Particle Physics

Review for Particle Physics

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0.1 Hamilton Formalism

0.1.1 Legendre Transformation

Now, remember that we obtained the **Hamiltonian** H of the system through the **Legendre transformation** of the Lagrangian L:

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{0.1}$$

where i represents each degree of freedom of the system.

In the field, the degree of freedom is infinite - each indexed by the spatial position $\vec{x}$, then,

Definition 0.1: Hamiltonian of a Field

The **Hamiltonian** of a field $\psi(\vec{x}, t)$ (whose canonical conjugate field is $\pi(\vec{x}, t)$) is defined as

$$H[\psi, \partial_i\psi, \pi] = \int d^3\vec{x} \left[\pi \cdot \partial_t\psi\right] - L = \int d^3\vec{x} \left[\pi \cdot \partial_t\psi - \mathcal{L}\right] \tag{0.2}$$

thus, we should define the **Hamiltonian density** $\mathcal{H}$ as:

Definition 0.2: Hamiltonian Density

Hamiltonian density $\mathcal{H}$ is defined as the Hamiltonian per unit volume of the field:

$$\mathcal{H}(\psi, \partial_i\psi, \pi, t) = \pi \cdot \partial_t\psi - \mathcal{L}(\psi, \partial_i\psi, \partial_t\psi, t) \tag{0.3}$$

which satisfies:

$$\int d^3\vec{x}\, \mathcal{H}(\psi, \partial_i\psi, \pi, t) = H[\psi, \partial_i\psi, \pi, t] \tag{0.4}$$

which makes the Hamiltonian H a functional of the field ψ, its spatial derivatives $\partial_i\psi$, and the conjugate field π.

Note:

Similarly to the discrete case, we can write the Lagrangian densty $\mathcal{L}$ in terms of the Hamiltonian density $\mathcal{H}$:

$$\mathcal{L}[\psi(\vec{x}, t), \pi(\vec{x}, t)] = \pi \cdot \partial_t\psi - \mathcal{H}[\psi(\vec{x}, t), \pi(\vec{x}, t)] \tag{0.5}$$

where

$$\partial_t\psi = \frac{\partial\mathcal{H}[\psi, \pi]}{\partial\pi} \tag{0.6}$$

0.1.2 Canonical Equations of Fields

We are also interested in the **canonical equations** of fields, which are derived from the variational principle:

$$\delta S = 0 \iff \delta \int dt \int d^3\vec{x}\mathcal{L} = \delta \int dt \int d^3\vec{x}\, \pi \cdot \partial_t\psi - \mathcal{H} \tag{0.7}$$

The variation on this integral can be expanded as follows:

$$\begin{aligned}
\delta S &= \int dt \int d^3\vec{x}\, \delta\pi\, \partial_t\psi + \pi\delta(\partial_t\psi) - \delta\mathcal{H} && (0.8)\\
&= \int dt \int d^3\vec{x}\, \delta\pi\, \partial_t\psi + \pi\, \partial_t(\delta\psi) - \frac{\partial\mathcal{H}}{\partial\psi}\delta\psi - \frac{\partial\mathcal{H}}{\partial\pi}\delta\pi - \frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\delta(\partial_i\psi) && (0.9)\\
&= \int dt \int d^3\vec{x}\left(\partial_t\psi - \frac{\partial\mathcal{H}}{\partial\pi}\right)\delta\pi + \left(\pi\, \partial_t(\delta\psi) - \frac{\partial\mathcal{H}}{\partial\psi}\delta\psi - \frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\, \partial_i(\delta\psi)\right) && (0.10)
\end{aligned}$$

the $\delta\psi$ term can be integrated by parts:

$$\begin{aligned}
&\int dt \int d^3\vec{x}\, \pi\, \partial_t(\delta\psi) - \frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\, \partial_i(\delta\psi) && (0.11)\\
&= \int dt \int d^3\vec{x}\ \partial_t(\pi\delta\psi) - \int dt \int d^3\vec{x}\ \partial_t\pi \cdot \delta\psi && (0.12)\\
&\quad - \int dt \int d^3\vec{x}\ \partial_i\left(\frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\delta\psi\right) + \int dt \int d^3\vec{x}\ \partial_i\left(\frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\right)\delta\psi \\
&= \int d^3\vec{x}\, [\pi\delta\psi]_{\text{boundary}} - \int dt \int d^3\vec{x}\ \partial_t\pi\delta\psi && (0.13)\\
&\quad - \int dt \int_{\text{boundary}} dS\, \frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\delta\psi + \int dt \int d^3\vec{x}\ \partial_i\left(\frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\right)\delta\psi
\end{aligned}$$

since $\delta\psi$ at the boundary is zero, first term vanishes, and we have:

$$\int dt \int d^3\vec{x}\, \pi\, \partial_t(\delta\psi) - \frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\, \partial_i(\delta\psi) = -\int dt \int d^3\vec{x}\ \partial_t\pi\delta\psi + \int dt \int d^3\vec{x}\ \partial_i\left(\frac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\right)\delta\psi \tag{0.14}$$

$$= -\int dt \int d^3\vec{x}\left[\partial_t\pi - \nabla\cdot\left(\frac{\partial\mathcal{H}}{\partial(\nabla\psi)}\right)\right]\delta\psi \tag{0.15}$$

and thus the variation of the action becomes:

$$\delta S = \int dt \int d^3\vec{x}\left(\partial_t\psi - \frac{\partial\mathcal{H}}{\partial\pi}\right)\delta\pi - \left(\partial_t\pi - \nabla\cdot\left(\frac{\partial\mathcal{H}}{\partial(\nabla\psi)}\right)\right)\delta\psi \tag{0.16}$$

for the action to be stationary, the integrand must vanish:

$$\begin{cases} \partial_t\psi - \dfrac{\partial\mathcal{H}}{\partial\pi} & = 0 \\ \partial_t\pi + \dfrac{\partial\mathcal{H}}{\partial\psi} - \partial_i\left(\dfrac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\right) & = 0 \end{cases} \iff \begin{cases} \dfrac{\partial\psi(\vec{x},t)}{\partial t} & = \dfrac{\partial\mathcal{H}[\psi,\pi]}{\partial\pi} \\ \dfrac{\partial\pi(\vec{x},t)}{\partial t} & = -\dfrac{\partial\mathcal{H}[\psi,\pi]}{\partial\psi} + \partial_i\left(\dfrac{\partial\mathcal{H}}{\partial(\partial_i\psi)}\right) \end{cases} \tag{0.17}$$

Thus we have the **canonical equations of fields**:

Theorem 0.1.1 Canonical Equations of Fields

The variational principle in Hamilton formalism leads to the **canonical equations of fields**:

$$\frac{\partial\psi(\vec{x},t)}{\partial t} = \frac{\partial\mathcal{H}(\psi,\partial_i\psi,\pi)}{\partial\pi}, \qquad \frac{\partial\pi(\vec{x},t)}{\partial t} = -\frac{\partial\mathcal{H}(\psi,\partial_i\psi,\pi)}{\partial\psi} + \partial_i\left(\frac{\partial\mathcal{H}(\psi,\partial_i\psi,\pi)}{\partial(\partial_i\psi)}\right) \tag{0.18}$$

Corollary 0.1.1 Canonical Equations of Fields using H

Using Theorem 1.1.1, the **canonical equations of fields** can be re-written using the Hamiltonian H instead of the Hamiltonian density $\mathcal{H}$:

$$\frac{\partial\psi(\vec{x},t)}{\partial t} = \frac{\delta H[\psi,\pi]}{\delta\pi}, \qquad \frac{\partial\pi(\vec{x},t)}{\partial t} = -\frac{\delta H[\psi,\pi]}{\delta\psi} \tag{0.19}$$

0.1.3 Poisson Bracket

In the discrete case, the time evolution of a physical quantity $X(q_i,p_i,t)$, $\dot{X}$ can be written as:

$$\dot{X} = \frac{dX}{dt} = \frac{\partial X}{\partial t} + \sum_i\left(\frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i\right) = \frac{\partial X}{\partial t} + \{X,\, H\} \tag{0.20}$$

In the continous case, a physical quantity X should be an integral of "density" $\tilde{X}$ over some volume:

$$X = \int_V d^3\vec{x}'\, \tilde{X}(\vec{x}',t) \tag{0.21}$$

and assume that $\tilde{X}$ is a function of the field $\psi(\vec{x},t)$ and its conjugate field $\pi(\vec{x},t)$ (which makes X a functional of the fields):

$$\implies \quad X[\psi,\pi,t] = \int_V d^3\vec{x}'\, \tilde{X}(\psi(\vec{x}',t),\pi(\vec{x}',t),t) \tag{0.22}$$

Then the time evolution of X can be written as:

$$\frac{dX[\psi,\pi,t]}{dt} = \frac{d}{dt}\int_V d^3\vec{x'}\,\tilde{X}(\psi(\vec{x'},t),\pi(\vec{x'},t),t) = \int_V d^3\vec{x'}\,\frac{d\tilde{X}(\psi(\vec{x'},t),\pi(\vec{x'},t),t)}{dt} \tag{0.23}$$

$$= \int_V d^3\vec{x'}\,\frac{\partial\tilde{X}(\psi,\pi,t)}{\partial t} + \frac{\partial\tilde{X}(\psi,\pi,t)}{\partial\psi}\,\partial_t\psi + \frac{\partial\tilde{X}(\psi,\pi,t)}{\partial\pi}\,\partial_t\pi \tag{0.24}$$

$$= \int_V d^3\vec{x'}\,\frac{\partial\tilde{X}}{\partial t} + \int_V d^3\vec{x'}\left(\frac{\partial\tilde{X}}{\partial\psi}\frac{\partial\mathcal{H}}{\partial\pi} - \frac{\partial\tilde{X}}{\partial\pi}\frac{\partial\mathcal{H}}{\partial\psi}\right) \tag{0.25}$$

Using Theorem 1.1.1, the partial derivatives can be replaced with functional derivatives:

$$\frac{\partial\tilde{X}}{\partial t} = \frac{\delta X}{\delta t}, \qquad \frac{\partial\mathcal{H}}{\partial\psi} = \frac{\delta H}{\delta\psi}, \qquad \frac{\partial\mathcal{H}}{\partial\pi} = \frac{\delta H}{\delta\pi} \tag{0.26}$$

Thus, we can write the time evolution of X as:

$$\frac{dX[\psi,\pi,t]}{dt} = \frac{\partial X[\psi,\pi,t]}{\partial t} + \int d^3\vec{x'}\left(\frac{\delta X}{\delta\psi}\frac{\delta H}{\delta\pi} - \frac{\delta X}{\delta\pi}\frac{\delta H}{\delta\psi}\right) \tag{0.27}$$

If we were to write the coordinates explicitly,

$$\frac{dX[\psi,\pi,t]}{dt} = \frac{\partial X}{\partial t} + \int d^3\vec{x'}\left(\frac{\delta X[\psi,\pi,t]}{\delta\psi(\vec{x'},t)}\frac{\delta H[\psi,\pi,t]}{\delta\pi(\vec{x'},t)} - \frac{X[\psi,\pi,t]}{\delta\pi(\vec{x'},t)}\frac{\delta H[\psi,\pi,t]}{\delta\psi(\vec{x'},t)}\right) \tag{0.28}$$

Comparing with the discrete case, we can define the **Poisson bracket** of two physical quantities X and Y as:

Definition 0.3: Poisson Bracket of a Field

The **Poisson bracket** of two physical quantities $X[\psi,\pi,t]$ and $Y[\psi,\pi,t]$ is defined as:

$$\{X,\,Y\} = \int d^3\vec{x'}\left(\frac{\delta X}{\delta\psi(\vec{x'},t)}\frac{\delta Y}{\delta\pi(\vec{x'},t)} - \frac{\delta X}{\delta\pi(\vec{x'},t)}\frac{\delta Y}{\delta\psi(\vec{x'},t)}\right) \tag{0.29}$$

Theorem 0.1.2 Time Evolution of a Physical Quantity

The time evolution of a physical quantity $X[\psi,\pi,t]$ can be written as:

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \{X,\,H\} \tag{0.30}$$

Theorem 0.1.3 Poisson Brackets of Fields

For ψ and its conjugate field π, the Poisson bracket satisfies the following properties:

$$\Big\{\psi(\vec{x},t),\, \pi(\vec{x'},t)\Big\} = \delta^3(\vec{x}-\vec{x'}) \tag{0.31}$$

$$\Big\{\psi(\vec{x},t),\, \psi(\vec{x'},t)\Big\} = 0 \tag{0.32}$$

$$\Big\{\pi(\vec{x},t),\, \pi(\vec{x'},t)\Big\} = 0 \tag{0.33}$$

Chapter 1 Mathematical Remarks

1.1 Functional Derivative

1.1.1 Definition

Consider a quantity I defined as follows:

$$I := \int_A^B dx\, F(x) \tag{1.1}$$

Notice that I is not really a function of x, but if you had to say, it is more a "function" of F - may be $F(x) = e^x$, or $F(x) = ax^2 + bx + c$, or, etc. So, to denote the dependence of I on the function F, we write

$$I[f] := \int_A^B dx\, F(x) \tag{1.2}$$

This is called a **(linear) functional**. Now, imagine that F is a function of f, for example, $F[f] = f(x)^2$. By chain rule, a small change in F, denoted as δF, can be expressed as:

$$\delta F[f] = F[f + \delta f] - F[f] \tag{1.3}$$

$$= \frac{\partial F}{\partial f}\,\delta f \tag{1.4}$$

so, similarly,

$$\delta I[F] = I[F + \delta F] - I[F] \tag{1.5}$$

$$= \int_A^B dx\; \delta F[f] \tag{1.6}$$

$$= \int_A^B dx\, \frac{\partial F}{\partial f}\,\delta f \tag{1.7}$$

Then, the **functional derivative** of I with respect to f, $\frac{\delta I}{\delta f}$, is defined as follows:

Definition 1.1: Functional Derivative

If a function $\phi(x)$ exists, such that

$$\delta I = \int_A^B dx\, \phi(x)\, \delta f(x), \tag{1.8}$$

we say that $\phi(x)$ is the **functional derivative** of I with respect to f, and denote it as

$$\frac{\delta I}{\delta f(x)} := \phi(x) \iff \delta I := \int_B^A dx\, \frac{\delta I}{\delta f(x)}\, \delta f(x). \tag{1.9}$$

Immidiately, by comparing Eq.(1.9) and Eq.(1.7), we see the following relation:

Theorem 1.1.1 Functional-density relation

For a quantity $I[F]$ and its density $F[f(x)]$, the functional derivative satisfies the following relation:

$$I = \int_A^B dx\, F[f(x)] \quad \Longrightarrow \quad \frac{\delta I}{\delta f(x)} = \frac{\partial F[f(x)]}{\partial f(x)} \tag{1.10}$$

1.1.2 Two function case

Consider a case where I is the functional of F, which is also a functional of f and g:

$$I[F[f,g]] = \int_B^A dx\, F[f(x), g(x)] \tag{1.11}$$

Or more generally, if a function $D(x)$ satisfies the following Now, let us add some small change of f, δf:

$$\begin{aligned} I[F[f+\delta f, g]] &= \int_B^A dx\, F[f(x)+\delta f(x), g(x)] && (1.12)\\ &= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f && (1.13) \end{aligned}$$

and similarly, by adding δg,

$$\begin{aligned} I[F[f, g+\delta g]] &= \int_B^A dx\, F[f(x), g(x)+\delta g(x)] && (1.14)\\ &= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial g}\,\delta g && (1.15) \end{aligned}$$

Combining these two, we have

$$I[F[f+\delta f, g+\delta g]] = \int_B^A dx\, F[f(x)+\delta f(x), g(x)+\delta g(x)] \tag{1.16}$$

$$= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.17}$$

$$\implies \quad I[F[f+\delta f, g+\delta g]] - I[F[f,g]] = \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.18}$$

In this case, we just like our normal derivative, we should denote the LHS as:

$$\delta I := \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.19}$$

or alternatively,

$$\delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\,\delta f + \frac{\delta I}{\delta g}\,\delta g \tag{1.20}$$

1.1.3 Euler-Lagrange Equation

For the two function case, especially consider that $g = \frac{df}{dx}$, and let us see what happens. Specifically, let us set that $\delta f(A) = \delta f(B) = 0$. Then, we have:

$$\frac{\delta I}{\delta g}\,\delta g = \frac{\delta I}{\delta \frac{df}{dx}}\,\delta \frac{df}{dx} \tag{1.21}$$

we can change the order of the derivative:

$$= \frac{\delta I}{\delta \frac{df}{dx}}\,\frac{d\,\delta f}{dx} \tag{1.22}$$

from the differentiation of a product, we have

$$= \frac{d}{dx}\left(\frac{\delta I}{\delta\left(\frac{df}{dx}\right)}\,\delta f\right) - \frac{d}{dx}\frac{\delta I}{\delta\left(\frac{df}{dx}\right)}\,\delta f \tag{1.23}$$

Now, subsituting this to the two function case, we get:

$$\delta I = \int_B^A dx\, \left[\left(\frac{\delta I}{\delta f} - \frac{d}{dx}\frac{\delta I}{\delta\left(\frac{df}{dx}\right)}\right)\delta f + \frac{d}{dx}\left(\frac{\delta I}{\delta\left(\frac{df}{dx}\right)}\,\delta f\right)\right] \tag{1.24}$$

the total derivative term is zero, since $\delta f(A) = \delta f(B) = 0$. Thus, we have

$$\delta I = \int_B^A dx\, \left(\frac{\delta I}{\delta f} - \frac{d}{dx}\frac{\delta I}{\delta\left(\frac{df}{dx}\right)}\right)\delta f \tag{1.25}$$

and since $I = \int_B^A dx\, F[f(x), g(x)]$, we can say that

$$\frac{\partial F}{\partial f} = \frac{\delta I}{\delta f}, \quad \frac{\partial F}{\partial g} = \frac{\delta I}{\delta g} \tag{1.26}$$

Then

$$\delta I = \int_B^A dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left(\frac{df}{dx}\right)} \right) \delta f \tag{1.27}$$

And if we somehow want to find a minimum of I, we can set $\delta I = 0$:

$$\implies \quad \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left(\frac{df}{dx}\right)} = 0 \tag{1.28}$$

This is called the **Euler-Lagrange equation**.

> **Theorem 1.1.2** Euler-Lagrange Equation
> For a functional $I\Big[F(f, \frac{df}{dx})\Big]$ to be stationary, ($\delta I = 0$), the **Euler-Lagrange equation** must be satisfied:
>
> $$\delta I\left[F(f, \frac{df}{dx})\right] \iff \frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial \left(\frac{df}{dx}\right)}\right) = 0 \tag{1.29}$$

1.1.4 Important Property

In general, consider that the functional F is a function of $f_1(t), f_2(t), \ldots, f_n(t)$:

$$F[f_1, f_2, \ldots, f_n] \implies \delta F = \sum_{j=1}^{n} \frac{\delta F}{\delta f_j(t)} \delta f_j(t) \tag{1.30}$$

If $F = f_i$, we expect that

$$\frac{\delta f_i}{\delta f_i} = 1 \implies \delta f_i = \sum_{j=1}^{n} \frac{\delta f_i}{\delta f_j} \delta f_j \implies \quad \frac{\delta f_i}{\delta f_j} = \delta_{ij} \tag{1.31}$$

Similarly, consider a continous case where F is a function of $f(x)$:,

$$F[f(x,t)] \implies \delta F = \int_A^B dx' \frac{\delta F}{\delta f(x',t)} \delta f(x',t) \tag{1.32}$$

Note the distinction between the variable x and the integration variable x'. This is because x is an "index" of $f(t)$: $f_i \to f(x)$. Then, if we set $F = f(x)$, we expect that

$$\frac{\delta f(x,t)}{\delta f(x,t)} = 1 \implies \delta f(x,t) = \int_A^B dx' \frac{\delta f(x,t)}{\delta f(x',t)} \delta f(x',t) \tag{1.33}$$

Comparing with this with the definition of **Dirac delta function**:

Definition 1.2: Dirac Delta Function

The **Dirac delta function** $\delta(x)$ is defined as a function that satisfies the following property:

$$\int dx'\, \delta(x'-x)\varphi(x') = \varphi(x), \quad {}^{\forall}\varphi(x') \in C^{\infty} \tag{1.34}$$

we have

Theorem 1.1.3 Property of Functional Derivative

For a functional $f_i(t)$ or $f(x,t)$, the functional derivative satisfies the following property:

$$\frac{\delta f_i(t)}{\delta f_j(t)} = \delta_{ij}, \quad \text{or} \quad \frac{\delta f(x,t)}{\delta f(x',t)} = \delta(x-x') \tag{1.35}$$

Also, when $I[F]$ and F are functionals of $f_i(x)$,

$$\delta I = \int_A^B dx'\ \delta F = \sum_i \int_A^B dx'\, \frac{\delta I}{\delta f_i(x')}\, \delta f_i(x') \tag{1.36}$$

$$= \sum_i \int_A^B dx'\, \frac{\partial F}{\partial f_i(x')}\, \delta f_i(x') \tag{1.37}$$

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