Particle Physics

Review for Particle Physics

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Contents

Chapter 1 Analytical Mechanics

1.1 Functional Derivative

1.1.1 Definition

Consider a quantity I defined as follows:

$$I := \int_A^B dx\, F(x) \tag{1.1}$$

Notice that I is not really a function of x, but if you had to say, it is more a "function" of F - may be $F(x) = e^x$, or $F(x) = ax^2 + bx + c$, or, etc. So, to denote the dependence of I on the function F, we write

$$I[f] := \int_A^B dx\, F(x) \tag{1.2}$$

This is called a functional. Now, imagine that F is a function of f, for example, $F[f] = f(x)^2$. By chain rule, a small change in F, denoted as δF, can be expressed as:

$$\delta F[f] = F[f + \delta f] - F[f] \tag{1.3}$$
$$= \frac{\partial F}{\partial f}\,\delta f \tag{1.4}$$

so, similarly,

$$\delta I[F] = I[F + \delta F] - I[F] \tag{1.5}$$
$$= \int_A^B dx\ \delta F[f] \tag{1.6}$$
$$= \int_A^B dx\,\frac{\partial F}{\partial f}\,\delta f \tag{1.7}$$

Then, the functional derivative of I with respect to f, $\frac{\delta I}{\delta f}$, is defined as follows:

Definition 1.1.1: Functional Derivative

If a function $X(x)$ exists, such that

$$\delta I = \int_A^B dx\, X(x)\, \delta f(x), \tag{1.8}$$

we say that $X(x)$ is the functional derivative of I with respect to f, and denote it as

$$\frac{\delta I}{\delta f} := X(x) \iff \delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\, \delta f. \tag{1.9}$$

1.1.2 Two function case

Consider a case where I is the functional of F, which is also a functional of f and g:

$$I[F[f,g]] = \int_B^A dx\, F[f(x), g(x)] \tag{1.10}$$

Or more generally, if a function $D(x)$ satisfies the following Now, let us add some small change of f, δf:

$$\begin{aligned} I[F[f+\delta f, g]] &= \int_B^A dx\, F[f(x)+\delta f(x), g(x)] && (1.11)\\ &= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f && (1.12)\end{aligned}$$

and similarly, by adding δg,

$$\begin{aligned} I[F[f, g+\delta g]] &= \int_B^A dx\, F[f(x), g(x)+\delta g(x)] && (1.13)\\ &= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial g}\,\delta g && (1.14)\end{aligned}$$

Combining these two, we have

$$\begin{aligned} I[F[f+\delta f, g+\delta g]] &= \int_B^A dx\, F[f(x)+\delta f(x), g(x)+\delta g(x)] && (1.15)\\ &= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g && (1.16)\\ \implies \quad I[F[f+\delta f, g+\delta g]] - I[F[f,g]] &= \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g && (1.17)\end{aligned}$$

In this case, we just like our normal derivative, we should denote the LHS as:

$$\delta I := \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.18}$$

or alternatively,

$$\delta I := \int_B^A dx \, \frac{\delta I}{\delta f} \, \delta f + \frac{\delta I}{\delta g} \, \delta g \tag{1.19}$$

1.1.3 Euler-Lagrange Equation

For the two function case, especially consider that $g = \frac{df}{dx}$, and let us see what happens. Specifically, let us set that $\delta f(A) = \delta f(B) = 0$. Then, we have:

$$\frac{\delta I}{\delta g} \, \delta g = \frac{\delta I}{\delta \frac{df}{dx}} \, \delta \frac{df}{dx} \tag{1.20}$$

we can change the order of the derivative:

$$= \frac{\delta I}{\delta \frac{df}{dx}} \frac{d \, \delta f}{dx} \tag{1.21}$$

from the differentiation of a product, we have

$$= \frac{d}{dx} \left(\frac{\delta I}{\delta \frac{df}{dx}} \, \delta f \right) - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}} \, \delta f \tag{1.22}$$

Now, subsituting this to the two function case, we get:

$$\delta I = \int_B^A dx \left[\left(\frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}} \right) \delta f + \frac{d}{dx} \left(\frac{\delta I}{\delta \frac{df}{dx}} \, \delta f \right) \right] \tag{1.23}$$

the total derivative term is zero, since $\delta f(A) = \delta f(B) = 0$. Thus, we have

$$\delta I = \int_B^A dx \left(\frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}} \right) \delta f \tag{1.24}$$

and since $I = \int_B^A dx \, F[f(x), g(x)]$, we can say that

$$\frac{\partial F}{\partial f} = \frac{\delta I}{\delta f}, \quad \frac{\partial F}{\partial g} = \frac{\delta I}{\delta g} \tag{1.25}$$

Then

$$\delta I = \int_B^A dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \frac{df}{dx}} \right) \delta f \tag{1.26}$$

And if we somehow want to find a minimum of I, we can set $\delta I = 0$:

$$\implies \quad \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \frac{df}{dx}} = 0 \tag{1.27}$$

This is called the Euler-Lagrange equation.

1.1.4 Important Property

In general, consider that the functional F is a function of $f_1, f_2, \dots, f_n$:

$$F[f_1, f_2, \dots, f_n] \implies \delta F = \sum_{j=1}^{n} \frac{\partial F}{\partial f_j} \delta f_j \tag{1.28}$$

$$\tag{1.29}$$

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1.2 Lagrange Formalism

1.2.1 Quick Recap: Newtonian Mechanics

In Newtonian mechanics, the motion of a particle is described through a few important quantities: for a particle of (inertial) mass m, position $\vec{r}$, we have

$$\text{velocity} : \vec{v} = \frac{d\vec{r}}{dt} \tag{1.30}$$

$$\text{acceleration} : \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{1.31}$$

$$\text{momentum} : \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{1.32}$$

and the relations between these quantities, in the presence of external forces $\vec{F}^{(i)}_{\text{ext}}$ acting on the particle, are given by Newton's second law:

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = \sum_i \vec{F}^{(i)}_{\text{ext}} \tag{1.33}$$

The work done by such forces is given by

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}}, \quad \text{where } l \text{ is the path of the particle.} \tag{1.34}$$

This is the energy change of the particle through the motion:

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}} \tag{1.35}$$

$$= \int_{t_i}^{t_f} dt\, \vec{v} \cdot m\frac{d\vec{v}}{dt} \tag{1.36}$$

$$= \int_{t_i}^{t_f} dt\, \frac{m}{2}\frac{d}{dt}\vec{v}^2 \tag{1.37}$$

$$= \frac{m}{2}\vec{v}_f^2 - \frac{m}{2}\vec{v}_i^2 \tag{1.38}$$

meaning that $m\vec{v}^2/2$ is the energy due to the motion of the particle: the kinetic energy T:

$$T = \frac{m}{2}\vec{v}^2 \tag{1.39}$$

Now, often, the external force acting on the particle is due to a potential V:

$$\vec{F}_{\text{ext}} = -\nabla V \tag{1.40}$$

For example, for a 1D spring, the potential is given by

Example 1.2.1 (1D spring/ Harmonic potential)

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -kx \tag{1.41}$$

or the electrostatic potential:

Example 1.2.2 (Electrostatic potential)

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r} \implies F_{\text{ext}} = -\nabla V = -\frac{q_1q_2}{4\pi\varepsilon_0 r^2}\hat{r} \tag{1.42}$$

Now, for a particle whose the external forces are given by a potential:

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla V \iff -m\frac{d^2\vec{r}}{dt^2} - \nabla V = 0 \tag{1.43}$$

This looks as if the forces $-\nabla V$ and $m\ddot{\vec{r}}$ are in equilibrium. So, if we move a particle by an infinitisimal distance $\delta\vec{r}$, the total work done by these forces must be zero:

$$\left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{1.44}$$

at any time t. We want to apply this for entire path of the motion of the particle, from t_i to t_f. Then, the integral of this equation over the time interval $[t_i, t_f]$ gives

$$\delta I = \int_{t_i}^{t_f} dt\left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{1.45}$$

now, we can apply integration by parts:

$$\begin{aligned}
& & \frac{d}{dt}\left[\dot{\vec{r}}\cdot\delta\vec{r}\right] &= \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\frac{d\,\delta\vec{r}}{dt} & (1.46)\\
& & &= \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} & (1.47)\\
&\iff & -m\ddot{\vec{r}} &= -\frac{d}{dt}\left(\dot{\vec{r}}\cdot\delta\vec{r}\right) + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} & (1.48)\\
& & &= -\frac{d}{dt}\left(m\dot{\vec{r}}\cdot\delta\vec{r}\right) + \delta\left(\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right), & (1.49)
\end{aligned}$$

where we used the commutitivity of $\frac{d(\cdot)}{dt}$ and $\delta(\cdot)$. Then the integral becomes

$$\begin{aligned} \delta I &= \int_{t_i}^{t_f} dt \left(\delta \left[\frac{m}{2} \left(\frac{d\vec{r}}{dt} \right)^2 \right] - \delta V \right) && (1.50) \\ &= \int_{t_i}^{t_f} dt \, (\delta T - \delta V) = 0 && (1.51) \\ \iff \quad \delta I &= \delta \int_{t_i}^{t_f} dt \, (T - V) = 0 && (1.52) \end{aligned}$$

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1.2.2 Lagrangian and Variational Principle

In Lagrangian mechanics, we will use a different approach to describe the motion of a particle than the Newtonian mechanics. Instead of using the usual Euclidean space, we will use a configuration space $\mathcal{C}$, which is the space of all possible positions of the particle. This space is spanned by the so-caled generalized coordinates q_i, and their time derivatives (or generalized velocity) $\dot{q}_i := \frac{dq_i}{dt}$. Now, let us define quantities called the Lagrangian L and action S:

$$L := T - V, \qquad S := \int_{t_i}^{t_f} dt \, L \tag{1.53}$$

For one particle, the Lagrangian in general contains the position $q_i(t)$ and velocity $\dot{q}_i(t)$ and time t:

$$L = L(q_i, \dot{q}_i, t) \tag{1.54}$$

The quantity called action is then given by

$$S = \int_{t_i}^{t_f} dt \, L(q_i(t), \dot{q}_i(t), t) = \int_{t_i}^{t_f} dt \, (T - V) \tag{1.55}$$

and the variation of the action δS is given by

$$\delta S = \delta \int_{t_i}^{t_f} dt \, (T - V) \tag{1.56}$$

which is an identical expression to (1.52). So, we postulate that the motion of the particle is such that the action is stationary:

Proposition 1.2.1 Variational Principle
The motion of a particle is such that the action S is stationary, i.e., $\delta S = 0$.

$$\delta S = \delta \int_{t_i}^{t_f} dt \, L(q_i(t), \dot{q}_i(t), t) = 0 \tag{1.57}$$

As to show why this maybe useful, let us compare between Eq. (1.45):

$$\delta I = \delta S[L] = \int_{t_i}^{t_f} dt \left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V \right) \cdot \delta\vec{r} = 0 \tag{1.58}$$

and Eq. (1.26):

$$\delta I = \delta S[f] = \int_{B}^{A} dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial \frac{df}{dx}} \right) \delta f = 0 \tag{1.59}$$

which immidiately gives that the EoM is the Euler-Lagrange equation, if we set $F = L(q_i, \dot{q}_i, t)$:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \iff m\frac{d^2\vec{r}}{dt^2} = -\nabla V \tag{1.60}$$

The generalization to multiple particles is straightforward:

$$L = \sum_n \frac{m_n}{2}\dot{\vec{r}}_n^2 - V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \tag{1.61}$$

The Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial \vec{r}_i^{(N)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i^{(N)}}\right) = 0 \tag{1.62}$$

1.2.3 Harmonic Oscillator

Let us consider a particle of mass m in a 1D harmonic potential:

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -\nabla V = -kx \tag{1.63}$$

then the Lagrangian is given by

$$L = T - V = \frac{m}{2}\dot{x}^2 - \frac{1}{2}kx^2 \tag{1.64}$$

The Euler-Lagrange equation yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \implies \frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{1.65}$$

$$\implies \quad -kx - \frac{d}{dt}(m\dot{x}) = 0 \tag{1.66}$$

$$\implies \quad m\frac{d^2x}{dt^2} = -kx \tag{1.67}$$

which is the equation of motion of a harmonic oscillator in Newtonian mechanics. Here, notice that

$$\frac{\partial L}{\partial x} = -kx = -\nabla V = F_{\text{ext}} \tag{1.68}$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p \tag{1.69}$$

these derivatives of the Lagrangian are the generalized force and conjugate momentum, respectively.

$$\frac{\partial L}{\partial q} = F_{\text{g}}, \quad \frac{\partial L}{\partial \dot{q}} = p_{\text{g}} \tag{1.70}$$

1.3 Hamiltonian Formalism

Hamiltonian formalism is another way to obtain the equation of motion. Sometimes, it is more convinient to have a coupled 1st order PDEs, rather than a 2nd order PDE, which is the case in Hamiltonian formalism and Lagrangian formalism respectively. Honestly, derivation of Hamiltonian formalism is motivated by not a clever foresight, but more by the clairvoyance from the people who already developed it.

1.3.1 Hamiltonian and Legendre Transform

Definition 1.3.1: Legendre Transform (in analytical mechanics)

Legendre transform

Chapter 2

Quantum Mechanics

Chapter 3

Physics of Fields

3.1 Variational Principle in Fluid Mechanics

3.1.1 Overview: Cauchy's Equation of Motion

The equation of motion of a point mass is given by Newton's second law:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} \tag{3.1}$$

The equation of motion of a fluid (in general) is given by Cauchy's equation of motion:

$$\int_V \rho\, dV\, \frac{D\vec{u}}{Dt} = \oint_S dS\, \sigma\vec{n} + \int_V \rho\, dV\, \vec{\tilde{F}} \tag{3.2}$$

where

- V: Volume of the fluid
- S: Surface of V
- $\rho(\vec{r}, t)$: Density of the fluid at position $\vec{r}$ and time t
- $\vec{u}(\vec{r}, t)$: Velocity of the fluid at position $\vec{r}$ and time t
- $\vec{n}$: Normal vector of the surface S pointing outward
- $\sigma(\vec{r}, t)$: Stress tensor of the fluid
- $\vec{\tilde{F}}$: External force per unit volume acting on the infinitisimal volume dV of fluid

the stress tensor shows how the forces act on any surface within a fluid.

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{3.3}$$

The Newtonian derivation of Cauchy's equation of motion is based on the conservation of the momentum of a fluid element:

$$\left(\frac{\partial p}{\partial t} + \Delta p \text{ flux per time}\right) = (\Delta p \text{ from the surface force}) + (\Delta p \text{ from the volume force}) \tag{3.4}$$

$$\implies \quad \rho\, dV \frac{D\vec{u}}{Dt} = \nabla \cdot (\sigma \vec{n}) + \rho\, dV \vec{\tilde{F}} \tag{3.5}$$

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3.1.2 Lagrange Derivative

Imagine that you want to measure the height of the water in a river at a certain point and time. Let us assume that you put a very light, small box with some measuring equipment on the water surface at some point $\vec{r}_i$ at time t_i. Then we put another box at another point $\vec{r}_f$ at the same time t_i, but we fix this box so that it does not move.

Since the water surface is not static, box 1 will be carried by the water flow. Let us denote the position of the box 1 at time t as $\vec{\xi}(\vec{r}_i, t; t_i)$, where the box 1 is placed at the point $\vec{r}_i$ at time t_i.

Let us denote the height of the water surface at a point $\vec{r}$ at time t by $X(\vec{r}, t)$. Then, we can write the height of the water surface at the positions of box 1 and 2 at time t as

$$X(\vec{\xi}, t) = X_1(t) \tag{3.6}$$

$$X(\vec{r}_f, t) = X_2(t) \tag{3.7}$$

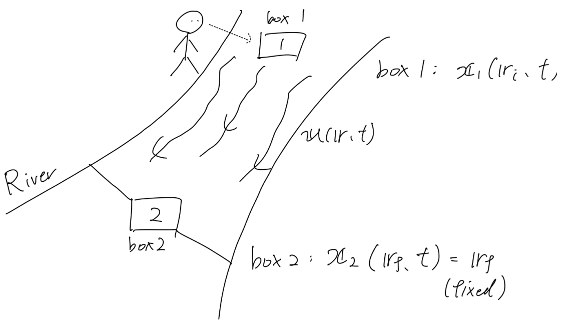

Figure 3.1: Boxes on a river

Finally, assume that the box 1 will reach $\vec{r}_f$ at time t_f.

Now, at initial time t_i, the change in height of the water surface is given by:

$$X(\vec{r}_f, t_i) - X(\vec{r}_i, t_i) = \nabla X(\vec{r}_i, t_i) \cdot (\vec{r}_f - \vec{r}_i) \tag{3.8}$$

$$= \nabla X(\vec{r}_i, t_i) \cdot \Delta \vec{r} \tag{3.9}$$

assuming that $\Delta \vec{r} := \vec{r}_f - \vec{r}_i$ is small enough. On the other hand, for box 2, the change in height of the water surface in a time interval $\Delta t = t_f - t_i$ is given by:

$$X(\vec{r}_f, t_f) - X(\vec{r}_f, t_i) = \frac{\partial X(\vec{r}_f, t_i)}{\partial t} \Delta t \tag{3.10}$$

Thus, the total change of X measured by box 1, during Δt, moving at the velocity $\vec{u}$ is given by

$$X(\vec{r}_f, t_f) - X(\vec{r}_i, t_i) = X(\vec{r}_f, t_f) \underbrace{-X(\vec{r}_f, t_i) + X(\vec{r}_f, t_i)}_{=0} - X(\vec{r}_i, t_i) \tag{3.11}$$

$$= \frac{\partial X}{\partial t}\Delta t + \nabla X(\vec{r}_i, t_i) \cdot \Delta \vec{r} \tag{3.12}$$

Now, since $\vec{r}_i = \vec{\xi}(\vec{r}_i, t_i)$ and $\vec{r}_f = \vec{\xi}(\vec{r}_i, t_f)$, we can rewrite the equation above:

$$X(\vec{\xi}(\vec{r}_i, t_f), t_f) - X(\vec{\xi}, t_i) = \frac{\partial X}{\partial t}\Delta t + \nabla X(\vec{\xi}, t_i) \cdot \Delta \vec{\xi} \tag{3.13}$$

where $\Delta\vec{\xi} = \vec{\xi}(\vec{r}_i, t_f) - \vec{\xi}(\vec{r}_i, t_i)$.

Using $\Delta t = t_f - t_i$, we can rewrite the equation above as:

$$\frac{X(\vec{\xi} + \Delta\vec{\xi}, t_i + \Delta t) - X(\vec{\xi}, t_i)}{\Delta t} = \frac{\partial X}{\partial t} + \nabla X(\vec{\xi}, t_i) \cdot \frac{\Delta\vec{\xi}}{\Delta t} \tag{3.14}$$

$$\tag{3.15}$$

Since the box 1 is a part of the fluid, the time derivative of $\vec{\xi}$ is the velocity of the fluid at the position $\vec{\xi}$:

$$\frac{\partial \vec{\xi}(\vec{r}_i, t)}{\partial t} = \vec{u}(\vec{\xi}, t) \tag{3.16}$$

Note

Note In this derivative, we fix the starting position $\vec{r}_i$, we only care about the time evolution from the position $\vec{r}_i$. This is different from the total derivative, which takes into account a change in the starting position $\vec{r}_i$.

thus, by taking $\Delta t \to 0 (\implies \Delta x \to 0)$, we can rewrite the equation above as:

$$\frac{DX(\vec{\xi}, t_i)}{Dt} := \frac{\partial X(\vec{\xi}, t_i)}{\partial t} + \vec{u}(\vec{\xi}, t_i) \cdot \nabla X(\vec{\xi}, t_i) \tag{3.17}$$

where $\frac{D}{Dt}$ indicates that we are tracking the box 1 and its measurement of X. This is caled the Lagrange derivative.

Now, define a new quantity called the "position function" $\vec{x}(\vec{r}, t)$:

$$\vec{x}(\vec{r}, t) := \vec{r} \tag{3.18}$$

If we measure this position function at the box 1 at $t = t_i$,

$$\vec{x}(\vec{r}_i, t_i) = \vec{r}_i \tag{3.19}$$

and notice that tracking the position of box 1 is equivalent to observing the time-evolved position $\xi(\vec{r}_i, t_i)$:

$$\frac{\partial \vec{\xi}}{\partial t} = \frac{D\vec{x}}{Dt} = \vec{u}(\vec{x}, t) \tag{3.20}$$

3.1.3 Derivation from Action Integral

In Newtonian mechanics, for a particle of mass m, the action integral S is given by:

$$S = \int dt\, L = \int dt\, m\mathcal{L}, \quad \mathcal{L} := \frac{L}{m} = \left[\frac{1}{2}\dot{\vec{x}}^2 - \tilde{V}(\vec{x})\right] \tag{3.21}$$

where $\mathcal{L}$ is the Lagrangian density, and

$$\tilde{V}(\vec{x}) = \frac{V(\vec{x})}{m} \tag{3.22}$$

by assuming a similar form of the Lagrangian density for a fluid, we can write the action integral for a fluid as:

$$S = \int dt\, L = \int dt \int_V \rho(\vec{x}, t)\, dV(\vec{x}, t)\, \mathcal{L}\left(\vec{x}, \frac{D\vec{x}}{Dt}\right) \tag{3.23}$$

where the Lagrangian density is

$$\mathcal{L}\left(\vec{x}(\vec{r}, t), \frac{D\vec{x}(\vec{r}, t)}{Dt}\right) = \frac{1}{2}\left(\frac{D\vec{x}(\vec{r}, t)}{Dt}\right)^2 - \tilde{V}(\vec{x}(\vec{r}, t)) + \frac{1}{\rho(\vec{r}, t)}\nabla \cdot \sigma(\vec{r}, t)\vec{x}(\vec{r}, t) \tag{3.24}$$

Then the Euler-Lagrange equation for this Lagrangian density is given by:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} - \frac{D}{Dt}\left(\frac{\partial \mathcal{L}}{\partial \left(\frac{D\vec{x}}{Dt}\right)}\right) = 0 \tag{3.25}$$

calculating the partial derivative gives:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} = \frac{1}{\rho}\nabla \cdot \sigma - \nabla\tilde{V}(\vec{x}) = \frac{1}{\rho}\nabla \cdot \sigma + \vec{\tilde{F}} \tag{3.26}$$

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{D\vec{x}}{Dt}\right)} = \frac{D\vec{x}}{Dt} = \vec{u}(\vec{x}, t) \tag{3.27}$$

hence the Euler-Lagrange equation becomes:

$$\frac{D\vec{u}}{Dt} = \frac{1}{\rho}\nabla \cdot \sigma + \vec{\tilde{F}} \tag{3.28}$$

by integrating over the volume V, we get

$$\int_V \rho\, dV\, \frac{D\vec{u}}{Dt} = \int_V dV\, \nabla \cdot \sigma + \int_V \rho\, dV\, \vec{\tilde{F}} \tag{3.29}$$

$$\Longrightarrow \quad \int_V \rho\, dV\, \frac{D\vec{u}}{Dt} = \int_V ds\, \vec{n} \cdot \sigma + \int_V \rho\, dV\, \vec{\tilde{F}} \tag{3.30}$$

3.1.4 Hamilton Formalism

Similarly to the point mass case, we should define the momentum density $\vec{\pi}(\vec{r}, t)$ as the derivative of the Lagrangian density with respect to the velocity:

$$\vec{\pi}(\vec{r}, t) = \frac{\partial \mathcal{L}}{\partial \left(\frac{D\vec{x}}{Dt}\right)} \tag{3.31}$$

then the Hamiltonian density should be defined by the Legendre transformation:

$$\mathcal{H}(\vec{\pi}, \vec{x}) = \vec{\pi} \cdot \frac{D\vec{x}}{Dt} - \mathcal{L}\left(\vec{x}, \frac{D\vec{x}}{Dt})\right) \tag{3.32}$$

$$= \frac{1}{2}\left(\frac{D\vec{x}}{Dt}\right)^2 + \tilde{V}(\vec{x}) - \frac{1}{\rho}\nabla \cdot \sigma\vec{x} \tag{3.33}$$

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3.2 1D String, Revisited