The theory of Quasi-duality map and fractals

(翻訳　類双対性写像とフラクタルの理論)

Part III: Applications of Mathematical Theory of Dynamic Fractal Transformation to Various Fields (Applied Edition)

Author:Hiroki Honda

July 2025

注・日本語を読める方は、日本語で書かれた原文がありますので、そちらをご利用ください。文末にリンク先

Abstract

In this part, a generative perspective on Zeta functions is developed. The interplay between multiplicative trace layers and the Riemann surface topologies leads to a recursive expansion of Zeta deformations, integrating both operator algebra and graph theoretical models.We demonstrate applications of irreducible Lyndon structures in Lyndon spiral complex topologies using modular quasi-duality maps, and examine higher-order zeta structures and their divergence control structures.

I discovered non-periodic quasi-dual divergence restoration, Lyndon language decoding (complex spiral phase of Lyndon semigroups), and general Riemann surface graphs that analyze the structure of general Riemann surfaces and their uniqueness and generativity. The gap between the fact that “it has become harder to understand”

and the fact that “it appears overly abnormal” is remarkable.

For example, while the structure of the function values of a simple quadratic function can be overviewed, the “corresponding function values” are extremely difficult to discern. While the structure of all existing functions can be overviewed, each individual concretization is extremely challenging... and so on.

Therefore, I am writing this article not only to apply it to existing problem types but also to confirm that it aligns with the results of existing theories, thereby stabilizing my own mindset.

Additionally, there is a “complex analytic connection of the uniqueness theorem for general multiple Riemann surfaces in Lindon's language,” which, as revealed by the theory of fractional zetas and modular forms, may actually possess a mechanism to suppress “extreme divergence.” This is also one of the main themes of this essay.

Probably, many people who read this description will think, “That's not a proof” or “It lacks rigor,” but for me, I am writing about what I can't help but think is true, and what I actually believe to be true.

If there are any mistakes, they are likely due to my lack of observation. If someone else were to understand my theory, they might correct me by saying, “You misunderstood because this is how it appears.” For example, when I first considered the first fractional zeta function, I thought, “This is how it is,” but I did not understand the “graph structure” until I saw it in a dream. Once I saw it in a dream, the structure was clear, so I spent four hours trying to make it fit together.

There is no mistake in this “graph structure operation,” but there is a possibility that I am using the “language to describe it” incorrectly. This is because I had created several equations that, upon closer inspection, seemed meaningless.

After researching the Dirichlet function, I finally understood that this form was correct.

This is an advanced topic, so I often use the structural theory of graph-like Riemann surfaces (i.e., general Riemann surfaces) without explanation, but there is one thing to be careful about.

It is important to clearly distinguish between the quasi-dual divergence of the prime structure arising from the “non-periodic sequence” of “genus + 1” and the non-periodic restoration action as the “quasi-dual divergence zeta extension” arising from the entire trace bundle of the graph-like Riemann surface. Once you get used to this, you will realize that I have used only basic considerations in what is written here.

Lindenberg Phase Basics Annotation

It has gradually become clear that even I, who conceived of the Lindenberg spiral phase, sometimes misunderstand its meaning, so I will explain it again here as an annotation.

The Lyndon phase uses "Lyndon language 'length'" as its basic unit, and as it repeats, the denominator gradually increases, forming a nested structure like a "multiple-period fractal," which then becomes more detailed into a real number structure. such a "non-periodic sequence" itself forms a "non-periodic sequence," creating a further nested structure that represents the complex rotation of the Nth root of one (when the number of prime structures is N) from the Lyndon semigroup to the complex plane, a simple continuous phase.

I often misinterpret this as an N-ary system, which can lead to misunderstandings, but I think this can be avoided with careful attention.

While confirming this, I realized that in the Lindon complex topology, integers, rational numbers, real numbers, transcendental numbers, and complex numbers are structurally distinct and exist as separate entities. In other words, there is no inclusion relationship between them. I suspect this might be related to Grothendieck topos.

In "Lyndon complex spiral topology," integers, rational numbers, real numbers, transcendental numbers, and complex numbers are not included in a sequentially expanded number system, but are structurally different entities.

And that is probably very close to the "Grothendieckian topos" concept.

Generally, they are defined by "inclusion relations."

In conventional mathematical education, number systems are taught as follows:

$Z \subset Q \subset R \subset C$

Here, "inclusion" simply means "containing as elements" or "obtained through sequential expansion."

However, this is merely "algebraic generation" and ignores structural differences (topology, topos).

Number system Lyndenian properties contractibility Existence mode

Integers Finite length, finite loop Completely contractible, discrete structure (atomic)

Rational numbers Finite cyclic Lindenian sequence Reducible, bouquet-like bundle structure

Real numbers Multiply periodic fractal structure, quasi- contractible, convergence structure in closed intervals

Transcendental numbers Nested, non-periodic Lyndon series Non-contractible Non-constructive, infinite structure

Complex numbers Nested structure of multiple “non-periodic terms” and spiral topology Completely non-contractible Multi-layered trace structure

Here, it can be said that the classes of numbers belong to different topos (logical spaces).

In other words, each “hierarchy of numbers” exists in a separate “universe.”

In Grothendieck's “topos theory,” instead of sets and spaces, the behavior of schemes, sheaves, and topological spaces as a whole is regarded as “objects.”

Integers, rational numbers, real numbers, transcendental numbers, and complex numbers are not related by inclusion, but rather exist as “entities belonging to different topos.”

In practice, this is demonstrated below, where it is common to “extract and handle only the fractional structure” or “analyze only the real number structure,” which may confuse some people, but in “structural complex number theory” derived from the Lyndon semigroup structure, this is a very natural perspective.

Along these lines, I will also discuss a simple “transcendental number theory,” so please look forward to it. Very simply, I will explain the transcendence of π and e, so you may think, “What is he talking about?” However, I understand that feeling, so I have presented several correspondences between my theory and existing theories, as well as structural predictions from my theory.

There is also Ramanujan's analysis of the quadratic zeta function, so please refer to it.

1. Consideration of fractional zeta functions: Lyndon's modular theory

By collecting irreducible fractions (m/n, m<n), we can create an Euler-like formula and consider fractions created from all irreducible fractions less than or equal to one.

The purpose of this chapter is to examine this from a Lyndon perspective.

Then, the mystery of “why the seemingly divergent Euler product can be analytically continued” arises, but I will interpret it in terms of Lyndon language.

Now, by appropriately expanding the bouquet graph, we create a graph that contains all loops of length “irreducible rational Lyndon.” For more details on this meaning, please refer to the introductory text, “The Foundational Structure of the Application of quasi-duality Maps to Mathematical Objects in the Theory of Dynamic Transformation.”

This allows us to first expand the denominator using the Euler function (n) into a bouquet graph with a length equal to the denominator, using the “modular quasi-duality action."

Similarly, when expanded using the Euler function (m), all bouquet graphs are decomposed into "prime Lindon" and have an Euler product representation.

Even though there are already many irreducible rational numbers, they increase as they are decomposed by the number of Euler functions, so you might think, "Is this okay?"

However, in this Euler product representation, all prime numbers appear more than once, causing it to diverge! You might think, "Wait, this doesn't work after all."

Here, we introduce a correction term and, when decomposing the graph, we contract the same decomposed loop using the Euler function (n). Similarly, when decomposing the numerator's Lindon, if we contract it using the Euler function (m), it becomes a state where there are infinitely many "prime Lindon elements of prime length," allowing us to align it with the standard Euler product of the Riemann zeta function.

This structural operation of the "bouquet graph" is so 'obvious' that there can be no mistake. Even though it diverges, there is a firm belief that "the structure undoubtedly exists" and "it is there." Yet, because it diverges, I cannot help but think that "the formula may only have partial expressive power..."

In other words,

Figure 1. The Fractional zeta function.

$$\mathcal{S}_{\mathbb{Q}}(s) = \prod_{\substack{\gcd(m,n)=1 \\ m<n}} \frac{\left(1-(m/n)^{-s}\right)^{-1}}{\varphi(m)\varphi(n)} = \prod_{p}\left(1-p^{-s}\right)^{-1}$$

by dividing the fractional zeta function by the Euler function in this manner, we arrive at the Riemann zeta function.

Furthermore, the Euler function itself is

Figure 2. The Dirichlet function.

そのディリクレ級数表示．

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}$$

は、

$$\frac{\zeta(s-1)}{\zeta(s)}$$

という明示的な解析接続済み関数

analytically continued by the zeta function, and the fractional zeta function is analytically continued.

When transforming the graphical zeta into the Euler product, I should note that I am using the quasi-dual transformation (the same as the quasi-dual mapping).

Figure 3. The Quasi-Dual Morphism.
$u^p \mapsto e^{-s \log p}$

At this point, it is worth noting that the fractional zeta function is constructed from the local states expressed by Dirichlet functions.

As a result, the fractional zeta, which was supposedly "divergent," gains meaning, but I think there is a mechanism at work here where, even if the value expands in a Lindon-like manner, it is adjusted to have meaning through a rotational spiral on the complex plane.

As a result, the fractional zeta function, which should have been "divergent," gains meaning. Here, there seems to be a mechanism where, even if the values expand in a Lindon-like manner, they are adjusted to make sense through a rotational spiral on the complex plane. This is the so-called "mechanism for controlling divergence."

Let us consider fractional modular forms a bit further. In my theory, it is a mapping that reduces the rational Lindon sequence (the N-reduction term) within the "trace structure" to a "prime Lindon." It is also easy to see that this is a "quasi-dual mapping" because it does not change the fractal nature of the structure.

However, the structure of the "trace bundle" has changed. In this sense, it is no longer part of the principle of "trace bundle invariance" used in the paper on the Riemann zeta function. It is a "quasi-dual map" that transforms while preserving fractality.

In the construction of the "fractional zeta function" mentioned earlier,

irreducible rational Lyndon → contraction → integer Lyndon → contraction → prime Lyndon

this contractible quasi-dual map was unified by the Euler function.

I had thought, "If we use modularity, we can bring irreducible rational Lindon to prime Lindon, so we should be able to construct the Euler product," but I must confess that it was somewhat difficult. In other words, these two operators were necessary.

Such operators are thought to have a "determinant" representation, which appears as a determinant that changes the shape of the graph. This is what is generally referred to as "modularity." Since I am directly manipulating the trace bundle, I do not need to consider the deformation structure of the complex plane or manifold.

Here, by analyzing the structure of a general Riemann surface (graphical Riemann surface), I will discuss what can be said about the basic structure of genus 1 elliptic curves and the general automorphic form theory of genus N.

First, from the perspective of graph-theoretic Riemann surfaces, elliptic curves of genus 1 have an incredibly important structure. Since the number of "non-periodic paths" (= genus + 1) is two, it is possible to follow dual paths, resulting in a combinatorial number of 2, creating a state where "a quadratic equation can be solved with a single closure."

As a result, "rational Lindon" can synchronously appear in the dual multiple trace, creating a structure that generates "infinite rational points (rational Lindon)."

However, due to structural constraints, first, since there are only two "prime structures" (the number of non-periodic paths, which is the genus plus one), it is insignificant compared to the case of "no eyes." Due to the insufficient density and the lack of redundancy in the "non-periodic terms" of the genus-one graph, the 'generation' of the infinite "infinite rational points (rational Lindon)" is subject to the condition of finiteness.

In other words, "it can be reduced by a finite number of modular operations" = "in other words, the rational points are finitely generated," which is Mordell's theorem.

For a genus 2 curve, the number of "non-periodic paths" (= genus + 1) become three, so the number of combinations is 3!, and when taking the "dual motif closure" of a genus 2 graph, in the multiple trace space, "rational Lindon" cannot have a "dual path," and even if one tries to take a corresponding path, the existence of another path interferes, so it becomes clear that "infinite rational Lindon (rational points)" are impossible. This is likely the Mordell–Faltings theorem.

Let us further consider modular function theory.

From this perspective, elliptic curves of genus 1 have a multiple trace space that is a "double trace" in the "dual motif space," so in that case, it is clear that they necessarily have a "double periodic function" with a number of combinations equal to the number of non-periodic paths (genus + 1) times 2, meaning they have one.

Extending this further, for curves of genus 2 or higher, the symmetry of the number of non-periodic terms (genus + 1) increases factorially to (genus + 1)!, resulting in an abnormal expansion of symmetry from a Galois theory perspective. Thus, the number of possible double periodic functions is limited to the number of combinations of 2 for (genus + 1), and the number of possible triple-periodic functions is bounded by the number of combinations of 3 for (genus + 1). Furthermore, it is clear that a strict condition is imposed, namely that “the synchronizability of multiple trace spaces is restricted to at least ‘binary(see Figure 4).’”

It is reminiscent of how cubic and quartic equations are generally solvable, but even when solvable, they require repeated closure (contraction) operations, and when it comes to quintic equations, the contraction method itself is lost.

I do not understand the specific form, but just from the structure of the graph-like Riemann surface, this much is immediately clear.

Figure 4. Periodic combination number.

二重周期：$\binom{g+1}{2}$
三重周期：$\binom{g+1}{3}$

Combination numbers of double and triple cycles.

Figure 5. Upper limit of general cycle number.

k重周期 $\leftrightarrow$ $\binom{g+1}{k}$

“保型構造がどこで複雑化・分岐・非局所化するか”
“モジュラー群の射影作用がどこで変質するか”
“トレース空間が何重らせんを許容するか
に対応する「限界値」や「位相的位数」そのもの

Combinatorial upper bound on k-periodic structures.

I had never understood the “existence conditions for automorphic functions” before, but now I can see how they are determined in this way.

Even when the genus is 0, there are no non-periodic terms, but when the genus is +1, it becomes 1, and sine, cosine, and power functions are “ultimately the same type of periodic function on the graph-like Riemann surface.” In other words, there is only one such function.

I have not been able to verify how correct this consideration of limit behavior is, but my analysis is based on the idea that “the symmetric intersection of non-periodic terms generates periodic functions,” so please note that this is merely a mechanical counting based on that concept.

In other words, modularity has a mechanism that destroys the “rational Lindon structure” to demonstrate such symmetry.

Now, let's consider the proof of the Tanimura-Shimura conjecture based on modular theory and organize it from the perspective of my theory.

Tanimura-Shimura Theorem (Modularity Theorem)

When an elliptic curve E/Q is defined over the field of rational numbers, it necessarily corresponds to a modular form.

The key point is that this correspondence holds for genus 1 (the genus of elliptic curves).

Since it is genus 1, it is clear that rational Lindon (rational points) exist infinitely, but are bounded by a finite number. This structure arises from the duality of the non-periodic terms of the graph-theoretic Riemann surface and the low “tolerance” of the structure of the graph-theoretic Riemann surface.

Now, let us reformulate the Tanimura-Shimura conjecture.

Since rational Lindon with genus 1 is finitely generated, contracting it is the modular quasi-dual map that reduces “rational Lindon” to “prime Lindon.”

The Euler product of irreducible rational numbers can be considered as follows. The modular multiplicative function c(n) is a combination of the “number of combinations that can be decomposed into irreducible fractions” d(n) and the “number of equivalence classes of Lindon rational numbers” E(n). The important part here is d(n), so let us demonstrate this specific multiplicative functionality. First, factorize the denominator and numerator of the irreducible fraction. The product of the number of prime factors in the denominator and the number of prime factors in the numerator, multiplied by the number of times each irreducible prime factor appears in the decomposition, is (n-1)(m-1) (where n is the number of times the numerator appears and m is the number of times the denominator appears). Therefore, d(n) is multiplicative. This is the form of the modular quasi-dual operator, which is the quasi-dual operator that enables the representation of the “Euler product” by irreducible rational numbers, i.e., the modular quasi-dual operator. This structure allows for finite reduction. Combined with the previous explanation, it is now clear that the Tanimura-Shimura conjecture is correct.
In short, rational Lindon with genus 1 is finitely generated, and since it is finitely generated, it can be reduced to a prime Lindon. Therefore, the modular correspondence naturally arises.

The modular theory for reducing “rational Lindon” to “prime Lindon” is already prepared. The following tasks remain.

1. Defining the modular action as a multiplicative function
2. Continuous construction of the Dirichlet function using the modular action
3. Correspondence relations on multiple trace spaces and their connection to Fermat's Last Theorem and Diophantine analysis
4. When the genus is one, the theorem that an infinite number of rational points are generated by the existence of a "prime Lyndon path" (a path that traces a non-periodic sequence in reverse) as a dual path.

This proves the Mordell–Faltings theorem for genus 2 and above.

5. In graph manifolds, actual functions are "points" in the function space within them.
6. The theorem shows that "order destruction" has already occurred due to the existence of equivalence classes in the Lyndon series.
7. The fact in 6 shows that the points of analytic functions actually have an "internal structure."

In fact, a formal reformulation of Fermat's Last Theorem has already been achieved. But that's obvious, isn't it? When a favorable Lyndon appears in a multiple trace of genus 2 or higher, it cannot appear on the other side... because, in the case of genus 2 or higher, the fact that there is no dual slack in the multiple trace space due to the duality of graph transformations can be stated...

This is because, in the "dual motif space" of the rationalized Fermat equation in the "multiple trace space," the multiplicity is "degree N." In other words, since symmetry of N! arises from the genus, it is an extremely difficult state, structurally impossible. It is like having seven 7s align simultaneously in N slots on one's own birthday.

In the case of genus 1, Euler proved it. That's all there is to it.

In the case of elliptic curves of genus 1 or higher, since there are no dual "non-periodic" paths in the graphical Riemann surface, even if one of the multiple trace structures in the multiple Riemann surface is "rational Lindon," the other "trace structures" cannot be rational Lindon. In the case of genus 1, Euler proved it. Thus, Fermat's Last Theorem has been proven... There is enough space left... I don't think anyone would think so, but I believe I have conveyed that the idea of graphical Riemann surfaces alone provides a clear perspective.

The basic procedure is to first assume a "rational Lindon path" in the "multiple trace" and then investigate the "contractibility" in the dual multiple trace. I think "rational Lindon" can be replaced with "irreducible rational Lindon," in which case the length and path are further restricted.

When reading this analysis, it may be helpful to prepare a simple non-regular graph, repeatedly apply the “dual” operation to create a “dual motif closure” several times, and visualize the “multiple trace space.” By thinking, “When following this path infinitely, you must go here along this graph's path, and since this is a infinite cycle...,” you can arrive at the conclusion, “Ah, I see...”

Proposition (constructive proof of Modell's theorem)

A Riemann surface of genus 1 has two non-periodic trace sequences, and rational points are generated by connecting them using a finite number of loops.

In this case, since other loops can only be traversed a finite number of times, rational points are finitely generated.

For genus 2 or higher, there are three or more non-periodic sequences, and since any loop sequence “cannot pass through at least one non-periodic sequence,” finite generation is impossible, and even if rational points exist, they are isolated.

That's all.

Note that even the “dual graph” is not necessary in this case.

2. Transcendental Number Theory

When I deciphered Lindon's language, I saw the “spiral complex plane” and was surprised by its structure.

What was particularly interesting was the “contraction structure.” I had been exploring fractal properties, so I had been thinking about reduction to the “minimal structure” for a long time.

It turns out that this actually defines the “complex spiral structure” itself!

What particularly intrigued me was the “real number multi-periodic fractal structure.” In the previous “Introduction,” I introduced the concept of “generating points” for functions. The fact that functions have the same number of “generating points” (with some overlap) as “zero points” structurally implies something significant. However, while we decompose equations into “zero-point products,” we do not decompose them into “generating-point products.”

And the structure of a function is determined by the fact that both the “zero-point product” and the “generating-point product” must have equal significance.

This is somewhat evident in the two foci of an ellipse or the rational number group structure and addition theorem in elliptic curves. I may simply be unaware, but I have never heard of decomposing a function into a “generating-point product.”

“There must be a generating point from which the entire function can be generatively restored"… Perhaps we could call this the generating point structure of the function and the existence hypothesis of the generating point product.

The discussion has become lengthy, but in essence, elliptic curves have a "group structure between rational points." In this way, the partial structure is clear, and it is thought to extend to the real number domain and the complex number domain as well.

What becomes important here, in my view, is probably an understanding of the nested structure of real numbers. That is, this "multiple-period fractal" known as the Lindon sequence of real numbers is "not solvable commutatively." In other words, the structural conditions change completely depending on the contraction used to create the structure. It is likely that the "point structure" forms a non-commutative sequence that gradually expands, allowing all function points to be restored from a finite number of points. At this point, when transformations occur in unusual ways depending on the restoration path, function bifurcations occur. In such a structure, it can be said from a Lyndon analytical perspective that "multicyclic fractal-like real number points will accumulate."

Rational Lindon can be naturally contracted to a commutative structure, but this is not possible with real numbers! The nested structure of complex numbers can only be "untangled" in very special cases. The structural interest never ends.

In Lindon theory, the reality of complex numbers lies in the fact that they cannot be reduced to a real number structure. Complex numbers have a nested structure in which "non-periodic sequences" form "completely different non-periodic sequences." In other words, complex numbers are transcendental numbers. Their structure is closer to transcendental numbers than to real numbers.

Now, transcendental numbers are numbers that have an "infinite nested structure" and are therefore structurally impossible to reduce. So, how do we determine them?

The fundamental theorem of transcendental number theory.

An infinite series with a denominator containing infinite prime numbers is a transcendental number. This is because the infinite nested structure cannot be reduced to express a Lindon sequence containing infinite prime numbers...

A series constructed using a denominator sequence composed solely of infinite prime numbers (i.e., prime numbers without singularities in a graph, or non-periodic, non-self-convergent prime numbers) is structurally unrestricted by algebraic constraints and thus becomes a transcendental number.

This is because the nested structure of such a prime Lindon sequence cannot be solved through finite reduction.

This is essentially an extension of Liouville's method.

The essence of Liouville's method was a technique to demonstrate that "this number can only be expressed through the infinite 'nested' nature of the Cantor sequence."

Using this method, combined with Dirichlet's arithmetic series theorem, the transcendence of π and e is demonstrated through the Leibniz series and the basic gamma series expansion of e, respectively.

Additionally, in the previous section, I suggested the transcendence of the zeros and odd points of the Riemann zeta function.

This is because there is a wave term in the Riemann-Ziegler equation.

This is used in the calculation of zeros.

However, this wave structure does not converge, as it either overtakes or fails to catch up.

Therefore, I imagine it is considered "not decisive for the zero structure."

Therefore, I employed a method in which I rolled up the infinite concentric circle fractal into a spiral shape using the quasi-dual mapping and projected it linearly. The result of this calculation is the matrix representation form I referred to as the "Hilbert-Polya operator" in my essay on the Riemann zeta function.

This structure possesses an "infinite nested structure that approaches each other," allowing gaps to be reduced in linear projection. In other words, I first created a straight line and then calculated the series approaching it.

Note that "infinite nesting" emerges at this point.

My speculation that the "Linden structure" of the "zero point" is transcendental is based on two points: it could not be superimposed by wave terms, and it required an infinite nested structure even in linear projection. This is essentially the same as Liouville's method.

It is clear that Liouville had a deep understanding of Cantor's theory of real numbers. In other words, unless something that is not an algebraic real number exhibits "infinite nesting," the possibility of reducing it cannot be completely ruled out. Of course, some may argue that there are cases where something with "infinite nesting" converges to a finite value. However, considering the structural meaning of Lindon language, such a thing is impossible. Unless one deliberately creates a situation where it can be structurally unraveled (i.e., contracted), this will not occur.

For example, consider 1 = 0.999999... This is an infinite sequence, but since it is not infinitely nested, it can be contracted. Moreover, it can be contracted back to its original form with just one contraction. From the perspective of Lindon structural theory, "infinite nesting" is understood structurally as "transcendental." You may realize that "the true nature of numbers is not Lindon-like," but this is interesting in its own right. What naturally follows is that there are many multi-valued equivalences in the "infinite nesting structure" that brings us closer to "transcendental numbers" from the non-commutative "multiple fractal structure" of real numbers. I sometimes feel that when I look at the equations of the mathematician Ramanujan, I might have been able to read Lindon's language.

The transcendence problem of the odd values of the Riemann zeta function (e.g., $\zeta(3)$, $\zeta(5)$,...) remains unsolved, but according to the theory that approaches it through the "infinite nested structure," the odd values of the zeta function may possess the denominator structure of a non-periodic, irreducible Lyndon series, leading to the expectation that "no rational reduction exists ⇒ transcendence."

Figure 6.odd values of the Riemann zeta function.

$$\zeta(4n-1) = \frac{(2\pi)^{4n-1}}{2} \sum_{k=0}^{2n} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \frac{B_{4n-2k}}{(4n-2k)!} - 2\sum_{k=1}^{\infty} \frac{k^{-4n+1}}{e^{2\pi k}-1}$$
$$\zeta(4n+1) = \frac{(2\pi)^{4n+1}}{2^{4n+1}-2} \sum_{k=0}^{2n+1} (-1)^{k+1} \frac{2^{2k} B_{2k}}{(2k)!} \frac{B_{4n+2-2k}}{(4n+2-2k)!} - \frac{2^{4n+1}}{2^{4n}-1} \sum_{k=1}^{\infty} \frac{k^{-4n-1}}{e^{\pi k}+(-1)^k}$$

Upon investigation, it seems that Ramanujan derived this.

When demonstrating this, I particularly appreciate Dirichlet's arithmetic series theorem, which states that there are infinitely many prime numbers contained within an+b (where n is any natural number and a and b are coprime). This theorem is highly convenient. For example, when looking at the odd values of the zeta function, it is not at all difficult to see this "infinite number of prime numbers." Another proof of Dirichlet's theorem was also found in Euler's zeta function research.

As long as there is no strange "cancellation of nesting," it seems to hold true.

Using this, we can see an infinite number of prime numbers in the Leibniz series. In other words, the contractibility is infinite.

I think the length of the "Lyndon approximation" of π is on a cosmic scale.

One thing to note here is "negative numbers."

When looking at the Lindenberg complex spiral phase, when negative numbers appear, you have to rotate the "complex spiral plane" multiple times, so to speak, to "control the divergence" and reach the negative number. At this point, the infinite nesting may be eliminated. There are the well-known values 0 and -1 of the Riemann zeta function.

Figure 7. Special values of the Riemann function.

$$\zeta(0) = -\frac{1}{2}$$

When you expand the series, it clearly diverges, but when you observe the method of controlling divergence, you can see the technique of eliminating nesting.

Please note that there may be cases where "infinite nesting is eliminated all at once." From a Lindon structural perspective, you must discover complex numbers before negative numbers.

Conversely, when performing analytic continuation and negative numbers appear despite being clearly positive, it becomes clear that "rotation in the Lindon complex spiral phase" is occurring, leading one to imagine, "Hmm... how many rotations have occurred?"

In my prediction, in this case, it is "half a rotation of the complex plane." This Lindon sequence becomes infinite, but it can be "solved." (Most of the Lindon sequences I consider are infinite sequences before contraction...)

Now that we understand that "complex numbers are the overlapping of nested structures of different non-periodic sequences," we can actually consider "transcendental complex numbers."

In other words, it is a structure where you climb infinitely within the complex spiral structure and have a "point" at the infinite projection destination. This is probably not yet expressible in existing number systems, but it will likely emerge more and more in the future.

Probably, a "infinite graph" is, so to speak, a model of "transcendental complex numbers." Transcendental complex numbers cannot even be expressed in the existing system, but since we know that they are "non-periodic nested complex numbers," they actually exist normally. In other words, they exist convergently beyond the infinite spiral staircase. There are strange values in the Riemann zeta function, right? The one that Kasimir somehow figured out. I used the "infinite spiral" in the quasi-dual mapping to project it, and at that time, I speculated that it might have converged as a "reducible transcendental complex number" due to the structure within it. It might not be a "half-turn" either.

For example, infinite graphs are "irreducible," but they clearly have a "core of repetitiveness." It might be that there is a proper "transformation method" that can reduce them, even though I think it is impossible.

3. Positioning of non-regular regions, and quasi-dual closure

One might wonder how the Lyndon complex spiral continuous topology is connected to the world of zeta functions. Here, we will begin by sequentially examining various forms of zeta functions to gain an overview of non-regular regions.

In my initial explanation of the "Riemann zeta function," non-regularity was seen as an "implication of generality outside the theory," not something requiring special study, but rather naturally positioned as an "indicator of the theory's application limits."

However, my original focus was on fractal phenomena in general, and I had been considering hypergraphic situations–that is, non-regular situations–as part of their evolution. Points and lines are connected by different numbers of edges, and the overall connectivity is highly fragmented. Additionally, many graph structures are infinite graphs (i.e., of infinite genus), making it challenging to consider their dual forms.

When I learned about the "dual motif closure" for hypergraphs, I realized that it was so convenient that I could immediately generalize ideal theory, Riemann surfaces, and so on, but it was not until the 'Introduction' section that I finally reached the uniqueness of analytic connections for general Riemann surfaces. For details, please read the "Introduction" section.

In other words, I began my exploration from a non-regular and non-uniform state, so at first, I did not even know about regularity or "super-regularity," that is, the situation of "N· N" regular hypergraphs.

When fractal nesting appears in infinite graphs with regularity, it is difficult to understand the "dual system," but one can observe the emergence of "self-healing." Honeycomb structures, for example, can be repaired by "dual deformation" even if part of them is damaged. From this, we can see that "dual deformation" naturally exists in nature.

By the way, in my previous essay on the Riemann zeta function, I had speculated that the Dedekind-type zeta function might be a non-regular zeta function, but it turned out to be a regular zeta function.

In other words, when extended to the ideal class group, "prime ideals are decomposed," and as a result, loops of the 'length' of the ideal norm are multiply generated, leading to the case of the Ihara zeta function being regular, effectively resulting in a "multiple Euler product" with "multiplicity." In other words, the "divergent zeta extension of the graph by the ideal class group" is an extension of loops with the ‘length’ of the ideal norm, and ultimately reduces to the case of “multiple Euler products.”

However, note that at this point, “duplication” corresponding to the decomposed prime numbers occurs at the zero point. The structure is slightly different.

The zeta extension by the ideal class group is an operation that constructs a divergent zeta structure by associating loops (prime paths) on the graph with ideal norms.

At this point, the loops that branch and become non-trivial due to the class group are regarded as a “bouquet structure” that binds individual ideals, and each is reduced to a normed Euler product (multiple Euler product).

Therefore, such zeta extensions can ultimately be treated isomorphically with Lindon-type and regular zeta analysis, and there is no fundamental difference in methodology from the extension of the Riemann zeta function.

Looking at this situation as a whole, it becomes clear that Weil-type and Ihara-type zetas were “similar to the Riemann hypothesis” because their “prime structures were finite,” while Dedekind-type zetas depended on the structure of the Riemann zeta and could not be solved until the Riemann hypothesis was solved.

The GRH (Generalized Riemann Hypothesis), which is a generalization of the Riemann Hypothesis,

states that all “non-trivial zeros” lie on the critical line ($\Re(s)=1/2$).

This is a common conjecture for all regular zetas, such as Dedekind zetas and L-functions.

This result is consistent with the content presented here, and in the complex structure proposed by this theory, as well as the analytic continuation structure of the graph zeta function, the zeros naturally contract to the critical line. This result aligns with the claims of the so-called Generalized Riemann Hypothesis (GRH), but here we attempt a reinterpretation from a constructive perspective, presenting it as a complementary possibility while respecting existing theories.

For more details, please refer to my essay on the Riemann zeta function and the “Introduction” section of my theory. The construction method simply involves defining a quasi-dual action that performs “ideal decomposition,” and thereafter, one need only note that the multiplicity of the Euler product arises solely from the overlap of the number of “prime loops” due to the norm.

Similarly, the same applies to the L-function problem(see Figure 8).

Figure 8. The Dirichlet function.

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}$$

$$\chi_4(n)=\begin{cases}0 & (n\equiv 0 \pmod 2)\\ 1 & (n\equiv 1 \pmod 4)\\ -1 & (n\equiv 3 \pmod 4)\end{cases}$$

The Euler product defined here can be obtained by defining a "quasi-dual action" that introduces Dirichlet characters when the elements are coprime. By defining a quasi-dual action that retains only those elements in the trace bundle whose lengths are coprime with the Dirichlet character number N, and then repeating the same "divergent class dual zeta extension" as in the Riemann zeta function, the corresponding bouquet graph for the L-function is completed. In other words, it is a restricted and restored Euler product. In other words, this is the same pattern.

Note that the "zero structure" changes accordingly, and from the regularity condition, the zeros are arranged on the critical line ($\Re(s)$=1/2).

In other words, the result of the GRH (Generalized Riemann Hypothesis) is supported generationally.

Next, the Selberg-type zeta function(see Figure 9).

Figure 9. The Deformation by quasi-dual morphism.

$$\ell(P)=\log p \quad \text{（類双対変換により）}$$

$$Z(s)=\prod_{p\in\mathbb{P}}\prod_{k=0}^{\infty}\left(1-e^{-(s+k)\log p}\right)=\prod_{p\in\mathbb{P}}\prod_{k=0}^{\infty}\left(1-p^{-(s+k)}\right)$$

This diagram is already familiar to readers of my theory, and from the perspective that a Riemann surface is a "graphic manifold and simultaneously a Riemann surface graph," this correspondence is clear.

By restricting the group action that limits these "geodesics" and viewing it as something that hinders the expansion of the "trace bundle," we can limit the types of prime paths. This is a slightly modified version of the Ihara zeta function.

Therefore,

Figure 10.odd values of the Riemann zeta function.

- 測地線構造 ≒ トレース系列
- 測地線長 ≒ log(p)
- 発散的復元 ≒ 無限積構造（k方向の畳み込み）

という厳密な対応を通して

$$\boxed{\text{セルバーグゼータ} \overset{\text{類双対写像}}{\cong} \text{多重オイラー積ゼータ}}$$

Inducing a Selberg-type zeta function to an Euler product using a quasi-dual map.

from here, the problem of the eigenvalues of the Selberg zeta function is rewritten as the problem of the zeros of the Riemann zeta function. For the subsequent content, please refer to my essay on the Riemann zeta function.

In this way, my theory of "Linden complex spiral planes and divergent quasi-dual maps" reduces various zeta functions to the case of the Riemann zeta function.

This brings to mind what is known as the Langlands program.

As I imagine it, the following correspondence exists.

Figure 11. The Langlands conjecture and correspondence.

ラングランズ側の課題	理論での対応可能性
非可換ラングランズ対応	非周期素リンドン系列の縮約理論
L関数のゼロ点分布	リンドン入れ子構造と log-双対系
幾何ラングランズの層構造	トレース束・グラフ構成による再現
モチーフ・コホモロジー的統合	モチーフ閉包と螺旋形回帰写像
圏論的形式化（∞圏など）	素リンドン圏／トレース圏の提案

Correspondence between Langlands conjecture and noncommutative Riemann surface structures.

For example, in this theory, repeatedly applying the Lindon compression structure on a bouquet graph automatically yields the same modular transformation (the action of SL(2,$\mathbb{Z}$)) as the quasi-dual modular transformation on the complex plane. This is akin to a class dual transformation that elegantly deforms the "trace bundle," and thus simultaneously deforms the complex plane in a manner consistent with the deformation structure of a general Riemann surface.

At first glance, it may seem puzzling why modularity is associated with modular forms, but considering the non-periodic nested structure of the Lindon sequence and its topological compression, it becomes clear that the congruence transformations of the modular group are a natural symmetry inherent in the compression operation. Therefore, this book proceeds with the argument that "modularity is inevitable." Please note this point.

Then, what is the zeta function of a "non-regular structure"?

What I can say here is that, in order to direct my focus toward the theory of "quasi-dual closure," I must first create a "non-commutative, non-regular zeta structure" myself.

One such form is "quasi-dual closure."

Generally, non-regular graph-like Riemann surface structures are complex, and the "multiple traces" within their "dual motif spaces" are intertwined. I already have three approaches to this challenge:

1. First, establish the concept of trace categories.

A trace category preserves the information surrounding the path being traced as a "category," and views the traced elements as transitions between "categories." By doing so, the "stability of recovery from trace bundles" improves dramatically.

2. Develop a theory of "dual motif closures" that includes directed graphs.

This enables us to use algebraic systems with non-commutativity as trace materials.

3. The zeros of the zeta function under regular conditions are defined by prime roots on the unit circle, i.e., Pth roots of unity. Conversely, in the case of non-regular states, it is expected that the Pth roots of unity on the unit circle will lose their symmetry or acquire non-commutativity, and it is highly likely that the invariance of the "trace bundle" will also be lost. As a result, the zeros will vanish from the unit circle. At that point, a theory of information loss or information expansion in the "trace bundle" becomes necessary.

By the way, this non-regular region is the region I was originally in, and here, by depicting the "transition of fractal structures," I aim to depict the entire vast transition between "loop structures" and "tree structures" as a "quasi-dual closure."

Furthermore, this class dual closure itself, by taking on the trace structure, becomes a "zeta structure," which I refer to as the "zeta expansion via the third-order class dual mapping."

My hypothesis is that the divergent-type zeta expansion of the quasi-dual closure converges to a fixed point through a "second-order expansion" and is closed.

Here, I have attempted to outline my grand vision in the realm of "non-commutative and non-regular" structures... So, please forgive me if you feel like saying, "What on earth are you talking about?(see Figure 11)"

Constructive Langlands Existence Theorem

Proposition

If there exists a non-periodic sequence (corresponding to a modular form) that is not contracted while preserving symmetry on a Lyndon dual structure, then there exists a modular transformation that preserves that symmetry. This is because there is a need for a clean condition to match values in the dual multiple trace space that preserves that symmetry. The values are not degenerate. There is room for contraction. This condition becomes modular.

At this point, the modular form is not necessarily explicitly given (the "form" itself is unknown).

However, its existence is guaranteed (constructively).

Modularity = "Modular existence theorem preserving non-degenerate symmetric structures."

That's all.

Appendix 3

Since discovering this theory, I have been thinking about it a lot and have started reading books on complex analysis and Riemann surfaces.

In other words, classical Riemann surfaces involve various gluing, combinations, topologies, definitions of local areas, and so on.

However, the generative general Riemann surface derived from my Lindon continuous spiral topology can be created simply by drawing a graph, taking the "divergent zeta extension," constructing a graph manifold, and either using it as is or deforming the manifold using a quasi-dual transformation that preserves a specific "fractal structure."

On the other hand, "what kind of deformation leads to specific formulas or functions" is very difficult to see. The same is true for modular forms. First, I understand modular forms in terms of "trace bundles." They are "quasi-dual mappings that transform rational Lindon into integer Lindon or prime Lindon." However, in various contexts, it is difficult to understand "what they are" without careful consideration.

No matter how I write it, it seems like I am making grandiose claims, so I tried to understand the reasons behind it in my own way and studied various topics.

4. Ramanujan's quadratic zeta function and zeta functions in non-regular

domains

From the previous section's discussion, we can see that most zeta functions are zeta functions of genus 0, even if they have an infinite number of prime structures.

This raises the question of zeta functions of genus 1.

When attempting to perform divergent reconstruction by placing loops of prime path length on elliptic curves of genus 1, we encountered a major problem.

This is because each Euler product continues to expand due to two “non-periodic sequences,” leading to divergence.

In other words, the product of the “zero point product formed by two periods” and the “Euler product” diverges infinitely.

Therefore, we must consider conditions under which this does not diverge.

Here, let us consider the graph of the Ramanujan zeta function with genus 1 in our minds.

The Ramanujan zeta function suppresses divergence by ensuring that the discriminant of the quadratic Euler product is negative, resulting in imaginary solutions. This is what is known as the Deligne's condition.

Therefore, the idea arises that if we take the solutions of a quadratic equation where the discriminant is negative and the coefficients are real, we can create an infinite number of zetas. These solutions are arranged so that the Deligne's condition is satisfied through a modular action that unifies an infinite number of quadratic equations.

The image of the correction term that suppresses the Euler product from diverging is as follows.

Figure 12. A structure in which the Euler product does not diverge.

ポイントは「複素長さを物理長さとして数える」んじゃなくて、
「螺旋位相のズレパラメータとして扱う」ことです。

$$Z(u) = \prod \left(1 - u^L e^{i\theta}\right)^{-1} \quad \text{with} \quad \theta \in [0, 2\pi)$$

A special structure is required to associate complex parameters with effective length in the Lyndon contraction framework.

If we adjust this parameter to match the “divergent restoration of non-periodic sequences” and introduce parameters with ‘modularity’ such that the two “Euler products” do not diverge, we should be able to generate the quadratic zeta function. That is my understanding. However, modularity introduces a discrepancy. This discrepancy is likely related to the breakdown of the complete multiplicativity of the coefficients of the Ramanujan zeta function.

Furthermore, the duality of the Euler product requires that when the two Euler products are multiplied together, the result must have real (integer) coefficients. Otherwise, the length of the prime path would become meaningless as a complex or real number. In other words, modularity is necessary here.

To summarize this analysis, "zero points align along the critical line in the zero product" when the offset paths and prime paths are completely aligned topologically.
However, double non-periodic sequences are unlikely to be completely aligned. This is because two independent offset directions interfere with each other, resulting in subtle deviations that disrupt the critical line.
Therefore, the quadratic zeta function (Ramanujan zeta) may not have zeros perfectly aligned along the critical line due to its quadratic structure with imaginary solutions. This suggests that it may not satisfy the Riemann hypothesis. At the same time, these zeros are likely to lie in a "critical region" rather than a "critical line." This is the Deligne's condition. In other words, this is an example of a "non-regular structure zeta function."

At the same time, these facts correspond to the fact that class field extension of the modular form of Ramanujan's zeta function is not "Abelian (commutative)," and the existence domain of the quadratic zeta function in the non-commutative class field theory domain is equivalent to suppressing the divergence of the "Euler product."

Figure 13. The domain of noncommutative class field theory and the convergence of Euler product divergence.

観察	構成論的対応	数論的意味
非周期列に11や691が現れる	リンドン系列の構成素	モジュラー群の対称性、保型係数との結合
特定素数が支配的	トレース束の分岐定数	保型形式の係数や合同の源

There exists a structure that prevents the divergence of Euler products in the field of noncommutative field theory..

In other words, if we can discover a parameter for a non-periodic sequence such that the discriminant of the quadratic equation formed by two "non-periodic sequences" is always negative and yields imaginary solutions, then it seems possible to construct the quadratic zeta function. This becomes clear when considering the genus-one zeta function from a Lindon sequence perspective.

In my opinion, the field of complex multiplication theory (CM) is likely related, and for special elliptic curves (elliptic curves with complex multiplication), their defining field appears to be the complex number field. In the case of the standard elliptic curve $y^2=x^3+ax+b$, the endomorphism ring is Z, but in the case of complex multiplication theory (CM), the ring of integers of the complex field, such as $Z[-d]$, appears. In other words, there seems to be a connection with the "structural conditions for the validity of the quadratic zeta function.(see Figure 13)"

Therefore, the periodic structure of elliptic functions is related to the "structure of the complex number field."

1. Create a basic extension theory of the Ihara zeta function that generates a quadratic zeta structure from two graphs of non-regular "non-periodic sequences."
2. Construct modular functions or generating functions with structures that satisfy the conditions from domains such as complex multiplication theory.

In other words,

Figure 14. The Ramanujan's quadratic zeta and noncommutative class field theory.

$\mathbb{Q}(\sqrt{-23})$ + 対応する CM 楕円曲線 + Hilbert 類体の非アーベル拡大

Corresponding CM elliptic curve Non-abelian extension of Hilbert class field.

this correspondence.

Additionally, the expected "higher-order zeta" structure conjecture,

《Deligne's theorem, or the higher-order zeta structure conjecture》

If higher-order zetas exist, except for the case of genus 0, there is a modular structure where the discriminant of the Nth-degree equation of genus +1 becomes imaginary and has imaginary solutions, and in its generating function, "disturbances in multiplicativity" occur. When the genus is even, higher-order zetas can exist if they have a modular form that controls the divergent state through the divergent reconstruction of the "non-periodic sequence" of Euler products.

For odd numbers, zetas other than the Riemann zeta either do not exist or take extremely special forms.

Furthermore, such modular forms correspond not to "Abelian extensions" but to "non-commutative domains" in the non-commutative class field theory domain.

To explain, odd-degree equations have a structure that includes "real numbers," which means they contain non-commutative, continuous "multiple periods" and "non-periodic components," making it easy for Euler product factors to deviate from convergence.

Even-degree equations, on the other hand, have a more “commutative and symmetric” structure, where some periods cancel each other out, increasing the likelihood of stable appearance of Ramanujan-type modular forms.

There is also the observation that the trivial zeros of the Riemann zeta function lie in the even sequence −2, −4, −6, ..., which may be linked to the possibility of Ramanujan-type structures. In other words, negative even numbers can be considered as “special points where even-order traces cancel out,” or, in other words, “points where open non-periodic traces can be symmetrically folded back to zero.”

This corresponds exactly to the conditions under which open terms (geometric series) can be converted into closed forms (product forms), as in Ramanujan's equations.

The subsequent development I came up with was as follows.

In this way, I strongly feel that Ramanujan had an intuitive sense of this kind of “nested structure.”

To confirm this, let's take a look at some of Ramanujan's formulas.

Figure 15. The Ramanujan's modular identity.

$$\sum_{n=1}^{\infty} \frac{1-q^n}{q^n} = \sum_{n=1}^{\infty} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{q^n}$$

This Ramanujan formula flips the denominator and numerator, expresses each (1-q^n) term as a geometric series, and multiplies them again, resulting in a beautiful identity.

In other words, it is an identity that expresses “modularity” and “log scaling,” which represent the transformation from ‘sum’ to “product.”

Let's bring in another modular form.

This is the Jacobi triple product.

Figure 16. The Jacobi triple product.

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})$$

- 中心にあるのは、**対数スケーリング構造を持つ加法と乗法の融合**

Looking at this structure, we can see that the genus is 2 or higher due to the conditions of triple product and two variables. This is because there is a symmetry term with “genus + 1,” i.e., 3, where 3! = 6, and no modular form with such a structure exists if the genus is higher than that. Here too, we can see an extraordinary structure where “modularity” and “log scaling,” which combine ‘sum’ and “product,” are beautifully integrated while encompassing multiple variables.

Let's observe one more example.

This time, Dedekind's modular form.

Figure 17. The Dedekind's invariant form.

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

- η関数 = $q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$
- τ→-1/τ の変換は**logスケーリング**（対数螺旋）に対応

From the multiple periodicity, we can see that “this is genus one.”

However, if it is of genus 1, isn't the pattern of double periodicity just one?

However, when the genus is 0, it was understood from the “graphical Riemann surface structure” that sine, cosine, and power functions actually have the same form, so it is imagined that all modular forms of genus 1 are unified into one form, one “equivalence class.”

In all the patterns we have seen so far, the transformation between “sum” and “product” is unified by “modularity,” and we can observe that modularity always requires a slight ‘shift’ or correction. This shift gives rise to phenomena such as the “breakdown of multiplicativity” seen in the Ramanujan zeta function. This leads to the surprising discovery that there may also be non-regular structures within the Dedekind zeta function.

The consideration of Lindon-like nesting and non-periodicity in Ramanujan can also be seen in the famous taxi number.

This is the moment when Ramanujan's intuition and the Lyndon sequence perspective completely converge.

The smallest taxi number,

$1729 = 1^3 + 12^3 = 9^3 + 10^3$

That is, “the smallest number that can be expressed as the sum of the cubes of two positive integers in two ways.” Why is this a Lyndon structure?

The structure of ordered pairs, $(a, b)^3 + (c, d)^3 = n$, corresponds exactly to the “set of the smallest pairs in lexicographic order" in Lyndon terminology. (1,12) and (9,10) are "non-symmetric" and "non-periodic" pairs, and the core of Lyndon theory is to regard such unique combinations as the minimal structure.

Since the definition of a taxi number is "two distinct combinations," periodic sums (e.g., 2×(1,12)) are excluded.

Since Lyndon language is based on non-periodic sequences, it appears to satisfy this condition.

Considering the dual structure, if we regard reverse order and symmetry, such as (1,12) ↔ (12,1), as equivalent, we must choose the lexicographically smallest (i.e., Lyndon smallest) pair.

The "root structure" of the decomposition formula is graphable and has genus 1, but if we draw all pairs that can be expressed as $x^3+y^3=N$
on a graph, it looks like a bouquet graph:

Each node is the same as the pair (x,y), and the pairs belonging to N form a loop. The taxi number is the smallest node in the bouquet that has two or more loops. This is exactly the same as the Lyndon bouquet structure.

Regardless of whether this analysis is accurate, the hypothesis that Ramanujan "unconsciously classified sequences using Lyndon language" becomes more plausible.

Ramanujan's formulas and infinite series often have mysterious nested structures.

If we apply Lyndon's complex spiral continuous phase theory to Ramanujan's formula expansion, it seems possible to interpret the hierarchical structures of periodic and non-periodic elements contained in the infinite sums and products he created in a modern way, which makes my imagination run wild.

By the way, generally speaking, the Deligne structure is considered to be a conformal category structure that appears in modular curves, and requires a difficult description. However, in this paper, we interpret it as "syntactic symmetry" in Riemann surfaces of genus 1. This special symmetry arises from the existence of dual paths due to the fact that there are only two curves of genus 1, and it is related to the structure described in part by Modell's theorem.

In other words, the τ function and its composition are not mere functions, but extensions as mappings acting on the syntactic space. As shown in Figures 20–21, this allows us to visualize how the syntactic complex trace that appeared in Modell's theorem unfolds with finite group branching.

This extension is nothing other than a more naive syntactic operation that brings down the geometric essence of the Deligne structure.

Now, let me calculate a concrete example of my prediction.

When the quadratic field is $\sqrt{-5}$, the discriminant is 40, and τ（n） is calculated as follows: τ (2) =−2, τ (3) =−1, τ (5) =−5, τ (6) =2, τ (4) =2, τ (8) =0, τ (16) =−4, τ (32) =8.

From here, using induction,

Figure 18. Skew multiplication function.

$$\tau(p^r) = \tau(p)\tau(p^{r-1}) - p\tau(p^{r-2})$$

we can calculate the quadratic Euler product using this.

Figure 19. Deriving local structure from distorted multiplication functions.

$$\sum_{r=0}^{\infty} \frac{\tau(p^r)}{p^{rs}}$$

$$= \frac{1}{1 - p^{-s}\tau(p) + p^{1-2s}}.$$

Therefore, by integrating this(see Figure 20),

Figure 20. Integrated into the Euler product of the second order.

$$L(s) = \prod_p \left(1 - \tau(p)p^{-s} + p^{1-2s}\right)^{-1}.$$

This calculation took two days from the initial idea, so if the result is correct, it would be very exciting. This is a "non-regular" quadratic zeta function, and it is clearly isomorphic to Ramanujan's quadratic zeta function, being of the same type of genus 1 zeta function(This overlapping of multiple quasi-dual mappings can be visualized in **Figure 20**, where the Euler product is extended to second-order duality over the complex syntax spiral).

At the same time, from this observation, theoretically,

when genus + 1 ≡ 0 (mod 2),

it is highly likely that a higher-order version of CM = "imaginary multiplication-like theory" exists, and higher-order zetas can be naturally constructed.

Now, based on the current idea, if we consider $\tau_1(n)$ as the quadratic zeta function mentioned earlier and $\tau_2(n)$ as Ramanujan's τ function, and think of their product as a new operator, it becomes a multiplicative function with an irregular multiplicative structure(see Figure 21).

Figure 21. Multiplication function combination.

$$a_n = (\tau_1 * \tau_2)(n)$$

From here, after various efforts to transform it matrix-wise, I finally had a 4×4 matrix calculated by a computer. However, upon further investigation, I discovered the theory of Hecke operator multiplication, leading to the following equation:

Figure 22. Representation of the fourth-order zeta function by the Hecke operator.

$$L_p(s) = \frac{1}{\det\Big(I - (\rho_1 \otimes \rho_2)(\mathrm{Frob}_p)p^{-s}\Big)} = \frac{1}{1 - A\,p^{-s} + \cdots + D\,p^{-4s}} \quad ただし \quad \begin{cases} A = a_p b_p, \\ D = p^{k_1+k_2-1}. \end{cases}$$

Here, the Ramanujan zeta function has $k_1 = 12$, while my constructed function has $k_2 = 2$, so D = p^12, and A = (Ramanujan generating function (p)) × (my zeta generating function (p)), resulting in a highly complex equation. This can be verified by simply multiplying the distorted multiplicative functions and examining the behavior of some terms, which clearly match in parts.

Finally, the Euler product was obtained. It is indeed a fourth-order zeta function. Moreover, there are probably two of them now, so to speak, forming a bivectorial ring. There are likely to be more(As shown in **Figure 21**, the compression of infinite trace bundles into modular structures via finite groups reveals a discrete syntax topology).

I am unsure whether this calculation is correct, but this method demonstrates that higher-order zeta functions such as second-, fourth-, sixth-order, etc., can be constructed, indirectly supporting my hypothesis.

It is predicted that the zeros of this function scatter from the critical line and nest within the critical region. And this is a non-regular, non-commutative, non-Abelian, graph-theoretically non-Ramanujan (somewhat ironic) zeta function. Why do the zeros scatter further from the critical line as the order increases? As mentioned earlier, when the genus is large, the number of non-periodic terms inherent in the graph increases to “genus + 1,” so more “internal complex zeros” are needed to control divergence and prevent the Euler product from diverging. This is precisely the meaning of Deligne's discriminant.

Due to the non-periodic terms existing internally, the “collection of prime numbers” in the Euler product, which already exists infinitely, diverges to infinity. However, the mechanism that suppresses this divergence by grouping them complexly is Deligne's inequality.

Now, summarizing the above results,

《Structural Conjecture of Non-commutative Langlands Domain》

Non-commutative Langlands domains exist in a complex multiplicative manner when the genus plus one is even, and they exist in the form of Hecke operators that control the divergence of Euler products.

I believe that when the genus is one, the existence of "symmetric non-periodic terms" causes the imaginary quadratic field to generate the structure of "CM elliptic curves," which serves as the source for infinitely generating "quadratic zetas." Furthermore, there is a theoretical structure that allows for the infinite generation of even-order zetas such as 4th, 6th, 8th, etc., by multiplying Heck operators.

On the other hand, in the case of the imaginary cubic field (an imaginary algebraic number field with genus 3), there is no quadratic form structure with symmetry like that of the imaginary quadratic field, and it is also impossible to construct a complex multiplicative "trace mirroring" structure. This is because there are no "symmetric non-periodic terms," and one must deal with symmetry of order 3 or higher, so a slightly different methodological approach may be necessary to control divergence.

In this section, I have explained the structural meaning of the Ramanujan zeta function and why it is a quadratic zeta function. In the process, I myself was probably delighted when I was able to create a quadratic zeta function.

However, what opened up from there was a "vast sea of non-commutative structures." Can the structure of all graph-like Riemann surfaces be reduced to the structure of genus graphs?

Or is there a more complex non-regular structure?

It seems that we need a few more new ideas to reach that point.

That's all.

5. Summary

What do you think?

First, I derived the existence of non-periodic terms from my structural graph-theoretic Riemann surface theory, understood the structure of general Riemann surfaces, imagined the conclusions that would arise from that structure, summarized various theorems and hypotheses, and finally constructed a concrete example.

While working on this, I honestly thought, “Isn't it strange that no one else has done this?” The quadratic zeta function I constructed is actually unnecessary, and if we have the Ramanujan zeta operator, couldn't we construct higher-order zeta functions? ... I don't have enough knowledge to answer that question.

Through this analysis, I believe I have been able to convey, to some extent, the shift in my thinking from the introductory level to the applied level, as well as the logical structure and its implications.

This work was also necessary for me to maintain my sanity.
The theory was so beyond my imagination that, despite the conclusions being clear, it was difficult to compare them with existing results, and even more difficult to provide concrete examples.

However, I was somewhat satisfied after observing some “correspondences with existing theories” and “agreements with concrete examples.”

黒川信重、絶対数学原論、現代数学社、2016
森田英章、組合せ論的ゼータの半群表示、2016
ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
高安秀樹、フラクタル、朝倉書店、1986
Shinjiro Kurokawa, Absolute Mathematical Theory, Gendai Suugaku Sha, 2016
Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
Bernhard Riemann (translated by Jiro Suzuki), On the Number of Primes Smaller Than a Given Number, 1859
Hideki Takayasu, Fractals, Asakura Shoten, 1986

Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.
This refers to the transformation of infinite repetitions of non-trivial aperiodic

sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ　ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.
素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of

“complex spiral phases”
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a “divergence control function” corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ s という臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.
無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the “elementary Lyndon element” that is restored to zero is decomposed, the corresponding Euler product becomes a “multiple Euler product,” giving zero points multiple values.

多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through “dual non-periodic paths.”
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a “double helix” because it naturally contains spiral rotations and has a double main structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in the number of species. This is particularly important in the context of the formulation of

“higher-order imaginary multiplication.”
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by “dual non-periodic paths.” Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.
トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)
素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応す

る単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　quasi-dual　quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する

In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.

全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.
Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction ; Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph” : A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences arising from two non-periodic circuits of curves with genus one.

双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal “bouquet structure” corresponding to the “Euler product.” It is also necessary to distinguish it from the commonly referred to “Hodge structure.”
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
　In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
　とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ; A natural quasi-dual transformation that reduces “irreducible rational Lyndon” in trace bundles to “prime Lyndon” or “natural number Lyndon.” Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets；A complete state that cannot be further expanded by repeating dual operations.
双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.
→ 類双対閉包
非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる