The theory of Quasi-duality map and fractals

(翻訳　類双対性写像とフラクタルの理論)

Part I:The Theory of the Lyndon Complex Spiral Phase

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Abstract

This chapter introduces the foundational concept of Lyndon words and their combinatorial and algebraic properties. It explores how non-periodic structures encode complex symmetry through recursive aperiodic traces, leading to the definition of fractal phase topology on the complex plane.We define analytic functions, multiple Riemann surfaces, general noncommutative norms, etc. in the structure named the Lindon complex spiral phase.

1. Introduction

This article is a further introduction to the theory described in the essays that follow, “Introduction,” “Applications,” and “Considerations on the Riemann Zeta Function,” or a glossary of terms.

I first understood that the “zero points of the Riemann zeta function” correspond to non-periodic sequences that are “prime Lyndon sequence.”
As I continued to think about this, I discovered the Lindon complex spiral phase, that is,

“Any infinite (or finite) Lindon sequence” → “A complex plane rotating spirally”

This “embedding structure” correspondence, or continuous phase structure.

However, it took me a full month to realize that this concept was surprisingly difficult to understand.

I had originally planned to write a brief explanation of one or two pages, but I felt obligated to explain it in greater detail.

Therefore, in this section, I have provided a more detailed explanation with concrete examples, and I have also included explanations of terms that might be unclear to some readers.

This “introduction” demonstrates a method by which anyone can start from the structure of the Lyndon semigroup, begin with the prime factorization of integers, and then move beyond rational numbers, real numbers, and complex numbers to reach the non-commutative, non-associative broken domains of quaternions, octonions, and 16-tonions in one fell swoop.

Everything progresses fractally.

In other words, it is a methodology for “how to compress information and extract meaningful information.”

2. First Principle: Lindon words are uniquely decomposable

A Lindon sequence is a semigroup, which is an ordered arrangement of various elements.

For example,

(01)(01)

Let's say there is a Lyndon sequence like this. As will be explained later, (01) is called a prime Lyndon, which is a “prime number” in the Lyndon sequence that cannot be further reduced, and McMahon's theorem states that all Lyndon sequences can be uniquely decomposed into prime Lyndon sequences.

For example, let's consider a random sequence.

0010010101011010100100010

You may find it hard to believe that this can be uniquely decomposed into "prime Lindon."

A "prime Lyndon" is "the smallest combination of numbers that does not have periodicity."

However, McMahon's theorem states that this is possible.

Figure 1. The Duval decomposition.

与えられた列

```
0010010101011010100100010
```

を **Duval アルゴリズム**に基づいて一意的に **素リンドン語**へ分解

```
['00100101010110101', '001', '0001', '0']
```

Decompose the given sequence into unique prime Lyndon words based on the Duval algorithm.

Furthermore, the length of a prime Lyndon includes at least one of every natural number.

In other words, for any natural number n≥1, there is at least one prime Lyndon word in the alphabet sequence of length n(see Figure 2).

Figure 2. Length of the original Lyndon sequence.

長さ N	素リンドン語（例）
1	`0`, `1`
2	`01`, `10`
3	`001`, `011`, `101`, `111`（ただし素なもののみ）
4	`0001`, `0011`, …

There exists a Lyndon word of length n.

It seems nothing short of mysterious.

The simplicity of the algorithm itself, which uniquely decomposes the sequence, is also surprising. 1. Look from left to right and extract the largest non-periodic sequence. 2. Remove it and then extract the next largest sequence in the same way. Just repeat this process(see Figure 1).

Now, let's consider a new rule.

Prime Lyndon length → Natural number

This is the correspondence.

However, the (01)(01) we initially considered is

(01)(01) → The same repeated Lindon sequence is reducible because its length 2 and repetition count 2 match, making it equivalent to (01), which is an integer with a value of "2" since its length is 2.

This is the "principle of information contraction."

In other words, the combination of prime Lindons is a natural number.

Note that the order is destroyed and a "value" is produced at this point.

