Supplemental Material For 10/02 Book Reading Seminar

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Abstract

This is the supplemental material for the book reading seminar.
Please refer to the textbook and the whiteboard for the main content.
If necessary, please also refer to the repository.
The textbook is available at here.
I recommend you to download the book from
"Nonsmooth Convex Optimization Pages 139-240 Download chapter PDF ".

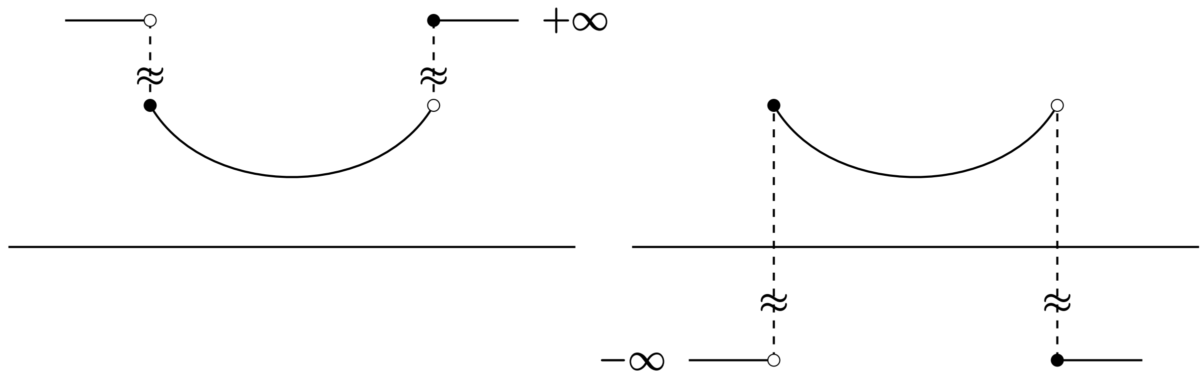
p.61 Def 2.1.1 / p.139-142 Preliminary

$$Q \text{ is convex} \iff \alpha x + (1-\alpha)y \in Q$$
$$(\forall x, y \in Q,\ \forall \alpha \in [0,1])$$

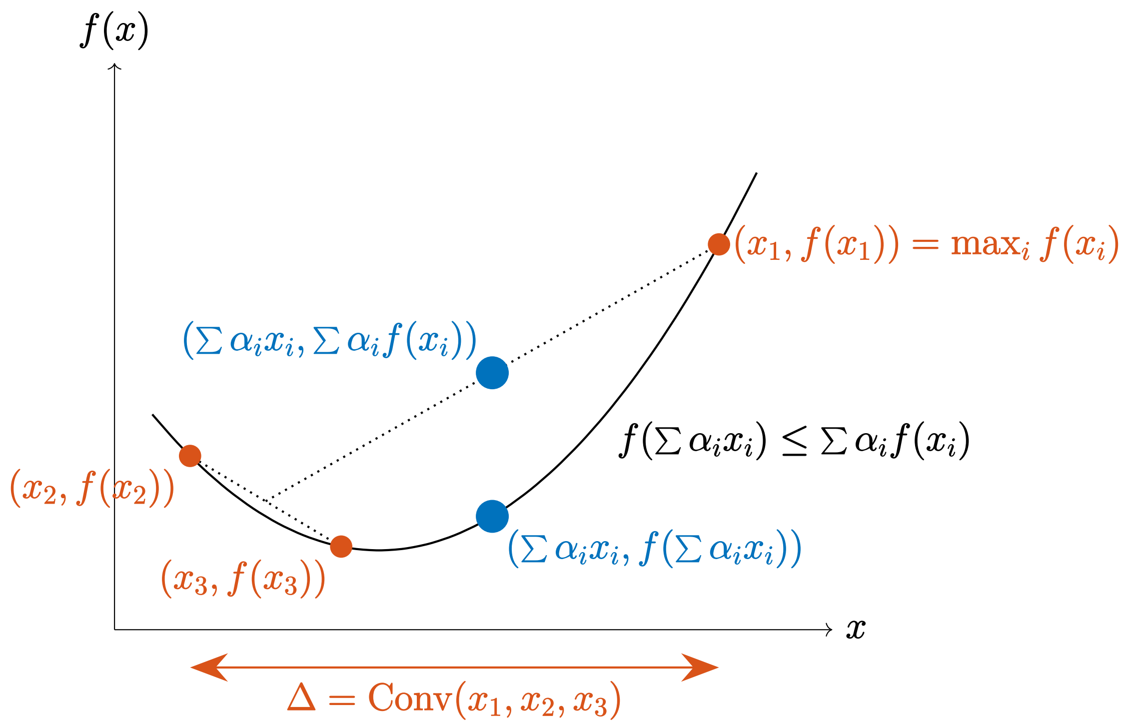
$$\mathrm{dom} f = \{x \in \mathbb{R}^n \mid |f(x)| < \infty\}$$

