The theory of Quasi-duality map and fractals

(翻訳　類双対性写像とフラクタルの理論)

Part II: The Basic Structure of the Application of Quasi-dual Morphism to Mathematical Objects in the Theory of Dynamic Transformation (Introduction)

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Abstract

This section formalizes the notion of quasi-dual morphisms, a generalization of quasi-duality in trace bundles. The theory focuses on divergent recoverability, where structural loops emerge from non-periodic trace contractions, constructing topological invariants linked to Zeta morphisms.Applying the graph-like Riemannian surface structure of the Lyndon complex spiral phase defined in the previous section, we will examine the basic concepts of fractal geometry.

This paper aims to explain the conceptual structure that was deliberately not explicitly stated in the previous paper on the Riemann zeta function (“Generative Approach to Zeta Structures via Quasi-dual morphism and Graph Zetas: Generative Deformations of the Riemann Zeta Function and the Construction of Their Zero Points via Quasi-dual Divergence Maps”) by applying it to familiar mathematical examples, thereby making it easier to understand.

In other words, this is both an introductory and an applied essay.

Furthermore, it touches on connections with theories that are attracting interest in various fields, as well as unsolved conjectures, so please look forward to it.

As a note, there are parts of this essay where explicit formulas or transformations are not immediately apparent.

However, this reflects the surprising structure that the graph itself actually contains a functional structure, and rather indicates that the "explicit formula" that appears in Ihara's zeta function is a rare exception.

In other words, in this theory, the "structure" already exists before the formula is written, and the analytical expression is "derived" from that structure.

In this paper, we see that the "uniqueness of the natural analytic connection of multiple Riemann surfaces" is constructively derived by the dual motif closure of the trace structure.

This uniqueness theorem suggests the possibility of redefining the geometric meaning of analytic connections in complex function theory from the perspective of graph theory and recursive reconstruction.

In this paper, we introduce trace series and Quasi-dual maps as a framework for generatively describing the divergent structure of the Riemann zeta function, and attempt to provide a generative description of the zero structure.

This involves constructing something that naturally seems "to be the case."

In particular, the "uniqueness theorem for multiple Riemann surfaces" redefines the spatial meaning of analytic continuation in a constructive manner and suggests that the geometric connectivity of critical lines can be reached from within the divergent series.

Those who find the argument difficult to understand should pay attention to the following points.

The main objective is to describe the construction theory of general Riemann surfaces graph-theoretically and to explore their divergent reconstruction and graph motif-based generation principles.

Through this framework, we define a hierarchy of non-periodic generating factors using prime Lyndon sequences and equate the number of generating factors with the genus of the Riemann surface, thereby clarifying the topological structure of multiple Riemann surfaces as graph manifolds analytically connected to the complex plane via the decoded Lyndon language continuous topology.

In this theory, Lyndon sequences play a role in imparting a natural topology to the analytic connection to the complex plane, so reading with this in mind may make it easier to understand.

1. Classification of non-periodic terms, zero points, rational points, real points, complex points, and transcendental points of function values

First, let us assume that there is a graph structure.

We can take a trace of it. The trace may be finite or infinite, but here we will consider only the infinite case.

If the graph forms a loop, the trace path will repeatedly loop around, resulting in an "infinitely repeated segment of the trace" that appears as an infinite repetition segment.

In this case, you can either "restore the loop structure" or "decompose it into a tree structure."

In short, it is simply a transformation that converts a loop structure into a tree structure and restores a tree structure into a loop structure. For example, factorization and infinite series. Integration and differentiation. Loop-type graphs and tree-type graphs. These appear in numerous places, and what connects them is the "trace bundle."

A "trace bundle" is, simply put, the collection of all possible "trace paths" that have been traced.

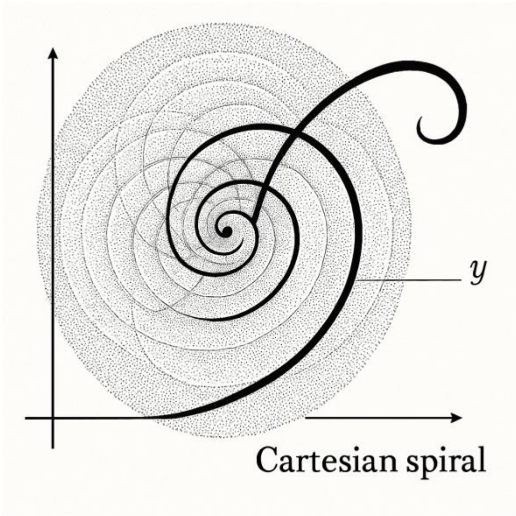
This quasi-dual morphism is fundamentally a non-commutative mapping and is also multivalued. In other words, the process involves first expanding into a "trace bundle" (akin to a quantum state) and then restoring it, during which various "restoration methods" emerge. Among these, the "divergent restoration" and the non-trivial "class dual divergent zeta extension" are particularly important.

For more details, please refer to my essay on the Riemann zeta function.

Here, think of it as "restoring the infinite repetition to a looped tree."

This is the basic principle of the quasi-dual morphism, represented by a diagram of infinite concentric circles and a Cartesian spiral. In this diagram, you can observe that the circles and spirals are divided at their intersection points, with each region corresponding one-to-one. This theory can be considered an extension of Cantor's theory.

Figure 1. The fundamental quasi-dual mapping.



Basic quasi-dual mapping: infinite concentric circles →← Cartesian spiral

Note that the infinite concentric circle structure and the Descartes spiral correspond in a one-to-one manner, ensuring the extendibility of this compression mechanism to arbitrarily high dimensions.

Look at the diagram above. When the concentric circles expand exponentially, the Cartesian spiral also expands exponentially, and each is cut off while maintaining a one-to-one correspondence. At this point, it is important to note that while the infinite concentric circles have a certain mapping, the Cartesian spiral is an infinite-scale or scale-free fractal.

From here, we consider the concept of "non-periodic terms."

Non-periodic terms are those that do not contain repeated elements, meaning they cannot be further compressed. Otherwise, even if we try to "restore the loop," it might not return to its original state because we would be restoring an "extremely long loop." However, even attempts to restore it properly involve multi-valued restorability, which is a very important property.

In my Riemann's zeta theory, this is called "non-periodic term-like quasi-dual divergent reconstruction zeta expansion."

McMahon's theorem states that these "non-cyclic terms" can be further reduced to the state of "prime Lindon (non-cyclic terms)."

In other words, it is a surprising theorem that "all non-cyclic terms can be uniquely decomposed by prime Lindon words."

By this, the restoration from the bouquet graph is naturally expanded to "a graph with a loop structure of the length of all prime numbers," and it can be seen as a "zeta expansion" that contracts to that area. I called this "Euler product hologram fractal restorability."

Now, this may seem abrupt, but it is important to first list the classification of “non-periodic terms” that are restored divergently, as we will use this in the subsequent discussion. However, this definition may seem abrupt. Its meaning is explained in the discussion of “zeta expansion” and will also be explained in the following text, so I recommend comparing the two.

First, the divergent reconstruction of “prime Lindon elements” corresponds to the “zero points” of functions.

This was utilized as the most important fact in the previous discussion on the “Riemann zeta function.”

Next, regarding non-periodic terms, “n-contractible non-periodic terms,” that is, non-periodic terms that can be decomposed into prime Lindon elements, “return to irreducible rational numbers.” This is a remarkable fact. The irreducible iterations within the structure decomposed into “prime Lindon” divide the integers, which are a collection of “prime Lindon” origins with “prime number lengths,” by their “prime” period numbers. As a result, they become rational numbers.

Furthermore, even among non-periodic terms, there are “non-periodic terms with multiple contraction.” For example, it is easy to understand if you imagine a root enclosed by roots. Such non-periodic terms return to real numbers.

To explain this a little further, first, according to McMahon's theorem, a Lindon sequence decomposed into “prime Lindons” is always uniquely reduced to a sequence of “irreducible rational Lindons.” Then, we can consider the “prime” repetition in the same way as we did for rational numbers, and we can take the ‘prime’ repetition of “prime irreducible rational Lindons” and “raise it to a power.” The Lindon sequence that has been “exponentiated” in this way is uniquely contracted to the world of “exponentiated” numbers, and again, according to McMahon's theorem, it is decomposed into “prime root-raised Lindon” numbers, which are multiplied by “root-raised” numbers in the same way, and this continues indefinitely.

What is important in this structure is that it appears to be a nested structure of “fractals with multiple cycles.” Compare this to the “multi-valued nature of roots” or the fact that “the nested structure of multiple roots ($\sqrt{}$) basically does not come apart.” This will clarify the image. It is strange that non-periodic terms are fractal.

Please note that as the scale increases, the details become finer.

And among the non-periodic terms, those that do not reduce to a “prime Lindon element” no matter how they are reduced are transcendental numbers. (I have prepared another essay on transcendental number theory, so please look forward to it.)

From the fact that these are “mixed in the trace sequence of the graph,” we can understand the surprising fact that all graphs are generated by a Cantor-like non-constructive principle that takes a "set of subsets."

You may be thinking, "What on earth is he talking about...?"

So, from here on, I will apply this to examples that everyone is familiar with to form concrete examples.

So, let's start by considering general Nth-degree equations and also consider the structure of Galois theory inherent in the fundamental theorem of algebra.

Figure 2. Recovery of non-periodic terms.

核心定理（生成論）

- **非周期的項の階層**
    - **ゼロ点**：完全縮約された素リンドン列
    - **有理点**：有限階層で縮約可能な非周期列
    - **実数**：多重縮約可能な非周期列（内部に無限擬似周期性）
    - **超越数**：最大限に縮約不能な非周期列
- **命題**：
  「任意の関数値は、非周期的トレース項の構造的復元によって一意に決まる」

ここで、関数の値を決めるのは級数ではなく、
非正則グラフ上の素経路（素リンドン）の階層縮約である

The prime Lyndon component in the divergently restored function evaluates to zero.

Now, those who have read this may be thinking, "Wait, what about complex numbers?"

As mentioned above, there are also "multiple non-periodic terms" in non-periodic terms. The sequence of "non-periodic terms" is further arranged in a "non-periodic" manner, and this is repeated infinitely. In this "nesting of non-periodic terms," the "structure of non-periodic terms" itself may differ.

To put it another way, since the nested structure of "non-periodic terms" itself is a "non-periodic term," this becomes a structure in which "non-periodic terms" are arranged in a semigroup. In other words, this is uniquely decomposed into "prime non-cyclic terms" by McMahon's theorem and contains a higher-order real number structure.

This can be considered to be a complex number. The multiplicity of non-cyclicity in this theory suggests a relationship with complex structures and topological rotational symmetry, and perhaps a relationship with the Nth root of unity, but this will be left as a topic for future consideration.

The concept of non-periodicity described in this section may seem abstract at first glance, but if you find it difficult to understand, please recall the “non-periodic structure” hidden in Euclid's proof of the infinity of prime numbers. “The cycle shifts and cannot return, leading to infiniteness.”

It is this non-periodicity within infinity that supports the core of the most basic image of the non-periodic sequence referred to here.

Finally, I will write down the “conjecture of topological mapping from Lyndon semigroups to the complex plane.”

“Conjecture of continuous spiral complex topological mapping of hierarchical non-periodic nested structures (Lyndon semigroup structure conjecture)”

The combination of prime Lyndon sequences gives a natural number structure, and its repetitive structure produces a rational number reduction structure. The fractal nested structure of this reduction repetition generates an irrational real number structure through root extraction, and when these hierarchical nested structures are further combined non-periodically, the hierarchical order is parameterized by the real number structure within the nested structures as a topological “rotation,” closing in a two-dimensional (complex plane) spiral manner.

Therefore, the infinite hierarchical nesting is bound by the rotation group on the complex plane and ultimately closes as a finite topological space (complex plane)(see Figure 2).

Note that this structural conjecture naturally produces a “spiral overlap of the complex plane” and naturally explains the multi-valued nature of the log function. Even if this mapping is not “unique,” it is clear that a topological structure similar to this structure exists.

To reiterate, the nested structure of the Lyndon semigroup ascends through the hierarchy of integers, rational numbers, and real numbers, but as the nested structure progresses further, it enters a rotational state, and the overlapping of the nested structures ascends this hierarchy again, circling back to the same complex plane.

Thus, the complex plane inherently has a spiral shape, which is believed to be the root cause of the multi-valued nature of the logarithmic function. We will analyze this further later.

Next, we will introduce a continuous phase, but it is unclear whether “this is the correct way to do it.” There may be other ways to introduce a continuous phase, but with this, we can construct a manifold of trace bundles where “continuity” holds.

《Deterministic Lyndon Continuous Complex Spiral Phase》

This is a method of treating the "length" of a Lyndon sequence as a basic topology, and then constructing a nested structure of rational numbers, real numbers, and complex numbers, while treating the "finely chopped" state as a continuous phase. Furthermore, this structure coincides with the spiral multivalued structure of a multiple Riemann surface and behaves as a continuous covering of the complex plane.

The following discussion will focus mainly on this continuous phase, so please be careful not to confuse the two. In fact, I often confuse them myself. It's strange because I don't make mistakes when I use them myself. This "continuous phase" is easy to use, has no discrepancies, and allows for various developments in a very natural way.

Let's call the infinite nested structure of this "Lyndon semigroup" with its natural contraction structure and the continuity of the structure of "unique decomposition by McMahon's theorem" the "Lyndon semigroup nesting theorem." If we think about this structure carefully, it also proves McMahon's theorem.

Lyndon semigroup nesting theorem

By considering "prime" iterations, the Lyndon semigroup generates natural irreducible rational numbers, and through natural reduction to those irreducible rational numbers, it can be further decomposed using McMahon's theorem, and then further refined by root extraction, resulting in an infinite McMahon nested structure. Furthermore, even in higher dimensions, it has another "Lyndon semigroup" called a "non-periodic sequence" of non-periodic sequences, which similarly the "phase of real numbers," that is, an infinite nested structure, which can be used to define rotation in the complex plane.

It is clear that there is a "continuation" to this structural theorem, because even in a semigroup reduced to complex numbers, there is a higher-order Lyndon semigroup called a "non-periodic sequence of non-periodic sequences," which probably corresponds to "quaternions."

Contractible structure of prime Lyndon sequences: natural numbers → rational numbers → real numbers

Rotation projection of non-periodic sequences: complex projection (ℂ) using arg

Non-periodic nesting of non-periodic sequences: multi-axis rotation ⇒ non-commutativity ⇒ quaternion structure (ℍ)

Furthermore, the nested structure of this hierarchical structure: the nesting itself

becomes nested ⇒ the structure diverges
As a result, the structure is

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset ???$$

All I understand is that it gradually breaks down.
What you need to be careful about here is that, in comparison to "it appears to be n-ary numeral system but is not the essence of complex spiral phase structure," the "naturalness of length phase" is demonstrated, so please read that part, or I plan to explain it in the "application section" that I am planning.

"The order of the number system resembles dictionary order in form, but its essence lies in the 'length of the orbit.'"

If you can think of this as "one way of interpreting" Lyndon language, that would be fortunate. If it is different, that is interesting, and even if it is different, you can still pay attention to the fact that the following discussion holds true. I apologize, but I will explain and prove multiple Riemann surfaces later.

Appendix 1: Equivalence classes of irreducible rational numbers of irreducible Lindon origins, rotation theory, p-adic graph manifolds

Here, I will focus on explaining the point I wrote in 1: "How do we know that the non-periodic sequence of N-reductions is rational?"

Let us assume that there are Lindon sequences repeated in the "center nucleus" that is n-contractible Lindon sequence, with lengths A, B, and C, which are prime numbers. Then, the Lindon sequence that is N-reduced must have a length that is a multiple of A, B, and C. Therefore, it is contractible.

As a result, the N-reducible repeated Lindon sequence is reduced to a state where the "number of repetitions" of any Lindon sequence length contained in the 'core' at the center is "coprime." In other words, this is precisely an "irreducible fraction." Interesting, isn't it?

Here, I suggested that the nested repetition of non-periodic sequences is a rotation in complex numbers, but let me explain why it is the Nth root of one.

In the case of a "prime structure of two," it finally rotates to the 'negative' state of the real axis. Then, in the second nesting, it returns to the real axis again. In other words, "real integers" are formed in this state of "two prime structures."

This corresponds to the fact that in the Riemann zeta function, when the "prime structure is infinite," the length of the prime structure is almost a prime number and is

odd, so it does not actually reach the "negative real axis."
In other words, it corresponds to the fact that the complete breakdown of the "inversion symmetry of the Riemann zeta function" occurs only on the negative real axis.
This is the part supplemented by the sine function in Riemann's formula.

In the case of odd prime numbers, no matter how large they become, they never reach the "point on the negative real axis."

This has significantly increased the likelihood that the "rotation level is the angle of the Nth root of the number of non-periodic sequences in the non-periodic sequence structure of the prime structure." The structural discontinuity and divergence of any ζ(s) on the real axis are consistent with the prime factor rotational angle structure of the Lindon series, and since a consistent explanation is difficult with other generative principles, introducing this rotational theory is believed to be the only reasonable reconstruction method.

In other words, it is a construction that explains the asymmetric structure of the Riemann zeta function on the real axis.

For more details, please refer to my essay on the Riemann zeta function.

Below, I will describe a method for decomposing graph manifolds into local fields based on a simple definition of the "Quasi-dual map" for the Lyndon series.

The graph manifold takes on the form of a "graph manifold" by incorporating continuous topology.

We define the "class dual map" as removing only elements of a certain "prime length P" or fixing them and observing the rest, without destroying this structure.

As a result, the "prime Lyndon structure" remains, and at this point, it becomes clear that the "residue field" of the graph manifold with respect to the "prime number P" (or prime structure N), i.e., the "P-adic graph manifold," is constructed. Note that a continuous topology is naturally embedded in this as well.

2. Correspondence between the fundamental theorem of algebra, Galois theory, and trace bundle theory

As described above, the view that the structure of a graph constitutes the "structure of function values" as a trace bundle demonstrates both the completion of functions to real numbers and the uniqueness of analytic continuation. In other words, this view itself represents the meaning of the term "analytic continuation," and its determinant representation by the Iihara zeta function is rather a problem of representation theory that accompanies it. This has become clear, but we will return to it later.

Let us consider a very simple cyclic graph with two loops.

It is sufficient to imagine two circles connected at a single point.

When considering this graph, one might wonder, "Where is the infinite repetition of non-periodic terms, even though there are only loops?" However, if we represent one loop as 0 and the other as 1, we can see that the order in which they rotate can be expressed as an ordered set of 0 and 1.

Interestingly, within this ordered set, there exists a non-periodic set of length N. Moreover, there exists an infinite non-periodic term that does not contain any periodic repetitions. With just two "prime structures," a "real completion" arises. And if the loop is directionless, there are two ways to rotate it, "front and back," meaning there are "two elementary structures." In other words, if there is a circular structure, "there exists an infinite repetition of non-periodic terms of length N, and there exists a graph structure that restores it."

This is a special and non-trivial "quasi-dual mapping" called the "quasi-dual divergent zeta extension," and it generates the "zero point set" that produces the reconstruction.

Thus, a graph with two loops expands continuously through such a divergent quasi-dual mapping, and under regular conditions, it expands into a "circular loop with paths of all prime numbers" (this is described in a discussion of the zeta function), so one might wonder, "Do the solutions to quadratic equations diverge?"

Here, a Galois-theoretic approach is necessary.

In Galois theory, "the solutions are permuted." That is, the elements of the "prime elements (or prime Lindon)" are permuted. At this point, the prime structure or prime Lindon structure is "decomposed" by this permutation.

That is, there is a "trace bundle" of a graph with two loops.

Here, we introduce a groupoid that "permutates the elements of the trace."

As a result, most "prime Lyndon" structures are decomposed, and the expansion stops. And as a result, "the elements are always reduced to two prime Lyndon structures"... This is the structure of the solutions to quadratic equations, and the structure of the "zero points." And at this point of reduction, since the round-trip path between the going and returning paths can be connected, the "symmetry of the zero-point solutions" can be understood.

This "Galois group-like groupoid" expands the diversity of "prime Lindon" structures by a factor of N! as the number of "prime structures" constituting the trace bundle increases. This is the expansiveness of the Galois group. And the fact that these prime Lindon structures can be ordered and reduced by prime numbers is the condition for the "solvability" of equations determined by the Galois group. Thus, it becomes clear that the meaning of “solvability” of an equation is that the prime Lindon structure is destroyed by the operation of “replacing elements” in the trace.

Furthermore, Gauss's fundamental theorem for general Nth-degree equations, known as the “fundamental theorem of algebra,” can now be stated with a clear understanding. That is, by introducing an Nth-order groupoid (an operation that swaps elements) into the “trace bundle” of a graph with N loop structures and observing its divergent restoration, we see that the prime Lyndon structure is “finitely generated” and can be reduced to N prime Lyndon elements. This is the meaning of the fundamental theorem of algebra.

Furthermore, these prime Lyndon elements possess symmetry corresponding to their number, and we can see that they are “conjugate elements.”

The “addition of solutions” of irreducible polynomials in Galois theory can be formulated as a quasi-dual deformation expansion of the “trace bundle,” but this is not the subject of this discussion.

a. The roots of a polynomial are represented as a prime Lindon series.

b. The action of the Galois group = permutation of the prime structure of the trace sequence.

c. The number of roots is finite due to the contractibility of non-periodic sequences.

Appendix 2: Contraction of Weil-type zeta functions to Ramanujan-type functions

After writing this, I suddenly thought, “The action of the Galois group I wrote about here is what is known as the cyclic Frobenius map...”

In other words, “my Galois action, which rearranges the prime elements of the trace and controls them so that their number does not expand excessively,” is completely the Frobenius map(see Figure 3),

Figure 3. The Frobenius mapping.

フロベニウス写像とは何か

有限体上では、フロベニウスは元を冪乗で写す自己同型写像。

その繰り返しが有限体拡大のガロア群を生成する。

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \mathrm{Frob} \rangle.$$

Quasi-dual Frobenius map.

that is, it is the Frobenius map itself.

In other words, by controlling the expansion of the bouquet graph, the deformation ultimately decomposes the non-periodic terms into a "finite" number of components, so the structural problem similar to the Riemann zeta function–that the zeros of the Ihara zeta function with Ramanujan-like properties in regular graphs have zeros on the critical line Re(s) = 1/2 (or in a circular region before transformation)—is resolved.

For those who find this difficult to understand, let me explain it in more detail.

Since the infinite trace bundle (prime Lindon spiral) is essentially divergent, I obtained a dual mapping that "rearranges the elements of the trace bundle to limit the length of the non-periodic terms" in order to consider the Nth-order equation. Ultimately, this is a collection of N! actions representing how to rearrange the N elements, so it is isomorphic to Frobenius, meaning that Frobenius acts as an operator that cyclically bundles them.

Thus, the spiral dynamically expanded by the quasi-dual map is converged into a finite closed orbit structure by Frobenius. The fact that it is isomorphic to the action of the Galois group is likely due to its regularity. The Frobenius is not merely a mapping but acts as a group action that permutes the generating nucleus.

Thus, the closed orbit structure of a finite field = the cyclicity of the Galois group = the Frobenius cycle are all the same.

As a result, the critical line situation and Ramanujan property are naturally derived. Since the divergent zeta extension is reduced to a Ramanujan-type regular graph by Frobenius, the "quasi-dual transformation,"

Figure 4. The Quasi-Dual Morphism
$u^p \mapsto e^{-s \log p}$

reduces it to the case of the finite Riemann zeta, and its spectral structure agrees with the critical line condition of the Riemann zeta.

In summary, "**The Frobenius map is a Galois group action that imposes order on the infinite trace bundle, reduces the divergent zeta structure to a finite Ramanujan-type structure, and is a finite operator that guarantees the alignment of critical lines.**" Thus, the problem of Weil-type zetas can be said to have already been addressed within the scope of the discussion.

This is because it has become clear that the Weil type is a special case that falls under the finite case in the problem of constructing the Riemann zeta function through divergent quasi-dual zeta extensions.

When the order of the cyclic group action by Frobenius is not prime, the prime Lyndon structure is split into partial orbits, and a zero element appears in the trace bundle, destroying the spiral loop. This is the condition for the destruction of the divergent zeta ordering and an example of a geometric explanation of the critical line condition, so I will mention it briefly at the end of the supplement.

3. In the case of non-regularity, the structural theory of trace bundles of elliptic curves and Mordell's theorem

In the case of general Nth-degree equations, divergent quasi-dual transformations from loop structures with N rings in regular ring graphs were obtained by restricting them with Galois group-like groupoids, i.e., operations that swap elements. It should also be noted that this "operation of swapping elements" itself is a "quasi-dual map."

Now, what happens in the case of non-regular elliptic curves?

For example, consider a graph structure with two loops, one of which is connected to the other by a path through the center. This becomes "graphically non-regular" in the connected part. In other words, the structure of the graph's connections is no longer "non-regularity." Conversely, there are 'uniform' parts and "non-uniform" parts.

At first glance, this graph appears to have "three" loops, but in fact, it has an infinite number of loops.

This is the same as the "non-periodicity" problem mentioned earlier.

If we denote which of the two paths of a single disconnected loop to take as "0" or "1," the loop generates "infinite prime paths" due to non-periodic elements.

Here, if we introduce group oids, i.e., operations that swap prime elements, and solve the cubic equation, we will notice that these infinite prime elements are actually "incomplete" due to the graph structure.

In other words, if we call the trace bundle consisting of non-periodic terms that would be formed from an infinite number of prime elements a "complete set," it becomes a set with a density far from "completeness." However, just as there are three solutions to the cubic equation, the prime Lindon decomposition becomes three, and there are "three zero points." Furthermore, since the structure of the "non-periodic terms" has become thinner from the complete state, analyzing the "prime Lindon elements of N-reduction" reveals that they are "completely controlled by a finite number of non-periodic terms," which can be inferred from the fact that the trace bundle is thinner than the "completeness" state.

In other words, when there are infinitely many prime structures, the structure of non-periodic terms in the complete state is demonstrated to be more restricted than that of non-periodic terms generated from elliptic graphs, so it does not become "infinite" but rather "finite and contained."

This is Mordell 's theorem that “rational points are finitely generated.”

This is the meaning of Mordell 's theorem, as demonstrated by the trace bundle's class duality theory.

To elaborate further, in a graph manifold of genus one (to be explained later), there are two generating factors for the non-periodic sequence, i.e., “genus + 1,” so there is a dual path. There are two possible paths depending on which side of the non-periodic sequence you start from, and they are interchangeable. This is the reason why rational Lyndons (rational points) are generated infinitely in “multiple traces” (to be explained later), and at the same time, the principle that restricts their “finite generation” is added here. In other words, it can be said from the analysis of the Lyndon structure that “it is just a repetition of a finite number of points, even though it is infinite.”

Incidentally, this is the reason why the Mordell–Faltings theorem holds in graph manifolds of genus 2 or higher where such “dual paths” do not exist.

Here, I will make a somewhat bold prediction: if we perform the “dual motif closure” of the graph with the structure described above and then perform a divergent quasi-dual expansion, the series will include all graphs of elliptic curves of genus 1.

That is my conjecture.

In this way, the operation of “replacing elements” is ultimately restricted by the limitations within each prime path, so the fact that “transformation to a general equation” and “transformation to a zeta function” are divergent is also a point of interest in “class dual transformation.”

The “quasi-dual closure” created by all these transformation operations is precisely the grand vision I am seeking.

Theorem (Prime Lindon Mordell Theorem)

In a non-regular graph corresponding to an elliptic curve, the rational point structure appearing on the trace path is not included in the complete prime Lindon series, but appears infinitely as a pseudo-periodic recursive sequence, and its entirety is finitely generated by the prime Lindon decomposition system.

Now, let's summarize.

a. Elliptic curves possess infinite non-periodic structures due to the overlap of prime paths.

b. However, within them, reducible parts (N-reduction sequences) yield rational points.

c. This finiteness agrees with Mordell's theorem.

4. Reinterpretation of the error terms in modular theory and inter-universe Teichmüller theory

The meaning of a graph being a function is not merely that an "Ihara zeta function" can be constructed from the graph, but rather that by applying appropriate "quasi-dual transformations" to the graph, structures such as Nth-degree equations also emerge. This general operation can naturally handle "non-regular states," as was demonstrated in the case of elliptic curves earlier.

Here, I will write about what can be understood by reexamining existing concepts from the perspective of "quasi-dual transformations and trace bundles."

First, by viewing the zero point as the reconstruction of a prime Lindon and the rational points as "N-reduced non-periodic terms," we saw that the theory of rational points of a function can also be handled constructively.

Now, the operation of reducing an "N-reduced non-periodic term" to a "prime Lindon" is indeed a "quasi-dual map," but what does this mean?

This is what is known as a modular group.

That is, the reduction method has both infinity and finiteness, as well as symmetry.

This is expressed as a modular group.

In other words, it is a "contraction" mapping from rational non-periodic terms (rational Lyndon) to zero-point non-periodic terms (prime Lyndon). This will be discussed in another essay, so please look forward to it.

This has a finite hierarchy, and for example, it will have a sequence that gradually reduces from non-periodic terms of real numbers to rational non-periodic terms.

Please note that by applying the "divergent quasi-dual mapping" and the "reduction mapping" in this way, the so-called "prime Lindon" structure is defined, and from this, the "uniqueness of prime Lindon decomposition" mentioned in McMahon's theorem (a fundamental theorem in semigroups that states that prime Lindon sequences can be uniquely decomposed) can be derived. This can be described as the "construction of a prime Lindon structure" in terms of quasi-dual ity.

Furthermore, I consider this "divergent quasi-duality" to be the extraction of the "prime Lindon structure" from the normal "infinite prime structure." Conversely,

semigroup structure → prime structure →

we can see a contraction structure "by log scaling."

To add a little more, in my essay,

Figure 5. Continuous deformation by quasi-dual morphism.

素リンドン半群 → log変換 → 素数構造 → さらにlog変換 → 乗数構造

Prime Lyndon semigroup → logarithmic transformation → prime structure → second logarithmic transformation → multiplicative hierarchy

a method for continuously handling this mutual transition appears.

Please refer to my essay on the "Riemann zeta function."

I constructed a generative "Hilbert-Polya" operator as a continuous mapping representation in such a quasi-duality mapping.

This means that I was able to give a continuous structure that fits this unclear discrete operation.

This immediately brought to mind the following.

I heard that in Shinichi Mochizuki's inter-universe Teichmüller theory, the evaluation of error terms becomes an issue in the loglogΘ structure. I think this is consistent with the problem of quasi-duality maps.

In other words, when considering the quasi-dual mapping, the "zero point set (prime Lindon elements)" and "prime number structure" naturally emerge as structural transformations of semigroups and commutative groups. This is a structure where the same trace bundle seems to encompass two universes, and I suspect that the quasi-dual modular mapping I previously examined also possesses a similar structure.

In other words, the concept of inter-universe Teichmüller theory has given rise to the possibility of reinterpreting it through "quasi-duality mapping theory and trace bundle theory."

For details, please refer to my essay on the Riemann zeta function, which explains the construction method.

The following is a summary.

Modular group = symmetry group of contraction operations

Modularity implies the covariant nature of finite-level contractions.

IUT error term = residual term of an incompletely constructed contraction sequence

Inter-universe comparison refers to the control of structural discrepancies via quasi-duality mappings.

5. Fractal Geometry Hypothesis

In this section, we will discuss the question, "If fractal structures are latent in nature and mathematical phenomena, what are the basic forms (points, lines, spirals, circles, waves, etc.) that constitute their smallest units?"

The reason for this is that while fractal structures possess infinite nested hierarchies, identifying the smallest “pattern elements” that constitute those hierarchies is not only critically important for understanding the nature of nesting and scalability but also aids in comprehending natural and mathematical structures in general.

In my discussion on the Riemann zeta function, I wrote about the “difficulty of spiral reconstruction.”

Specifically, the question is whether there is a distinction between walking through a gentle spiral structure and walking along a straight line for someone inside it. This will likely pose a significant problem when that “person inside” attempts to “reconstruct” the structure based on their experience.

After considering various possibilities, I first realized that a spiral is composed of multiple concentric circles. A circle is dual to a point, so it also consists of multiple points.

As I continued to think about this, a graphical reconstruction began to emerge.
In other words, it resembles a circular graph.

However, a single node (point) is replaced by a chain (circle), and through this, the memory of having passed through that “circle” is sequentially recorded in the trace.

From this, I realized that a “circle” is a graph element that is neither a ‘point’ nor a “line.” In fact, in the bouquet graph, the circular structure was depicted as a “structure without a forward or reverse path.”

Now, how many types of dual elements determine the graph structure (i.e., the function structure)? This is the core of my fractal geometric conjecture.

《Fractal Geometric Conjecture》

The constituent elements of any structure capable of dual transformation are closed by points, lines, circles, spirals, and waves.

Non-periodic structures centered on prime Lyndon series are actually completely generated by these basic geometric forms.

In other words, as the ultimate geometric “basis” of function structures, even infinite non-periodic sequences ultimately
points (zero-point structures)
lines (prime paths, trace bundles)
circles (closed cycles, ring graphs “as well as” chains)
spirals (divergent connections via log scaling, possessing circular points)
waves (overlapping pseudo-periodicity through infinite hierarchical repetition, capable

of containing infinite point sequences)

These five elements. Is this how fractal structures are constructed?

The spiral-shaped graph model constructed using "point, line, circle" is one "dual form" of the 'spiral' structure, and it makes more sense to consider the "spiral" as a separate entity.

Based on this theory, any function structure can be expressed as a finite geometric element of "point, line, circle, spiral, and wave" by closing the divergent structure of the prime Lindon sequence contained in the trace bundle via log scaling (a single quasi-dual mapping). This means that any infinite structure in non-regular regions can also be described as finitely generated in a constructive manner.

In my theory, in the previous discussion, the quasi-dual map that converts "non-commutative structures" into "commutative structures" proceeded in the direction of "infinite concentric circles" → "spiral-type expansion" → "projection onto a straight line."

It can be said that "when the order is broken, a linear projection occurs."

This reminds me of the "reflection formula" of the gamma function.

"Waves" may be difficult to understand, but I once worked on creating a puzzle-like "N·N" regular graph as part of a theory of dynamic transformation. At that time, I clearly remember that edges often emerged that picked up points on a straight line in a wave-like manner in order to maintain fractal properties.

Additionally, in my discussion on the "Riemann zeta function," I analyzed the approach using the "wave term" in Riemann-Siegel.

If approximation using infinite geometric series is considered a "linear approximation" as a method for approximating infinite bases, then Fourier approximation can be described as a "wave-type approximation."

Setting that aside, let us return to the structural meaning of fractal geometric conjectures.

Let us examine this through two examples.

Many high school students may have felt a strange sensation when integrating y=1/x. This is actually due to the multi-valued nature of the logarithmic function, which we will study later.

Looking at y=1/x, at the point 0, it diverges into a circular pattern, with a spiral winding inside the circle, and this winding pattern appears as multi-valuedness within the integral. First, the "indefinite constant" in the integral is also a strange concept. "What is this divergence?"

However, at this point, the "zero point" has already appeared. As mentioned earlier in the “spiral restoration,” there is a “circle” structure within the circular graph, and this structure lifts the trace structure within the circle into a spiral shape while winding it up. This structure is already present within the integral of the logarithm.

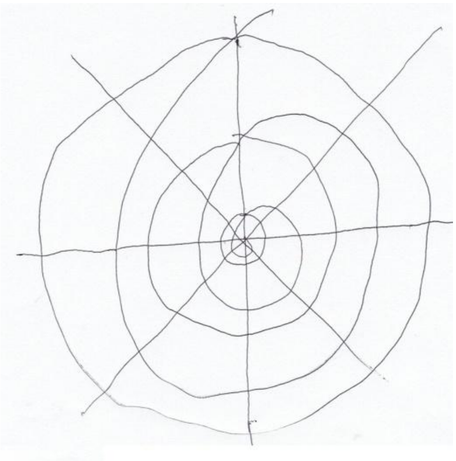
By interpreting the multi-valued nature of the logarithm from the perspective of differential calculus, the multi-valued nature of the integral can be interpreted as a spiral-shaped manifestation. This is also evident from the fact that the zero-point-like, concentric circular structures of the zeta function are unfolded into a spiral shape, overlapping with the graphical representation of the spiral-shaped structure... That is the idea.

Similarly, I used a method where Ihara’s “non-commutativity to commutativity” Quasi-dual mapping from a state where primitive p-th root of unity on the unit circle are embedded in infinitely overlapping unit circles, unfolded each one spirally on a logarithmic scale, and projected them all at once onto a straight line, the so-called “critical line.” What is striking here is that everything except the wave is acting.

That is why I connect the Riemann surface graph of the log function in a circular pattern with a single “zero” (circle).

The pole of the Riemann zeta function at X=1 clearly acts as a point that wraps up infinite concentric circles and infinite spirals. Therefore, only here does “symmetry breaking” occur.

Figure 6. Overlapping fractal structures.



Fractal structures can be stacked! Waves can also be stacked!

The figure above combines points, lines, circles, and spirals. For example, when a spiral is made wavy, waves are introduced, but the “fractal nature” is maintained. In other words, it is quasi-dualit.

Fractal nature is rare, but here are some examples that can be unified through operations involving "lines" and "circles":

Figure-eight: A one-dimensional structure created by crossing lines, connected in a curved manner.

Möbius strip: A single line segment (strip) twisted into a ring.

Torus: A circle (one-dimensional circumference) rotated into a ring.

Klein bottle: Created by attaching a ring along a circle again. ...

It can be decomposed into basic elements.

6. Graphical Riemann surfaces

We have explained that "a graph is a function," and we have shown examples of how to functionize it, such as the example of the Iihara zeta function and the method I described in this essay using the "Galois group" (a mapping that swaps elements within the trace).

In other words, I see the "structure of a graph-like Riemann surface" emerging at this point.

At this point, the regular structure is almost entirely composed of "chains of rings," and while the structure is uniform, when it becomes an elliptic curve, the graph structure suddenly undergoes a drastic change, and even without restoring the "prime Lindon" using the divergent quasi-dual map, the "prime structure" itself diverges to infinity.

Non-regularity can be described as the infinite branching structure of the "prime structure" resulting from non-uniform elements infiltrating the finite graph.

However, I still cannot treat this "graphical Riemann surface" as something that is generated theoretically, but rather as something that I can intuitively grasp from the beginning.

I do not understand the generative principle whereby, given a function, a Riemann surface gradually emerges generatively from it, fits into a graph-type structure, and represents all "regular structures" and "irregular structures."

Let us call the object created by the restoration of the divergent "Lyndon element" quasi-dual map a "graph manifold." This manifold shows the "values" of all functions. In this case, in non-regular situations, the graph type and "trace bundle" structure change, meaning that structural bifurcations are expected to occur within the graph manifold.

Is it possible to construct something like this using a finite set of closed graph structures? This is the problem of graph-like Riemann surfaces.

Here, the "motif closure of a graph" is likely to play an important role in the

formulation, and in particular, the theory of the "motif closure" of directed graphs will be important. This is because trace bundles have a structure that is almost ordered.
Furthermore, let us state an important conjecture, or rather, a conjecture that can be partially confirmed.

I have stated that "a graph is a function." I have also described how to obtain "values." However, some may wonder, "Isn't a function supposed to map one value to another?"
The following "Uniqueness Theorem for Multiple Riemann Surfaces" describes this mechanism.

Theorem

Any multiple Riemann surface is uniquely determined by the dual motif closure of the corresponding trace sequence.

Let me explain its content.

In graph structures, there is a dual operation called "swapping points and lines." Under 'regular' conditions, this dual structure is unique. That is, the "function values" of a graph correspond to the "function values" of its dual graph.

However, in the case of non-regular graphs, the graph structure becomes multi-valued, and the "values" trace synchronously, resulting in "multiple traces." This is the "multi-valuedness" of functions.

In other words, the "trace" bundle structure transformed by the dual operation determines the correspondence of functions.

This is an analytic function.

As mentioned earlier, in the case of non-regular graphs, this "dual operation" exhibits multi-valuedness, forming a "category of graphs organized into sets" called the "dual motif closure." In other words, "function values contain multi-valuedness and multiplicity."

Consider the case of Log. If the graph of Log is a circle, the dual operation does not change it. A circle is the most beautiful regular graph. Therefore, the symmetry of the "circle" must be broken, and this is the principle of the multivalued nature of Log, which "diverges without eyes" through the dual operation.

This demonstrates that the analytic function is "naturally and uniquely connected within the multiple graph surfaces." What a beautiful theorem! A multiple Riemann surface, which appears to branch infinitely, can be completely described by a single generating principle. There is no need to "patch together" from the outside. This is because the trace sequences within the graph function surfaces, i.e., multiple graph manifolds, are connected while synchronized.

This synchronized structure is already inherent within the trace structure.

A simple proof:

1. The values of each graph are continuously embedded within the divergent restoration.
2. The trace bundles of each graph are completely synchronized within the motif closure.
3. Therefore, the movement of the function uniquely determines the internal structure of the Riemann surface through these two factors.

Now, in the extension of the "complex spiral continuous topology" of the Lyndon semigroup, I discovered that the "higher Lyndon term images" further nest the Lyndon semigroup. This reveals an even more astonishing fact.

Theorem (Unique Continuous Analytic Connection Theorem for Non-commutative Riemann Surfaces)

In any directed graph with a finite loop structure (prime Lyndon generators), when it has a non-periodic trace sequence, the corresponding contraction map series is naturally mapped uniquely to a topological rotational structure (spiral) on the complex plane through the hierarchical structure of the Lyndon semigroup.

Therefore, a complex projection system based on an arbitrary non-periodic Lyndon structure uniquely generates a continuous, one-to-one, rotational correspondence structure (i.e., a non-commutative Riemann surface) with a dual plot system.

This uniqueness extends to quaternionic structures, hexadecimal structures, and beyond...

That concludes my presentation.

7. Proposal of genus and generating points based on graph Riemann surfaces (simple comparison with Riemann surfaces)

From the above considerations, it is clear that the problem of Riemann surfaces of Nth-degree equations can be summarized in the form of a graph with N rings.

Here, since these rings can be attached when there are "outward and return paths," the following ring function theorem and generating point theorem hold.

Theorem: Structure of Nth-degree equations and generating points

Let us describe a very simple application of the above considerations.

The trace graph of an Nth-degree equation is a structural ring with N "origins," so whether or not a loop can be formed within it, including self-intersections and folds, determines the "circular function." That is, when N is 2, the forward and return paths overlap once; when N is 3, the forward and return paths overlap twice; when N is 4, there are three ways in which they overlap, but it is the same as when N is 2; when N is 5 or 6, it is the same as when N is 4; that is, when N is even, it is N/2, and when N is odd, it is N/2 + 1/2.

Furthermore, an Nth-degree equation has N "generating points" from which all points are generated, and in the case of a circle, these are repeated roots, and in the case of an ellipse, they are foci.

This can be understood solely from the structure of the Riemann surface graph, and it is very simple, but it clearly does not match the existing definition of "genus." Although the shapes are similar.

In the case of the fourth degree, it is possible that all the rings overlap, but in this case, it is considered to be in a "multiple trace state." Typically, the Riemann surface corresponding to a fourth-degree equation can be interpreted as a covering state with four branch points and a forward and backward path.

Here, we consider that the genus of a Riemann surface is the number of "holes."

Is there a concept that aligns with this?

For example, when considering the case of an elliptic curve of genus 1, a structure emerged where the "non-periodicity of prime paths" branches into 0 and 1. This has a lower density than infinite branching.

It may be possible to count the number of branches of prime paths. This could be read as the genus in a graph manifold, representing the degree of infinite divergence of prime paths due to non-periodic terms of prime paths.

Theorem (Structural Definition of Graphical Genus)

In a graphical Riemann surface, when there are k independent non-periodic prime Lindon terms that are not mutually reducible, the graphical degree of freedom (≒ structural genus) of this space is at least k.

This is related to the concentration of divergent non-periodic terms, and as the number of non-periodic prime Lindon terms increases, the concentration increases, potentially providing an upper bound on the usual topological genus.

In the case of the log function, due to the spiral restoration, it becomes an "∞-multiple trace," which is the same situation as being ∞-valued. However, the graph of the log function is spiral-shaped, and while it may have a multiple trace structure, it lacks non-periodicity. The genus is 0.

Additionally, as can be seen from the non-periodic prime Lyndon structure,

the nested structure of the log function and the "nested disappearance" theorem

The non-periodic solutions of the log trace branch into multiple stages, with the trace recursively nested. As mentioned earlier, due to the spiral shape of the graph structure, it separates into a "multiple trace space" and generates multi-valuedness.

However, when the discriminant corresponds to a quadratic equation with a positive discriminant, the trace is simplified in a single stroke, meaning that only real solutions exist, so there is no nested structure in the non-periodic Lyndon series.

The structural necessity of such a structure within the solution can be understood from the theory of Riemann surface graphs, but it is not so clear from other perspectives.

From the structure of non-periodic Lyndon semigroups, the following can also be said. For more details, please refer to the paper on "zeta."

Transcendence of zeta zeros and odd values (conjecture)

The zeros and odd values of zeta cannot be solutions to any algebraic equation.

This can be said because it is impossible for non-periodic Lyndon series to converge to some nested structure. Indeed, it can be confirmed that this includes existing results.
I think that by writing this far, it should be clear that "Riemann surfaces are graphs and can be classified by graph structure," but what do you think? The following can be expected.

Proposition (Classification of Riemann surfaces by graphs)

Any Riemann surface uniquely corresponds to a trace graph based on the contraction and nested structure of a prime Lindon series, and the properties of the Riemann surface (covering structure, zero point distribution, transcendence, analytic connectivity) are completely classified by the structural characteristics of the graph (graphic genus, self-intersection, periodicity, nested depth).

As graph manifolds, multiple Riemann surfaces that ensure continuity and uniqueness provide a very simple view, and it becomes clear that when they are regular, they follow a very simple pattern. How does the classification based on infinite divergence due to "non-periodicity" in graph manifolds differ from the classification based on the genus of Riemann surfaces? I still don't understand this.

Graph (its motif closure) → trace bundle → divergent quasi-dual map → Riemann surface → function

This flow is easy to understand.

With this idea, when looking at the divergent quasi-dual reconstruction of a graph, it is possible to transform a genus 0 quadratic curve into a quartic curve without changing the structure of the trace bundle. In other words, the degree of non-periodic divergence supports the diversity of the trace bundle itself. At that point, the prime structure clearly becomes redundant (it becomes a reducible non-periodic reconstruction). It becomes possible to view such "functions on graph-like Riemann surfaces" as reconstructions that do not change the trace bundle.

Appendix 3: Correspondence between graph manifolds and general Riemann surfaces and genus calculation

Think of the shape of a regular Riemann surface doughnut.

The doughnut has as many holes as there are genus numbers.

When considering the paths, if we express which path to take as "0, 1," we can create a non-periodic sequence, which corresponds to the structure of a Riemann surface of genus one.

In other words, if there is a Riemann surface shaped like a double doughnut, the concept of genus aligns with the branching number that generates the non-periodic sequence.

This shows that the "branching number of non-periodic sequences" of graph-like Riemann surfaces and the genus of general Riemann surfaces are consistent, demonstrating the same thing. At the same time, this creates "genus equivalence in graph structures" and reveals that the structural information of the structure that creates "non-periodic branching" in graph manifolds is lost in the genus information.

This means that a theory that describes the "way loops overlap" is necessary.

First, let us consider a doughnut-shaped graph with N holes.

This corresponds to a graph obtained by cutting a circle with N-1 lines.

In this case, it is obvious that the genus is N.

Now, regarding the structural conjecture, if we do not consider that any Riemann surface graph structure of genus N can be constructed by applying motif deformation repeatedly and repeatedly applying "quasi-dual divergent graph deformation" to this regular but not "N·N" regular graph structure, then what is the purpose of the genus information? Therefore, the conjecture is:

《Conjecture on the genus calculation method for general Riemann surfaces》

If we take the dual motif closure of a doughnut-shaped graph with N holes and repeatedly apply divergent Quasi-dual morphism, the structure of any graph of genus N can be generated and holographically reconstructed.

If this does not hold, it is certain that information other than genus will become important. At the same time, this marks the transition from the theory of "graph manifolds" to the theory of "general Riemann surfaces."

One theoretical point worth noting here is that when attempting to construct a model of an "N· N" regular graph, it is often necessary to use a "a compact Riemann surface of genus n," which frequently forms a nested system. In extreme cases, this can even result in a fractal system of "genus doughnut graphs."

This is an intriguing phenomenon.

I imagine that Ramanujan, for example, recognized such nested structures and formulated equations based on them. This is fascinating, and I have analyzed such situations by constructing a "multiple matrix ring" and discussed the theory of its "infinite nested expansion" in my essay on the "Riemann zeta function," so please refer to it for further details.

The most straightforward "N· N" regular graph is, I believe, the "3· 3" regular infinite graph formed by crossing three lines in a "honeycomb structure" (the shape of a beehive). I think it would be best to use that as a reference for creation. The structure that naturally "reconstructs" itself through various transformations is truly remarkable.

8. Analytic Connection Theory of F1 Geometric Multiplication Functions

In my discussion of the Riemann zeta function, I constructed the "overlap number" b(n) while examining the gamma factor. At that time, I realized that by infinitely summing the Nth power sum of the Riemann zeta function, I could also set the coefficient field F1 geometrically, and constructively analyze the multiplicative function b(n) using the zeta function.

In other words,

Figure 7. The F_1–geometric multiplication Function(see Figure 7)

$$\zeta(s)+\zeta(s)^2+\zeta(s)^3+\cdots=\left(\sum_{k=1}^{\infty}\zeta(s)^k\right)=\frac{\zeta(s)}{1-\zeta(s)}$$

this is the "generating function" of the function(see Figure 7). It is clear that it is naturally analytically connected by the zeta function. Note that it diverges in general.

The conclusion that can be imagined from this is quite simple.

Conjecture

All multiplicative functions can be analytically connected by the zeta function in F_1 geometry.

If we express this in a more general form, it might look like this.

Figure 8. The F_1-Geometric form of multiplication function.

F_1ゼータ解析接続命題

任意の乗法関数 $f(n)$ に対して、対応する構成論的係数列 $a_k \in \mathbb{Z}_{F_1}$ が存在して、

$$f(n)=\sum_{k=1}^{\infty} a_k \cdot \zeta(k)^n \quad \text{（解析接続として解釈される）}$$

またはその生成関数として：

$$F(s)=\sum_{n=1}^{\infty} f(n)s^n=\sum_{k=1}^{\infty}\frac{a_k\cdot s\cdot\zeta(k)}{1-s\cdot\zeta(k)}$$

ここで $a_k \in \mathbb{Z}_{F_1}$ は、**素リンドン系列**、**トレース束**、**類双対変換**などの構成論的幾何から自然に生まれる整数

Consider multiplicative functions in F1 coefficient fields.

Furthermore, this problem is known to have significant implications for the functionalization of quasi-duality maps.

Regarding this meaning, as I have written in my essay on the "Riemann zeta function," I will not repeat it here, but I will introduce it briefly.

9. Summary

This paper aims to clarify the roles of quasi-duality maps and non-periodic structures, which have been treated vaguely in the theory of dynamic transformation, using as simple mathematical examples as possible, and to serve as a foundational guide connecting to zeta structure theory, elliptic geometry, and the Riemann zeta function. This theory still has many unresolved aspects, and this short paper does not provide a complete proof. Nevertheless, if the perspective presented here is taken up by someone else and leads to the development of a new fractal theory or a deeper understanding of the mathematical structures pioneered by Professor Kurokawa and Professor Morita, I could ask for nothing more as the author.

Though much remains unorganized, I hope that this perspective will one day be developed by someone else and take shape as a modest expression of gratitude to the professors.

At the root of this theory lies my own initial observation that the wave noise from streetlights exhibits the same fractal scaling structure as the sun and moon. In other words, “within the fractal, the distance space is nullified.” I hope that this theory, born from that observation, will eventually lead to a “foundational theory of noise phenomena.”

黒川信重、絶対数学原論、現代数学社、2016
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ベルンハルト・リーマン（鈴木治郎訳）、与えられた数より小さな素数の個数について、1859
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Shinjiro Kurokawa, Absolute Mathematical Theory, Gendai Suugaku Sha, 2016
Hideaki Morita, Semigroup Representation of Combinatorial Zetas, 2016
Bernhard Riemann (translated by Jiro Suzuki), On the Number of Primes Smaller Than a Given Number, 1859
Hideki Takayasu, Fractals, Asakura Shoten, 1986

Glossary(用語集)

1, Related to the Lyndon series(リンドン列関連)

Aperiodic sequence:A sequence with an order that does not contain periodic elements throughout the sequence.
非周期列：内部に周期的な要素を含まない順序を持つ列。

Contraction: The unique decomposition and reduction of non-periodic Lyndon sequences into a minimal trace structure.　.

This refers to the transformation of infinite repetitions of non-trivial aperiodic sequences within trace bundles into loop-type or tree-type structures.
縮約　非周期的リンドン列を最小のトレース構造に分解し、簡約化する独自の過程。
これは、トレース束内の非自明な非周期的列の無限反復を、ループ型またはツリー型構造に変換するプロセスを指す。

Contraction morphism: An operation that performs structural deformation on a trace sequence, trace bundle, or structure in a class-dual manner while preserving fractality.
縮約写像　あるトレース列やトレース束、または構造体を類双対的に、フラクタル性を保ちつつ、構造的変形を行う縮約の操作

Dual Lyndon words ;Corresponding to the reverse order of Lyndon sequences, Lyndon sequence decomposition structures contribute to the stability of the existence of inverses in graph-like Riemann surfaces.
双対リンドン列リ ンドン列の逆順に対応する、リンドン列分解構造、グラフ的リーマン面では逆元の存在の安定性に寄与する。あるリンドン列に対応するグラフの双対構造
→Part II, Part III

Lyndon series reduction; Trace contraction of a non-periodic Lyndon sequence. Note that there are two types of Lyndon series Contraction.
リンドン系列の縮約　非周期的なリンドン列のトレース縮約。注意：リンドン系列の縮約には 2 種類あります。

prime Lyndon word:Shorthand for the smallest unit of a non-periodic sequence. It is uniquely determined by McMahon's theorem and Duval decomposition algorithm.
素リンドン語非周期列の最小の単位。マクマホンの定理や Duval 分解アルゴリズムによって一意的に定まる

Prime Lyndon sequence: An indivisible non-periodic sequence serving as the fundamental unit of contraction.
素リンドン：収縮の基本単位として機能する、分割不能で非周期的な列。単純に、「既約」ではなく、最小単位。

2, Quasi-dual morphism and Zeta(類双対写像とゼータ)

Complex spiral integration: Terms referring to the differential and integral structures of "complex spiral phases"
Although it is not yet clear, it is gradually becoming apparent that as the number of species increases, there is a "divergence control function" corresponding to complex spiral phases, and that there are conversions to higher-order structures and lower-order structures corresponding to this.
複素位相積分　「複素螺旋位相」の微分・積分構造に言及する語
まだ明らかにはなっていないが、種数が増えていくたびに、複素螺旋位相に対応する、「発散制御機能」があり、それに対応して、高次構造への変換や低次構造への変換が存在していることが次第に明らかになっている

Critical line symmetry: Symmetry on the critical line s→←1-s, mainly seen in the Riemann zeta function.
臨界線上の対称性
主にリーマンゼータに見られる s→←１－ｓという臨界線上の対称性

Ideal class motif : The ideal concept also undergoes a process of restoring higher-order structures by first extending a single structure to infinity and then contracting it. This is structurally similar to the graph-theoretic dual motif closure and the structure of class-dual divergent restoration in my theory.
イデアル概念も一旦単一的な構成を無限性へと引き伸ばしてから、縮約するという過程を伴って、高次構造を復元する過程をとる。これは、グラフ論的双対モチーフ閉包と、あるいは、僕の理論における類双対的発散的復元の構造と構造的に類似している。このことから、「一般非可換イデアル論」などの構成が示唆されている。

Infinite compression operator:This refers to the Möbius compression structure, which is an abstract description of the integral kernel that includes rotation, inversion, and spiral convergence. It has a mechanism that controls the divergence of the zeta structure of genus 0 in a spiral rotation, and arranges the structure symmetrically along the critical line of the Riemann zeta function.
無限圧縮作用素　Möbius 的圧縮構造のことで、回転・反転・スパイラル的収束を含む積分核の抽象記述。種数０のゼータ構造の発散を螺旋回転的に制御する仕組みを持っており、リーマンゼータの臨界線に沿って、左右対称に構造を鏡像的に配置する

Multiplicity of zero: When the "elementary Lyndon element" that is restored to zero is decomposed, the corresponding Euler product becomes a "multiple Euler product,"

giving zero points multiple values.
多重零点 ゼロ点へと復元される「素リンドン元」が分解されるときに、それに対応するオイラー積は、「多重オイラー積」になって、ゼロ点にも多重性を与える。

The basic quasi-dual mapping: One-to-one correspondence between infinite concentric circle fractals and Cartesian spirals. Pure transitions between loop shapes and tree shapes can be seen naturally.
基本類双対写像　無限同心円フラクタルとデカルト螺旋との一対一対応。ループ形とツリー形の純粋な移行が自然に見られる

Divergent-density completion:Denotes the state where an infinite set of prime-like structural elements achieves a density such that further divergent reconstructions cause no structural deformation.
発散密度完備 無限の素数類似構造要素の集合が、さらに発散する再構成が構造的変形を引き起こさないような密度を達成した状態を指す。

Divergent restoration: The operation of recovering a potentially infinite structure from contractions by non-closed quasi-dual e morphisms.
発散的復元：非自明な非周期列を復元する類双対写像を用いて、収縮的縮約から潜在的に無限の構造を回復する操作。

Effect of imaginary number multiplication:Imaginary multiplication realized through motif-aligned rotations. In this theory, the divergent structure of Euler products is controlled through "dual non-periodic paths."
虚数乗法の作用　この理論では「双対非周期的経路」を通じてのオイラー積の発散的構造を制御する構造

Lyndon complex spiral continuous phase:A continuous complex phase that is uniquely determined for a Lyndon sequence, which is a semigroup. It is sometimes referred to as a "double helix" because it naturally contains spiral rotations and has a double main structure.
半群であるリンドン列に対して、一意的に定まる連続複素位相。自然に螺旋形の回転を含んでいるところ、二重の縮約的構造を持っているところなどから、「二重螺旋」と表現することもある。

Genus expansion: An expression for structural development accompanied by changes in

the number of species. This is particularly important in the context of the formulation of “higher-order imaginary multiplication.”
In other words, it can be understood that the Hecke operator of higher-order zeta functions acts as an operator that changes the structure of graph-like Riemann surfaces, allowing for the interpretation that this is a comprehensive integral of Riemann surfaces.
種数の拡張　種数の変化を伴う構造展開に対する表現。とくに「高次虚数乗法」の定式化文脈で重要。
つまり、グラフ的リーマン面の構造を高次元に変化させる作用素として、高次ゼータのヘッケ作用素が作用していることが分かるために、これはリーマン面の包括的積分である、という解釈を許す

Non-regular zeta structure：An extension of the zeta function with genus and loop structure. It naturally appears when constructing the quadratic zeta function in Dedekind's zeta function. The zero points probably extend beyond the critical line, and their Euler product divergence is prevented by “dual non-periodic paths.” Higher orders are also possible.
非正則ゼータ構造　種数・ループ構造をもつゼータ関数の拡張。デデキントのゼータで、二次のゼータを構成する時に自然に出てくる。ゼロ点はおそらく臨界線上からはみ出し、「双対非周期経路」によって、そのオイラー積の発散が防がれている。より、高次化も可能。

Spiral development:Spiral expansion representing recursive quasi-dual morphism. Used when bundling the infinite concentric circle structure of the Zeta function into a spiral shape and projecting it linearly.
螺旋的展開 再帰的類双対写像を表現する螺旋展開。ゼータ関数の無限同心円構造を螺旋形に束ねて、直線的に射影するときに使われる

Trace bundle: The structure generated by repeated contractions and expansions of Lyndon sequences.
トレース束　構造体の全経路を集約した構造。それぞれのトレースは、リンドン列と一意対応。
→全体（特に Part I, III）

Primitive p-th root of unity:Primitive p-th root of on the unit circle (associated with a prime p)

素数 p に対応する単位円状の一乗根　p は素数。「素数に対応する無限同心円の上に対応する単位乗根」という意味

Quasi-dual morphism: A mapping that transforms fractals into fractals, transforming trace bundles into either loop-type or tree-type structures.A morphism that resembles duality but inherently resists full closure.　　quasi-dual　quasi-dual morphism
フラクタルをフラクタルへと変形する写像、トレース束をループ型のほうか、ツリー型のほうへと変形する
In this theory, we define quasi-dual operations as dual-like transformations that lack formal duality properties such as closure or invertibility, yet govern recursive, non-commutative constructions within trace structures.
全体（とくに Part II）

Recursive quasi-duality: A structure that repeatedly performs class dual operations. A concept connected to the category zeta structure in particular.
When repeating class dual transformations, it is necessary to determine whether the structure is invariant or not, while noting that it is non-commutative and multivalued, in order to find the restorability of a specific structure.
類双対操作を反復的に繰り返す構造。特に圏的ゼータ構造に接続する概念。類双対変形を繰り返すときそれが非可換であり、多値であることに注意しつつ、構造の不変性を変えているのか、変えていないのかを見ながら、特定の構造への復元性を見つけないといけない。

u^p→e^-slogp; One of the quasi-dual maps, often used in deformations such as the Ihara zeta function.
u^p→e^-slogp; 類双対写像の一つで、伊原ゼータ関数などの変形においてよく用いられる。

Zeta deformation process:When fractally deforming the zeta function, there is always “multivalueness,” so it is necessary to find an appropriate deformation method that corresponds to such “diverse deformation possibilities.” For this reason, I am attempting four types of deformation methods in my essay.
Just pay attention to scaling and discrete/continuous properties.
ゼータ変形　ゼータ関数をフラクタルや類双対写像で変形するプロセス。ゼータ関数をフラクタル的に変形するときには、必ず「多値性」があるので、そのような「多様な変形可能性」に応じて、適切な変形方法を探らないといけない。そのため、僕は論考の中で４種類の変形方法を試みている。スケーリングや離散・連続性に注意すればいい。

3, Fractal restoration theory(フラクタル復元理論)

Fractal reconstruction；Mainly by continuously applying divergent quasi-dual mappings, the internal completeness of the structure is constructed. If there are two prime structures, for example, one Euler product, then naturally all Euler products across all prime numbers can be restored.
The prime Lyndon elements contain all natural numbers,but the prime path lengths in the bouquet graph lack ordering,and this absence leads to a contraction to the prime number structure,corresponding to the Euler product.
フラクタル復元　部分構造から全体を生成する写像操作。主に発散的類双対写像の連続適用によって、構造体の内部的な完備性を構成する。素構造が 2 つあれば、たとえば、ひとつのオイラー積などは自然にすべての素数に渡るオイラー積が復元可能
「オイラー積に対応する」伊原ゼータの花束グラフを復元するときに、「素リンドン元にはすべての自然数が含まれる」けど、「素経路の長さ」には順序性がないから、「素数」へと縮約される,という「非可換」→「可換」という変換に注意。
→Part I, Part IV

4. Structures, graphs, and Riemann surfaces(構造体・グラフ・リーマン面)

“Bouquet graph”：A wedge sum of n circles, i.e., a single vertex with multiple attached loops. This structure serves as the minimal model for the trace contraction in the graphical Riemann surface.
花束グラフ　n 個の円からなるウェッジ和を指し、すなわち、複数のループが接続された単一の頂点からなる構造。この構造は、グラフ的リーマン面におけるトレース収縮の最小モデルとして機能します。

Deligne's condition:Unlike general Deligne cohomology, here we refer to the divergence control structure resulting from the combination of dual non-periodic paths and imaginary multiplication circuits as the Deligne structure. Structures that satisfy Ramanujan's inequality
ドリーニュの構造　一般のドリーニュコホモロジーの意味とは異なり、ここでは双対非周期的経路と虚数乗算回路の組み合わせから生じる発散制御構造をドリーニュ構造と呼ぶ。ラマヌジャンの不等式を満たす構造のこと。

Dual non-periodic paths :The dual structure of extremely simple non-periodic sequences

arising from two non-periodic circuits of curves with genus one.
双対的非周期回路　種数一の曲線の非周期的回路が 2 つであるところから生じる、極度に単純な非周期列の双対的構造

Hodge bouquet: A collection of Riemannian surface graphs with the same number of seeds, arranged in a bouquet graph. Note that it also has a normal "bouquet structure" corresponding to the "Euler product." It is also necessary to distinguish it from the commonly referred to "Hodge structure."
ホッジの花束　種数一のリーマン面グラフを花束グラフ状に束ねたもの。「オイラー積」に対応する通常の「花束構造」をも持っていることに注意。また、通常言われている「ホッジ構造」との区別が必要。

Trace bundle : This refers to the entire set of all paths (traces) that pass through the interior of a given structure, including both finite and infinite lengths.
In particular, when the components of the path can be uniquely distinguished, this set can be one-to-one corresponding with the entire Lyndon sequence (and its infinite repetition).
トレース束　ある構造体の内部を通過するすべての経路（トレース）を、有限長・無限長のいずれの場合も含めて集めた集合全体をいう。
とくに、その経路の構成要素が一意に区別可能なとき、この集合はリンドン列全体（およびその無限反復）と一対一に対応しうる。

quasi-modular trace ;　A natural quasi-dual transformation that reduces　"irreducible rational Lyndon"　in trace bundles to　"prime Lyndon"　or　"natural number Lyndon."　Note that this can be performed even without a specific form, as long as a trace bundle is available. In that case, it can be expressed as a geometric operation as a deformation of the graph.
トレース束における「既約有理リンドン」を「素リンドン」や「自然数リンドン」へ縮約する自然な類双対変形。特に明示的形式がなくてもトレース束があれば行えることに注意。その場合、グラフの変形として、幾何学的操作の一環として、表現できるだろう。

Regularity:A function is regular when the local structure of its graph is uniform and orderly.non-regularity
正則性　関数が正則、グラフの局所構造が一様で整っていること
非正則性 グラフの局所構造が一様ではなく、正則でない構造、ゼロ点配置が乱れているなど

Non-regularity Irregularity: The local structure of the graph is not uniform but sparse. The zeros of the zeta function are scattered along the critical line.
非正則性　グラフの局所構造が一様ではなく、まばらであること。ゼータのゼロ点が臨界線からばらばらになる。

5,公理・写像・圏的表現

Collections of dual motif-closed sets；A complete state that cannot be further expanded by repeating dual operations.
双対モチーフ閉包　双対操作を繰り返すことによってこれ以上拡大しない圏的な完備状態

Fractal-based logic; Since quasi-duality transformations transform fractals into fractals, fractal properties are normally preserved even with normal restoration or reduction, as well as with divergent restoration or reduction. Note that there are times when the structure of the "trace bundle" remains unchanged and times when it undergoes structural changes. A language is needed to describe the structural changes of the trace bundle.
フラクタル準拠論理　類双対性変換はフラクタルをフラクタルへと変形するので、通常の復元や縮約でも、発散的復元や縮約でも、普通にフラクタル性が保たれていること。そして、そのとき、「トレース束」の構造が不変であるときと構造論的な変化をする時があることに注意。トレース束の変化構造を記述する言語が必要。

quasi-dual morphism
→ 類双対写像

Quasi-duality closure ;A noncommutative, multivalued, quasi-dual transformation that cycles through all transformations between the maximum loop structure and the maximum tree structure until it reaches a state that cannot be further expanded. This becomes a zeta structure of a categorical structure.
→ 類双対閉包
非可換で、多値的な、類双対変換が、最大ループ構造と最大ツリー構造の間の変換をすべて巡らせて、これ以上拡大し得ない状態へと達すること 圏的構造のゼータ構造体になる