In a sense, the "contraction of order" is the first reduction.

Figure 3. Order of contraction.

```
(123)(123)(123) → (123)
```

In this case, it is reducible because the "length" 3 and the "repetition count" 3 overlap. Such a structure that allows compression is "multiplicative" and poses no problem, but the "additive structure" is partially broken, losing its order, and must be restored later(see Figure 3).

This is related to structures like F_1.

Furthermore,

Overlap of "length" and "repetition count" in prime Lindon sequences → Overlap of information → Compression

Contraction of structure → Crushed by compression mapping

This can be **understood as monoid algebraic behavior on F_1.**

3. Construction of natural numbers

When prime Lindon sequences are combined, they form “natural numbers.”

It should be noted that “natural numbers” do not have an order structure, but prime Lindon sequences do. There are infinitely many “prime Lindon sequences,” and they can be decomposed uniquely.

For example, if there were (23)(568)(23), the order would be broken, so it would revert to 2 × 3 × 2. In other words, it would become the “value” 12, but even if the order changes, it is still the same “equivalence class.”

In this way, since there are prime Lindon sequences of all “prime number” lengths, all natural numbers are assigned a “value.”

The reason why the overlap between the “length” and “repetition number” of prime Lindon sequences is reduced is that the repetition of 3 in a prime Lindon sequence of length 3 is reduced to 3 due to the overlap between, for example, 3 × 3 and 3 + 3 in natural numbers. ...However, in the case of the “prime Lindon” of 6 and 9, it can be expressed differently. As I will explain later, since 6/2 exists within the “irreducible rational Lindon,” the combination of 3 (prime Lindon) + 3 (repetition) is restored within the “irreducible rational Lyndon sequence.”

4. Fractions and the composition of irreducible fractions

At first, I called it “N-reducible,” but it seems to be unusually difficult to understand, so here's what it means.

If there were a Lindon language like (23)(568)(23), I said it would become 12. However, the following Lindon sequence, which continues five times,

(23)(568)(23)(23)(568)(23)(23) (568) (23) (23) (568) (23) (23) (568) (23)

Since 5 is neither a multiple of 2 nor a multiple of 3, it cannot be contracted.

In other words, the “product of the lengths of the constituent elements (e.g., 12)” is determined as an irreducible ratio due to the inconsistency with the number of repetitions (5 in this case). This is because 2, 3, and 5 are “prime” numbers.

Therefore, it returns the “value” 12/5 as an irreducible rational number.

At this point, every Lindon sequence has a unique contraction to an “irreducible rational Lindon” through prime Lindon decomposition. In other words, a Lindon sequence has a unique mapping to a new “irreducible rational Lindon semigroup” and is transformable.

Theorem

A Lindon sequence has a unique contraction mapping to a new "irreducible rational Lindon semigroup."

Furthermore, if there is a sequence of "irreducibly repeated Lindon sequences," it becomes additive.

The Addition Theorem of Lyndon Languages

Addition is the additive alignment of Lyndon languages.

Note that reductions such as "3+3=3" are restored here.

Figure 4. The Additive contraction.

$$3+3=\frac{6}{2}+\frac{6}{2}=\frac{12}{2}=6$$

Additive contraction.

They are restored as an "irreducible rational Lyndon sequence." However, 6 is a "prime Lyndon" element of length 6. 2 cannot have a repetition that includes 6. Because it is only a divisor(see Figure 4).

In other words, since "the order of addition is broken," it is okay to restore it later.

5. Real numbers

Please think of all the symbols we will deal with from now on as "irreducible rational Lindon sequences."

By the theorem we wrote earlier, we can consider the collection of "irreducible rational Lindon sequences" as a new "Lindon sequence." This is because any given Lindon sequence can be transformed by "prime Lyndon decomposition" → irreducible rational Lyndon contraction map → "new Lyndon sequence by irreducible rational Lyndon."

Any Lyndon sequence can be uniquely contracted to an "irreducible rational Lyndon sequence."

Since there are infinitely many irreducible rational Lyndons themselves, they can be regarded as "elements of Lyndon language," and can be uniquely decomposed into "prime irreducible rational Lyndons" by McMahon's theorem.

Then, the combinations of irreducible rational Lyndons themselves can be reduced in the same way as before.

However, as before, the combination of prime irreducible Lindons cannot be reduced unless the length of the prime irreducible Lindons is a prime number. Conversely, when reduction is not possible, the number of repetitions of the prime irreducible Lindons is successively **root-extracted according to the number of iterations.**

Therefore, for example, the golden number can be expressed as

two repetitions of the sequence of Lindons of 1 and Lindons of root 5

In other words, all solutions to algebraic equations can be expressed in this way.
In this way, the operation continues indefinitely, and the reduced "root Lindon" can be uniquely decomposed into a "prime radical Lyndon" by McMahon's theorem... This repetition constitutes real numbers.

In other words, **each time a "root" is stacked, a reduction map to a new "Lyndon semigroup" is uniquely defined**.

When it becomes a repetitive state in which different elements overlap, such as a "multicyclic fractal," it cannot be reduced. It becomes non-commutative, and the root cannot be easily solved.

That is a real number.

Figure 5. An image of Lyndon- real numbers.

たとえばこういうものを思い浮かべる

$$x = \sqrt{2 + \sqrt{3 + \sqrt{5 + \sqrt{7 + \cdots}}}}$$

または、もう少し数論的に：

$$x = \sqrt[3]{2 + \sqrt[5]{3 + \sqrt[7]{5 + \cdots}}}$$

理論の語彙で言えば

「入れ子になった**既約有理リンドン列**の非周期列」

「縮約不可能な素構造の密な重なり」

「式として"定義できる"が、解析的には閉じない値」

Image of Lyndon-real number in Lyndon sequence.

We can observe this nested structure of roots(see Figure 5).

We know that such overlapping roots are "real numbers," but we do not understand what kind of "value" they actually represent. Similarly, in a multi-periodic fractal manner, when roots are nested and the repeated number is treated as a root, the same 'value' is returned as a "real number."

Here, it is important to note that while Repetitions of a prime Lyndon are treated as divisions under the criterion of irreducibility, the iteration of an irreducible rational Lindon or real Lindon does not disappear, but rather "has meaning if it is irreducible." This is because they return to the 'power' of the rational Lindon sequence or real Lindon sequence, respectively, rather than disappearing. In other words, it becomes a "to take the n-th root of x." Furthermore, a sequence of non-irreducible rational Lindon or real Lindon corresponds to situations where "rational numbers become integers" or "real number roots can be solved." If the sequence cannot to be contracted to any compressible periodicity, it is a real number.

Here, it is important to note that such quasi-periodic "non-contractible structures" with "multiple periodicity" are, with few exceptions, "finite Lindon sequences."

In other words, they can have "infinite recursive structures."

Note that this reduction map from the Lindon semigroup to the Lindon semigroup forms a nested fractal that continues uniquely, refining the "values" in detail. As the scale increases, the values become finer... **This is a principle of information theory.**

Also, if the solution to an algebraic equation is within an algebraically closed field, it can always be rationalized. From this, it follows that even irrational numbers can be expressed using this method, as they can be reduced to rational numbers in a finite number of steps.

Summarizing the above(see Figure 6),
Repetition of natural number Lindon → division
Repetition of irreducible rational Lindon → to take the n-th root of x
Such things are arranged → addition
Repetition of additive structure → division
Repetition of additive structure in irreducible rational state → to take the n-th root of x

Figure 6. Types of Lyndon sequence.

リンドンの種類	例	繰り返しに対する振る舞い	解釈
素リンドン列	`(0101)`	繰り返しは**冗長 → 縮約される**	構造単位そのもの。情報の最小単位。F_1的縮約
既約有理リンドン列	`(01)(01)(01)`	繰り返しが**比率的 → 意味を持つ**	有理数としての「反復構造」を保つ（3回なら3/1）
無理数（実数）リンドン列	`(01)(011)(010011)…`	既約有理リンドン列が「既約」な「反復を持っている状態」=累乗	ズレが累積し、縮約不能な実数位相へ
超越的リンドン構造	混合・入れ子・自己非縮約列	無限の入れ子 → 完全に縮約不能	高次の情報構造、複素・回転・多価性を形成

The hierarchical structure of the Lyndon sequence.

Please note that everything is structured in an "information theory" manner.

6. Complex numbers

Perhaps the most common question that arises when hearing this story is, "Why is that so?" Complex numbers are probably the most common example.

From this point on, the elements can be natural numbers, irreducible rational numbers, or real numbers.

If these elements are arranged in an "aperiodic sequence" manner, the "aperiodic sequence" itself can be regarded as a single "value."

In other words, it is a nested structure of "an aperiodic sequence."

This nested structure is also a "semigroup," so it can be uniquely decomposed by McMahon's theorem. In other words, based on the previous discussion, we can say that it "can have a nested real number structure," so this higher-order nested structure has the "values" of natural numbers, irreducible rational numbers, or real numbers.

Theorem

Every Lyndon sequence has a contraction map in the form of "non-periodicity of non-periodicity," and it is unique. In other words, there is a unique mapping from "any Lyndon sequence" to "non-periodicity of non-periodicity" to "new Lyndon sequence."

This applies to complex numbers, natural number Lindon sequences, irreducible Lindon sequences, and real number Lindon sequences. Depending on the value, if the number of prime elements is N, the angle is determined by the Nth root of N, and the "non-periodic nested structure" is rotated by the "real number" value indicated by the semigroup. Note that this "non-periodic sequence" can be continued infinitely. In other words, this rotation continues forever, spiraling endlessly across the complex plane.

This is what I call the “Linden complex spiral continuous phase(see Figure 7).”

In other words, if the real number itself forms a “non-periodic sequence,” it becomes a rotation on the complex plane, and even if it is a natural number Linden, it rotates. This is the complex spiral plane.

This “non-periodic sequence nested within a non-periodic sequence” can also be a “finite Lindon” or an “infinite Lindon.”

From this concept, we can understand that “negative numbers” are quite rare. In fact, when a “negative value” appears in the zeta function, it can be seen as a spiral in which the divergence is contracted.

Isn't this frighteningly nested structure of semigroups and McMahon's continuous mapping beautiful?

7. Divergent reconstruction and transcendental numbers

When such a “Lyndon sequence plotted on a complex spiral plane” is “infinitely repeated,” its “value” returns.

For example, suppose there is a graph with two loops.

Then, if we assign the numbers 0 and 1 to the loops and take infinite traces on all graphs, all “Lyndon sequences” created from ‘0’ and “1” will be generated.

In other words, if the trace bundle of a graph has “infinite repetition,” it will return the ‘value’ of the “finite Lyndon,” and even if there are some that are not “infinite repetition,” it will return “transcendental numbers.”

Transcendental numbers cannot be contracted by any finite reduction method. In other words,

complex Lindon → real Lindon → irreducible rational Lindon → natural number Lindon

Such a “contraction” can be defined, but anything that cannot be reduced to a “finite-length Lindon sequence” by such a finite reduction is considered a “transcendental number.”

Figure 7. Rotation by repeating nested structures.

対象	リンドン構造	幾何的意味	結果
実数 r	有限リンドン列 L_r	点の集合（直線上）	実軸上の数
非周期的な L_r の反復	入れ子·ズレのある列	螺旋構造を作る	回転 = arg = 複素数
回転構造	実数 × 方向	単位円周上の点	複素数

Complex Lyndon sequence indicates rotation.

8. Analytic functions

As before, let us consider a graph with two loops.

We will name these loops 0 and 1.

Then, if we take any infinite trace bundle of this graph, it will contain all sequences such as 00010100101001010101... In other words, when condensed, there is always a "plot on the complex plane" based on a prime Lyndon.

At the same time, this "trace path" also synchronizes in the same way in the "dual graph" of that graph.

The dual operation is, simply put, an operation that swaps "points and lines." This operation changes the structure of the graph, so the "trace structure" also changes. As a result, the same trace in the graph plots different numbers. In other words, **"a certain complex number" and "a certain complex number" correspond one-to-one**. This is an analytic function(see Figure 8).

Simply by existing, the graph generates an "analytic function" on the entire complex plane.

From this fact, it is possible to extract any general Riemann surface structure from any graph structure, but the explanation and proof of this are discussed in the "Introduction" section.

Figure 8. Mechanism of Graph-like Riemann surfaces.

（グラフ構造）
↓ トレース列
（リンドン縮約）
↓ ズレ構造
（回転角度）
↓ 複素点
↓
（双対グラフからもう一つの複素点）
↓
（z ↦ w の対応）＝解析関数
↓
（関数が重なり合っていく）＝リーマン面

Graph → Lyndon contraction → rational numbers/real numbers → complex numbers → dual graph → analytic functions → overlapping of graph-like Riemann surfaces (multiple Riemann surfaces)

9. Divergent Restoration

As explained above, even with just two loop structures, there is a continuous complex spiral structure that is "divergently restored" from the graph.

This is referred to as "divergent restoration."

In a different sense, there is also a "divergence of prime structures" inherent in the graph.

For example, if two loops overlap, the choice of which loop to follow results in a "0, 1" branching.

Due to this branching, the "trace bundle" contains the structure "0101010101010010100...," resulting in an infinite number of "prime structures."

Such structures are referred to as "non-periodic paths" within the graph.

When there are two such non-periodic paths, it is called a Riemann surface of "genus 1."

The counting method is the same. When there are three, it is "genus 2."

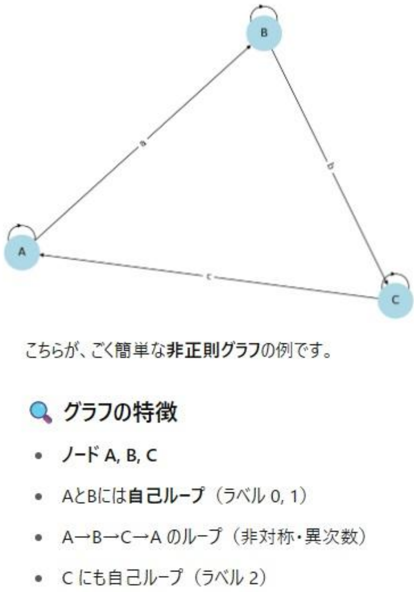
In other words, normally, "loop structures" do not have divergence.

However, when there are "non-periodic paths," divergence occurs in the trace structure. This makes it more complex. Therefore, the degree of complexity is represented by the number "genus."

At this point, it becomes important to distinguish between the usual "divergent restoration" and the divergence caused by the "non-cyclic path" of "genus + 1" within this graph.

10. Synchronous structure in the dual space

Figure 9. Example of a triangular graph with self-loops.



Self-looped triangular graph.

Let's assume there is a graph (see Figure 9).

The self-loops of A and B are assumed to have generators of the prime symbols “0” and “1,” respectively.

The route A→B→C→A functions as the symbol sequence “a → b → c,” but the self-loop (2) at C introduces asymmetry and irregularity.

The trace sequence obtained from this entire graph produces a structure that mixes periodicity and non-periodicity, such as:

0 0 a b 1 b c 2 2 0 1 1 c a b …

.

The “trace Lyndon sequence” of a graph with such a “self-loop” structure is characterized by its high divergence, and it becomes important in proving the “fixed point” during “divergent expansion.”

On the other hand, when considering the dual graph corresponding to this graph, if there are correspondences such as “0↔1” and “a↔c,” the same trace sequence will have different plot values.

Therefore, different complex numbers are assigned to the same trace sequence between “one graph” and “its dual graph.”

This is precisely the definition of the analytic function $z \mapsto f(z)$.

Furthermore, when multiple prime symbols overlap in a loop and an irreducible non-periodic sequence appears, the graph produces a unique plot on the complex plane. In other words, it expresses all continuity, including transcendental numbers.

11. Dual motif closure

Incidentally, the dual operation of “swapping points and lines” is not unique.

When it becomes a hypergraph with many points connected by many lines, there are many ways to swap them, and we write down all of them.

Then, we take the dual operation of “swapping points and lines” on those transformed graphs... and continue this operation. In the case of a finite graph, this operation ends after a finite number of steps and completely closes the “dual operation.” In other words, the “motif closure” is an operation that replaces the “multivalued nature” of dual transformations with the “relationship between sets of graphs.”

There is no need to understand this deeply. For example, a quadratic function has one value, but if you flip it, it becomes two values. This is also a dual transformation (taking the inverse function). If you swap them again, it returns to the original. However, in more complex Riemann surfaces, it naturally becomes multivalued.

12. “Multiple Traces” and “Multiple Trace Space”

“Multiple trace space” refers to **a structure in which the same trace sequence in the entire motif closure proceeds simultaneously (i.e., synchronously) in multiple motif structures (including duals) while having different interpretations of contraction, projection, and prime structures.**

There were many dual-transformed graphs, but their “points and lines” corresponded to each other.

Therefore, when “taking the trace of a single graph,” a “trace” synchronized with it runs in all dual graphs.

This is “multiple trace.”

And the important point is that the “trace sequence” is “plotting points on the complex spiral plane.”

This means that in the various graph shapes of these dual graphs, “different ‘values’” are simultaneously plotted on the complex spiral plane.

This is “continuous,” meaning that among the “trace bundles divergently reconstructed on any complex spiral plane,” each trace has a “value.” Thus, it becomes clear that these deformations are “continuously connected.”

This is the uniqueness of analytic connection in graph-like Riemann surfaces, or more generally, in Riemann surfaces. You will realize that the question “Why does analytic connection occur on Riemann surfaces when extended to the complex number domain?” is no longer necessary.

The same trace sequence has different meaning structures in different motif closures (i.e., different prime Lindon reduction structures). Each is mapped to different complex numbers z_1, z_2, z_3..., and these mapping relationships give rise to multivalued mappings (Riemann surfaces) and analytic function systems.

With the addition of the concept of “multiple trace spaces,” graph structures begin to take on spatial meaning, such as “projective systems,” rather than merely generating mappings, and function spaces become families of projective mappings generated by motif dependence(see Figure 10).

Phenomena such as zeta functions, modular forms, and automorphisms also begin to appear in different guises within “multiple trace synchronization.”

Figure 10. Structural level of graph-like Riemann surfaces

構成レベル	名称	意味	主な構成要素
① ローカル	トレース列	グラフGの1つの経路	リンドン語列、非周期列など
② ミドル	モチーフ閉包	グラフGの双対グラフを考えることで、多値性を制御した閉包構造	点と線がそれぞれ対応しながら変形されたさまざまなグラフの集合体
③ グローバル	多重トレース空間	複数のモチーフ閉包（双対・派生）にわたるトレース列の **同期的構造**	同一トレース列の複数プロット、関数対応 ($z \mapsto w$)

Local → Trace sequence (trace bundle) Middle → Dual motif closure Global → Multiple trace space

13. General Riemann surfaces are not “patchworks” but “inevitable overlaps.”

When there is a synchronous structure between the “complex plane” in the Lindon spiral continuous topology and the “multiple trace space,” there exist infinite non-periodic Lindon sequences (i.e., sequences with discrepancies in construction density).

Each of these is plotted as a rotation point (e^{2πiθ/n}) on the complex plane (where n is the number of prime structures in the Lyndon sequence). In other words, each of the deformed graphs on the motif closure has a “value.”

Furthermore, since there are many motif closures/dual graphs, the same sequence is mapped to multiple plots (multiple traces).

A “natural continuous mapping” (synchronization) occurs between these multiple structures, and **then all discrepancies are self-synchronously complemented, and all plots are uniquely connected by a continuous mapping, thus connecting the complex plane and covering it "without gaps."**

This is the "uniqueness of analytic continuation in multivariate functions."

14. To the higher algebraic domain

From McMahon's decomposition theorem, through rational reduction, we can hierarchize and further hierarchize real numbers on fractals as roots of powers in the same way, and from the higher-order non-trivial "Lyndon semigroup" called "non-periodic terms of non-periodic terms," complex numbers are extracted in exactly the same way, They form a spiral, and quaternions are extracted from the "higher-order structure" of that complex structure, rotating again to a different axis, which continues to hexadecimals, and gradually the algebraic system breaks down... I partially described this flow.

15. The order of multiplication and addition is broken, but it will be restored later.

In the Lyndon contraction structure, multiplication, addition, and their order first continue to exist, so they cannot be extracted without first being restored by the contraction map.

In the initial structure, addition does not exist, and only the contraction repetition dominates.

However, when non-repeatable Lyndon structures are combined, addition "emerges" in parallel. .

This addition is not the addition of natural numbers, but rather emerges through the rearrangement of hierarchical trace-repetition structures.

Therefore, addition in this theory is **a non-commutative order structure** that is constructively reconstituted.

Is there a "zero element" within this?

It is likely the "infinite repetition state of a prime Lyndon structure."

In terms of information content, it has no meaning. Each "prime Lyndon" functions as an identity element and possesses a multiplication structure until it is restored.

When restored, the "order" becomes addition.

This is a non-commutative order structure. It has no meaning in terms of information. Each "prime Lindon" functions as a unit element and has a multiplicative structure until it is restored.

When restored, the "sequence" becomes addition.

This corresponds to the fact that the order of operations from multiplication to addition is fixed.

This is natural in terms of information theory and is fractal.

16. Fractality as the most important principle: Three principles

In this system, if I were to dare to cite axioms, they would be

1. McMahon's uniqueness decomposition theorem

All Lyndon sequences can be decomposed into "prime Lyndon sequences" (including sequences of all natural numbers).

2. Uniqueness of reduction maps to irreducibility

All Lyndon sequences have a unique map to Lyndon sequences that are more "irreducible" than their current state.

3. Uniqueness of contraction maps to "non-periodic sequences of non-periodic sequences"

All Lyndon sequences have a higher-order nested structure, and within that nested structure, they have a contraction map as a "Lyndon semigroup."

These are the only three principles.

The complex spiral continuous phase is almost entirely composed of continuous repetitions of these three principles, that is, a "nested structure," and is therefore complete.

In particular, let us call the two types of contraction maps inherent in the Lyndon semigroup the "Lyndon double spiral structure."

When a Lyndon sequence has an irreducible contraction series and a non-periodic nested series, and these are isomorphic while breaking the non-commutative order structure, we can call this syntactic structure the Lyndon double spiral.

17. Proof of the existence of dual Lyndon sequences

Duval decomposition is performed from the left.

However, it can also be performed from the right.

What is of interest is the existence of a reverse decomposition of a "prime Lyndon structure," or in other words, the existence of a "dual Lyndon sequence."

Let us prove its existence.

By definition, a non-periodic sequence is non-periodic when viewed in reverse. Suppose it can be decomposed into another non-periodic sequence. Then, "different non-periodicities" would be inherent in the same structure, leading to a contradiction. Therefore, "dual Lindon elements must be decomposed in the same way."

Thus, dual Lindon sequences must exist.

Note that the “dual Lindon sequence” in this Lindon decomposition naturally defines the “inverse of the trace sequence.” In other words, when returning a “value,” “the same general Riemann surface structure is reached regardless of which trace path is used as a reference.”

This is related to the structure of inverses in Lindon algebraic structures.

In other words, the current proof establishes that **“the value of the inverse function in the general Riemann surface is uniquely determined**(see Figure 10 for syntactic dual mappings).**”**

18, Non-commutative Rotational Norm Theorem

On the infinite “spiral phase” defined for Lyndon sequences,which can be described as a “non-periodicity of non-periodicity”:

This spiral phase compresses the inherent “rotation” into a real value,thus uniquely determining a norm.In other words, the structure rotates while being slightly extended—a spiral that never fully closes.

Theorem (Non-commutative Rotational Norm)

Every trace bundle, composed of multi-nested irreducible Lyndon structures,admits a unique contraction into a real-valued norm,which inherently carries a discrete rotational phaseand an infinitesimal spiral deviation.

Figure 11. The Noncommutative rotational norm.

$$|x| = (r, \theta) \in \mathbb{R}_{\geq 0} \times S^1, \quad \theta = \frac{2\pi k}{N},\ k \in \mathbb{Z}_N,\ r = r_0 + \varepsilon.$$

Formally, the rotational phase originates from the residual structure of the quasi-dual morphism,while the infinitesimal shift ε represents the non-closed spiral deformation of the trace bundle.

Therefore, despite the trace bundle being non-commutative and infinitely nested,
its contracted norm is uniquely determined—
accompanied by a discrete rotation and a subtle spiral extension.

Probably no one who reads this theorem would doubt
that the “nesting” itself is the “rotation”.

−The Cartesian Spiral…

19. Points to note when reading

Basically, the most commonly used structure is that when “prime Lindon” is restored in a divergent manner, it returns to the “zero point.”

For this reason, the zeta function has two forms of expression: the “Euler product (prime number structure)” and the “zero point product (zero point structure).”

Once this structural transition is recognized, the theory of “irreducible rational Lindon” is occasionally utilized, but the realm of “real Lindon” remains largely unexplored. This is an area where “non-periodic fractals” and “non-separable” properties—that is, “non-commutative reduction methods are the only ones available”—come into play, akin to the “non-commutativity of modularity.”

This involves extremely difficult problems, so even I cannot fully grasp it.

While this theory owes much to the combinatorial groundwork of MacMahon and the decomposition algorithm of Duval, it diverges sharply by constructing a syntactic topological space where Lyndon words act not merely as sequences, but as operators shaping the structure of complex spiral manifolds

The quasi-dual transform $u\hat{}p \to e\hat{}{-s\log p}$ is the Mandala Core’s purest spiral bridge — uniting power structures, logarithmic phases, and the non-commutative analytic continuation of the zeta.

An algebraically closed field of infinite dimension…

黒川信重、絶対数学原論、現代数学社、2016
森田英章、組合せ論的ゼータの半群表示、2016
ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
高安秀樹、フラクタル、朝倉書店、1986
Shinjiro Kurokawa, Absolute Mathematical Theory, Gendai Suugaku Sha, 2016
Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
Bernhard Riemann (translated by Jiro Suzuki), On the Number of Primes Smaller Than a Given Number, 1859
Hideki Takayasu, Fractals, Asakura Shoten, 1986

Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.
This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.

素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of "complex spiral phases"
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a "divergence control function" corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ s という臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.

無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the "elementary Lyndon element" that is restored to zero is decomposed, the corresponding Euler product becomes a "multiple Euler product," giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through "dual non-periodic paths."
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a "double helix" because it naturally contains spiral rotations and has a double main

structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in the number of species. This is particularly important in the context of the formulation of “higher-order imaginary multiplication.”
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by “dual non-periodic paths.” Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.

トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)
素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　 quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する
In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.
全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.

Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy

Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal "bouquet structure" corresponding to the "Euler product." It is also necessary to distinguish it from the commonly referred to "Hodge structure."
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ;　A natural quasi-dual transformation that reduces　"irreducible rational Lyndon"　in trace bundles to　"prime Lyndon"　or　"natural number Lyndon."
Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets ; A complete state that cannot be further expanded by repeating dual operations.
双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.

→ 類双対閉包

非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる