The theory of Quasi-duality map and fractals

(翻訳　類双対性写像とフラクタルの理論)

Part I:The Theory of the Lyndon Complex Spiral Phase

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July 2025

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Abstract

This chapter introduces the foundational concept of Lyndon words and their combinatorial and algebraic properties. It explores how non-periodic structures encode complex symmetry through recursive aperiodic traces, leading to the definition of fractal phase topology on the complex plane.We define analytic functions, multiple Riemann surfaces, general noncommutative norms, etc. in the structure named the Lindon complex spiral phase.

1. Introduction

This article is a further introduction to the theory described in the essays that follow, “Introduction,” “Applications,” and “Considerations on the Riemann Zeta Function,” or a glossary of terms.

I first understood that the "zero points of the Riemann zeta function" correspond to non-periodic sequences that are "prime Lyndon sequence."
As I continued to think about this, I discovered the Lindon complex spiral phase, that is,

"Any infinite (or finite) Lindon sequence" → "A complex plane rotating spirally"

This "embedding structure" correspondence, or continuous phase structure.

However, it took me a full month to realize that this concept was surprisingly difficult to understand.

I had originally planned to write a brief explanation of one or two pages, but I felt obligated to explain it in greater detail.

Therefore, in this section, I have provided a more detailed explanation with concrete examples, and I have also included explanations of terms that might be unclear to some readers.

This "introduction" demonstrates a method by which anyone can start from the structure of the Lyndon semigroup, begin with the prime factorization of integers, and then move beyond rational numbers, real numbers, and complex numbers to reach the non-commutative, non-associative broken domains of quaternions, octonions, and 16-tonions in one fell swoop.

Everything progresses fractally.

In other words, it is a methodology for "how to compress information and extract meaningful information."

2. First Principle: Lindon words are uniquely decomposable

A Lindon sequence is a semigroup, which is an ordered arrangement of various elements.

For example,

(01)(01)

Let's say there is a Lyndon sequence like this. As will be explained later, (01) is called a prime Lyndon, which is a "prime number" in the Lyndon sequence that cannot be further reduced, and McMahon's theorem states that all Lyndon sequences can be uniquely decomposed into prime Lyndon sequences.

For example, let's consider a random sequence.

00100101010110101001000100

You may find it hard to believe that this can be uniquely decomposed into "prime Lindon."

A "prime Lyndon" is "the smallest combination of numbers that does not have periodicity."

However, McMahon's theorem states that this is possible.

Figure 1. The Duval decomposition.

与えられた列

```
00100101010110101001000100
```

を **Duval アルゴリズム**に基づいて一意的に **素リンドン語**へ分解

```
['00100101010110101', '001', '0001', '0']
```

Decompose the given sequence into unique prime Lyndon words based on the Duval algorithm.

Furthermore, the length of a prime Lyndon includes at least one of every natural number.

In other words, for any natural number n≥1, there is at least one prime Lyndon word in the alphabet sequence of length n(see Figure 2).

Figure 2. Length of the original Lyndon sequence.

長さ N	素リンドン語（例）
1	`0`, `1`
2	`01`, `10`
3	`001`, `011`, `101`, `111` （ただし素なもののみ）
4	`0001`, `0011`, …

There exists a Lyndon word of length n.

It seems nothing short of mysterious.

The simplicity of the algorithm itself, which uniquely decomposes the sequence, is also surprising. 1. Look from left to right and extract the largest non-periodic sequence. 2. Remove it and then extract the next largest sequence in the same way. Just repeat this process(see Figure 1)**.**

Now, let's consider a new rule.

Prime Lyndon length → Natural number

This is the correspondence.

However, the (01)(01) we initially considered is

(01)(01) → The same repeated Lindon sequence is reducible because its length 2 and repetition count 2 match, making it equivalent to (01), which is an integer with a value of "2" since its length is 2.

This is the "principle of information contraction."

In other words, the combination of prime Lindons is a natural number.

Note that the order is destroyed and a "value" is produced at this point.

In a sense, the "contraction of order" is the first reduction.

Figure 3. Order of contraction.

```
(123)(123)(123) → (123)
```

In this case, it is reducible because the "length" 3 and the "repetition count" 3 overlap. Such a structure that allows compression is "multiplicative" and poses no problem, but the "additive structure" is partially broken, losing its order, and must be restored later(see Figure 3).

This is related to structures like F_1.

Furthermore,

Overlap of "length" and "repetition count" in prime Lindon sequences → Overlap of information → Compression

Contraction of structure → Crushed by compression mapping

This can be **understood as monoid algebraic behavior on F_1.**

3. Construction of natural numbers

When prime Lindon sequences are combined, they form “natural numbers.”

It should be noted that “natural numbers” do not have an order structure, but prime Lindon sequences do. There are infinitely many “prime Lindon sequences,” and they can be decomposed uniquely.

For example, if there were (23)(568)(23), the order would be broken, so it would revert to 2 × 3 × 2. In other words, it would become the “value” 12, but even if the order changes, it is still the same “equivalence class.”
In this way, since there are prime Lindon sequences of all “prime number” lengths, all natural numbers are assigned a “value.”

The reason why the overlap between the “length” and “repetition number” of prime Lindon sequences is reduced is that the repetition of 3 in a prime Lindon sequence of length 3 is reduced to 3 due to the overlap between, for example, 3 × 3 and 3 + 3 in natural numbers. ...However, in the case of the “prime Lindon” of 6 and 9, it can be expressed differently. As I will explain later, since 6/2 exists within the “irreducible rational Lindon,” the combination of 3 (prime Lindon) + 3 (repetition) is restored within the “irreducible rational Lyndon sequence.”

4. Fractions and the composition of irreducible fractions

At first, I called it “N-reducible,” but it seems to be unusually difficult to understand, so here's what it means.

If there were a Lindon language like (23)(568)(23), I said it would become 12. However, the following Lindon sequence, which continues five times,

(23)(568)(23)(23)(568)(23)(23) (568) (23) (23) (568) (23) (23) (568) (23)

Since 5 is neither a multiple of 2 nor a multiple of 3, it cannot be contracted.

In other words, the “product of the lengths of the constituent elements (e.g., 12)” is determined as an irreducible ratio due to the inconsistency with the number of repetitions (5 in this case). This is because 2, 3, and 5 are “prime” numbers.

Therefore, it returns the “value” 12/5 as an irreducible rational number.

At this point, every Lindon sequence has a unique contraction to an “irreducible rational Lindon” through prime Lindon decomposition. In other words, a Lindon sequence has a unique mapping to a new “irreducible rational Lindon semigroup” and is transformable.

Theorem

A Lindon sequence has a unique contraction mapping to a new "irreducible rational Lindon semigroup."

Furthermore, if there is a sequence of "irreducibly repeated Lindon sequences," it becomes additive.

The Addition Theorem of Lyndon Languages

Addition is the additive alignment of Lyndon languages.

Note that reductions such as "3+3=3" are restored here.

Figure 4. The Additive contraction.

$$3 + 3 = \frac{6}{2} + \frac{6}{2} = \frac{12}{2} = 6$$

Additive contraction.

They are restored as an "irreducible rational Lyndon sequence." However, 6 is a "prime Lyndon" element of length 6. 2 cannot have a repetition that includes 6. Because it is only a divisor(see Figure 4).

In other words, since "the order of addition is broken," it is okay to restore it later.

5. Real numbers

Please think of all the symbols we will deal with from now on as "irreducible rational Lindon sequences."

By the theorem we wrote earlier, we can consider the collection of "irreducible rational Lindon sequences" as a new "Lindon sequence." This is because any given Lindon sequence can be transformed by "prime Lyndon decomposition" → irreducible rational Lyndon contraction map → "new Lyndon sequence by irreducible rational Lyndon."

Any Lyndon sequence can be uniquely contracted to an "irreducible rational Lyndon sequence."

Since there are infinitely many irreducible rational Lyndons themselves, they can be regarded as "elements of Lyndon language," and can be uniquely decomposed into "prime irreducible rational Lyndons" by McMahon's theorem.

Then, the combinations of irreducible rational Lyndons themselves can be reduced in the same way as before.

However, as before, the combination of prime irreducible Lindons cannot be reduced unless the length of the prime irreducible Lindons is a prime number. Conversely, when reduction is not possible, the number of repetitions of the prime irreducible Lindons is successively **root-extracted according to the number of iterations.**

Therefore, for example, the golden number can be expressed as

two repetitions of the sequence of Lindons of 1 and Lindons of root 5

In other words, all solutions to algebraic equations can be expressed in this way.
In this way, the operation continues indefinitely, and the reduced “root Lindon” can be uniquely decomposed into a “prime radical Lyndon” by McMahon's theorem... This repetition constitutes real numbers.

In other words, **each time a “root” is stacked, a reduction map to a new “Lyndon semigroup” is uniquely defined**.

When it becomes a repetitive state in which different elements overlap, such as a “multicyclic fractal,” it cannot be reduced. It becomes non-commutative, and the root cannot be easily solved.

That is a real number.

Figure 5. An image of Lyndon- real numbers.

たとえばこういうものを思い浮かべる

$$x = \sqrt{2 + \sqrt{3 + \sqrt{5 + \sqrt{7 + \cdots}}}}$$

または、もう少し数論的に：

$$x = \sqrt[3]{2 + \sqrt[5]{3 + \sqrt[7]{5 + \cdots}}}$$

理論の語彙で言えば

「入れ子になった**既約有理リンドン列**の非周期列」

「縮約不可能な素構造の密な重なり」

「式として“定義できる”が、解析的には閉じない値」

Image of Lyndon-real number in Lyndon sequence.

We can observe this nested structure of roots(see Figure 5).

We know that such overlapping roots are “real numbers,” but we do not understand what kind of “value” they actually represent. Similarly, in a multi-periodic fractal manner, when roots are nested and the repeated number is treated as a root, the same ‘value’ is returned as a “real number.”

Here, it is important to note that while Repetitions of a prime Lyndon are treated as divisions under the criterion of irreducibility, the iteration of an irreducible rational Lindon or real Lindon does not disappear, but rather “has meaning if it is irreducible.” This is because they return to the ‘power’ of the rational Lindon sequence or real Lindon sequence, respectively, rather than disappearing. In other words, it becomes a “to take the n-th root of x.” Furthermore, a sequence of non-irreducible rational Lindon or real Lindon corresponds to situations where “rational numbers become integers” or “real number roots can be solved.” If the sequence cannot to be contracted to any compressible periodicity, it is a real number.

Here, it is important to note that such quasi-periodic “non-contractible structures” with “multiple periodicity” are, with few exceptions, “finite Lindon sequences.”

In other words, they can have “infinite recursive structures.”

Note that this reduction map from the Lindon semigroup to the Lindon semigroup forms a nested fractal that continues uniquely, refining the “values” in detail. As the scale increases, the values become finer... **This is a principle of information theory.**

Also, if the solution to an algebraic equation is within an algebraically closed field, it can always be rationalized. From this, it follows that even irrational numbers can be expressed using this method, as they can be reduced to rational numbers in a finite number of steps.

Summarizing the above(see Figure 6),
Repetition of natural number Lindon → division
Repetition of irreducible rational Lindon → to take the n-th root of x
Such things are arranged → addition
Repetition of additive structure → division
Repetition of additive structure in irreducible rational state → to take the n-th root of x

Figure 6. Types of Lyndon sequence.

リンドンの種類	例	繰り返しに対する振る舞い	解釈
素リンドン列	`(0101)`	繰り返しは**冗長 → 縮約される**	構造単位そのもの。情報の最小単位。F_1的縮約
既約有理リンドン列	`(01)(01)(01)`	繰り返しが**比率的 → 意味を持つ**	有理数としての「反復構造」を保つ（3回なら3/1）
無理数（実数）リンドン列	`(01)(011)(010011)…`	既約有理リンドン列が「既約」な「反復を持っている状態」=累乗	ズレが累積し、縮約不能な実数位相へ
超越的リンドン構造	混合・入れ子・自己非縮約列	無限の入れ子 → 完全に縮約不能	高次の情報構造、複素・回転・多価性を形成

The hierarchical structure of the Lyndon sequence.

Please note that everything is structured in an “information theory” manner.

5. Complex numbers

Perhaps the most common question that arises when hearing this story is, “Why is that so?” Complex numbers are probably the most common example.

From this point on, the elements can be natural numbers, irreducible rational numbers, or real numbers.

If these elements are arranged in an “aperiodic sequence” manner, the “aperiodic sequence” itself can be regarded as a single “value.”

In other words, it is a nested structure of “an aperiodic sequence.”

This nested structure is also a “semigroup,” so it can be uniquely decomposed by McMahon's theorem. In other words, based on the previous discussion, we can say that it “can have a nested real number structure,” so this higher-order nested structure has the “values” of natural numbers, irreducible rational numbers, or real numbers.

Theorem

Every Lyndon sequence has a contraction map in the form of “non-periodicity of non-periodicity,” and it is unique. In other words, there is a unique mapping from “any Lyndon sequence” to “non-periodicity of non-periodicity” to “new Lyndon sequence.”

This applies to complex numbers, natural number Lindon sequences, irreducible Lindon sequences, and real number Lindon sequences. Depending on the value, if the number of prime elements is N, the angle is determined by the Nth root of N, and the “non-periodic nested structure” is rotated by the “real number” value indicated by the semigroup. Note that this “non-periodic sequence” can be continued infinitely. In other words, this rotation continues forever, spiraling endlessly across the complex plane.

This is what I call the “Linden complex spiral continuous phase(see Figure 7).”

In other words, if the real number itself forms a “non-periodic sequence,” it becomes a rotation on the complex plane, and even if it is a natural number Linden, it rotates. This is the complex spiral plane.

This “non-periodic sequence nested within a non-periodic sequence” can also be a “finite Lindon” or an “infinite Lindon.”

From this concept, we can understand that “negative numbers” are quite rare. In fact, when a “negative value” appears in the zeta function, it can be seen as a spiral in which the divergence is contracted.

Isn't this frighteningly nested structure of semigroups and McMahon's continuous mapping beautiful?

6. Divergent reconstruction and transcendental numbers

When such a “Lyndon sequence plotted on a complex spiral plane” is “infinitely repeated,” its “value” returns.

For example, suppose there is a graph with two loops.

Then, if we assign the numbers 0 and 1 to the loops and take infinite traces on all graphs, all “Lyndon sequences” created from ‘0’ and “1” will be generated.

In other words, if the trace bundle of a graph has “infinite repetition,” it will return the ‘value’ of the “finite Lyndon,” and even if there are some that are not “infinite repetition,” it will return “transcendental numbers.”

Transcendental numbers cannot be contracted by any finite reduction method. In other words,

complex Lindon → real Lindon → irreducible rational Lindon → natural number Lindon

Such a “contraction” can be defined, but anything that cannot be reduced to a “finite-length Lindon sequence” by such a finite reduction is considered a “transcendental number.”

Figure 7. Rotation by repeating nested structures.

対象	リンドン構造	幾何的意味	結果
実数 r	有限リンドン列 L_r	点の集合（直線上）	実軸上の数
非周期的な L_r の反復	入れ子･ズレのある列	螺旋構造を作る	回転 = arg = 複素数
回転構造	実数 × 方向	単位円周上の点	複素数

Complex Lyndon sequence indicates rotation.

7. Analytic functions

As before, let us consider a graph with two loops.

We will name these loops 0 and 1.

Then, if we take any infinite trace bundle of this graph, it will contain all sequences such as 00010100101001010101... In other words, when condensed, there is always a "plot on the complex plane" based on a prime Lyndon.

At the same time, this "trace path" also synchronizes in the same way in the "dual graph" of that graph.

The dual operation is, simply put, an operation that swaps "points and lines." This operation changes the structure of the graph, so the "trace structure" also changes. As a result, the same trace in the graph plots different numbers. In other words, **"a certain complex number" and "a certain complex number" correspond one-to-one**. This is an analytic function(see Figure 8).

Simply by existing, the graph generates an "analytic function" on the entire complex plane.

From this fact, it is possible to extract any general Riemann surface structure from any graph structure, but the explanation and proof of this are discussed in the "Introduction" section.

Figure 8. Mechanism of Graph-like Riemann surfaces.

```
（グラフ構造）
    ↓ トレース列
（リンドン縮約）
    ↓ ズレ構造
（回転角度）
    ↓ 複素点
    ↓
（双対グラフからもう一つの複素点）
    ↓
（z ↦ w の対応）＝解析関数
    ↓
（関数が重なり合っていく）＝リーマン面
```

Graph → Lyndon contraction → rational numbers/real numbers → complex numbers → dual graph → analytic functions → overlapping of graph-like Riemann surfaces (multiple Riemann surfaces)

8. Divergent Restoration

As explained above, even with just two loop structures, there is a continuous complex spiral structure that is “divergently restored” from the graph.
This is referred to as “divergent restoration.”

In a different sense, there is also a “divergence of prime structures” inherent in the graph.

For example, if two loops overlap, the choice of which loop to follow results in a “0, 1” branching.

Due to this branching, the “trace bundle” contains the structure “0101010101010010100...,” resulting in an infinite number of “prime structures.”

Such structures are referred to as “non-periodic paths” within the graph.

When there are two such non-periodic paths, it is called a Riemann surface of “genus 1.”

The counting method is the same. When there are three, it is “genus 2.”

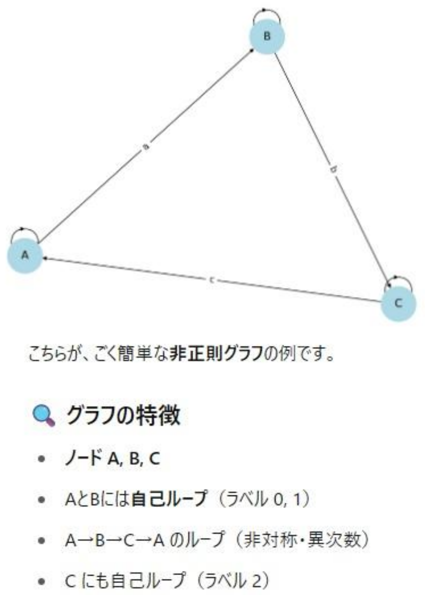
In other words, normally, “loop structures” do not have divergence.

However, when there are “non-periodic paths,” divergence occurs in the trace structure. This makes it more complex. Therefore, the degree of complexity is represented by the number “genus.”

At this point, it becomes important to distinguish between the usual “divergent restoration” and the divergence caused by the “non-cyclic path” of “genus + 1” within this graph.

9. Synchronous structure in the dual space

Figure 9. Example of a triangular graph with self-loops.



Self-looped triangular graph.

Let's assume there is a graph (see Figure 9).

The self-loops of A and B are assumed to have generators of the prime symbols “0” and “1,” respectively.

The route A→B→C→A functions as the symbol sequence “a → b → c,” but the self-loop (2) at C introduces asymmetry and irregularity.

The trace sequence obtained from this entire graph produces a structure that mixes periodicity and non-periodicity, such as:

0 0 a b 1 b c 2 2 0 1 1 c a b ...

.

The “trace Lyndon sequence” of a graph with such a “self-loop” structure is characterized by its high divergence, and it becomes important in proving the “fixed point” during “divergent expansion.”

On the other hand, when considering the dual graph corresponding to this graph, if there are correspondences such as “0↔1” and “a↔c,” the same trace sequence will have different plot values.

Therefore, different complex numbers are assigned to the same trace sequence between "one graph" and "its dual graph."

This is precisely the definition of the analytic function $z \mapsto f(z)$.

Furthermore, when multiple prime symbols overlap in a loop and an irreducible non-periodic sequence appears, the graph produces a unique plot on the complex plane. In other words, it expresses all continuity, including transcendental numbers.

10. Dual motif closure

Incidentally, the dual operation of "swapping points and lines" is not unique.

When it becomes a hypergraph with many points connected by many lines, there are many ways to swap them, and we write down all of them.

Then, we take the dual operation of "swapping points and lines" on those transformed graphs... and continue this operation. In the case of a finite graph, this operation ends after a finite number of steps and completely closes the "dual operation." In other words, the "motif closure" is an operation that replaces the "multivalued nature" of dual transformations with the "relationship between sets of graphs."

There is no need to understand this deeply. For example, a quadratic function has one value, but if you flip it, it becomes two values. This is also a dual transformation (taking the inverse function). If you swap them again, it returns to the original. However, in more complex Riemann surfaces, it naturally becomes multivalued.

11. "Multiple Traces" and "Multiple Trace Space"

"Multiple trace space" refers to **a structure in which the same trace sequence in the entire motif closure proceeds simultaneously (i.e., synchronously) in multiple motif structures (including duals) while having different interpretations of contraction, projection, and prime structures.**

There were many dual-transformed graphs, but their "points and lines" corresponded to each other.

Therefore, when "taking the trace of a single graph," a "trace" synchronized with it runs in all dual graphs.

This is "multiple trace."

And the important point is that the "trace sequence" is "plotting points on the complex spiral plane."

This means that in the various graph shapes of these dual graphs, "different 'values'" are simultaneously plotted on the complex spiral plane.

This is "continuous," meaning that among the "trace bundles divergently reconstructed on any complex spiral plane," each trace has a "value." Thus, it becomes clear that these deformations are “continuously connected.”

This is the uniqueness of analytic connection in graph-like Riemann surfaces, or more generally, in Riemann surfaces. You will realize that the question “Why does analytic connection occur on Riemann surfaces when extended to the complex number domain?” is no longer necessary.

The same trace sequence has different meaning structures in different motif closures (i.e., different prime Lindon reduction structures). Each is mapped to different complex numbers z_1, z_2, z_3..., and these mapping relationships give rise to multivalued mappings (Riemann surfaces) and analytic function systems.

With the addition of the concept of “multiple trace spaces,” graph structures begin to take on spatial meaning, such as “projective systems,” rather than merely generating mappings, and function spaces become families of projective mappings generated by motif dependence(see Figure 10).

Phenomena such as zeta functions, modular forms, and automorphisms also begin to appear in different guises within “multiple trace synchronization.”

Figure 10. Structural level of graph-like Riemann surfaces

構成レベル	名称	意味	主な構成要素
① ローカル	トレース列	グラフGの1つの経路	リンドン語列、非周期列など
② ミドル	モチーフ閉包	グラフGの双対グラフを考えることで、多値性を制御した閉包構造	点と線がそれぞれ対応しながら変形されたさまざまなグラフの集合体
③ グローバル	多重トレース空間	複数のモチーフ閉包（双対・派生）にわたるトレース列の **同期的構造**	同一トレース列の複数プロット、関数対応 ($z \mapsto w$)

Local → Trace sequence (trace bundle) Middle → Dual motif closure Global → Multiple trace space

12. General Riemann surfaces are not “patchworks” but “inevitable overlaps.”

When there is a synchronous structure between the “complex plane” in the Lindon spiral continuous topology and the “multiple trace space,” there exist infinite non-periodic Lindon sequences (i.e., sequences with discrepancies in construction density).

Each of these is plotted as a rotation point (e^{2πiθ/n}) on the complex plane (where n is the number of prime structures in the Lyndon sequence). In other words, each of the deformed graphs on the motif closure has a “value.”

Furthermore, since there are many motif closures/dual graphs, the same sequence is mapped to multiple plots (multiple traces).

A “natural continuous mapping” (synchronization) occurs between these multiple structures, and **then all discrepancies are self-synchronously complemented, and all plots are uniquely connected by a continuous mapping, thus connecting the complex plane and covering it "without gaps."**

This is the "uniqueness of analytic continuation in multivariate functions."

13. To the higher algebraic domain

From McMahon's decomposition theorem, through rational reduction, we can hierarchize and further hierarchize real numbers on fractals as roots of powers in the same way, and from the higher-order non-trivial "Lyndon semigroup" called "non-periodic terms of non-periodic terms," complex numbers are extracted in exactly the same way, They form a spiral, and quaternions are extracted from the "higher-order structure" of that complex structure, rotating again to a different axis, which continues to hexadecimals, and gradually the algebraic system breaks down... I partially described this flow.

14. The order of multiplication and addition is broken, but it will be restored later.

In the Lyndon contraction structure, multiplication, addition, and their order first continue to exist, so they cannot be extracted without first being restored by the contraction map.

In the initial structure, addition does not exist, and only the contraction repetition dominates.

However, when non-repeatable Lyndon structures are combined, addition "emerges" in parallel. .

This addition is not the addition of natural numbers, but rather emerges through the rearrangement of hierarchical trace-repetition structures.

Therefore, addition in this theory is **a non-commutative order structure** that is constructively reconstituted.

Is there a "zero element" within this?

It is likely the "infinite repetition state of a prime Lyndon structure."

In terms of information content, it has no meaning. Each "prime Lyndon" functions as an identity element and possesses a multiplication structure until it is restored.

When restored, the "order" becomes addition.

This is a non-commutative order structure. It has no meaning in terms of information. Each "prime Lindon" functions as a unit element and has a multiplicative structure until it is restored.

When restored, the "sequence" becomes addition.

This corresponds to the fact that the order of operations from multiplication to addition is fixed.

This is natural in terms of information theory and is fractal.

15. Fractality as the most important principle: Three principles

In this system, if I were to dare to cite axioms, they would be

1. McMahon's uniqueness decomposition theorem

All Lyndon sequences can be decomposed into "prime Lyndon sequences" (including sequences of all natural numbers).

2. Uniqueness of reduction maps to irreducibility

All Lyndon sequences have a unique map to Lyndon sequences that are more "irreducible" than their current state.

3. Uniqueness of contraction maps to "non-periodic sequences of non-periodic sequences"

All Lyndon sequences have a higher-order nested structure, and within that nested structure, they have a contraction map as a "Lyndon semigroup."

These are the only three principles.

The complex spiral continuous phase is almost entirely composed of continuous repetitions of these three principles, that is, a "nested structure," and is therefore complete.

In particular, let us call the two types of contraction maps inherent in the Lyndon semigroup the "Lyndon double spiral structure."

When a Lyndon sequence has an irreducible contraction series and a non-periodic nested series, and these are isomorphic while breaking the non-commutative order structure, we can call this syntactic structure the Lyndon double spiral.

16. Proof of the existence of dual Lyndon sequences

Duval decomposition is performed from the left.

However, it can also be performed from the right.

What is of interest is the existence of a reverse decomposition of a "prime Lyndon structure," or in other words, the existence of a "dual Lyndon sequence."

Let us prove its existence.

By definition, a non-periodic sequence is non-periodic when viewed in reverse. Suppose it can be decomposed into another non-periodic sequence. Then, "different non-periodicities" would be inherent in the same structure, leading to a contradiction. Therefore, "dual Lindon elements must be decomposed in the same way."

Thus, dual Lindon sequences must exist.

Note that the "dual Lindon sequence" in this Lindon decomposition naturally defines the "inverse of the trace sequence." In other words, when returning a "value," "the same general Riemann surface structure is reached regardless of which trace path is used as a reference."

This is related to the structure of inverses in Lindon algebraic structures.

In other words, the current proof establishes that **"the value of the inverse function in the general Riemann surface is uniquely determined**(see Figure 10 for syntactic dual mappings).**"**

17, Non-commutative Rotational Norm Theorem

On the infinite "spiral phase" defined for Lyndon sequences,which can be described as a "non-periodicity of non-periodicity":

This spiral phase compresses the inherent "rotation" into a real value,thus uniquely determining a norm.In other words, the structure rotates while being slightly extended—a spiral that never fully closes.

Theorem (Non-commutative Rotational Norm)

Every trace bundle, composed of multi-nested irreducible Lyndon structures,admits a unique contraction into a real-valued norm,which inherently carries a discrete rotational phaseand an infinitesimal spiral deviation.

Figure 11. The Noncommutative rotational norm.

$$|x| = (r, \theta) \in \mathbb{R}_{\geq 0} \times S^1, \quad \theta = \frac{2\pi k}{N},\ k \in \mathbb{Z}_N,\ r = r_0 + \varepsilon.$$

Formally, the rotational phase originates from the residual structure of the quasi-dual morphism,while the infinitesimal shift ε represents the non-closed spiral deformation of the trace bundle.

Therefore, despite the trace bundle being non-commutative and infinitely nested,
its contracted norm is uniquely determined—
accompanied by a discrete rotation and a subtle spiral extension.

Probably no one who reads this theorem would doubt
that the "nesting" itself is the "rotation".

−The Cartesian Spiral…

18. Points to note when reading

Basically, the most commonly used structure is that when “prime Lindon” is restored in a divergent manner, it returns to the “zero point.”

For this reason, the zeta function has two forms of expression: the “Euler product (prime number structure)” and the “zero point product (zero point structure).”

Once this structural transition is recognized, the theory of “irreducible rational Lindon” is occasionally utilized, but the realm of “real Lindon” remains largely unexplored. This is an area where “non-periodic fractals” and “non-separable” properties—that is, “non-commutative reduction methods are the only ones available”—come into play, akin to the “non-commutativity of modularity.”

This involves extremely difficult problems, so even I cannot fully grasp it.

While this theory owes much to the combinatorial groundwork of MacMahon and the decomposition algorithm of Duval, it diverges sharply by constructing a syntactic topological space where Lyndon words act not merely as sequences, but as operators shaping the structure of complex spiral manifolds

The quasi-dual transform u^p→e^−$s$log$p$ is the Mandala Core’s purest spiral bridge — uniting power structures, logarithmic phases, and the non-commutative analytic continuation of the zeta.

An algebraically closed field of infinite dimension…

黒川信重、絶対数学原論、現代数学社、2016
森田英章、組合せ論的ゼータの半群表示、2016
ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
高安秀樹、フラクタル、朝倉書店、1986
Shinjiro Kurokawa, Absolute Mathematical Theory, Gendai Suugaku Sha, 2016
Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
Bernhard Riemann (translated by Jiro Suzuki), On the Number of Primes Smaller Than a Given Number, 1859
Hideki Takayasu, Fractals, Asakura Shoten, 1986

Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure. .
This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.

素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of "complex spiral phases"
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a "divergence control function" corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ｓという臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.

無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the “elementary Lyndon element” that is restored to zero is decomposed, the corresponding Euler product becomes a “multiple Euler product,” giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through “dual non-periodic paths.”
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a “double helix” because it naturally contains spiral rotations and has a double main

structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in the number of species. This is particularly important in the context of the formulation of "higher-order imaginary multiplication."
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by "dual non-periodic paths." Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.

トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)
素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する
In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.
全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.

Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy

Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal “bouquet structure” corresponding to the “Euler product.” It is also necessary to distinguish it from the commonly referred to “Hodge structure.”
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace；　A natural quasi-dual transformation that reduces　“irreducible rational Lyndon”　in trace bundles to　“prime Lyndon”　or　“natural number Lyndon.”　Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets；A complete state that cannot be further expanded by repeating dual operations.
双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.

→ 類双対閉包

非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる

Part II: The Basic Structure of the Application of Quasi-dual Morphism to Mathematical Objects in the Theory of Dynamic Transformation (Introduction)

Author:Hiroki Honda

July 2025

注・日本語を読める方は、日本語で書かれた原文がありますので、そちらをご利用ください。文末にリンク先

Abstract

This section formalizes the notion of quasi-dual morphisms, a generalization of quasi-duality in trace bundles. The theory focuses on divergent recoverability, where structural loops emerge from non-periodic trace contractions, constructing topological invariants linked to Zeta morphisms.Applying the graph-like Riemannian surface structure of the Lyndon complex spiral phase defined in the previous section, we will examine the basic concepts of fractal geometry.

This paper aims to explain the conceptual structure that was deliberately not explicitly stated in the previous paper on the Riemann zeta function ("Generative Approach to Zeta Structures via Quasi-dual morphism and Graph Zetas: Generative Deformations of the Riemann Zeta Function and the Construction of Their Zero Points via Quasi-dual Divergence Maps") by applying it to familiar mathematical examples, thereby making it easier to understand.

In other words, this is both an introductory and an applied essay.

Furthermore, it touches on connections with theories that are attracting interest in various fields, as well as unsolved conjectures, so please look forward to it.

As a note, there are parts of this essay where explicit formulas or transformations are not immediately apparent.

However, this reflects the surprising structure that the graph itself actually contains a functional structure, and rather indicates that the “explicit formula” that appears in Ihara's zeta function is a rare exception.

In other words, in this theory, the “structure” already exists before the formula is written, and the analytical expression is “derived” from that structure.

In this paper, we see that the “uniqueness of the natural analytic connection of multiple Riemann surfaces” is constructively derived by the dual motif closure of the trace structure.

This uniqueness theorem suggests the possibility of redefining the geometric meaning of analytic connections in complex function theory from the perspective of graph theory and recursive reconstruction.

In this paper, we introduce trace series and Quasi-dual maps as a framework for generatively describing the divergent structure of the Riemann zeta function, and attempt to provide a generative description of the zero structure.

This involves constructing something that naturally seems “to be the case.”

In particular, the “uniqueness theorem for multiple Riemann surfaces” redefines the spatial meaning of analytic continuation in a constructive manner and suggests that the geometric connectivity of critical lines can be reached from within the divergent series.

Those who find the argument difficult to understand should pay attention to the following points.

The main objective is to describe the construction theory of general Riemann surfaces graph-theoretically and to explore their divergent reconstruction and graph motif-based generation principles.

Through this framework, we define a hierarchy of non-periodic generating factors using prime Lyndon sequences and equate the number of generating factors with the genus of the Riemann surface, thereby clarifying the topological structure of multiple Riemann surfaces as graph manifolds analytically connected to the complex plane via the decoded Lyndon language continuous topology.

In this theory, Lyndon sequences play a role in imparting a natural topology to the analytic connection to the complex plane, so reading with this in mind may make it easier to understand.

1. Classification of non-periodic terms, zero points, rational points, real points, complex

points, and transcendental points of function values

First, let us assume that there is a graph structure.

We can take a trace of it. The trace may be finite or infinite, but here we will consider only the infinite case.

If the graph forms a loop, the trace path will repeatedly loop around, resulting in an "infinitely repeated segment of the trace" that appears as an infinite repetition segment.

In this case, you can either "restore the loop structure" or "decompose it into a tree structure."

In short, it is simply a transformation that converts a loop structure into a tree structure and restores a tree structure into a loop structure. For example, factorization and infinite series. Integration and differentiation. Loop-type graphs and tree-type graphs. These appear in numerous places, and what connects them is the "trace bundle."

A "trace bundle" is, simply put, the collection of all possible "trace paths" that have been traced.

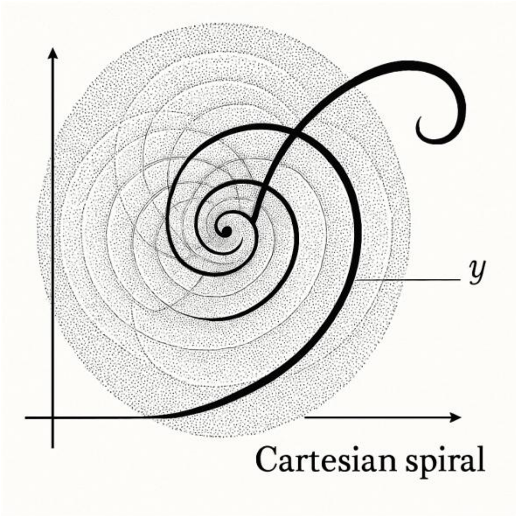
This quasi-dual morphism is fundamentally a non-commutative mapping and is also multivalued. In other words, the process involves first expanding into a "trace bundle" (akin to a quantum state) and then restoring it, during which various "restoration methods" emerge. Among these, the "divergent restoration" and the non-trivial "class dual divergent zeta extension" are particularly important.

For more details, please refer to my essay on the Riemann zeta function.

Here, think of it as "restoring the infinite repetition to a looped tree."

This is the basic principle of the quasi-dual morphism, represented by a diagram of infinite concentric circles and a Cartesian spiral. In this diagram, you can observe that the circles and spirals are divided at their intersection points, with each region corresponding one-to-one. This theory can be considered an extension of Cantor's theory.

Figure 1. The fundamental quasi-dual mapping.



Basic quasi-dual mapping: infinite concentric circles →← Cartesian spiral

Note that the infinite concentric circle structure and the Descartes spiral correspond in a one-to-one manner, ensuring the extendibility of this compression mechanism to arbitrarily high dimensions.

Look at the diagram above. When the concentric circles expand exponentially, the Cartesian spiral also expands exponentially, and each is cut off while maintaining a one-to-one correspondence. At this point, it is important to note that while the infinite concentric circles have a certain mapping, the Cartesian spiral is an infinite-scale or scale-free fractal.

From here, we consider the concept of "non-periodic terms."

Non-periodic terms are those that do not contain repeated elements, meaning they cannot be further compressed. Otherwise, even if we try to "restore the loop," it might not return to its original state because we would be restoring an "extremely long loop." However, even attempts to restore it properly involve multi-valued restorability, which is a very important property.

In my Riemann's zeta theory, this is called "non-periodic term-like quasi-dual divergent reconstruction zeta expansion."

McMahon's theorem states that these "non-cyclic terms" can be further reduced to the state of "prime Lindon (non-cyclic terms)."

In other words, it is a surprising theorem that "all non-cyclic terms can be uniquely decomposed by prime Lindon words."

By this, the restoration from the bouquet graph is naturally expanded to "a graph with a loop structure of the length of all prime numbers," and it can be seen as a "zeta expansion" that contracts to that area. I called this "Euler product hologram fractal restorability."

Now, this may seem abrupt, but it is important to first list the classification of "non-periodic terms" that are restored divergently, as we will use this in the subsequent discussion. However, this definition may seem abrupt. Its meaning is explained in the discussion of "zeta expansion" and will also be explained in the following text, so I recommend comparing the two.

First, the divergent reconstruction of "prime Lindon elements" corresponds to the "zero points" of functions.

This was utilized as the most important fact in the previous discussion on the "Riemann zeta function."

Next, regarding non-periodic terms, "n-contractible non-periodic terms," that is, non-periodic terms that can be decomposed into prime Lindon elements, "return to irreducible rational numbers." This is a remarkable fact. The irreducible iterations within the structure decomposed into "prime Lindon" divide the integers, which are a collection of "prime Lindon" origins with "prime number lengths," by their "prime" period numbers. As a result, they become rational numbers.

Furthermore, even among non-periodic terms, there are "non-periodic terms with multiple contraction." For example, it is easy to understand if you imagine a root enclosed by roots. Such non-periodic terms return to real numbers.

To explain this a little further, first, according to McMahon's theorem, a Lindon sequence decomposed into "prime Lindons" is always uniquely reduced to a sequence of "irreducible rational Lindons." Then, we can consider the "prime" repetition in the same way as we did for rational numbers, and we can take the 'prime' repetition of "prime irreducible rational Lindons" and "raise it to a power." The Lindon sequence that has been "exponentiated" in this way is uniquely contracted to the world of "exponentiated" numbers, and again, according to McMahon's theorem, it is decomposed into "prime root-raised Lindon" numbers, which are multiplied by "root-raised" numbers in the same way, and this continues indefinitely.

What is important in this structure is that it appears to be a nested structure of "fractals with multiple cycles." Compare this to the "multi-valued nature of roots" or the fact that "the nested structure of multiple roots ($\sqrt{\ }$) basically does not come apart." This will clarify the image. It is strange that non-periodic terms are fractal.

Please note that as the scale increases, the details become finer.

And among the non-periodic terms, those that do not reduce to a "prime Lindon element" no matter how they are reduced are transcendental numbers. (I have prepared another essay on transcendental number theory, so please look forward to it.)

From the fact that these are "mixed in the trace sequence of the graph," we can understand the surprising fact that all graphs are generated by a Cantor-like non-constructive principle that takes a "set of subsets."

You may be thinking, "What on earth is he talking about...?"

So, from here on, I will apply this to examples that everyone is familiar with to form concrete examples.

So, let's start by considering general Nth-degree equations and also consider the structure of Galois theory inherent in the fundamental theorem of algebra.

Figure 2. Recovery of non-periodic terms.

核心定理（生成論）

- **非周期的項の階層**
    - **ゼロ点**：完全縮約された素リンドン列
    - **有理点**：有限階層で縮約可能な非周期列
    - **実数**：多重縮約可能な非周期列（内部に無限擬似周期性）
    - **超越数**：最大限に縮約不能な非周期列
- **命題**：
  「任意の関数値は、非周期的トレース項の構造的復元によって一意に決まる」

ここで、関数の値を決めるのは級数ではなく、
非正則グラフ上の素経路（素リンドン）の階層縮約である

The prime Lyndon component in the divergently restored function evaluates to zero.

Now, those who have read this may be thinking, "Wait, what about complex numbers?"

As mentioned above, there are also "multiple non-periodic terms" in non-periodic terms. The sequence of "non-periodic terms" is further arranged in a "non-periodic" manner, and this is repeated infinitely. In this "nesting of non-periodic terms," the "structure of non-periodic terms" itself may differ.

To put it another way, since the nested structure of "non-periodic terms" itself is a "non-periodic term," this becomes a structure in which "non-periodic terms" are arranged in a semigroup. In other words, this is uniquely decomposed into "prime non-cyclic terms" by McMahon's theorem and contains a higher-order real number structure.

This can be considered to be a complex number. The multiplicity of non-cyclicity in this theory suggests a relationship with complex structures and topological rotational symmetry, and perhaps a relationship with the Nth root of unity, but this will be left as a topic for future consideration.

The concept of non-periodicity described in this section may seem abstract at first glance, but if you find it difficult to understand, please recall the "non-periodic structure" hidden in Euclid's proof of the infinity of prime numbers. "The cycle shifts and cannot return, leading to infiniteness."

It is this non-periodicity within infinity that supports the core of the most basic image of the non-periodic sequence referred to here.

Finally, I will write down the "conjecture of topological mapping from Lyndon semigroups to the complex plane."

" Conjecture of continuous spiral complex topological mapping of hierarchical non-periodic nested structures (Lyndon semigroup structure conjecture)"

The combination of prime Lyndon sequences gives a natural number structure, and its repetitive structure produces a rational number reduction structure. The fractal nested structure of this reduction repetition generates an irrational real number structure through root extraction, and when these hierarchical nested structures are further combined non-periodically, the hierarchical order is parameterized by the real number structure within the nested structures as a topological "rotation," closing in a two-dimensional (complex plane) spiral manner.

Therefore, the infinite hierarchical nesting is bound by the rotation group on the complex plane and ultimately closes as a finite topological space (complex plane)(see Figure 2).

Note that this structural conjecture naturally produces a "spiral overlap of the complex plane" and naturally explains the multi-valued nature of the log function. Even if this mapping is not "unique," it is clear that a topological structure similar to this structure exists.

To reiterate, the nested structure of the Lyndon semigroup ascends through the hierarchy of integers, rational numbers, and real numbers, but as the nested structure progresses further, it enters a rotational state, and the overlapping of the nested structures ascends this hierarchy again, circling back to the same complex plane.
Thus, the complex plane inherently has a spiral shape, which is believed to be the root cause of the multi-valued nature of the logarithmic function. We will analyze this further later.

Next, we will introduce a continuous phase, but it is unclear whether "this is the correct way to do it." There may be other ways to introduce a continuous phase, but with this, we can construct a manifold of trace bundles where "continuity" holds.

《Deterministic Lyndon Continuous Complex Spiral Phase》

This is a method of treating the "length" of a Lyndon sequence as a basic topology, and then constructing a nested structure of rational numbers, real numbers, and complex numbers, while treating the "finely chopped" state as a continuous phase. Furthermore, this structure coincides with the spiral multivalued structure of a multiple Riemann surface and behaves as a continuous covering of the complex plane.

The following discussion will focus mainly on this continuous phase, so please be careful not to confuse the two. In fact, I often confuse them myself. It's strange because I don't make mistakes when I use them myself. This "continuous phase" is easy to use, has no discrepancies, and allows for various developments in a very natural way.

Let's call the infinite nested structure of this "Lyndon semigroup" with its natural contraction structure and the continuity of the structure of "unique decomposition by McMahon's theorem" the "Lyndon semigroup nesting theorem." If we think about this structure carefully, it also proves McMahon's theorem.

Lyndon semigroup nesting theorem

By considering "prime" iterations, the Lyndon semigroup generates natural irreducible rational numbers, and through natural reduction to those irreducible rational numbers, it can be further decomposed using McMahon's theorem, and then further refined by root extraction, resulting in an infinite McMahon nested structure. Furthermore, even in higher dimensions, it has another "Lyndon semigroup" called a "non-periodic sequence" of non-periodic sequences, which similarly the "phase of real numbers," that is, an infinite nested structure, which can be used to define rotation in the complex plane.

It is clear that there is a "continuation" to this structural theorem, because even in a semigroup reduced to complex numbers, there is a higher-order Lyndon semigroup called a "non-periodic sequence of non-periodic sequences," which probably corresponds to "quaternions."

Contractible structure of prime Lyndon sequences: natural numbers → rational numbers → real numbers

Rotation projection of non-periodic sequences: complex projection (ℂ) using arg

Non-periodic nesting of non-periodic sequences: multi-axis rotation ⇒ non-commutativity ⇒ quaternion structure (ℍ)

Furthermore, the nested structure of this hierarchical structure: the nesting itself

becomes nested ⇒ the structure diverges
As a result, the structure is

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset ???$$

All I understand is that it gradually breaks down.
What you need to be careful about here is that, in comparison to "it appears to be n-ary numeral system but is not the essence of complex spiral phase structure," the "naturalness of length phase" is demonstrated, so please read that part, or I plan to explain it in the "application section" that I am planning.

"The order of the number system resembles dictionary order in form, but its essence lies in the 'length of the orbit.'"

If you can think of this as "one way of interpreting" Lyndon language, that would be fortunate. If it is different, that is interesting, and even if it is different, you can still pay attention to the fact that the following discussion holds true. I apologize, but I will explain and prove multiple Riemann surfaces later.

Appendix 1: Equivalence classes of irreducible rational numbers of irreducible Lindon origins, rotation theory, p-adic graph manifolds

Here, I will focus on explaining the point I wrote in 1: "How do we know that the non-periodic sequence of N-reductions is rational?"

Let us assume that there are Lindon sequences repeated in the "center nucleus" that is n-contractible Lindon sequence, with lengths A, B, and C, which are prime numbers. Then, the Lindon sequence that is N-reduced must have a length that is a multiple of A, B, and C. Therefore, it is contractible.

As a result, the N-reducible repeated Lindon sequence is reduced to a state where the "number of repetitions" of any Lindon sequence length contained in the 'core' at the center is "coprime." In other words, this is precisely an "irreducible fraction." Interesting, isn't it?

Here, I suggested that the nested repetition of non-periodic sequences is a rotation in complex numbers, but let me explain why it is the Nth root of one.

In the case of a "prime structure of two," it finally rotates to the 'negative' state of the real axis. Then, in the second nesting, it returns to the real axis again. In other words, "real integers" are formed in this state of "two prime structures."

This corresponds to the fact that in the Riemann zeta function, when the "prime structure is infinite," the length of the prime structure is almost a prime number and is odd, so it does not actually reach the "negative real axis."

In other words, it corresponds to the fact that the complete breakdown of the "inversion symmetry of the Riemann zeta function" occurs only on the negative real axis.

This is the part supplemented by the sine function in Riemann's formula.

In the case of odd prime numbers, no matter how large they become, they never reach the "point on the negative real axis."

This has significantly increased the likelihood that the "rotation level is the angle of the Nth root of the number of non-periodic sequences in the non-periodic sequence structure of the prime structure." The structural discontinuity and divergence of any ζ(s) on the real axis are consistent with the prime factor rotational angle structure of the Lindon series, and since a consistent explanation is difficult with other generative principles, introducing this rotational theory is believed to be the only reasonable reconstruction method.

In other words, it is a construction that explains the asymmetric structure of the Riemann zeta function on the real axis.

For more details, please refer to my essay on the Riemann zeta function.

Below, I will describe a method for decomposing graph manifolds into local fields based on a simple definition of the "Quasi-dual map" for the Lyndon series.

The graph manifold takes on the form of a "graph manifold" by incorporating continuous topology.

We define the "class dual map" as removing only elements of a certain "prime length P" or fixing them and observing the rest, without destroying this structure.

As a result, the "prime Lyndon structure" remains, and at this point, it becomes clear that the "residue field" of the graph manifold with respect to the "prime number P" (or prime structure N), i.e., the "P-adic graph manifold," is constructed. Note that a continuous topology is naturally embedded in this as well.

2. Correspondence between the fundamental theorem of algebra, Galois theory, and trace bundle theory

As described above, the view that the structure of a graph constitutes the "structure of function values" as a trace bundle demonstrates both the completion of functions to real numbers and the uniqueness of analytic continuation. In other words, this view itself represents the meaning of the term "analytic continuation," and its determinant representation by the Iihara zeta function is rather a problem of representation theory that accompanies it. This has become clear, but we will return to it later.

Let us consider a very simple cyclic graph with two loops.

It is sufficient to imagine two circles connected at a single point.

When considering this graph, one might wonder, "Where is the infinite repetition of non-periodic terms, even though there are only loops?" However, if we represent one loop as 0 and the other as 1, we can see that the order in which they rotate can be expressed as an ordered set of 0 and 1.

Interestingly, within this ordered set, there exists a non-periodic set of length N. Moreover, there exists an infinite non-periodic term that does not contain any periodic repetitions. With just two "prime structures," a "real completion" arises. And if the loop is directionless, there are two ways to rotate it, "front and back," meaning there are "two elementary structures." In other words, if there is a circular structure, "there exists an infinite repetition of non-periodic terms of length N, and there exists a graph structure that restores it."

This is a special and non-trivial "quasi-dual mapping" called the "quasi-dual divergent zeta extension," and it generates the "zero point set" that produces the reconstruction.

Thus, a graph with two loops expands continuously through such a divergent quasi-dual mapping, and under regular conditions, it expands into a "circular loop with paths of all prime numbers" (this is described in a discussion of the zeta function), so one might wonder, "Do the solutions to quadratic equations diverge?"
Here, a Galois-theoretic approach is necessary.

In Galois theory, "the solutions are permuted." That is, the elements of the "prime elements (or prime Lindon)" are permuted. At this point, the prime structure or prime Lindon structure is "decomposed" by this permutation.

That is, there is a "trace bundle" of a graph with two loops.

Here, we introduce a groupoid that "permutates the elements of the trace."

As a result, most "prime Lyndon" structures are decomposed, and the expansion stops. And as a result, "the elements are always reduced to two prime Lyndon structures"... This is the structure of the solutions to quadratic equations, and the structure of the "zero points." And at this point of reduction, since the round-trip path between the going and returning paths can be connected, the "symmetry of the zero-point solutions" can be understood.

This "Galois group-like groupoid" expands the diversity of "prime Lindon" structures by a factor of N! as the number of "prime structures" constituting the trace bundle increases. This is the expansiveness of the Galois group. And the fact that these prime Lindon structures can be ordered and reduced by prime numbers is the condition for the "solvability" of equations determined by the Galois group. Thus, it becomes clear that the meaning of "solvability" of an equation is that the prime Lindon structure is destroyed by the operation of "replacing elements" in the trace.

Furthermore, Gauss's fundamental theorem for general Nth-degree equations, known as the "fundamental theorem of algebra," can now be stated with a clear understanding. That is, by introducing an Nth-order groupoid (an operation that swaps elements) into the "trace bundle" of a graph with N loop structures and observing its divergent restoration, we see that the prime Lyndon structure is "finitely generated" and can be reduced to N prime Lyndon elements. This is the meaning of the fundamental theorem of algebra.

Furthermore, these prime Lyndon elements possess symmetry corresponding to their number, and we can see that they are "conjugate elements."

The "addition of solutions" of irreducible polynomials in Galois theory can be formulated as a quasi-dual deformation expansion of the "trace bundle," but this is not the subject of this discussion.

a. The roots of a polynomial are represented as a prime Lindon series.

b. The action of the Galois group = permutation of the prime structure of the trace sequence.

c. The number of roots is finite due to the contractibility of non-periodic sequences.

Appendix 2: Contraction of Weil-type zeta functions to Ramanujan-type functions

After writing this, I suddenly thought, "The action of the Galois group I wrote about here is what is known as the cyclic Frobenius map..."

In other words, "my Galois action, which rearranges the prime elements of the trace and controls them so that their number does not expand excessively," is completely the Frobenius map(see Figure 3),

Figure 3. The Frobenius mapping.

フロベニウス写像とは何か

有限体上では、フロベニウスは元を冪乗で写す自己同型写像。

その繰り返しが有限体拡大のガロア群を生成する。

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \mathrm{Frob} \rangle.$$

Quasi-dual Frobenius map.

that is, it is the Frobenius map itself.

In other words, by controlling the expansion of the bouquet graph, the deformation ultimately decomposes the non-periodic terms into a "finite" number of components, so the structural problem similar to the Riemann zeta function–that the zeros of the Ihara zeta function with Ramanujan-like properties in regular graphs have zeros on the critical line Re(s) = 1/2 (or in a circular region before transformation)—is resolved.

For those who find this difficult to understand, let me explain it in more detail.

Since the infinite trace bundle (prime Lindon spiral) is essentially divergent, I obtained a dual mapping that "rearranges the elements of the trace bundle to limit the length of the non-periodic terms" in order to consider the Nth-order equation. Ultimately, this is a collection of N! actions representing how to rearrange the N elements, so it is isomorphic to Frobenius, meaning that Frobenius acts as an operator that cyclically bundles them.

Thus, the spiral dynamically expanded by the quasi-dual map is converged into a finite closed orbit structure by Frobenius. The fact that it is isomorphic to the action of the Galois group is likely due to its regularity. The Frobenius is not merely a mapping but acts as a group action that permutes the generating nucleus.

Thus, the closed orbit structure of a finite field = the cyclicity of the Galois group = the Frobenius cycle are all the same.

As a result, the critical line situation and Ramanujan property are naturally derived. Since the divergent zeta extension is reduced to a Ramanujan-type regular graph by Frobenius, the "quasi-dual transformation,"

Figure 4. The Quasi-Dual Morphism

$$u^p \mapsto e^{-s \log p}$$

reduces it to the case of the finite Riemann zeta, and its spectral structure agrees with the critical line condition of the Riemann zeta.

In summary, "**The Frobenius map is a Galois group action that imposes order on the infinite trace bundle, reduces the divergent zeta structure to a finite Ramanujan-type structure, and is a finite operator that guarantees the alignment of critical lines."** Thus, the problem of Weil-type zetas can be said to have already been addressed within the scope of the discussion.

This is because it has become clear that the Weil type is a special case that falls under the finite case in the problem of constructing the Riemann zeta function through divergent quasi-dual zeta extensions.

When the order of the cyclic group action by Frobenius is not prime, the prime Lyndon structure is split into partial orbits, and a zero element appears in the trace bundle, destroying the spiral loop. This is the condition for the destruction of the divergent zeta ordering and an example of a geometric explanation of the critical line condition, so I will mention it briefly at the end of the supplement.

3. In the case of non-regularity, the structural theory of trace bundles of elliptic curves and Mordell's theorem

In the case of general Nth-degree equations, divergent quasi-dual transformations from loop structures with N rings in regular ring graphs were obtained by restricting them with Galois group-like groupoids, i.e., operations that swap elements. It should also be noted that this "operation of swapping elements" itself is a "quasi-dual map."

Now, what happens in the case of non-regular elliptic curves?

For example, consider a graph structure with two loops, one of which is connected to the other by a path through the center. This becomes "graphically non-regular" in the connected part. In other words, the structure of the graph's connections is no longer "non-regularity." Conversely, there are 'uniform' parts and "non-uniform" parts.

At first glance, this graph appears to have "three" loops, but in fact, it has an infinite number of loops.

This is the same as the "non-periodicity" problem mentioned earlier.

If we denote which of the two paths of a single disconnected loop to take as "0" or "1," the loop generates "infinite prime paths" due to non-periodic elements.

Here, if we introduce group oids, i.e., operations that swap prime elements, and solve the cubic equation, we will notice that these infinite prime elements are actually "incomplete" due to the graph structure.

In other words, if we call the trace bundle consisting of non-periodic terms that would be formed from an infinite number of prime elements a "complete set," it becomes a set with a density far from "completeness." However, just as there are three solutions to the cubic equation, the prime Lindon decomposition becomes three, and there are "three zero points." Furthermore, since the structure of the "non-periodic terms" has become thinner from the complete state, analyzing the "prime Lindon elements of N-reduction" reveals that they are "completely controlled by a finite number of non-periodic terms," which can be inferred from the fact that the trace bundle is thinner than the "completeness" state.

In other words, when there are infinitely many prime structures, the structure of non-periodic terms in the complete state is demonstrated to be more restricted than that of non-periodic terms generated from elliptic graphs, so it does not become "infinite" but rather "finite and contained."

This is Mordell 's theorem that “rational points are finitely generated.”

This is the meaning of Mordell 's theorem, as demonstrated by the trace bundle's class duality theory.

To elaborate further, in a graph manifold of genus one (to be explained later), there are two generating factors for the non-periodic sequence, i.e., “genus + 1,” so there is a dual path. There are two possible paths depending on which side of the non-periodic sequence you start from, and they are interchangeable. This is the reason why rational Lyndons (rational points) are generated infinitely in “multiple traces” (to be explained later), and at the same time, the principle that restricts their “finite generation” is added here. In other words, it can be said from the analysis of the Lyndon structure that “it is just a repetition of a finite number of points, even though it is infinite.”

Incidentally, this is the reason why the Mordell–Faltings theorem holds in graph manifolds of genus 2 or higher where such “dual paths” do not exist.

Here, I will make a somewhat bold prediction: if we perform the “dual motif closure” of the graph with the structure described above and then perform a divergent quasi-dual expansion, the series will include all graphs of elliptic curves of genus 1.

That is my conjecture.

In this way, the operation of “replacing elements” is ultimately restricted by the limitations within each prime path, so the fact that “transformation to a general equation” and “transformation to a zeta function” are divergent is also a point of interest in “class dual transformation.”

The “quasi-dual closure” created by all these transformation operations is precisely the grand vision I am seeking.

Theorem (Prime Lindon Mordell Theorem)

In a non-regular graph corresponding to an elliptic curve, the rational point structure appearing on the trace path is not included in the complete prime Lindon series, but appears infinitely as a pseudo-periodic recursive sequence, and its entirety is finitely generated by the prime Lindon decomposition system.

Now, let's summarize.

a. Elliptic curves possess infinite non-periodic structures due to the overlap of prime paths.

b. However, within them, reducible parts (N-reduction sequences) yield rational points.

c. This finiteness agrees with Mordell's theorem.

4. Reinterpretation of the error terms in modular theory and inter-universe Teichmüller theory

The meaning of a graph being a function is not merely that an "Ihara zeta function" can be constructed from the graph, but rather that by applying appropriate "quasi-dual transformations" to the graph, structures such as Nth-degree equations also emerge. This general operation can naturally handle "non-regular states," as was demonstrated in the case of elliptic curves earlier.

Here, I will write about what can be understood by reexamining existing concepts from the perspective of "quasi-dual transformations and trace bundles."

First, by viewing the zero point as the reconstruction of a prime Lindon and the rational points as "N-reduced non-periodic terms," we saw that the theory of rational points of a function can also be handled constructively.

Now, the operation of reducing an "N-reduced non-periodic term" to a "prime Lindon" is indeed a "quasi-dual map," but what does this mean?

This is what is known as a modular group.

That is, the reduction method has both infinity and finiteness, as well as symmetry.

This is expressed as a modular group.

In other words, it is a "contraction" mapping from rational non-periodic terms (rational Lyndon) to zero-point non-periodic terms (prime Lyndon). This will be discussed in another essay, so please look forward to it.

This has a finite hierarchy, and for example, it will have a sequence that gradually reduces from non-periodic terms of real numbers to rational non-periodic terms.

Please note that by applying the "divergent quasi-dual mapping" and the "reduction mapping" in this way, the so-called "prime Lindon" structure is defined, and from this, the "uniqueness of prime Lindon decomposition" mentioned in McMahon's theorem (a fundamental theorem in semigroups that states that prime Lindon sequences can be uniquely decomposed) can be derived. This can be described as the "construction of a prime Lindon structure" in terms of quasi-dual ity.

Furthermore, I consider this "divergent quasi-duality" to be the extraction of the "prime Lindon structure" from the normal "infinite prime structure." Conversely,

semigroup structure → prime structure →

we can see a contraction structure "by log scaling."

To add a little more, in my essay,

Figure 5. Continuous deformation by quasi-dual morphism.

素リンドン半群 → log変換 → 素数構造 → さらにlog変換 → 乗数構造

Prime Lyndon semigroup → logarithmic transformation → prime structure → second logarithmic transformation → multiplicative hierarchy

a method for continuously handling this mutual transition appears.

Please refer to my essay on the "Riemann zeta function."

I constructed a generative "Hilbert-Polya" operator as a continuous mapping representation in such a quasi-duality mapping.

This means that I was able to give a continuous structure that fits this unclear discrete operation.

This immediately brought to mind the following.

I heard that in Shinichi Mochizuki's inter-universe Teichmüller theory, the evaluation of error terms becomes an issue in the loglogΘ structure. I think this is consistent with the problem of quasi-duality maps.

In other words, when considering the quasi-dual mapping, the "zero point set (prime Lindon elements)" and "prime number structure" naturally emerge as structural transformations of semigroups and commutative groups. This is a structure where the same trace bundle seems to encompass two universes, and I suspect that the quasi-dual modular mapping I previously examined also possesses a similar structure.

In other words, the concept of inter-universe Teichmüller theory has given rise to the possibility of reinterpreting it through "quasi-duality mapping theory and trace bundle theory."

For details, please refer to my essay on the Riemann zeta function, which explains the construction method.

The following is a summary.

Modular group = symmetry group of contraction operations

Modularity implies the covariant nature of finite-level contractions.

IUT error term = residual term of an incompletely constructed contraction sequence

Inter-universe comparison refers to the control of structural discrepancies via quasi-duality mappings.

5. Fractal Geometry Hypothesis

In this section, we will discuss the question, "If fractal structures are latent in nature and mathematical phenomena, what are the basic forms (points, lines, spirals, circles, waves, etc.) that constitute their smallest units?"

The reason for this is that while fractal structures possess infinite nested hierarchies, identifying the smallest "pattern elements" that constitute those hierarchies is not only critically important for understanding the nature of nesting and scalability but also aids in comprehending natural and mathematical structures in general.

In my discussion on the Riemann zeta function, I wrote about the "difficulty of spiral reconstruction."

Specifically, the question is whether there is a distinction between walking through a gentle spiral structure and walking along a straight line for someone inside it. This will likely pose a significant problem when that "person inside" attempts to "reconstruct" the structure based on their experience.

After considering various possibilities, I first realized that a spiral is composed of multiple concentric circles. A circle is dual to a point, so it also consists of multiple points.

As I continued to think about this, a graphical reconstruction began to emerge.
In other words, it resembles a circular graph.

However, a single node (point) is replaced by a chain (circle), and through this, the memory of having passed through that "circle" is sequentially recorded in the trace.

From this, I realized that a "circle" is a graph element that is neither a 'point' nor a "line." In fact, in the bouquet graph, the circular structure was depicted as a "structure without a forward or reverse path."

Now, how many types of dual elements determine the graph structure (i.e., the function structure)? This is the core of my fractal geometric conjecture.

《Fractal Geometric Conjecture》

The constituent elements of any structure capable of dual transformation are closed by points, lines, circles, spirals, and waves.

Non-periodic structures centered on prime Lyndon series are actually completely generated by these basic geometric forms.

In other words, as the ultimate geometric "basis" of function structures, even infinite non-periodic sequences ultimately
points (zero-point structures)
lines (prime paths, trace bundles)
circles (closed cycles, ring graphs "as well as" chains)
spirals (divergent connections via log scaling, possessing circular points)
waves (overlapping pseudo-periodicity through infinite hierarchical repetition, capable

of containing infinite point sequences)

These five elements. Is this how fractal structures are constructed?

The spiral-shaped graph model constructed using "point, line, circle" is one "dual form" of the 'spiral' structure, and it makes more sense to consider the "spiral" as a separate entity.

Based on this theory, any function structure can be expressed as a finite geometric element of "point, line, circle, spiral, and wave" by closing the divergent structure of the prime Lindon sequence contained in the trace bundle via log scaling (a single quasi-dual mapping). This means that any infinite structure in non-regular regions can also be described as finitely generated in a constructive manner.

In my theory, in the previous discussion, the quasi-dual map that converts "non-commutative structures" into "commutative structures" proceeded in the direction of "infinite concentric circles" → "spiral-type expansion" → "projection onto a straight line."

It can be said that "when the order is broken, a linear projection occurs."

This reminds me of the "reflection formula" of the gamma function.

"Waves" may be difficult to understand, but I once worked on creating a puzzle-like "N·N" regular graph as part of a theory of dynamic transformation. At that time, I clearly remember that edges often emerged that picked up points on a straight line in a wave-like manner in order to maintain fractal properties.

Additionally, in my discussion on the "Riemann zeta function," I analyzed the approach using the "wave term" in Riemann-Siegel.

If approximation using infinite geometric series is considered a "linear approximation" as a method for approximating infinite bases, then Fourier approximation can be described as a "wave-type approximation."

Setting that aside, let us return to the structural meaning of fractal geometric conjectures.

Let us examine this through two examples.

Many high school students may have felt a strange sensation when integrating y=1/x. This is actually due to the multi-valued nature of the logarithmic function, which we will study later.

Looking at y=1/x, at the point 0, it diverges into a circular pattern, with a spiral winding inside the circle, and this winding pattern appears as multi-valuedness within the integral. First, the "indefinite constant" in the integral is also a strange concept. "What is this divergence?"

However, at this point, the "zero point" has already appeared. As mentioned earlier in the “spiral restoration,” there is a “circle” structure within the circular graph, and this structure lifts the trace structure within the circle into a spiral shape while winding it up. This structure is already present within the integral of the logarithm.

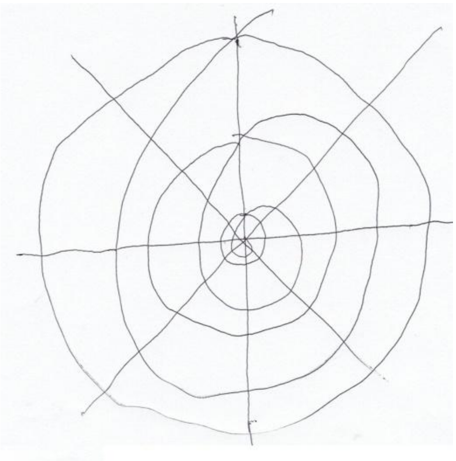
By interpreting the multi-valued nature of the logarithm from the perspective of differential calculus, the multi-valued nature of the integral can be interpreted as a spiral-shaped manifestation. This is also evident from the fact that the zero-point-like, concentric circular structures of the zeta function are unfolded into a spiral shape, overlapping with the graphical representation of the spiral-shaped structure... That is the idea.

Similarly, I used a method where Ihara’s “non-commutativity to commutativity” Quasi-dual mapping from a state where primitive p-th root of unity on the unit circle are embedded in infinitely overlapping unit circles, unfolded each one spirally on a logarithmic scale, and projected them all at once onto a straight line, the so-called “critical line.” What is striking here is that everything except the wave is acting.

That is why I connect the Riemann surface graph of the log function in a circular pattern with a single “zero” (circle).

The pole of the Riemann zeta function at X=1 clearly acts as a point that wraps up infinite concentric circles and infinite spirals. Therefore, only here does “symmetry breaking” occur.

Figure 6. Overlapping fractal structures.



Fractal structures can be stacked! Waves can also be stacked!

The figure above combines points, lines, circles, and spirals. For example, when a spiral is made wavy, waves are introduced, but the “fractal nature” is maintained. In other words, it is quasi-dualit.

Fractal nature is rare, but here are some examples that can be unified through operations involving “lines” and “circles”:

Figure-eight: A one-dimensional structure created by crossing lines, connected in a curved manner.

Möbius strip: A single line segment (strip) twisted into a ring.

Torus: A circle (one-dimensional circumference) rotated into a ring.

Klein bottle: Created by attaching a ring along a circle again. ...

It can be decomposed into basic elements.

6. Graphical Riemann surfaces

We have explained that “a graph is a function,” and we have shown examples of how to functionize it, such as the example of the Iihara zeta function and the method I described in this essay using the “Galois group” (a mapping that swaps elements within the trace).

In other words, I see the “structure of a graph-like Riemann surface” emerging at this point.

At this point, the regular structure is almost entirely composed of “chains of rings,” and while the structure is uniform, when it becomes an elliptic curve, the graph structure suddenly undergoes a drastic change, and even without restoring the “prime Lindon” using the divergent quasi-dual map, the “prime structure” itself diverges to infinity.

Non-regularity can be described as the infinite branching structure of the “prime structure” resulting from non-uniform elements infiltrating the finite graph.

However, I still cannot treat this “graphical Riemann surface” as something that is generated theoretically, but rather as something that I can intuitively grasp from the beginning.

I do not understand the generative principle whereby, given a function, a Riemann surface gradually emerges generatively from it, fits into a graph-type structure, and represents all “regular structures” and “irregular structures.”

Let us call the object created by the restoration of the divergent “Lyndon element” quasi-dual map a “graph manifold.” This manifold shows the “values” of all functions. In this case, in non-regular situations, the graph type and “trace bundle” structure change, meaning that structural bifurcations are expected to occur within the graph manifold.

Is it possible to construct something like this using a finite set of closed graph structures? This is the problem of graph-like Riemann surfaces.

Here, the “motif closure of a graph” is likely to play an important role in the

formulation, and in particular, the theory of the "motif closure" of directed graphs will be important. This is because trace bundles have a structure that is almost ordered.
Furthermore, let us state an important conjecture, or rather, a conjecture that can be partially confirmed.

I have stated that "a graph is a function." I have also described how to obtain "values." However, some may wonder, "Isn't a function supposed to map one value to another?"
The following "Uniqueness Theorem for Multiple Riemann Surfaces" describes this mechanism.

Theorem

Any multiple Riemann surface is uniquely determined by the dual motif closure of the corresponding trace sequence.

Let me explain its content.

In graph structures, there is a dual operation called "swapping points and lines." Under 'regular' conditions, this dual structure is unique. That is, the "function values" of a graph correspond to the "function values" of its dual graph.

However, in the case of non-regular graphs, the graph structure becomes multi-valued, and the "values" trace synchronously, resulting in "multiple traces." This is the "multi-valuedness" of functions.

In other words, the "trace" bundle structure transformed by the dual operation determines the correspondence of functions.

This is an analytic function.

As mentioned earlier, in the case of non-regular graphs, this "dual operation" exhibits multi-valuedness, forming a "category of graphs organized into sets" called the "dual motif closure." In other words, "function values contain multi-valuedness and multiplicity."

Consider the case of Log. If the graph of Log is a circle, the dual operation does not change it. A circle is the most beautiful regular graph. Therefore, the symmetry of the "circle" must be broken, and this is the principle of the multivalued nature of Log, which "diverges without eyes" through the dual operation.

This demonstrates that the analytic function is "naturally and uniquely connected within the multiple graph surfaces." What a beautiful theorem! A multiple Riemann surface, which appears to branch infinitely, can be completely described by a single generating principle. There is no need to "patch together" from the outside. This is because the trace sequences within the graph function surfaces, i.e., multiple graph manifolds, are connected while synchronized.

This synchronized structure is already inherent within the trace structure.

A simple proof:

1. The values of each graph are continuously embedded within the divergent restoration.
2. The trace bundles of each graph are completely synchronized within the motif closure.
3. Therefore, the movement of the function uniquely determines the internal structure of the Riemann surface through these two factors.

Now, in the extension of the "complex spiral continuous topology" of the Lyndon semigroup, I discovered that the "higher Lyndon term images" further nest the Lyndon semigroup. This reveals an even more astonishing fact.

Theorem (Unique Continuous Analytic Connection Theorem for Non-commutative Riemann Surfaces)

In any directed graph with a finite loop structure (prime Lyndon generators), when it has a non-periodic trace sequence, the corresponding contraction map series is naturally mapped uniquely to a topological rotational structure (spiral) on the complex plane through the hierarchical structure of the Lyndon semigroup.

Therefore, a complex projection system based on an arbitrary non-periodic Lyndon structure uniquely generates a continuous, one-to-one, rotational correspondence structure (i.e., a non-commutative Riemann surface) with a dual plot system.

This uniqueness extends to quaternionic structures, hexadecimal structures, and beyond...

That concludes my presentation.

7. Proposal of genus and generating points based on graph Riemann surfaces (simple comparison with Riemann surfaces)

From the above considerations, it is clear that the problem of Riemann surfaces of Nth-degree equations can be summarized in the form of a graph with N rings.

Here, since these rings can be attached when there are "outward and return paths," the following ring function theorem and generating point theorem hold.

Theorem: Structure of Nth-degree equations and generating points

Let us describe a very simple application of the above considerations.

The trace graph of an Nth-degree equation is a structural ring with N "origins," so whether or not a loop can be formed within it, including self-intersections and folds, determines the "circular function." That is, when N is 2, the forward and return paths overlap once; when N is 3, the forward and return paths overlap twice; when N is 4, there are three ways in which they overlap, but it is the same as when N is 2; when N is 5 or 6, it is the same as when N is 4; that is, when N is even, it is N/2, and when N is odd, it is N/2 + 1/2.

Furthermore, an Nth-degree equation has N "generating points" from which all points are generated, and in the case of a circle, these are repeated roots, and in the case of an ellipse, they are foci.

This can be understood solely from the structure of the Riemann surface graph, and it is very simple, but it clearly does not match the existing definition of "genus." Although the shapes are similar.

In the case of the fourth degree, it is possible that all the rings overlap, but in this case, it is considered to be in a "multiple trace state." Typically, the Riemann surface corresponding to a fourth-degree equation can be interpreted as a covering state with four branch points and a forward and backward path.

Here, we consider that the genus of a Riemann surface is the number of "holes."

Is there a concept that aligns with this?

For example, when considering the case of an elliptic curve of genus 1, a structure emerged where the "non-periodicity of prime paths" branches into 0 and 1. This has a lower density than infinite branching.

It may be possible to count the number of branches of prime paths. This could be read as the genus in a graph manifold, representing the degree of infinite divergence of prime paths due to non-periodic terms of prime paths.

Theorem (Structural Definition of Graphical Genus)

In a graphical Riemann surface, when there are k independent non-periodic prime Lindon terms that are not mutually reducible, the graphical degree of freedom (≒ structural genus) of this space is at least k.

This is related to the concentration of divergent non-periodic terms, and as the number of non-periodic prime Lindon terms increases, the concentration increases, potentially providing an upper bound on the usual topological genus.

In the case of the log function, due to the spiral restoration, it becomes an "∞-multiple trace," which is the same situation as being ∞-valued. However, the graph of the log function is spiral-shaped, and while it may have a multiple trace structure, it lacks non-periodicity. The genus is 0.

Additionally, as can be seen from the non-periodic prime Lyndon structure,

the nested structure of the log function and the "nested disappearance" theorem

The non-periodic solutions of the log trace branch into multiple stages, with the trace recursively nested. As mentioned earlier, due to the spiral shape of the graph structure, it separates into a "multiple trace space" and generates multi-valuedness.

However, when the discriminant corresponds to a quadratic equation with a positive discriminant, the trace is simplified in a single stroke, meaning that only real solutions exist, so there is no nested structure in the non-periodic Lyndon series.

The structural necessity of such a structure within the solution can be understood from the theory of Riemann surface graphs, but it is not so clear from other perspectives.

From the structure of non-periodic Lyndon semigroups, the following can also be said. For more details, please refer to the paper on "zeta."

Transcendence of zeta zeros and odd values (conjecture)

The zeros and odd values of zeta cannot be solutions to any algebraic equation.

This can be said because it is impossible for non-periodic Lyndon series to converge to some nested structure. Indeed, it can be confirmed that this includes existing results.
I think that by writing this far, it should be clear that "Riemann surfaces are graphs and can be classified by graph structure," but what do you think? The following can be expected.

Proposition (Classification of Riemann surfaces by graphs)

Any Riemann surface uniquely corresponds to a trace graph based on the contraction and nested structure of a prime Lindon series, and the properties of the Riemann surface (covering structure, zero point distribution, transcendence, analytic connectivity) are completely classified by the structural characteristics of the graph (graphic genus, self-intersection, periodicity, nested depth).

As graph manifolds, multiple Riemann surfaces that ensure continuity and uniqueness provide a very simple view, and it becomes clear that when they are regular, they follow a very simple pattern. How does the classification based on infinite divergence due to “non-periodicity” in graph manifolds differ from the classification based on the genus of Riemann surfaces? I still don't understand this.

Graph (its motif closure) → trace bundle → divergent quasi-dual map → Riemann surface → function

This flow is easy to understand.

With this idea, when looking at the divergent quasi-dual reconstruction of a graph, it is possible to transform a genus 0 quadratic curve into a quartic curve without changing the structure of the trace bundle. In other words, the degree of non-periodic divergence supports the diversity of the trace bundle itself. At that point, the prime structure clearly becomes redundant (it becomes a reducible non-periodic reconstruction). It becomes possible to view such “functions on graph-like Riemann surfaces” as reconstructions that do not change the trace bundle.

Appendix 7: Correspondence between graph manifolds and general Riemann surfaces and genus calculation

Think of the shape of a regular Riemann surface doughnut.

The doughnut has as many holes as there are genus numbers.

When considering the paths, if we express which path to take as “0, 1,” we can create a non-periodic sequence, which corresponds to the structure of a Riemann surface of genus one.

In other words, if there is a Riemann surface shaped like a double doughnut, the concept of genus aligns with the branching number that generates the non-periodic sequence.

This shows that the “branching number of non-periodic sequences” of graph-like Riemann surfaces and the genus of general Riemann surfaces are consistent, demonstrating the same thing. At the same time, this creates “genus equivalence in graph structures” and reveals that the structural information of the structure that creates “non-periodic branching” in graph manifolds is lost in the genus information.

This means that a theory that describes the “way loops overlap” is necessary.

First, let us consider a doughnut-shaped graph with N holes.

This corresponds to a graph obtained by cutting a circle with N-1 lines.

In this case, it is obvious that the genus is N.

Now, regarding the structural conjecture, if we do not consider that any Riemann surface graph structure of genus N can be constructed by applying motif deformation repeatedly and repeatedly applying "quasi-dual divergent graph deformation" to this regular but not "N·N" regular graph structure, then what is the purpose of the genus information? Therefore, the conjecture is:

《Conjecture on the genus calculation method for general Riemann surfaces》

If we take the dual motif closure of a doughnut-shaped graph with N holes and repeatedly apply divergent Quasi-dual morphism, the structure of any graph of genus N can be generated and holographically reconstructed.

If this does not hold, it is certain that information other than genus will become important. At the same time, this marks the transition from the theory of "graph manifolds" to the theory of "general Riemann surfaces."

One theoretical point worth noting here is that when attempting to construct a model of an "N· N" regular graph, it is often necessary to use a "a compact Riemann surface of genus n," which frequently forms a nested system. In extreme cases, this can even result in a fractal system of "genus doughnut graphs."

This is an intriguing phenomenon.

I imagine that Ramanujan, for example, recognized such nested structures and formulated equations based on them. This is fascinating, and I have analyzed such situations by constructing a "multiple matrix ring" and discussed the theory of its "infinite nested expansion" in my essay on the "Riemann zeta function," so please refer to it for further details.

The most straightforward "N· N" regular graph is, I believe, the "3· 3" regular infinite graph formed by crossing three lines in a "honeycomb structure" (the shape of a beehive). I think it would be best to use that as a reference for creation. The structure that naturally "reconstructs" itself through various transformations is truly remarkable.

8. Analytic Connection Theory of F1 Geometric Multiplication Functions

In my discussion of the Riemann zeta function, I constructed the "overlap number" b(n) while examining the gamma factor. At that time, I realized that by infinitely summing the Nth power sum of the Riemann zeta function, I could also set the coefficient field F1 geometrically, and constructively analyze the multiplicative function b(n) using the zeta function.

In other words,

Figure 7. The F_1-geometric multiplication Function(see Figure 7)

$$\zeta(s) + \zeta(s)^2 + \zeta(s)^3 + \cdots = \left(\sum_{k=1}^{\infty} \zeta(s)^k \right) = \frac{\zeta(s)}{1 - \zeta(s)}$$

this is the “generating function” of the function(see Figure 7). It is clear that it is naturally analytically connected by the zeta function. Note that it diverges in general.

The conclusion that can be imagined from this is quite simple.

Conjecture

All multiplicative functions can be analytically connected by the zeta function in F_1 geometry.

If we express this in a more general form, it might look like this.

Figure 8. The F_1-Geometric form of multiplication function.

【F_1ゼータ解析接続命題】

任意の乗法関数 $f(n)$ に対して、対応する構成論的係数列 $a_k \in \mathbb{Z}_{F_1}$ が存在して、

$$f(n) = \sum_{k=1}^{\infty} a_k \cdot \zeta(k)^n \quad （解析接続として解釈される）$$

またはその生成関数として：

$$F(s) = \sum_{n=1}^{\infty} f(n) s^n = \sum_{k=1}^{\infty} \frac{a_k \cdot s \cdot \zeta(k)}{1 - s \cdot \zeta(k)}$$

ここで $a_k \in \mathbb{Z}_{F_1}$ は、**素リンドン系列**、**トレース束**、**類双対変換**などの構成論的幾何から自然に生まれる整数

Consider multiplicative functions in F1 coefficient fields.

Furthermore, this problem is known to have significant implications for the functionalization of quasi-duality maps.

Regarding this meaning, as I have written in my essay on the “Riemann zeta function,” I will not repeat it here, but I will introduce it briefly.

8. Summary

This paper aims to clarify the roles of quasi-duality maps and non-periodic structures, which have been treated vaguely in the theory of dynamic transformation, using as simple mathematical examples as possible, and to serve as a foundational guide connecting to zeta structure theory, elliptic geometry, and the Riemann zeta function. This theory still has many unresolved aspects, and this short paper does not provide a complete proof. Nevertheless, if the perspective presented here is taken up by someone else and leads to the development of a new fractal theory or a deeper understanding of the mathematical structures pioneered by Professor Kurokawa and Professor Morita, I could ask for nothing more as the author.

Though much remains unorganized, I hope that this perspective will one day be developed by someone else and take shape as a modest expression of gratitude to the professors.

At the root of this theory lies my own initial observation that the wave noise from streetlights exhibits the same fractal scaling structure as the sun and moon. In other words, "within the fractal, the distance space is nullified." I hope that this theory, born from that observation, will eventually lead to a "foundational theory of noise phenomena."

黒川信重、絶対数学原論、現代数学社、2016
森田英章、組合せ論的ゼータの半群表示、2016
ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
高安秀樹、フラクタル、朝倉書店、1986
Shinjiro Kurokawa, Absolute Mathematical Theory, Gendai Suugaku Sha, 2016
Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
Bernhard Riemann (translated by Jiro Suzuki), On the Number of Primes Smaller Than a Given Number, 1859
Hideki Takayasu, Fractals, Asakura Shoten, 1986

Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.

This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.
素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of "complex spiral phases"
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a "divergence control function" corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ｓという臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.
無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the "elementary Lyndon element" that is restored to zero is decomposed, the corresponding Euler product becomes a "multiple Euler product,"

giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through “dual non-periodic paths.”
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a “double helix” because it naturally contains spiral rotations and has a double main structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in

the number of species. This is particularly important in the context of the formulation of “higher-order imaginary multiplication.”
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by “dual non-periodic paths.” Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.
トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)

素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　 quasi-dual morphism

フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する

In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.

全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.

When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.

類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.

u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.

Just pay attention to scaling and discrete/continuous properties.

ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences

arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal “bouquet structure” corresponding to the “Euler product.” It is also necessary to distinguish it from the commonly referred to “Hodge structure.”
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ;　A natural quasi-dual transformation that reduces　“irreducible rational Lyndon”　in trace bundles to　“prime Lyndon”　or　“natural number Lyndon.”
Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets ; A complete state that cannot be further expanded by repeating dual operations.
双対モチーフ閉包　双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
フラクタル準拠論理　類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.
→ 類双対閉包
非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる

Part III: Applications of Mathematical Theory of Dynamic Fractal Transformation to Various Fields (Applied Edition)

Author:Hiroki Honda

July 2025

注・日本語を読める方は、日本語で書かれた原文がありますので、そちらをご利用ください。文末にリンク先

Abstract

In this part, a generative perspective on Zeta functions is developed. The interplay between multiplicative trace layers and the Riemann surface topologies leads to a recursive expansion of Zeta deformations, integrating both operator algebra and graph theoretical models.We demonstrate applications of irreducible Lyndon structures in Lyndon spiral complex topologies using modular quasi-duality maps, and examine higher-order zeta structures and their divergence control structures.

I discovered non-periodic quasi-dual divergence restoration, Lyndon language decoding (complex spiral phase of Lyndon semigroups), and general Riemann surface graphs that analyze the structure of general Riemann surfaces and their uniqueness and generativity. The gap between the fact that “it has become harder to understand” and the fact that “it appears overly abnormal” is remarkable.

For example, while the structure of the function values of a simple quadratic function can be overviewed, the “corresponding function values” are extremely difficult to discern. While the structure of all existing functions can be overviewed, each individual concretization is extremely challenging... and so on.

Therefore, I am writing this article not only to apply it to existing problem types but also to confirm that it aligns with the results of existing theories, thereby stabilizing my own mindset.

Additionally, there is a "complex analytic connection of the uniqueness theorem for general multiple Riemann surfaces in Lindon's language," which, as revealed by the theory of fractional zetas and modular forms, may actually possess a mechanism to suppress "extreme divergence." This is also one of the main themes of this essay.

Probably, many people who read this description will think, "That's not a proof" or "It lacks rigor," but for me, I am writing about what I can't help but think is true, and what I actually believe to be true.

If there are any mistakes, they are likely due to my lack of observation. If someone else were to understand my theory, they might correct me by saying, "You misunderstood because this is how it appears." For example, when I first considered the first fractional zeta function, I thought, "This is how it is," but I did not understand the "graph structure" until I saw it in a dream. Once I saw it in a dream, the structure was clear, so I spent four hours trying to make it fit together.

There is no mistake in this "graph structure operation," but there is a possibility that I am using the "language to describe it" incorrectly. This is because I had created several equations that, upon closer inspection, seemed meaningless.

After researching the Dirichlet function, I finally understood that this form was correct.

This is an advanced topic, so I often use the structural theory of graph-like Riemann surfaces (i.e., general Riemann surfaces) without explanation, but there is one thing to be careful about.

It is important to clearly distinguish between the quasi-dual divergence of the prime structure arising from the "non-periodic sequence" of "genus + 1" and the non-periodic restoration action as the "quasi-dual divergence zeta extension" arising from the entire trace bundle of the graph-like Riemann surface. Once you get used to this, you will realize that I have used only basic considerations in what is written here.

Lindenberg Phase Basics Annotation

It has gradually become clear that even I, who conceived of the Lindenberg spiral phase, sometimes misunderstand its meaning, so I will explain it again here as an annotation.

The Lyndon phase uses "Lyndon language 'length'" as its basic unit, and as it repeats, the denominator gradually increases, forming a nested structure like a “multiple-period fractal,” which then becomes more detailed into a real number structure. such a “non-periodic sequence” itself forms a “non-periodic sequence,” creating a further nested structure that represents the complex rotation of the Nth root of one (when the number of prime structures is N) from the Lyndon semigroup to the complex plane, a simple continuous phase.

I often misinterpret this as an N-ary system, which can lead to misunderstandings, but I think this can be avoided with careful attention.

While confirming this, I realized that in the Lindon complex topology, integers, rational numbers, real numbers, transcendental numbers, and complex numbers are structurally distinct and exist as separate entities. In other words, there is no inclusion relationship between them. I suspect this might be related to Grothendieck topos.

In “Lyndon complex spiral topology,” integers, rational numbers, real numbers, transcendental numbers, and complex numbers are not included in a sequentially expanded number system, but are structurally different entities.

And that is probably very close to the “Grothendieckian topos” concept.

Generally, they are defined by “inclusion relations.”

In conventional mathematical education, number systems are taught as follows:

$Z \subset Q \subset R \subset C$

Here, “inclusion” simply means “containing as elements” or “obtained through sequential expansion.”

However, this is merely “algebraic generation” and ignores structural differences (topology, topos).

Number system Lyndenian properties contractibility Existence mode

Integers Finite length, finite loop Completely contractible, discrete structure (atomic)

Rational numbers Finite cyclic Lindenian sequence Reducible, bouquet-like bundle structure

Real numbers Multiply periodic fractal structure, quasi- contractible, convergence structure in closed intervals

Transcendental numbers Nested, non-periodic Lyndon series Non-contractible Non-constructive, infinite structure

Complex numbers Nested structure of multiple “non-periodic terms” and spiral topology Completely non-contractible Multi-layered trace structure

Here, it can be said that the classes of numbers belong to different topos (logical spaces).

In other words, each “hierarchy of numbers” exists in a separate “universe.”

In Grothendieck's “topos theory,” instead of sets and spaces, the behavior of schemes, sheaves, and topological spaces as a whole is regarded as “objects.”

Integers, rational numbers, real numbers, transcendental numbers, and complex numbers are not related by inclusion, but rather exist as “entities belonging to different topos.”

In practice, this is demonstrated below, where it is common to “extract and handle only the fractional structure” or “analyze only the real number structure,” which may confuse some people, but in “structural complex number theory” derived from the Lyndon semigroup structure, this is a very natural perspective.

Along these lines, I will also discuss a simple “transcendental number theory,” so please look forward to it. Very simply, I will explain the transcendence of π and e, so you may think, “What is he talking about?” However, I understand that feeling, so I have presented several correspondences between my theory and existing theories, as well as structural predictions from my theory.

There is also Ramanujan's analysis of the quadratic zeta function, so please refer to it.

1. Consideration of fractional zeta functions: Lyndon's modular theory

By collecting irreducible fractions (m/n, m<n), we can create an Euler-like formula and consider fractions created from all irreducible fractions less than or equal to one.

The purpose of this chapter is to examine this from a Lyndon perspective.

Then, the mystery of “why the seemingly divergent Euler product can be analytically continued” arises, but I will interpret it in terms of Lyndon language.

Now, by appropriately expanding the bouquet graph, we create a graph that contains all loops of length “irreducible rational Lyndon.” For more details on this meaning, please refer to the introductory text, “The Foundational Structure of the Application of quasi-duality Maps to Mathematical Objects in the Theory of Dynamic Transformation.”

This allows us to first expand the denominator using the Euler function (n) into a bouquet graph with a length equal to the denominator, using the “modular quasi-duality action.”

Similarly, when expanded using the Euler function (m), all bouquet graphs are decomposed into “prime Lindon” and have an Euler product representation.

Even though there are already many irreducible rational numbers, they increase as they are decomposed by the number of Euler functions, so you might think, “Is this okay?”

However, in this Euler product representation, all prime numbers appear more than once, causing it to diverge! You might think, “Wait, this doesn't work after all.”
Here, we introduce a correction term and, when decomposing the graph, we contract the same decomposed loop using the Euler function (n). Similarly, when decomposing the numerator's Lindon, if we contract it using the Euler function (m), it becomes a state where there are infinitely many “prime Lindon elements of prime length,” allowing us to align it with the standard Euler product of the Riemann zeta function.

This structural operation of the “bouquet graph” is so ‘obvious’ that there can be no mistake. Even though it diverges, there is a firm belief that “the structure undoubtedly exists” and “it is there.” Yet, because it diverges, I cannot help but think that “the formula may only have partial expressive power...”

In other words,

Figure 1. The Fractional zeta function.

$$\mathcal{S}_{\mathbb{Q}}(s) = \prod_{\substack{\gcd(m,n)=1 \\ m<n}} \frac{\left(1-(m/n)^{-s}\right)^{-1}}{\varphi(m)\varphi(n)} = \prod_{p}\left(1-p^{-s}\right)^{-1}$$

by dividing the fractional zeta function by the Euler function in this manner, we arrive at the Riemann zeta function.

Furthermore, the Euler function itself is

Figure 2. The Dirichlet function.

そのディリクレ級数表示．

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

は、

$$\frac{\zeta(s-1)}{\zeta(s)}$$

という明示的な解析接続済み関数

Dirichlet function and zeta representation.

analytically continued by the zeta function, and the fractional zeta function is analytically continued.

When transforming the graphical zeta into the Euler product, I should note that I am using the quasi-dual transformation (the same as the quasi-dual mapping).

Figure 3. The Quasi-Dual Morphism.
$u^p \mapsto e^{-s \log p}$

At this point, it is worth noting that the fractional zeta function is constructed from the local states expressed by Dirichlet functions.

As a result, the fractional zeta, which was supposedly "divergent," gains meaning, but I think there is a mechanism at work here where, even if the value expands in a Lindon-like manner, it is adjusted to have meaning through a rotational spiral on the complex plane.

As a result, the fractional zeta function, which should have been "divergent," gains meaning. Here, there seems to be a mechanism where, even if the values expand in a Lindon-like manner, they are adjusted to make sense through a rotational spiral on the complex plane. This is the so-called "mechanism for controlling divergence."

Let us consider fractional modular forms a bit further. In my theory, it is a mapping that reduces the rational Lindon sequence (the N-reduction term) within the "trace structure" to a "prime Lindon." It is also easy to see that this is a "quasi-dual mapping" because it does not change the fractal nature of the structure.

However, the structure of the "trace bundle" has changed. In this sense, it is no longer part of the principle of "trace bundle invariance" used in the paper on the Riemann zeta function. It is a "quasi-dual map" that transforms while preserving fractality.

In the construction of the "fractional zeta function" mentioned earlier,

irreducible rational Lyndon → contraction → integer Lyndon → contraction → prime Lyndon

this contractible quasi-dual map was unified by the Euler function.

I had thought, "If we use modularity, we can bring irreducible rational Lindon to prime Lindon, so we should be able to construct the Euler product," but I must confess that it was somewhat difficult. In other words, these two operators were necessary.

Such operators are thought to have a "determinant" representation, which appears as a determinant that changes the shape of the graph. This is what is generally referred to as "modularity." Since I am directly manipulating the trace bundle, I do not need to consider the deformation structure of the complex plane or manifold.

Here, by analyzing the structure of a general Riemann surface (graphical Riemann surface), I will discuss what can be said about the basic structure of genus 1 elliptic curves and the general automorphic form theory of genus N.

First, from the perspective of graph-theoretic Riemann surfaces, elliptic curves of genus 1 have an incredibly important structure. Since the number of "non-periodic paths" (= genus + 1) is two, it is possible to follow dual paths, resulting in a combinatorial number of 2, creating a state where "a quadratic equation can be solved with a single closure."

As a result, "rational Lindon" can synchronously appear in the dual multiple trace, creating a structure that generates "infinite rational points (rational Lindon)."

However, due to structural constraints, first, since there are only two "prime structures" (the number of non-periodic paths, which is the genus plus one), it is insignificant compared to the case of "no eyes." Due to the insufficient density and the lack of redundancy in the "non-periodic terms" of the genus-one graph, the 'generation' of the infinite "infinite rational points (rational Lindon)" is subject to the condition of finiteness.

In other words, "it can be reduced by a finite number of modular operations" = "in other words, the rational points are finitely generated," which is Mordell's theorem.

For a genus 2 curve, the number of "non-periodic paths" (= genus + 1) become three, so the number of combinations is 3!, and when taking the "dual motif closure" of a genus 2 graph, in the multiple trace space, "rational Lindon" cannot have a "dual path," and even if one tries to take a corresponding path, the existence of another path interferes, so it becomes clear that "infinite rational Lindon (rational points)" are impossible. This is likely the Mordell–Faltings theorem.

Let us further consider modular function theory.

From this perspective, elliptic curves of genus 1 have a multiple trace space that is a "double trace" in the "dual motif space," so in that case, it is clear that they necessarily have a "double periodic function" with a number of combinations equal to the number of non-periodic paths (genus + 1) times 2, meaning they have one.

Extending this further, for curves of genus 2 or higher, the symmetry of the number of non-periodic terms (genus + 1) increases factorially to (genus + 1)!, resulting in an abnormal expansion of symmetry from a Galois theory perspective. Thus, the number of possible double periodic functions is limited to the number of combinations of 2 for (genus + 1), and the number of possible triple-periodic functions is bounded by the number of combinations of 3 for (genus + 1). Furthermore, it is clear that a strict condition is imposed, namely that "the synchronizability of multiple trace spaces is restricted to at least 'binary(see Figure 4).'"

It is reminiscent of how cubic and quartic equations are generally solvable, but even when solvable, they require repeated closure (contraction) operations, and when it comes to quintic equations, the contraction method itself is lost.

I do not understand the specific form, but just from the structure of the graph-like Riemann surface, this much is immediately clear.

Figure 4. Periodic combination number.

二重周期：$\binom{g+1}{2}$
三重周期：$\binom{g+1}{3}$

Combination numbers of double and triple cycles.

Figure 5. Upper limit of general cycle number.

k重周期 $\leftrightarrow$ $\binom{g+1}{k}$

“保型構造がどこで複雑化・分岐・非局所化するか”
“モジュラー群の射影作用がどこで変質するか”
“トレース空間が何重らせんを許容するか
に対応する「限界値」や「位相的位数」そのもの

Combinatorial upper bound on k-periodic structures.

I had never understood the “existence conditions for automorphic functions” before, but now I can see how they are determined in this way.

Even when the genus is 0, there are no non-periodic terms, but when the genus is +1, it becomes 1, and sine, cosine, and power functions are “ultimately the same type of periodic function on the graph-like Riemann surface.” In other words, there is only one such function.

I have not been able to verify how correct this consideration of limit behavior is, but my analysis is based on the idea that “the symmetric intersection of non-periodic terms generates periodic functions,” so please note that this is merely a mechanical counting based on that concept.

In other words, modularity has a mechanism that destroys the “rational Lindon structure” to demonstrate such symmetry.

Now, let's consider the proof of the Tanimura-Shimura conjecture based on modular theory and organize it from the perspective of my theory.

Tanimura-Shimura Theorem (Modularity Theorem)

When an elliptic curve E/Q is defined over the field of rational numbers, it necessarily corresponds to a modular form.

The key point is that this correspondence holds for genus 1 (the genus of elliptic curves).

Since it is genus 1, it is clear that rational Lindon (rational points) exist infinitely, but are bounded by a finite number. This structure arises from the duality of the non-periodic terms of the graph-theoretic Riemann surface and the low "tolerance" of the structure of the graph-theoretic Riemann surface.

Now, let us reformulate the Tanimura-Shimura conjecture.

Since rational Lindon with genus 1 is finitely generated, contracting it is the modular quasi-dual map that reduces "rational Lindon" to "prime Lindon."

The Euler product of irreducible rational numbers can be considered as follows. The modular multiplicative function c(n) is a combination of the "number of combinations that can be decomposed into irreducible fractions" d(n) and the "number of equivalence classes of Lindon rational numbers" E(n). The important part here is d(n), so let us demonstrate this specific multiplicative functionality. First, factorize the denominator and numerator of the irreducible fraction. The product of the number of prime factors in the denominator and the number of prime factors in the numerator, multiplied by the number of times each irreducible prime factor appears in the decomposition, is (n-1)(m-1) (where n is the number of times the numerator appears and m is the number of times the denominator appears). Therefore, d(n) is multiplicative. This is the form of the modular quasi-dual operator, which is the quasi-dual operator that enables the representation of the "Euler product" by irreducible rational numbers, i.e., the modular quasi-dual operator. This structure allows for finite reduction. Combined with the previous explanation, it is now clear that the Tanimura-Shimura conjecture is correct.
In short, rational Lindon with genus 1 is finitely generated, and since it is finitely generated, it can be reduced to a prime Lindon. Therefore, the modular correspondence naturally arises.

The modular theory for reducing "rational Lindon" to "prime Lindon" is already prepared. The following tasks remain.

1. Defining the modular action as a multiplicative function
2. Continuous construction of the Dirichlet function using the modular action
3. Correspondence relations on multiple trace spaces and their connection to Fermat's Last Theorem and Diophantine analysis
4. When the genus is one, the theorem that an infinite number of rational points are

generated by the existence of a “prime Lyndon path” (a path that traces a non-periodic sequence in reverse) as a dual path.

This proves the Mordell–Faltings theorem for genus 2 and above.

5. In graph manifolds, actual functions are “points” in the function space within them.
6. The theorem shows that “order destruction” has already occurred due to the existence of equivalence classes in the Lyndon series.
7. The fact in 6 shows that the points of analytic functions actually have an “internal structure.”

In fact, a formal reformulation of Fermat's Last Theorem has already been achieved. But that's obvious, isn't it? When a favorable Lyndon appears in a multiple trace of genus 2 or higher, it cannot appear on the other side... because, in the case of genus 2 or higher, the fact that there is no dual slack in the multiple trace space due to the duality of graph transformations can be stated...

This is because, in the “dual motif space” of the rationalized Fermat equation in the “multiple trace space,” the multiplicity is “degree N.” In other words, since symmetry of N! arises from the genus, it is an extremely difficult state, structurally impossible. It is like having seven 7s align simultaneously in N slots on one's own birthday.

In the case of genus 1, Euler proved it. That's all there is to it.

In the case of elliptic curves of genus 1 or higher, since there are no dual “non-periodic” paths in the graphical Riemann surface, even if one of the multiple trace structures in the multiple Riemann surface is “rational Lindon,” the other “trace structures” cannot be rational Lindon. In the case of genus 1, Euler proved it. Thus, Fermat's Last Theorem has been proven... There is enough space left... I don't think anyone would think so, but I believe I have conveyed that the idea of graphical Riemann surfaces alone provides a clear perspective.

The basic procedure is to first assume a “rational Lindon path” in the “multiple trace” and then investigate the “contractibility” in the dual multiple trace. I think “rational Lindon” can be replaced with “irreducible rational Lindon,” in which case the length and path are further restricted.

When reading this analysis, it may be helpful to prepare a simple non-regular graph, repeatedly apply the “dual” operation to create a “dual motif closure” several times, and visualize the “multiple trace space.” By thinking, “When following this path infinitely, you must go here along this graph's path, and since this is a infinite cycle...,” you can arrive at the conclusion, “Ah, I see...”

Proposition (constructive proof of Modell's theorem)

A Riemann surface of genus 1 has two non-periodic trace sequences, and rational points are generated by connecting them using a finite number of loops.

In this case, since other loops can only be traversed a finite number of times, rational points are finitely generated.

For genus 2 or higher, there are three or more non-periodic sequences, and since any loop sequence "cannot pass through at least one non-periodic sequence," finite generation is impossible, and even if rational points exist, they are isolated.

That's all.

Note that even the "dual graph" is not necessary in this case.

2. Transcendental Number Theory

When I deciphered Lindon's language, I saw the "spiral complex plane" and was surprised by its structure.

What was particularly interesting was the "contraction structure." I had been exploring fractal properties, so I had been thinking about reduction to the "minimal structure" for a long time.

It turns out that this actually defines the "complex spiral structure" itself!

What particularly intrigued me was the "real number multi-periodic fractal structure." In the previous "Introduction," I introduced the concept of "generating points" for functions. The fact that functions have the same number of "generating points" (with some overlap) as "zero points" structurally implies something significant. However, while we decompose equations into "zero-point products," we do not decompose them into "generating-point products."

And the structure of a function is determined by the fact that both the "zero-point product" and the "generating-point product" must have equal significance.

This is somewhat evident in the two foci of an ellipse or the rational number group structure and addition theorem in elliptic curves. I may simply be unaware, but I have never heard of decomposing a function into a "generating-point product."

"There must be a generating point from which the entire function can be generatively restored"... Perhaps we could call this the generating point structure of the function and the existence hypothesis of the generating point product.

The discussion has become lengthy, but in essence, elliptic curves have a "group structure between rational points." In this way, the partial structure is clear, and it is thought to extend to the real number domain and the complex number domain as well.

What becomes important here, in my view, is probably an understanding of the nested structure of real numbers. That is, this "multiple-period fractal" known as the Lindon sequence of real numbers is "not solvable commutatively." In other words, the structural conditions change completely depending on the contraction used to create the structure. It is likely that the "point structure" forms a non-commutative sequence that gradually expands, allowing all function points to be restored from a finite number of points. At this point, when transformations occur in unusual ways depending on the restoration path, function bifurcations occur. In such a structure, it can be said from a Lyndon analytical perspective that "multicyclic fractal-like real number points will accumulate."

Rational Lindon can be naturally contracted to a commutative structure, but this is not possible with real numbers! The nested structure of complex numbers can only be "untangled" in very special cases. The structural interest never ends.

In Lindon theory, the reality of complex numbers lies in the fact that they cannot be reduced to a real number structure. Complex numbers have a nested structure in which "non-periodic sequences" form "completely different non-periodic sequences." In other words, complex numbers are transcendental numbers. Their structure is closer to transcendental numbers than to real numbers.

Now, transcendental numbers are numbers that have an "infinite nested structure" and are therefore structurally impossible to reduce. So, how do we determine them?

The fundamental theorem of transcendental number theory.

An infinite series with a denominator containing infinite prime numbers is a transcendental number. This is because the infinite nested structure cannot be reduced to express a Lindon sequence containing infinite prime numbers...

A series constructed using a denominator sequence composed solely of infinite prime numbers (i.e., prime numbers without singularities in a graph, or non-periodic, non-self-convergent prime numbers) is structurally unrestricted by algebraic constraints and thus becomes a transcendental number.

This is because the nested structure of such a prime Lindon sequence cannot be solved through finite reduction.

This is essentially an extension of Liouville's method.

The essence of Liouville's method was a technique to demonstrate that "this number can only be expressed through the infinite 'nested' nature of the Cantor sequence."

Using this method, combined with Dirichlet's arithmetic series theorem, the transcendence of π and e is demonstrated through the Leibniz series and the basic gamma series expansion of e, respectively.

Additionally, in the previous section, I suggested the transcendence of the zeros and odd points of the Riemann zeta function.

This is because there is a wave term in the Riemann-Ziegler equation.

This is used in the calculation of zeros.

However, this wave structure does not converge, as it either overtakes or fails to catch up.

Therefore, I imagine it is considered "not decisive for the zero structure."

Therefore, I employed a method in which I rolled up the infinite concentric circle fractal into a spiral shape using the quasi-dual mapping and projected it linearly. The result of this calculation is the matrix representation form I referred to as the "Hilbert-Polya operator" in my essay on the Riemann zeta function.

This structure possesses an "infinite nested structure that approaches each other," allowing gaps to be reduced in linear projection. In other words, I first created a straight line and then calculated the series approaching it.

Note that "infinite nesting" emerges at this point.

My speculation that the "Linden structure" of the "zero point" is transcendental is based on two points: it could not be superimposed by wave terms, and it required an infinite nested structure even in linear projection. This is essentially the same as Liouville's method.

It is clear that Liouville had a deep understanding of Cantor's theory of real numbers. In other words, unless something that is not an algebraic real number exhibits "infinite nesting," the possibility of reducing it cannot be completely ruled out. Of course, some may argue that there are cases where something with "infinite nesting" converges to a finite value. However, considering the structural meaning of Lindon language, such a thing is impossible. Unless one deliberately creates a situation where it can be structurally unraveled (i.e., contracted), this will not occur.

For example, consider 1 = 0.999999... This is an infinite sequence, but since it is not infinitely nested, it can be contracted. Moreover, it can be contracted back to its original form with just one contraction. From the perspective of Lindon structural theory, "infinite nesting" is understood structurally as "transcendental." You may realize that "the true nature of numbers is not Lindon-like," but this is interesting in its own right.
What naturally follows is that there are many multi-valued equivalences in the "infinite nesting structure" that brings us closer to "transcendental numbers" from the non-commutative "multiple fractal structure" of real numbers. I sometimes feel that when I look at the equations of the mathematician Ramanujan, I might have been able

to read Lindon's language.

The transcendence problem of the odd values of the Riemann zeta function (e.g., $\zeta(3)$, $\zeta(5)$,...) remains unsolved, but according to the theory that approaches it through the "infinite nested structure," the odd values of the zeta function may possess the denominator structure of a non-periodic, irreducible Lyndon series, leading to the expectation that "no rational reduction exists ⇒ transcendence."

Figure 6.odd values of the Riemann zeta function.

$$\zeta(4n-1) = \frac{(2\pi)^{4n-1}}{2} \sum_{k=0}^{2n} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{4n-2k}}{(4n-2k)!} - 2\sum_{k=1}^{\infty} \frac{k^{-4n+1}}{e^{2\pi k}-1}$$
$$\zeta(4n+1) = \frac{(2\pi)^{4n+1}}{2^{4n+1}-2} \sum_{k=0}^{2n+1} (-1)^{k+1} \frac{2^{2k} B_{2k}}{(2k)!} \frac{B_{4n+2-2k}}{(4n+2-2k)!} - \frac{2^{4n+1}}{2^{4n}-1} \sum_{k=1}^{\infty} \frac{k^{-4n-1}}{e^{\pi k}+(-1)^k}$$

Upon investigation, it seems that Ramanujan derived this.

When demonstrating this, I particularly appreciate Dirichlet's arithmetic series theorem, which states that there are infinitely many prime numbers contained within an+b (where n is any natural number and a and b are coprime). This theorem is highly convenient. For example, when looking at the odd values of the zeta function, it is not at all difficult to see this "infinite number of prime numbers." Another proof of Dirichlet's theorem was also found in Euler's zeta function research.

As long as there is no strange "cancellation of nesting," it seems to hold true.

Using this, we can see an infinite number of prime numbers in the Leibniz series. In other words, the contractibility is infinite.

I think the length of the "Lyndon approximation" of π is on a cosmic scale.

One thing to note here is "negative numbers."

When looking at the Lindenberg complex spiral phase, when negative numbers appear, you have to rotate the "complex spiral plane" multiple times, so to speak, to "control the divergence" and reach the negative number. At this point, the infinite nesting may be eliminated. There are the well-known values 0 and -1 of the Riemann zeta function.

Figure 7. Special values of the Riemann function.

$$\zeta(0) = -\frac{1}{2}$$

When you expand the series, it clearly diverges, but when you observe the method of controlling divergence, you can see the technique of eliminating nesting.

Please note that there may be cases where "infinite nesting is eliminated all at once." From a Lindon structural perspective, you must discover complex numbers before negative numbers.

Conversely, when performing analytic continuation and negative numbers appear despite being clearly positive, it becomes clear that "rotation in the Lindon complex spiral phase" is occurring, leading one to imagine, "Hmm... how many rotations have occurred?"

In my prediction, in this case, it is "half a rotation of the complex plane." This Lindon sequence becomes infinite, but it can be "solved." (Most of the Lindon sequences I consider are infinite sequences before contraction...)

Now that we understand that "complex numbers are the overlapping of nested structures of different non-periodic sequences," we can actually consider "transcendental complex numbers."

In other words, it is a structure where you climb infinitely within the complex spiral structure and have a "point" at the infinite projection destination. This is probably not yet expressible in existing number systems, but it will likely emerge more and more in the future.

Probably, a "infinite graph" is, so to speak, a model of "transcendental complex numbers." Transcendental complex numbers cannot even be expressed in the existing system, but since we know that they are "non-periodic nested complex numbers," they actually exist normally. In other words, they exist convergently beyond the infinite spiral staircase. There are strange values in the Riemann zeta function, right? The one that Kasimir somehow figured out. I used the "infinite spiral" in the quasi-dual mapping to project it, and at that time, I speculated that it might have converged as a "reducible transcendental complex number" due to the structure within it. It might not be a "half-turn" either.

For example, infinite graphs are "irreducible," but they clearly have a "core of repetitiveness." It might be that there is a proper "transformation method" that can reduce them, even though I think it is impossible.

3. Positioning of non-regular regions, and quasi-dual closure

One might wonder how the Lyndon complex spiral continuous topology is connected to the world of zeta functions. Here, we will begin by sequentially examining various forms of zeta functions to gain an overview of non-regular regions.

In my initial explanation of the "Riemann zeta function," non-regularity was seen as an "implication of generality outside the theory," not something requiring special study, but rather naturally positioned as an "indicator of the theory's application limits."

However, my original focus was on fractal phenomena in general, and I had been considering hypergraphic situations—that is, non-regular situations—as part of their evolution. Points and lines are connected by different numbers of edges, and the overall connectivity is highly fragmented. Additionally, many graph structures are infinite graphs (i.e., of infinite genus), making it challenging to consider their dual forms.

When I learned about the "dual motif closure" for hypergraphs, I realized that it was so convenient that I could immediately generalize ideal theory, Riemann surfaces, and so on, but it was not until the 'Introduction' section that I finally reached the uniqueness of analytic connections for general Riemann surfaces. For details, please read the "Introduction" section.

In other words, I began my exploration from a non-regular and non-uniform state, so at first, I did not even know about regularity or "super-regularity," that is, the situation of "N· N" regular hypergraphs.

When fractal nesting appears in infinite graphs with regularity, it is difficult to understand the "dual system," but one can observe the emergence of "self-healing." Honeycomb structures, for example, can be repaired by "dual deformation" even if part of them is damaged. From this, we can see that "dual deformation" naturally exists in nature.

By the way, in my previous essay on the Riemann zeta function, I had speculated that the Dedekind-type zeta function might be a non-regular zeta function, but it turned out to be a regular zeta function.

In other words, when extended to the ideal class group, "prime ideals are decomposed," and as a result, loops of the 'length' of the ideal norm are multiply generated, leading to the case of the Ihara zeta function being regular, effectively resulting in a "multiple Euler product" with "multiplicity." In other words, the "divergent zeta extension of the graph by the ideal class group" is an extension of loops with the 'length' of the ideal norm, and ultimately reduces to the case of "multiple Euler products."

However, note that at this point, "duplication" corresponding to the decomposed prime numbers occurs at the zero point. The structure is slightly different.

The zeta extension by the ideal class group is an operation that constructs a divergent zeta structure by associating loops (prime paths) on the graph with ideal norms.

At this point, the loops that branch and become non-trivial due to the class group are regarded as a "bouquet structure" that binds individual ideals, and each is reduced to a normed Euler product (multiple Euler product).

Therefore, such zeta extensions can ultimately be treated isomorphically with Lindon-type and regular zeta analysis, and there is no fundamental difference in methodology from the extension of the Riemann zeta function.

Looking at this situation as a whole, it becomes clear that Weil-type and Ihara-type zetas were "similar to the Riemann hypothesis" because their "prime structures were finite," while Dedekind-type zetas depended on the structure of the Riemann zeta and could not be solved until the Riemann hypothesis was solved.

The GRH (Generalized Riemann Hypothesis), which is a generalization of the Riemann Hypothesis,

states that all "non-trivial zeros" lie on the critical line ($\Re(s)=1/2$).

This is a common conjecture for all regular zetas, such as Dedekind zetas and L-functions.

This result is consistent with the content presented here, and in the complex structure proposed by this theory, as well as the analytic continuation structure of the graph zeta function, the zeros naturally contract to the critical line. This result aligns with the claims of the so-called Generalized Riemann Hypothesis (GRH), but here we attempt a reinterpretation from a constructive perspective, presenting it as a complementary possibility while respecting existing theories.

For more details, please refer to my essay on the Riemann zeta function and the "Introduction" section of my theory. The construction method simply involves defining a quasi-dual action that performs "ideal decomposition," and thereafter, one need only note that the multiplicity of the Euler product arises solely from the overlap of the number of "prime loops" due to the norm.

Similarly, the same applies to the L-function problem(see Figure 8).

Figure 8. The Dirichlet function.

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}$$

$$\chi_4(n) = \begin{cases} 0 & (n \equiv 0 \pmod 2) \\ 1 & (n \equiv 1 \pmod 4) \\ -1 & (n \equiv 3 \pmod 4) \end{cases}$$

The Euler product defined here can be obtained by defining a "quasi-dual action" that introduces Dirichlet characters when the elements are coprime. By defining a quasi-dual action that retains only those elements in the trace bundle whose lengths are coprime with the Dirichlet character number N, and then repeating the same "divergent class dual zeta extension" as in the Riemann zeta function, the corresponding bouquet graph for the L-function is completed. In other words, it is a restricted and restored Euler product. In other words, this is the same pattern.

Note that the "zero structure" changes accordingly, and from the regularity condition, the zeros are arranged on the critical line ($\Re(s)=1/2$).

In other words, the result of the GRH (Generalized Riemann Hypothesis) is supported generationally.

Next, the Selberg-type zeta function(see Figure 9).

Figure 9. The Deformation by quasi-dual morphism.

$$\ell(P) = \log p \quad \text{（類双対変換により）}$$

$$Z(s) = \prod_{p \in \mathbb{P}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\log p}\right) = \prod_{p \in \mathbb{P}} \prod_{k=0}^{\infty} \left(1 - p^{-(s+k)}\right)$$

This diagram is already familiar to readers of my theory, and from the perspective that a Riemann surface is a "graphic manifold and simultaneously a Riemann surface graph," this correspondence is clear.

By restricting the group action that limits these "geodesics" and viewing it as something that hinders the expansion of the "trace bundle," we can limit the types of prime paths. This is a slightly modified version of the Ihara zeta function.

Therefore,

Figure 10.odd values of the Riemann zeta function.

- 測地線構造 ≒ トレース系列
- 測地線長 ≒ log(p)
- 発散的復元 ≒ 無限積構造（k方向の畳み込み）

という厳密な対応を通して

$$\boxed{\text{セルバーグゼータ} \overset{\text{類双対写像}}{\cong} \text{多重オイラー積ゼータ}}$$

Inducing a Selberg-type zeta function to an Euler product using a quasi-dual map.

from here, the problem of the eigenvalues of the Selberg zeta function is rewritten as the problem of the zeros of the Riemann zeta function. For the subsequent content, please refer to my essay on the Riemann zeta function.

In this way, my theory of "Linden complex spiral planes and divergent quasi-dual maps" reduces various zeta functions to the case of the Riemann zeta function.

This brings to mind what is known as the Langlands program.

As I imagine it, the following correspondence exists.

Figure 11. The Langlands conjecture and correspondence.

ラングランズ側の課題	理論での対応可能性
非可換ラングランズ対応	非周期素リンドン系列の縮約理論
L関数のゼロ点分布	リンドン入れ子構造と log-双対系
幾何ラングランズの層構造	トレース束・グラフ構成による再現
モチーフ・コホモロジー的統合	モチーフ閉包と螺旋形回帰写像
圏論的形式化（∞圏など）	素リンドン圏／トレース圏の提案

Correspondence between Langlands conjecture and noncommutative Riemann surface structures.

For example, in this theory, repeatedly applying the Lindon compression structure on a bouquet graph automatically yields the same modular transformation (the action of SL(2,$\mathbb{Z}$)) as the quasi-dual modular transformation on the complex plane. This is akin to a class dual transformation that elegantly deforms the "trace bundle," and thus simultaneously deforms the complex plane in a manner consistent with the deformation structure of a general Riemann surface.

At first glance, it may seem puzzling why modularity is associated with modular forms, but considering the non-periodic nested structure of the Lindon sequence and its topological compression, it becomes clear that the congruence transformations of the modular group are a natural symmetry inherent in the compression operation.

Therefore, this book proceeds with the argument that "modularity is inevitable." Please note this point.

Then, what is the zeta function of a "non-regular structure"?

What I can say here is that, in order to direct my focus toward the theory of "quasi-dual closure," I must first create a "non-commutative, non-regular zeta structure" myself.

One such form is "quasi-dual closure."

Generally, non-regular graph-like Riemann surface structures are complex, and the "multiple traces" within their "dual motif spaces" are intertwined. I already have three approaches to this challenge:

1. First, establish the concept of trace categories.

A trace category preserves the information surrounding the path being traced as a "category," and views the traced elements as transitions between "categories." By doing so, the "stability of recovery from trace bundles" improves dramatically.

2. Develop a theory of "dual motif closures" that includes directed graphs.

This enables us to use algebraic systems with non-commutativity as trace materials.

3. The zeros of the zeta function under regular conditions are defined by prime roots on the unit circle, i.e., Pth roots of unity. Conversely, in the case of non-regular states, it is expected that the Pth roots of unity on the unit circle will lose their symmetry or acquire non-commutativity, and it is highly likely that the invariance of the "trace bundle" will also be lost. As a result, the zeros will vanish from the unit circle. At that point, a theory of information loss or information expansion in the "trace bundle" becomes necessary.

By the way, this non-regular region is the region I was originally in, and here, by depicting the "transition of fractal structures," I aim to depict the entire vast transition between "loop structures" and "tree structures" as a "quasi-dual closure."

Furthermore, this class dual closure itself, by taking on the trace structure, becomes a "zeta structure," which I refer to as the "zeta expansion via the third-order class dual mapping."

My hypothesis is that the divergent-type zeta expansion of the quasi-dual closure converges to a fixed point through a "second-order expansion" and is closed.

Here, I have attempted to outline my grand vision in the realm of "non-commutative and non-regular" structures... So, please forgive me if you feel like saying, "What on earth are you talking about?(see Figure 11)"

Constructive Langlands Existence Theorem

Proposition

If there exists a non-periodic sequence (corresponding to a modular form) that is not contracted while preserving symmetry on a Lyndon dual structure, then there exists a modular transformation that preserves that symmetry. This is because there is a need for a clean condition to match values in the dual multiple trace space that preserves that symmetry. The values are not degenerate. There is room for contraction. This condition becomes modular.

At this point, the modular form is not necessarily explicitly given (the “form” itself is unknown).

However, its existence is guaranteed (constructively).

Modularity = “Modular existence theorem preserving non-degenerate symmetric structures.”

That's all.

Appendix 3

Since discovering this theory, I have been thinking about it a lot and have started reading books on complex analysis and Riemann surfaces.

In other words, classical Riemann surfaces involve various gluing, combinations, topologies, definitions of local areas, and so on.

However, the generative general Riemann surface derived from my Lindon continuous spiral topology can be created simply by drawing a graph, taking the “divergent zeta extension,” constructing a graph manifold, and either using it as is or deforming the manifold using a quasi-dual transformation that preserves a specific “fractal structure.”

On the other hand, “what kind of deformation leads to specific formulas or functions” is very difficult to see. The same is true for modular forms. First, I understand modular forms in terms of “trace bundles.” They are “quasi-dual mappings that transform rational Lindon into integer Lindon or prime Lindon.” However, in various contexts, it is difficult to understand “what they are” without careful consideration.

No matter how I write it, it seems like I am making grandiose claims, so I tried to understand the reasons behind it in my own way and studied various topics.

3. Ramanujan's quadratic zeta function and zeta functions in non-regular domains

From the previous section's discussion, we can see that most zeta functions are zeta functions of genus 0, even if they have an infinite number of prime structures.

This raises the question of zeta functions of genus 1.

When attempting to perform divergent reconstruction by placing loops of prime path length on elliptic curves of genus 1, we encountered a major problem.

This is because each Euler product continues to expand due to two “non-periodic sequences,” leading to divergence.

In other words, the product of the “zero point product formed by two periods” and the “Euler product” diverges infinitely.

Therefore, we must consider conditions under which this does not diverge.

Here, let us consider the graph of the Ramanujan zeta function with genus 1 in our minds.

The Ramanujan zeta function suppresses divergence by ensuring that the discriminant of the quadratic Euler product is negative, resulting in imaginary solutions. This is what is known as the Deligne's condition.

Therefore, the idea arises that if we take the solutions of a quadratic equation where the discriminant is negative and the coefficients are real, we can create an infinite number of zetas. These solutions are arranged so that the Deligne's condition is satisfied through a modular action that unifies an infinite number of quadratic equations.

The image of the correction term that suppresses the Euler product from diverging is as follows.

Figure 12. A structure in which the Euler product does not diverge.

ポイントは「複素長さを物理長さとして数える」んじゃなくて、
「螺旋位相のズレパラメータとして扱う」ことです。

$$Z(u) = \prod \left(1 - u^L e^{i\theta}\right)^{-1} \quad \text{with} \quad \theta \in [0, 2\pi)$$

A special structure is required to associate complex parameters with effective length in the Lyndon contraction framework.

If we adjust this parameter to match the “divergent restoration of non-periodic sequences” and introduce parameters with ‘modularity’ such that the two “Euler products” do not diverge, we should be able to generate the quadratic zeta function. That is my understanding. However, modularity introduces a discrepancy. This discrepancy is likely related to the breakdown of the complete multiplicativity of the coefficients of the Ramanujan zeta function.

Furthermore, the duality of the Euler product requires that when the two Euler products are multiplied together, the result must have real (integer) coefficients. Otherwise, the length of the prime path would become meaningless as a complex or real number. In other words, modularity is necessary here.

To summarize this analysis, “zero points align along the critical line in the zero product” when the offset paths and prime paths are completely aligned topologically. However, double non-periodic sequences are unlikely to be completely aligned. This is because two independent offset directions interfere with each other, resulting in subtle deviations that disrupt the critical line.

Therefore, the quadratic zeta function (Ramanujan zeta) may not have zeros perfectly aligned along the critical line due to its quadratic structure with imaginary solutions. This suggests that it may not satisfy the Riemann hypothesis. At the same time, these zeros are likely to lie in a “critical region” rather than a “critical line.” This is the Deligne's condition. In other words, this is an example of a “non-regular structure zeta function.”

At the same time, these facts correspond to the fact that class field extension of the modular form of Ramanujan's zeta function is not “Abelian (commutative),” and the existence domain of the quadratic zeta function in the non-commutative class field theory domain is equivalent to suppressing the divergence of the “Euler product.”

Figure 13. The domain of noncommutative class field theory and the convergence of Euler product divergence.

観察	構成論的対応	数論的意味
非周期列に11や691が現れる	リンドン系列の構成素	モジュラー群の対称性、保型係数との結合
特定素数が支配的	トレース束の分岐定数	保型形式の係数や合同の源

There exists a structure that prevents the divergence of Euler products in the field of noncommutative field theory..

In other words, if we can discover a parameter for a non-periodic sequence such that the discriminant of the quadratic equation formed by two “non-periodic sequences” is always negative and yields imaginary solutions, then it seems possible to construct the quadratic zeta function. This becomes clear when considering the genus-one zeta function from a Lindon sequence perspective.

In my opinion, the field of complex multiplication theory (CM) is likely related, and for special elliptic curves (elliptic curves with complex multiplication), their defining field appears to be the complex number field. In the case of the standard elliptic curve $y^2=x^3+ax+b$, the endomorphism ring is Z, but in the case of complex multiplication theory (CM), the ring of integers of the complex field, such as $Z[-d]$, appears. In other words, there seems to be a connection with the "structural conditions for the validity of the quadratic zeta function.(see Figure 13)"

Therefore, the periodic structure of elliptic functions is related to the "structure of the complex number field."

1. Create a basic extension theory of the Ihara zeta function that generates a quadratic zeta structure from two graphs of non-regular "non-periodic sequences."
2. Construct modular functions or generating functions with structures that satisfy the conditions from domains such as complex multiplication theory.

In other words,

Figure 14. The Ramanujan's quadratic zeta and noncommutative class field theory.

$\mathbb{Q}(\sqrt{-23})$ + 対応する CM 楕円曲線 + Hilbert 類体の非アーベル拡大

Corresponding CM elliptic curve Non-abelian extension of Hilbert class field.

this correspondence.

Additionally, the expected "higher-order zeta" structure conjecture,

《Deligne's theorem, or the higher-order zeta structure conjecture》

If higher-order zetas exist, except for the case of genus 0, there is a modular structure where the discriminant of the Nth-degree equation of genus +1 becomes imaginary and has imaginary solutions, and in its generating function, "disturbances in multiplicativity" occur. When the genus is even, higher-order zetas can exist if they have a modular form that controls the divergent state through the divergent reconstruction of the "non-periodic sequence" of Euler products.

For odd numbers, zetas other than the Riemann zeta either do not exist or take extremely special forms.

Furthermore, such modular forms correspond not to "Abelian extensions" but to "non-commutative domains" in the non-commutative class field theory domain.

To explain, odd-degree equations have a structure that includes "real numbers," which means they contain non-commutative, continuous "multiple periods" and "non-periodic components," making it easy for Euler product factors to deviate from convergence.

Even-degree equations, on the other hand, have a more "commutative and symmetric" structure, where some periods cancel each other out, increasing the likelihood of stable appearance of Ramanujan-type modular forms.

There is also the observation that the trivial zeros of the Riemann zeta function lie in the even sequence −2, −4, −6, ..., which may be linked to the possibility of Ramanujan-type structures. In other words, negative even numbers can be considered as “special points where even-order traces cancel out,” or, in other words, “points where open non-periodic traces can be symmetrically folded back to zero.”

This corresponds exactly to the conditions under which open terms (geometric series) can be converted into closed forms (product forms), as in Ramanujan's equations.

The subsequent development I came up with was as follows.
In this way, I strongly feel that Ramanujan had an intuitive sense of this kind of “nested structure.”

To confirm this, let's take a look at some of Ramanujan's formulas.

Figure 15. The Ramanujan's modular identity.

$$\sum_{n=1}^{\infty} \frac{1-q^n}{q^n} = \sum_{n=1}^{\infty} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{q^n}$$

This Ramanujan formula flips the denominator and numerator, expresses each (1-q^n) term as a geometric series, and multiplies them again, resulting in a beautiful identity.

In other words, it is an identity that expresses “modularity” and “log scaling,” which represent the transformation from ‘sum’ to “product.”

Let's bring in another modular form.

This is the Jacobi triple product.

Figure 16. The Jacobi triple product.

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})$$

- 中心にあるのは、**対数スケーリング構造を持つ加法と乗法の融合**

Looking at this structure, we can see that the genus is 2 or higher due to the conditions of triple product and two variables. This is because there is a symmetry term with “genus + 1,” i.e., 3, where 3! = 6, and no modular form with such a structure exists if the genus is higher than that. Here too, we can see an extraordinary structure where “modularity” and “log scaling,” which combine ‘sum’ and “product,” are beautifully integrated while encompassing multiple variables.

Let's observe one more example.

This time, Dedekind's modular form.

Figure 17. The Dedekind's invariant form.

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

- η関数 = $q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$
- τ→-1/τ の変換は**logスケーリング**（対数螺旋）に対応

From the multiple periodicity, we can see that “this is genus one.”

However, if it is of genus 1, isn't the pattern of double periodicity just one?

However, when the genus is 0, it was understood from the “graphical Riemann surface structure” that sine, cosine, and power functions actually have the same form, so it is imagined that all modular forms of genus 1 are unified into one form, one “equivalence class.”

In all the patterns we have seen so far, the transformation between “sum” and “product” is unified by “modularity,” and we can observe that modularity always requires a slight ‘shift’ or correction. This shift gives rise to phenomena such as the “breakdown of multiplicativity” seen in the Ramanujan zeta function. This leads to the surprising discovery that there may also be non-regular structures within the Dedekind zeta function.

The consideration of Lindon-like nesting and non-periodicity in Ramanujan can also be seen in the famous taxi number.

This is the moment when Ramanujan's intuition and the Lyndon sequence perspective completely converge.

The smallest taxi number,

$1729 = 1^3 + 12^3 = 9^3 + 10^3$

That is, “the smallest number that can be expressed as the sum of the cubes of two positive integers in two ways.” Why is this a Lyndon structure?

The structure of ordered pairs, $(a, b)^3 + (c, d)^3 = n$, corresponds exactly to the “set of the smallest pairs in lexicographic order” in Lyndon terminology. (1,12) and (9,10) are “non-symmetric” and “non-periodic” pairs, and the core of Lyndon theory is to regard such unique combinations as the minimal structure.

Since the definition of a taxi number is “two distinct combinations,” periodic sums (e.g., 2×(1,12)) are excluded.

Since Lyndon language is based on non-periodic sequences, it appears to satisfy this condition.

Considering the dual structure, if we regard reverse order and symmetry, such as (1,12) ↔ (12,1), as equivalent, we must choose the lexicographically smallest (i.e., Lyndon smallest) pair.

The “root structure” of the decomposition formula is graphable and has genus 1, but if we draw all pairs that can be expressed as $x^3+y^3=N$
on a graph, it looks like a bouquet graph:

Each node is the same as the pair (x,y), and the pairs belonging to N form a loop. The taxi number is the smallest node in the bouquet that has two or more loops. This is exactly the same as the Lyndon bouquet structure.

Regardless of whether this analysis is accurate, the hypothesis that Ramanujan “unconsciously classified sequences using Lyndon language” becomes more plausible.

Ramanujan's formulas and infinite series often have mysterious nested structures.

If we apply Lyndon's complex spiral continuous phase theory to Ramanujan's formula expansion, it seems possible to interpret the hierarchical structures of periodic and non-periodic elements contained in the infinite sums and products he created in a modern way, which makes my imagination run wild.

By the way, generally speaking, the Deligne structure is considered to be a conformal category structure that appears in modular curves, and requires a difficult description. However, in this paper, we interpret it as “syntactic symmetry” in Riemann surfaces of genus 1. This special symmetry arises from the existence of dual paths due to the fact that there are only two curves of genus 1, and it is related to the structure described in part by Modell's theorem.

In other words, the τ function and its composition are not mere functions, but extensions as mappings acting on the syntactic space. As shown in Figures 20–21, this allows us to visualize how the syntactic complex trace that appeared in Modell's theorem unfolds with finite group branching.

This extension is nothing other than a more naive syntactic operation that brings down the geometric essence of the Deligne structure.

Now, let me calculate a concrete example of my prediction.

When the quadratic field is $\sqrt{-5}$, the discriminant is 40, and $\tau(n)$ is calculated as follows: $\tau(2)=-2$, $\tau(3)=-1$, $\tau(5)=-5$, $\tau(6)=2$, $\tau(4)=2$, $\tau(8)=0$, $\tau(16)=-4$, $\tau(32)=8$.

From here, using induction,

Figure 18. Skew multiplication function.

$$\tau(p^r) = \tau(p)\tau(p^{r-1}) - p\tau(p^{r-2})$$

we can calculate the quadratic Euler product using this.

Figure 19. Deriving local structure from distorted multiplication functions.

$$\sum_{r=0}^{\infty} \frac{\tau(p^r)}{p^{rs}}$$
$$= \frac{1}{1 - p^{-s}\tau(p) + p^{1-2s}}.$$

Therefore, by integrating this(see Figure 20),

Figure 20. Integrated into the Euler product of the second order.

$$L(s) = \prod_p \left(1 - \tau(p)p^{-s} + p^{1-2s}\right)^{-1}.$$

This calculation took two days from the initial idea, so if the result is correct, it would be very exciting. This is a "non-regular" quadratic zeta function, and it is clearly isomorphic to Ramanujan's quadratic zeta function, being of the same type of genus 1 zeta function(This overlapping of multiple quasi-dual mappings can be visualized in **Figure 20**, where the Euler product is extended to second-order duality over the complex syntax spiral).

At the same time, from this observation, theoretically,
when genus + 1 $\equiv$ 0 (mod 2),

it is highly likely that a higher-order version of CM = "imaginary multiplication-like theory" exists, and higher-order zetas can be naturally constructed.

Now, based on the current idea, if we consider $\tau_1(n)$ as the quadratic zeta function mentioned earlier and $\tau_2(n)$ as Ramanujan's τ function, and think of their product as a new operator, it becomes a multiplicative function with an irregular multiplicative structure(see Figure 21).

Figure 21. Multiplication function combination.

$$a_n = (\tau_1 * \tau_2)(n)$$

From here, after various efforts to transform it matrix-wise, I finally had a 4×4 matrix calculated by a computer. However, upon further investigation, I discovered the theory of Hecke operator multiplication, leading to the following equation:

Figure 22. Representation of the fourth-order zeta function by the Hecke operator.

$$L_p(s) = \frac{1}{\det\left(I - (\rho_1 \otimes \rho_2)(\mathrm{Frob}_p)p^{-s}\right)} = \frac{1}{1 - A\,p^{-s} + \cdots + D\,p^{-4s}} \quad \text{ただし} \quad \begin{cases} A = a_p b_p, \\ D = p^{k_1+k_2-1}. \end{cases}$$

Here, the Ramanujan zeta function has k_1 = 12, while my constructed function has k_2 = 2, so D = p^12, and A = (Ramanujan generating function (p)) × (my zeta generating function (p)), resulting in a highly complex equation. This can be verified by simply multiplying the distorted multiplicative functions and examining the behavior of some terms, which clearly match in parts.

Finally, the Euler product was obtained. It is indeed a fourth-order zeta function. Moreover, there are probably two of them now, so to speak, forming a bivectorial ring. There are likely to be more(As shown in **Figure 21**, the compression of infinite trace bundles into modular structures via finite groups reveals a discrete syntax topology).

I am unsure whether this calculation is correct, but this method demonstrates that higher-order zeta functions such as second-, fourth-, sixth-order, etc., can be constructed, indirectly supporting my hypothesis.

It is predicted that the zeros of this function scatter from the critical line and nest within the critical region. And this is a non-regular, non-commutative, non-Abelian, graph-theoretically non-Ramanujan (somewhat ironic) zeta function. Why do the zeros scatter further from the critical line as the order increases? As mentioned earlier, when the genus is large, the number of non-periodic terms inherent in the graph increases to "genus + 1," so more "internal complex zeros" are needed to control divergence and prevent the Euler product from diverging. This is precisely the meaning of Deligne's discriminant.

Due to the non-periodic terms existing internally, the "collection of prime numbers" in the Euler product, which already exists infinitely, diverges to infinity. However, the mechanism that suppresses this divergence by grouping them complexly is Deligne's inequality.

Now, summarizing the above results,

《Structural Conjecture of Non-commutative Langlands Domain》

Non-commutative Langlands domains exist in a complex multiplicative manner when the genus plus one is even, and they exist in the form of Hecke operators that control the divergence of Euler products.

I believe that when the genus is one, the existence of "symmetric non-periodic terms" causes the imaginary quadratic field to generate the structure of "CM elliptic curves," which serves as the source for infinitely generating "quadratic zetas." Furthermore, there is a theoretical structure that allows for the infinite generation of even-order zetas such as 4th, 6th, 8th, etc., by multiplying Heck operators.

On the other hand, in the case of the imaginary cubic field (an imaginary algebraic number field with genus 3), there is no quadratic form structure with symmetry like that of the imaginary quadratic field, and it is also impossible to construct a complex multiplicative "trace mirroring" structure. This is because there are no "symmetric non-periodic terms," and one must deal with symmetry of order 3 or higher, so a slightly different methodological approach may be necessary to control divergence.

In this section, I have explained the structural meaning of the Ramanujan zeta function and why it is a quadratic zeta function. In the process, I myself was probably delighted when I was able to create a quadratic zeta function.

However, what opened up from there was a "vast sea of non-commutative structures." Can the structure of all graph-like Riemann surfaces be reduced to the structure of genus graphs?

Or is there a more complex non-regular structure?

It seems that we need a few more new ideas to reach that point.

That's all.

4. Summary

What do you think?

First, I derived the existence of non-periodic terms from my structural graph-theoretic Riemann surface theory, understood the structure of general Riemann surfaces, imagined the conclusions that would arise from that structure, summarized various theorems and hypotheses, and finally constructed a concrete example.

While working on this, I honestly thought, "Isn't it strange that no one else has done this?" The quadratic zeta function I constructed is actually unnecessary, and if we have the Ramanujan zeta operator, couldn't we construct higher-order zeta functions? ... I don't have enough knowledge to answer that question.

Through this analysis, I believe I have been able to convey, to some extent, the shift in my thinking from the introductory level to the applied level, as well as the logical structure and its implications.

This work was also necessary for me to maintain my sanity.
The theory was so beyond my imagination that, despite the conclusions being clear, it was difficult to compare them with existing results, and even more difficult to provide concrete examples.

However, I was somewhat satisfied after observing some "correspondences with existing theories" and "agreements with concrete examples."

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森田英章、組合せ論的ゼータの半群表示、2016
ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
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Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
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Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.
This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。

これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.
素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of "complex spiral phases"
Although it is not yet clear, it is gradually becoming apparent that as the number of

species increases, there is a “divergence control function” corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ｓ という臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.
無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the “elementary Lyndon element” that is restored to zero is decomposed, the corresponding Euler product becomes a “multiple Euler product,” giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through "dual non-periodic paths."
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a "double helix" because it naturally contains spiral rotations and has a double main structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in the number of species. This is particularly important in the context of the formulation of "higher-order imaginary multiplication."
In other words, it can be understood that the Hecke operator of higher-order zeta

functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by “dual non-periodic paths.” Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.
トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)
素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　 quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する

In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.

全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.
Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings,

the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal "bouquet structure" corresponding to the "Euler product." It is also necessary to distinguish it from the commonly referred to "Hodge structure."
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
　In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
　とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ;　A natural quasi-dual transformation that reduces "irreducible rational Lyndon" in trace bundles to "prime Lyndon" or "natural number Lyndon." Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.

非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets；A complete state that cannot be further expanded by repeating dual operations.
双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.
→ 類双対閉包
非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる

Chapter 3: Deformations of fractals and a generative approach to zeta structures

—Generative Deformations of the Riemann Zeta Function from Graph Zetas via Quasi-dual Divergent Maps and the Construction of Their Zero Points—(A Study on the Riemann Zeta Function)

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July 2025

注・日本語を読める方は、日本語で書かれた原文がありますので、そちらをご利用ください。文末にリンク先

Abstract

This final part examines the half-critical region and the non-regular structures of the Riemann Zeta function. Using spiral graphs, ideal class motifs, and topological genus transitions, the text proposes a novel geometric interpretation of the critical line and the continuum hypothesis.We analyze the critical line of the Riemann zeta function from various perspectives using “quasi-duality maps,” and discuss the mechanism of Hilbert-Polya action and the mechanism of “symmetric divergence control” in the analytic continuation of Euler products.

Summary and Explanation of This Document

The important point is that this began with the exploration of noncommutative structures in the mathematical modeling of noise phenomena.

What I consider most important is the “interpretation of zero points in divergent zeta constructions” in quasi-dual maps. In other words, the zeros of the zeta function are structural objects that can be reconstructed from the trace of a non-periodic ordered prime number structure. The essence of this construction method is that when a conceptual object exists, by considering its trace and performing various transformations using the trace bundle, a special “divergent reconstruction” occurs, which is shown to possess the natural properties of a “zeta structure.”

In summary, by considering a method of “contracting loops or expanding trees” in what might be called “infinite repetition within an infinitely continuing path,” I discovered the possibility of constructing a zeta structure. It was an extremely abstract theory, and I found a vivid example of this theory in the Ihara zeta. The Ihara zeta function is the very transformation that converts any “loop structure” into a “tree structure.” As a result, through the determinant representation of the Ihara zeta function, a function with a structure similar to the Riemann zeta function appears, and it seems possible to reexamine its zero point structure from the perspective of regularity.

Finally, this paper includes a chapter that outlines a constructive solution to the Hilbert-Polya operator problem using a matrix ring based on a prime number structure.

In this summary, I will focus on how the zeros and function expressions of Riemann zeta-like formulas, which appear as a consequence of the theory of class duality maps in dynamic transformation, are generated.

First, in the regular Ramanujan graph structure of the Ihara zeta function, the zeros are arranged on a circle, and in particular, in the “bouquet zeta function,” they are arranged on the unit circle. Repeatedly applying the “quasi-dual zeta extension” of this bouquet graph leads to a state where “the length of the path is a prime number” (a state where all prime paths exist) only under regular conditions.

Then, in this state, the matrix obtained by expressing the “prime Lyndon” expansion of the Ihara zeta function based on paths in a “determinant representation” is transformed by the class-dual transformation “u^p→e^-slogp,” which decomposes the path information into “infinite prime power circles” for each path, and then shifts it to the critical line (Re=1/2). Then, from the regular Ramanujan condition of the Ihara zeta function, it can be seen that this zeta function satisfies the Riemann hypothesis-like condition, but in fact, this transformation converts it into the Riemann zeta function.

The quasi-dual transformation “u^p→e^-slogp” connects this infinite concentric circle structure in a tensor-like manner based on the “prime structure (cycle),” creating a structure where there are infinitely many prime numbers. This is a basic yet special regular graph, but even in this graph, it is shown that the zero points are properly arranged on the unit circle. Applying the quasi-dual transformation "u^p→e^-slogp" to this graph results in the same structure, so the quasi-dual transformation "u^p→e^-slogp" invariant, i.e., it leads to a fixed point. Then, by defining a "matrix" that satisfies the conditions of the quasi-dual transformation at each scale logp of the concentric circles and infinitely combining them, it interacts with the determinant representation of the Ihara zeta function and acts as a Hilbert-Polya operator.

The above is an overview, and I will insert an explanation including the motivation later.

1. Starting Point: Noise Phenomena and Noncommutative Structures

The starting point for this exploration is the "mathematical structuring of noise phenomena."

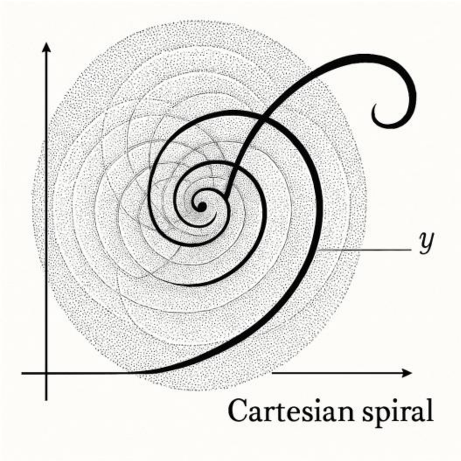
My interest in the "mathematical modeling of semi-subjective phenomena" known as noise phenomena arose when I learned about the "error backpropagation method" used in today's popular generative artificial intelligence.

In the "error backpropagation method," there is a mechanism for "generating meaningful images from noise." However, I had already noticed that, similarly, in "noise phenomena," through training, one can observe "non-trivial and involuntary meaningful visual or audIharary images (generally perceptual images) within the noise." I named this "intuitive images." For example, in literary expression, Aldous Huxley's "The Doors of Perception" describes a similar experiential structure. I won't explain this here, but since this noise phenomenon generally has a fractal-like structure with movement, I decided to explore how fractal structures transform while maintaining their fractal nature.

The structural similarities between "human observation methods" and "artificial intelligence generation methods" do not end there. For example, there are generative intelligences capable of "enhancing the resolution of old photos, converting them to color," or "removing human figures from photos." In fact, through "noise observation," I had already experienced this sensation. Therefore, I recognized that "images are created by contracting fractals (after expanding the world fractally)" and named this process "fractal contraction," referring to the operation as "quasi-dual transformation" or "quasi-dual mapping."

In its simplest form, this manifests as "mapping a Cartesian spiral fractal to an infinite concentric circle fractal in a one-to-one correspondence." This diagram is important, so let me illustrate it with a figure. It is a schematic diagram, but imagine it as “a spiral expanding outward, cutting out each concentric circle one by one.”

Figure 1. The Infinite concentric circle fractal and The Cartesian spiral.



Basic quasi-dual mapping: infinite concentric circles →← Cartesian spiral

What is important is that by “simultaneously deforming” the “infinite repetitive structure” created by the enlargement mapping that appears in fractals, various fractal structures can be deformed while maintaining their fractal nature. When considering such matters, it is helpful to trace the structure and examine the “infinitely repeating portion” of the trace. In other words, “looping” involves compressing and connecting the infinitely repeating paths, while ‘treeing’ involves expanding them. Note that these are one-to-one correspondences. This is a local deformation, but we will also consider the case of “continuous mappings” later.

Such quasi-dual transformations are generally non-commutative, and the transformations themselves are multivalued and possess multiplicity. Regarding the name “quasi-dual transformation,” there is first the “dual transformation.” For example, using this, we can construct a “dual motif category,” which is a closed structure defined by a deformation structure, from a generalization of a graph structure called a hypergraph (which can be considered to show correspondence between multivaluedness and multivaluedness). This can be described as an operation that “expands graphs set-theoretically, consolidates multiplicity, and restores uniqueness.” This “dual operation” overlaps with “quasi-dual transformations” in very simple situations, such as regular graphs.

Therefore, we named this “quasi-dual mapping” as something similar to the dual operation. Since it is essentially non-commutative and multi-valued, considering the “closure structure” to handle it requires considering the round-trip of the operation transformation, and the categorical structure created by this operation can be described as a series of transformations from "loop-type structure" to "tree-type structure," which we call "quasi-dual closure."

Note that the essence of this transformation series is revealed in the simple mutual transformation state of "concentric circle fractal →← Cartesian spiral."

These transformations are defined by "quasi-dual transformations," and when considering what they are applied to, if there is a structure, we can take the path that traces its elements, i.e., the "trace." These traces include both finite and infinite ones, but here we will only consider the infinite ones.

When "infinite repetition" appears in the trace, the transition from loop-type structure to tree-type structure can be expressed through the operation "concentric circle fractal →← Cartesian spiral." We will look at a concrete example of this later in the form of the Ihara zeta function.

This operation itself can be performed as many times as desired where "traces and the trace bundles that collect them" are established, but when I learned that the construction method of "Ihara zeta" and its structure are identical to those of an infinite regular tree, I realized that it could actually be formalized in special cases, which became a major motivation for writing this paper. In other words, I understood that Ihara Zeta is a special, commutative, (semi-)unique, special class dual transformation example of "(regular) finite graph" (loop structure) →← "regular infinite tree" (tree structure).

However, it is important to note that the term "semi-unique" is only partially valid, as there actually exists a "non-trivial infinite repetitive structure" that naturally "divergently" transforms a finite graph into a graph structure with infinite loops. In other words, this is the structure called the "zeta structure," and it is important to note that this is connected to the "zero product," and that the product is first defined "non-complexly" and then "analytically connected to the complex domain" through the "Ihara zeta determinant representation."

In other words, while the Ihara zeta function derived from a (regular) finite graph is expressed as a "rational function," the determinant representation derived from the infinite loop graph transformed through the trace bundle formally becomes an infinite matrix, which can be interpreted as reflecting the possibility that the zeta function derived from a finite graph may have an infinite number of zeros. However, many divergent zetas also fall into the finite type, and the tolerance of this finiteness is shown to be related to regularity. But this will be explained later.

In other words, through the quasi-dual map, which deforms the trace bundle, and the exploration of the noncommutative transformation structure that emerges there, the possibility of describing the zeros that appear in the zeta function not as mere numerical properties but as the result of structural trace restoration deformation was accidentally discovered.

2. Reconstruction of the “structure” of zeros using quasi-duality maps and “divergent quasi-duality transformations”

Now, let us explain the “divergent zeta construction in quasi-duality maps” and the “structural interpretation of zeros” that naturally appears in it.

A quasi-dual map is an operation that “transforms one fractal into another fractal.”

It involves collecting traces from a structure and examining a multivalued structure that can be constructed and reconstructed from them. When there is “infinite repetition” in the trace bundle, it is summarized and reduced to a “loop structure.”

Here, for non-periodic trace structures (∞ Lyndon series), the proposition that zero points structurally emerge by identifying higher-order traces (i.e., zero-point products) that reconstruct them was proposed. This is considered crucial.

Let me explain.

First, the Ihara zeta function traces all possible “paths traversing the graph.” This involves considering a “closed loop” as a “primitive structure” and constructing all possible paths as non-commutative combinations of these.

By doing so, the “loop structure” of a finite graph can be “transformed” into the form of a “regular infinite tree.” This seems to be due to the observation of a person named Sale. The term ‘transformation’ is from my perspective, while the “quasi-dual map” is from the perspective of transforming the structure. Note that in this transformation, it is usually the “trace bundle itself” that is transformed. There is a difference between the “trace bundle” of the original structure and the “trace bundle” after transformation, which can be interpreted as a loss or addition of information.

It is important to convert the concept of “closed path” into the concept of “trace bundle.” The concept of a closed circuit uses concepts such as equivalence and reduction to extract “loopiness.” However, if we shift to the idea of restoring “infinite repetition” in the “trace bundle,” these equivalence concepts for “extracting loop structures” become unnecessary. Strictly speaking, various problems arise, but I will not discuss them here, as they are not necessary for this discussion.

Furthermore, by replacing the concept of “closed path” with that of “trace bundle,” it becomes important to extract “non-trivial loop structures.”

For example, if there are two or more “prime structures,” let us represent them as 0 and 1. Then, we can see that there are repetitive structures such as ‘01’ and “001.” In other words, it becomes clear that “non-trivial loop structures can be constructed by combining prime structures into non-periodic structures with order,” and this is “necessarily contained within the trace bundle.” If we denote the length of this repetitive structure as n, it can be stated that a “non-periodic structure” of length n necessarily exists, and that “non-periodic repetitive structures” exist infinitely. Specifically, this becomes clear when examining the construction method of sequences that do not contain repetitions.

By treating this “non-periodic structure” as a “higher-order prime structure” and repeatedly performing “quasi-dual restoration,” it becomes clear that it is possible to restore an “infinite structure graph with infinitely divergent loop structures.” However, redundancy always occurs, and if this is reduced, it becomes clear that this divergent graph is actually a finite system. The rigor of this will be demonstrated later in the analysis of Lyndon language. Finite graphs typically have “paths that trace back along the same route,” so it is important to note that “the number of elementary structures is almost always two or more.” Of course, it is possible to construct an Ihara zeta function for this “infinite structure graph,” and it can be seen that there is a determinant representation. Furthermore, it can be seen that the infinite graph reconstructed from the finite graph shares the “trace structure” as much as possible and can be expressed by an infinite matrix, which corresponds to the fact that the zeros of the “(regular) finite graph zeta function” are at most infinitely many. However, as mentioned earlier, due to “redundancy,” it ultimately becomes “finitely generated,” and due to its remarkable expansive nature, it demonstrates hologram-fractal-like restorability, expanding to an “infinite graph where the length of each path is a distinct prime number p” under regularity conditions.

This special quasi-dual transformation to a divergent infinite graph can be called a “zeta transformation.”

“Divergent quasi-dual transformation” = zeta transformation.

And this “zeta transformation” provides a natural construction and interpretation for the existence of the zero product.

Let's organize the diagram of this “zeta transformation” a little more. Note that there is a “nested structure” here.

(Figure of zeta expansion: usually restored from the repetition of “prime elements,” but restoration from “non-periodic terms of prime elements” is also possible, and these have fixed points)

Note that this “divergent quasi-dual transformation” can be constructed one by one by actually combining several “non-periodic terms” to show the actual zero point structure in a determinant expression. This sequence is considered to be non-additive, and the matrix can be pointed out to have some symmetries based on the structure of the graph. Before actually constructing this “non-periodic term” as a Lyndon sequence, there is something called an “infinite non-periodic sequence,” and some people may think that this does not fit into the determinant representation because it appears to have no “end point.” However, “zeta extensions” have a hierarchical structure. That is, there exists a zeta extension that ‘swallows’ the “non-periodic term,” and it can be shown that the “non-periodic term” exists as an infinitely repetitive term within that extension. In other words, this hierarchical structure has a “nested structure” that infinitely swallows itself, and after the final extension, it forms a “fixed point.” Note that a “non-countable continuous topos,” which differs from ordinary rational numbers in that it can only observe “local structures” but can properly observe “adjacency relations,” naturally emerges. Moreover, it can be expressed as a “nested structure.” In rational numbers, there is no such thing as a “neighboring rational number.” However, this non-periodic term has a “neighbor.” This is thought to be a topic in “category theory,” which will not be discussed here, but those who are interested should pay attention to this difference.

Let us specifically construct the infinite graph structure resulting from this “divergent quasi-dual deformation” one by one.

The elementary structures are loops of various lengths... Since they are loops, they are connected, and this structure is reflected in the determinant representation. If these elementary structures are 0, 1, 2, 3... N, we connect them in a non-repetitive, ordered manner. For example, if there are only 0 and 1, it could be 1101. This is not a repetition of any length of sequence. We translate this into an actual structure and create a loop structure as a “new elementary structure.” Then we create another elementary structure. It could be 01. We also translate this into the actual structure and consider it as a new elementary structure to create a loop.

We collect such structures and arrange them infinitely... Note that this is “uncountable.” With this, we can gradually construct the determinant side of Ihara's “divergent zeta extension.”

Then, as in the construction of Ihara's zeta function, we can translate the “connectivity” between a new prime structure and a new prime structure into 0 or 1 and create a determinant representation.

If the characteristic equation of this determinant can be solved within the range of complex numbers, we can see that Ihara's zeta function is naturally analytically continued to the complex number domain and has a complex function representation based on zero points (new higher-order prime structures). These eigenvalues can be considered "zero points." It is important to note that if an Euler product (loop-type structure) exists, it can be transformed almost automatically into "loop-type structure → tree-type structure → zeta structure." At this point, it is also important to note that the "content of the prime structure" is not important, and only "the connection of the prime structure, the uniqueness of decomposition, and the power (repetition) of the prime structure" have structural significance.

This is the determinant representation showing the zero-point structure constructed from the graph structure.

In this construction, if we note that the quasi-dual transformation in Ihara's zeta function is commutative and unique because the graph structure is uniform (the combination of points and edges is the same) and regular Ramanujan-like, then the question arises, "In cases where this is not the case, are there generally multiple zero structures?" For example, "Does it become a Dedekind-type zeta function when it is non-regular?" This is because quasi-dual transformations typically have multiple transformation series and form a non-commutative category structure. However, this is not an issue here, so we will just note it. The conditions that the Ihara zeta function is rational and finite, and that the tree structure it expands into is regular (Ramanujan-like), can be summarized as such.

We reinterpret this "non-periodic term" as a semigroup structure called the Lyndon language.

Now, we refer to Hideaki Morita's "Semigroup Representation of Combinatorial Zetas."

If we consider a finite number of "prime structures" as the alphabet of Lyndon languages, we can see that their ordered combinations "decompose uniquely into non-periodic terms" and "have an operation structure without inverses." At the same time, we can also see that "the structure of non-periodic terms was already hidden in the zeta construction of finite graphs." In other words, the repetition of "non-repetitive terms" can be uniquely expressed as a combination of prime elements with non-repetitive properties called prime Lyndon words. And if we consider these "prime Lyndon words" to exist as loops, we can restore the trace path in its entirety. This is McMahon's fundamental theorem. In other words, we can see that "prime structures" have a "semigroup order structure" as higher-order prime structures.

In summary

All trace structures can be described as combinations of prime Lyndon sequences.

The repetition of these prime structures can be finite or infinite, and in the case of infinity, it forms a non-cyclic trace path and becomes a higher-order constituent factor involved in the generation of zero points in the zeta structure.

(The theorem that the diversity of trace bundles in finite graphs is equal to the diversity of trace structures in trace bundles in infinite graphs)

To put it clearly, this means that there are multiple prime decomposition structures in trace bundles.

The "infinite cycle structure" in trace bundles allows for two or more different types of decomposition methods at the same time: decomposition into prime Lyndon languages and decomposition into prime structures. This has a structure similar to that of unique decomposition in ideal theory in commutative factor rings, and is thought to correspond to the phenomenon of hierarchical structures appearing in decomposability in noncommutative ideal theory.

To summarize once again, the "decomposition representation theorem for trace bundles" is as follows.

Any finite trace bundle *T* can be decomposed into either of the following forms:

(1) Lexical decomposition by prime Lyndon words

(2) Prime structural cycle decomposition

These decompositions are mutually intersecting, resulting in a noncommutative hierarchical ideal decomposition structure.

The recursive trace of a prime Lyndon sequence is the basis in the quasi-dual category.

The loop structure that is restored may actually be infinite. It may be restored to an infinite graph. However, that is redundant.

The loop structure that can be restored may actually be infinite. It may be restored to an infinite graph. However, this is redundant.

In other words, in a noncommutative structure, there are (at least) two Gödel-generating bases obtained by prime structural decomposition. Gödel arranged prime numbers and inserted an "infinite decimal" structure into them to construct a one-to-one correspondence with natural numbers.

The question of how many "prime Lyndon elements" are needed to express all "non-periodic terms" will be discussed later, when we examine the relationship between "prime structures" and "prime Lyndon elements" derived from the fixed-point property in the quasi-dual map of (regular) finite graphs, at the same time as the zero-point ring structure.

The fact that the "expressibility of the trace bundle becomes multiple" allows for two structural representations: the Euler product and the zero product. This non-commutative structure enables the two graph structures, despite having different dimensions, to mutually transform into each other. They undergo "quasi-dual transformations." The two graph structures are "reconstructed" from the trace bundle.

This is the most important content of this article.

Note that the infinite loop structure appears not as an isomorphism class of self-loops but as a noncommutative expansion structure, and that there is an order structure, which also includes cases where "prime structures continue infinitely."

This self-loop-like zeta inclusion is thought to give rise to the essential nested structure of zero points.

This will become an issue later when constructing the Riemann zeta from the graph zeta.

The decomposition of natural numbers into prime numbers has a structure where "order" is absent or broken. We introduce the gamma structure into this broken part and follow the procedure of normalization. In this way, we transition from a structure with an order structure to one without an order structure.

3. Regular Graphs and Zero Points: The Cyclicity of Zero Point Structures in Bouquet Structures

It was from Shigenobu Kurokawa's Absolute Mathematics: Introduction to Absolute Mathematics and the method of constructing Ihara zetas described therein that it became clear that the quasi-duality mapping, which is part of the theory of abstract structures, can be expressed as a concrete formula in the form of a zeta structure. The Ihara zeta function can be considered as the zeta function that counts all possible trace paths within a graph-theoretic structure.

In Shigenobu Kurokawa's *Absolute Mathematics: Introduction to Absolute Mathematics *, there is a section where the Ihara zeta function in a graph with a bouquet structure is presented as an example where the Euler product of the absolute zeta function can be constructed.

In regular graphs of the Ihara zeta function, particularly those with a "bouquet graph" structure, it has been shown that the zeros lie on the unit circle, providing a foundation for the "structural Riemann hypothesis."

Even when bouquet graphs are connected via an infinite tensor product (Kurokawa?) and constructed as non-regular graphs with infinite prime structures, the zeros remain on the circle. Regular graphs with such circular zeros are summarized as having Ramanujan-like characteristics.

Here, we first consider the zeta structure of graphs and how their zero structures form circular configurations.

Now, let us first consider the finite graph structure (regular).

The class-dual mapping transformation of the Ihara zeta function had a remarkable property. That is, the "structure of the trace bundle" remains invariant, regardless of whether the transformation is from loop-type to tree-type, or even in the case of "divergent quasi-dual zeta transformations (extensions)." This can be described as an "invariant structure in transformations."

Here, we will note only that "the graph structure itself that is transformed is infinite." From this fact, we will later see that "the prime number structure is restored within the graph structure by the quasi-dual zeta expansion."

Let us explain the "fixed point structure of the quasi-dual mapping."

First, we define the quasi-dual mapping as follows.

Definition (quasi-dual map):

For a trace sequence with an infinite recursive structure, a map that deforms the local structure (prime structure) and the global structure (trace bundle) in a noncommutative manner, such that even when the structure is deformed by the operation, the trace bundle generated from the structure returns to the same trace sequence structure, thereby becoming a fixed point of the structure as a whole.

In this case, the "structure that becomes a fixed point" is a structure that cannot be further transformed by the quasi-dual map. In other words, it preserves the "infinite recursive structure" as a "structural closure," and as a result, the zero point is stably mapped to the loop structure, which is why it is called the "fixed point structure = cyclic zero point structure."

Why does it become a "cyclic zero point structure"?

Let us consider a very simple situation, such as the structure of a bouquet graph or a (2,2)-regular graph.

These graphs have the following properties:

All nodes are isomorphic (regularity), all trace paths are closed loops, and they have a structure that leads to infinite repetition. This trace structure is classifiable by a prime Lyndon word (contracted structure), and in such a structure, even if the trace bundle is deformed, the same structure reappears.

This is a structure that remains invariant under repeated application of the quasi-duality map.

This demonstrates that the matrix representation and eigenvalues in the construction of the Ihara zeta function actually indicate that “the determinant structure remains unchanged” because the trace bundle does not change.

From the above, in the Ihara zeta function, the trace paths corresponding to each prime loop (cycle) are:

Figure 2. The Ihara-type zeta and the “prime” length of paths.

各素ループ（サイクル）に対応するトレース経路は、

$$\prod_{[P]}(1-u^{\ell(P)})^{-1}$$

の形で表される（ここで $\ell(P)$ はループの長さ）

Ihara-type zeta function of bouquet graphs: Integration of loops of prime length.

If the graph is regular and Ramanujan-type, the eigenvalues of the corresponding matrix (such as the infinite Laplacian) are concentrated on the unit circle.

At this point, the zeros of the zeta function also appear on the unit circle.

What is important here is the fact that under the transformation “u→q＾-s,” the zeros are mapped to “Re＝1/2” and satisfy a condition similar to the Riemann hypothesis.

Let us introduce an auxiliary structural theorem.

The Iihara zeta function of a regular finite graph converges to a rational type. That is, there exists a homomorphism or isomorphic structure between “prime structures” (prime cycles) (= prime numbers) and “prime Lyndon elements” (prime Lyndon loops) (= zeros).

In other words, the number of minimal loops in the “quasi-dual divergent zeta structure” and the structure of the “combinatorial non-periodic terms” derived from them possess an infinite ideal structure generated by “prime Lyndon elements.” In other words, since the determinant representation of the Iihara zeta function of a regular finite graph converging to a rational type is finite, the number of prime structures is finite, and the same number of “zero points” arise, but due to the “duplication of zero points,” the following equation holds.

Number of prime structures ≥ number of prime Lyndon elements (number of zero points) (reduced by the number of zero point duplicates)

This means that, in the case of the highly restricted structure of regular graphs, the number of prime structures allows us to infer the number and structure of prime Lyndon elements, from which all trace structures can be reconstructed. Can we call this the finiteness of prime Lyndon elements in regular finite graphs?

This demonstrates an important structural condition in the non-commutative "zeta structure" generated by "quasi-dual divergent deformation."

It can be decomposed into a "zero product" in the same rational (finite) form.

Furthermore, it shows that the Iihara zeta structure has a "zero point" on the unit circle defined by the length of the prime Lyndon element. In other words, this means that regular graphs with Ramanujan-like properties have a "zero point structure" on the unit circle.

In summary,

Auxiliary proposition

The Ihara zeta function defined for graph structures that are fixed points of the quasi-dual map (e.g., regular bouquet structures) has a zero point structure on the unit circle.

Furthermore, this zero point structure is isomorphic to the structural stability of the trace bundle.

[Visual representation] Construction diagram (schematic)

[Trace bundle (infinite repetition)]

↓ Quasi-dual map

[Tree-like expansion (decomposition)]

↓ Closure reconstruction

[Restoration as a loop structure (circle)]

↓ Zeta construction

[Determinant zeta] − Eigenvalue → [Zero point on the unit circle]

The "circular zero point structure" is the result of the infinite loop structure as a fixed point of the quasi-dual map being analytically represented, and its essence is guaranteed by the regularity of the graph, the commutativity of the quasi-dual map, the closure structure of the trace bundle, and the stability of the eigenvalues of the characteristic equation of the zeta determinant. This results in the appearance of a circular structure as a stable topological structure for the hierarchy of quasi-dual transformations.

The phenomenon itself that the zero-point structure remains unchanged is the analytical shadow of the fixed-point structure in the quasi-dual map, which constitutes the decisive construction theorem of this cyclic zero-point structure.

Is “not changing” decisive?

To recap, the quasi-dual map is

a map that sublimates the infinite recursive structure into “elementary structure → elementary Lyndon sequence → higher-order structure,” temporarily transforms it into a tree-like expansion, and then recursively restores the structure in a closed manner.

However, in general, this transformation is non-commutative and non-associative, causing complex structural transformations. Since “trace bundles” are typically transformed, this situation indicates the existence of a special “fixed-point situation” or “structural kernel.”

In the theoretically constructed “divergent quasi-dual zeta,” the zero points indeed moved as the prime structures were transformed. Therefore, conversely, zero points that do not move even when transformed are

= structural fixed points

= stable category in the quasi-dual category

= essence of the circular zero-point structure

This demonstrates the “structural theorem.”

Auxiliary theorem

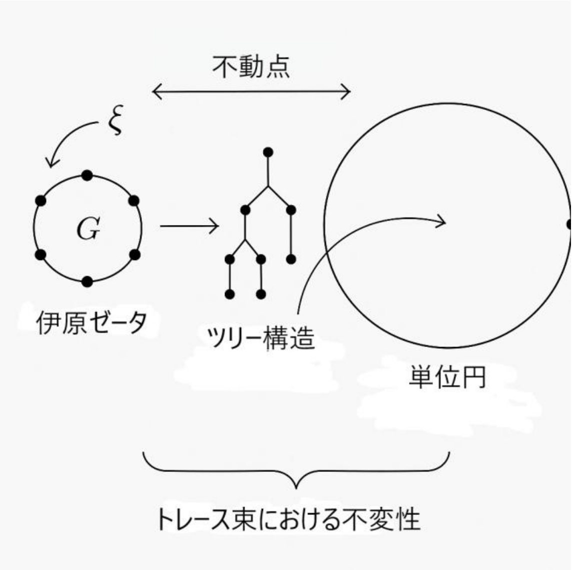
Circular zero-point structure as a quasi-dual fixed point

The Ihara zeta structure based on regular bouquet graphs is a trace bundle fixed point with respect to the quasi-dual mapping. In this case, the zero-point structure of the zeta function exists stably on the unit circle. This circular structure is also stable with respect to the closure of the iterative structure in the trace bundle and becomes a structural fixed point with respect to the iterative expansion by the quasi-dual map(see Figure 3).

With this structural theorem, the following can be said.

It becomes possible to consider “why the zero points form a circle” geometrically, algebraically, and categorically. The “boundary between when the zero points move and when they do not” becomes visible. As a concrete example, we can connect “stability of the quasi-dual map” with “Ramanujan property of the zeta structure.”

Figure 3. The relationship between Iihara Zeta and Trace Bundles.



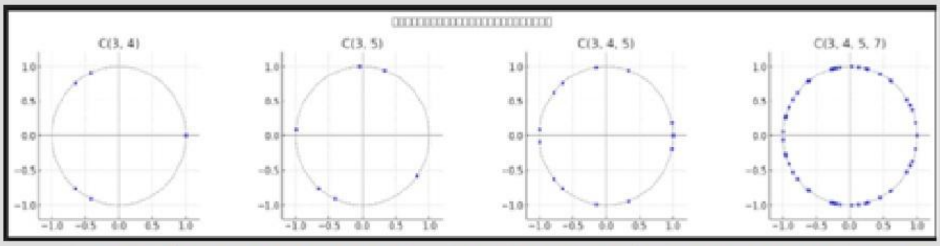
In the Ihara-type Zeta, the “trace bundle” structure is preserved.

In this divergent reconstruction, it is important to note that the reconstructed loop paths are basically noncommutative and have an ordered (directed) structure. However, at the zero point, a symmetric element (conjugate element) appears in the circularity. When it comes to the Nth root of 1, the conjugate appears slightly less than half of N. At this point, two possibilities arise in the restoration. That is, when there is a complex zero point symmetric to the real axis, a loop may have directedness in both forward and reverse directions and can be combined into a single loop. In this case, the directed edges lose their ordered structure. It is important to distinguish this from “multiple roots.” In most cases, it is difficult to distinguish between the two reconstructions structurally. They can be separated or combined.

The “cyclic nature” of the zeros of the graph-theoretic Ihara zeta function of regular graphs has been clarified, and by connecting these zeta structures, it is possible to construct a zeta expansion with the same regular q-Ramanujan-like tree-type expansion, and that its trace structure also has a regular q-Ramanujan-like tree-type structure, it becomes clear that they simultaneously have “numerous, and in an extended sense, infinite, zero points constructed on the unit circle.”

The actual calculation at this point is shown in the figure below.

Figure 4. Zero points of Ramanujan-type graphs.



各円環グラフCₙのテンソル積構造に基づいて得られる固有値の分布（= ゼータ構造に対応する値）は、複素平面上の単位円内部に自然に配置されました。

The zero points of Ramanujan-type graphs lie on the unit circle.

The meaning of this structure will be discussed later in the “Zero Point Structure Theorem.”

4. Gamma normalization and order breaks

Now, using the Ihara zeta function related to the (regular) graph zeta function as a concrete example, we have described how a special “divergent deformation” in quasi-dual transformations constitutes a zeta structure. At this point, we notice that, metaphorically speaking, “=” appears to have a noncommutative action.

(Loop structure) = (←quasi-dual transformation→) = (tree structure)

At this point, “=” moves back and forth between the left and right sides, but each time it does so, it subtly deforms the “trace bundle” of the structure. We can see that when this deformation is a “commutative structure,” it is a “general equal.” This commutative structure, as can be seen from the previous discussion, means the same thing as “the structure of the trace bundle reaching an invariant fixed point through a quasi-dual transformation.” A quasi-dual mapping can generally be described as a non-commutative transformation that “moves back and forth between loops and trees.” The structure of the “trace bundle” is generally deformed by deformation. Fractal structures, such as the human vascular system, are both cyclic and tree-like in the peripheral vascular system. In other words, it is noteworthy that in many cases, fractal structures are “mixed structures of loops and tree structures.”

In such non-commutative transformations, the “Euler product” of the Ihara zeta function → “generating function notation that converges to rational form” appears to be unique, but in general, for example, “zeta deformation,” that is, zeta structures that have undergone divergent quasi-dual deformation, can they be restored to the structure of the Ihara zeta function as they are? In my opinion, the answer to this inverse problem is "they are likely to be multivalued." In other words, it is not considered to be uniquely reversible. However, I do not think this multi-valuedness will be an issue in this discussion.

Therefore, the divergent deformation observed in the quasi-dual "zeta deformation" of the Iihara zeta function can be regarded as the shadow of this multi-valuedness.

In this way, the "trace bundle" can be imagined as a "quantum existence" between structural deformations. It is a structure that has not yet taken shape before deformation is applied. The fact that 'loops' (waves), "trees" (particles), and class-dual transformations are fundamentally non-commutative and multivalued is somehow meaningful.

By the way, such graph zetas naturally possess what might be called "ordered prime factorization." That is, they "preserve the order in which the paths are traced." In this section, we proceed with the motivation that "if we can somehow break this order structure, we should be able to construct a Riemann zeta-like structure from the graph zeta."

And the first step in breaking this order is thought to be the gamma factor. "Ordered" means that "the order in which loops in the graph are traversed is preserved." If we can handle this, does it mean that "a graph zeta with N prime structures can be transformed into a factorial zeta"?

Here, we will take a brief look at the F_1 geometric structure through the multiplicative function b(n).

This gamma factor is b(n), which is constructed as a multiplicative function, and by acting on the graph zeta via a determinant representation, it is thought that analytic continuation to a Riemann zeta-like structure becomes possible. Note that this is also an example of how considering multiplicative functions naturally leads to F_1 geometric situations.

Now, let us consider this concretely.

First, as preparation, we introduce the multiplicative factorial function b(n) that indicates the number of overlaps.

In the graph zeta function, the order in which loops around prime structures are traversed is preserved. In other words, it is "ordered factorization." Due to the universality of the "trace bundle," there is isomorphism between "ordered prime factorization" and "ordered prime Lyndon factorization." We proceed with the discussion while noting this point first.

Counting the number of overlaps(see Figure 5),

Figure 5. Number of overlaps.

```
 1 : 1
 2 : 1
 3 : 1
 4 : 2        (2^2 → 2! = 2)
 5 : 1
 6 : 1        (2×3 → 1!×1! = 1)
 7 : 1
 8 : 6        (2^3 → 3! = 6)
 9 : 2        (3^2 → 2! = 2)
10 : 1
11 : 1
12 : 2        (2^2×3 → 2!×1! = 2)
13 : 1
14 : 1
15 : 1
16 : 24       (2^4 → 4! = 24)
17 : 1
18 : 2        (2×3^2 → 1!×2! = 2)
19 : 1
```

From the above observations,

Figure 6. Multiplication function of overlapping numbers.

$$b(n) = \prod_{p^e \| n} \Gamma(e+1)$$

it can be predicted that

This is a multiplicative function, as can be seen by taking only the terms of prime number p raised to the power of n from n, counting their overlaps as n!, and amplifying the number of overlaps of the remaining numbers. This can be understood by mathematical induction over prime numbers.

Intuitively, we might expect the generating function representation,

Figure 7. Divergence of generating function.

$$\zeta(s) + \zeta(s)^2 + \zeta(s)^3 + \cdots = \left(\sum_{k=1}^{\infty} \zeta(s)^k\right) = \frac{\zeta(s)}{1-\zeta(s)}$$

to hold, but in fact it diverges. However, here we introduce an F_1-geometric idea.

For example, let us count the number of overlaps for $2^3 = 8$.

$8 = [(1, 2, 2, 2), (2, 2, 2), (1, 2, 4), (2, 4), (4, 2), (1, 8)]$ $3! = 6$

In other words, if we count the identity element 1 as an absorbing element, the generating function of the infinite sum of the zeta function above does not diverge, and it is clear that it holds as the generating function of b(n). This idea naturally arises in the construction of the Kurokawa tensor product and in the theory of zero point structure. There, the pole of the Riemann function at S = 1 is interpreted as an "absorbing element."

Formally, it is as follows.

$4 \cdot 1 + 1 \cdot 4 = 1 \cdot 4 \quad 4 \neq 1 \cdot 4$ ($4 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 4$ does not diverge)

It may be possible to interpret this as "partial non-commutativity remaining in the unit element," but such considerations will be postponed for later.

In other words, the natural idea arises of taking the coefficient field of the multiplicative function from the F_1 geometric space, but this will be discussed later, so I will leave it at a suggestion here.

The gamma factor is important, but in fact, in this discussion, its existence is demonstrated intrinsically.

5. Quasi-duality, gamma normalization, and mapping to the critical line

Originally, I was not interested in zeta functions, or even mathematics, but rather in structures that seemed far removed from number theory, such as "noise phenomena," "trace structures," "dynamic transformation," "fractals," and "quasi-duality." However, I naturally came to discover a kind of zeta structure. I may write about this somewhere else. I was aware that I was constructing a highly abstract structure that possessed zeta-like properties.

I knew that the Ihara zeta was related to "loop structures," but I could not imagine that the structure in my mind (quasi-dual closure) could be expressed in any formulaic way, so the relationship remained unclear, and I considered it to be something like a non-commutative zeta structure.

However, when I began researching Iihara zeta, I realized that it was an example of the structure I had in mind. Furthermore, I discovered that it was possible to "transform the zeta divergence."

As I continued to transform these graph-like zeta functions, I realized that they possessed a zero configuration that satisfied the Riemann hypothesis, and that this configuration could be constructed through an explicit determinant structure.

I will refer to this structural phenomenon as the "virtual Riemann zeta structure."

It possesses a structural self-reference that mirrors the essential properties of the

Riemann zeta function, and it describes the transformation process from discrete, constructive elements such as prime numbers and Lyndon elements to the zero structure on the critical line.

Regarding the property that the zero structure of the graph zeta function is arranged on the unit circle, by performing a variable transformation (u→q^-s), this zero sequence can be mapped to the critical line (Re(s)=1/2). This allows the function to be constructed without breaking the symmetry of the inversion formula of the correction term.

At this point, there exists a matrix representation (complex matrix) that applies a quasi-dual transformation to all eigenvalues of the matrix, and this operation constructs the transition from the "Ramanujan-type circle structure" of the zeta function to the "Riemann-type critical structure."

Figure 8. Transformation of graph paths using divergent quasi-dual mappings.

伊原ゼータ関数はグラフの閉路（primitive loop）を対象にし、

$$Z(u) = \prod_{[P]} \left(1 - u^{\ell(P)}\right)^{-1}$$

と定義されます。

ここで、Möbius変換に類似した操作は：

- **各ループ P を、他のループとの合成／変形で再構成すること**
- これは、**トレース系列の長さ構造（距離）を変形する操作**

たとえば、長さ ℓ のループ列を $\frac{a\ell+b}{c\ell+d}$ のように写すなら、

- 対応する u^{ℓ} の変数が、$u^{f(\ell)}$ に変わり、
- ゼータ関数の項全体が「再定式化」されます

これは、「伊原ゼータに対する変数変換的な操作」＝Möbius写像の**離散版**に相当します。

Transformation that deforms loops in Iihara-type zeta functions using a class dual mapping = Möbius-like.

Generally, variable transformations alter the structure of loops or decomposition structures.

At this point, the question arises: Is there a variable transformation that deforms the zero points while preserving their decomposition structure, without altering the structural conditions of the graph, as in the quasi-dual transformation? For example, the Möbius transformation has a function similar to that of a quasi-dual mapping, changing a circle into a line, for instance. The quasi-dual mapping had the function of "mapping an infinite concentric circle fractal into a Cartesian spiral." In my theory, a spiral is homeomorphic to a line. (This means that they are currently indistinguishable theoretically, though there are interpretations that distinguish them.)

At this point, the intuition arises that "perhaps the quasi-dual mapping itself functions as a quasi-dual transformation, akin to an ∞-Möbius transformation." This will be actually implemented later.

At this point, we utilize the fact that the lengths of the prime paths corresponding to the different "prime Lyndon loops" of the Ihara zeta function are identified with the prime numbers under regularity, and in the regular bouquet zeta function, we separate the zero points that overlap as the Nth roots of unity on the unit circle, as it were, for each prime number, and infinitely decompose them into a circular structure of prime powers. We then consider the operation of "quasi-dual transformation" on this infinitely decomposed concentric circle structure. I will explain this in detail later, but this picture will be useful.

At this point, I will explain the details later, but we consider the Kurokawa tensor product and the noncommutative zero structure.

From the "invariance of the trace bundle under class duality transformation" in the regular graph zeta function, if we perform a "quasi-dual divergent zeta transformation" on an "infinite bouquet graph" (a graph with infinitely many distinguishable edges attached to each node), under the regular condition, the following holds.

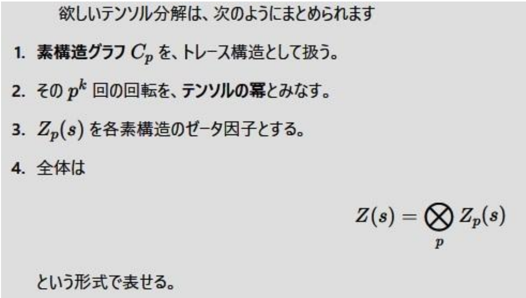
That is, the prime Lyndon factorization appearing in the "trace path formed by an infinite number of prime structures" is isomorphic to the "prime structure" (i.e., prime numbers) and the ordered combination of prime Lyndons (i.e., the decomposition of prime numbers into ordered natural numbers). This follows from the fact that the trace bundle remains unchanged in the "quasi-dual transformation," and thus the prime structure decomposition and the prime Lyndon decomposition coincide. This justifies the correspondence between "prime Lyndon loops" and "primes" mentioned earlier. Note that this does not hold in the case of non-regularity.

From this, the following can be derived.

Quasi-dual invariance of trace bundles In infinite flower bundle graphs, ordered infinite Kurokawa tensor representations are isomorphic to the structure of "trace bundles" in ordered "prime Lyndon loop structures."

Let me explain.

Figure 9. Viewing graph joins as tensor products.



欲しいテンソル分解は、次のようにまとめられます

1. **素構造グラフ** C_p を、トレース構造として扱う。
2. その p^k 回の回転を、**テンソルの冪**とみなす。
3. $Z_p(s)$ を各素構造のゼータ因子とする。
4. 全体は

$$Z(s)=\bigotimes_p Z_p(s)$$

という形式で表せる。

Using a divergent quasi-dual mapping, the Ihara zeta graph is reconstructed, allowing the tensor representation of the zeta structure to be recovered.

Consider the tensor product of such a graph structure. This tensor product can be reinterpreted as a structure in which each component is regular and connected by a single node or infinitely overlapping F_1 geometric “absorbing elements.”

And from the previous consideration, under regular conditions, the infinite Kurokawa tensor product, which connects prime Lyndon loop structures with lengths across all prime numbers, corresponds to the structure of ordered “prime factorization.”

In other words, there is an infinite series in which “quasi-dual divergent zeta extensions” are performed repeatedly in various ways on bouquet graph zeta.

At this point, an infinite graph structure can be restored, but according to the “uniqueness of prime Lyndon decomposition” (McMahon's theorem), if the path length n can be factored, the path length can be further reduced (zeta expansion) to a path that is a divisor of that factor, and ultimately, the path length will be reduced to pass through all prime numbers, i.e., 2, 3, 5, 7, etc.

This can be called “fractal restorability of prime number structures by zeta expansion.”

This remarkable property is an astonishing fact, and the fact that this zeta expands rapidly and shows a certain pattern is reminiscent of Shigenobu Kurokawa's theory of zeta conjugation by Kurokawa tensor product.

Now, this occurs because the graph is “Ramanujan regular,” i.e., “regular.” What would happen if it were ‘irregular’? Perhaps the “prime Lyndon structure” would expand further, exhibiting the structural characteristics of a certain ideal class group, and the graph zeta function would become a shadow of the Dedekind-type zeta function? Such a conjecture arises.

We will revisit this point later.

Here, the structural theorem from the previous chapter, which states that the zeros of a regular bouquet graph are arranged in a circular configuration, becomes important.

That is, for all prime numbers, the zeros are embedded within the unit circle, and the ordered structure where there are infinitely many overlaps only at the unit element is the essential structure of the Kurokawa tensor product and the ordered prime Lyndon loop structure.

As will become important later, note that due to the oddness of prime numbers, the configuration of zero points is extremely biased toward the negative complex plane. Furthermore, these configurations do not have multiplicative overlaps outside the unit element, depending on the properties of prime numbers. We will refer to this as "**anti-idealness.**" And, as it were, across an infinite number of prime numbers, it can be observed that finite fields are F_1-structural structures in which the additive structure is broken only in the overlapping part of the unit element.

This operation structure can be expressed by a transformation that rotates around the unit circle.

We consider the following quasi-dual map that transforms this rotating operation structure from a "circular structure" to a "linear structure (half-critical line)."

By considering this transformation, the zero points on the unit circle with prime Lyndon operation structures are all transformed onto the line with real part s = 1/2, while simultaneously preserving the transformation $s \to 1 - s$ satisfied by the Riemann zeta function.

Note that the "order structure" is preserved within the graph zeta structures indicating prime structures or prime Lyndon structures.

Additionally, it is important to note that the "infinite prime numbers overlap with multiple prime Lyndon zero points on the unit circle structure" has a quasi-dual transformation aspect of "transforming from an infinite circle to a line."

This graph zeta has a determinant representation by the Ihara zeta, and as shown earlier, since the class dual transformation also has characteristics, it can be seen that this transformed ordered zeta structure also has a determinant representation and a transformation structure that transfers it.

The question is whether it is possible to create a function that satisfies the structure predicted by the Riemann zeta function structure, as it were, "forgetting" this order.

In other words, let us first consider the zeta structure decomposed by the Kurokawa tensor product of infinite flower bundle graphs.

Consider the Ihara zeta function Z(u), take the logarithm in the same way, and convert it to a generating function notation. The form of the Ihara zeta function reduced to a rational system is written below(see Figure 10). Note that both of these are "combinatorial representations of all paths traced in a regular finite graph."

Figure 10. Basic transformation of the Ihara-type zeta function.

$$\log Z(u) = \sum_{[P]} \sum_{m=1}^{\infty} \frac{1}{m} u^{m\ell(P)} = \sum_{k=1}^{\infty} \frac{N_k}{k} u^k.$$

$$Z(u) = \prod_{[P]} \frac{1}{1 - u^{\ell(P)}}.$$

Here, it is clear from the "invariance of the trace bundle" that p can be either a prime Lyndon element or a prime structure. This Ihara zeta has a determinant representation in both prime structures and prime Lyndon elements, but the determinant is still arranged "anti-ideally" on the unit circle and geometrically in F_1.

As mentioned earlier, by repeatedly performing "quasi-dual zeta extensions" in various forms, l(p) reduces to a structure that passes through the prime numbers 2, 3, 5, etc. Please note this.

Here, we apply a quasi-dual transformation T(A) similar to an operation that infinitely applies a Möbius transformation that moves a circle to a line.

Then, the expansion formula of this matrix is transformed into a structure similar to the ordered Riemann zeta function.

In summary, the process is as follows.

Figure 11. Flow of quasi-dual morphism.

花束ゼータ → 類双対変換 → ガンマ正規化 → オイラー積 → リーマンゼータ

Bouquet Zeta → Quasi-dual transformation → Gamma normalization → Euler product → Riemann zeta

First, we define the quasi-dual transformation as a continuous deformation that maps a non-commutative discrete structure to a continuous space as follows.

That is, we consider the quasi-dual deformation that maps the prime power p^s = e^(-slog(p)) as an infinite concentric circle structure scaled by a prime number, and then maps it to a spiral and a straight line.

Figure 12. Explanation of quasi-dual morphism.

これを「類双対変換」と見ると：

$$\boxed{u^p \longrightarrow p^s \longrightarrow \bar{e}^{s\log p}.}$$

つまり、

- u^p は閉路構造（生成論的）
- p^s は解析的指数（積分構造）
- e^{s　log p} は螺旋（位相展開）

The quasi-dual morphism log P corresponds to a spiral component.

Note that this is the exact representation of the quasi-dual mapping shown in the first diagram(see Figure 12 and Figure 1), “Plotting the infinite concentric circle structure as a Cartesian spiral.” However, while the concentric circle structure consists solely of “prime circles,” the Cartesian spiral, as a scaling-invariant fractal structure, serves as an example of the quasi-dual mapping where “non-commutative fractal structures are transformed into commutative continuous fractal structures.”

To explain what is happening here, the zeros of the Ihara zeta function that were originally arranged along the unit circle are first “decomposed into an infinite concentric circle fractal at scale logp” in this process. Then, the decomposed infinite concentric circle fractal is transferred to the Riemann zeta critical line (Re = 1/2) as described in the “regular Ramanujan Ihara zeta theorem,” while retaining the “zero points” on the circle. In other words, this zeta satisfies the Riemann hypothesis-like condition.

To summarize what we have done so far, please note that we have been repeatedly performing “quasi-dual transformations.”

In other words, starting with a bouquet graph and performing a “quasi-dual divergent zeta extension,” the prime number structure that constitutes natural numbers was restored within the graph structure. Then, by performing the same “quasi-dual transformation” again, a new zeta function is obtained. In other words, the quasi-dual transformation is a mapping that transforms a fractal into another fractal, and the zeta function possesses fractal characteristics.

By this process, the non-commutative structure is transformed into a continuous structure while preserving its fractal structure.

Specifically,

Figure 13. Projection onto a straight line by quasi-dual morphism.

素数べきの位相

$$z_{p^m} = p^{im} = e^{i\,m\log p}.$$

螺旋パラメータ

$$r = a\,\theta,\quad \theta = m\,\log p. \quad \Longrightarrow \quad r_{p^m} = a\,m\,\log p.$$

臨界線への漸近

無限べきの積分極限で、
素数 log スケールが Re(s)=1/2 に線形射影される。

through this transformation, the "infinite prime Lyndon element (zero point)" expansion of the (regular) bouquet zeta function is transferred to the Euler product.

Figure 14. The Euler product transformation in quasi-duality.

類双対変換でオイラー積に移す

- 類双対変換：

$$u^p \longrightarrow p^{-s} \longrightarrow e^{-s\log p}.$$

- 花束ゼータのリンドン積は、

$$Z_{\text{花束}}(u) = \prod_P (1-u^{\ell(P)})^{-1} \quad \longrightarrow \quad Z_{\mathrm{Euler}}(s) = \prod_p (1-p^{-s})^{-1}.$$

- ここで、素閉路の長さ $\ell(P)$ が $\log p$ に対応するので、
無限素リンドン構造が素数構造に写る。

Euler product can be obtained by quasi-dual morphism.

At this point, note that in the case of the bouquet zeta function, the order structure remains and is preserved, so the multiplicative function b(n) for the overlap number is likely implicitly embedded. In other words, it is as follows.

Figure 15. The Gamma normalization.

ガンマ正規化

- 花束ゼータのリンドン積は、多重度が階乗的に発散する：

$$Z_{\text{花束}}(s) \sim \sum_{n} b(n)\, u^{n}.$$

- これを **ガンマ正規化**（階乗分割）で除去する：

$$Z_{\Gamma}(s) = \frac{Z_{\text{花束}}(s)}{\Gamma(b(n))} \quad \text{（概念的）}.$$

- これにより、重なりを除去して、
素閉路の積が純粋なオイラー積に一致する形に整う。

By removing the "number of overlaps" in the "prime Lyndon product expansion" using this multiplicative function, it is thought that the Riemann zeta-like Euler product is generated in a constructive manner.

However, when performing the above transformation, the result reaches the Euler product.

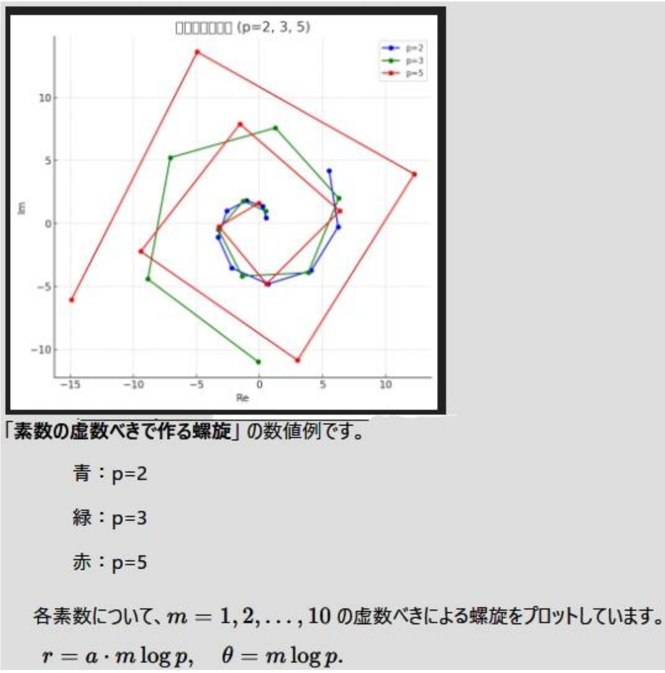
Figure 16. The Euler product of the Riemann zeta function.

$$Z_{\text{Euler}}(s) = \prod_{p} (1 - p^{-s})^{-1}.$$

At this point, it is thought that analytical continuation naturally occurs through continuous transformations in the quasi-dual mapping, and that the movement of zeros also takes place. Strictly speaking, this only considers the "discrete action" part, and it is unclear at this stage how the "continuous action" part occurs. However, the points on the concentric circles of the prime number scale are, in other words, the transformation from "prime Lyndon elements" to "zero points."

I have summarized this continuous plot on a computer, so please take a look at the figure.

Figure 17. Calculation of prime number spirals.



Prime spiral calculated by computer.

At this point, through quasi-dual transformations, the overlap of "prime Lyndon decompositions" and "zero point decompositions" demonstrates that the Riemann zeta function also possesses "prime number structures" and "higher-order prime number structures" (i.e., zero point structures), revealing an example of higher-order prime structures in commutative domains. The non-commutative overlap numbers are reduced and transferred to commutative structures.

In other words, the "gamma factor" was actually inherent in the variable transformation.

And this is precisely supported by the transition from

(Euler product) →← (Hadamard (zero point) product).

Also, note that due to the "prime nature" of prime numbers, this zero-point structure has no overlap and maintains "anti-idealness." The concentric circles of prime numbers are prime roots of 1, so none of them overlap and are thought to have a structure similar to that of a finite field. However, in reality, the identity element forms "infinite overlap" in this infinite concentric circle, acting as an "absorbing element," and partially breaks addition.

This is reflected in the Riemann zeta function, where there is a pole at s=1 and no zero points at -1, which can be seen as a reflection of “symmetry breaking.” As a result, it possesses what is known as an “F_1 geometric operation structure.”

Here, the non-commutative Kurokawa tensor decomposition is mapped to the commutative Kurokawa tensor decomposition (Euler product). Moreover, the construction of the “ gamma normalization ” b(n) when establishing this correspondence is calculated by considering the Nth power sum of the Riemann zeta function as the coefficient field of the F_1-geometric “absorbing element,” that is, through the F_1-geometric action. This is an interesting fact.

In other words,

the “Kurokawa infinite tensor product” of the bouquet graph zeta function ≅ zero-point structure (F_1 geometric)

From this isomorphism, one might imagine structural theorems such as the finiteness of rational points of elliptic functions.

This generative Riemann zeta function was analytically continued by being mapped via a scaling-invariant Cartesian spiral, but this raises two questions.

Namely, does this quasi-dual transformation and gamma normalization procedure provide a method for calculating the zeros of the Riemann zeta function from the “prime Lyndon zeros” of the bouquet zeta function, which are already computable as nth roots on the unit circle?

Furthermore, what is the “operator algebra” that this quasi-dual transformation acts upon? Intuitively, it is imagined to be something like a “fractal category” that deforms fractals into fractals while preserving fractality.

This is because there are various ways to transform a fractal into another fractal. Fundamentally, it is a combination of points, lines, circles, waves, and spirals, and whether all of these can be constructed through such combinations is an important topological problem in fractal theory.

Another question is why, through such a construction, the zero points plotted on an infinite concentric circle are transferred to the critical region of one-half(1/2).

Looking at the variable transformation known as the conjugate quasi-dual transformation, we can see that it is a mapping that transforms rotational states into lines on the complex plane. Considering this along with the symmetry of the Riemann zeta function under $s \to 1 - s$, the following is assumed:

Figure 18. Projection of zero points onto critical line by quasi-dual transformations.

$\boxed{\text{類双対変換：} u^p \longrightarrow p^{2\pi s} \longrightarrow e^{s\,2\pi \log p} \quad \text{は、} \quad \text{位相＝螺旋＝解析接続の対称軸} \quad \text{を } \mathrm{Re}(s) = 1/2 \text{ に整列させる。}}$

- 「螺旋は位相（波数）」
- 「直線プロットは log スケール」
- 「解析接続の汎関数等式が対称軸を Re(s)=1/2 に決める」

This can be rephrased as follows: "The Riemann zeta function, which is constructed generatively, has a structure in which its zero points are shifted to the critical line, i.e., the region where Re = 1/2, through a quasi-dual transformation." This is what I mentioned earlier as the "intrinsic nature of the gamma factor."

In other words, the "inversion formula" of the gamma factor is inherent in this transformation.

With this, I believe I have provided an overview of one approach to constructing a class dual mapping for a "generative Riemann zeta-like structure."

From the regular Ramanujan graph theorem, the Riemann hypothesis, i.e., Re = 1/2, holds.

Now, the problem is that the quasi-dual mapping

Figure 19. Relationship between quasi-dual morphism and Euler products.

類双対変換でオイラー積に移す

- 類双対変換：

$$u^p \longrightarrow p^{-s} \longrightarrow e^{-s \log p}.$$

- 花束ゼータのリンドン積は、

$$Z_{\text{花束}}(u) = \prod_P (1 - u^{\ell(P)})^{-1} \quad \longrightarrow \quad Z_{\mathrm{Euler}}(s) = \prod_p (1 - p^{-s})^{-1}.$$

- ここで、素閉路の長さ $\ell(P)$ が $\log p$ に対応するので、
無限素リンドン構造が素数構造に写る。

is "discrete." And it does not clearly demonstrate proper "fractal nature(see Figure 16)."

In other words, it is necessary to incorporate a continuous structure into the quasi-dual mapping so that it maintains scaling and satisfies fractal properties through a fractal enlargement mapping.

To achieve this, we must reinterpret the concentric circle structure appearing as "prime powers" such as 2, 3, 5, 7, etc., using log scaling, which is said to be indicated by the increasing sequence of prime numbers, and consider a fractal enlargement mapping within that framework. the “fractal nature of the infinite prime number concentric circle structure and the infinite prime number spiral structure in log scaling” must be transferred to a straight line using “quasi-dual deformation.”

In other words, while it has already been demonstrated that the “Euler product” can be constructed from the discrete distribution of prime power concentric circle structures, the goal is to “connect the structures between circles with a continuous structure,” i.e., to analytically connect them.

In practice, if one simply uses a “Cartesian spiral,” the terms collapse, rendering the transformation meaningless.

In other words, the task is to compress the “fractal structure” with its power-law expansion using logarithmic scaling and then insert a new “fractal structure” into it. By doing so, it becomes possible to continuously connect the “concentric circle structures expanding through prime powers.”

By doing so, the “zero points on the prime number infinite concentric circle structure” are transferred to a “non-Cartesian logarithmic scaling spiral” rather than a Cartesian spiral, and if it is shown that they align on a straight line, that is the endpoint in the construction of my generative Riemann zeta function.

This is an example of the general construction method:

non-commutative, discrete zeta structure → commutative, continuous zeta structure

.

This type of quasi-dual transformation can be seen as a Möbius transformation that maps “circle → line,” or more precisely, “infinite concentric circles → spiral shape → line (critical line Re = 1/2),” which is akin to an “∞-Möbius transformation.”

In other words, it is a transformation that maps the infinite expansion of concentric circles, along with the surrounding space, into a continuous spiral-shaped expansion (or local space) while preserving fractal scaling. In this case, the “fixed point” would be the invariant part within the “minimal structure” of the fractal structure.

And the core is this.

Figure 20. Projection of zero points in a trace bundle onto a straight line.

$\mathrm{Tr}(\text{類双対作用}) = \zeta(s)$ の臨界線上零点 $\Re(s) = \frac{1}{2}$.

Translate the zero point on the circumference to the critical line (Re = 1/2) using a quasi-dual morphism. The spiral structure reflects the logarithmic length of prime cycles, showing alignment with ζ(s) zeros.

6. What is the Hilbert-Polya operator?

Let us return to the Iihara zeta function, which is a regular bouquet graph structure that restores all prime numbers constituting natural numbers.

The determinant representation of this zeta function can be constructed by calculating the non-periodic terms, prime Lyndon elements, and their connectivity, in accordance with the definition of the Ihara zeta function.

Considering this meaning, it seems that there is a kind of decay between prime numbers and the density of zero points,

Prime numbers ← log ← Zero points (density)

However, this is not a rigorous construction. The basic idea is that they are homeomorphic, meaning that there is a sense of being nested within a log-like structure. This will become important later.

Now, a quasi-dual transformation is applied to the matrix representing Ihara's zeta function.

Figure 21 . Continuation of local quasi-dual maps.

定義（類双対写像）

定義 1.4

素閉路変数 u^p に作用する**類双対写像** $\mathcal{D}$ を

$$\mathcal{D}(u^p) := e^{-s\log p} \quad (s \in \mathbb{C})$$

で与える。この作用素は全素数についてのテンソル積で拡張される：

$$\mathcal{D} := \bigotimes_{p\in\mathbb{P}} \mathcal{D}_p.$$

This quasi-dual transformation was previously defined only locally for each prime number, so we will extend it to the entire structure. This is illustrated in the figure below.

If we can apply such a tensor product to the Ihara matrix, it will become what is known as the “Hilbert-Polya” operator(see Figure 23).

When considering this, the previous idea is useful.

Figure 23. Structural diagram of continuous quasi-dual morphism.

定理（構成論的ヒルベルト・ポリヤ作用素）

定理 1.5

花束ループグラフ G のゼータ拡大を発散的類双対写像で写像するとき、
その行列式表示は次のように与えられる：

$$Z_G(u) = \prod_{[P]}(1 - u^{|P|})^{-1} \quad \xrightarrow{\mathcal{D}} \quad \zeta(s) = \prod_{p}(1 - p^{-s})^{-1}.$$

このとき、

- 変数写像は素閉路の次数を対数スケーリングに写す Mellin 核に一致する。
- 得られる行列式は自己随伴作用素のスペクトルを持ち、その固有値は臨界線上に配置される。

First, keep the following flow in mind.

In this way, the non-commutative order structure is controlled commutatively by the gamma factor.

Figure 23. Structural diagram of continuous quasi-dual morphism.

$$\underbrace{\text{素閉路}}_{\text{局所}} \Longrightarrow \underbrace{\text{行列ブロック}}_{\text{表現}} \Longrightarrow \underbrace{\text{入れ子複行列環}}_{\text{順序構造の統合}} \Longrightarrow \underbrace{\text{類双対変換}}_{\text{臨界線写像}}$$

In other words,

Figure 24. Block Matrix Materials.

素数ごとに構成されるブロック行列

$$M_p = \begin{pmatrix} 0 & \log p \\ \log p & 0 \end{pmatrix} \quad \text{で} \quad M = \bigoplus_p M_p$$

Note: This initial Hilbert–Pólya correspondence is structural only and will be revisited in the Appendix 1.

consider the following matrix.

First, since the diagonal components are zero, the prime number structure itself does not remain unchanged (non-triviality). The off-diagonal components are log$p$, and the logarithmic scale of prime numbers becomes the nested “jump width.” Symmetry exists, and reverse paths and dual paths are treated equally (quasi-duality).

The concentric circle structure is determined by the prime number p, with the “circumference” being log$p$. Therefore, the log$p$ of the prime number can be said to correspond to the phase angle and the angular velocity of the spiral.

This block matrix can be considered as an operator that rotates the basis with a spiral structure. And, in other words, it binds all the concentric circle structures of prime numbers into one large operator space by infinite direct sum. This has the effect of breaking the order structure (the independence of prime numbers is mixed by symmetric nesting), demonstrating the intrinsic nature of the gamma factor.

As a result, the spiral structure is mapped to the critical line (preserving the geometric symmetry of points, lines, circles, and spirals in the quasi-dual transformation while maintaining the scaling invariance of the structure) (see Figure 20).

Figure 25. The Final form of continuous quasi-dual morphism.

作用素の核は

$T : u^p \mapsto e^{-s \log p}$ かつ (全素数) p についてテンソル積

として定義できます。

1つの閉路 → Mellin型核
素閉路列 → 素リンドン分解
全素閉路 → 素リンドンテンソル積

この写像が**自己随伴性**を持ち、かつ全体で**固有値が臨界線上に並ぶ**

The kernel of the dual transformation is a Mellin-type transformation that incorporates gamma factors, transitions to elementary Lyndon structures for each elementary structure, and integrates their effects.

As a result, it transports infinite concentric circular information, along with the surrounding circumstances, into a linear form while maintaining fractal scaling.

This is the continuous version of the “quasi-dual mapping.”

In summary,

the Hilbert-Polya operator integrated by this result is

Figure 26. Incorporate into the determinant representation of the Iihara-type zeta function.

行列式構造

伊原ゼータの行列式表示と同じ：

$$Z(u) = \det(I - uA + u^2Q)^{-1}$$

ここで：

- A = アジャセンシー行列（経路情報）
- Q = 次数補正行列（自己ループ補正）

これに **ブロック行列** M を組み込むと：

$$\det(I - uMA + u^2MQ)$$

みたいな構造が自然に現れる。

Thus, here too, nested structures emerge, and it can be observed that fractal structures dominate the entire system through scaling. In short,

Figure 27. Constructive Hilbert-Polya operator.

補題（構成的ヒルベルト・ポリヤ作用素）

各素数pに対して、複行列

$$M_p = \begin{pmatrix} 0 & \log p \\ \log p & 0 \end{pmatrix}$$

を考える。これらは自己共役である。

全素数にわたる無限テンソル積

$$\mathcal{M} = \bigotimes_{p \in \mathbb{P}} M_p$$

は、ヒルベルト空間 $\mathcal{H} = \bigotimes_p \mathbb{C}^2$ 上の自己共役作用素である。

このとき、$\mathcal{M}$ のスペクトル構造は、素数構造と log p によるスケーリングを通じて、リーマンゼータ関数のゼロ点の対称構造と一致する。

特に、素リンドン経路の類双対変換（＝logスケールによる指数変換）を通じて、
同心円的配置のゼロ点が臨界線に写像されることにより、構成論的ヒルベルト・ポリヤの作用素として解釈される。

Hilbert-Polya action (D).

explaining each step, this "Hilbert-Polya operator" controls the "non-commutative prime number (i.e., prime structure) structure (zero-point structure, i.e., prime Lyndon)" created by graph-theoretic structures locally via "gamma factors," the "Möbius transformation" that maps circles to lines is bundled for each prime number, forming an infinite Möbius strip, and instantly maps the infinite concentric circle fractal, along with its surrounding space, to a line.

Furthermore, it possesses the structural characteristic of an “infinite matrix” with an infinite nested structure, and its contents are self-adjoint, maintaining self-adjointness on the infinite Hilbert space where they are infinitely connected. These requirements are underpinned by a “simple analytical” operation that involves repeatedly receiving various elements derived from the deformation of the “trace bundle” via the “quasi-dual mapping” and then deforming them again as “quasi-dual mappings.”

A matrix is, in a sense, a nested matrix that inherently contains various operational patterns. While this will be briefly touched upon in another chapter, here it manifests itself as a conjugate infinite Möbius structure, specifically in the form of a prime spiral nested structure.

As can be seen from the previous discussion, the crucial operation is the “quasi-dual map.” The impetus for this analysis was the discovery of the “quasi-dual divergent zeta extension” via the prime Lyndon path, which is a non-trivial restoration path. However, the revelation that the structure of prime numbers actually possesses holographic fractal restoration properties was surprising.

However, the “quasi-dual map” is highly multi-valued and intuitive, and I have not yet fully grasped its precise, overarching role. I have touched on this briefly, and in fact, it has been an inherent problem all along. For example, the question of whether to restore a line or a spiral from a trace bundle is actually a very difficult problem. For instance, can a person walking on an extremely gentle spiral staircase be distinguished from a person walking on a straight path in terms of their intrinsic state, that is, their subjective state? This restoration problem involves concepts such as the “categorization” of the trace path itself (i.e., layers? meaning the retention of local information) or the difference between infinite and finite immanence, but while there are partial ideas, a complete solution is still far off.

The concept of the “trace category” will undoubtedly become important when the trace path becomes “non-regular,” and it is already known that this will generate another restoration path.

In the continuous “quasi-dual map” introduced in this chapter, there are “prime-type spirals,” “prime-type infinite concentric circles,” and “critical lines,” and since these are all controlled simultaneously in this case, there is no problem, but there will be situations where problems arise.

In fact, an infinite Möbius structure was necessary to “wind the spiral.” Is there nothing simpler? Everyone must think so.

The quasi-dual map, which I call the “theory of dynamic transformation,” seems to be a world that has only just begun, in my understanding.

7. Continuation of fractal deformation of dual mapping

How many such deformations are there?

What are the methods of construction?

I will not delve into such issues in this essay, but in general, it can be assumed that there are many. For example, consider the dynamic transformation I often think about (which can also be described as observing noise), in which "the Cartesian spiral disintegrates into point fractals in each local state." This is also a type of "continuous transformation." And this "continuous transformation" will be mapped into an infinite concentric circle fractal via a "continuous quasi-dual transformation." The fact that there are likely to be many such decomposition methods can be imagined from this example. Furthermore, when the "scaling infinite structure" of the Cartesian spiral—that is, the fractal structure that remains unchanged no matter how much it is enlarged−is transferred into the infinite concentric circle structure, it becomes restricted to a "self-similar mapping" based on the size of the concentric circles and undergoes changes. In other words, while the fractal structure is preserved, the structure itself changes. The problem of Quasi-dual transformations also includes the issue of changes in the structure's information and relationships accompanying structural deformation. It goes without saying that this reveals the fundamental problem of "non-commutative deformation."

In this chapter, we will examine an example of how adding periodic waves to a spiral-shaped fractal is also quasi-dual.

Let us organize the problem.

First, through graph zeta-based generation theory based on prime number structures (trace bundles, prime Lyndon sequences, and fractal doubling maps), we constructed a Riemann zeta-like zeta structure with commutative Euler products. However, the "quasi-dual transformation" we performed holds in the discrete domain of infinite prime-powered concentric circles, so unless we "continuously extend" it, we end up in a situation where we cannot determine how various structures are ultimately mapped onto a "line" via the quasi-dual transformation. It might not even be a "line." In that case, what I constructed would result in something that differs from the known structural characteristics of the Riemann zeta function.

Therefore, I first considered constructing it as a continuous mapping corresponding to the log scale, and the idea arose to introduce a continuous topology that satisfies the fractal scaling transformation based on log scaling, corresponding to the discrete spread of prime number concentric circles.

By doing so, we can consider introducing a new scaling structure (log scaling) that satisfies the fractal nature by stretching and contracting the “concentric circle fractal with power-law expansion and Cartesian spiral structure” through logarithmic scaling. It is well known that the scaling where prime numbers appear is log scaling.

First, the basic log scaling,

Figure 28. Transform prime powers logarithmically.

- $\ell(P)$：巡回構造 P の「長さ」や「次数」
- p：それに対応する素数（写像によって対応）
- $\log p$：その対数スケール

つまり、

$$\ell(P) \quad \longmapsto \quad p \quad \longmapsto \quad \log p$$

and then extend this relationship to

Figure 28. Transform prime powers logarithmically.

素数からログへの写像（離散 → 連続）

次に、この素数列を連続変数へと写す「スケール変換」が：

$$p_n \quad \mapsto \quad \log p_n$$

これは自然対数写像（連続）です。

by doing so, we consider incorporating continuous log scaling into the discrete structure of prime power concentric circles. We then add an integral expression as follows, incorporating logarithmic spiral structures into the expansion of prime power concentric circle structures, and gradually consider “logarithmic scaling-based quasi-dual transformations” as continuous structures.

Figure 29. Successive transformation per prime.

$$Z[f] = \sum_{P \in \mathcal{P}} f(\log p_{\ell(P)})$$

や、連続化して：

$$Z[f] = \int_0^\infty f(\log p(x))\,dx \quad \text{with} \quad p(x) = \text{n番目の素数}$$

この「$p(x)$ の連続化」にはチュビンガーの素数近似式 $p_n \sim n \log n$ を使えば、

$$Z[f] \approx \int_0^\infty f(\log(n \log n))\,dn$$

のようにも変形できます。

In this way, by transforming the discrete graph traversal structure (Lyndon sequence) into a continuous log-scale space, the logarithmic spiral structure is naturally generated.

Functions that constitute such continuity connecting discrete structures are called functional extension, and a well-known example is the factorial function N! (equivalent to the gamma function) with introduced continuity.

However, simply applying log scaling to the Cartesian spiral and aligning it with the expansiveness of prime numbers is insufficient; it was found that a "correction term" must be added alongside it. This correction term is a vibrating term resembling a wave oscillating up and down. By adding such a correction term and deforming the log-scaled Cartesian spiral, we can refer to it as a "non-Cartesian spiral."

Conversely, the fact that "adding wave terms to a spiral shape preserves fractal self-similarity" is considered to be one aspect of the multi-valued nature of the essential "quasi-dual mapping" and "quasi-dual transformation." Simply put, visually, imagine growing various capillary-like fractal trees along the Cartesian spiral.

Summarizing the above considerations, it can be summarized as follows.

The fractal-like expanding structure is deformed by the wave correction while maintaining self-similarity.

Figure 30. Correct by wave motion.

波動補正入り積分核：

$$K(\theta,p)=e^{-(b+i)\theta}\,\Phi(\theta,p).$$

拡大構造の本体：

$$e^{-(b+i)\theta}\qquad \text{（これが log スケールでの拡大核）.}$$

波動補正：

$$\Phi(\theta,p)=1+\text{（小さな補正項）.}$$

この形から分かるように、拡大核（フラクタル構造）自体は指数項だけで支配されている。

When considering what form the correction term should take, for example, the Riemann–Siegel formula, which is famous for Riemann's calculation of the zeros of the Riemann zeta function as a complex analytic function, was devised by Riemann to approximate the zeta function near the critical line and later systematized by Siegel. Using this(see Figure 31),

ζ (s) is roughly

Figure 31. The Shape of wave correction.

臨界線 $\Re(s)=1/2$ での $\zeta(s)$ は大まかに：

$$\zeta\Big(\frac{1}{2}+it\Big)=2\sum_{n=1}^{N}n^{-1/2}\cos\big(t\log n-\theta(t)\big)+R(N,t).$$

ここで：

- $\theta(t)$ はガンマ関数由来の位相補正項
  $\theta(t)=\arg\Gamma\Big(\frac{it}{2}+\frac{1}{4}\Big)-\frac{t}{2}\log\pi.$

and in this case, the correction term we are looking for is

Figure 32. Correction term on the critical line of the Riemann zeta function.

螺旋核：

$$K(\theta,p)=e^{-(b+i)\theta},\quad \theta=\log p.$$

に対して、

リーマン–ジーゲル型の波動補正：

$$\Phi(\theta,t)=e^{\,i[\,t\,\theta-\theta(t)\,]}.$$

を掛け合わせる。

Note that the wave correction behaves wave-like in log scaling and does not destroy fractality.

Substituting this,

Figure 32. Correction term on the critical line of the Riemann zeta function.

$$Z(t) = \sum_p K(\log p, p, t) = \sum_p e^{-b \log p} \cdot e^{i[(t-1)\log p - \theta(t)]}.$$

$$Z(t) = e^{-i\theta(t)} \sum_p p^{-b} \cdot p^{-i(t-1)}.$$

$$\boxed{Z(t) = e^{-i\theta(t)} \sum_p p^{-b} p^{-i(t-1)} = e^{-i\theta(t)} \sum_p p^{-(b+i(t-1))}.}$$

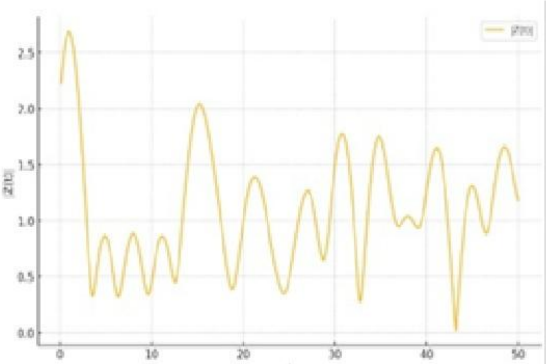
we obtain the desired equation.

This wave-corrected spiral kernel sum, when b = 1/2, gives wave-like oscillations along prime numbers and includes the gamma phase correction of analytic continuation, so it has a structure where Iharascillates linearly along the critical line Re = 1/2 due to the wave term(**Figure 31** illustrates the wave-like correction layers of dual Lyndon structures, stacked fractally along the complex Möbius spiral).

In the previous chapter, there was a "Hilbert–Polya operator" that acts as an operator controlling such a structure, and it shifted the zero point to the critical line. It functions like a wave operator. Note that the critical line carries the "prime Lyndon element" (i.e., zero point) brought from the graph zeta.

When this is calculated by a computer,

Figure 34. Behavior of zero points calculated using finite prime numbers.



小さな素数列（2, 3, 5, 7, 11, 13）で計算した
「螺旋核＋リーマン-ジーゲル波動補正」の
$|Z(t)|$ の挙動

it becomes clear that the real axis Re = 1/2 is already oscillating from the initial stage. The multi-valued nature of the continuous (functional) form of the quasi-dual mapping, aligned with the prime number (log) scale, has gradually progressed. This indicates that the "quasi-dual mapping," which geometrically continuousizes the prime number structure, can be explicitly expressed as an integral kernel, and that when P is taken to infinity, it converges at the critical line.

And although only an approximate formula has been derived so far, the following is already clear.

Define an enlargement mapping (spiral mapping) that preserves fractality in accordance with prime concentric circles on a logarithmic scale. This constitutes the continuous form of the quasi-dual mapping.

While this method has not yet yielded a clear explicit expression, it establishes the commutative, multivalued quasi-dual mapping as the kernel of a divergent zeta structure, and incorporates its multivalued nature into a prime Lyndon structure (infinite prime cycles), and further continuous it using a fractal enlargement operator on the log scale, we were able to approximately construct a fractal functional extension that maps the spiral (concentric circles) to the critical line.

At this point, based on "anti-ideality," or the fact that there is no overlap at the zero point, the "zero point" transferred to the spiral straight line moves back and forth along the critical line on the wave. From a structural perspective rather than a generative perspective, the structure of the zero point can be considered to have the symmetry s→1-s (Riemann's formula). In other words, it becomes clear within the wave nature that the "zero-point structure transported from the infinite bouquet zeta" will not be arranged in such a symmetrical form. Of course, if the operator is in an undetermined state, it may exhibit abnormal behavior, but considering the situation in the previous chapter, it is understood that such a thing will not occur at the generative construction stage.

I believe the objective of explaining how the zero-point structure is generated based on the elementary structure through the self-similarity of divergent quasi-dual transformations has been achieved.

7. Functional of the basic quasi-dual map and generative construction of deep Riemann structures

The subject of my essay was whether it would be possible to construct a Riemann zeta-like structure by creating a divergent zeta structure using my own class dual map, a noncommutative, multivalued operation, such that its multivalued nature naturally manifests as prime structures and higher prime structures (zero point structures). Through this effect, I sought to concretize a “theory of dynamic transformation” that discusses noise phenomena.

This generative task has, for the time being, been formally achieved up to the previous chapter... However, when defining an enlargement map that preserves fractal properties in the logarithmic scale aligned with prime number concentric circles, the extension of the continuity of the quasi-dual map has reached a stage where it can only be described as “there are various possibilities.”

This chapter will further discuss this point.

As can be seen from my analysis, the “quasi-dual mapping” that appears everywhere is what connects loop structures and tree structures via a “trace bundle.” In the previous chapter, I attempted to “continuously analyze” it as a functional, so to speak, and had to reconstruct the troublesome “fractal structure” within the transformation using log scaling.

However, those who read this must have naturally wondered, “What about the continuous analysis, or functionalization, of the ordinary form of the quasi-dual map, i.e., the simple power-law concentric circle fractal and spiral fractal?” At least, I did. I had never considered this possibility until I drew it.

However, this problem leads to a new method for constructing generative zeta structures.

Let's summarize.

The zeta construction method I have shown has the following generative principle: “When a prime structure exists, deformation introduces multi-valuedness, and within that, the construction of a divergent zeta structure naturally leads to a zeta structure.” Let's apply this to a simple concentric circle fractal, for example.

This approach may provide an alternative solution to Hilbert-Polya’s operator. While it may not be a proper realization of the conjecture, this issue was discussed in Chapter 6.

To summarize again, in the previous chapter, we used the wave term of the Riemann-Ziegler function, applied logarithmic scaling, and introduced fractal topology into the prime power concentric circle structure. However, upon further reflection, since the quasi-dual transformation between the power-law infinite concentric circle structure and the Cartesian spiral is also discrete, it seems possible to continuousize it using a “functional extension.” I couldn't help but realize that I was unaware of such an obvious perspective and had been tackling a more complicated problem.

Now, setting aside the log analysis of prime scales for the time being, I want to consider a functional that continuously connects the “infinite scale-free mapping of the unit circle to form concentric circle fractals and the Cartesian spiral” in a more pure manner. How should this be done?

First, we organize the structure using a power-of-two concentric circle fractal structure.

Figure 35. Quasi-dual morphism from a single Euler product.

等倍同心円の構造

- 半径 $r_k = 2^k$ （2倍拡大）
- 中心は同じで、円環的に配置
- $k \in \mathbb{Z}_{\geq 0}$

このとき、「無限同心円フラクタル」は、

$$S = \bigcup_{k=0}^{\infty} C_{2^k} \quad \text{where} \quad C_{2^k} = \{(r, \theta) \mid r = 2^k,\, \theta \in [0, 2\pi)\}$$

デカルト螺旋

螺旋は：

$$r(\theta) = a\, e^{b\theta}.$$

ここでは、**等倍同心円構造と対応させるには**

螺旋がちょうど「回転角度」で倍々拡大する形にしたい。

つまり、

円周の位相 と **螺旋の角度** が一致するとき、

螺旋の半径が 2^k に一致するようにする。

$$r(\theta) = a\, e^{b\theta} = 2^k.$$

したがって

$$b\theta = k \ln 2 - \ln a. \quad \Longrightarrow \quad \theta = \frac{k \ln 2 - \ln a}{b}$$

Now that we have determined the discrete structure, we will scale it continuously.

Figure 36. Nucleus of a local quasi-dual map.

$$Z[f] = \int_0^{\infty} f\big(r(u),\, \theta(u)\big)\, du.$$

$$\boxed{Z[f] = \int_0^{\infty} f\left(2^u,\, \theta(u) = \frac{u \ln 2 - \ln a}{b}\right) du.}$$

This can be considered the desired general functional extension.

Now, the question arises: can we express this in a more explicit form, including the wave term? As we encountered with prime-number concentric circles, there are many patterns in “function systems that satisfy fractal scaling.”

Such “equal-scale infinite concentric circles × spiral × wave correction” functionals are thought to be able to form countless patterns depending on how the wave term is incorporated.

Now, based on the current “prime-free, simple equal-scale structure” form, let us combine several more explicit representative patterns that include the Riemann-Siegel-like wave term considered during log scaling.

Figure 36. Nucleus of a local quasi-dual map.

等倍 × 螺旋 × 波動補正の汎関数

もっとも汎用的な形は：

$$\boxed{Z = \int_0^\infty \exp\left[\, i\,(\,k\,\theta(u) + \omega\,u\,\ln 2 + \Phi(u)\,)\right] du, \quad \theta(u) = \frac{u\ln 2 - \ln a}{b}.}$$

ここで：

- k：螺旋の波動数
- ω：等倍スケールの対数波動数
- $\Phi(u)$：追加の波動補正（Riemann–Siegel 相や他の共鳴項）

Each combines discrete power-of-two concentric circle fractals, adjusted Cartesian spirals, and Riemann-Siegel-type wave terms that adjust them wave-like.

What exactly is the effect of this “quasi-dual transformation generic function”?

This is precisely the core role of the homotopic mapping: “mapping the (topological, closed-loop) state on the concentric circles to a linear critical region through a continuous mapping.”

In other words, the original topological space is an infinite number of concentric circles (generated by a quasi-dual mapping and forming a fractal structure). The Cartesian spiral unfolds this topological state into a straight line using an angular parameter. Wave corrections (such as the Siegel term) are responsible for “refining the mapping and aligning conditions,” striving to maintain this linear state.

Ultimately, the images converge in the critical region (e.g., Re(s) = 1/2).

At this point, recall that the basic discrete quasi-dual transformation considered in the previous chapter converted the concentric circle structure into an Euler product. In this case, the Euler product is (1-2^-s)^-1, which is the Euler product of the prime number 2. In other words, in this case, the “zero points” collected in this critical region are plotted from the circumference onto the critical line.

This brings to mind the core of Professor Kurokawa's "Deep Riemann Hypothesis," which involves "extracting each prime number from the Euler product and examining whether it corresponds to the critical conditions of the Riemann Hypothesis on its own." In other words, the process we just performed—the 2-power concentric circle fractal × spiral nucleus × wave correction—maps only the p=2 portion of the Euler product onto the critical line, demonstrating this as a functional.

Note that this is an operator called a "quasi-dual transformation."

In other words, a "quasi-dual transformation" is a non-commutative transformation, and in this case, it maps points on the concentric circles (including the zero point) of the concentric circle fractal to the spiral fractal and plots them on the critical line. Note that in this case, through the divergent zeta extension of Ihara's zeta function and the bouquet zeta function, a correspondence with the commutative prime sequence (i.e., ordinary primes when the order is ignored) appears via the prime Lyndon element. And in the case of 2, the square root of 1 was the zero point. This zero point is thought to disappear from the critical line due to asymmetry and form a pole at s=1. This pole acted as an "absorbing element" in the infinite tensor product, where infinite zero points were superimposed.

In principle, the radius scaling of concentric circles is exponential because it is a power function.

The spiral mapping linearly solves the exponential structure using the angle parameter.

Figure 37. Continuous scaling.

$$r(\theta) = a\,e^{b\theta} \qquad \Longrightarrow \qquad \theta = \tfrac{1}{b}\ln(r/a).$$

Furthermore, the power-law concentric circle structure is

Figure 38. Local continuous scaling.

$$r_k = 2^k. \qquad \Longrightarrow \qquad \theta_k = \frac{k\ \ln 2 - \ln a}{b}.$$

discrete, so θk is a discrete sequence, but when continuous, $k \mapsto u$ causes the infinite concentric circle structure to fall onto a smooth $\theta(u)$ axis. As seen in the previous chapter, this is approximated wave-like and gradually falls onto a straight line.

Where this line ends up depends on the zeta structure where this "quasi-dual transformation" acts, and since this zeta structure has mirror symmetry with respect to s→1-s, it converges to Re=1/2.

In other words, by reorganizing the discrete action of the quasi-dual mapping into a continuous quasi-dual mapping via a topological infinite concentric circle structure (non-commutative, multi-valued) and a spiral, and combining it with critical line alignment via wave correction, the infinite overlapping structure of discrete closed loops is mapped to a linear region under a continuous operator, forming a "generative quasi-dual mapping."

As can be seen by substituting numerical values, this is composed of prime numbers such as 3, 5, 7, etc., and from these, the Euler product is constructed through the quasi-dual transformation. In other words, by combining all prime numbers, the Euler product of the Riemann zeta function is constructed.

This can be described as a generative "deep Riemann hypothesis," which states that if the local closed-loop structure of each prime number is spiraled by the quasi-dual mapping and guaranteed to be mapped to the critical line by wave correction, then their product (i.e., the entire zeta) should have zeros aligned on the critical line.

In other words, the form of the Hilbert-Polya operator can be approximately written as follows.

Figure 39. Bundle all prime numbers.

積分核で束ねれば、

$$\prod Z_p[f] \quad \text{with} \quad Z_p[f] = \int^{\infty} \exp\left[\, i\,(k\,\theta_p(u) + \Phi_p(u))\right] du.$$

This is thought to be the operator of the Kurokawa tensor product, which tensor-bundles the functions for each prime number. This operation approximately bundles the infinite prime power circles onto the critical line. At the same time, this is considered to be equivalent to the "class dual transformation" described in the previous chapter, which is log-scaled.

Generatively, this means that if the local structure of each prime number falls on the critical line, the entire product is regularly aligned on the critical line. This can certainly be considered the framework of a generative proof in the form of the deep Riemann hypothesis.

The idea is to support Kurokawa Shigenobu's statement that "the Riemann hypothesis naturally follows from the deep Riemann hypothesis" from the perspective of generative theory.

Furthermore, it becomes clear that the "continuous, functional-type quasi-dual transformation," which seems to have an action akin to the ∞-Möbius transformation, is precisely the desired operator form, and that it possesses a tensor-like structure determined by the structure of each prime number. One might imagine this as an action that stacks infinity upon infinity.

The matrix form of the Hilbert-Polya operator is also such.

Let us summarize.

First, we replaced the role of a single linear fractional transformation called the Möbius transformation with a functional called a "quasi-dual mapping" for each infinite prime number.

This means that "the local zeta structure of each prime number itself is generated by a topological operation called a quasi-dual mapping."

In addition, I think the following can be said.

1. Each prime number independently possesses a "local spiral kernel."

$Zp[f]= \int \exp[i(k\theta p(u)+\Phi p(u))]du$. (The integral is from 0 to ∞.)

Here, the scale unique to the prime number p is woven in.

2. The product that binds them together coincides with the Kurokawa tensor product.

$Z = \otimes pZ\ [f]$. (p passes through all prime numbers)

This is not a simple commutative product, but rather a local structure that acts in an operator-like manner.

3. It incorporates multi-valuedness (quasi-duality), which could not be achieved by the "single linear fractional operation" of the Möbius transformation,

into the functional integral.

It is undoubtedly true that the observation methods of concentric circle fractals and Cartesian spiral fractals are the root cause of the generative zeta structure deformation. Additionally, the deformation characteristic that concentric circles have a doubling map while Cartesian spirals are infinite in scale is also important.

8. The Concept of Multiple Matrix Rings and the Elementary Lyndon Decomposition of Matrix Representations in Ihara Zeta Functions: A Generative Approach to Deep Riemannian Structures

I was originally exploring objects that change and evolve with mobility, such as noise phenomena, as noncommutative mobility.

In the process, I encountered graph structures, particularly hypergraph structures. This hypergraph is a generalization of graph structures, where the correspondence between "points (nodes)" and "lines (edges)" is not fixed, but varies locally, with 'N' and "M" (where N and M are different) points and lines. In other words, there are many points within a line, and many lines within a point. This is called non-regular, and it became necessary to develop a theory of the correspondence between variables of different numbers.

Part of the solution to this problem is the concept of "motifs," which involves applying dual operations to hyperedges, transforming them, and compressing them into "structured sets." This allows the multiplicity and multi-valued nature of structural transformations to be "pseudo-uniquely" defined, much like ideal theory or Riemann surface theory. It can also be proven that this forms a "closed structure" (i.e., motif closure).

As a slight digression, when using non-commutative algebras as "trace material," the theory of "motif closures" including "directed edges" will be necessary. The path is cut off, or suddenly divided. I do not yet fully understand this problem.

Please note that this essay focuses on the structure within a highly ordered situation, "regularity." This is why the invariance of the "trace bundle" could be used.

In this process, I encountered operations involving (2×2) matrices and (3×3) matrices. These are non-regular non-commutative operations with different degrees. When considering this(see Figure 40),

I noticed that within two (3×3) matrices, there are three (2×2) matrices, and when considering matrix operations, this can be extended to calculate (6×6) matrices in a nested manner. This can be called "nested" matrix multiplication. What is important is the abundance of "equivalence classes" in the case of infinite nesting. Contraction is also possible.

Even if it is not regular, any pattern of "A × B" and "C × D" can be applied, and the degree is easy to understand.

There is also a "combination operation."

For example, suppose there are a (2 × 2) matrix and a (4 × 4) matrix. Take out two columns from the latter. There are six possibilities.

Then, we take two rows from the four rows in those two columns. This also gives us six possibilities.

As a result, there are 36 possible operations between (2×2) and (2×2), which yields a (12×12) matrix. Although multiplicity is not an issue due to symmetry, this is a semigroup, and there may be cases where the "dimension" does not exist, resulting in a region where inverses cannot be found.

In the reverse case, you can simply divide the acting side. If the length of the rows is insufficient, be careful to calculate by considering permutations and combinations from the reverse side.

Figure 40. Examples of calculations using Multiple Matrix.

① (2,2) 行列 M

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

② (3,3) 行列 N

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

複行列の定義

複行列 $M \boxtimes N$ を、

> 「M の各成分 m_{ij} を係数として、全体をスケールした N を 4 ブロックに配置」

とする。すると、$M \boxtimes N \in \mathbb{C}^{6\times 6}$：

$$M \boxtimes N = \begin{bmatrix} 1\cdot N & 2\cdot N \\ 3\cdot N & 4\cdot N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 4 & 0 \\ 3 & 0 & 3 & 4 & 0 & 4 \\ 0 & 3 & 0 & 0 & 4 & 0 \end{bmatrix}$$

Examples of Multiple Matrix calculations.

I named this the “multiple matrix ring.”

When the number of operations is insufficient, you can simply expand it by asking, “How many can fit?” and extend the number of characters in the matrix. By generalizing this, you can define operations for “N × M” matrices and “L × O” matrices. This is also a situation that naturally arises when acting on matrices as vectors.

Note that this nested structure is “fractal-like.”

The following prediction can be made.

The “motif closure” structure of hypergraphs is embedded “closure-wise” into the operation structure of the matrix ring.

Now, let us consider Iihara's “divergent zeta expansion.”

A (2,2) regular graph becomes a circular loop graph. This has “two prime structures,” but since there are forward and reverse directions, two conjugates appear in the determinant.

The minimal form of a loop graph with “one prime structure” is a structure with one (point) node and an indistinguishable self-loop, which Shigenobu Kurokawa considered to be related to F1 geometry. If we consider this to be distinguishable, then there exists a “quasi-dual divergent zeta extension” for all (2,2) regular graph structures of prime length, meaning that class-dually, “circular graphs of prime length are, so to speak, quasi-dual.” In this case, “what about natural numbers? They are prime Lyndon decomposable.” Now, how does this prime Lyndon decomposition occur?

That is, “every circular graph with a prime number structure has a quasi-dual divergent zeta extension to every graph with a natural number structure.” In other words, every circular graph with a prime number length is homeomorphic to a circular graph with a natural number structure that combines prime number structures. This is the holographic fractal restorability of prime numbers.

This is equivalent to saying that if you have one Euler product, you can reconstruct all Euler products across all natural numbers, demonstrating the fractal and holographic nature of natural number structures.

It is understood that this corresponds to the prime number combination of the Euler product, which means that the local information of prime numbers acts as the nucleus for generating the overall structure. In other words, it can be expressed as “the Euler product is not closed locally, but it is closed within the matrix ring.”
Some may also note that the matrix itself inherently contains a potential natural number structure.

From the correspondence between decomposition in “multiple matrix rings” and “prime Lyndon decomposition in the determinant of Ihara's zeta,” a situation arises in which “only the prime number concentric circle structure needs to be considered” when constructing the analytic zeta from the graph zeta.

Now, when considering the structure of zeros, the operation of constructing the analytic zeta function from the graph zeta function resembles the process of creating a structure by tensorially combining an infinite number of (2,2) matrix elements.

One could also consider an operation where an infinite nesting is initially expanded infinitely in a certain “dimension,” maintained in a self-similar form, and then a state where “something cannot be contracted” is considered.

Such a matrix ring corresponds to the “∞-Mobius transformation” in the sense that the “quasi-dual transformation” considered in the previous chapter, which transfers the correspondence between concentric circle fractals and spiral fractals to a line, has an action similar to that of the “∞-Mobius transformation.” In other words, the nested structure of the matrix action corresponds to the quasi-dual “zeta construction,” i.e., the “analytic connection.” Note that this can be expressed as an operator.

At this point, note that each “regular complex matrix ring” of prime dimension has a nested structure and possesses an infinite-dimensional matrix operation structure. This nested structure corresponds to the “fractal nature” in the quasi-dual transformation.

In other words, it is an expression of the “∞-Möbius transformation” property in quasi-dual transformations.

Furthermore, its expansion as an operator is thought to exist as an approximate limIharaf the structure created by combining the quasi-dual functional structure analyzed in the previous chapter, namely “equivalence × spirality × wave nature.”

Note that the analytical expansion of this determinant is the Euler product, which is an analytical expression of the so-called deep Riemann hypothesis, where one Euler product contains all the information of all Euler products, and the zero point of one Euler product is determined by the relationship with Euler products across all prime numbers.

This possesses an essential “non-commutativity” and further an essential “non-regularity.” Non-commutativity means “cannot be rearranged,” and non-regularity means “multiple objects of different numbers are combined.” This suggests that essential non-commutativity and non-regularity are potentially contained within commutative and analytical objects such as the Riemann zeta structure.

9. Considerations on what appears to be the Hilbert-Polya operator expressed as a matrix

Here, when we observe the Hilbert-Polya operator expressed as a tensor product across multiple prime numbers, we find that it has remarkable characteristics.

To summarize the key points, even a single Euler product has a structure that can be extended to “all prime numbers” via a quasi-dual divergent zeta extension, which is the tensor product structure of a matrix ring, taking the form of the local structure of one prime number multiplied by the infinite product of all prime numbers.

The infinite product is concretized as an “infinite nested Möbius structure” in the operator.

This is connected to the “self-adjointness of the matrix ring” and the “closure property of determinants (infinite determinants).”

From this, we can conclude that the structure of the Euler product itself expresses the symmetry and entanglement that give rise to the conditions for the arrangement of zero points (the balance between divergence and cancellation).

As a result, “Re(s) = 1/2 is naturally aligned.”

To arrive at this conclusion, we will now reconsider the Riemann zeta function.

Figure 41. Imaginary component of the Riemann zeta function.

$$Z(t) = \sum_{p \in \text{Primes}} p^{-s} = \sum_{p} p^{-(\sigma+it)}$$

We will transform this already analytically continued Riemann zeta function using logarithmic scaling.

Figure 42. Logarithmic scaling.

$$Z(t) = \sum_p e^{-s \log p} = \sum_p e^{-(\sigma + it) \log p}$$

We re-evaluate this in integral form.

Figure 43. Evaluated in integral form.

$$Z(t) = \int e^{-it \log p} \cdot e^{-\sigma \log p} \, d\pi(p) \quad x = \log p.$$

We regard this as the kernel of the "Hilbert-Polya operator,"

Figure 44. Extracting Hilbert-Polya kernels.

$$Z(t) = \hat{\mu}_\sigma(t), \quad \mu_\sigma(x) = e^{-\sigma x} d\pi(x) \quad (\text{素数分布}).$$

and consider the change in the value of σ.

Figure 45. The physical behavior at Re = 1/2 is most stable.

フーリエ波動と指数減衰の干渉が最大化する。

ここが本質で：

$$e^{-\sigma \log p} = p^{-\sigma} \quad \text{と} \quad e^{-it \log p} = p^{-it} \quad \Longrightarrow \quad p^{-s} = p^{-\sigma} p^{-it}$$

- $\sigma = 0$ だと発散する。
- $\sigma = 1$ だと指数的に消えすぎる。
- $\sigma = 1/2$ が最も「エネルギー密度が残りつつ、波動が干渉する臨界点」。

1/2 The physical behavior of the critical line is most stable.

This is the summarized diagram.

Let us summarize the key points.

Using logarithmic scaling of prime numbers, the irregularity of the prime number sequence is mapped onto a "continuous kernel" (Hilbert-Polya kernel).

In graph zeta (Ihara zeta, etc.), it was shown in the previous chapter that prime cycles correspond perfectly to prime factorization.

Analytically interpreting this using “prime Lyndon decomposition,”

directed cycles ≈ Lyndon languages ≈ prime factorization

thus, it becomes clear that the Euler product of graph zeta closes the natural number structure as a “circle.”

This is considered to be the concrete form of “prime number scale self-similarity (spiral kernel),” which is expressed in the form of a matrix and as the “Hilbert-Polya operator.”

Therefore, as long as it has a graph zeta-like Euler product structure, the zero points depend on the closed path structure (trace bundle) and are arranged along the critical line.

Even for a single prime number product, the logarithmic kernel already contains an infinite structure.

Since all infinite prime structures interfere with each other, self-similarity converging to Re(s) = 1/2 appears in every prime term. This can also be understood from the structure of the ∞-Möbius transformation and the quasi-dual transformation.

This is the “consistency between the local (prime) and the whole (zeta),” and it is clear that this forms the foundation of the self-adjoint structure of the matrix ring.

Even with a single Euler product, we have actually seen that it can be extended to “all primes” through a quasi-dual divergent zeta extension, as demonstrated by the operation of the “quasi-dual divergent zeta extension” on the Ihara zeta function.

This matrix ring retains the local structure of each prime number (prime Lyndon decomposition, infinite Möbius nesting structure) internally, and by tensor-combining them across all prime numbers, it concretizes the Hilbert-Polya operator and provides a theoretical framework that generates zeta zeros as its eigenvalues.

In conclusion, the structure of the Euler product itself generates the conditions for the arrangement of zeros (balance between divergence and cancellation), and as a result, the eigenvalues naturally align at Re(s) = 1/2.

The concentration at Re(s) = 1/2 appears even without the wave term because the main scaling of the eigenvalues (or traces) of the matrix at each prime number p is based on the logarithmic scale logp and consists of an exponential decay term p^-s. This suggests that the energy density is naturally maximized at Re(s) = 1/2.

The above reasoning does not include a specific method for determining the zero points.

Of course, this is purely a generative structure. That was the purpose.

However, the structural necessity that the zero point "must exist on the critical line Re(s) = 1/2"—the conclusion that "the zero point must only exist on the critical line" can be derived from constructive principles without knowing where the zero point is—is precisely what the tensor construction using a matrix ring possesses, as constructed in the previous chapter.

As a conclusion to this chapter, summarizing this paper while looking at it as a whole, what was demonstrated in this paper was the fact that the operator anticipated by Hilbert-Polya is generatively constructed by a matrix ring that includes the logarithmic scale structure of prime numbers, prime Lyndon decomposition, and infinite Möbius nesting structure.

While the direct numerical determination of the zero point remains, the structural necessity of why Iharanly appears at Re(s) = 1/2 is naturally explained by the infinite adjunction and self-similarity of the matrix ring.

Some may be confused by the unfamiliar concepts used here, but these originally stemmed from various observations and conceptual forms I developed to interpret the sensory phenomenon of "noise phenomena" exhibiting fractal properties. It was only recently that I mathematically formalized these concepts and, coincidentally, discovered that they could be expressed as equations through the construction method of the Ihara zeta function. Until then, it was largely conceptual "play with clay."

I never imagined that my former conceptual play would intersect with the core of the Riemann zeta function in this way. However, this discovery demonstrates that number-theoretic structures can coalesce into a single image in physical, geometric, and generative terms.

The challenge moving forward is to translate this generative structure into a concrete finite-dimensional approximation and confirm the numerical consistency of eigenvalues. Furthermore, the structure suggested by this matrix ring is likely to be extendable to noncommutative geometry, tensor categories, and nested supur graph theory, but this is not the subject of this paper.

This paper demonstrates that the zero structure of the Riemann zeta function can be "generatively constructed" within the matrix ring and its infinite tensor product.
As for applied theory (or rather, that was the beginning, and it wasn't even mathematics to begin with). The beginning of this mathematical exploration was

1. The fact that the error backpropagation method generates "images" from noise, which was completely isomorphic to my "noise method"... Also, from that perspective, I realized that various sensory training methods, such as extinction hallucinations, are implemented in artificial intelligence.

2. The common property of noise and artificial intelligence is "fractal contraction," and I was enlightened to the relationship between fractals and zeta structures. ...

That's it.

For those interested in the "exploration of concrete phenomena," I've written a brief note at the end, though it strays from the main topic.

10. Structural Conjectures for Complex Matrix Rings and the Restoration of Various Linear Algebraic Concepts

In this section, we consider propositions about complex matrix rings as a kind of appendix.

Conjectures on matrix rings

By viewing the nested structure of matrix rings of prime degree as a power, the tensor product of matrix rings can be decomposed in the same way as natural numbers and prime numbers, with the same structure as the unique decomposition of natural numbers and prime numbers. Within this structure, concepts such as the usual matrix ring structure, eigenvalues, and adjoint operators are naturally defined as "prime-normalized" structures.

This conjecture states that any complex matrix can be decomposed into a complex matrix of prime order, and the decomposition within the prime complex matrix ring is defined as a nested matrix ring obtained by repeatedly applying the nested structure of matrices as powers. For each prime number p, we define a local complex matrix ring A(p). Then, *A*(*p*) is a regular complex matrix ring of order (p, p), naturally containing self-adjointness and eigenvalue structures.

In other words, this local complex matrix ring corresponds isomorphically to the prime factorization of natural numbers.

Figure 46. Decomposition of Multiple Matrix ring.

$$N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \quad \Longleftrightarrow \quad A_N = A_{p_1}^{\otimes a_1} \otimes A_{p_2}^{\otimes a_2} \otimes \cdots \otimes A_{p_k}^{\otimes a_k}.$$

In this localized and normalized complex matrix ring of prime powers, concepts such as trace, eigenvalues, and adjoint are expected to be naturally defined.

Furthermore, as a global structure via tensor product, the entire complex matrix ring is constructed as an infinite product of primes.

Figure 47. Tensor product for each prime number.

$$O_{HP} := \bigotimes_{p \in P} A_p.$$

This structure is thought to correspond to the Euler product, and the combination of each local structure generates the overall picture of the Riemann zeta function.

Each prime matrix ring has a nested structure and an infinite Möbius-like construction. That is, each $A(p)$ contains an infinite Möbius action as a nested structure.

This infinite nesting corresponds to the prime cycle decomposition (prime Lyndon structure) in the graph zeta function and is thought to reflect the fractal nature of the natural number structure. At the same time, it is also thought to extend the structure of natural numbers.

1 = 0.999...

Is this the same?

"It is the same."

Furthermore, in the non-commutative and non-regular matrix ring we considered earlier, there exists a broader "equivalence," which I have termed "representational multi-valuedness." This can be understood by observing the structure that allows an (2,2) matrix to be infinitely expanded into an infinitely nested (2,2) matrix. This overlaps with the infinite nested Möbius structure of multiple matrices and is thought to guarantee the "multivaluedness of quasi-dual representations" necessary for generating zero points. It is an infinite extension of 1.

1 = 0.999... can be considered a simplified model of the equivalence motif space in the multiple matrix structure.

Representation multiplicity and the structure theorem of equivalent motif spaces

The matrix ring has a local nested structure for each prime number, but the specific values of the operators are not uniquely determined by "representation multiplicity." However, this multiplicity is not arbitrary; within the class dual transformation and infinite nested fractal structure, all representations converge to equivalent motif spaces.

Appendix 1.Numerical Experiment at Re(s) = 1/2 and the Hodge Bouquet

Here, I record my attempt to compute some concrete numerical results by letting the computer test my idea.

Figure 48.The value of the real axis 1/2 of the Riemann zeta function.

$$\zeta\left(\frac{1}{2}\right) \approx -1.4603545088\ldots$$

It is known that the Euler product converges only for Re(s) > 1.

So, to see how the values behave near Re(s) = 1/2,I compute the finite prime structure of the Ihara zeta function for primes p ≤ 11.

Figure 49.Composed of graphs by Zeta Ihara.

$$U^{-|P|} \longrightarrow e^{-s\log p} \quad \text{かつ} \quad |P| = \log p.$$

だから、有限積で

$$Z(U) = \prod_{|P|\le \log P_N} \frac{1}{1-U^{-|P|}} \quad \Longrightarrow \quad \prod_{p\le P_N} \frac{1}{1-e^{-s\log p}}.$$

つまり：

$$\frac{1}{1-e^{-s\log p}} = \frac{1}{1-p^{-s}}.$$

Figure 50.Calculation from the origin on Iihara Zeta's graph.

$$\zeta_5(1/2) = \prod_{p\le 11} \frac{1}{1-p^{-1/2}}.$$

By observing how this value changes, I try to imagine how the “analytic continuation” (via the so-called quasi-dual mapping) might be realized.

Figure 51.The ordinary Euler product diverges.

Prime	Factor $1/(1-p^{-1/2})$
2	3.41421356
3	2.36602540
5	1.80901699
7	1.60762522
11	1.43166248

This infinite product factor ultimately diverges,but the infinite bouquet structure with its endless closed loops undergoes a spiral-like class dual deformation,twists back Möbius-like to reverse direction, and tends toward negative values.

I believe this mechanism produces the negative values of the Riemann zeta function at negative integers.

In other words, this infinite Möbius can be thought of as generating the gamma factor and the self-dual structure (s → 1−s) in the analytic continuation of the Riemann zeta function.

Thus, the locally positive product structure is turned over spiral-wise (by an infinite prime-bundled spiral) through the class dual mapping,creating a “topological core” that generates negative values via analytic continuation.

Local positive → class dual mapping → spiral → analytic continuation → negative value

That’s the flow.

In this sense, the spiral Möbius twist appears to reverse the sign of the exponent term,turning −log p into +log p.

This sign reversal explains how the analytic extension generates negative values and matches the self-duality symmetry (s → 1−s).

Here, I recall an idea I once abandoned:the matrix that swaps the sign,

Figure 52.Modified Möbius action.

$$M = \begin{bmatrix} 0 & \log p \\ -\log p & 0 \end{bmatrix}.$$

When I actually let the computer compute this,it became clear that it works to suppress the divergent factor precisely. In other words, the correction factor is 1.

Figure 53.Spiral-shaped dispersion suppression.

Prime	log p	exp(+i s log p)	exp(−i s log p)
2	0.6931	0.9405 + 0.3397i	0.9405 − 0.3397i
3	1.0986	0.8529 + 0.5221i	0.8529 − 0.5221i
5	1.6094	0.6933 + 0.7206i	0.6933 − 0.7206i
7	1.9459	0.5629 + 0.8266i	0.5629 − 0.8266i
11	2.3979	0.3633 + 0.9317i	0.3633 − 0.9317i

I thought, “Perhaps this was the right direction after all…” and had the machine calculate further.

Figure 54.Negative values appear in the correction term.

$\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)$ を定義

s	$\zeta(s)$	$\chi(s)$	$\zeta(1-s)$	$\chi(s)\cdot\zeta(1-s)$	差
0.5	−1.4603545088...	1.0	−1.4603545088...	−1.4603545088...	0.0
−1	−1/12 ≈ −0.08333	−0.05066...	$\pi^2/6 \approx 1.64493...$	−0.0833333...	≈ 0（誤差 ~10^{-32}）

Then I realized how this mechanism could indeed lead to negative values(see Figure 54 illustrating how gamma correction leads to negative values).

Without a computer, I would never have understood this structure.

At the time, I had a feeling that the sign of log p was “indifferent.”

This arose from the essential sense that “rotation drives convergence” in the trace structure,and I believed that some degrees of freedom remained.

Now that I have confirmed it, I feel that this matrix M can naturally be defined as a “infinite compression operator” — an operator structure that embodies rotation.

Figure 55.Rotational Möbius action.

$$\exp(tM) = \begin{bmatrix} \cos(t\log p) & -\sin(t\log p) \\ \sin(t\log p) & \cos(t\log p) \end{bmatrix}$$

Next, I recall that the genus-1 “dual aperiodic path” can be seen, from the perspective of graph-theoretic Riemann surfaces,as precisely suppressing the divergence of the Euler product and constructing the quadratic zeta.

What I originally saw as the “infinite Möbius spiral” was the way log p aligns itself structurally.

But when it is “bundled,” for some reason, a correction factor appears to protect the whole from divergence.

Now I understand that this is the action of the genus-1 dual aperiodic path —in other words, an “imaginary multiplication trace compression.”

This note was written first, but various later ideas have connected back,recovering the hidden thread of this infinite Möbius structural understanding.

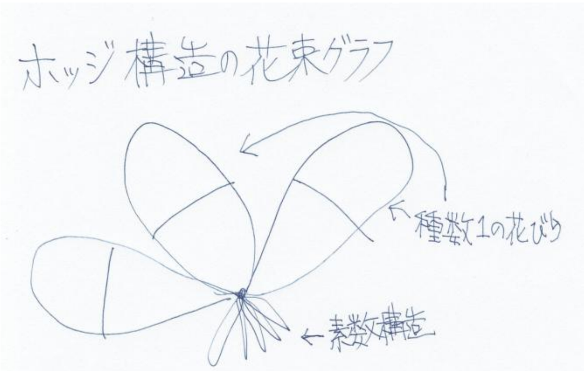
In short, the genus-0 “divergence control circuit” is spiral,while the genus-1 “divergence control circuit” is a “dual aperiodic circuit,”and both appear to be related to the imaginary multiplication circuit —that is the hypothesis this leads to.

Therefore, this matrix M can naturally be defined as the “infinite compression operator for divergence control” in the the Lyndon Complex Spiral Phase.

Analytic continuation is, in other words, a “divergence control circuit” formed by a “rotational spiral structure.”

Consequently, by considering Ramanujan’s quadratic zeta structure,I naturally arrived at what I now call the Hodge Bouquet Graph.

Figure 56.Hodge's bouquet.



To explain this Hodge Bouquet: when you attach a genus-1 petal to the prime structure,the number of aperiodic circuits becomes two,which tends to make divergence more likely,but the “dual aperiodic circuit” works in an imaginary-multiplication-like way to suppress divergence.

Through this, the quadratic zeta — and then the 4, 6, 8 and higher multiple Hecke operators —in other words, via multiple Hecke operators, an endlessly higher structure is generated.

In summary:

Genus 0: local rotational compression by spiral Möbius

Genus 1: non-commutative topological compression by dual aperiodic paths

Note that the number of “aperiodic circuits” in this bouquet structure takes the form $2n-2$.

In this way, you can see that connecting many “dual aperiodic circuits” creates a “cushion”that prevents divergence more effectively.

In other words, the “prime structure” becomes less prone to divergence.

From here, is it possible to build a theory of higher-dimensional “divergence control circuits”…?

Proposition

The real number line is not structurally complete.

This is because it does not contain a nested hierarchy of non-periodic sequences within itself.

Only through divergent quasi-dual recovery can an infinite Möbius bouquet be generated, elevating the bare continuum into a self-dual complex topological structure.

This is the origin of the self-duality ($s \leftrightarrow 1-s$)and the topological projection of the complex spiral syntax plane.

This theorem first appeared to me in a dream.

When I woke up, I initially rejected it as absurd.

However, it perfectly aligned with the structure I had been building.

What I had thought was a mistake was actually the key.

This is the final proposition of this essay.

What appears as meaningless concentric circles—flat and infinite—only becomes the grammar of a spiral when lifted through divergent embedding.

This is what it means to “wind infinite concentric circles into a spiral.”

The double helix structure of fractals…

That is, consider an infinite bouquet graph with branches representing real numbers.

When attempting to reconstruct this graph through “divergent restoration,” its value takes on a structure that ascends each level of the infinite nested structure.

In other words, to reproduce the structure of the infinite bouquet graph, it must demonstrate higher-dimensional nestedness not through convergence but through construction, utilizing the “infinite Möbius structure” it contains.

This is the proof of the existence of infinite Möbiusness.

Appendix 2

As a summary, I will organize the proof of the generation theory of the Riemann zeta function, including side paths, and future prospects.

1. The graph zeta function based on prime cycles in bouquet graphs is uniquely determined by prime Lyndon decompositions.

2. By divergent zeta extensions, the recursive decomposition of loops coincides with the infinite prime number structure.

3. The quasi-dual map $u^p \mapsto e^{-s \log p}$ is , which is equal to the product kernel of the Mellin transform.

4. By the regular Ramanujan condition, the eigenvalues of the graph determinant have Hilbert-Polya-type self-adjointness.

5. Therefore, the prime cycle sequence coincides with the analytic structure of the Euler product, and the zero structure is mapped onto the critical line.

The flow is very simple, and if there are any leaps or gaps, they are likely to be found in the logical structure.

In the class duality mapping theory related to zeta structures, another idea that comes to mind is that the prime Lyndon structure of "non-regular graphs" is "Dedekind-type," i.e., the prime Lyndon structure of non-regular graphs is Dedekind-type.

To summarize, it is as follows.

In regular (Ramanujan) graphs, the prime Lyndon decomposition of loops is locally finitely generated, and the eigenvalues (prime Lyndon) distribution align neatly, resulting in the zero points being arranged along the critical line.

This also suggests the "closure property of the zeta function, which is finite and closed, through quasi-dual divergent extensions." This is the same as restoring all prime paths under regular conditions.

On the other hand, in non-regular graphs, infinitely generated branches (forks) intrude into the local structure, and the decomposition of prime loops becomes "non-unique" or "redundant." It is also possible that a "trace category" will be needed to express the "local structure." This is also to increase the possibility of restoration.

Furthermore, in Dedekind domains, "class groups" that do not fit into principal ideals appear. Similarly, in "irregular trace bundles," the decomposition of prime cycles allows for multiple paths as "non-principal."

In other words, there is a correspondence between the redundancy of prime Lyndon structures → quasi-dual group bifurcations → the non-triviality of Dedekind-type zetas.

In regular Ramanujan graphs, the "Riemann hypothesis" and in irregular (Dedekind-type) graphs, the "quasi-dual group-containing zeta" correspond to each other in the quasi-duality map.

The Riemann hypothesis in regular Ramanujan graphs and the zeta function with group structure in non-regular (Dedekind-type) graphs may be unified by the difference in the structure of trace bundles in the quasi-duality map. The invariance of trace bundles is likely to be broken, and its deformation may become important.

Finally, a few points that were not explained.

The prime Lyndon sequence in bouquet graphs coincides with the finiteness of the trace bundle.

The limIharaf the zeta extension is isomorphic to the structure of the Kurokawa tensor product.

Non-regular zeta functions incorporate quasi-dual group structures due to redundant branching of prime Lyndon series.

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Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.

This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ　ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.
素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of “complex spiral phases”
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a “divergence control function” corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ｓという臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.
無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the “elementary Lyndon element” that is restored to zero is decomposed, the corresponding Euler product becomes a “multiple Euler product,”

giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through “dual non-periodic paths.”
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a “double helix” because it naturally contains spiral rotations and has a double main structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in

the number of species. This is particularly important in the context of the formulation of "higher-order imaginary multiplication."
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by "dual non-periodic paths." Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.
トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)

素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する
　In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.
　全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.
Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences

arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal “bouquet structure” corresponding to the “Euler product.” It is also necessary to distinguish it from the commonly referred to “Hodge structure.”
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.

In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。

とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ; A natural quasi-dual transformation that reduces “irreducible rational Lyndon” in trace bundles to “prime Lyndon” or “natural number Lyndon.” Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets；A complete state that cannot be further expanded by repeating dual operations.
双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.
→ 類双対閉包
非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる