UNIVERSITY OF OSNABRÜCK

MASTER THESIS

Theoretical considerations towards deep convolutional neural networks from Fourier analysis

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Declaration of Authorship

"Whether gods exist or not, there is no way to get absolute certainty about ethics. Without absolute certainty, what do we do? We do the best we can. "

Richard Stallman

UNIVERSITY OF OSNABRÜCK

Abstract

Faculty Name
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by Hiroshi SAWADA

Deep convolutional neural network exhibits great power to capture nonlinear mapping having dominant position for image processing [1], speech processing [21], and natural language processing [22] in recent years. However, it has not yet well revealed the reasons the algorithm holds extraordinary capabilities to solve these problems. Thereby, this thesis aims to describe intrinsic properties that this algorithm has from an aspect.

Besides, the network of deep learning needs a lot of computation to be trained and its memory for formed weights has considerable amount of size. Slow convergence and requirement of large amount of memory makes this algorithm difficult to be trained as online-learning.

One of the approach to overcome this problem is having the network only binary values, either +1 or -1 [13]. The target object which turns to be binary means only weights or both weights and activations of nodes. The way of computation which reveals state-of-art outcome is introduced in [17]. I came up with slightly different computation of convolution from the approach of this paper, and this way of computation is displayed on chapter2.

Chapter1 describes basic knowledge of back propagation which is most fundamental idea of convolutional neural networks.
Chapter2 pays attention of convolutional neural networks itself, and new ways of computation for binary neural networks is introduced here.
Chapter3 states the relevance of dimensional reduction and compression. Binary neural network can be seen as sort of quantized figure of normal convolutional neural networks which should have some relevance to lossy compression.
Chapter4 contains description of Fourier analysis to understand binary input data and binary weights better.
Chapter5 shows you analysis of weights which come from training of binary neural networks, and conclude all chapters.

This paper holds many exemplifications of displayed ideas with codes.
It partially because this thesis is not a paper which notes only results, but a thesis that I shall show what I have studied.
From a different point of view, it is because I suppose making things clear and deliberated to the detail is needed particularly about algorithms named "deep learning". The reason is what is occurring in CNN tends to be black boxed for many developers. This comes partially from natures of CNN but driven by usefulness of pre-released library.

Codes on chapter1-4 are written by Haskell.
I acknowledge this makes rather difficult to be read for many readers.
Nevertheless, it gives ones very well structured representation of code that fit a thesis. I believe, also practically, Haskell is going to be introduced for machine learning tasks due to its support of parallel computing.
Codes on chapter5 are written in c++ and lua.

I declare I have written every lines of codes with my understanding. Comprehensive figures of code can be seen in [9]

I wish that reading this thesis guides readers more profound considerations of this area.

Acknowledgements

Thanks to my solid academic training of cognitive science for my two years master in Osnabrueck, Germany.

Stepping back to myself just after entering this institute, I have realized that this master course of cognitive science could provide most of the answers towards my questions about cognition which I have wondered in my early life. Also, a lot of challenging tasks which should be pursued during the rest of my life came out from different perspectives which I haven't focused ever. This period in Osnabrueck literally turned to be my turning point in my life.

The two major I picked up, Neuro-informatics and artificial intelligence are one of the most exciting fields where you could apply what I have studied to the various challenging problems in industries. It was pretty fortunate that I could have studied these fields from great professors with those who holds curiosity which is more or less shared with me.

Specifically, I would like to express my sincere gratitude to four professors in Neuro-informatics group and Artificial intelligence group.

Prof. Dr.-Ing. Gunther Heidemann gave me solid training of machine learning and computer vision, where some part of my thesis is addressed to what I have learned then.

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Chapter 1

Back propagation

1.1 Overview

In this section, I will introduce most basic algorithm for training neural networks which is back propagation [5] particularly for two layers perceptron. This algorithm is necessary to introduce CNN in the following chapter. However, back propagation is not only for CNN, and most generally explained with multilayer perceptron. Thereby, I will as well explain it with multilayer perceptron.

Back propagation algorithm consists of two steps mathematically

1) feed-forward
2) assignment of derivatives with respect to each weights

Result of second calculation tells you as an implementation, should be divided into two distinct operations, which consists of three in total

1) feed-forward
2) error-propagation
3) weights update

1.2 Feed-forward process

1.2.1 Formalization

Feed Forward process is regarded as a nonlinear mapping from dimensions of data to teacher class. This is denoted as following.

$$n_{d2j} = \phi_1((w_{1ij} + b_j) \cdot n_{d1i}) \tag{1.1}$$

$$n_{d3k} = \phi_2((w_{2jk} + b_k) \cdot n_{d2j}) \tag{1.2}$$

where $i \in I, j \in J, k \in K d \in D$,
I = number of input neurons,
J = number of hidden neurons,
K = number of output neurons,
b = bias ,
D = number of training data,
n_{dai} = activation of each neurons,
a = number of layer,
w_{abc} = activation of weight,
b = index of node in previous layer which forms a vertex of weight,
c = index of node in subsequent layer which forms another vertex of weight,
ϕ_a = activation function ,

Weights can be included into the neurons of a front layer which is denoted as following.

$$h_j = \phi_1(w_{1ij} \cdot (g_i + b_0)) \tag{1.3}$$

$$o_k = \phi_2(w_{2jk} \cdot (h_j + b_0)) \tag{1.4}$$

This states following.

Input Nodes ; n_{d1i}

- Input nodes except bias terms are equivalent to training data. But, Input dimension have to be a vector even given data has more than two Rank. For instance, image can be treated as a vector as well as signal data.
- Each data can be either in serial (Online learning) or in parallel (Batch learning) passed to input node. As of online learning, it is assumed unit of data-set for optimization is changed if you get more data. It makes difficult to train a network in a online learning because the performance is worse if the network is trained with another data-set after optimized with certain data-set than the one trained by every data-set initially. But its benefit is one does not need to prepare all of data before training, but training process is slower than batch learning since there is a dependency in the order of feeding data in.

Hidden Nodes ; n_{d2j}

- Hidden layer is an only layer whose number of nodes can be picked up by ones to implement it.
- Number of hidden neurons is considered with the number of data, the number of teacher class, intrinsic dimension of data, and complexity of task. There is no deterministic estimation to get most proper N, thereby determined by inductively. Complexity of task is derived from the combination of training data and teacher data. For instance, discrimination of XandY data is regarded as an easier task than XOR data given Euclidean distance between pairs of data which is supposed to be in the same class.

Output Nodes ; n_{d3k}

- Number of output neurons have to be equal to the number of class which you would like to classify if the task is classification.
- Note that number of input node is determined by descriptive dimension + bias, whereas, the number of output node is absolutely no relevant to the input dimension, since this comes from teacher data which is generally less than number of data. For instance, if the dimension of data is 1, and the number of output class is 100, Ninput, hidden, output is N 2,X,100

Weights ; w_{1ij}, w_{2ij}

- Weights are often initially randomized, and its values are floating values. Algorithm itself is deterministic, however, initial condition ends up with different result.
- One of weights have to be independent from activation of previous layer's neurons so that the network could represent the item which is not attributed from input neurons.

Activation function ϕ_a

- There are some well known activation function such as sigmoid, tanh , softmax, and rectified linear unit.
- One of the properties of activation function(sigmoid,hyperbolic tangent,softmax) in feed-forward process is preventing overshooting of fires of neurons, and squashing inner dot of previous weights and nodes to certain values.
- activation function of hidden layers and output layers can be different.

1.2.2 Code representation

In this section, I will put python and Haskell code because comparing both gives us small insights about parallel computing.

Following is the feed-forward process in python without using matrix library.

```
for i in range(hiddenNum):
    for j in range(inputNum):
        hidden_value[i] += activationF(w1[j][i] * input_value[j])
```

Since the process is inner product with vector and matrix whose rank is two, it can be written as 2 loop assuming batch size is 1.

Next, is Haskell code.

```
data Weight = Weight [[Double]] deriving (Show)
data Node   = Node    [Double]  deriving (Show)

--weight initialize
wInit :: (Int,Int) -> Weight
--vector*vector
vv  :: [Double] ->  [Double]  -> [Double]
--vector*matrix
vm  :: [Double] -> [[Double]] -> [Double]
--sigmoid function
sigmoid :: Double -> Double
--feed-forward integration function
feedForward :: (Double -> Double) -> Node -> Weight -> Node

wInit (a,b) = Weight [[0.1,0.7],[-0.4,0.1]]
vv a b = foldr (\(x,y) z -> x * y : z) [] (zip a b)
vm a b  = map (\x ->foldl (+) 0 $ vv a x ) b
feedForward f (Node a) (Weight b) = Node $ map f $ vm a b
sigmoid x = 1 / (1+exp(-x))

let w1 = wInit (2,2)
let w2 = wInit (2,2)

let hidden = feedForward sigmoid input  w1
let output = feedForward sigmoid hidden w2
```

Inner product is written as combination of map, zip, and reduce(foldl and foldr in Haskell).
In skeletal parallel computing [3] , it is indispensable to distinct generalized loop into higher order functions especially map and reduce.

Map is applying a function from a list to another list whose element is 1:1 relation which does not affect other mapping at all. This is a loop which enables us completely separate on different processors or different threads.
Zip is receiving two lists and create two dimensional tuple appending two elements of the list.
Reduce is applying a function to a list and generates results which caused from elements from an original list.

Coming back to the Haskell code of back propagation, "feedForward" is defined by multiplication of a matrix and a vector, in it, there is a multiplication operation of a vector and a vector.

Haskell type of both is defined as

```
map :: (a -> b) -> [a] -> [b]
zip :: [a] -> [b] -> [(a, b)]
reduce (foldr) :: (a -> b -> b) -> b -> t a -> b
```

Each computational cost can be estimated by

Given
number of processors = p
number of list elements = n
cost of applying function = t

$map\ f = (\frac{n}{p})t_f$
$zip \quad = \frac{n}{p}t$
$reduce(\otimes) = ((\frac{n}{p}) + log_2 p)t(\otimes)$

The computation quantity of "reduce" follows $log_2 p$ because it is assumed that data structure is going to be a balanced binary tree as list elements are mutually associative. For instance, summation of element of list does not affect the order of its calculation.

By the way, summation of inner product can be written by using another interesting higher order function which is $scan$. Scan is a $reduce$ which stores all of elements on its calculation process. For instance,

```
scan (+) 0 [3,2,1,4,5] == [3,5,6,10,15]
```

This is pretty useful when you calculate summation of partial element of array over and over again. For instance, if you want to calculate sums from 2nd element to 4th which is 2+1+4 you can calculate subtract 1st element from 4th element on integral array which is 10 - 3.

This is called integral image in computer vision but useful in parallel programming.

1.2.3 Notation with function application

Codes above in Haskell can be translated to more simple way which means not just succession of function which is succeded "$" but nested function application. The Result after translation is following.

```
type Weight = [[Double]]
type Node   =  [Double]

--vv a b = foldr (\(x,y) z -> x * y : z) [] $ zip a b
vv = (foldr (\(x,y) z -> x * y : z) [] .) . zip

--vm a b = map (\x ->foldl (+) 0 $ vv a x ) b
vm = map . ((foldl (+) 0 .) . vv)
```

```
--feedForward f (Node a) (Weight b) = Node $ map f $ vm a b
feedForward f = (map f .) . vm
```

First, I replaced all of data constructors to the form without any data constructors by "type" which allows me to erase arguments.
Next, I have removed all of arguments of $vv, vm, feedForward$. How could you assure the transformation is correct?
Here is the operation of transformation for vv.

```
vv a b = foldr (\(x,y) z -> x * y : z) [] $ zip a b
vv a b = (foldr (\(x,y) z -> x * y : z) [] . zip a) b
vv a = (foldr (\(x,y) z -> x * y : z) [] . zip a)
vv a = ((foldr (\(x,y) z -> x * y : z) [] .) $ zip a)
vv a = ((foldr (\(x,y) z -> x * y : z) [] .) $ zip a)
vv = ((foldr (\(x,y) z -> x * y : z) [] .) . zip )
vv = (foldr (\(x,y) z -> x * y : z) [] .) . zip
```

The main idea in it is an operator (\$) can be translated in a following way.

$a \, \$ \, b \, c \, d == (a \, . \, b \, c) \, d$

1.3 Calculation of error derivatives

1.3.1 Error derivatives for the weights on a second layer

Error propagation is denoted as following.

Objective function of back propagation is following an error minimization function

$$E_{total} = \frac{1}{D} \sum_{d=1}^{D} E_d \tag{1.5}$$

E_d is defined as

$$E_d = \frac{1}{2} \sum_{k=0}^{K} (n_{d2k} - t_{dk})^2 \tag{1.6}$$

where t_{dk} = 1 of k vectors which are orthonormal
The derivatives of each weights are

$$w'_{lij} = w_{lij} - \rho \frac{\partial E_d}{\partial w_{lij}} \tag{1.7}$$

where ρ = learning rate.
l = index of the layer

$\frac{\partial E_d}{\partial w_{lij}}$ means distribution of overall error to every single weights in any layers.

w_{lij} consists of two which are w_{1ij} and w_{2jk} given two layers in the network.

w_{2jk} is able to be unfolded in a following way using chain rule.

$$\frac{\partial E_d}{\partial w_{2ij}} = \frac{\partial E_d}{\partial n_{d2k}} \frac{\partial n_{d2k}}{\partial w_{2jk}} \tag{1.8}$$

$\frac{\partial E_d}{\partial n_{d2k}}$ can be calculated as

$$\frac{\partial E_d}{\partial n_{d2k}} = \frac{\partial \frac{1}{2} \sum_l (n_{d2k} - t_{dk})^2}{\partial n_{d2k}} = n_{d2k} - t_{dk} \tag{1.9}$$

However, $n_{d2k} = \phi y_{d2k}$ needs to be composed to one step smaller elements for calculation of its derivative.

Note that each layers are calculated as a sequence of inner product and multiplication of activation function.

$$n_{d2k} = \phi(y_{d2k}) \tag{1.10}$$

where $y_{d2k} = (w_{2jk} + b_j) \cdot n_{d1j}$

thereby, above can be written as

$$\frac{\partial E_d}{\partial w_{2ij}} = \frac{\partial E_d}{\partial n_{d2k}} \frac{\partial n_{d2k}}{\partial \phi y_{d2k}} \frac{\partial \phi y_{d2k}}{\partial w_{2jk}} \tag{1.11}$$

In this condition, $\frac{\partial n_{d2k}}{\partial \phi y_{d2k}}$ and $\frac{\partial \phi y_{d2k}}{\partial w_{2jk}}$ can be calculated as

$$\frac{\partial n_{d2k}}{\partial \phi y_{d2k}} = \frac{\partial \phi_k(y)}{\partial y}|_{y=y_{d2k}} = \phi' \tag{1.12}$$

$$\frac{\partial y_{d2k}}{\partial w_{2jk}} = n_{d1j} \tag{1.13}$$

Above two mean derivative of activation function and activation of nodes which is in a previous layer.

Substituting (1.9), (1.12), (1.13) to (1.11),

$$\frac{\partial E_d}{\partial w_{2ij}} = (n_{d2k} - t_{dk}) {\phi'}_2 n_{d1j} \tag{1.14}$$

This means a weight matrix on a second layer should be updated reflected activation of nodes in a previous layer which is connected to the weight and difference between teacher signal and node values and derivatives of adopted activation function.

1.3.2 Error derivatives for the weights on the first layer

$w1$ is not directly interconnected to output nodes thereby needs to be propagated to the more forward in a following way.

$$\frac{\partial E_d}{\partial w_{1ij}} = \frac{\partial E_d}{\partial n_{d2k}} \frac{\partial n_{d2k}}{\partial y_{d2k}} \frac{\partial y_{d2k}}{\partial n_{d1j}} \frac{\partial n_{d1j}}{\partial y_{d1j}} \frac{\partial y_{d1j}}{\partial w_{1ij}} \tag{1.15}$$

The first derivative ($\frac{\partial E_d}{\partial n_{d2k}}$) and the second derivative ($\frac{\partial n_{d2k}}{\partial y_{d2k}}$) are derivatives which have already been calculated by (1.9) and (1.12).

The third derivative ($\frac{\partial y_{d2k}}{\partial n_{d1j}}$) of (1.15) is

$$\frac{\partial y_{d2k}}{\partial n_{d1j}} = w_{2jk} \tag{1.16}$$

because

$$\because y_{d2k} = w_{2jk} \cdot n_{d1j}$$

The fourth ($\frac{\partial n_{d1j}}{\partial y_{d1j}}$) of (1.15) is

$$\frac{\partial n_{d1j}}{\partial y_{d1j}} = \phi'_1 \tag{1.17}$$

because

$$\because n_{d1j} = \phi'_1(y_{d1j})$$

and The fifth ($\frac{\partial y_{d1j}}{\partial w_{1ij}}$) of (1.15) is

$$\frac{\partial y_{d1j}}{\partial w_{1ij}} = n_{d0i} \tag{1.18}$$

because

$$\because y_{d1j} = w_{1ij} \cdot n_{d0i}$$

To feed (1.9),(1.12),(1.16),(1.17),(1.18) into (1.15)

$$\begin{aligned} \frac{\partial E_d}{\partial w_{1ij}} &= (n_{d2k} - t_{dk})\phi'_2 w_{2jk}\phi'_1 n_{d0i} \\ &= ((n_{d2k} - t_{dk})\phi'_2 \cdot w_{2jk})\phi'_1 n_{d0i} \end{aligned} \tag{1.19}$$

$(n_{d2k} - t_{dk})\phi'_2 \cdot w_{2jk}$ in (1.19) is calculated in (1.14) and it means propagating error to the backward passing through the same weights which have been made use of in a forward pass.

1.3.3 Interpretation of an error derivative calculation

(1,14) and (1,19) tells that derivatives of error with respect to each weights are composed by following procedures.

- derivative of activation function given firings of its neurons in feed-forward process
- one step forward values of neurons
- accumulated errors which has been propagated from backwards layers.

These deduce following statements.

Letting forward node of w_{lfb} N_{df} , backward node N_{db}

A Punishment towards the excitement of input N_{df}
: if more excited the activation of the input value to the weight is (absolute value of neuron is larger), more error would be propagated.

A Punishment towards the amount of error propagation N_{db}
: if the amount of error propagated from backward is larger, updating values would be larger.

1.3.4 Derivatives of sigmoid function

By the way, so as for network weights to hold every information for nonlinear mapping between data and its teacher label, they have to have a functionality which saves patterns which have already learned when new data is fed in. I call this anti-plasticity because it prevents violating acquired mapping from another training data. This functionality appears in back propagation algorithm specifically derivatives of activation function.

Suppose activation function is sigmoid function, let us take a look at its character.

$$S(u) = \frac{1}{1+\exp(-u)} \tag{1.20}$$

Its derivatives are calculated as following.

$$\begin{aligned}
S'(u) &= \frac{dS}{du} \\
&= \frac{d}{du}(1+\mathrm{e}^{-u})^{-1} \\
&= -1(1+\mathrm{e}^{-u})^{-2}\frac{d\mathrm{e}^{-u}}{du} \\
&= -1(1+\mathrm{e}^{-u})^{-2}(-\mathrm{e}^{-u}) \\
&= \frac{\mathrm{e}^{-u}}{(1+\mathrm{e}^{-u})^2} \\
&= \frac{1+\mathrm{e}^{-u}-1}{(1+\mathrm{e}^{-u})^2} \\
&= \frac{1}{1+\mathrm{e}^{-u}}(1-\frac{1}{1+\mathrm{e}^{-u}}) \\
&= S(u)(1-S(u))
\end{aligned}$$

Figure 1.1 represents a mapping by sigmoid function. Figure 1.2 represents a mapping via derivatives of sigmoid function.

It tells you when you let activation function sigmoid, all of the values of nodes have values from 0 to 1, and its derivatives are maximized when the value is 0.5. If you note that derivatives of activations are calculated feeding into the values of neurons in feed-forward step, it says exactly about anti-plasticity. If one firing of neuron is close to 0 or 1, the error is less likely to be propagated to the node than a node whose value is close to 0.5. In other words, if the activation of neurons are already fixed by past training

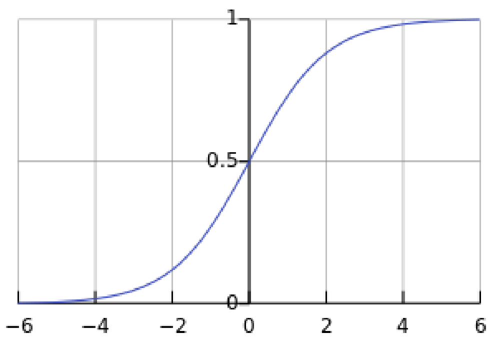


FIGURE 1.1: sigmoid function

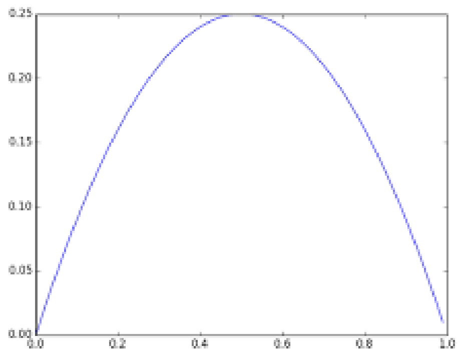


FIGURE 1.2: derivative of sigmoid

regardless of current input values, errors are not be propagated to the node, rather be flowed to the another nodes whose values have never been fixed. hyperbolic tangent has similar property that sigmoid has, the firing of neurons can be negative in this case, though.

To summarise them, Back propagation algorithm can be divided into two parts, which are

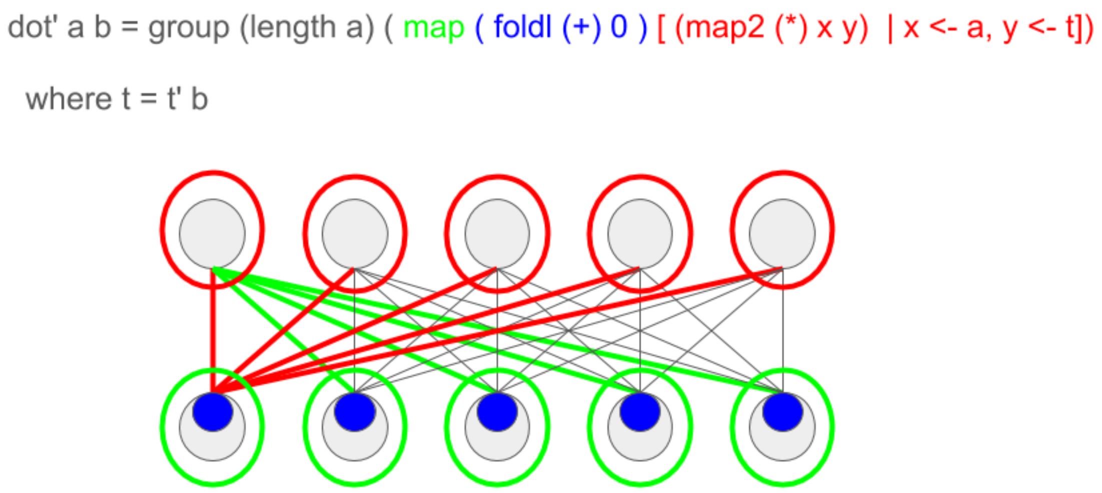
Plasticity N_{df}
: If more excited the activation of the input value to the weight is (absolute value of neuron is larger), more error would be propagated.

Anti-plasticity N_{db}
: If the amount of error propagated from backward is larger, updating values would be larger.

1.3.5 Code representation

```
--activation function
sigmoid :: Double -> Double
devSig  :: Double -> Double

--calcErrors input : (Derivative of activation function,
--Teacher node , Output node, Output error)
calcErrors :: (Double -> Double) -> Node -> Node -> Node

--sum of square errors
calcErrorSigma :: Node ->  Double

--backpropagate error (Derivative of activation function, Node,
--Node, Weight between 1st & 2nd layer, hiddenError)
hiddenError  :: (Double -> Double) -> Node -> Node -> Weight -> Node

--weight update (learning rate, input node, output node,
--weight before update, weight after update)
--I let the node two dimensional to multiply element wise multiplication
adjustWeight :: Double -> [Node] -> [Node] -> Weight -> Weight

--square mean of error in final layer
calcErrorSigma = (foldl (+) 0 . map (\x->x*x))

--backpropagation to hidden layer
hiddenError f h = ((foldr (\(x,y) z-> x*y:z) [] . zip (map f h)) . ) . vm
```

```
--weight adjustment in two layer
adjustWeight lr = (map2 (map2 (+)) . ) . dot'

sigmoid x = 1 / (1+exp(-x))
devSig x  = x * (1 - x)

--map2
map2 :: (a -> b -> c) -> [a] -> [b] -> [c]
map2 _  a []= []
map2 _ [] a = []
map2 f (h1:t1) (h2:t2) = f h1 h2 : map2 f t1 t2

-- matrix inner product.
-- more proper than vv and vm which is defined above
-- because vector is regarded as two dimensional matrix.
-- This is necessary to calculate (vector.T dot vector -> matrix)
dot' :: (Num a) => [[a]] -> [[a]] -> [[a]]
dot' a b = group (length a) ( map ( foldl (+) 0 )
  [ (map2 (*) x y)  | x <- a, y <- t])
  where t = t' b

--below is functions which is called from above.

--transposition of matrix
t' :: [[a]] -> [[a]]
t' ([]:_) = []
t' x = (map head x) : t' (map tail x)

--take some elements before index
take' :: Int -> [a] -> [a]
take' a [] = []
take' 0 a  = []
take' i (h:t) = h : (take' (i-1) t)

take'' :: Int -> [a] -> [a]
take'' a l = reverse
$ foldl (\x y -> if length x < 2 then y : x else x) [] l

--drop some elements after index
drop' :: Int -> [a] -> [a]
drop' 0 a  = a
drop' i (h:t) = drop' (i-1) t

--group is combination of take & drop
group :: Int -> [a] -> [[a]]
group  _ [] = []
group i t = take' i t : group i (drop' i t)
```



```
dot' a b = group (length a) ( map ( foldl (+) 0 ) [ (map2 (*) x y)  | x <- a, y <- t])
 where t = t' b
```

FIGURE 1.3: dot product and feed-forward

Chapter 2

Convolutional Neural Network

2.1 Prelude

In this section, I will introduce convolutional neural network[8],[12]. As we had a look at in the last chapter, every nodes of multi-layer perceptron are interconnecting to the every nodes of the next layer. It states that each dimension of multi-layer perceptron should not have any local dependencies in it, in other words, asked to be mutually independent. This is why an unsupervised learning algorithm for dimensional reduction is adopted before feeding data into multi-layer perceptron.

CNN is different from multi-layer perceptron in a sense that it is in charge of dimensional reduction as well as classification. Concretely, CNN sets a constraint towards connection between values on a dimension in a data and values on another dimension in the same data. If the distance between two dimensions are longer than certain values, the connection in between them are out of scope that the network takes into its consideration. The maximum distance is determined by filter size which is one of its parameters that one can pick up by themselves. For instance, if the filter size is just 1, it means there is no interconnection between each dimensions in a same channel but only corresponding elements in different channels. If the number of dimensions are as many as length of input dimension, it means all of dimensions interconnect to dimensions on a subsequent layer just as the case of multi-layer perceptron.

2.2 Drawback of training in more than 4 layers perceptron

For some data-set, CNN which has more than 2 hidden layers shows better performance than the one whose layer is just 1. However, from a following reason, it is difficult to propagate every weights as it should be by back-propagation if activation function is sigmoid. This is called vanishing gradient problem on error propagation step. Namely, error derivative is going to be smaller if the node is closer to forward. This is explained as following.

Error propagation step to the first weight w_{1ij} had be written as following.

$$\frac{\partial E_d}{\partial w_{1ij}} = (n_{d2k} - t_{dk})\phi'_2 w_{2jk}\phi'_1 n_{d0i} \tag{2.1}$$

If the network has more than 4 layers, error propagation to the 1st weight can be written as

$$\frac{\partial E_d}{\partial w_{1ij}} = (n_{d3l} - t_{dl})\phi'_3 w_{3kl}\phi'_2 w_{2jk}\phi'_1 n_{d0i} \tag{2.2}$$

This tells you backward process is completely linear with respect to the errors of the last layer. Besides,

$$\phi = \frac{1}{1+\exp(-u)}$$
$$\phi'(u) = \phi(u)(1-\phi(u))$$

$$0 < u < 1$$

$$0 < \phi'(u) < 0.25$$

When derivatives of errors in the last layer come back to the forward layer, error is going to be reduced by passing through multiplication of derivatives of multiplication.

To summarise it, normal back-propagation with more than 4 layers is trained, forward weight cannot be well tuned.

This gives you a motivation to introduce different activation function which is rectified linear unit introduced later on.

2.3 Feed-forward process of convolutional neural network

In this section, I will explain feed-forward process of CNN. Main architecture of Feed-forward process of CNN consists of 4, linear combination of convolution, activation function, pooling, and fully connected layers. There are many types of CNN which is modified this mainstream. In this chapter, I will have a look at the ones except fully connected layer because it is identical with multilayer perceptron in the previous chapter.

2.3.1 1D Feed-forward summation of convolution

CNN has different architecture based on dimension of input data contrary to multilayer perceptron. Here I will have a look at 1 dimensional and 2 dimensional CNN respectively.

Feed-forward process of convolution network operation is defined by following which is 1st convolution and 2nd, summing up all of corresponding elements.

$$b_{djn} = \sum_{i=0}^{I-1} \sum_{s=0}^{S-1} a_{di(s+n)} w_{ijs} \tag{2.3}$$

where S = kernel size
$s \in S$
$n \in N$
N = dimension of data
I = number of input feature
$i \in I$
$j \in J$
J = number of hidden feature
$a_{di(s+n)}$ = input data set
b_{djn} = a feature in hidden layer
w_{ijs} = weight

First of all, dimensions of nodes and weights are incremented just one up than the ones for multilayer perceptron, namely, two dimensions for nodes, three dimensions for weights.

Specifically, regarding nodes, one hidden node consists of smaller unit of nodes whose dimension is the same of original data or dimension - 2 (assuming no boundary compensation) and there are multiple hidden nodes. Here, I call larger unit of hidden node as feature, and smaller unit of node an individual hidden node. For instance, $b_{1,5,3}$ of b_{djn} means a hidden node in the 1st data of data-set, in the 5th feature, and 3rd value on the data.

A Weight between one layer and subsequent one has also three distinct index. $w_{1,6,3}$ of w_{ijs} means a weight which connects 1st feature in previous layer and 6th feature in the subsequent layer, and the 3rd value on the filter.

Input feature in a first layer is mostly just single unless it has multiple dimension originally such as RGB value on image data, thereby first hidden layer does not sum up multiple firing of neurons but just supply multiple features as they are.

2.3.2 Code representation

If you implement linear combination of convolution straightforwardly in Haskell, the code is going to be like following.

```
--argument
  --1st : activation function
  --2nd : previous node
  --3rd : weights in between
  --return : Node on next layers
  feedForward1D :: (Double->Double) -> Node1D -> Weight1D -> Node1D

  --second map from left will iterate different features on same nodes,
  --then, rightest map2 puts pairs of input and weights on convolution calculation,
  --then, the matrix is transposed so it can be added by element-wise.
  feedForward1D f n = map $ map g . t' . map2 convolve1D n
     where g = f . foldl (+) 0
```

This operation is overall scheme of summation of convolution which have passed through activation function.
"map2 convolve1D n" calculates convolution with input node and weights on each filter. They are going to be transposed because we would like to calculate element wise addition in each dimension. After the transposition, each dimension is going to be added together. Most outside "map" operation is the distribution to the next feature from one of the features on input node.

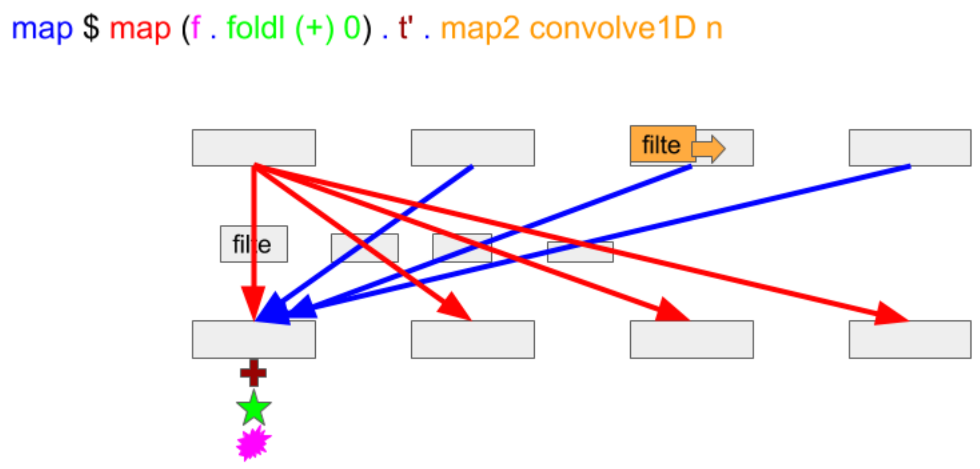


FIGURE 2.1: One dimensional CNN feed-forward process

convolve1D is defined as following, and its explanation is described by figure 2.1

```
--1 dimensional convolution
  convolve1D :: [Double] -> Kernel1D -> [Double]

  --apply boundry compensation to the top of list and the bottom of list
  --iterate 1d list
  boundry1D :: Int -> [Double] -> [Double]

  --take first element and insert on top of the list
  --iterate 1d list
  filling1D  :: Int -> [Double] -> [Double]

  --this function is just insertion N value on top of list
  -- N is first argument
  filling'   :: Int -> Double -> [Double]

  --actual operation of convolution
  convolve1D' :: [Double] -> Kernel1D -> [Double]

  convolve1D d k = convolve1D' dd k
  -- get the half of kernel size to insert boundry compensation
  -- for convolution beforehand
    where s = (length k) `div` 2
  -- apply boundry compensation
          dd = boundry1D s d


  boundry1D s a = dt
  --boundry compensation here is taking nearest elements on the list,
  --and insert it to the top and bottom of the list.
    where r  = (reverse . filling1D s . reverse) a
          dt = filling1D s r


  filling1D s a = b
  --take first element and insert on top of given list
    where rh = filling' s (head a)
          b  = (++) rh a

  convolve1D' d k
  --if the size of remained vector is less than kernel size, stop convolution.
    | lk > ld    = []
  --convolution is done by taking first N element from the list
  --and multiply with kernel and then sum up
  --first N element is kernel size
    | otherwise = foldl (+) 0 ( map2 (*) k (take' lk d)) : (convolve1D' (tail d) k)
    where ld = length d
          lk = length k
```

That is, input nodes are going to be scaled bigger according to the size of kernel. If the kernel size is 3, you need to fill a value on the top and the bottom of the list.

2.3.3 Inner product based feed-forward

If the kernel size is 1, it can be intuitively understood that convolution operation can be regarded as inner product in each dimension, and the number of the procedure is equivalent to number of dimension. This is because summation of convolution can be omitted because kernel is single. This is useful to compare normal multilayer perceptron and CNN because feed-forward process of multilayer perceptron is successive

```
boundary1D sh st a = filling1D sh ((reverse . filling1D st . reverse) a)

filling1D s a = (++) (filling' s (head a)) a

filling' i a = a : filling' (i-1) a

convolve1D’ = foldl (+) 0 ( map2 (*) k (take' lk d)) : (convolve1D' (tail d) k)

convolve1D d k = convolve1D' (boundary1D s s d) k  ///  where s = (length k) `div` 2
```

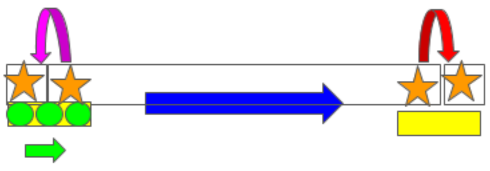


FIGURE 2.2: One dimensional convolution (boundary compensation)

application of inner product and nonlinear activation function. Different point over them is multi-layer perceptron has inter connection over different dimensions. However, convolution whose kernel is just 1 does not have any interconnection over multiple dimension at all.

If the kernel size is more than 2, convolution operation can be seen as summation of inner product over different features. The reason is following. Convolution of more than 2 kernel size can be seen as linear combination of convolution operation whose kernel size is just 1. As I explained before, convolution operation for single kernel can be seen as inner product. In addition, summation for convolution and summation for over multi-features after each convolution can be commutative. Therefore, convolution operation in this case is linear combination of inner product. Mathematically, the logic is described as following

$$\begin{aligned} b_{djn} &= \sum_{i=0}^{I-1}\sum_{s=0}^{S-1} a_{di(s+n)} w_{ijs} \\ &= \sum_{s=0}^{S-1}\sum_{i=0}^{I-1} a_{di(s+n)} w_{ijs} \\ &= \sum_{s=0}^{S-1} (a_{di(s+n)} \cdot w_{ijs}) \end{aligned} \tag{2.4}$$

That means, instead of convolution with filters, different input features (a_{di}) is added together after letting them pass through multiplication of different weights, then, they are summed up according to the size of kernel. I name this operation inner product based feed-forward.

Algorithmically, however, the operation is not going to be combinations of simple inner products because this is an operation for two dimensional matrix and three dimensional matrix. The process can be interpreted as following figure2.3 and figure2.4.

Figure 2 explains the process of distribution from one feature to multiple features on the next layer. Figure 3 explains flow of computation in case there are multiple input features.

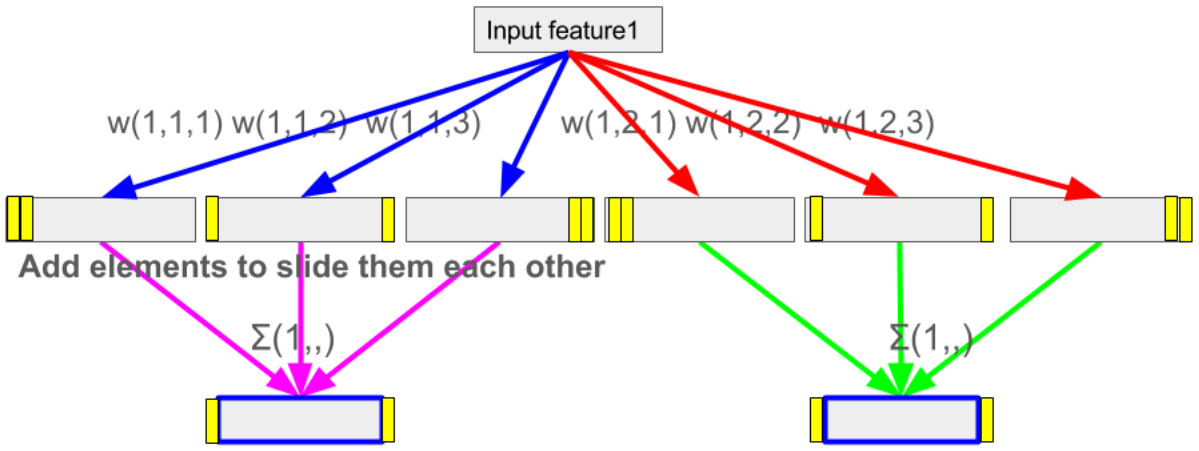

Input feature1
w(1,1,1) w(1,1,2) w(1,1,3)
w(1,2,1) w(1,2,2) w(1,2,3)
Add elements to slide them each other
Σ(1,,)
Σ(1,,)

FIGURE 2.3: one dimensional CNN feed-forward process

Inner based feed-forward operation can be distinct with two operations. First is multiplication and summation over multi-dimensions. Second is another summation enlarging input features according to the size of filters, and let the size as it was in the end, namely cutting out center of it.

The first operation consists of four in terms of its necessary iterations.
The First operation is iterating one input feature with one of weights on a filter. Then, the second process is repeating first operation setting the weights different on the same filter. The third operation is iterating different input features over previous operation. And the fourth is distributing features for the next features over every operations till 3rd step. To summarize it,

1. iterate on single feature and multiply with given single weight.
2. iterate weights on a filter
3. iterate multiple input features with different weights
4. iterate output features (or distribute features for the next layer)

Note that every step has own finite loop and certain conditions to go to the next step. This operation is not sequential rather nested.

After the end of third step, every corresponding elements of list which acquired from 1st step should be added with corresponding elements of list in third dimension.

For instance,

given

[[[1,2,3,4,5] [2,3,4,5,6]], [[-1,-2,-3,-4,-5] [1,2,2,2,1]]]

you should get

[1-1,2-2,3-3,4-4,5-5],[2+1,3+2,4+2,5+2,6+1]

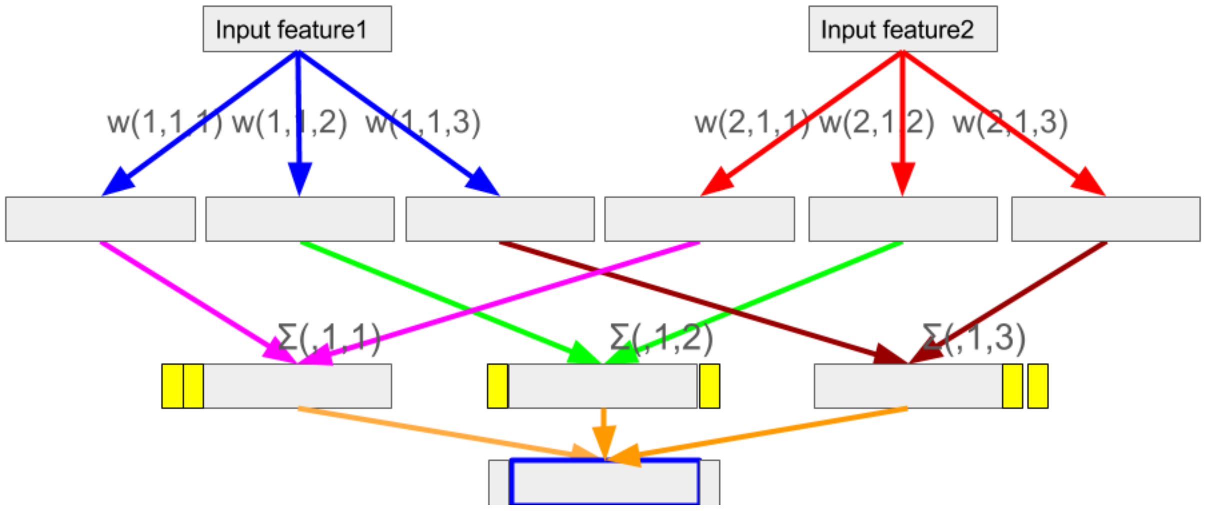


FIGURE 2.4: one dimensional CNN feed-forward process

Algorithmically, first operation is represented as following.

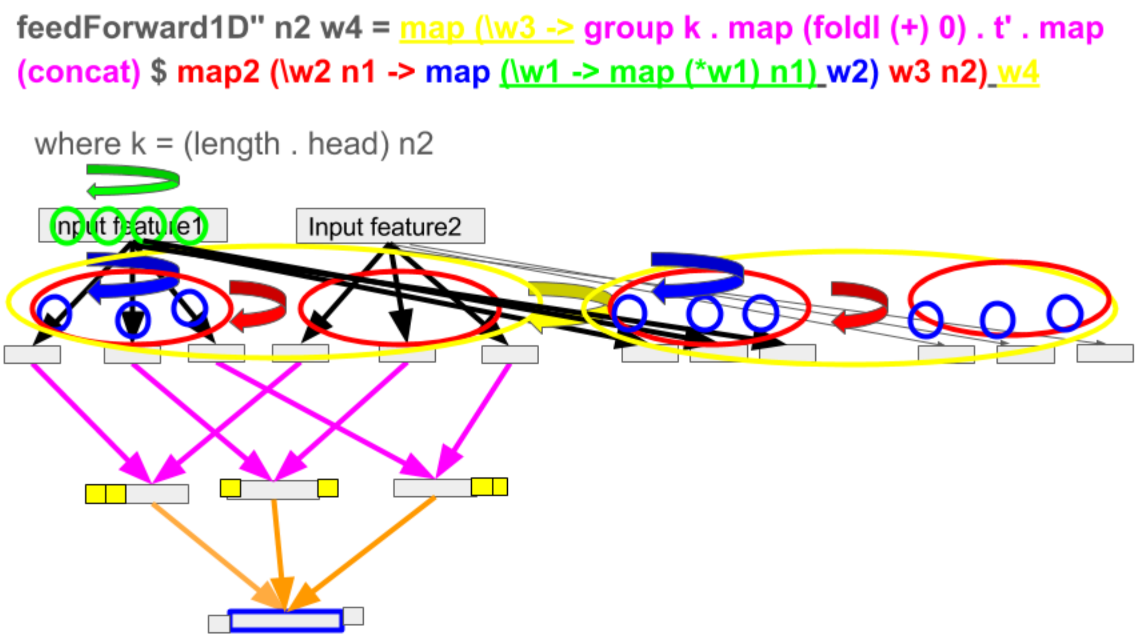
```
--feedForward in a different way,
  feedForward1D' :: Node1D -> Weight1D -> Node1D

--sub function of
  feedForward1D''  :: Node1D -> Weight1D -> [Node1D]

  feedForward1D' n w = feedForward1D''' (feedForward1D'' n w) k
     where k = (length . head . head) w

  feedForward1D'' n2 w4 = map (\w3 -> group k
     . map (foldl (+) 0) . t' . map (concat)
     $ map2 (\w2 n1 -> map (\w1 -> map (*w1) n1) w2) w3 n2) w4
     where k = (length . head) n2
```

Figure2.5 explains correspondence of nested structure.

One thing which is not self-explanatory is summation over different features.

Element-wise summation can be written as combination of "foldl (+) 0 " and "t'"(transposition). In this case, it is element-wise multiplication, but elements of 1st dimension and 3rd dimension, not the adjacent dimension. Thereby, you need to insert concatenation of list before transposition and grouping after summation.

```
  feedForward1D''' :: [Node1D] -> KernelSize -> Node1D


  feedForward1D''' n k = map (\x-> pickUp x kk kk) s
```



```
feedForward1D'' n2 w4 = map (\w3 -> group k . map (foldl (+) 0) . t' . map
(concat) $ map2 (\w2 n1 -> map (\w1 -> map (*w1) n1) w2) w3 n2) w4

where k = (length . head) n2
```

FIGURE 2.5: first part of CNN feed-forward (altered version)

```
where kk = k `div` 2
      s  = map (\x -> map (foldl (+) 0) . t'
      \$ map2 (\(h,t) xx-> boundry1D t h xx)
      (zip [0..2*kk] (reverse [0..2*kk])) x ) n
```

Second step of inner product based feed-forward is summation over output of first operation according to the number of weights on single filter. All of values which multiplied with weights on single filter should be added together.
For instance, given you have output from first operation.

$$\begin{pmatrix} 0.1 & -0.2 & -0.2 & 0.3 & 0.1 & 0.3 & 0.1 & -0.3 \\ 0.2 & -0.3 & -1 & -0.4 & 0.5 & 0.0 & 0.1 & -0.1 \\ -0.2 & -0.2 & -0.4 & 0.2 & 0.3 & -0.1 & 0.2 & 0.4 \end{pmatrix}$$
$$\begin{pmatrix} -0.2 & -0.4 & -0.5 & 0.1 & 0.6 & 0.2 & -0.1 & 0.3 \\ 0.4 & -0.1 & -0.2 & -0.4 & 0.5 & 0.4 & 0.1 & 0.2 \\ -0.6 & -0.6 & -0.1 & 0.2 & 0.3 & 0.1 & -0.2 & 0.1 \end{pmatrix}$$

and filter is

$$[[0.2, -0.1, 0.3], [-0.1, 0.2, -0.1]], [[-0.2, 0.3, 0.1], [0.2, -0.1, 0.4]]$$

Column of matrix corresponds dimensions of data.
Row of matrix represents Nth of weights on a filter which are added.
And, third dimension is the result of Nth filters.(N is numbers of filter.)

Column of the matrix is enlarged by following.

$$\begin{pmatrix} 0.1 & 0.1 & 0.1 & -0.2 & -0.2 & 0.3 & 0.1 & 0.3 & 0.1 & -0.3 \\ 0.2 & 0.2 & -0.3 & -1 & -0.4 & 0.5 & 0.0 & 0.1 & -0.1 & -0.1 \\ -0.2 & -0.2 & -0.4 & 0.2 & 0.3 & -0.1 & 0.2 & 0.4 & 0.4 & 0.4 \end{pmatrix}$$

2 dimensions are added in such a manner. 2 reflects the fact that filter size is 3. And, element whose value is equal to adjacent element is added either top of the list or bottom of the list. This compensation can be done in other ways such as filling "0" and so on. The position of inserting elements is determined by the row of matrix. If the list is in first row, it means these values are outcomes after multiplying Nth weights on a filter. Come to think of convolution process, multiplication between last element of a data and first weights on a kernel is discarded or simply not calculated. By contrast, elements of "-1"th element and first weights on a filter is used. It suggests that same calculation can be done by sliding elements of list, sums them up, and retrieve central part.For instance, the result of multiplication between last element means values of (row,column) = (1,10) which does not need to be calculated. Instead, -1 nth values on the first list can be represented as the values of (row, column) = (1,1) which should be included as final result.

Concretely, edge of above matrix is cut leaving center

$$\begin{pmatrix} 0.1 & 0.1 & -0.2 & -0.2 & 0.3 & 0.1 & 0.3 & 0.1 \\ 0.2 & -0.3 & -1 & -0.4 & 0.5 & 0.0 & 0.1 & -0.1 \\ -0.2 & -0.4 & 0.2 & 0.3 & -0.1 & 0.2 & 0.4 & 0.4 \end{pmatrix}$$

and, added by element-wisely.

$$\begin{pmatrix} 0.1 & -0.6 & -0.3 & -1.0 & 0.7 & 0.3 & 0.8 & 0.4 \end{pmatrix}$$

```
feedForward1D''' n k = map (\x-> pickUp x kk kk) s

where s  = map (\x -> map (foldl (+) 0) . t' $ map2 (\(h,t) xx-> boundry1D t h
xx) (zip [0..2*kk] (reverse [0..2*kk])) x ) n
```

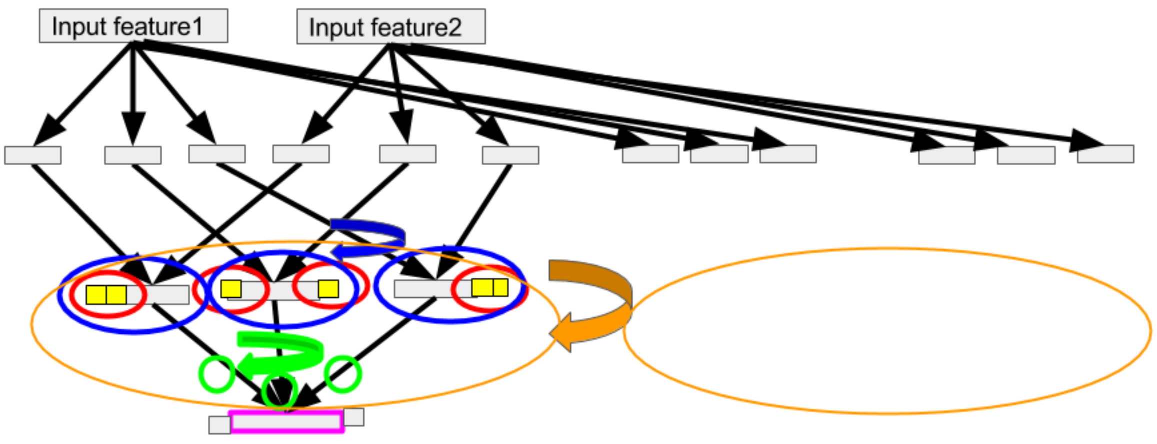


FIGURE 2.6: second part of CNN feed-forward (altered version)

2.3.4 Two dimensional linear summation of feed-forward

Two dimensional feed-forward process of convolution can be written as following.

$$b_{djmn} = \sum_{i=0}^{I-1} \sum_{v=0}^{V-1} \sum_{h=0}^{H-1} a_{di(m+v)(n+h)} w_{ijvh} \tag{2.5}$$

where V = kernel vertical size
$v \in V$
H = kernel vertical size
$h \in H$
$m \in M$
M = vertical dimension of data
$n \in N$
N = horizontal dimension of data
I = number of input feature
$i \in I$
$j \in J$
J = number of hidden feature
$a_{di(m+v)(n+h)}$ = input data set
b_{djmn} = a feature in hidden layer
w_{ijvh} = weight

In 2D convolutional neural network, dimension of node and weight is incremented from 1D convolutional neural network.

Node is 4 dimensional whose dimension is position of data-set, position of feature, vertical position of data, horizontal position of data .

Weight is also defined by 4 dimension which is position of feature in previous layer for one vertex,position of feature in subsequent layer for another vertex, vertical position on kernel, horizontal position on kernel

2.3.5 Code representation of two dimensional linear summation of convolution

```
--dimension is one higher than 1D
feedForward2D :: Node2D -> Weight2D -> Node2D

--only difference between 1D and 2D is two.
--One is convolving by 2D and another is covering over it
--with one more "map"
feedForward2D n = map $ map g . t' . map2 convolve2D n
   where g = map (foldl (+) 0)

--the way not to use signature "$" is following
--feedForward2D f n = map (map (map (f . foldl (+) 0))
  . t' . map2 (convolve2D) n)
```

Convolution for two dimensions can be written as following.

```
--2 dimensional convolution
  convolve2D  :: [[Double]] -> Kernel2D -> [[Double]]

  --apply boundry compensation to the top of list and the bottom of list
  --iterate 2d list
  boundry2D :: Int -> Int -> [[Double]] -> [[Double]]

  --take first element and insert on top of the list
  --iterate 2d list
```

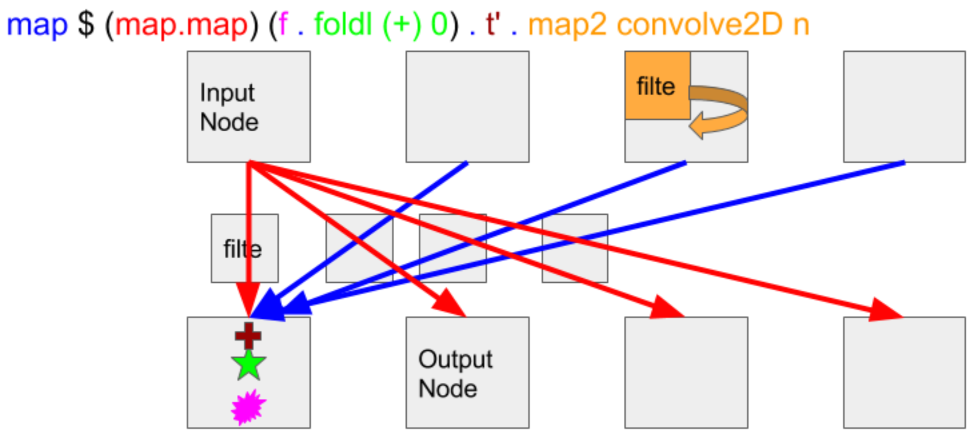


FIGURE 2.7: two dimensional CNN feed-forward

```
filling2D  :: Int -> [[Double]] -> [[Double]]

--actual operation of 2d convolution
convolve2D'  :: [[Double]] -> Kernel2D -> [[Double]]
convolve2D'' :: [[Double]] -> Kernel2D ->  [Double]

convolve2D d k = convolve2D' dd k
--for 2d convolution, boundry compensation have to be done
--twice for rows and columns.
  where s  = (length k) `div` 2
        dd = (boundry2D s s . boundry2D s s) d


boundry2D hs ts a = dt
--boundry2d is identical to boundry 1d except input list is 2d
  where r  = (map (reverse) . filling2D ts . map (reverse)) a
        dt = (t' . filling2D hs) r


filling2D s a = b
--filling1d is identical to filling 1d except input list is 2d
  where rh = map (\x -> filling' s (head x)) a
        b  = map2 (++) rh a

--according to the kernel size, you fill up nearest values as window.
--filling' 3 [2,3,4,5,6] -> [2,2,2]
filling' 0 _ = []
filling' i a = a : filling' (i-1) a

--this operation is to iterate different rows
convolve2D' d k
--if number of rest element is less than kernel size, stop
  | lk > ld    = []
--apply column convolution to the each list and append them
  | otherwise = (convolve2D'' d k) : (convolve2D' (tail d) k)
  where lk = length k
        ld = length d

--this operation is to iterate different columns
```

```
convolve2D'' d k
--if number of rest element is less than kernel size, stop
  | lk > ld    = []
--actual convolution operation.
--two dimensional list is concatenated to 1d and then,
--elementwise multiplication is done.
--finally, sums up everything.
  | otherwise = foldl (+) 0 ( map2 (*) (concat k)
  (concatMap (take' lk) d) ) :(convolve2D'' (map (tail) d) k)
  where lk = length $ head k
        ld = length $ head d
```

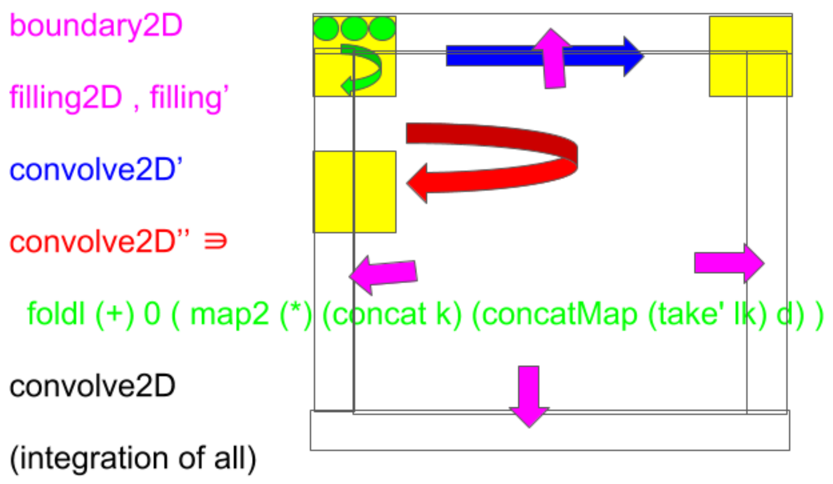


FIGURE 2.8: Two dimensional convolution (boundary compensation)

2.3.6 Approaches for accerelation of convolution calculation

For spatial convolution, NVIDIA provided cuDNN library in 2014 which makes a convolution operations much faster and keeps less memory during execution. The main ideas on it are presented in a paper [18] which are lowering and tiling.

Lowering is converting 2 dimensional filter to 1 dimension, and replace a convolution computation to matrix multiplication which is often more optimized in terms of its speed. Since numbers of feature maps are much larger than kernel size, letting the feature map index bottom of the multi-dimensional array will decrease random access to a memory. In that sense, element-wise summation of convolution turns out to be single MAC (multiply and accumulation) operation. Visually, input nodes and kernels are regarded as 3D volumes, where element wise multiplication is done sliding 3D kernels on 3D nodes.

Tiling is dividing 3D input nodes to smaller scales so that each nodes can be processed in parallel. For instance, input-nodes which contains all of feature map in a layer tends to be 4byte(float) $\times resolution \times numbersofchannelswhichisthesizethatshouldbeprocessedinparallel.ThesizeoftileshouldbedependentonGPUmemorytoopt$

cuDNN is constantly very fast approach even batch size is more or less small, but if batch size is bigger there is another approach which is faster than cuDNN: proposed as maxDNN [11]. maxDNN set bottom dimensions of a multi-dimensional array as batch-size assuming batch-size is larger than input feature numbers. On top of it, their method is written as assembly on CUDA which is called SGEMM to compute matrix multiplication faster.

There are another approaches which is not simulating precise calculations of convolutions, but by and large show equivalent performance towards original implementations. One is the method which make use of similar idea of fast FFT [2] but much faster than computing FFT itself. Another is letting weight matrix low rank [15].

However, these methods are assumed weights are expressed as 32bit floating point number.

2.3.7 Another approach of convolution

2.3.8 Activation function

Activation function for one dimensional case can be written as

$$c_{djn} = \phi b_{djn} \tag{2.6}$$

Note that activation function is applied after the summation of convolved values which is exactly identical to the case for multi-layer perceptron.

Most often used activation function for CNN is rectified linear unit (Relu) which is defined as follows.

$$Relu(x) = 0 \;\; (x \leq 0)$$

$$Relu(x) = x \;\; (x > 0) \tag{2.7}$$

Note that this activation function does not have any squashing output values under "1" unlike sigmoid or hyperbolic tangent.
The good feature of Relu is explained in the section of its derivative which comes later on.

2.3.9 Pooling layer

Convolution step and activation nonlinear function can be followed by pooling layer more than often not. Pooling layer is defined by following. There are some way to calculate pooling. Most frequently used pooling is max pooling, which is defined as following.

$$d_p = \max\left(c_{(lp+s)}\right) \tag{2.8}$$

where $s \in [0,\, l]$
p output dimension
l reduction scale

The reason that max pooling is an indispensable tool for convolutional neural network is acquiring different scaling of the data.

```
--argument
  --1st : node
  --return : node
  maxPooling1D  :: Node1D -> Node1D

  --argument
  --1st : pooling size (default 2)
  --2nd : node
  --return : node
  maxPooling1D' :: Int -> Node1D -> Node1D

  --same with 1D
  maxPooling2D  :: Node2D -> Node2D

  --same with 1D
  maxPooling2D' :: Int -> Node2D -> Node2D

  --assume pooling size is 2
  maxPooling1D = maxPooling1D' 2

  --pooling is done by following.
  --first group list with every N elements such as (N==2) , [2,3,4,5]-> [[2,3],[4,5]]
  --then, calculating max value among them
  maxPooling1D' s = map (map (foldr max 0) . group s)

  --assume pooling size is 2
  maxPooling2D = maxPooling2D' 2

  --only difference between 1D and 2D pooling is again two.
  --One is grouping by 2D and another is covering over it with one more "map"
  maxPooling2D' s = map (map (map (foldr max 0)) . group2D s)
```

After successive operation of multiple convolution layer and pooling layer, there is a case that full connected layer is stack in the end of the network. Formalization of this process is omitted since it is identical to the process of normal multi layer perceptron forward process. Its necessity and role is suppose to be discussed in further chapter.

2.4 Calculation of error derivatives

2.4.1 Difference between multilayer perceptron and back-propagation

What is different between normal multilayer perceptron and CNN is dimension of its weight. The former has 2 dimensions at most. Latter has either 3 (in case for 1 dimensional CNN) or 4 (in case for 2 dimensional CNN). Why is the dimension of CNN incremented ? That is because CNN sets a constraint towards connection of different dimension, on the other hand, normal multilayer perceptron does not have any constraints about inter-connectivity between different dimension.

In the chapter 1, objective function of BP for normal multilayer perceptron was described as following.

$$\frac{\partial E_d}{\partial w_{lij}} \tag{2.9}$$

where l the number of layer
i the index of hidden(input) neurons

j the index of output(hidden) neurons

Practically, weights are two dimensional if you separate them layer by layer,
Objective function of BP for 1D convolutional neural network is

$$\frac{\partial E_d}{\partial w_{lijn}} \tag{2.10}$$

where l = the number of layer
i = the index on hidden(input) neurons
j = the index on output(hidden) neurons
n = the index on the kernel

2.4.2 Error derivatives of element-wise summation after convolution

As I explained in the feed-forward process, there are 4 types of operation in CNN, convolution followed by summation, activation function, pooling, and fully connected inner product operation. I will explain those of three except fully connected inner product which is showed in the last chapter.

Convolution was written as following.

$$b_{djn} = \sum_{i=0}^{I-1}\sum_{s=0}^{S-1} a_{di(s+n)} w_{ijs} \tag{2.3}$$

Derivative of element-wise summation after convolution can be written as following.

$$\frac{\partial b_d}{\partial a_{di(s+n)}} = \sum_{i=0}^{I-1}\sum_{s=0}^{S-1} b_{dj(s+n)} w_{ijs} \tag{2.11}$$

Which means, to put it simply, derivative of an element-wise summation after convolution is an element-wise summation of convolution.
With more details, product between input node and weights have replace to the product between output node and weights. The reason that derivative of convolution is going to be so simple and symmetric is the fact that convolution operation is composed by linear combination of weights and neurons of forward layer as well as inner product. That makes sense that its derivative is going to be identical to feed-forward layer.

2.4.3 Derivatives of activation function

Derivatives of activation function is up to adopted nonlinear function.

As of most frequently used activation function, rectified linear unit, its derivative can be as following.

$$\frac{dRelu(x)}{dx} = \begin{cases} 0 & (x \le 0) \\ 1 & (x > 0) \end{cases} \tag{2.12}$$

Which means, there is no error propagation if an input value is negative. If an input value is positive, just let it pass through after multiplying with value of firing of nodes.

Applying Relu overcomes vanishing gradient problems in comparison with sigmoid function because it just let input pass through when the derivative is positive.

2.4.4 Derivatives of pooling

Derivative of pooling derives from an adopted pooling operation. If it is max-pooling, it is described as following.

$$\frac{\partial Pooling}{\partial x_{(li+s)(lj+t)}} = \begin{cases} \frac{\partial Pooling}{\partial a_{ij}} & if\ a_{ij} = x_{(li+s)(lj+t)} \\ 0 & otherwise \end{cases} \tag{2.13}$$

That means, simply you propagate errors to the node which only has max value and neglect rest of it.

2.4.5 Distribution of overall error derivatives with respect to each errors

To sum them up, what is going to be the distribution of errors to the each weights on different layers and layers ?

The overall scheme of back propagation is similar to the propagation of multilayer perceptron. From chapter 1, assignment of overall errors was written as following.

$$\frac{\partial E_d}{\partial w_{1ij}} = \frac{\partial E_d}{\partial n_{d2k}} \frac{\partial n_{d2k}}{\partial y_{d2k}} \frac{\partial y_{d2k}}{\partial n_{d1j}} \frac{\partial n_{d1j}}{\partial y_{d1j}} \frac{\partial y_{d1j}}{\partial w_{1ij}} \tag{2.14}$$

Given that, let us assume we have simple CNN whose composition operations are, in order, convolution with element-wise summation, activation function, pooling, fully-connected layer. Distribution of error to a weight on the first layer can be written as following.

$$\frac{\partial E_d}{\partial w_{1ijs}} = \underline{\frac{\partial E_d}{\partial y_{df}} \frac{\partial y_{df}}{\partial p_{dfs}} \frac{\partial p_{dfs}}{\partial a_{dfs}} \frac{\partial a_{dfs}}{\partial c_{dfs}}} \frac{\partial c_{dfs}}{\partial w} \tag{2.15}$$

where y = Node just before fully connected layer
p = Node just before pooling layer
a = Node just before passing through activation function
c = Node just before convolution with element-wise summation
f = Index of features
s = Index on a kernel
d = Index on data-set

Derivatives except rightest one of (2.15) is clear. The Leftest term is derived from chapter (1). The Second left one which is derivative of pooling operation is derived from (2.14). The Third left one which is derivatives of activation function is derived from (2.12). The Fourth left one which is derivatives of summation after convolution is derived from (2.11). Actual calculation depends on adopted activation function and way of pooling. But generally can be written as following.

$$\frac{\partial E_d}{\partial c_{dfs}} = {\phi'}_2 \sum_{i=0}^{I-1} \sum_{s=0}^{S-1} b_{dj(s+n)} w_{ijs} {\phi'}_1 \sum_{i=0}^{I-1} \sum_{s=0}^{S-1} b_{dj(s+n)} w_{ijs} \tag{2.16}$$

To summarise them, the operation which is marked as the underline is regarded as almost identical operation of feed-forward process . Only difference of them is activation function is refer to the values of nodes on feed-forward process not the back-propagated error. That means, error back-propagation itself is again linear, and negative errors can be flowed as opposed to feed-forward process.

What is still not clear is operation to update weights after propagating errors to the node.

This can be derived from following.

$$\frac{\partial c_d}{\partial w_{lijn}} = a_{di(s+n)} \cdot c_{dj(s+n)} \tag{2.17}$$

where i = index on input features to connect this weight
j = index on features of propagated errors to connect this weight
n = index on dimension

The result is almost identical with updates for multilayer perceptron. This is because inner product and convolution is both linear combination of each weight. Difference between them is for multi-layer perceptron, a weight is used only one time, but for CNN weights are multiplied with multiple values on different dimensions. In either case, updates function can be written as inner product between previous values of nodes and propagated errors in the next layer.

As we have realized in the calculation of error derivatives with regards to each weights, this process can be divided into two operations, which are error propagation and updating weights after propagation. Of course, you can calculate error derivatives integrating both operation. However, calculating error derivative for each weights individually is not computationally efficient because computation for error propagation is always identical over calculations for error derivatives towards different weights.

Therefore, error propagation step and updating weights should be distinct, and latter step should wait until you get the propagated values for a weight.

2.4.6 Code representation of error propagation

Error propagation code can be written as following.

```
--argument
  --1st : derivative of activation function
  --2nd : Node on previous layer
  --3rd : Error on subsequent layer
  --4th : weights in between
  backPropagateError1D :: (Double->Double) -> Node1D
  -> Error1D -> Weight1D -> Error1D

  --same with 1D
  backPropagateError2D :: (Double->Double) -> Node2D
  -> Error2D -> Weight2D -> Error2D

  --back propagation step of CNN is very similar to forward step.
  --however, there is a difference in terms of multiplying activation function.
  --In backward process, input for activation function is previous values of
    nodes not the propagated error.
```

```
--that is why (*df xx)) is used.
--Then, propagated error is multiplied by it
backPropagateError1D df n =
   map2 (\x -> map2 (\xx -> (*(df xx)) . foldl (+) 0)
   x . t' . map2 (convolve1D) n)


--same with 1D except one more map2 and convolution by 2D
backPropagateError2D df n =
  map2 (\x -> map2 (\xx -> map2 (\xxx -> (*(df xxx)) . foldl (+) 0) xx )
  x . t' . map2 (convolve2D) n)
```

Propagation of errors is element-wise summation after convolution which is exactly same of forward process. There is another way of calculation which have been introduced in the section for inner product based Feed-forward. Of course, it can be applied to error-propagation step. I will consider the meaning in a following section.

2.4.7 Code representation of updating weights

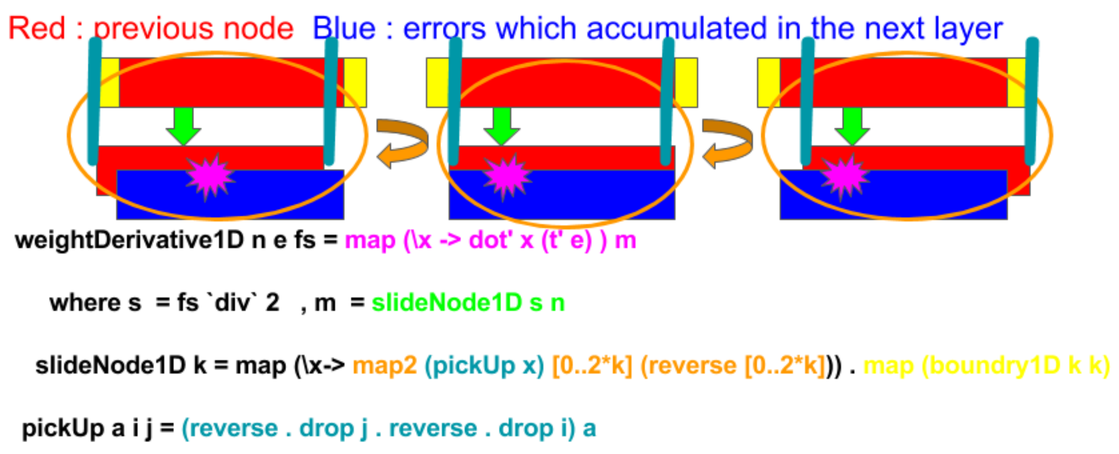
From (2.17), updating process can be written as inner product between input nodes and output nodes. But, we have to note that in convolution process, same weights are used as many times as the input nodes dimension. Therefore, input nodes does not need to be transposed unlike multilayer perceptron so as to keep its length of column as dimension of data.

In order to compute different weights on a filter respectively, dimension of input nodes and errors on subsequent layer requires to be slided according with the position of weights on a filter. That process is written as sliding input nodes according to kernel size before feeding them into dot product.

As piece of code, it can be realized by following in case for one dimensional CNN, and Figure2.9 represents basic idea of the computation.

```
--argument
--1st : Node on previous layer
--2nd : Error on subsequent layer
--3rd : Kernel size (this is necessary for sliding method which I adopt)
--return : derivative of weight
weightDerivative1D  :: Node1D -> Error1D -> KernelSize -> Weight1D

slideNode1D :: KernelSize -> Node1D -> [Node1D]

-- process to calculate derivatives of weights in each layer
-- weights are updated by values of previous node
-- and accumulated error in the next layer
-- since forward process is convolution, weights needs
-- to be calculated by iterating all of
-- pairs of node and errors. This operation can be done by dot product.
weightDerivative1D n e fs = map (\x -> dot' x (t' e) ) m
    where s  = fs `div` 2
          m  = slideNode1D s n


--first let size of list longer so as to reflect
--kernel size of the convolution.
--then, pick up all of possible combinations of filters
```

```
slideNode1D k = map (\x-> map2 (pickUp x) [0..2*k]
(reverse [0..2*k])) . map (boundry1D k k)

pickUp  :: [a] -> Int -> Int -> [a]

--this function pick up part of list dropping first I element,
--and last j element.
pickUp a i j = (reverse . drop j . reverse . drop i) a
```



```
weightDerivative1D n e fs = map (\x -> dot' x (t' e) ) m

  where s  = fs `div` 2  , m  = slideNode1D s n

 slideNode1D k = map (\x-> map2 (pickUp x) [0..2*k] (reverse [0..2*k])) . map (boundry1D k k)

pickUp a i j = (reverse . drop j . reverse . drop i) a
```

FIGURE 2.9: Updating weights for 1D CNN

For two dimensional CNN, weights are updated as following. Figure2.10 provides visualization of codes, and Figure2.11 describes visual combination of slided input nodes and errors on subsequent layer.

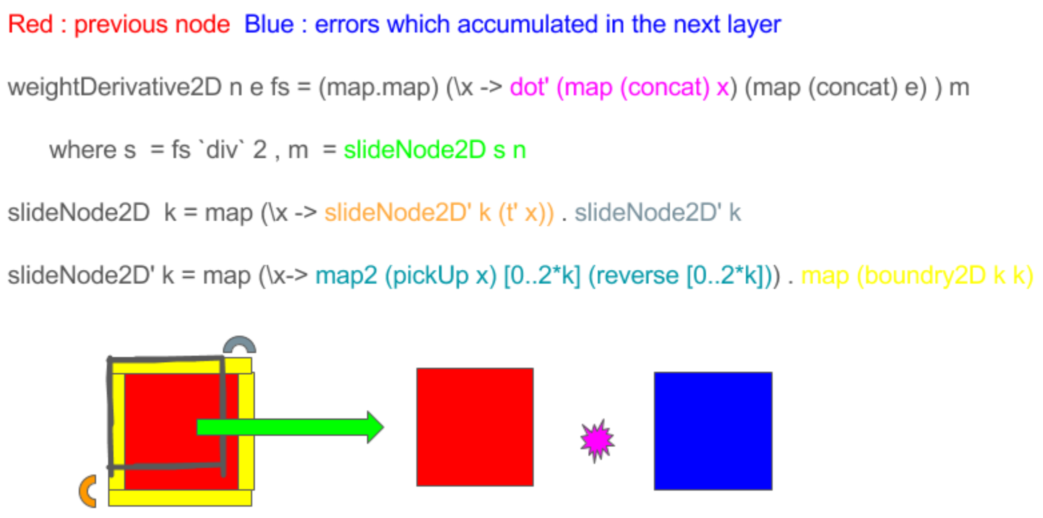
```
--same with 1D
weightDerivative2D  :: Node2D -> Error2D -> KernelSize -> Weight2D

slideNode2D  :: KernelSize -> Node2D -> [[Node2D]]
slideNode2D' :: KernelSize -> Node2D ->  [Node2D]

--basic idea is same with corresponding 1D version,
--note m is going to be five dimension,
--and most inner word dimension should be concatenated
--to be fed into the dot product.
weightDerivative2D n e fs = (map.map)
    (\x -> dot' (map (concat) x) (map (concat) e) ) m
    where s  = fs `div` 2
          m  = slideNode2D s n

--size of twoD node is going to be larger in any directions,
--reflecting convolution kernel,
  --then, cut out all of combinations of the candidate list,
  --for instance, if the kernel size is three,
  --nice rectangular nodes should be cut.
  --this step can be done by applying pickUp function
  --both vertically and horizontally.
  slideNode2D  k = map (\x -> slideNode2D' k (t' x))
      . slideNode2D' k

  --2D version of slideNode1D
  slideNode2D' k = map (\x-> map2 (pickUp x) [0..2*k]
      (reverse [0..2*k])) . map (boundry2D k k)



--argument
  --1st : learning rate
  --2nd : derivative of weight
  --3rd : weight before update
  --4th : updated weight
  updateWeight1D :: LearningRate -> Weight1D
    -> Weight1D -> Weight1D

  --same with 1D
  updateWeight2D :: LearningRate -> Weight2D
    -> Weight2D -> Weight2D


--update weight can be done
  updateWeight1D lr = (map2.map2.map2) (\x y-> x*y*lr)
  updateWeight2D lr = (map2.map2.map2.map2) (\x y-> x*y*lr)
```

2.5 Interpretation of Inner product based feed-forward / feed-back

I have introduced another way of feed-forward and feed-back calculation which I name inner based product feed-forward and feed-back. What is the incentive to consider making use of this method ? In terms of saving computation, it is unlikely to reduce it significantly because computation numbers of multiplication and addition for both methods are same. If it could reduce, it attributes whether it could let the network more parallel.

The biggest pros to consider this computation is that it gives you another representation which is unlike convolution.
By exchanging the order of computation of two summation, each weights are respectively multiplied with every values on the data. For instance, if the input data is image, this operation means the degree which is how much extent the original image would be emphasized or weaken. Then, each emphasized or weaken channel would be integrated together after being slided to certain direction.

Recently, the idea of 1 by 1 convolution is introduced and mixed with 3 by 3 filters from Network in Network [16] . 1 by 1 convolution means, from inner product based feed-forward's point of view, multiplying same weights to the input features and adding up element-wisely. Second addition after sliding each windows are omitted because there is just no addition for convolution. Thereby, 1 by 1 convolution never allows different dimensions in data-set to interact each other, but lets correspond dimensions in each



Red : previous node Blue : errors which accumulated in the next layer

```
weightDerivative2D n e fs = (map.map) (\x -> dot' (map (concat) x) (map (concat) e) ) m

    where s  = fs `div` 2 , m  = slideNode2D s n

slideNode2D  k = map (\x -> slideNode2D' k (t' x)) . slideNode2D' k

slideNode2D' k = map (\x-> map2 (pickUp x) [0..2*k] (reverse [0..2*k])) . map (boundry2D k k)
```

FIGURE 2.10: Updating weights for 2D CNN

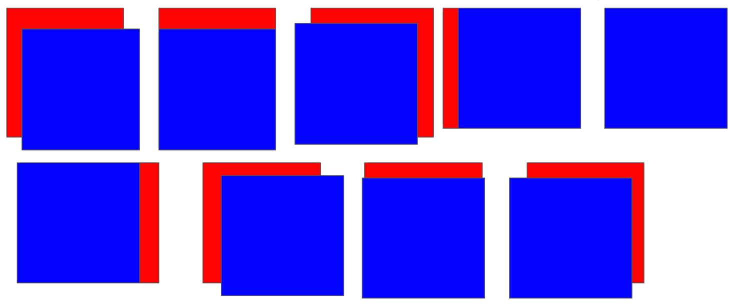


FIGURE 2.11: Updating weights for 2D CNN (all kernel)

features mix with.

Weights values can be negative. If it is negative, it contributes to pull integrated values to negative. If all weights are negative, it does not pass any information to the next layer if the activation function is rectified linear unit. If they are all positive, it neither reduces information nor retrieves features. Therefore, positive and negative weights needs to be mixed in a degree that could capture respective features.

Average or summation of all of weights on a filter lets you estimate how much information is going to be reduced by applying these weights. If average is positive, it is more likely that information quantity is not so much reduced by the filtering. However, negative average does not necessarily attribute all information is rubbed out. Figure2.13 shows convolution after applying certain weights. Titles for each picture denotes elements of non-zero component. For instance, $[0, 0] = -1.0, [1, 1] = 0.8$ means on 3 by 3 kernels, its element are $[[-1, 0, 0], [0, 0.8, 0], [0, 0, 0]]$. Average of weights are negative in case for left-top one, as all weights are $[[-1, 0, 0], [0, 0.8, 0], [0, 0, 0]]$. However, it retrieves some edges. Same things can be applied to different filters.

Also recently, binary neural network which lets values of weights either +1 or -1 has remarkable accuracy of classification in spite of its simplicity. It suggests following. First, in many cases, there are multiple combinations of weights to leave as best performance as the other weights could achieve. Second, pairs of adjacent weights which enables it to maximally retrieve features could be done by the combinations of

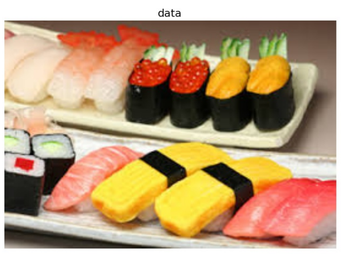


FIGURE 2.12: original data

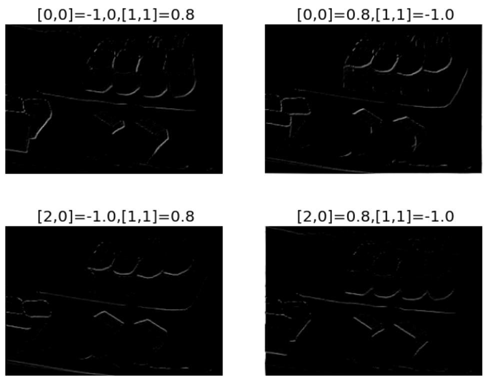


FIGURE 2.13: representation after convolution

weights whose value is either -1 or 1, assuming maximum boundary of absolute weights is 1. For instance, edge detection for image classification, edge is maximally retrieved by 1 by 1 adjacent filters whose values is large and signs are different. In binary network, there is no ways that weights could be "0". It might be better in terms of computation efficiency because weights whose value is "0" has no influence towards different dimensions, meaning that it does not at all help to retrieve features for classification inside of data. If you would let a feature retrieval to the more backward layer, you can set the weight pairs as [+1,+1]. If you would discard them, you could set [-1,-1], which will turn out that activation function can delete its influence.

What can we say about sliding windows for achieving summation of convolution in the last step? It means, dimension in the data is only mixed with some certain local area which is specified by the size of kernel. If CNN is two dimensional and the size of filter is 9, dimensions which are referred each other are the ones aligned with less than one intervals in between. That locality preference does take into account of relationship over more than filter size stacking multiple convolution.

2.6 Computation for binary input and weight with inner product based feed-forward / feed-back

Binary neural network is quite promising approach not only in terms of computation efficiency but understanding of CNN. It sets weights as binary or both inputs and weights binary. Some ways to compute convolution of binary input data with binary weights has been come up with recently. Inner product based feed-forward which I proposed in this chapter allows its computation to turn to binary operation in a pretty elegant way.

Let us assume network size and network weights are both binary, with precisely, input data is represented as "0" or "1", and weights are either "1" or "-1" as a functionality, but "-1" is represented as "0" and "1" as it is so that it can be binary data.

Operation of first multiplication can be covering a bit-mask. However, since weight "-1" turns to "0" which erases result of operation in either input data is "0" or "1", it should be converted to "1" in a way. To realize this, multiplication can be divided into two sub function which one of them is "AND" operation without flipping and another is "AND" operation after flipping operation. By this step, first multiplication step of binary step had been done.

Second, the value in the same dimension but in a different dimension needs to be summed up. This step is done by "Bit count" operation. It just calculates either which bit either "0" or "1" exceeds the other one in terms of its number. If the number of "1" exceeds "0", the overall value turns out to be "1". Otherwise, the overall value become "0".

After above step, all of nodes which is covered by bit-mask in a filter needs to be added by each element after sliding to certain direction. This sliding operation can be done by "bit-shift" operation. After bit-shift operation, again "bit-count" operation should be applied so that it along with summation of bit.

Replacing addition by "bit counting" contains quantization. But, in order to let every information be 1 bit, that should be a most proper way of computation.

To summarize it, overall algorithm can be written as following and which is also written as a diagram on Figure2.14 and Figure2.15.

1. Bit mask operation by weights on each filter.
2. Bit count operation for getting together distribution from different input features.
3. Bit shift operation (to the direction which is determined by weights).
4. Bit count operation for combining distribution from weights on a filter.

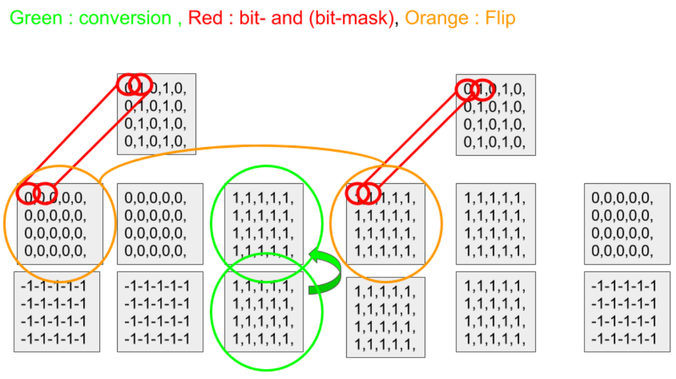

Green : conversion , Red : bit- and (bit-mask), Orange : Flip
0,1,0,1,0,
0,0,0,0,0,
1,1,1,1,1,
-1-1-1-1-1

FIGURE 2.14: binary convolution 1

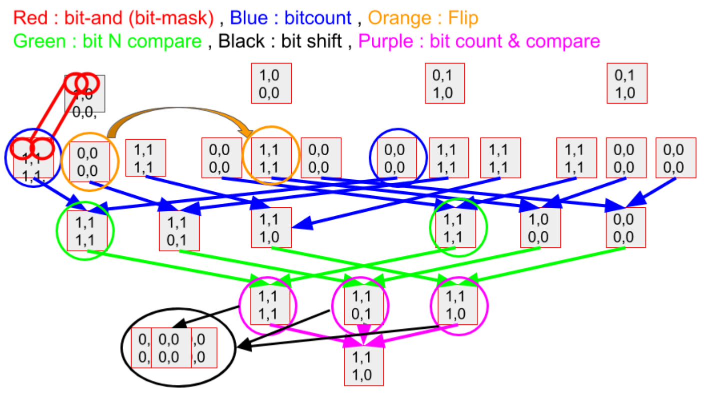

Red : bit-and (bit-mask) , Blue : bitcount , Orange : Flip
Green : bit N compare , Black : bit shift , Purple : bit count & compare
1,0
0,0
0,1
1,0
1,1
1,1
0,0
0,0

FIGURE 2.15: binary convolution 2

Binary neural network is comprehensively explained in the chapter 5.

2.7 Further discussion

In this section, I introduced representation of CNN with Haskell and proposed another way of calculation with it. This calculation is not for computational cost reduction, but analysis of weights for convolution step and calculation for binary input features and weights.

However, still there are very few points to reveal superiority of CNN which I can make after suggesting this way of computation. Then, is there any other approaches to give new perspective or consideration?

In order to investigate further, in the next chapter, I will step back to the basic questions which are, what is dimension of data, what is dimensional reduction, having look at adjacent area, namely compression.

Chapter 3

Compression for dimensional reduction

3.1 Overview

In this chapter, I will discuss what is meant by reducing dimension of data because the process in CNN layer by layer can be regarded as successive reduction of information quantity while retrieving features.

First, I will show the shared points between dimensional reduction and compression.

Second, I will introduce some compression algorithms in a range that they can contribute to explain dimensional reduction which is happening in CNN.

Overall objective of this chapter is to provide general points of view to investigate CNN better, and it is not directly connecting to binary neural networks.

3.2 Dimensional reduction and compression

3.2.1 Data Number and Dimensions

Data for machine learning has two different axis to be analyzed.

One is number of data-set. Another is dimension. If you take both as a vector. Data is represented as a two dimensional matrix where you can analyze on the scheme of linear algebra. If you let each data as a function, you can regard the dimension as terms of a polynomial function, and see its combination as overall sequential equations.

Tasks for machine learning can be categorized with two.

One is reducing data's dimension. Another is categorization (classification) of data set.

Both can be called dimensional reduction. Note that, in this thesis, "dimension" means numbers of properties that a data has.

Categorization of data-set needs dimensional reduction if original data set has more or less redundant dimensions. Formally, categorization requires data's dimension to be linear independent.

If there is a data-set just has 1 dimension, classification is pretty easy by adopting some metrics over data. However, if data has multiple dimensions which are far from being independent each other, descriptive dimension unvails intrinsic features, thereby counting on original data dimensions will deteriorate classifications.

3.2.2 Independent and sequential dimensions

There are two distinct types of data in terms of its dimensions.

First type of data is order of dimension can be arbitrary. It is data which is like user profile information at customer service department in one of companies. The order of column does not affect anything about data itself. The data source is called memoryless source because the information in between data has 0 which means every dimension is independent.

Second type of data is the order of each dimension is indispensable information and cannot be commutative each other. This is data such as natural language, speech,and image data. The source of information has some memory in its context, and thereby called finite or infinite context model. Furthermore, there are two models of information source. One is Markov information source which is often used for data whose type is character. Another is Fourier transformation based which is used for data for double type.

If the data has some context between each dimension, it have to be captured, and redundant dimension should be integrated. In the end, Second type of data which has dependency inside of it should ideally reduce to former types of data which each dimension is independent.

3.2.3 Compression and dimensional reduction

When reducing dimensions of sequential data, there are two distinct case. First case is data is just one. In that case, the way to compress is called compression algorithm. Second case is data is more than 2. In this case, algorithm is categorized as one of machine learning method. First case is independent from individual difference of data and measured by self-information quantity assuming dimension is mutually independent. On the other hand, second case is example dependent, and two data's similarity is measured by mutual information quantity. Note mutual information is also assumed independence of dimensions. On data whose dimension is independent, it is difficult to measure its information quantity and minimum length of code which is deduced from it.

Then, how could we scoop up orders of dimension as well as dimension together in order to find intrinsic feature?
To answer the question, I will devote this chapter for compression algorithm in terms of capturing context between data.

To summarise this section, supervised learning task ; for instance, classification task which is solved by CNN, needs two steps which are

1. sequential information inference (compression)

2. dimensional reduction between multiple data

3.2.4 Dimensional reduction on CNN

It is often said that convolutional layers in CNN are in charge of dimensional reductions for dependent data, and fully connected layer does for classification for data which has independent dimensions.

However, a recent network showed that each process does not need to be stepped sequentially, but can be simultaneously.

SqueezeNet[7] does not have any fully connected layers subsequent to convolutional layers, but it does not lower its classification accuracy. Instead, it has 1 by 1 convolutional layer which processed with 3 by 3 convolutions in parallel. 1 by 1 convolution is firstly introduced in a paper[16] and assumes each dimensions on a data are independent but each corresponding elements on difference channels are dependent. When the data-set are aligned, comparing elements in different data in a same dimension are often introduced such as EigenFace[10]. If it could remove fully connected layer, it means the functions of fully connected layer and the one of 1 by 1 convolution are more or less similar. As operations, dot product could be written that the kernel is as large as input-features, which means both of operations are distinct. However, the fact that fully connected layers may be replaced to 1 by 1 convolutions in a forward layer tells you there is no need to set the interconnections between elements whose distance in between exceeds kernel size in a data.

SqueezeNet is interesting in terms of a network compression as well. Compressed AlexNet for ImageNet whose original data size is 350MB comes down to 470KB which is enough small to be some FPGA.

By the way, weights which are extracted from trained model are also dimensionally dependent. Potentially, method which I will touch in this chapter is useful for compression of weights which was proved the [Deep compression][20]. The idea connects both SqueezeNet[7] and binaryNet[14] very closely.

3.3 Dimensional reduction for independent data

3.3.1 Run length encoding

Given that we have signal data = [0,0,0,1,1,1,1,0,0,0].
how could I reduce dimension, in this case information quantity?

One of the primitive algorithm is run length encoding which just putting same and sequential symbols together.

```haskell
class RunLength where
    runLength :: (Eq a) => [a] -> [(a,Int)]

instance RunLength where
  --runLength h = map (head &&& length) $ group' h
  runLength = map (\x -> (head x,length x)) . group'


take'  :: Int -> [a] -> [a]
take'' :: Int -> [a] -> [a]

drop' :: Int -> [a] -> [a]


--this group needs one argument, let successive symbols integrate.
--e.g.."sssaaaavvvxxx"-> " "sss","aaaa","vvv","xxx" "
group' :: (Eq a) => [a] -> [[a]]

take'' a l = reverse $ foldl (\x y -> if length x < 2 then y : x else x) [] l

drop' 0 a   = a
drop' i (h:t) = drop' (i-1) t

group' [] = []
group' h = (take l h) : group' (drop l h)
    where s = accord' h
          l = length s
```

3.3.2 Huffman coding

However, if the data is not binary but has very few sequential symbols such as [0,1,2,1,2,1,2,1,2,1,2], run length encoding rather increase the size of data. For such data, Huffman coding is more useful.

The concept of Huffman coding is assigning length of bits according to the frequency of symbol. Since frequently appeared symbol will get shorter bits, on the other hand, hardly seen symbol will get longer bits, to sum them up, it is expected that the size of bits after coding is going to be smaller.

For functional language, hash is implemented by a binary tree preventing side effects, and reduce computation resource for search $(O(N^2) - > O(logN))$.

Since data structure which entails encoding information as well can be treated as a binary tree which is biased to either rightest side or leftest side for making them prefix code representing figure3.1, it is well suited with the way for functional programming language to treat hash.

```haskell
import MyLib.LeafTree
import MyLib.TreeHash
```

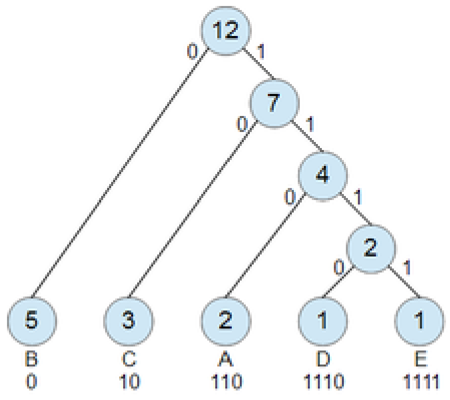


FIGURE 3.1: Huffman Tree

```
class Huffman where

  --this is integration of all code.
  huffman :: (Ord a) => [a] -> (String,[(a,String)])

  --make dictionary for encoding
  makeDict  :: LeafTree k v -> [(k,String)]
  makeDict' :: String -> LeafTree k v -> [(k,String)]

  --1st arugment: signal
  --2nd arugment: tree which has encode information
  encode  :: (Ord k) => [k] -> LeafTree k Int -> String
  encode' :: (Ord k) => String -> LeafTree k v -> k -> String


instance Huffman where

  --huhman coding integration code.
  --1 : cumulativeFromList ( TreeMap )
  --get cumulative frequency given input list in a tree structure,
  --2 : treeToList
  --change the tree into list,
  --3 : insertsVL ( LeafTree )
  --again change to tree but in this time, leaf tree
  -- leaf tree are allowed us to put anything on their intermidiate node.
  --4. create dictionary
  --4. encode given created leaf tree

  huffman l = (encode l a, makeDict a)
    where a = ((insertsVL . treeToList) . cumulativeFromList) l


  makeDict = makeDict' ""
  makeDict' _ None = []
  makeDict' s (LNode (Leaf k v) r) = (k,t) : makeDict' u r
    where t = s ++ "0"
          u = s ++ "1"


  --encode' _ None = []
  encode l tree = foldr (\x y -> (encode' "" tree x) ++ y) "" l
  encode' s (LNode (Leaf k v) r) e
    | e == k    = t
    | otherwise = encode' u r e
    where t = s ++ "0"
          u = s ++ "1"
```

Encoding and making hash table for decoding can be done by traversing the Huffman tree which is represented as figure3.1.
However, since tree as hash and tree for encoding is different in a sense that node except edge of the tree is able to hold information or not, tree as hash is converted list, and then insert again along with the values to tree of figure3.1.

Tree hash, specifically, the inside of function " freqTableFromList " is described as following.

```
data TreeMap k v  = Nil | Node k v (TreeMap k v) (TreeMap k v) deriving (Show)

class TreeHash where

  -- cumurative frequency
  -- iterating list
  cumulativeFromList :: Ord k => [k] -> TreeMap k Int

  --adding elements given one query
  cumulative :: Ord k => k -> TreeMap k Int -> TreeMap k Int

  --tree To list
  treeToList :: TreeMap k v -> [(k, v)]

  --traverse
  --compareL :: TreeMap k v -> (k,v)

instance TreeHash where

  -- input  : list
  -- output : Tree
  freqTableFromList = foldl (\x y -> cumulative y x) Nil

  -- if there are no elements at the point you come to a leaf, add new node
  cumulative a  Nil = Node a 1 Nil Nil
  -- if there are same elements in the list, add +1 on top of its index
  cumulative a (Node k v l r)
    | a == k    = Node k (v+1) l r
    | a < k     = Node k v (cumulative a l) r
    | otherwise = Node k v l (cumulative a r)
    -- search by key, go depth guided by key

  --this is search function
  search _ Nil = Nothing
  search x (Node k v l r)
    | x == k    = Just v
    | x < k     = search x l
    | otherwise = search x r

  --change tree to list
  treeToList tree = traverse tree [] where
    traverse Nil xs = xs
    traverse (Node k v l r) xs = traverse l ((k, v) : traverse r xs)
```

Another tree (Huffman Tree) which has the coding information as the path to the each node is described by following.

```haskell
--LeafTree is only leaves are able to hold pairs of key and value.
--Note LNode has just two argument unlike Node
--which is defined inside of TreeMap
data LeafTree k v = None | Leaf k v
   | LNode (LeafTree k v) (LeafTree k v) deriving (Show)

--This is class for functions which is for LeafTree
class FuncForLeafTree where
  --insert elements sorting by values
  insertsVL :: Ord v => [(k,v)] -> LeafTree k v
  insertVL  :: Ord v => k -> v -> LeafTree k v -> LeafTree k v


instance FuncForLeafTree where

  --this is just looping list of elements
  insertsVL = foldl (\a (k, v) -> insertVL k v a) None

  --this creates left-shifted tree
  insertVL k v None = LNode (Leaf k v) None
  insertVL k v (LNode (Leaf lk lv) r)
    | v > lv    = LNode (Leaf k v) (LNode (Leaf lk lv) r)
    | otherwise = (LNode (Leaf lk lv) (insertVL k v r))
```

Average coding length which is provided by Huffman coding cannot be reduced under self entropy. In other words, minimum coding length is determined by self entropy.

$$H(X) = -\sum_{x \in z\chi} p(x) log_2 p(x) \tag{3.1}$$

However, the compression limitation assumes memoryless source which every single data that come from is independent from each other.
If data-set does not hold independency assumption, there is a case that average coding length can be smaller than entropy in case of memoryless information source.

Nevertheless, the notion is still valuable because many compression algorithm adopts Huffman coding in the last process assuming redundant dimension can be reduced to independent base dimension where one can see its source as a memoryless information source.

3.4 Dimensional reduction for context dependent data

3.4.1 Compression for non-symbolical data (JPEG algorithm)

In the previous section, we took a look at an algorithm for data which is generated by Markov source. However, a problem of this model of source is when the data is represented as numerical values, the difference between multiple values are not going to be taken into consideration. Concretely, every distance between each byte sequence of ASCII code can be treated equal, this is proper. Whereas, difference of pixel values cannot be seen as having equal distance by this model. That is, Markov model assumes simplest unit can be treated as mutually independent, but it is applied only for symbolical data.Then, we need to have another context dependent model for compressing non-symbolical data.

In this section, I will take a look at one example algorithm; JPEG compression algorithm, to consider what means by compressing non-symbolical data.

There are some different ways of encoding, but Most dominantly used JPEG algorithm(JFIF encoding) consists of

1. Color space transformation (RGB to YRB)
2. Down-sampling (Resolution reduction)
3. Block splitting
4. Discrete Cosine transformation
5. Quantization (Pixel values reduction)
6. Jigzag scan
7. Separation of DC / AC component
8. Encoding difference of DC component
9. Zero-length Encoding for DC component
10. (Huffman coding)
11. Conversion to bit sequence

As you can see, it holds many steps but most crucial step is 4. Discrete Cosine transformation. I will have a brief look at each step.

1. Color space transformation

Input image is converted from RGB into a different color space called YCBCR (or, informally, YCbCr). It has three components Y′, C_B and C_R : the Y′ component represents the brightness of a pixel, and the CB and CR components represent the chrominance (split into blue and red components).

```haskell
data Image = RGB [[RGBPixel]] | YRB [[YRBPixel]]

data RGBPixel = RGBPixel
  {
    red   :: Double,
    green :: Double,
    blue  :: Double
  } deriving (Show)


data YRBPixel = YRBPixel
  {
    intensity :: Double,
    hueRed    :: Double,
    hueBlue   :: Double
  } deriving (Show)


  --First step is changing from rgb to yrb
  --image data type has three channels
  rgbToyrb :: Image -> Image

  --pixel has just three double values
  --data type is going to be changed from RGB to YRB
  rgbToyrb':: RGBPixel -> YRBPixel

  --each 2D picture is converted to YRB
  rgbToyrb (RGB a) = YRB (map (map (rgbToyrb')) a)

  --actual operation as a pixel level is done here.
  --this transformation is defined by Jpeg.
  rgbToyrb' a = YRBPixel {intensity = i,hueRed = hr,hueBlue = hb}
     where i  =   0.299 *(red a) + 0.587 *(green a) + 0.114 *(blue a) - 128
           hr = - 0.1687*(red a) - 0.3313*(green a) + 0.5   * (blue a) + 128
           hb =   0.5    *(red a) - 0.4187*(green a) - 0.0813*(blue a) + 128

--each 2D picture is converted to YRB
  rgbToyrb (RGB a) = YRB (map (map (rgbToyrb')) a)

  --actual operation as a pixel level is done here.
  --this transformation is defined by Jpeg.
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     where i  =   0.299 *(red a) + 0.587 *(green a) + 0.114 *(blue a) - 128
           hr = - 0.1687*(red a) - 0.3313*(green a) + 0.5   * (blue a) + 128
           hb =   0.5    *(red a) - 0.4187*(green a) - 0.0813*(blue a) + 128
```

Motivation to convert into YRB is this three channel represent more characteristics of data. The reason that hue of green is eliminated is it can be represented as middle of red and blue.

2. Down-sampling

Due to the densities of color- and brightness-sensitive receptors in the human eye, humans can see considerably more fine detail in the brightness of an image (the Y′ component) than in the hue and color saturation of an image (the Cb and Cr components). Using this knowledge, encoders can be designed to compress images more efficiently.

```
  --downsampling is reducing resolution.
  --Normally ration of resolution -> y(intensity) : hueRed : hueBlue
  --is going to be 4:2:2, but in this case,
  --4:4:4 which is often used indigital camera
  downSampling :: Image -> [[[Double]]]

  --In the process of downsampliing, data type is changed to list of double
  yrbToDouble :: YRBPixel -> [Double]

--scaling down resolution of Hue-red and Hue-green with respect to intensity
  --in this case, just chaging data type from my own "YRB" to Double
  downSampling (YRB a) = map (map (yrbToDouble)) a

  --this is type conversion
  yrbToDouble a = [i,hr,hb]
     where i  = intensity a
           hr = hueRed  a
           hb = hueBlue a
```

In most case, down-sampling is done by pooling resolution of hue red channel and hue blue channel. But, in this code, I will let it be because there are some cases that omit down-sampling.

3. Block splitting

After subsampling, each channel must be split into 88 blocks. Depending on chroma subsampling.

```
  --next step of down-sampling is block splitting
  --which each channel is truncated with 8 by 8
  blockSpliting :: [[[Double]]] -> [[[[Double]]]]

  --block splitting is splitting whole resolutions
  --into 8 by 8 blocks
  blockSpliting a = map (map (group 8)) a
```

Block splitting is better because without splitting, spectrum of coefficients tends to have very steep curve, which makes quantization step difficult.
The size of block ;8 may be derived from statistical experiments and partially because it it even.

4. Discrete cosine transform

Before computing the DCT of the 8 by 8 block, its values are shifted from a positive range to one centered on zero. For an 8-bit image, each entry in the original block falls in the range [0 , 255] [0,255] [0,255]. The midpoint of the range (in this case, the value 128) is subtracted from each entry to produce a data range that is centered on zero, so that the modified range is [128 , 127] [-128,127] [-128,127]. This step reduces the dynamic range requirements in the DCT processing stage that follows.

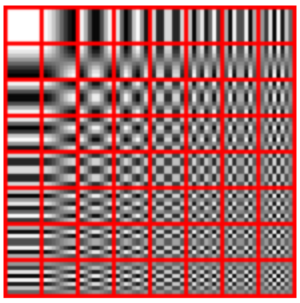


FIGURE 3.2: base representation of DCT

```
class DCT where

  dct1D :: [Double] -> [[Double]]

  dct2D :: [[Double]] -> [[Double]]

  dct2D'  :: [[Double]] -> [[Double]]
  dct2D'' :: [[Double]] -> [[Double]]

  base' :: (Double -> Double) -> Double -> [[Double]]

instance DCT where

  dct1D a = dot' b c
     where b = atLeast2d a
           l = fromIntegral (length a) / 1.0
           c = base' cos l

  dct2D = dct2D'' . dct2D'

  dct2D' a = dot' a c
     where l = fromIntegral (length a) / 1.0
           c = base' cos l


  dct2D'' a  = dot' b c
     where l = fromIntegral (length a) / 1.0
           b = t' a
           c = base' cos l
```

```
--base function generation
--sub function of base generator
base' f a = map (\y -> map (\x -> f ( 2 * pi * x * y / a ))
  [0..b] ) [0..b] where b = a - 1
```

1d and 2d discrete cosine transformation can be written as inner product (dot'), which is also used by neural network. Base function is two dimensional matrix and its coefficients(Figure3.2) are symmetrical. Its details are explained with Fourier transformation in the next chapter.

5. Quantization

The human eye is good at seeing small differences in brightness over a relatively large area, but not so good at distinguishing the exact strength of a high frequency brightness variation. This allows one to greatly reduce the amount of information in the high frequency components. This is done by simply dividing each component in the frequency domain by a constant for that component, and then rounding to the nearest integer. This rounding operation is the only lossy operation in the whole process (other than chroma subsampling) if the DCT computation is performed with sufficiently high precision. As a result of this, it is typically the case that many of the higher frequency components are rounded to zero, and many of the rest become small positive or negative numbers, which take many fewer bits to represent.

```
--next step after dct is quantization in each pixel.
--As a matter of fact, its process is just
--divided by values of quantization tables
--in a floating manner.
--values of elements on quantization table is set by Jpeg standard.

quantization :: [[Double]] -> [[Double]] -> [[Int]]
quantizationTable :: [[Double]]

--quantization is dividing by values of quantization tables and converting integer.
quantization = map2 (map2 (((round .) . (/)) ))

--quantization is defined by Jpeg standard
quantizationTable =
  [
    [16,11,10,16,24,40,51,61],
    [12,12,14,19,26,58,60,55],
    [14,13,16,24,40,57,60,56],
    [14,17,22,29,51,87,80,62],
    [18,22,37,56,68,109,103,77],
    [24,35,55,64,81,104,113,92],
    [49,64,78,87,103,121,120,101],
    [72,92,95,98,112,100,103,99]
  ]
```

The value of quantization table might also be derived from accumulated experiments to consider both information loss in this process and reduction of information quantity in the end.

6. Jigzag scan

This step is represented as figure3.3.

```
--next step after quantization is jigzag scan
--which traverse two dimensional tables in a jigzag way.
--It will let 2D tables 1D
jigzag  :: [[Int]] -> [Int]
```

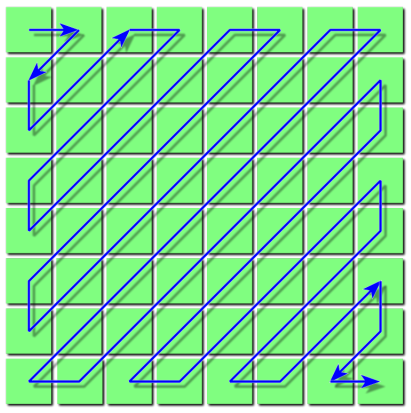


FIGURE 3.3: Jig-zag scan

```
jigzag' :: Int -> Int -> [[Int]] -> [Int]

--two dimensional blocks are converted into 1d 64 elements list
jigzag a = jigzag' ((length a)*2) 0 a

--jigzag process has 3 arugments
--argument is (depth of jigzag(8by8->16),current depth,2D list).
jigzag' m a b
-- current depth is the end of jigzag, prepare empty list
  | m == a    = []
-- if the current depth is even, append the first element
--of extracted list as usual
  | even a    = map (head) c ++ r
-- if the current depth is odd, append as reversed
--to let the traversal jigzag
  | otherwise = (reverse $ map (head) c) ++ r
-- after half step of jigzag, it has empty list which needs to be excluded
  where b' = (zip [0..] . filter (/=[])) b
-- according to the depth, lists are extracted
        c = (snd . unzip . filter (\(x,y)->x<a))  b'
-- none-extracted lists is going to brought as next arguments
        d = (snd . unzip . filter (\(x,y)->x>=a))  b'
        r = jigzag' m (a+1) ((map (tail) c) ++ d)
```

This scan aims to make as many alignment of zero as possible. This is because, diagonally neighboring elements has alike coefficients.

7. Separation of DC / AC

Direct component represents overall brightness of image and is represented as an independent coefficients from others. which is explained in also next chapter.

Alternated component represent how many split patterns is hidden of this image.

Since direct component tends to have by far bigger coefficients.

```
--next step is separating DC component and AC component.
  -- in 64 elements of block, a first value denotes DC component.
  -- and rest of them is AC component
  dcComponent :: [[[Int]]] ->  [[Int]]
  acComponent :: [[[Int]]] -> [[[Int]]]

--after jigzag scan, DC & AC component is divided.
```

```
dcComponent = map (map (head))
acComponent = map (map (tail))
```

8. Encoding difference of direct component

Since direct component is average intensity of its block, neighboring direct component might have similar values. To decode correctly, first direct component have to be encoded as it is.

```
--For DC component, the difference of previous values
--over multiple blocks are stored,
--except values of first block in each channel.
takeDifference  :: [Int] ->  [Int]
takeDifference' :: [Int] ->  [Int]

--DC component is encoded by the value of its difference.
takeDifference (h:t) = h : takeDifference' t
takeDifference' [h]    = []
takeDifference' (h:t) = h - (head t) : takeDifference' t
```

9. Zero-length Encoding for Alternated component

Elements of alternated component after jig-zag scan is expected to have many zero especially in the end of list which means high frequency coefficients. To reduce information maximally, each element of zero should not be coded as it is. Plus, to decode correctly, each length of running code, and element itself have to have fixed code length. However, coefficients in frequency domain are likely to have very big values in some cases even after quantization step. Thereby, JFIF standard lets numbers of bit to represent each coefficient variable code length.

Concretely, zero length yield following data format.

–[((A,B),C)]
–A = number of zero before this value (4 bit fixed)
–B = minimum bits enough to express coefficient value (4 bit fixed)
–C = its value

```
--For AC component, things are more complicated..
--All of non-zero code will be coded.
--[((A,B),C)]
--A = number of zero before this value (4 bit fixed)
--B = minimum bits enough to express coefficient value (4 bit fixed)
--C = its value
zeroLength  :: [Int] -> [((Int,Int),Int)]
zeroLength' :: [Int] -> Int -> [((Int,Int),Int)]

--AC component is only non-zero component is stored unless it has
--sequence more than 15 zero.

zeroLength a = zeroLength' a 0

--
zeroLength' [] _ = []
--1st argument is encoded list, second argument is variables
--to store previous successive number of 0
zeroLength' (h:t) s
--if it comes zero, you will jump it storing number of zero
  | h == 0    = zeroLength' t (s+1)
--if it comes non zero, you will store it with numbers of
```

```
  --minimum bits for this value
    | otherwise = ((s,bit+1),h) : zeroLength' t 0
  --minimum bits is calculated as following.
    where bit = round $ logBase 2 $ fromIntegral (h::Int)
```

10. (Huffman coding)

Normally, after doing zero length encoding, Huffman coding follows. But, I will omit this code since I described it already.

11. Conversion to bit sequence

Last step is letting sequence of integers to sequence of bits.

```
 --integer is going to be transformed into bits
 --in this case sequence of char type
  --first argument is expected length of bits
  bitsTransform  :: Int -> Int -> String
  bitsTransform' :: Int -> String

  --this is sub component of bits transform
  zeroPrepare :: Int -> String

  --this is a tool for feeding AC component
  --into bitsTransform function.
  acBits :: ((Int,Int),Int) -> String

  --integer turns into bits but as a string format
  --1st argument is length of bits showing discrimination for decoding
  bitsTransform a b = (zeroPrepare (a-l)) ++ c
    where c = (reverse . bitsTransform') b
          l = length c

  --int -> bit are done by dividing by 2 repeatedly.
  --if it comes zero, it will be ended
  bitsTransform' 0 = []
  bitsTransform' b
  --if mode b 2 == 0, then, encode as 0
    | mod b 2 == 0 = "0" ++ (bitsTransform' (b `div` 2))
  --if mode b 2 != 0, then, encode as 1
    | otherwise    = "1" ++ (bitsTransform' (b `div` 2))

  --if there is more space to be filled in, insert "0"
  zeroPrepare 0 = ""
  zeroPrepare a = "0" ++ (zeroPrepare (a-1))

  --AC component is fed into transform and concatenate.
  acBits ((a,b),c) = (bitsTransform 4 a) ++
    (bitsTransform 4 b) ++ (bitsTransform b c)
```

To summerise every step, overall pipe line of Jpeg encoder can be written as following.

```
 --this jpegEncoder is all integration of functions listed below.
  --it will change from image to byte string
  --(for visualization of final representation, output is char)
  jpegEncoder :: Image -> String
```

```haskell
--this is all of process when image is encoded as .jpg or .jpeg
jpegEncoder a = dc ++ ac
   where b  = map (map jigzag)
   $ map (map (quantization quantizationTable)) $ map (map dct2D)
   $ blockSpliting $ downSampling $ rgbToyrb a
         dc = concatMap (concatMap (bitsTransform 8))
              $ map (takeDifference) $ dcComponent b
         ac = concatMap (concatMap (concatMap acBits))
              $ map (map zeroLength) $ acComponent b
```

3.4.2 Consideration towards Jpeg encoding algorithm

Jpeg is known as one of the best compression algorithms which compress image data especially the ones which has general object, and essential idea is lying on discrete Fourier transform.

The underlying idea of DCT is treating data as a function, and the function can be represented as a combination of simple base functions like triangle functions. This approach is widely used for compression algorithm not only for image but sound because these data can be in some case better made sense in a decomposed representation by human.

For instance, human can easily make difference between sound of white noise and sound of pink noise distinct. However, analyzing both waves in real space given entropy provided by memoryless information source model and Markov information source model does not makes sense at all. By contrast, the difference between white noise and pink noise is obvious if you compute both frequency spectrum.

Note here that I do not claim that data which is transformed in frequency have always less information quantity. As a matter of fact, there are as many number of waves that increase information quantity in a frequency space as the ones which is decreasing. For instance, in case for 2d waves, images of binary character or digits are more compressed in a way which is operated not in frequency space such as PNG algorithm , with respect to Jpeg algorithm.

What I claim is, statistically speaking, data of sounds and general objects have less information quantity in frequency domain. This can be translated into that human cognition system is more likely to make sense of them if the data is well represented as combination of base function not as it is. This is because state-of-art compression algorithm consider human cognition process.

By the way, this should be distinct with arguments about if information is lost or not after encoding information by some algorithm. There is a remark that lossy compression cut off cognitively less awared information, to compress data. It is true for explaining about sampling and quantization process. But, what is more important is data itself such as image and sound which human think those are useful to be saved originally exhibit more structures in frequency space regardless of application of compression algorithm.

Contrast to minimum coding length given by entropy assuming infinite information source, compression limitation for non-symbolical data is quite difficult to be set. Being difficult in the sense is not brought by lossy compression, namely a discussion about how much extent human does not aware reduced information from original, but fundamentally, non-uniqueness of the way of capturing context between each dimension makes its estimation hard. Compression algorithm is applied to many different data in a same way. However, there are N different cases to reduce N different data.For instance, sine or cosine wave has least information given MPC algorithm, but square wave might have more information quantity by discrete cosine transform rather than applying wavelet transform.

3.4.3 Summery

In this chapter, we took a look at some compression algorithms to analyze dimensional reduction. Compression and dimensional reduction are not often discussed together. However, if the data-set is going to be bigger and get generality, its purpose is close to each other. The reason is following. Compression algorithm are required to have sort of generality which should have high compression rate in any data. By contrast, dimensional reduction can be very specific if the data-set is small and has small variance. But, if data-set is bigger with variety enough to represent generality of its groups, feature extraction for dimensional reduction approaches compression. Conversely, dimensional reduction may be regarded as compression for specific training data. But, most of dimensional reduction algorithm such as PCA is not assured reversibility to the original input from compressed figure unless possible all of error trajectories are preserved and back traced.

Chapter 4

Fourier analysis of CNN

4.1 Prelude

In the last chapter, we came to a conclusion that in order to compress data such as natural image, we are able to find more characteristics of the data in frequency domain rather than space domain, thereby it enables us to reduce its information quantity. This means hidden dimension between each elements of one data can be revealed better into the frequency domain in some cases.

In this chapter, we will take a look at representation of weights by Fourier transformation after introducing basic knowledge. But, why is Fourier transformation useful to understand the filters? This is because it gives us a new representation of convolution operation from different point of view which I showed in the chapter 2. The idea is based on convolutional theory which states that convolution operation between two functions can be element-wise multiplication of two functions after Fourier transformation.

Analysis in this chapter will succeed to the next chapters.

4.2 Fourier Transformation

Fourier transformation is described as following.

$$F(k) \; = \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-ikx} x \tag{4.1}$$

where x = original index defined such as time / space.
f = frequency
f(x) = original function, data-set
F(k) = signal component in frequency domain

Inverse Fourier transformation is defined by following.

$$f(x) \; = \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \, e^{ikx} k \tag{4.2}$$

Discrete Fourier transform is defined by following.

$$F(k) \; = \; \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} f(x) \, e^{-ikx} x \tag{4.3}$$

Code of one dimensional discrete Fourier transformation in Haskell is following.

```
import numpy as np

def dft_1d (self,start, x, N):
    "1d dft"
    X = [0.0] * N
    for k in range(N):
        "iterate different base function"
        for n in range(N):
            "itereate data itself with same base function "
            real = np.cos(2 * np.pi * k * n / N)
            imag = - np.sin(2 * np.pi * k * n / N)
            X[k] += x[start + n] * complex(real, imag)

    X = np.array(X)
    return X
```

In case for one dimensional Fourier transformation, base function, namely e^{-ikx} in (4.3) , or " complex(real,imag) " in python code is 2 dimensional array.

For instance, if you put data-set whose length is 4, its base would be

$$\begin{pmatrix} \cos(0)-i\sin(0) & \cos(0)-i\sin(0) & \cos(0)-i\sin(0) & \cos(0)-i\sin(0) \\ \cos(0)-i\sin(0) & \cos(\frac{1}{4}(2\pi))-i\sin(\frac{1}{4}(2\pi)) & \cos(\frac{2}{4}(2\pi))-i\sin(\frac{2}{4}(2\pi)) & \cos(\frac{3}{4}(2\pi))-i\sin(\frac{3}{4}(2\pi)) \\ \cos(0)-i\sin(0) & \cos(\frac{2}{4}(2\pi))-i\sin(\frac{2}{4}(2\pi)) & \cos(\frac{4}{4}(2\pi))-i\sin(\frac{4}{4}(2\pi)) & \cos(\frac{6}{4}(2\pi))-i\sin(\frac{6}{4}(2\pi)) \\ \cos(0)-i\sin(0) & \cos(\frac{3}{4}(2\pi))-i\sin(\frac{3}{4}(2\pi)) & \cos(\frac{6}{4}(2\pi))-i\sin(\frac{6}{4}(2\pi)) & \cos(\frac{9}{4}(2\pi))-i\sin(\frac{9}{4}(2\pi)) \end{pmatrix} \tag{4.4}$$

To put it simply, Fourier transformation would be inner product between this base matrix and input function. If you consider about that, these codes can be written in Haskell as following.

```haskell
import MyLib.MyComplex

--Fourier transformation
--Input   : transformed function can be defined by real number
--Output  : two dimensional complex number
-- (why two dimensional? Answer : matrix can be more easily treated than vector.)
transform :: [Double] -> [[Complex']]

--Fourier transformation
--Fourier transformation is inner product of complex number

--3. Compute inner dot between input and base function
transform a = iDot b c
    -- Let input complex data type
    where b = (atLeast2d . map (toComplex)) a
    -- get length of input, this is sampling frequency
          l = fromIntegral (length a) / 1.0
          c = (map (map (conjuguate)) . base) l
    --  prepare base function according to sampling frequency
    ----(Do not forget to take conjuguate; sin has negative from its definition.)
```

As you can see, Fourier transformation is defined by "iDot" which is dot product for complex number I defined as following and its argument is data itself, and base function.

function " Base " is defined by following.

```
 --base function preparation
  base  :: Double -> [[Complex']]

  --sub function of base generator
  base' :: (Double -> Double) -> Double -> [[Double]]

  --base function preparation
  base  a  = map (\s -> map (\(x,y) -> Complex' (x,y)) s ) d
    where b = base' cos' a
          c = base' sin a
          d = map2 (zip) b c


  --base function generation
  --sub function of base generator
  base' f a = map (\y -> map (\x -> f ( 2 * pi * x * y / a )) [0..b] )
             [0..b] where b = a - 1
```

And, "iDot" and "Conjuaguate" is defined by following.

```
data Complex' = Complex' (Double,Double) deriving (Show)

class MyComplex where

  --basic operation on complex space
  iAdd :: Complex' -> Complex' -> Complex'
  iSub :: Complex' -> Complex' -> Complex'
  iMul :: Complex' -> Complex' -> Complex'
  iDiv :: Complex' -> Complex' -> Complex'

  --inner product for complex number
  iDot :: [[Complex']] -> [[Complex']] -> [[Complex']]


  --change real to complex
  toComplex  :: Double -> Complex'

  --change real to complex
  toReal  :: Complex' -> Double


  --get conjuguate
  conjuguate :: Complex' -> Complex'



instance MyComplex where

  --basic operation on complex space
  iAdd (Complex' (k1,v1)) (Complex' (k2,v2)) = Complex' (k1+k2,v1+v2)
  iSub (Complex' (k1,v1)) (Complex' (k2,v2)) = Complex' (k1-k2,v1-v2)
  iMul (Complex' (k1,v1)) (Complex' (k2,v2)) = Complex' (k1*k2-v1*v2,k1*v2+k2*v1)
  iDiv (Complex' (k1,v1)) (Complex' (k2,v2)) = Complex' (k1/k2,v1/v2)

  --inner product
  iDot a b = group (length a) ( map ( foldl (iAdd) (Complex' (0,0)) )
     [ (map2 (iMul) x y)  | x <- a, y <- t])
     where t = t' a
```

```
--change real to complex
toComplex a = Complex' (a,0)

--change complex to real( just discard imaginary part)
toReal (Complex'(a,b)) = a

--take conjuguate
conjuguate (Complex'(a,b)) = Complex' (a,-b)
```

Since basic operation of complex number is distinct from operation for real number, I described comprehensive figure.

As you can see the definition of iDot, it is defined by just extension to complex number of dot product which I showed in first chapter.

If you note that Fourier transformation is unitary transformation, inverse Fourier transformation can be written in a way that we defined for Fourier transformation.

```
--Inverse Fourier transformation
--Input  : Coefficients of Fourier Series
--Output : one dimensional real number
invTransform :: [[Complex']] -> [Double]

--inverse Fourier transformation
--inverse Fourier transformation is almost identical with Fourier transformation
--Note

-- # Base do not need to take conjuguate from its definition.
-- # Do not forget multiply devide by length of input in the end

invTransform a = concatMap
    (map (\x -> (toReal . iDiv x) (Complex' (l,1)) )) (iDot b c)
    -- iDot again, but do not forget divide by sampling frequency
    --and let them real and flatten

    -- input needs to be transposed after Fourier transformation.
    where b = t' a
          l = fromIntegral (length a) / 1.0 -- get sampling frequency
          c = base l -- get base but do not conjuguate in this time
```

4.3 Fast Fourier transformation

Base matrix for Fourier transformation has regularity which came from frequency by its definition.

From matrix (1), considering constant frequency, $a = \cos(2n\pi) + a$ and $a = \sin(2n\pi) + a$

$$\begin{pmatrix} \cos(0) - i\sin(0) & \cos(0) - i\sin(0) & \cos(0) - i\sin(0) & \cos(0) - i\sin(0) \\ \cos(0) - i\sin(0) & \cos(\frac{1}{4}(2\pi)) - i\sin(\frac{1}{4}(2\pi)) & \cos(\frac{2}{4}(2\pi)) - i\sin(\frac{2}{4}(2\pi)) & \cos(\frac{3}{4}(2\pi)) - i\sin(\frac{3}{4}(2\pi)) \\ \cos(0) - i\sin(0) & \cos(\frac{2}{4}(2\pi)) - i\sin(\frac{2}{4}(2\pi)) & -\cos(0) + i\sin(0) & -\cos(\frac{2}{4}(2\pi)) + i\sin(\frac{2}{4}(2\pi)) \\ \cos(0) - i\sin(0) & \cos(\frac{3}{4}(2\pi)) - i\sin(\frac{3}{4}(2\pi)) & -\cos(\frac{2}{4}(2\pi)) + i\sin(\frac{2}{4}(2\pi)) & \cos(\frac{1}{4}(2\pi)) - i\sin(\frac{1}{4}(2\pi)) \end{pmatrix}$$

Given

$\cos(0) - i\sin(0) = 1$
$\cos(\frac{1*\pi}{4}) - i\sin(\frac{1*\pi}{4}) = \frac{\sqrt{2}}{2}(1-i)$
$\cos(\frac{2*\pi}{4}) - i\sin(\frac{2*\pi}{4}) = -i$
$\cos(\frac{3*\pi}{4}) - i\sin(\frac{3*\pi}{4}) = -\frac{\sqrt{2}}{2}(1-i)$

Actual value is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{\sqrt{2}}{2}(1-i) & -i & -\frac{\sqrt{2}}{2}(1-i) \\ 1 & -i & -1 & i \\ 1 & -\frac{\sqrt{2}}{2}(1-i) & i & \frac{\sqrt{2}}{2}(1-i) \end{pmatrix}$$

As you can see, base matrix has certain symmetrical structure in it.

if you swap 2nd column and 3rd column, the matrix can be divided into following two.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{2}}{2}(1-i) \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & \frac{\sqrt{2}}{2}(1-i) \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -i \end{pmatrix}$$

Matrix are divided into succession of product of matrix whose size is half. By this operation, the number of multiplication of Fourier transformation is going to be reduced from O(N2) to $N\log(N)$. Specifically, if the Rank of base matrix is 4, times of multiplication decrease from 4×4 to $4\log 4$. Logarithm means succession of matrix inner product application forms tree structure, namely, first, multiply most frequent wavelength, and add less frequent wavelength later on. In this example, right matrix is application of coefficients given frequency is 2, and right matrix is application of coefficients given frequency is 4.

division to two matrix

$$\begin{pmatrix} 1 & 0 & \cos(0) - i\sin(0) & 0 \\ 0 & 1 & 0 & \cos(\frac{1*\pi}{4}) - i\sin(\frac{1*\pi}{4}) \\ 1 & 0 & \cos(\frac{2*\pi}{4}) - i\sin(\frac{2*\pi}{4}) & 0 \\ 0 & 1 & 0 & \cos(\frac{3*\pi}{4}) - i\sin(\frac{3*\pi}{4}) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \cos(0) - i\sin(0) & \cos(0) - i\sin(0) & 0 & 0 \\ \cos(0) - i\sin(0) & \cos(\frac{2*\pi}{4}) - i\sin(\frac{2*\pi}{4}) & 0 & 0 \\ 0 & 0 & \cos(0) - i\sin(0) & \cos(0) - i\sin(0) \\ 0 & 0 & \cos(0) - i\sin(0) & \cos(\frac{2*\pi}{4}) - i\sin(\frac{2*\pi}{4}) \end{pmatrix}$$

4.4 Convolutional theory

This is the formalism when size of convolution is non-discrete and infinite.

$$f*g(x) \equiv \int_{-\infty}^{\infty} f(y)\, g(x-y) y \tag{4.5}$$

Convolution operation is translated as element wise multiplication after Fourier transformation of both equation.

$$\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g] \tag{4.6}$$

Following is the proof.

$$
\begin{aligned}
\mathcal{F}[f*g] &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)\, g(x-y)dy\right)\, e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(x-y)\, e^{-ikx}dx\right) dy \\
&= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(t)\, e^{-ik(t+y)}dt\right) dy \\
&= \int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(t)\, e^{-ikt}dt\right)\, e^{-iky}dy \\
&= \int_{-\infty}^{\infty} f(y)\mathcal{F}[g]\, e^{-iky}dy \\
&= \int_{-\infty}^{\infty} f(y)\, e^{-iky}dy\, \mathcal{F}[g] \\
&= \mathcal{F}[f]\, \mathcal{F}[g]
\end{aligned}
\tag{4.7}
$$

What is done here is
1st : Changing the order over integral of convolution and integral of Fourier transformation
2nd : Letting convolution inner product by transforming variables.
3rd : There you can see multiplication of two inner product.

To put it simply, Fourier transformation can be seen as an inner product between original function and base function.

Inner product between (convolution A and B) and base = multiplication (inner product between input A and base) (inner product between B and base)

This transformation (4.7) is significantly important when you think about difference between convolution and normal inner product.

What should be noted here is Fourier transform after multiplication of two function is equivalent to convolution of both function after Fourier transformation.

$$
\mathcal{F}[f]*\mathcal{F}[g] \;=\; 2\pi\, \mathcal{F}[fg] \tag{4.8}
$$

because

$$\begin{aligned}
\mathcal{F}[fg] &= \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} f(x) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(k')e^{ik'x}k'\right] e^{-ikx}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k') \left[\int_{-\infty}^{\infty} f(x)\, e^{-i(k-k')x}dx\right] dk' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k')\, F(k-k')dk' \\
&= \frac{1}{2\pi}\,(G{*}F)(k) \\
&= \frac{1}{2\pi}\,\mathcal{F}[f]{*}\mathcal{F}[g] \qquad (4.9)
\end{aligned}$$

The definition described here can be applied to combination of discrete Fourier transformation and discrete convolution which we encounter when considering convolutional neural network.

That is, convolution operation in original domain is expressed as multiplication in frequency domain. Whereas, multiplication operation such as passing through activation function is convolution operation in frequency domain.

4.5 Representation of signals after Fourier transformation

From the previous chapter, convolution operation is described as element-wise multiplication between Fourier transformation of data and Fourier transformation of a filter.

Before proceeding to the weight and filter analysis, let me introduce two functions (Delta function and Sinc function) which is useful to analyze weights and filters.

4.5.1 Delta function

When you think about mapping between original space and Fourier space, examples tell you a lot.
For instance, when you put data which has same values regardless of original dimension, how is it represented in Fourier space?
That is the case you have an image which has infinite resolution which is utterly black except in the midst which is white.

Given

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

$$\delta(0) = \infty$$

$$\delta(x) = 0(x \neq 0)$$

$$\int_{-\infty}^{\infty} \delta(x)e^{-ikx}x \;=\; e^{0} \;=\; 1 \qquad (4.10)$$

In the Fourier domain, Fourier transformation of delta function which has all same coefficients in spite of its terms (Figure4.1). That means, it has same values in all of frequency. Inverse Fourier transformation operation of 1 turns out to be δ function, which I omit the proof.

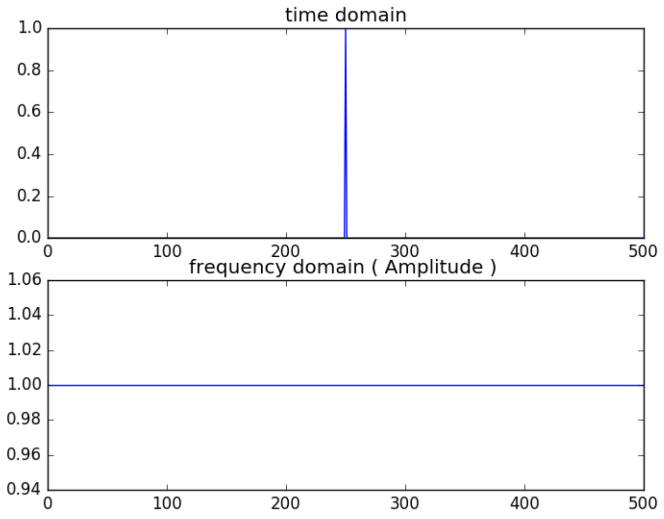


FIGURE 4.1: FFT of $\delta()$

Likewise, considering Fourier transformation pair, Fourier transformation of signal which has same values in all its domain, such as sheer white image, is going to be $\delta()$, and inverse Fourier transformation of $\delta()$ is 1 in frequency domain (Figure4.2).

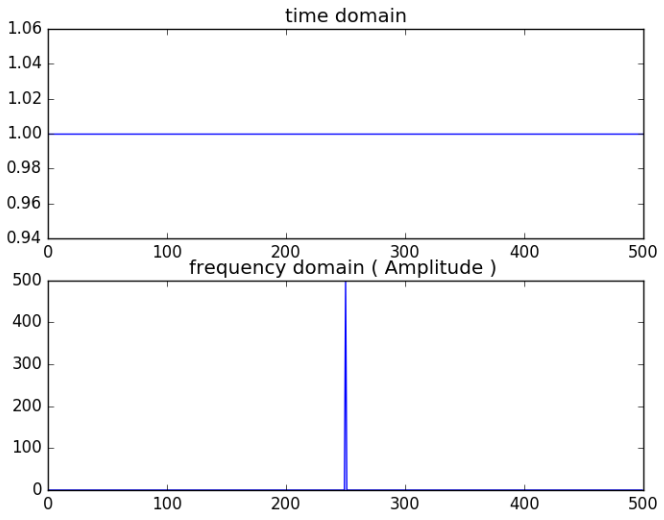


FIGURE 4.2: FFT of $\delta()$

4.5.2 Sinc function

Suppose you let the resolution of previous example finite, what kind of signal is going to be appeared in frequency domain ?

As the image has just two values which is white or black, the image can be represented as combinations of square waves which heads for different directions.

One signal which pass through center of pixel can be described as

$$r_{-a,a}(t) = 1 \; (-a \leq t \leq a)$$

$$r_{-a,a}(t) = 0 \; (t < -a \;||\; a < t) \tag{4.11}$$

which is called rectangular function.

Its Fourier transformation is

$$\begin{aligned} F[r_{-a,a}](t) \quad &= \textstyle\int_{-\infty}^{\infty} r_{-a,a}(t)\mathrm{e}^{-ikt}dt \\ &= \textstyle\int_{-a}^{a} \mathrm{e}^{-ikt}dt \end{aligned} \tag{4.12}$$

Calculating integral of natural logarithm

$$\begin{aligned} F[r_{-a,a}](t) \quad &= \left[\tfrac{1}{-it}\mathrm{e}^{-ikt}\right]_{-a}^{a} \\ &= \tfrac{1}{ik}(\mathrm{e}^{ika} - \mathrm{e}^{-ika}) \end{aligned} \tag{4.13}$$

Making use of $sin(a)k = \frac{\mathrm{e}^{ika} - \mathrm{e}^{-ika}}{2i}$

$$\begin{aligned} F[r_{-a,a}](t) \quad &= \tfrac{2}{k}\tfrac{\mathrm{e}^{ika} - \mathrm{e}^{-ika}}{2i} \\ &= \tfrac{1}{-ik}\sin(a)k \end{aligned} \tag{4.14}$$

(4.14) tells you spectral of Frequency domain of square wave is multiplication of inverse proportion of frequency and sin function.

Note that direct component(k = 0) is not infinite unlike delta function, but

$$\begin{aligned} F[r_{-a,a}](t) \mid (t = 0) \quad &= \textstyle\int_{-a}^{a} \mathrm{e}^{-i0x}dt \\ &= \textstyle\int_{-a}^{a} dt \\ &= 2a \end{aligned} \tag{4.15}$$

Direct component means, for instance in image, average intensity of the pixel. If classification should be robust to general object,the difference of direct component should be out of consideration, in other words, acquire property of intensity invariancy.

Figure4.3 is the signal whose value is 1 in the range from 190 to 210, and rest of them is just 0.

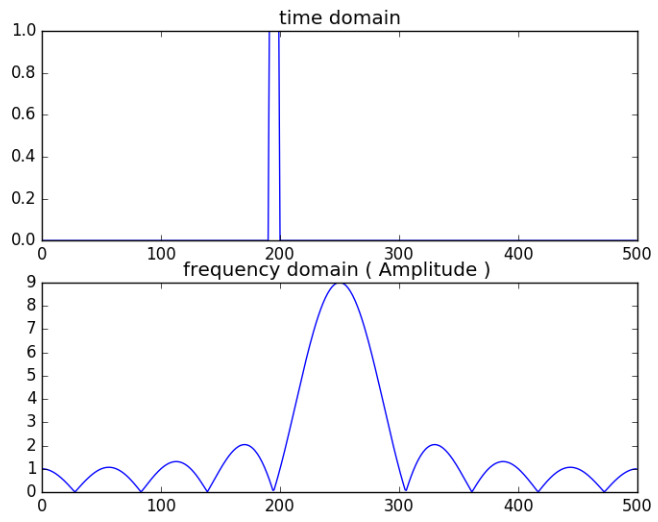


FIGURE 4.3: FFT of square wave

The below graph of Figure4.3 represents only amplitude in the frequency domain which is provided by following definition.

$$\mid F(k) \mid = \sqrt{\cos(x)^2 + \sin(y)^2} \tag{4.16}$$

When you shift signal to right or left, it does not affect amplitude, but phase, which is defined by following.

$$I(k) = \tan^{-1}(\sin(y)/\cos(x)) \tag{4.17}$$

This is because base function of Fourier series is represented as addition of $\cos$ and $i \sin$ and both can be either positive or negative. Since the derivatives of all four patterns of base functions which are ($+\cos$, $+\sin$,$-\cos$, $-\sin$) can be circular relationship, difference between original signal and shifted signal whose frequency is same but phase is different would be each values of coefficient and stayed on each frequencies.

Thereby, when we consider only amplitude, translation invariance is assured in frequency domain. Figure4.4 is a shifted signal of Figure4.3, bus shows same spectrum of Figure4.3.

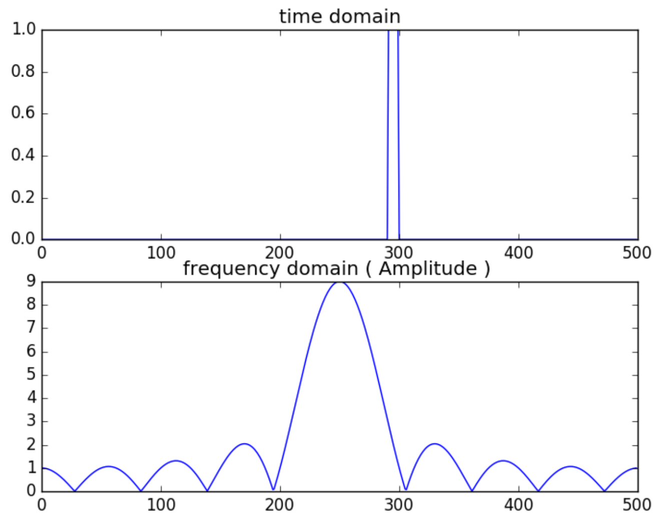


FIGURE 4.4: FFT of shifted square wave

Of course, if you set square wider, low frequency is going to be more dominant, which is on Figure4.5.

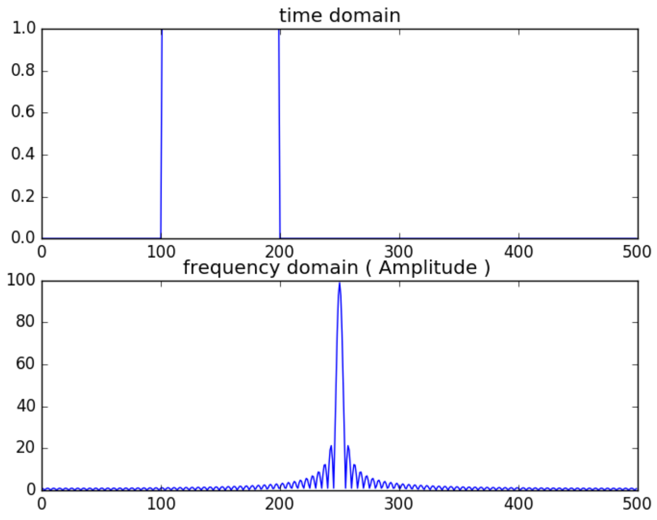


FIGURE 4.5: FFT of δ

4.6 Frequency domain of filters

4.6.1 Filter size

Signals of speech or general object are pretty difficult to be analyzed due to its complexity. On the other hand, filters which is often used for CNN is by far interpretable because it is rather much smaller. In CNN, network has relatively small size such as 3 or 5 reveal good performance. These filters can be represented as Sinc function in frequency domain.

For instance, Sobel filter which calculates first differential in original domain is represented in frequency domain as

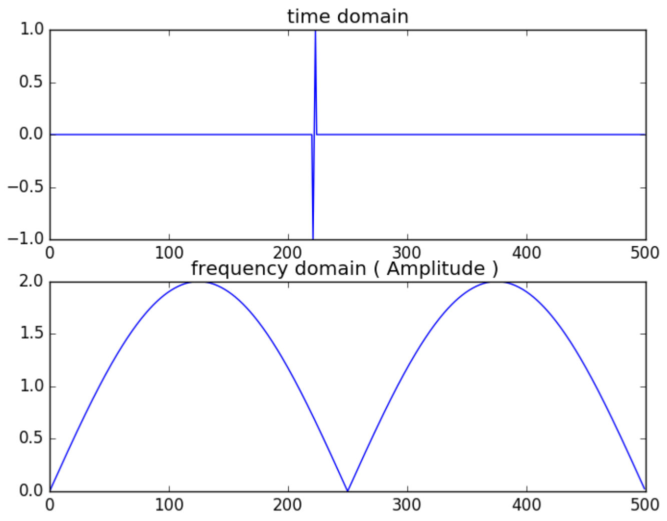


FIGURE 4.6: Sobel : signal (-1,1,-1)

That says, Sobel filter is working as bandpass filter.

Laplacian which is second derivative on original domain is defined as follows.

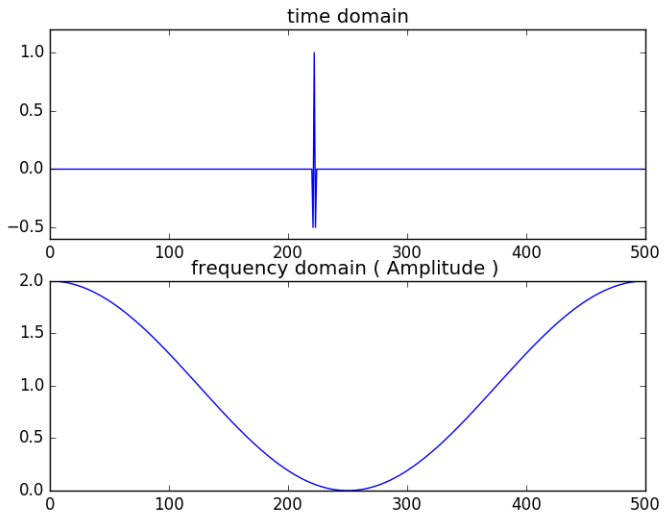


FIGURE 4.7: Laplacian : signal (-0.5,1,-0.5)

which is worked as low block filter or high pass filter.

This Sobel and Laplacian has filter whose size is 3. Sobel has (..,-1,0,1,..) Laplacian has (..,-0.5,1,-0.5,..) .. means rest of component which is 0.

Every filters including these two filters whose filter size is N can be represented as multiplication of coefficient a and combination of rectangular function $(-1 \leq a \leq 1)$ whose size is less than N. Formally,

$$f(N,k,t) = \sigma n = 0Nr_{-n,n}(t) * a \tag{4.18}$$

where

$$r_{-n,n}(t) = 1\ (-n \leq t \leq n)$$

$$r_{-n,n}(t) = 0\ (t < -n \,||\, n < t)$$

Fourier transformation of $r_{-n,n}(t)$ is defined by (25), and since Fourier transformation is linear transformation, multiplication of k and addition inside sigma is preserved after transformation, hence,

$$F[r_{-n,n}](t) = \sigma n = 0N\frac{1}{-i}\sin(n)a \tag{4.19}$$

From (4.19), you will know frequency of this combination of Sinc function (note "frequency in this context does not mean index of frequency domain after transformation") is determined by kernel size, not each values on filters, and its largest frequency is kernel size.

This means longer the filter size is, larger the frequency of Sinc function is. In other words, filter size determines complexity of spectrum in frequency domain. Considering why CNN which reveals best performance ever like [Residual net] has many layers but small filter size, it makes sense because each Sinc function in each layer can be regarded as sort of base function of overall transformation, and each base function does not need to be complex as itself, simple filter rather be able to capture more flexibility of different filters.

4.6.2 Fourier transformation of 1D filters on which binary weights are put

Let us consider the representation of filters whose weights are binary [1,-1], 1D, and its size is 3.
Figure4.8 4̃.10 represents Fourier transformation of binary weights. In this case, numbers of possible combinations are 2^3.

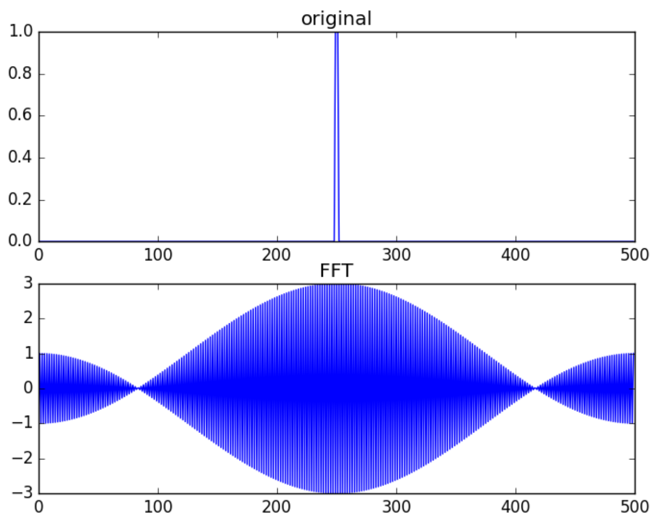


FIGURE 4.8: low-path : signal (1,1,1)

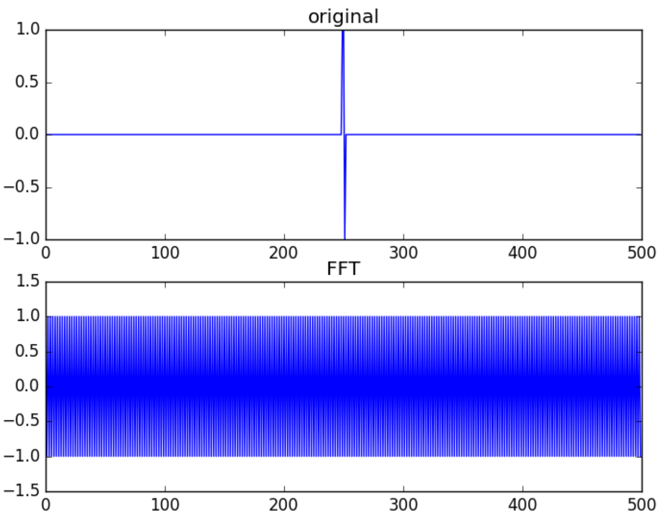


FIGURE 4.9: band(high)-pass : signal (1,-1,1)

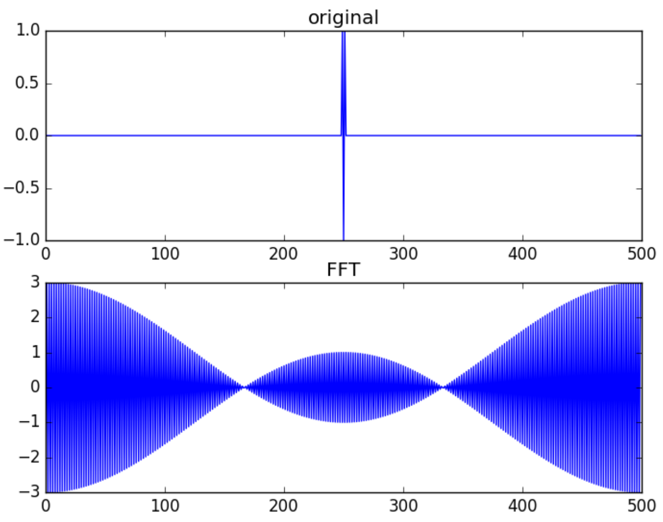


FIGURE 4.10: high-pass : signal (1,1,-1)

But, some of them shares almost identical representation in frequency domain.

For instance, filters whose weights are [1,-1,1] has just subtle difference with [-1,1,-1] which just flipped positive/negative. Same things can be said about the one between [1,1,1] and [-1,-1,-1]. Non-symmetrical filter based on its middle such as [-1,-1,1] has equivalent representation with mirror-rotated [1,-1,-1].
If you observe its spectrum,

$[1, 1, 1], [-1, -1, -1]$: low pass filter.
$[1, -1, -1], [-1, -1, 1], [1, 1, -1], [-1, 1, 1]$: constant pass filter
$[1, -1, 1], [-1, 1, -1]$: high pass filter

and its frequency of spectrum is defined by filter size as we saw in the formal section.

To conclude it, binary filters on 1D convolution have only three roles which is low pass, bandpass, or high pass, and its combinations form more complex classifications.

4.6.3 Fourier transformation of 2D binary filters

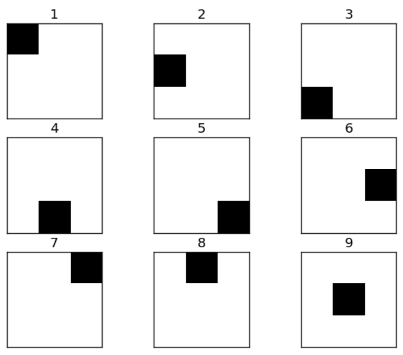
2D kernels cannot be classified as simple either low-pass / constant-pass / high-pass filters because they are able to contain various different orientations of 1D filters. For instance, it might be the case that a 2D filter has vertically low pass, horizontally high pass, and diagonally bandpass filters.

However, you can set some analysis with how much complexity each filters contain counting the number of 1 among a kernel. For instance, kernels whose dimension is 3 by 3 and 4 of 9 weights are 1 and the rest of them is -1 can be said as a more complex filter than a kernel whose weights are -1 except one weight on which 1 is put.

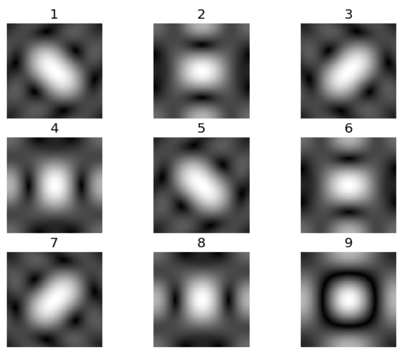
Since there is $\sum_{k=0}^{9} {}_9C_k$ patterns and some of them are overlapped in its representations on a frequency domain, I mapped part of them classifying proportion of 1 among -1. Figure4.11 Figure4.14 represents original weights and figures on a frequency domain.

Black rectangle indicates 1 and white does -1. Figure 4.11 says these filters works low pass but also catches middle or high frequency of certain orientations by the disposition of 1. This filter is called local binary descriptors and used in computer vision.

If you look around the figures from 4-12 to 4-14, you would realize that each corresponding filter works as a very complicated filter. It is very likely those functions of each filters are quite overlapped. Further analysis is going to be done in the last chapter.

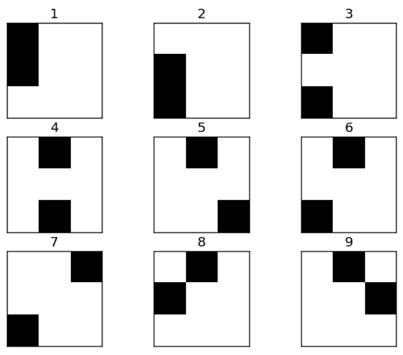


(A) Kernel analysis : 1 of 9

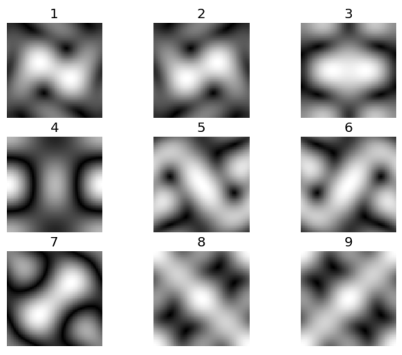


(B) Frequency in Fourier domain

FIGURE 4.11: Fourier analysis of weight (1)

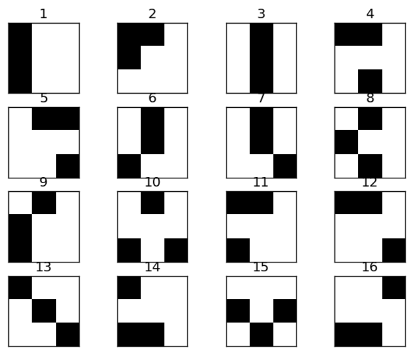


(A) Kernel analysis : 2 of 9

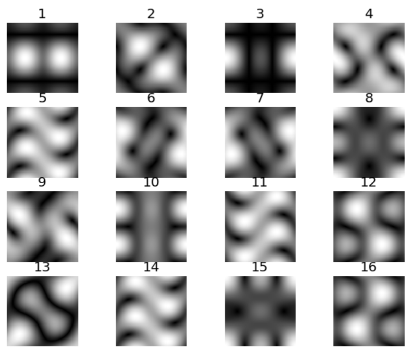


(B) Frequency in Fourier domain

FIGURE 4.12: Fourier analysis of weight (2)

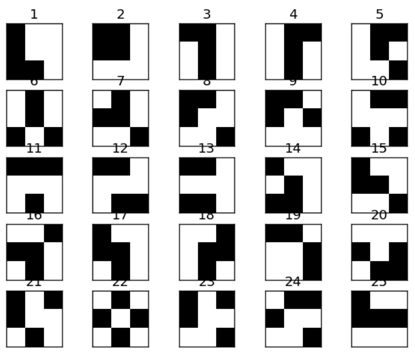


(A) Kernel analysis : 3 of 9

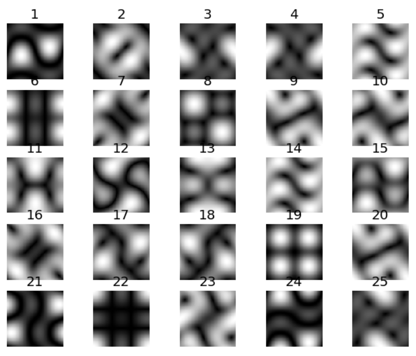


(B) Frequency in Fourier domain

FIGURE 4.13: Fourier analysis of weight (3)



(A) Kernel analysis : 4 of 9



(B) Frequency in Fourier domain

FIGURE 4.14: Fourier analysis of weight (4)

Chapter 5

Analysis of Binary neural network

5.0.1 Prelude

In this last chapter, I firstly introduce binary neural network algorithm, then show you the experiments which I did, and come to a conclusion.

Binary neural network is the network which part of the sets of operations which are defined in a model are replaced to bit operation due to data reduction of data type of activations of nodes and weights. Data types which sustain deep learning framework or algorithm is mostly float32(32bit) or Double(64bit). However, recent research revealed that data type could be replaced to smaller scale one such as unsigned int8(8bit) or even extremely just down to 1 bit. The benefit of this algorithm is not only to be able to shrink the model size $\frac{32}{1}$ but let basic unit of computation much faster replacing float dot product or convolution to binary operation such as Xnor, pop count, and shift.

This idea itself was provided some years ago, but lately well formalized by [23] [13], [14].

5.1 Binary neural network algorithm

The basic flow of this algorithm is following.

Parameter

E : epoch
L : number of layers
a : activation
a^b : binary valued activation
a^f : float valued activation but not floating point but fixed point number
g : gradient
r : learning rate

```
initialization
memory allocation
while $e \leq E$ do
    Forward
    for k = 1 to L do
        $a_k^b = quantize(a_k)$
        # above line is not necessary for inference
        $W_k^b = quantize(W_k)$
        # above line is not necessary for inference
        $a_k^b = XNor(a_{k-1}^b, W_k^b)$
        $a_k^b = PopCount(a_k^b)$
        $a_k^f = Act(a_k)$
    end
    Calculate Error
    Backward
    for k = L-1 to 2 do
        $a_k^b = quantize(a_k)$
        $W_b^k = quantize(W_k)$
        $g_k^b = XNor(g_{k+1}^b, transpose(W_k^b))$
        $a_k^b = PopCount(a_k^b) g_k^b = Act(a_f) * g * k$
    end
    Update Weights
    for k = 1 to L do
        $w_k^f + = r * Dot(transpose(a_k^f), g_k^f)$
        # I found that following is also worked out
        $w_k^b + = r * Dot(transpose(a_k^b), g_k^b)$
    end
end
```

Algorithm 1: binary neural network algorithm

As of forward and backward operations, there are three components which are

1) quantization
2) Xnor operations
3) pop count operation
4) activation function

Each detailed explanation are following.

5.1.1 Quantization

This operation has two targets, activations of nodes and weights.

As of quantization of nodes, it is needed in case that activation function on previous layer produced more longer bit length than bit length of nodes after quantization. If you find it on the first layer. Some previous researches [13] [17] showed that quantization to the first layer should be refrained due to the drop of accuracy by doing so. Nevertheless, you could in principle quantize input nodes on which there are data-itself. For instance, if the task is OCR, there should be very few reluctancy to prevent first layer quantization.

Regarding quantization of weights, it is pretty simple as we need just taking Sign of the original values. if the original values are fixed floating points, the operations turns to be retrieving bit on the left most digit.

5.1.2 Binarize operations

BNN paper suggested that dot product and convolution operations are replaced with bit operations such as Xnor(or Xor given with flip later on) operations and pop count operations. Activation functions is kept, but should be consistent with the way to calculate binary operations which is explained later on. After binary operations are fed into activation function, whose type of output turns to be float values but many research reveals that floating point number could be replaced to fixed point number using logarithmic expressions. [4]

Assuming there are two input for dot product and all of the elements in the matrix are represented as binary values, dot product can be approximated by bit operation. On binary neural networks, activations of nodes and values of weights are expressed by bit representation since they are quantized in the previous operations.

TABLE 5.1: multiplication of bit values

node	weight	output
-1	-1	1
-1	1	-1
1	-1	-1
1	1	1

TABLE 5.2: bit representation

node	weight	output
0	0	1
0	1	0
1	0	0
1	1	1

Table5.1 shows multiplication of two bit inputs and its outcome. if you regard -1 as 0, multiplication can be replaced by Xnor operations (Table5.2).

If you would like to let every input values non-negative in case you pick up rectifier linear unit or sigmoid function as activation function, you need "and" operations and flip operations instead of XNor operations.

TABLE 5.3: multiplication

node	weight	output
0	-1	0
0	1	0
1	-1	-1
1	1	1

TABLE 5.4: bit representation

node	weight	output
0	0	0
0	1	0
1	0	0
1	1	1

TABLE 5.5: bit representation

node	weight	output
0	1	0
0	0	0
1	1	1
1	0	0

To emulate operations described in table5.3, bit "and" operation have to be done twice in parallel, one is with original weights, another is with the one whose weights are flipped (Table5.4, Table5.5). Then, either of the output needs to be subtracted from another one.

As dot product of two matrix and convolutions are MACC(Multiply and Accumulation) operations, we need to simulate accumulation operation by bit operations. This is realized by bit count operation, so called pop count operation.

5.1.3 Pop count operation

Following is Xnor operations and pop count operations in c++.

```
int pop_count(uint32_t i)
{
  i = i - ((i >> 1) & 0x55555555);
  i = (i & 0x33333333) + ((i >> 2) & 0x33333333);
  return (((i + (i >> 4)) & 0x0F0F0F0F) * 0x01010101) >> 24;

}
```

Since the output of 0 on Xnor operations denotes -1, the values after pop count makes sense with the length of vectors which fed into pop count operation.

Set of this operation unit, namely activations and pop count operations over one output channels needs to be done x times, in which x is multiplication between number of bits which represents activations of nodes and output channels. As of numbers of bits about weights Recent paper Dorefanet[19] revealed that weights can be reduced to 1 bit, but activations of nodes and gradients cannot be tolerate to be represented as down to 1 bit due to drop of accuracy, but more than 2 or 3 bits.

5.1.4 Activation function

Applying activation functions can be translated into following operations.
1st is ignoring leftmost k bits assuming k is length of vectors on which pop count operation had been done. This is done by left shift. This is because output of pop count makes sense with length of input vector.
2nd is keeping p bits from left edge and removing the rest of it. p is the length of digit for fix point float. This is done by right shift. This is because quantization is needed to express nodes as fixed point float.
If you represent a node as multiple bits, you need to do q times bit left shift where q is the index of digit. Then, different binary outputs are added.

For instance, if the size of vector (maximum of pop count) is 32, and output of pop count is 20, 2^6 (= 32), the bits which are under 2^5 of 20, namely 10100 is going to be first output. then, given 3 bit size, rightmost 00 is removed, and 101 is left. Note first bit says sign of the activation of this node, and the rest of them are under 1.

For only inference, it does not makes sense to calculate bits except a sign bit because quantization process in the next layer will remove them. However, training, these fixed point float nodes are used for update weights.
The overall process except quantization is written as following in c++.

Case for Dot product

```
// x and y are 2D matrix which contains bit.
// so as to let required memory as small as possible,
// you should set every values on uint32 or uint64,
// then, do bit operations together in the unit.


template<typename T>
  auto bitDot<T>::operator()(auto x,auto y,auto z) {
    int unit = sizeof(T) * 8;
    int r1  = x->get_shape().dimensions[0];
    //int row = x1.get_shape().dimensions[1];
    int r2 = y->get_shape().dimensions[0];
    int c2 = y->get_shape().dimensions[1];
    int nbits = 0;int c = 0;
    int t = 0;int16_t all = 0;
    T tmp = 0;T stock = 0;T val;

    for(int o = 0 ; o < c2 ; o++){
      all = 0 ; t = 0;
      //different bit unit in a same pairs of row & column
      for(int i = 0 ; i < r2 ; i++){
        tmp = ~ (x->data[i] ^ y->data[i+r2*o]);
        t += pop_count(tmp);
        all += unit;
      }
      //this is when you have switched to next column
      stock += ((all - 2 * t) >= 0) ? pow(2,(o % unit)) : 0;
      //this is when you feed a value to the next bit array
      if( (o+1) % unit == 0 || (o+1) == c2 ){
        z->data[c] = stock;
        c ++;stock = 0;
      }
    }

  };
```

Case for Convolution

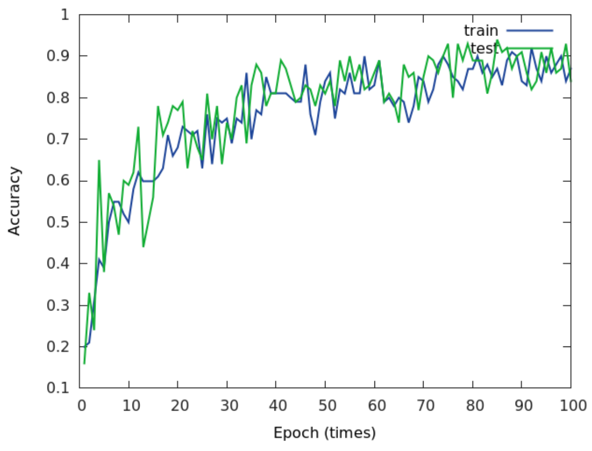
```
// Input should be 3 dimension without batch.
// Kernel should be 4 dimension.
//for details, please take a look at the code itself.

template<typename T>
auto bitConv<T>::operator()(auto input,auto kernel,auto output){

    for(int o = 0 ; o < kernelN ; o++){
      //index of input 3D node ( which is lowered to 1D after all ).
      currentPinInput = 0;
      //patch is the resolution
      for(size_t p = 0; p < patches; p++) {
            size_t out = 0;
            for(size_t k = 0; k < kernelBit; k++) {
                for(size_t i = 0; i < inputBit; i++) {
                    size_t nbits = 0;
                    for(size_t j = 0; j < unit; j++){
                        nbits += pop_count
                        ( ~(input->data[ currentPinInput + (i * 2) + j]
                        ^ kernel->data[(k * 2) + j + o*kernel3DeN]) );
                    }
                out += nbits;
              }
            }

            //if the kernel have to be switched vertically...
            currentPinInput = ( (p + 1) % (inputFmap * width) == 0 )
            ? currentPinInput + inputFmap * width : currentPinInput + inputFmap;

            output->data[p*outFmap+o/(sizeof(T)*8)]
            += (unit - 2 * out >= 0)? 0 : pow(2,(o%(sizeof(T)*8)));
        }
        }
}
```

5.2 Experiments for BNN with different bit numbers

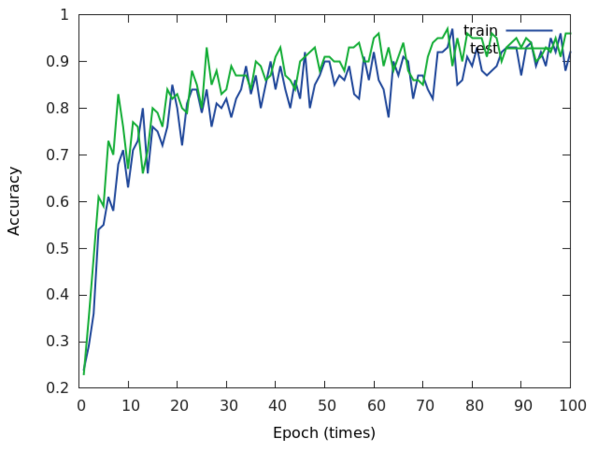
I have implemented binary neural network in two different languages, c++ and lua. The one in c++ was purely implemented without any help of existing deep learning libraries. It aims purely acceleration by bit operation which is not supported by neither deep learning libraries nor GPU. This attempt is not at all reckless because there have been many IP Cores for hardware implementations on deep learning chip whose aim is pretty similar. However, due to lack of knowledge for optimization towards specific CPU, implementation in c++ is not able to provide faster execution time.

Instead, I have implemented in lua with the help of only multi-dimensional array (so called tensor) provided from Torch. Aiming to acquire fully customization of backward-pass without any interference of computational graph, automatic differentiation function in Torch does not be used. Since tensor on Torch does not provide neither bit operations, bit type under 8, nor fixed point number, I let fixed point number float-point number and a proposed activation function in the previous section a normal activation function.

I put results of 10 different experiments letting forward-node / backward-node / weight either 1 bit or float(32bit). Following is the results.

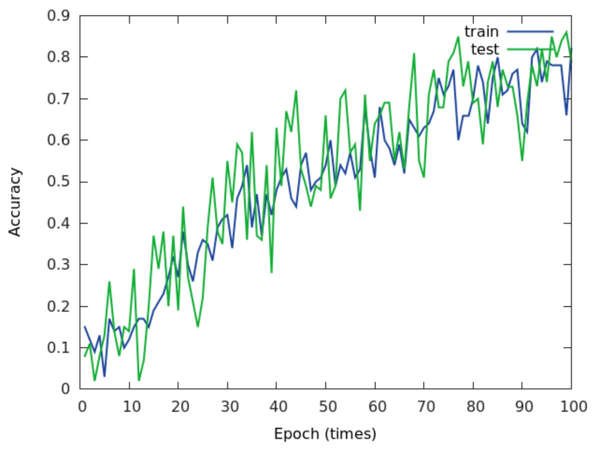


(A) F : B : W = 32 : 32 : 32

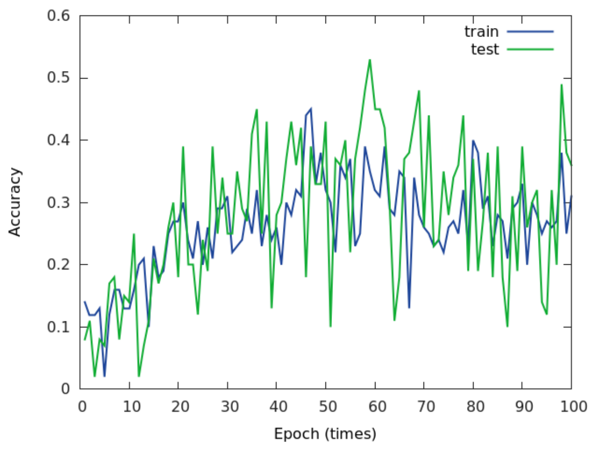


(B) F : B : W = 1 : 32 : 32

FIGURE 5.1: Forward / Backward / Weight bit numbers

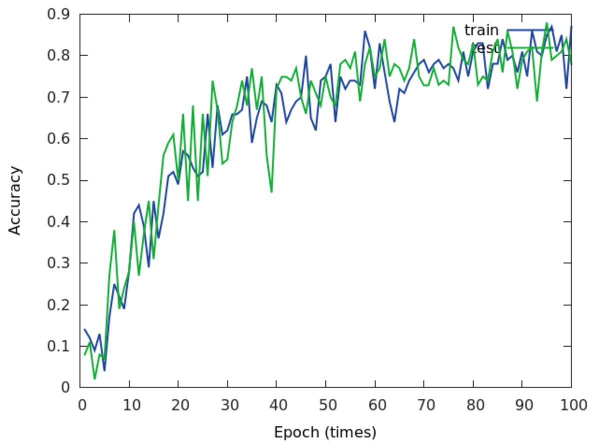


(A) F : B : W = 32 : 1 : 32

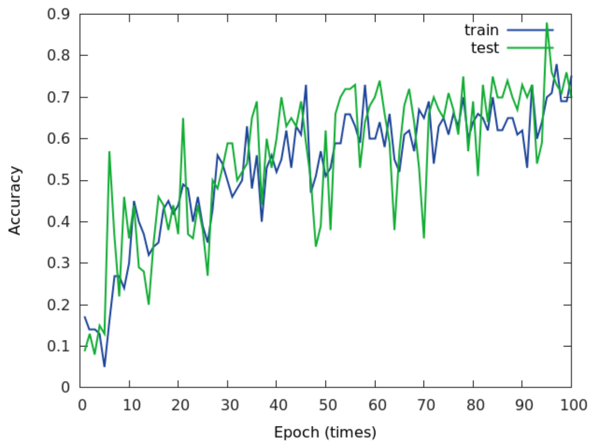


(B) F : B : W = 32 : 32 : 1

FIGURE 5.2: Forward / Backward / Weight bit numbers

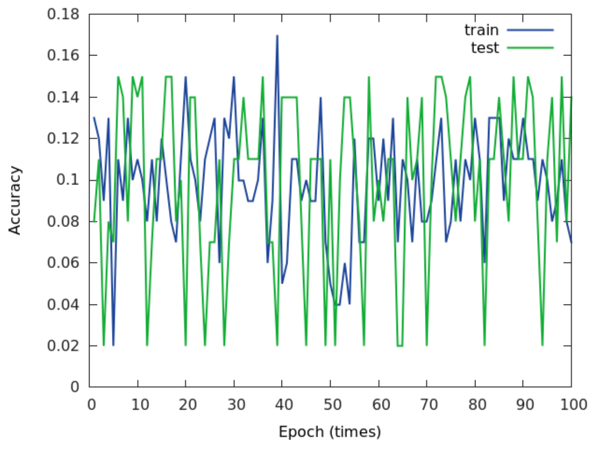


(A) F : B : W = 1 : 1 : 32

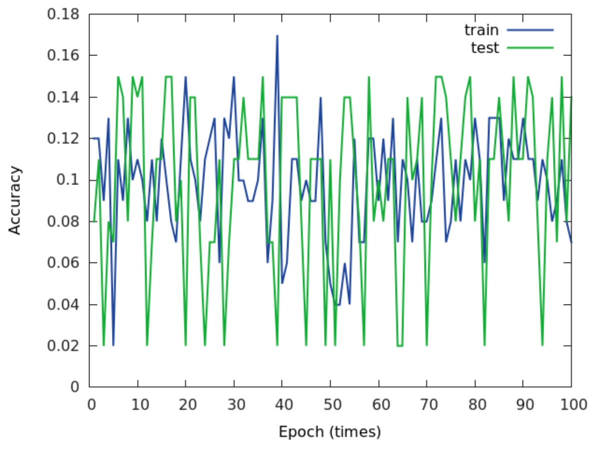


(B) F : B : W = 1 : 32 : 1

FIGURE 5.3: Forward / Backward / Weight bit numbers



(A) F : B : W = 32 : 1 : 1



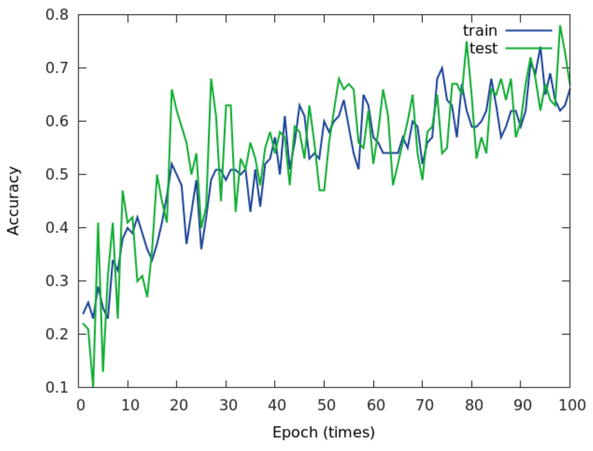
(B) F : B : W = 1 : 1 : 1

FIGURE 5.4: Forward / Backward / Weight bit numbers

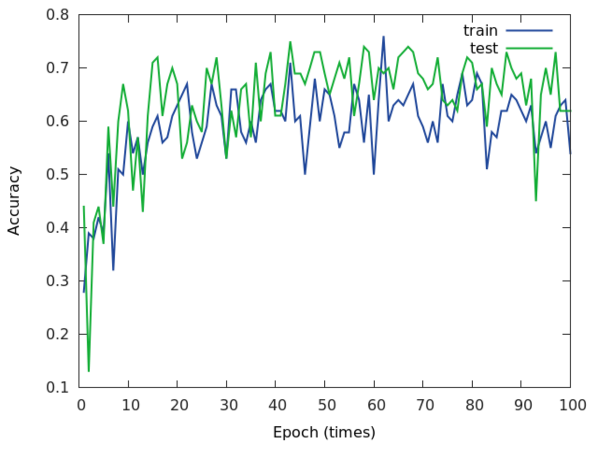
The data-set is first 100 data of MNIST. It is too small for validation of model, but at least must be useful to compare error transition till its convergences of each model. 1 Epoch means iterations of $\frac{1}{10}$ of data-set which is in this case per every 10 images. This is 4 layers multi-layer perceptron whose width of nodes are, in order, $[28 \times 28, 500, 300, 10]$.

The experiments on my implementations somewhat differed from the results of a paper [DorefaNet] [19] which did similar experiments on TensorFlow. The result of Dorefanet was weight could be the values which can be expressed as smallest bit; namely 1 bit, then gradient (backward node) largest, and forward-node: middle. On 4 layers MLP, forward node turns out to be the values which should be expressed as 1 bit (actually it is better than complete float for some reason). Complete binary model failed to be trained at all, but leaving either backward-nodes or weights binary was in the scope of conditions where it can reach to convergence.

Furthermore, two interesting additional results could be found. Left one of Figure5.5 shows the result that quantization of forward /backward nodes before weight-updates has been done. I assumed that updates is the only part which should not be binarized, but it turns out that quantization of them does not lower accuracy. Right one shows the result of 3-layer MLP which all of three are fully quantized. It says the network is trained well if the layer is 3 not 4 with just 1 bit.



(A) F : B : W = 1 : 32 : 1 (update quantized)



(B) F : B : W = 1 : 1 : 1 (layer num = 3)

FIGURE 5.5: Forward / Backward / Weight bit numbers

5.3 Experiments on Caltech101,256 CIFAR10,100

My first experiments does not prove validation of BNN itself because data-set is small. Instead, I have trained the algorithm which is proposed on [14] with different data-set, and additionally analyzed its trained weights.

The following is the result.(Figure5.1 - Figure5.4)

4 Different data-set had been tested for confirmation of effectiveness and generality that binary neural network has.

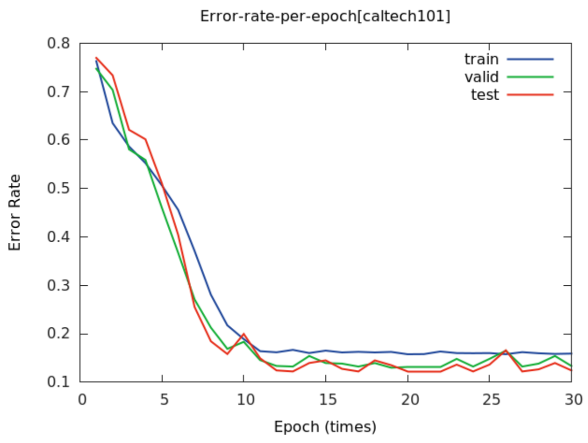
First is Caltech101. The Pictures of general objects belongs to 102, and each category has about 40 to 800 images. The size of each image is roughly 300 x 200 pixels, but I have resized input resolution down to 32 by 32 to prevent large parameter of architecture. As there is no distinction between training, validation, test, I have allocated 10 % of data to test, 10 % of the rest 90 % belongs to validation data and the rest of it comes into training data-set.

Second is Caltech256.It has 257 different categories, and overall number of images are 30607. I have resized its resolution down to 32 by 32 as well as Caltech101 as one of pre-processing operation. Since there is much fewer data-set than Cifar100 and some of the background of images is sheer white, task itself is rather easier than Cifar100 in spite of larger output categories.

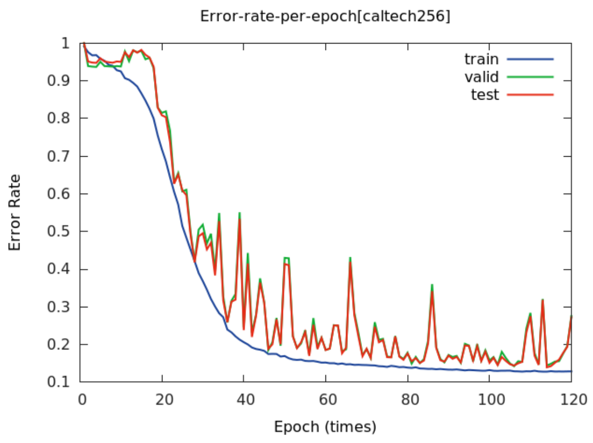
Third is Cifar10. It consists of 60000 32x32 colour images in 10 classes, with 6000 images per class. There are 50000 training images and 10000 test images.

And the last is Cifar100 whose output categories are 100, and the rest of information is as same as Cifar10, which is most difficult classification task among them.

The results are following.

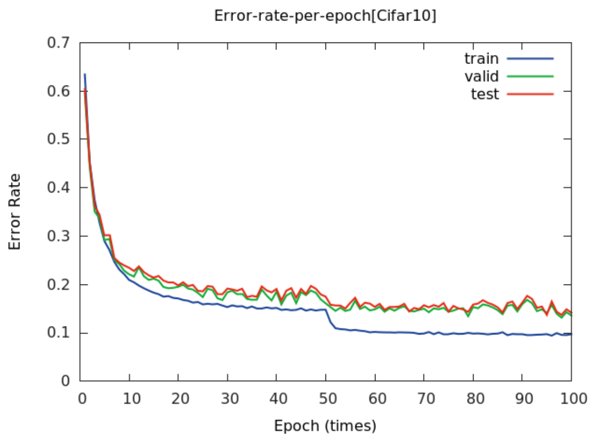


(A) Result : Caltech101

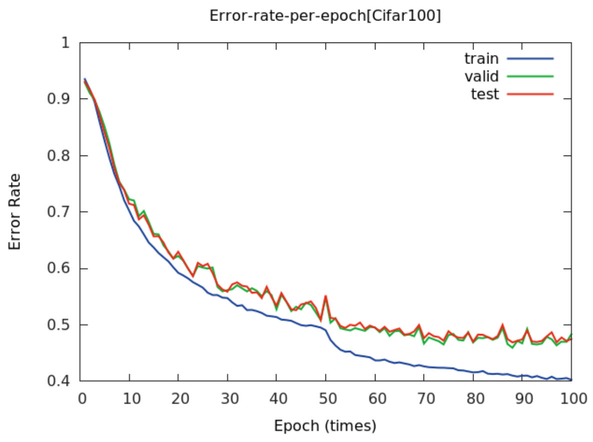


(B) Result : Caltech256

FIGURE 5.6: Results : Caltech



(A) Result : Cifar10



(B) Result : Cifar100

FIGURE 5.7: Results : Cifar

As you can see Figure5.1-5.4, there are 4 different experiments. Each network is binarized and hyper parameter is following. The network architecture described in table5.6 is determined according to the complexity of data-set.

Both of the nodes and weights are quantized down to 1 bit, but the nodes on first and last layer is not quantized due to prevention for dropping of accuracy rate. The value after quantization is described not floating point number but fixed point number as torch tensor does not support array on fixed point number by default. Regardless of bold quantization down to 1 bit, each network is trained very well to nearly state-of-art compared with training with no quantization. It proves quantization does not hinder training, providing possibility to acceleration.

TABLE 5.6: network architecture (number of channels by layer)

Caltech101	Caltech256	Cifar10	Cifar100
3	3	3	3
128	256	128	179
128	256	128	179
256	512	256	358
256	512	256	358
non	1024	512	716
non	1024	512	716
non	non	512	non
non	non	512	non
fully	fully	fully	fully
fully	fully	fully	fully
fully	fully	fully	fully
102	257	10	100

5.4 Weights analysis

On previous experiments all of kernel size in a convolutional layer was input channel by output channel by 3 by 3, and its elements were expressed as binary values. Therefore, as I analyzed in the previous chapter, trained weights have only $\sum_{k=0}^{9} {}_9C_k$ combinations on 3 by 3 kernels. I mapped distribution about numbers of 1 among elements which are either 1 or -1 on each kernels. Figure5.3 shows initial randomized distribution of kernels whose input features are [3,128,128,256,256,512,512]. Since they are generated by uniform random distribution, the propotion of 1 to -1 among elements of 2D kernels are likely to be half, theoretically its distribution should be binomial distribution.

After enough iterations enough to classify each dataset, some of weights on previous experiments had been changed, and its distribution is put on Figure5.4 for Caltech, Figure5.5 for Cifar.

As you can see them, all of the deviations of each distributions became larger, that means on 2D kernel, sign of each elements of 2D kernels tends to be same. The reason that I guess towards the fact that kernels on 1st layer in experiments towards Caltech has all -1 or all 1 is following. Some of the background of an object on Caltech is just sheer white, on the other hand, there is no such images on Cifar data-set.

As I discussed in the previous chapter, 2D binary kernels can be low pass, bandpass, and high pass filter by and large, and if 1 to -1 ratio of elements on 2D kernel come close to equal, its Fourier representation get to hold rather complexity, departing from simple low pass, bandpass, high pass filter. In other words, well trained network may form kernels which functions as more simple filter in Frequency domain.

Recent research [Local Binary Convolutional Neural Networks][6] revealed that with sparse filters inspired from Local binary patterns (LBP) descriptor which is fixed during training process is able to contribute faster convergence to the state-of-art result. The idea suggests that each kernels does not need to be so complex because it might cause over learning. In other words, combinations of very simple kernels but enough to be effective to protect certain frequency is more likely to contribute to form a robust and flexible classifier.

5.5 Summary

5.5.1 Possible applications of this approach

This paper was more or less written for provision of underlying explanations towards most promising approach about computation reduction about convolutional neural network. As of now, in 2016, a lot of company jumped into this challenging hardware acceleration of deep learning with binary neural network

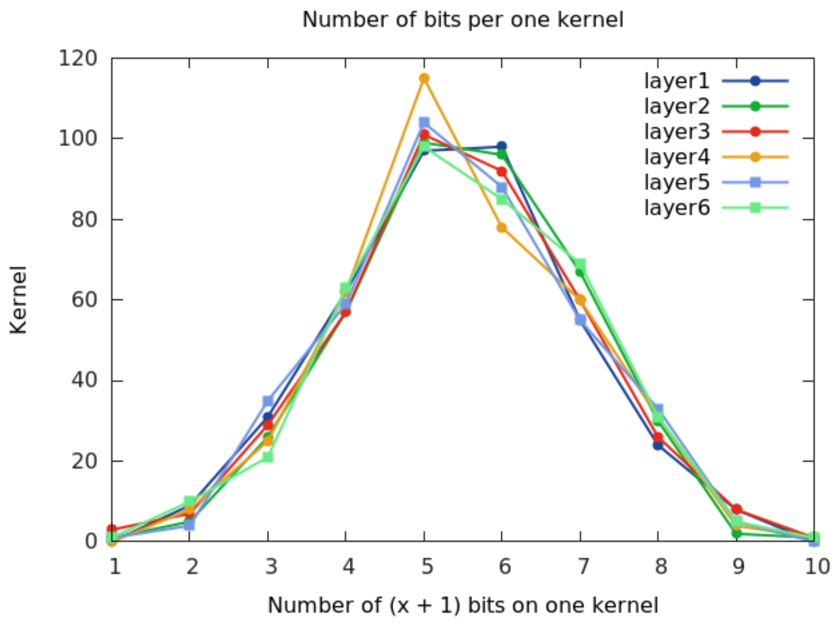
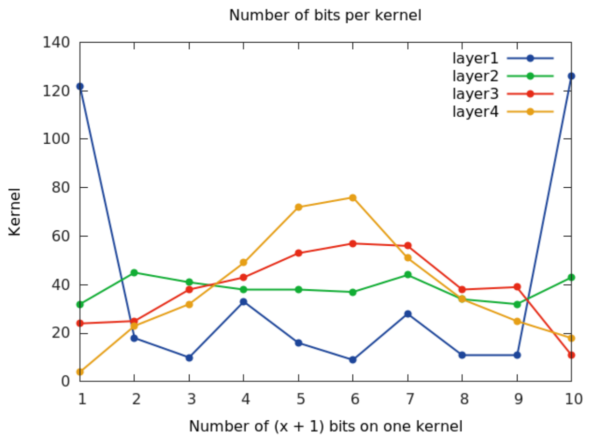
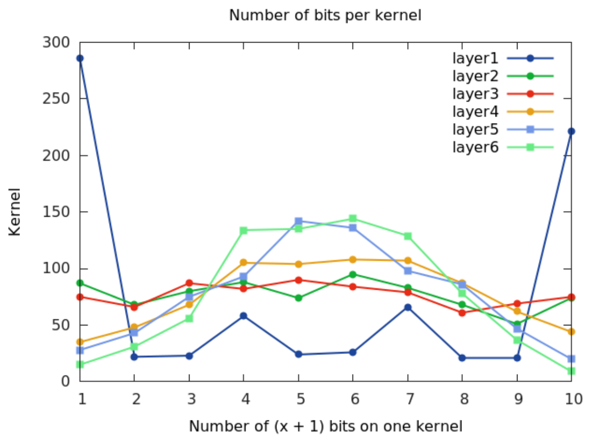


FIGURE 5.8: initial weight



(A) Kernel analysis : Caltech101



(B) Kernel analysis : Caltech256

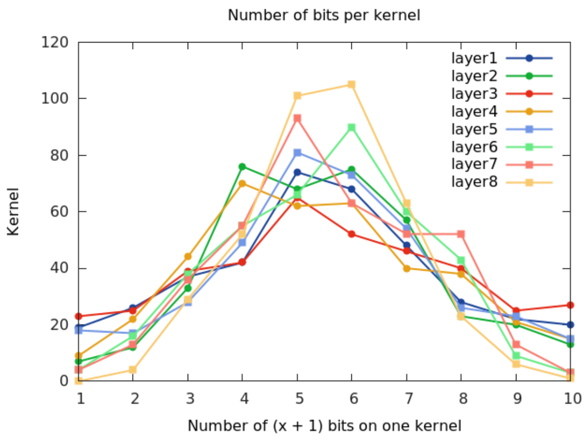
FIGURE 5.9: Kernel analysis : Caltech

and network compression technique.

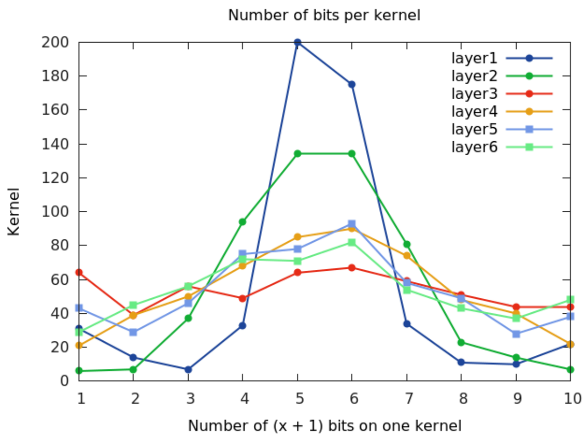
The biggest market of this area which is likely to be opened is object recognition for inference of automatic driving car. If a CMOS sensor provides new input image give its fps is 60, inference have to be done within 1image/(m)second. Inference duration on AlexNet of modern fastest GPU for deep learning,for instance, on Tegra X1 in FP16 is 45 img/sec/W without any batch processing. That says it is still a little bit slow to keep up with successive images which comes 60 per second with most high-spec hardware, much less realistically you must take into account of durations of object detection algorithm.

Assuming 3D convolution is done in itself, times of MAC (multiply and accumulation) operations in one layer is (resolution $times$ output channel) , for instance, given 100 by 100 size input-image, and 100 output channel, MAC operation is performed 10^6 times. Since AlexNet has 7 layers and a bifurcation, the overall MAC calculations are nearly 10^7 times.

If you let all of multiplication XNor bit operation, and accumulation pop-count operation 10 7 operation are performed less cycle on CPU. What is more, since model size is shrinked to $\frac{1}{32}$ at maximum, it is much



(A) Kernel analysis : Cifar10



(B) Kernel analysis : Cifar100

FIGURE 5.10: Kernel analysis : Cifar

less likely to be encountered with cash hit miss, in other words, more data can be on cash memories, which consequently reduce much latency.

Hardware which needs inference is not only cars but, drones, robots, surveillance cameras, and so on. if basic unit of calculation is done not by convolution, but combinations of bit operations, it is very likely that computation for inference is processed on edge but cloud which helps to prevail this algorithm into a society ubiquitously.

Meanwhile, training should be done by a cloud with hundreds of batch size. However, what makes developers perplex when you introduce purely new products with image recognition function is preparation of training data and gap between training data and inference. If the edge side could have enough capacity to compute training, it lightens the burden of developers which leads this algorithm next stage. So far, the performance of online learning is not as good as the one of batch learning. I suppose that bit operations for convolution is going to be more introduced with following a couple of years for reducing electricity consumption of computation.

5.5.2 Future works ahead on this thesis

There are a couple of staffs which needs to be done.
One is acceleration for inference optimized for FPGA and ARM CPU.

Another is improvements of training convergence retaining enough performance. I have discussed compressions in the 3rd chapter and briefly touched the architecture called [7] which provides smallest model retaining accuracy. To me, it is interesting to investigate binarization of the SqueezeNet architecture in terms of computational speed.

There have been almost 4 years after AlexNet[1] performs very well in ILSVRC in 2012, but a lot of tasks are still left in this field.

Bibliography

[1] Geoffrey E. Hinton Alex Krizhevsky Ilya Sutskever. "ImageNet Classification with Deep Convolutional Neural Networks". In: (2012).

[2] Scott Gray Andrew Lavin. "Fast Algorithms for Convolutional Neural Networks". In: (2015).

[3] M. Cole. "Algorithmic skeletons : A structured approach to the management of parallel computation. Research Monographs in Parallel and Distributed". In: (1989).

[4] Edward H Lee Daisuke Miyashita and Boris Murmann. "Convolutional neural networks using logarithmic data representation". In: (2016).

[5] Ronald J. Williams David E. Rumelhart Geoffrey E. Hinton. "Learning representations by back-propagating errors". In: (1986).

[6] Marios Savvides Felix Juefei-Xu Vishnu Naresh Boddeti. "Local Binary Convolutional Neural Networks". In: (2016).

[7] Matthew W. Moskewicz Khalid Ashraf William J. Dally Kurt Keutzer Forrest N. Iandola Song Han. "SqueezeNet: AlexNet-level accuracy with 50x fewer parameters and <0.5MB model size". In: (2016).

[8] Fukushima. "Neocognitron: A self-organizing neural network model for a mechanism of pattern recognition". In: (1980).

[9] Hiroshi Sawada. *comprehensive codes*. https://github.com/Hiroshi123.

[10] L. Sirovich; M. Kirby. "Low-dimensional procedure for the characterization of human faces". In: (1987).

[11] Andrew Lavin. "maxDNN: An Efficient Convolution Kernel for Deep Learning with Maxwell GPUs". In: (2015).

[12] J. S. Denker D. Henderson R. E. Howard W. Hubbard LeCun B. Boser and L. D. Jackel. "Backpropagation applied to handwritten zip code recognition. Neural Computation". In: (1989).

[13] Daniel Soudry Ran El-Yaniv Yoshua Bengio Matthieu Courbariaux Itay Hubara. "Binarized Neural Networks: Training Neural Networks with Weights and Activations Constrained to +1 or 1". In: (2016).

[14] Daniel Soudry Ran El-Yaniv Yoshua Bengio Matthieu Courbariaux Itay Hubara. "Quantized Neural Networks: Training Neural Networks with Low Precision Weights and Activations". In: (2016).

[15] Andrew Zisserman Max Jaderberg Andrea Vedaldi. "Speeding up Convolutional Neural Networks with Low Rank Expansions". In: (2014).

[16] Shuicheng Yan Min Lin Qiang Chen. "Network In Network". In: (2015).

[17] Joseph Redmon Ali Farhadi Mohammad Rastegari Vicente Ordonez. "XNOR-Net: ImageNet Classification Using Binary Convolutional Neural Networks". In: (2016).

[18] Philippe Vandermersch Jonathan Cohen-John Tran Bryan Catanzaro Evan Shelhamer Sharan Chetlur Cliff Woolley. "cuDNN: Efficient Primitives for Deep Learning". In: (2014).

[19] Xinyu Zhou He Wen-Yuxin Wu Shuchang Zhou Zekun Ni and Yuheng Zou. "Dorefa-net:Training low bitwidth convolutional neural networks with low bitwidth gradients". In: (2016).

[20] William J. Dally Song Han Huizi Mao. "Deep Compression: Compressing Deep Neural Networks with Pruning, Trained Quantization and Huffman Coding". In: (2015).

[21] Brian Kingsbury1 Bhuvana Ramabhadran1 Tara N. Sainath1 Abdel-rahman Mohamed2. "DEEP CONVOLUTIONAL NEURAL NETWORKS FOR LVCSR". In: (2012).

[22] Brian Kingsbury1 Bhuvana Ramabhadran1 Tara N. Sainath1 Abdel-rahman Mohamed2. "Deep Convolutional Neural Networks for Sentiment Analysis of Short Texts". In: (2014).

[23] Roland Memisevic Yoshua Bengio Zhouhan Lin Matthieu Courbariaux. "NEURAL NETWORKS WITH FEW MULTIPLICATIONS". In: (2016).