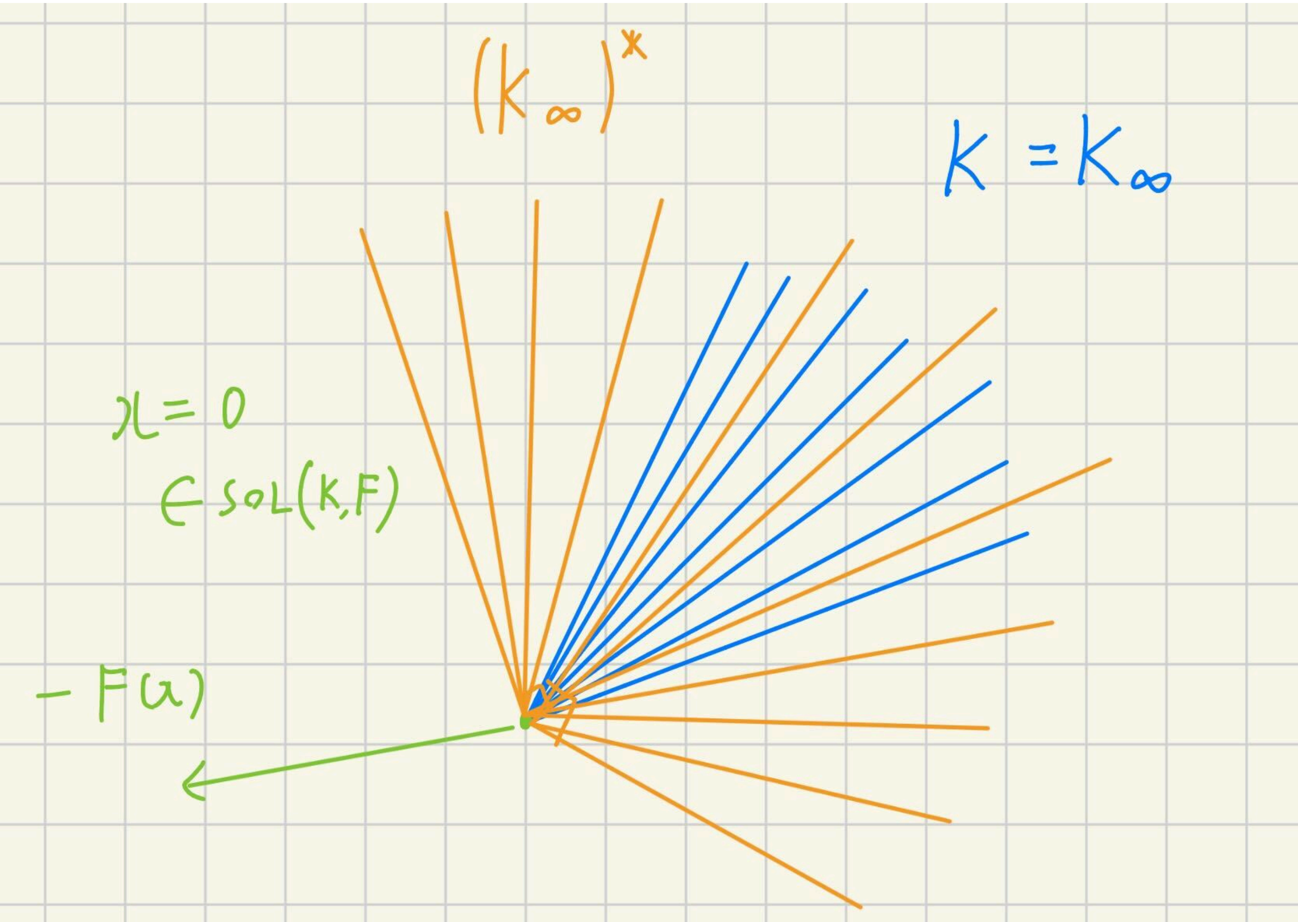
that is, the inclusion

$$F(\mathrm{SOL}(K,F)) \subseteq (K_\infty)^* \tag{2.3.4}$$

$$x \in \mathrm{SOL}(K,F) \iff (y-x)^\top F(x) \geq 0 \ \forall y \in K$$

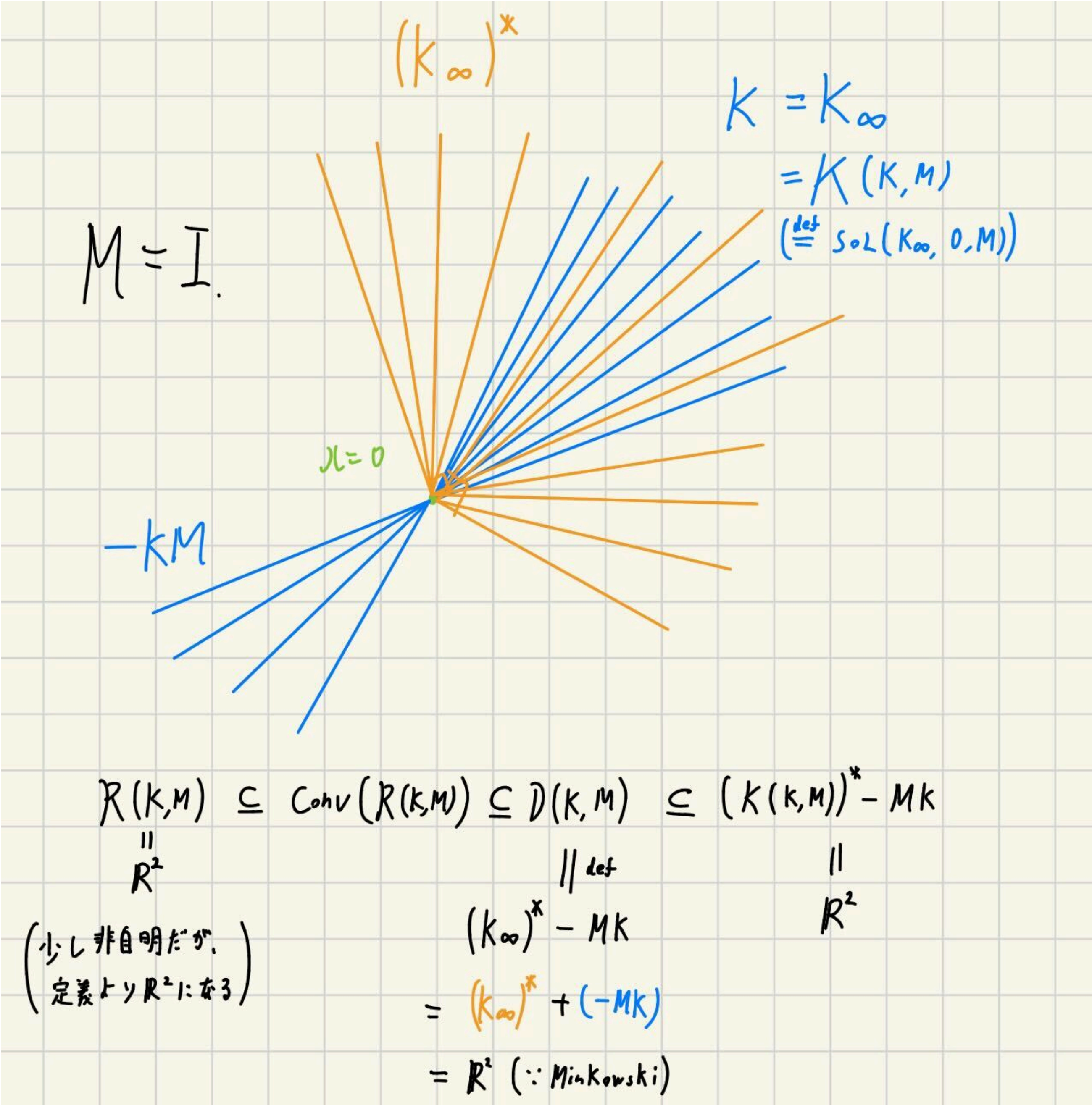
$$d \in (K_\infty)^* \iff v^\top d \geq 0 \ \forall v \in K_\infty$$

任意のK_∞の元 v(recession direction)に対し、あるyが存在して、$y-x=v$ ($t \geq 0$)となるのは定義自体。



(2.5.12)の説明

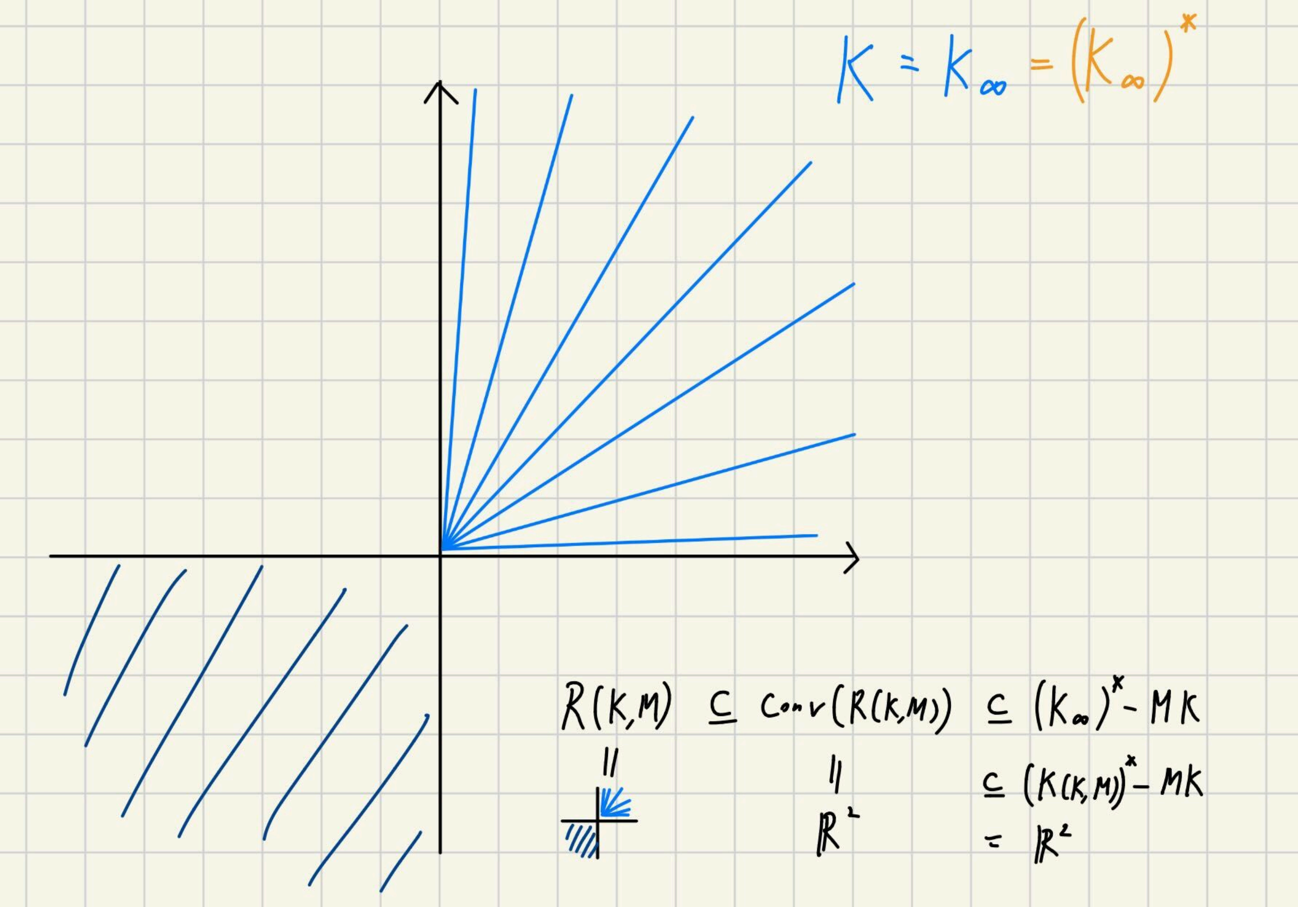
$$\begin{aligned}
&\mathcal{R}(K,M) = \{q \in \mathbb{R}^n \mid \mathrm{SOL}(K,q,M) \neq \emptyset\} \\
&\subseteq \mathrm{conv}(\mathcal{R}(K,M)) \\
&\subseteq \mathcal{D}(K,M) = (K_\infty)^* - MK \\
&\subseteq (\mathcal{K}(K,M))^* - MK = \mathrm{SOL}(K_\infty,0,M)^* - MK
\end{aligned}$$



Prop. 2.5.18

(2.5.1): $-MK \subseteq \mathcal{R}(K, M)$

(2.5.12)の等号がすべて成立するのは、(大まかにいえば)$\mathcal{R}(K, M)$が凸のときに限る。



counter example: $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $K = \mathbb{R}^2_+$

右から2番目において、$\mathcal{K}(K, M)$でなく、$(\mathcal{K}(K, M))^*$であることを除いて、証明は明らか。

Prop. 2.5.18の意味

Def 1.1.2を思い出すと、

- solvable set: q such that $\mathrm{SOL}(K, q, M) \neq \emptyset$
- feasible set: q such that $x \in K$ and $Mx + q \in K^*$

が一致している。節の名前にもある通り、"Solvability in terms of feasibility"である。

この節の残り

この節の残りは、Kがpolyhedral setのときに、同様の主張が成り立つことを示している。既に主張の大枠はつかめているが、"not as easy as the case of a cone K; ... M is copositive plus on the recession cone K_∞"とあるとおり、証明はかなり面倒、かつ、昔に読んだ道具を使い、更なる復習が必要。ここでは飛ばす。

(Lemma 2.5.19などは、単に定義通りで良い。Thm. 2.5.20は、先ほども扱ったpolyhedral setの具体的な表式を扱いながら式変形したり、いろいろと細かい部分のcheckをしたり、大変そうな感じ。詳細までは理解せず。)

(ここまで30分想定)

Sec 2.6

- Sec. 2.4ではpseudo monotone
- Sec. 2.5ではAffine を、それぞれFに対して課してきた。

ここでは、その両方の仮定を外す。

Thm. 2.6.1

いくつかの命題の基となる、一種の抽象的な命題である。

natural mapとnormal mapの復習

1.5.8 Proposition. Let $K \subseteq \mathbb{R}^n$ be closed convex and $F : K \to \mathbb{R}^n$ be arbitrary. It holds that:

$$[\, x \in \mathrm{SOL}(K, F) \,] \ \Leftrightarrow \ [\, \mathbf{F}_K^{\mathrm{nat}}(x) = 0 \,],$$

where

$$\mathbf{F}_K^{\mathrm{nat}}(v) \equiv v - \Pi_K(v - F(v)).$$

Proof. The defining inequality for the VI (K, F) is:

$$(\, y - x \,)^T F(x) \geq 0, \quad \forall\, y \in K,$$

which can be rewritten as:

$$(\, y - x \,)^T (\, x - (x - F(x))\,) \geq 0, \quad \forall\, y \in K.$$

By (1.5.8), the last inequality is equivalent to:

$$x = \Pi_K(x - F(x)),$$

or equivalently $\mathbf{F}_K^{\mathrm{nat}}(x) = 0$. □

From Proposition **1.5.8**, we can derive an alternative nonsmooth equation formulation of the VI.

1.5.9 Proposition. Let $K \subseteq \mathbb{R}^n$ be closed convex and $F : K \to \mathbb{R}^n$ be arbitrary. A vector x belongs to $\mathrm{SOL}(K, F)$ if and only if there exists a vector z such that $x = \Pi_K(z)$ and $\mathbf{F}_K^{\mathrm{nor}}(z) = 0$, where

$$\mathbf{F}_K^{\mathrm{nor}}(v) \equiv F(\Pi_K(v)) + v - \Pi_K(v).$$

Thm. 2.2.1の復習

2.2.1 Theorem. Let $K \subseteq \mathbb{R}^n$ be closed convex and $F : \mathcal{D} \supseteq K \to \mathbb{R}^n$ be continuous on the open set $\mathcal{D}$. Let $\mathbf{F}_K^{\rm nat}$ and $\mathbf{F}_K^{\rm nor}$ denote respectively the natural map and normal map of the pair (K, F). The following two statements hold:

(a) if there exists a bounded open set $\mathcal{U}$ satisfying $\mathrm{cl}\,\mathcal{U} \subseteq \mathcal{D}$ and such that $\deg(\mathbf{F}_K^{\rm nat}, \mathcal{U})$ is well defined and nonzero, then the VI (K, F) has a solution in $\mathcal{U}$;

(b) if there exists a bounded open set $\mathcal{U}'$ such that $\deg(\mathbf{F}_K^{\rm nor}, \mathcal{U}')$ is well defined and nonzero, then the VI (K, F) has a solution x such that $x - F(x) \in \mathcal{U}'$.

R0 pairの復習

Clearly, if M is strictly copositive on K_∞, then $\mathcal{K}(K, M) = \{0\}$. In general, we say that the pair (K, M) is an *R_0 pair* if $\mathcal{K}(K, M) = \{0\}$.

証明

以上を組み合わせて使うだけ

Cor. 2.6.2

解の存在性に対する、異なる特徴づけ。自明。

Cor. 2.6.3

function. In general, a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is *co-coercive* on a subset S of $\mathbb{R}^n$ if there exists a constant $c > 0$ such that

$$(F(x) - F(y))^T (x - y) \geq c \| F(x) - F(y) \|_2^2, \quad \forall\, x, y \in S.$$

Since the right-hand side is clearly nonnegative, we deduce that a co-coercive function must be *monotone*; that is,

$$(F(x) - F(y))^T (x - y) \geq 0, \quad \forall\, x, y \in S.$$

co-coercive

証明は計算するだけ

Cor. 2.6.4

同様にほぼ計算するだけ

Prop. 2.6.5

殆どの主張はこれまでの結果。飛ばします。

Sec 2.7

We consider a discrete linear elastic, small displacement, planar contact problem under a standard Coulomb friction law.

標準クーロン摩擦則のもとにおける離散線形弾性、小変位、平面接触問題を考察する。

$$Mu + C_n^T p_n + C_t^T p_t \;=\; f^{\mathrm{ext}} \tag{2.7.1}$$

$$0 \;\le\; p_n \;\perp\; g - C_n u \;\ge\; 0 \tag{2.7.2}$$

$$C_t(\,u - u^{\mathrm{ref}}\,) \;=\; \lambda_t^+ - \lambda_t^- \tag{2.7.3}$$

$$\left.\begin{array}{lcl} 0 \;\le\; \lambda_t^+ & \perp & p_t - \mu\, p_n \;\le\; 0 \\ 0 \;\le\; \lambda_t^- & \perp & -p_t - \mu\, p_n \;\le\; 0 \end{array}\right\}. \tag{2.7.4}$$

これの具体的な可解性の議論(Prop. 2.7.1, Prop. 2.7.3, Prop. 2.7.4)をしている。

最適化と殆ど関係ないので、飛ばす。 (もしご興味のある方が多ければ、次回にてお願いします。)

"more importantly, this solution can be computed by the well-known Lemke almost comple- mentary pivotal method"とあり、具体的なアルゴリズムはLemke法を使うことが出来るようである。

https://qiita.com/zhidao/items/ccc388e432624184c009