$$f \text{ is convex} \iff f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$
$$(\forall x, y \in \mathrm{dom} f : \text{convex},\ \forall \alpha \in [0,1])$$

$$f \text{ is concave} \iff -f \text{ is convex}$$



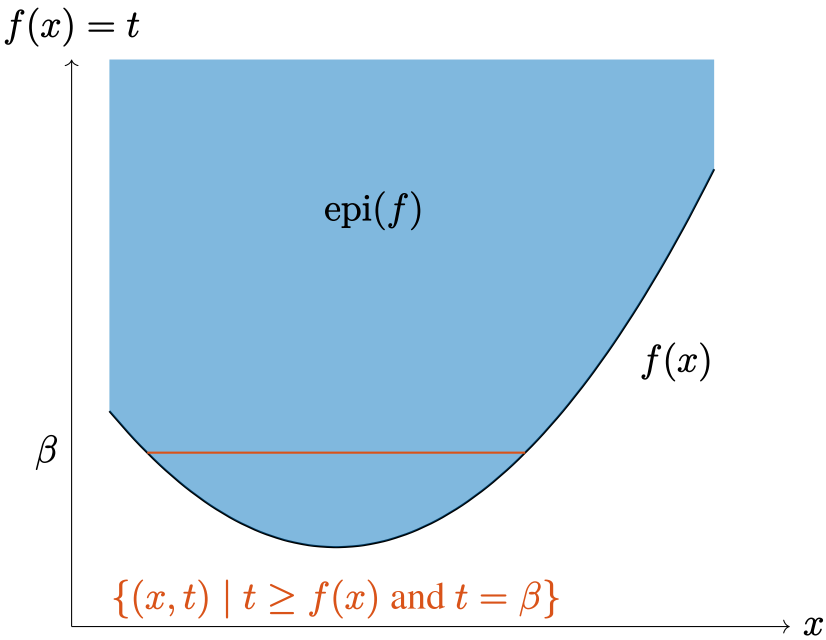
p.141 Lem 3.1.1 Jensen's Inequality, Col 3.1.1, 3.1.2, Thm 3.1.1



$\because$ induction on n.

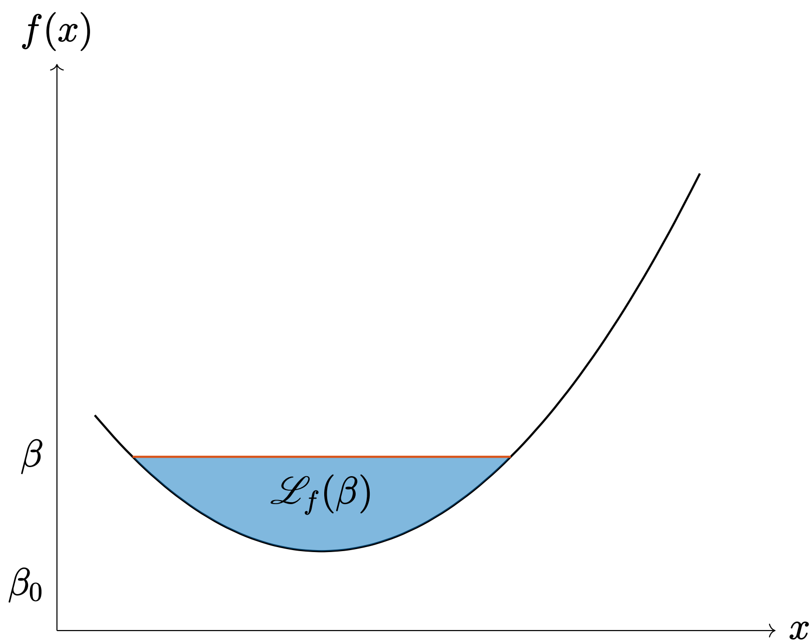
p.142 Thm 3.1.2

$$f \text{ is convex} \iff \text{epi}(f) \text{ is a convex set.}$$



p.143 Thm 3.1.3

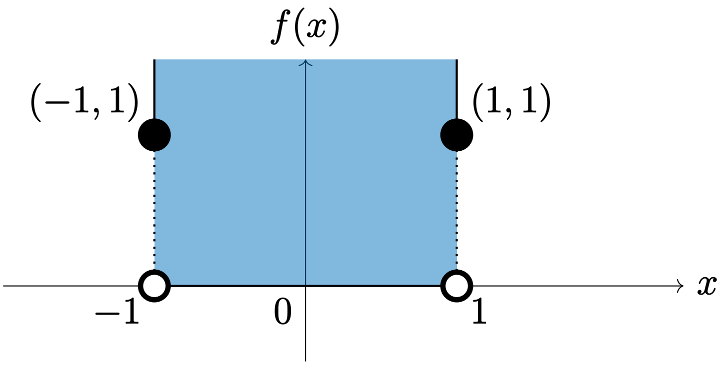
All level sets $\mathscr{L}_f(\beta)=\{x\in \mathrm{dom} f | f(x)\leq\beta\}$ are either convex or empty.



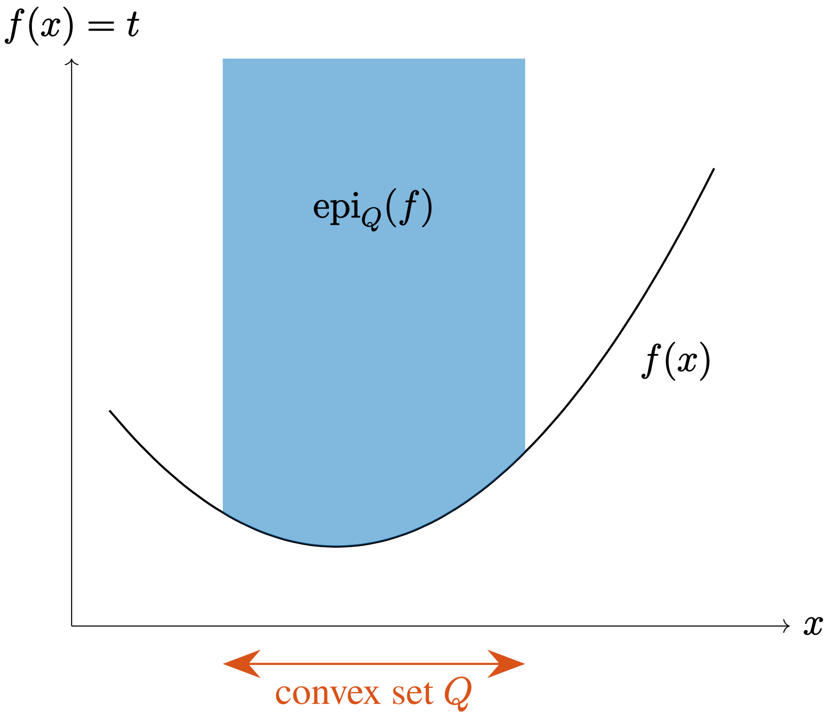
p.143 Def 3.1.2

1d version of Example 3.1.1(6):

$$f(x)=\begin{cases}0 & (-1\leq x\leq 1)\\ 1 & (x=-1 \text{ or } x=1)\\ \infty & (\text{otherwise})\end{cases}.$$



To exclude such a [1]pathological case, we define the constrained epigraph set $\mathrm{epi}_Q(f)$ and closed convex function.



p.144 Lem 3.1.2

subset of closed convex set.

Thm 2.2.8

Let $Q_1 \subseteq \mathbb{R}^n$ and $Q_2 \subseteq \mathbb{R}^n$ be closed convex sets, and $\mathscr{A}(\cdot)$ be a linear operator:

$$\mathscr{A}(x) = Ax + b : \mathbb{R}^n \to \mathbb{R}^m.$$

1. The intersection of two sets($m = n$), $Q_1 \cap Q_2 = \{x \in \mathbb{R}^n | x \in Q_1, x \in Q_2\}$, is convex and closed.

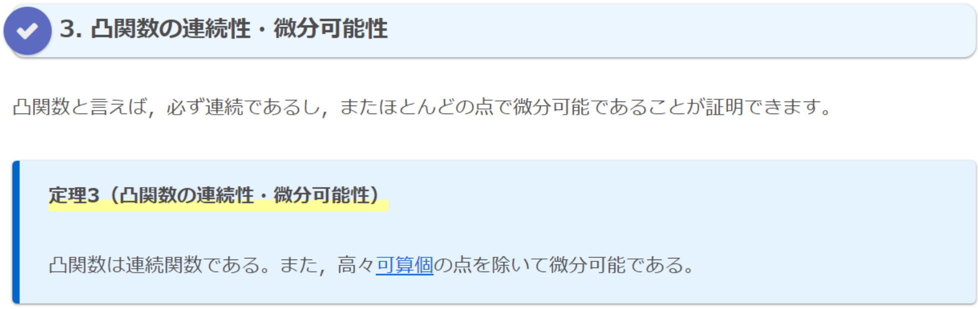
2. ...

[1]病的な

p.144 Thm 3.1.4

1

I think that Thm 3.1.4.1 is correct but quite confusing. Sometimes, the convex function is treated as a continuous function (example,Lem 3.1.4), but in this book, that is not the case.


3. 凸関数の連続性・微分可能性

凸関数と言えば，必ず連続であるし，またほとんどの点で微分可能であることが証明できます。

定理3（凸関数の連続性・微分可能性）

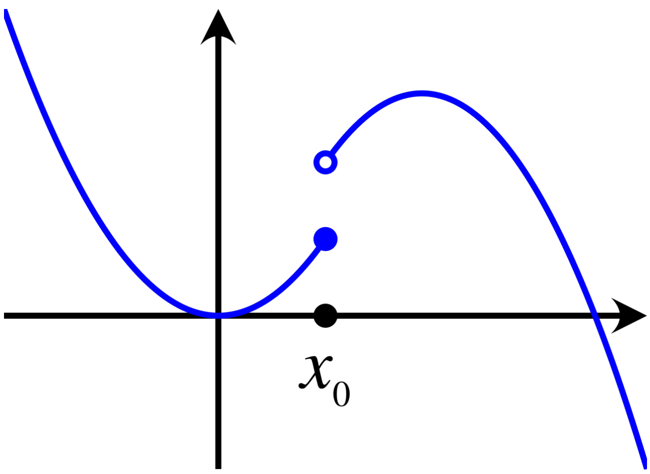
凸関数は連続関数である。また，高々可算個の点を除いて微分可能である。

This is because the definition is different. In this book, the following functions are convex. In particular, the function $f_1(x)$ is **not continuous** at $x = 0$ but **lower semi-continuous** as Thm 3.1.4.1 asserts.

$$f_1(x) = \begin{cases} 1 & (x \geq 0) \\ \infty & (x < 0) \end{cases} \quad \left(\text{e.g. } f\left(\frac{-2+1}{2}\right) = \infty \leq \frac{f(-2)+f(1)}{2} = \infty\right)$$

$$f_2(x) = \begin{cases} 1/x & (x > 0) \\ \infty & (x \leq 0) \end{cases} \quad \left(\text{e.g. } f\left(\frac{-2+1}{2}\right) = \infty \leq \frac{f(-2)+f(1)}{2} = \infty\right)$$

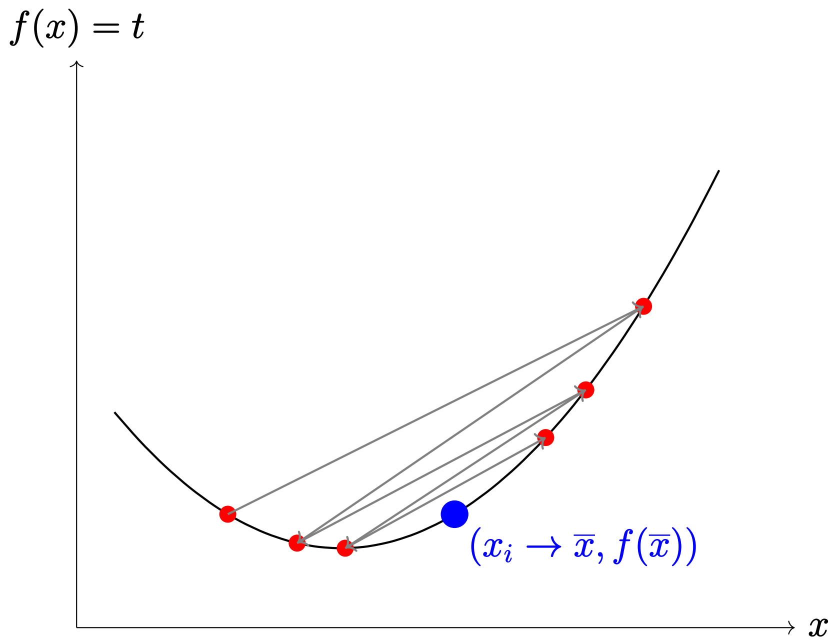
The lower semi-continuous by wikipedia:



We use the following facts to prove the theorem. See here for proof.

- If the terms in the sequence are real numbers, the limit superior and limit inferior always exist. (i.e. $\liminf_{k\to\infty} f(x_k) \in \mathbb{R} \cup \{\pm\infty\}$)

- If $\liminf_{k\to\infty} f(x_k) \to \beta$ where $\beta \in \mathbb{R}$, then there exists a subsequence $\{x_{k_j}\}$ such that $\lim_{j\to\infty} f(x_{k_j}) = \beta$.
- The assertion above implies that if there are no convergent subsequences, then $\liminf_{k\to\infty} f(x_k) \in \{\pm\infty\}$.



ここで，数列 $\{\sup_{k\geq n} a_k\}_n, \{\inf_{k\geq n} a_k\}_n$ はそれぞれ単調減少・単調増加ですから，**上極限・下極限は $\pm\infty$ も含めると必ず存在する**ことに注意しましょう(→上に有界な単調増加数列は収束することの証明)。極限とは違う性質ですね。

定理の主張〜有界な単調増加列の収束〜

定理（上に有界な単調増加数列は収束する）

実数の数列 $\{a_n\}$ は(広義)単調増加(すなわち $a_n \leq a_{n+1}$)とし，かつ上に有界(すなわち$a_n < K$) とする。
このとき，この数列は収束する。

なお，$\{-a_n\}$ を考えることで，下に有界な単調減少数列が収束することもわかります。

証明

上に有界なので，$\alpha = \sup\{a_n \mid n \geq 1\} < \infty$ と定める。

sup の定義より，任意の $\varepsilon > 0$ に対して，ある $N \geq 1$ が存在して，

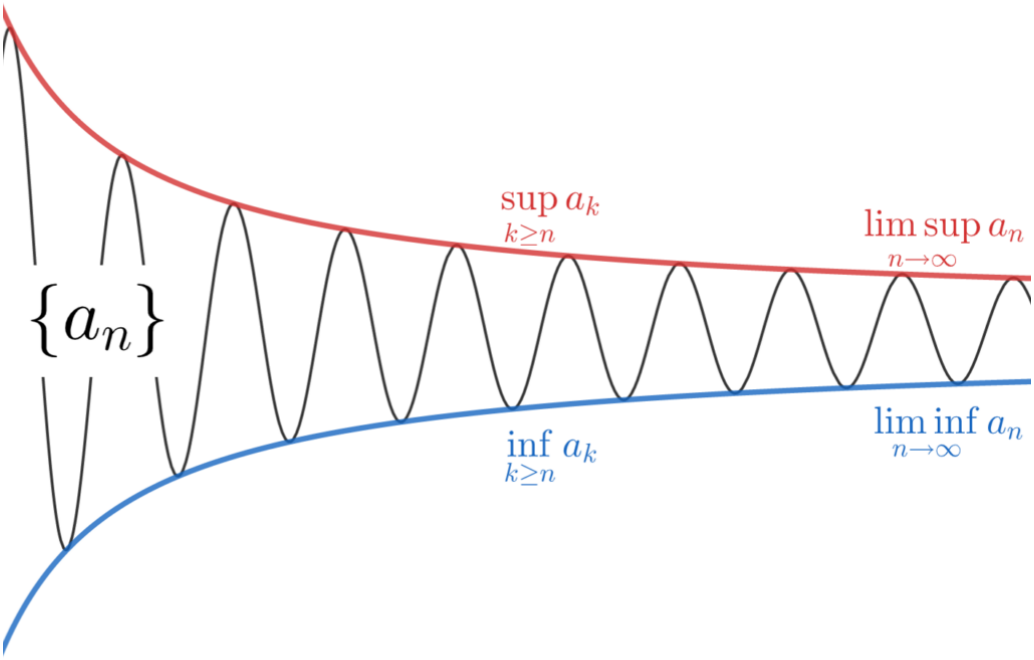
$$a_N > \alpha - \varepsilon$$

が成り立つ(→[上限,下限(sup,inf)の定義と最大,最小(max,min)との違い])。式変形して，$0 \leq \alpha - a_N < \varepsilon$ となる。
$\{a_n\}$ は単調増加であったから，

$$n \geq N \implies 0 \leq \alpha - a_n < \varepsilon$$

である。これは，$\lim_{n\to\infty} a_n = \alpha$ を意味する。

証明終



定理（上極限・下極限に収束する部分列の存在）

$\{a_n\}$ を実数列とし，$\limsup_{n\to\infty} a_n = \alpha, \liminf_{n\to\infty} a_n = \beta$ と定める。

このとき，ある部分列 $\{a_{\varphi_1(n)}\}, \{a_{\varphi_2(n)}\}$ が存在して，

$$\lim_{n\to\infty} a_{\varphi_1(n)} = \alpha, \quad \lim_{n\to\infty} a_{\varphi_2(n)} = \beta$$

とできる。

証明

$\varphi_1(1) = 1$ と定め $\varphi_1(n-1)$ まで定まったとして，$\varphi_1(n)$ を帰納的に定めよう。
極限の定義により，ある $N_n \geq 1$ が存在して，

$$N \geq N_n \implies \left|\sup_{k \geq N} a_k - \alpha\right| < \frac{1}{n}$$

とできる。とくに，$N'_n = \max\{N_n, \varphi(n-1)\}$ と定めると，

$$\left|\sup_{k \geq N'_n} a_k - \alpha\right| < \frac{1}{n}$$

である。

上限(sup)の定義により，ある $K_n \geq N'_n$ が存在して，

$$0 \leq \sup_{k \geq N'_n} a_k - a_{K_n} \leq \frac{1}{n}$$

とできる。$\varphi_1(n) = K_n$ と定めよう。すると，

$$\begin{aligned}
&|a_{\varphi_1(n)} - \alpha| \\
&= |a_{K_n} - \alpha| \\
&\leq \left|a_{K_n} - \sup_{k \geq N'_n} a_k\right| + \left|\sup_{k \geq N'_n} a_k - \alpha\right| \\
&\leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.
\end{aligned}$$

よって $\lim_{n \to \infty} a_{\varphi(n)} = \alpha$ が成立する。

証明終

2

whiteboard.

3

Please see thm 3.1.3.

4

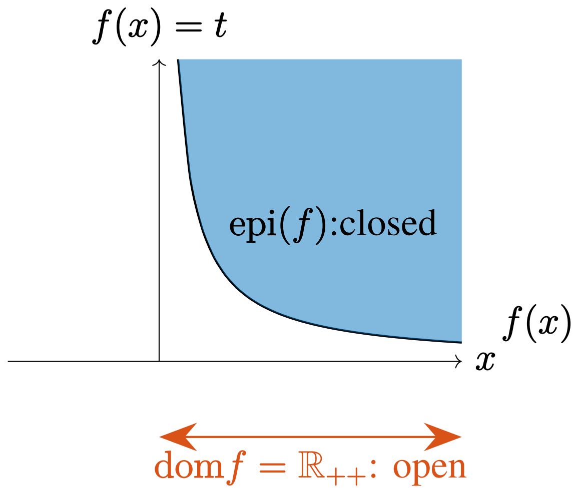
whiteboard.

5

whiteboard.

p.145 Example 3.1.1

4



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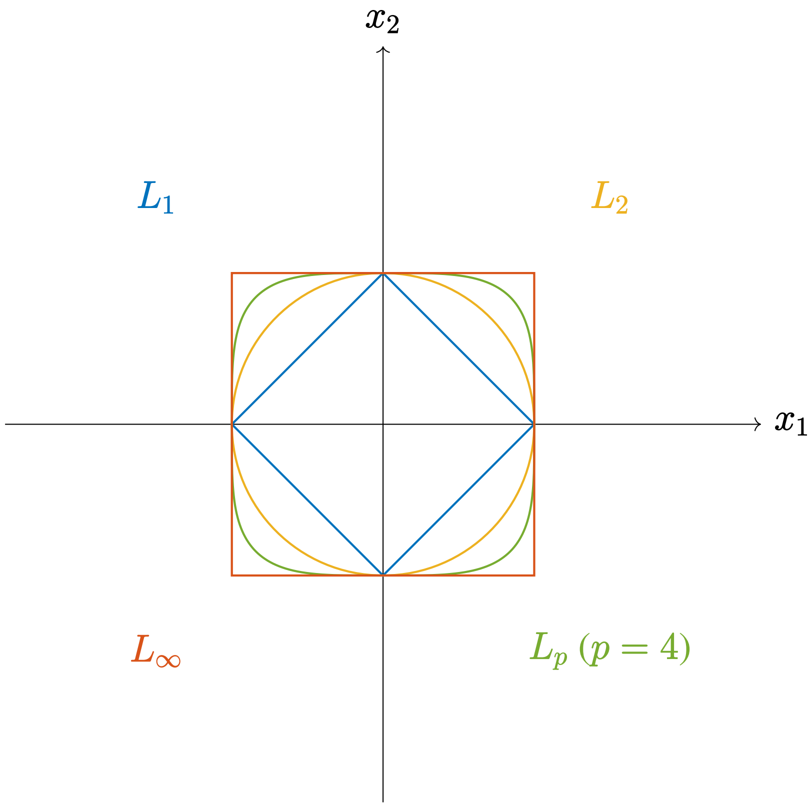


6

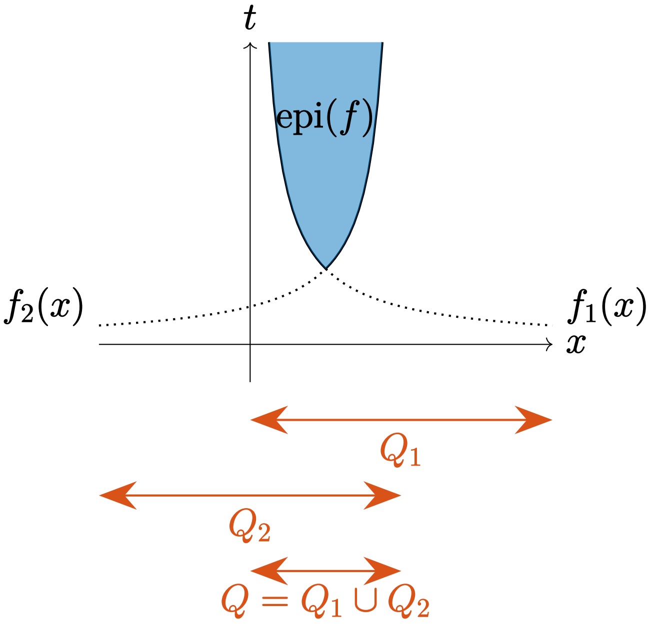
Please see def 3.1.2.

Let's have a break!

5 or 10 minutes break. Please feel free to ask me any questions.

p.147 Thm 3.1.5

3



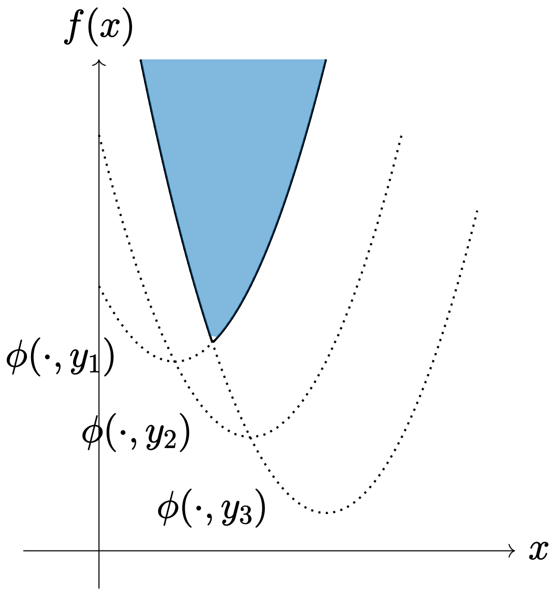
p.148 Thm 3.1.6

affine-invariant. whiteboard.

p.149 Thm 3.1.7

inf of $\phi(x, y)$. whiteboard.

p.149 Thm 3.1.8



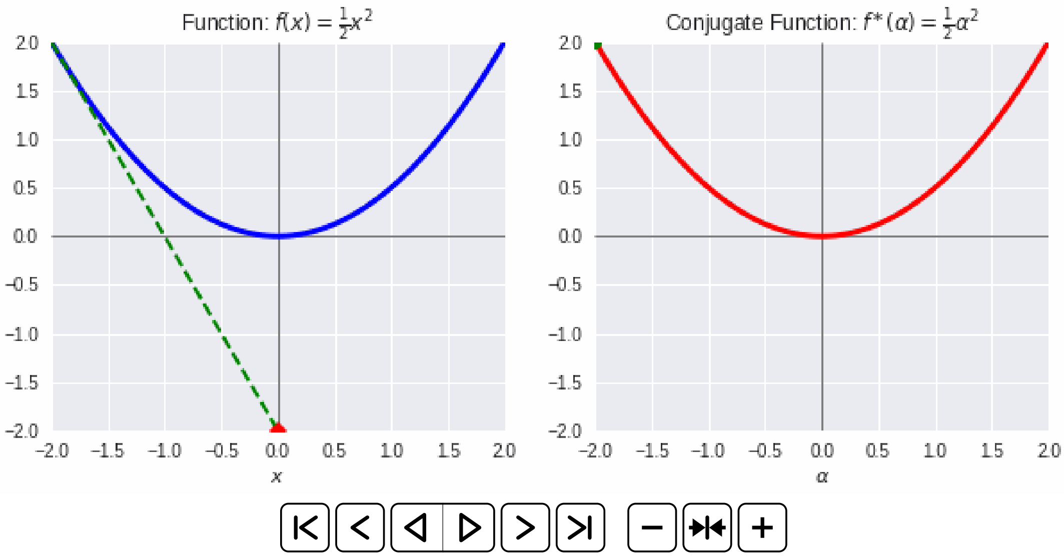
p.150 Thm 3.1.9

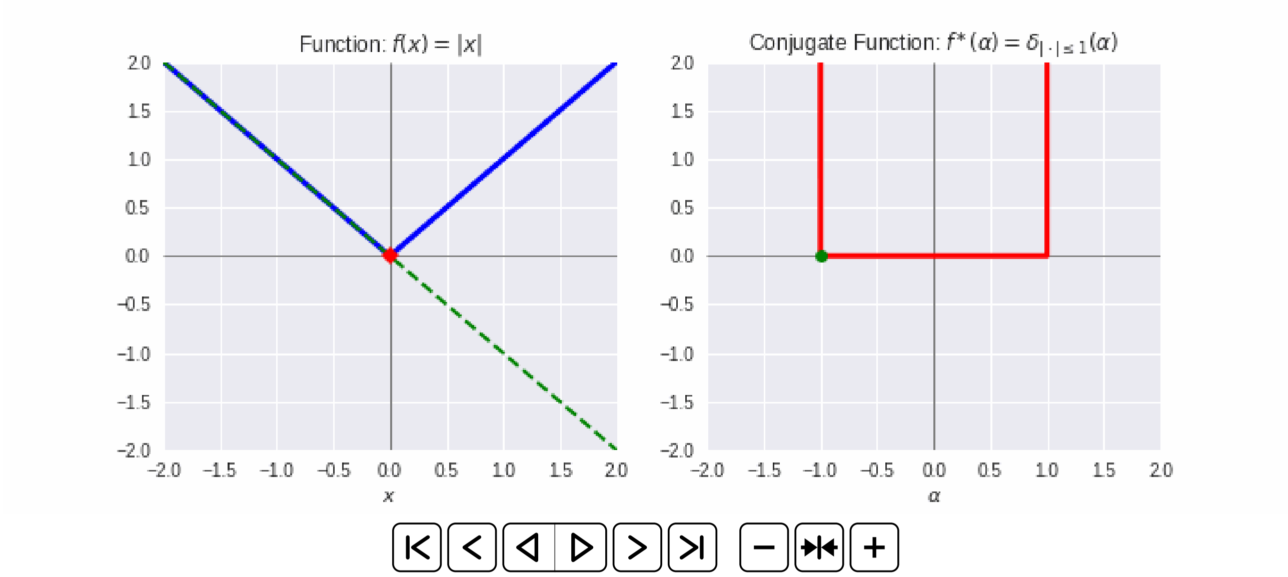
composition of convex functions. whiteboard.

p.150 Example 3.1.2

1

My favorite explanation of Fenchel conjugate: link





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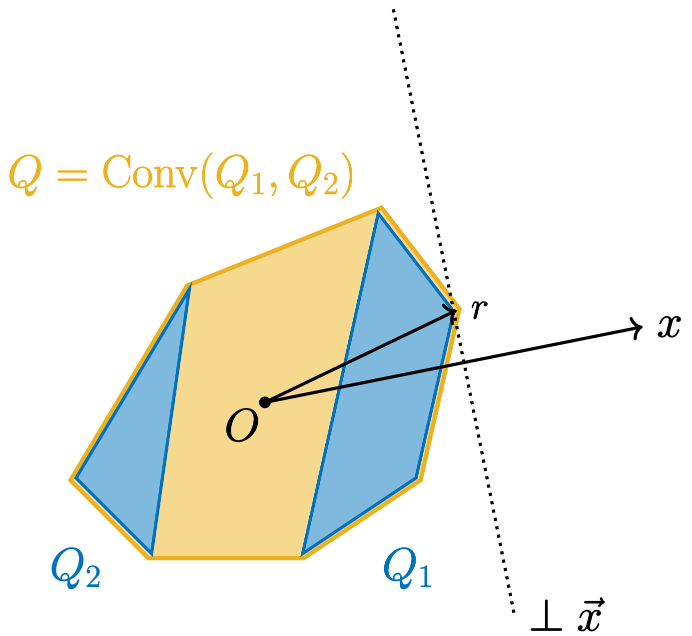


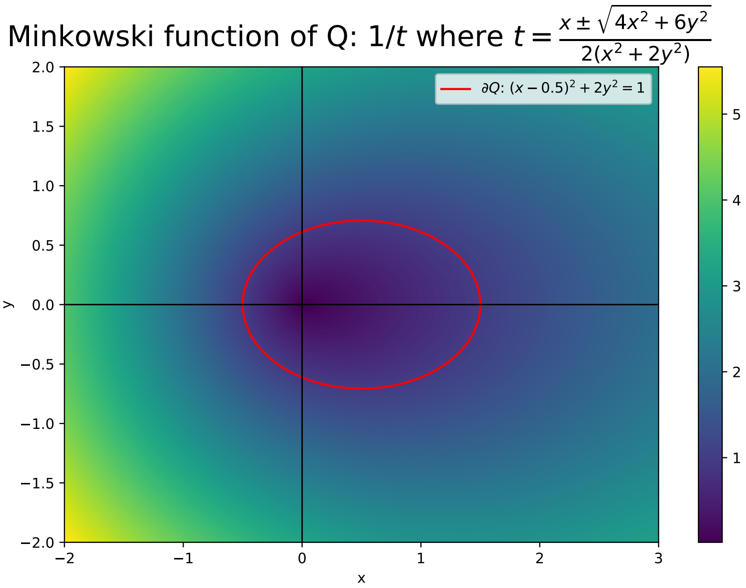
4

Minkowski function.

For example, let us consider the case the convex function Q is $Q = \{(x, y) \in \mathbb{R}^2 | (x - 0.5)^2 + 2y^2 \leq 1\}$. The Minkowski function is defined as:

$$
\begin{aligned}
f(x, y) &= \min_{\tau \geq 0} \{\tau : (x, y) \in \tau Q\} \\
&= \min_{\tau \geq 0} \{\tau : (1/\tau x - 0.5)^2 + 2(1/\tau y)^2 \leq 1\} \\
&= \max_{t \geq 0} \{t : (tx - 0.5)^2 + 2(ty)^2 \leq 1\} \\
&= \max_{t \geq 0} \{t : t^2(x^2 + 2y^2) - tx - 0.75 \leq 0\}
\end{aligned}
$$

$$= \frac{\max(x \pm \sqrt{4x^2 + 6y^2})}{2(x^2 + 2y^2)}.$$



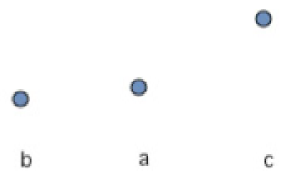
This is actually a convex function.

p.153 Lem 3.1.4

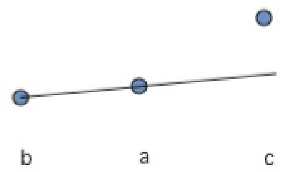
Please compare with lem 3.1.4.1.

visual proof link:

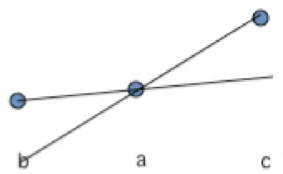
Suppose you want to prove continuity at a. Choose points b, c on either side. (This fails at an endpoint, in fact the result itself fails at an endpoint.)



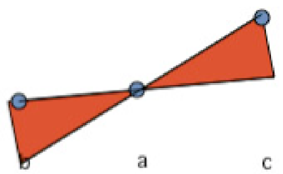
By convexity, the c point is above the a, b line, as shown:



Again, the b point is above the a, c line, as shown:



The graph lies inside the red region,



so obviously we have continuity at a.