Particle Physics

Review for Particle Physics

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Contents

Chapter 1

Analytical Mechanics

1.1 Functional Derivative

1.1.1 Definition

Consider a quantity I defined as follows:

$$I := \int_A^B dx\, F(x) \tag{1.1}$$

Notice that I is not really a function of x, but if you had to say, it is more a "function" of F - may be $F(x) = e^x$, or $F(x) = ax^2 + bx + c$, or, etc. So, to denote the dependence of I on the function F, we write

$$I[f] := \int_A^B dx\, F(x) \tag{1.2}$$

This is called a **functional**. Now, imagine that F is a function of f, for example, $F[f] = f(x)^2$. By chain rule, a small change in F, denoted as δF, can be expressed as:

$$\delta F[f] = F[f + \delta f] - F[f] \tag{1.3}$$

$$= \frac{\partial F}{\partial f}\,\delta f \tag{1.4}$$

so, similarly,

$$\delta I[F] = I[F + \delta F] - I[F] \tag{1.5}$$

$$= \int_A^B dx\ \delta F[f] \tag{1.6}$$

$$= \int_A^B dx\, \frac{\partial F}{\partial f}\,\delta f \tag{1.7}$$

Then, the **functional derivative** of I with respect to f, $\frac{\delta I}{\delta f}$, is defined as follows:

Definition 1.1: Functional Derivative

If a function $X(x)$ exists, such that

$$\delta I = \int_A^B dx\, X(x)\, \delta f(x), \tag{1.8}$$

we say that $X(x)$ is the **functional derivative** of I with respect to f, and denote it as

$$\frac{\delta I}{\delta f} := X(x) \iff \delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\, \delta f. \tag{1.9}$$

1.1.2 Two function case

Consider a case where I is the functional of F, which is also a functional of f and g:

$$I[F[f, g]] = \int_B^A dx\, F[f(x), g(x)] \tag{1.10}$$

Or more generally, if a function $D(x)$ satisfies the following Now, let us add some small change of f, δf:

$$\begin{aligned} I[F[f + \delta f, g]] &= \int_B^A dx\, F[f(x) + \delta f(x), g(x)] &(1.11)\\ &= \int_B^A dx\, F[f, g] + \frac{\partial F}{\partial f}\, \delta f &(1.12) \end{aligned}$$

and similarly, by adding δg,

$$\begin{aligned} I[F[f, g + \delta g]] &= \int_B^A dx\, F[f(x), g(x) + \delta g(x)] &(1.13)\\ &= \int_B^A dx\, F[f, g] + \frac{\partial F}{\partial g}\, \delta g &(1.14) \end{aligned}$$

Combining these two, we have

$$\begin{aligned} I[F[f + \delta f, g + \delta g]] &= \int_B^A dx\, F[f(x) + \delta f(x), g(x) + \delta g(x)] &(1.15)\\ &= \int_B^A dx\, F[f, g] + \frac{\partial F}{\partial f}\, \delta f + \frac{\partial F}{\partial g}\, \delta g &(1.16)\\ \implies \quad I[F[f + \delta f, g + \delta g]] - I[F[f, g]] &= \int_B^A dx\, \frac{\partial F}{\partial f}\, \delta f + \frac{\partial F}{\partial g}\, \delta g &(1.17) \end{aligned}$$

In this case, we just like our normal derivative, we should denote the LHS as:

$$\delta I := \int_B^A dx\, \frac{\partial F}{\partial f}\, \delta f + \frac{\partial F}{\partial g}\, \delta g \tag{1.18}$$

or alternatively,

$$\delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\, \delta f + \frac{\delta I}{\delta g}\, \delta g \tag{1.19}$$

1.1.3 Euler-Lagrange Equation

For the two function case, especially consider that $g = \frac{df}{dx}$, and let us see what happens. Specifically, let us set that $\delta f(A) = \delta f(B) = 0$. Then, we have:

$$\frac{\delta I}{\delta g}\, \delta g = \frac{\delta I}{\delta \frac{df}{dx}}\, \delta \frac{df}{dx} \tag{1.20}$$

we can change the order of the derivative:

$$= \frac{\delta I}{\delta \frac{df}{dx}} \frac{d\, \delta f}{dx} \tag{1.21}$$

from the differentiation of a product, we have

$$= \frac{d}{dx}\left(\frac{\delta I}{\delta \frac{df}{dx}}\, \delta f \right) - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}}\, \delta f \tag{1.22}$$

Now, subsituting this to the two function case, we get:

$$\delta I = \int_B^A dx \left[\left(\frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}} \right) \delta f + \frac{d}{dx}\left(\frac{\delta I}{\delta \frac{df}{dx}}\, \delta f \right) \right] \tag{1.23}$$

the total derivative term is zero, since $\delta f(A) = \delta f(B) = 0$. Thus, we have

$$\delta I = \int_B^A dx \left(\frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \frac{df}{dx}} \right) \delta f \tag{1.24}$$

and since $I = \int_B^A dx\, F[f(x), g(x)]$, we can say that

$$\frac{\partial F}{\partial f} = \frac{\delta I}{\delta f}, \quad \frac{\partial F}{\partial g} = \frac{\delta I}{\delta g} \tag{1.25}$$

Then

$$\delta I = \int_B^A dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left(\frac{df}{dx}\right)} \right) \delta f \tag{1.26}$$

And if we somehow want to find a minimum of I, we can set $\delta I = 0$:

$$\implies \quad \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left(\frac{df}{dx}\right)} = 0 \tag{1.27}$$

This is called the **Euler-Lagrange equation**.

Theorem 1.1.1 Euler-Lagrange Equation

For a functional $I\left[F(f, \frac{df}{dx})\right]$ to be stationary, ($\delta I = 0$), the **Euler-Lagrange equation** must be satisfied:

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial\left(\frac{df}{dx}\right)} = 0 \tag{1.28}$$

1.1.4 Important Property

In general, consider that the functional F is a function of $f_1(t), f_2(t), \ldots, f_n(t)$:

$$F[f_1, f_2, \ldots, f_n] \implies \delta F = \sum_{j=1}^{n} \frac{\delta F}{\delta f_j(t)} \delta f_j(t) \tag{1.29}$$

If $F = f_i$, we expect that

$$\frac{\delta f_i}{\delta f_i} = 1 \implies \delta f_i = \sum_{j=1}^{n} \frac{\delta f_i}{\delta f_j} \delta f_j \implies \quad \frac{\delta f_i}{\delta f_j} = \delta_{ij} \tag{1.30}$$

Similarly, consider a continous case where F is a function of $f(x)$:,

$$F[f(x,t)] \implies \delta F = \int_A^B dx' \, \frac{\delta F}{\delta f(x',t)} \delta f(x',t) \tag{1.31}$$

Note the distinction between the variable x and the integration variable x'. This is because x is an "index" of $f(t)$: $f_i \to f(x)$. Then, if we set $F = f(x)$, we expect that

$$\frac{\delta f(x,t)}{\delta f(x,t)} = 1 \implies \delta f(x,t) = \int_A^B dx' \, \frac{\delta f(x,t)}{\delta f(x',t)} \delta f(x',t) \tag{1.32}$$

Comparing with this with the definition of **Dirac delta function**:

Definition 1.2: Dirac Delta Function

The **Dirac delta function** $\delta(x)$ is defined as a function that satisfies the following property:

$$\int dx' \, \delta(x'-x)\varphi(x') = \varphi(x), \quad {}^\forall\varphi(x') \in C^\infty \tag{1.33}$$

we have

Theorem 1.1.2 Property of Functional Derivative

For a functional $f_i(t)$ or $f(x,t)$, the functional derivative satisfies the following property:

$$\frac{\delta f_i(t)}{\delta f_j(t)} = \delta_{ij}, \quad \text{or} \quad \frac{\delta f(x,t)}{\delta f(x',t)} = \delta(x-x') \tag{1.34}$$

[1]

1.2 Lagrange Formalism

1.2.1 Quick Recap: Newtonian Mechanics

In Newtonian mechanics, the motion of a particle is described through a few important quantities: for a particle of (inertial) mass m, position $\vec{r}$, we have

$$\text{velocity} : \vec{v} = \frac{d\vec{r}}{dt} \tag{1.35}$$

$$\text{acceleration} : \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{1.36}$$

$$\text{momentum} : \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{1.37}$$

and the relations between these quantities, in the presence of external forces $\vec{F}^{(i)}_{\text{ext}}$ acting on the particle, are given by *Newton's second law*:

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = \sum_i \vec{F}^{(i)}_{\text{ext}} \tag{1.38}$$

The work done by such forces is given by

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}}, \quad \text{where } l \text{ is the path of the particle.} \tag{1.39}$$

This is the energy change of the particle through the motion:

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}} \tag{1.40}$$

$$= \int_{t_i}^{t_f} dt\, \vec{v} \cdot m\frac{d\vec{v}}{dt} \tag{1.41}$$

$$= \int_{t_i}^{t_f} dt\, \frac{m}{2}\frac{d}{dt}\vec{v}^2 \tag{1.42}$$

$$= \frac{m}{2}\vec{v}_f^2 - \frac{m}{2}\vec{v}_i^2 \tag{1.43}$$

meaning that $m\vec{v}^2/2$ is the energy due to the motion of the particle: the **kinetic energy** T:

$$T = \frac{m}{2}\vec{v}^2 \tag{1.44}$$

Now, often, the external force acting on the particle is due to a potential V:

$$\vec{F}_{\text{ext}} = -\nabla V \tag{1.45}$$

For example, for a 1D spring, the potential is given by

Example 1.2.1 (1D spring/ Harmonic potential)

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -kx \tag{1.46}$$

or the electrostatic potential:

Example 1.2.2 (Electrostatic potential)

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \implies F_{\text{ext}} = -\nabla V = -\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r} \tag{1.47}$$

Now, for a particle whose the external forces are given by a potential:

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla V \iff -m\frac{d^2\vec{r}}{dt^2} - \nabla V = 0 \tag{1.48}$$

This looks as if the forces $-\nabla V$ and $m\ddot{\vec{r}}$ are in equilibrium. So, if we move a particle by an infinitisimal distance $\delta\vec{r}$, the total work done by these forces must be zero:

$$\left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{1.49}$$

at any time t. We want to apply this for entire path of the motion of the particle, from t_i to t_f. Then, the integral of this equation over the time interval $[t_i, t_f]$ gives

$$\delta I = \int_{t_i}^{t_f} dt\left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{1.50}$$

now, we can apply integration by parts:

$$\begin{aligned}
\frac{d}{dt}\left[\dot{\vec{r}}\cdot\delta\vec{r}\right] &= \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\frac{d\,\delta\vec{r}}{dt} && (1.51)\\
&= \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} && (1.52)\\
\iff \quad -m\ddot{\vec{r}} &= -\frac{d}{dt}\left(\dot{\vec{r}}\cdot\delta\vec{r}\right) + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} && (1.53)\\
&= -\frac{d}{dt}\left(m\dot{\vec{r}}\cdot\delta\vec{r}\right) + \delta\left(\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right), && (1.54)
\end{aligned}$$

where we used the commutitivity of $\frac{d(\cdot)}{dt}$ and $\delta(\cdot)$. Then the integral becomes

$$\delta I = \int_{t_i}^{t_f} dt \left(\delta\left[\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right] - \delta V\right) \tag{1.55}$$

$$= \int_{t_i}^{t_f} dt\,(\delta T - \delta V) = 0 \tag{1.56}$$

$$\iff \quad \delta I = \delta \int_{t_i}^{t_f} dt\,(T - V) = 0 \tag{1.57}$$

[2]

1.2.2 Lagrangian and Variational Principle

In Lagrangian mechanics, we will use a different approach to describe the motion of a particle than the Newtonian mechanics. Instead of using the usual Euclidean space, we will use a *configuration space* $\mathcal{C}$, which is the space of all possible positions of the particle. This space is spanned by the so-called **generalized coordinates** q_i, and their time derivatives (or generalized velocity) $\dot{q}_i := \frac{dq_i}{dt}$. Now, let us define quantities called the **Lagrangian** L and **action** S:

Definition 1.3: Lagrangian L and Action S

The Lagrangian L is defined as the difference between the kinetic energy T and potential energy V of a particle:

$$L := T - V, \qquad S[L] := \int_{t_i}^{t_f} dt\, L \tag{1.58}$$

Now, **variation** of the action δS is given by

$$\delta S = \delta \int_{t_i}^{t_f} dt\,(T - V) \tag{1.59}$$

which is an identical expression to (1.57). So, we postulate that the motion of the particle is such that the action is stationary:

Principle 1.1: Variational Principle

The motion of a particle is such that the action S is stationary, i.e., $\delta S = 0$.

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q_i(t), \dot{q}_i(t), t) = 0 \tag{1.60}$$

As to show why this maybe useful, let us compare between Eq. (1.50):

$$\delta I = \delta S[L] = \int_{t_i}^{t_f} dt \left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right) \cdot \delta\vec{r} = 0 \tag{1.61}$$

and Eq. (1.26):

$$\delta I = \delta S[f] = \int_B^A dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \frac{df}{dx}} \right) \delta f = 0 \tag{1.62}$$

which immidiately gives that the EoM is the Euler-Lagrange equation, if we set $F = L(q_i, \dot{q}_i, t)$:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \iff m\frac{d^2\vec{r}}{dt^2} = -\nabla V \tag{1.63}$$

The generalization to multiple particles is straightforward:

$$L = \sum_n \frac{m_n}{2} \dot{\vec{r}}_n^2 - V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \tag{1.64}$$

The Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial \vec{r}_i^{(N)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i^{(N)}} \right) = 0 \tag{1.65}$$

1.2.3 Harmonic Oscillator

Let us consider a particle of mass m in a 1D harmonic potential:

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -\nabla V = -kx \tag{1.66}$$

then the Lagrangian is given by

$$L = T - V = \frac{m}{2}\dot{x}^2 - \frac{1}{2}kx^2 \tag{1.67}$$

The Euler-Lagrange equation yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \implies \frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{1.68}$$

$$\implies \quad -kx - \frac{d}{dt}(m\dot{x}) = 0 \tag{1.69}$$

$$\implies \quad m\frac{d^2x}{dt^2} = -kx \tag{1.70}$$

which is the equation of motion of a harmonic oscillator in Newtonian mechanics. Here, notice that

$$\frac{\partial L}{\partial x} = -kx = -\nabla V = F_{\text{ext}} \tag{1.71}$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p \tag{1.72}$$

these derivatives of the Lagrangian are the **generalized force** and **conjugate momentum**, respectively.

$$\frac{\partial L}{\partial q} = F_{\mathrm{g}}, \quad \frac{\partial L}{\partial \dot{q}} = p_{\mathrm{g}} \tag{1.73}$$

1.3 Hamiltonian Formalism

1.3.1 Legendre Transform

Consider a function $f(x, y)$. The total differential of f is given by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{1.74}$$

Often we want to find another function g such that

$$g = p \cdot y - f(x, y) \tag{1.75}$$

and the total differential of g is given by

$$\begin{aligned} dg &= \frac{\partial g}{\partial p}\,dp + \frac{\partial g}{\partial y}\,dy - df && (1.76) \\ &= y\,dp + p\,dy - \frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy && (1.77) \\ &= y\,dp - \frac{\partial f}{\partial x}\,dx + \left(p - \frac{\partial f}{\partial y}\right)dy && (1.78) \end{aligned}$$

for this function g to be a function of x and p, we need to have

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}. \tag{1.79}$$

This is called the **Legendre transform** of f.

Definition 1.4: Legendre transform(Analytical Mechanics)

The Legendre transform of a function $f(x, y)$ is defined as

$$g(p, x) = p \cdot y - f(x, y) \tag{1.80}$$

where

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x} \tag{1.81}$$

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[3]

1.3.2 Hamiltonian and Canonical Equations

Now, we define Hamiltonian $H = H(q, p)$ as the Legendre transform of Lagrangian $L = L(q, \dot{q})$:

Definition 1.5: Hamiltonian H

The Hamiltonian $H(q, p)$ is defined as the Legendre transform of the Lagrangian $L(q, \dot{q})$ $(\dot{q} \to p)$:

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}) \tag{1.82}$$

where

$$p = \frac{\partial L}{\partial \dot{q}}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \tag{1.83}$$

Note:

In Lagrange formalism, generalized coordinate span the configuration space, while in Hamilton formalism, generalized coordinates and **generalized momenta** or **conjugate momenta** span the **phase space**.

This is actually a physically intuitive quantity. Since p is defined as the derivative of Lagrangian L:

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{q}} && (1.84) \\ &= \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right) = m\dot{q} && (1.85) \end{aligned}$$

and we can rewrite the Hamiltonian as

$$\begin{aligned} H(q,p) &= p \cdot \dot{q} - L(q,\dot{q}) && (1.86)\\ &= m \cdot \dot{q}^{\,2} - \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) && (1.87)\\ &= \frac{1}{2}m\dot{q}^{\,2} + V(q) = T + V && (1.88)\end{aligned}$$

This is the total energy of the system, for the case of non-velocity dependent potential $V(q)$.

This relation is useful because we can obtain Lagrangian L from the Hamiltonian H as well:

$$L(q,\dot{q}) = \dot{q} \cdot p - H(q,p) \implies \dot{q} = \frac{\partial H}{\partial p} \tag{1.89}$$

and the Euler-Lagrange equation can be rewritten in terms of Hamiltonian:

$$\begin{aligned} &\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 && (1.90)\\ \implies \quad &-\frac{\partial H}{\partial q} - \frac{d}{dt}(p) = 0 \iff \quad \dot{p} = -\frac{\partial H}{\partial q} && (1.91)\end{aligned}$$

These two equations are called the **canonical equations** or **Hamilton's equations**:

Theorem 1.3.1 Canonical Equations

The relationship between a mechanical variable q and its canonical conjugate variable p is given by the canonical equations:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{1.92}$$

For multiple degrees of freedom, we can write the Hamiltonian as

$$H(\{q_i\},\{p_i\},t) = \sum_i p_i\dot{q}_i - L(\{q_i\},\{\dot{q}_i\},t) \tag{1.93}$$

and the canonical equations as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.94}$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \tag{1.95}$$

1.3.3 Poisson Bracket

Now, consider how a physical quantity $X(q_i, p_i, t)$ changes with time:

$$\frac{dX(q_i, p_i, t)}{dt} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i \tag{1.96}$$

from the canonical equations, the second part becomes:

$$\frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i = \frac{\partial X}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{1.97}$$

Since this term only depend on X and H, given q_i and p_i, we can define a new quantity called the **Poisson bracket**:

Definition 1.6: Poisson Bracket

The Poisson bracket of two physical quantities A and B is defined as

$$\{A,\, B\} := \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \tag{1.98}$$

If a physical quantity X does not explicitly depend on time, we can rewrite the time derivative as:

$$\frac{dX(q_i, p_i)}{dt} = \{X,\, H\} \tag{1.99}$$

Now, what happens if X happens to be q_i or p_i? A physical quantity q_i or p_i can be written as a function of q_i and p_i - simply as itself (noting that no explicit time dependence is present):

$$q_i(q_i, p_i) = q_i, \quad p_i(q_i, p_i) = p_i \tag{1.100}$$

Then, the Poisson bracket of q_i and H is given by

$$\{q_i,\, H\} = \sum_j \left(\frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right) \tag{1.101}$$

$$= \sum_j \left(\delta_{ij}\frac{\partial H}{\partial p_j} - 0\right) = \frac{\partial H}{\partial p_i} \tag{1.102}$$

where Eq. (1.30) is used. Similarly, the Poisson bracket of p_i and H is given by

$$\{p_i,\, H\} = \sum_j \left(\frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right) \tag{1.103}$$

$$= \sum_j \left(0 - \delta_{ij}\frac{\partial H}{\partial q_j}\right) = -\frac{\partial H}{\partial q_i} \tag{1.104}$$

Thus, we can rewrite the canonical equations in terms of Poisson bracket:

Theorem 1.3.2 Canonical Equations with Poisson Bracket

The canonical equations can be rewritten in terms of Poisson bracket as follows:

$$\dot{q}_i = \{q_i,\, H\}\,, \quad \dot{p}_i = \{p_i,\, H\} \tag{1.105}$$

Finally, note that the Poisson bracket of q_i and p_j gives:

$$\{q_i,\, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_i}{\partial q_k} \tag{1.106}$$

$$= \sum_k \delta_{ik}\delta_{jk} - 0 = \delta_{ij} \tag{1.107}$$

Theorem 1.3.3 Canonical Commutation Relation in Analytical Mechanics

The Poisson bracket of the generalized coordinate q_i and its conjugate momentum p_j is given by

$$\{q_i,\, p_j\} = \delta_{ij} \tag{1.108}$$

1.4 Summary

Lagrangian and action integral are defined as follows:

$$L := \sum T - V, \quad S[L] := \int_{t_i}^{t_f} dt\, L(q_i, \dot{q}_i, t) \tag{1.109}$$

we define the Hamiltonian as the Legendre transform of the Lagrangian:

$$H(q_i, p_i, t) := \sum_i p_i \cdot \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{1.110}$$

The action integral is stationary under the variation of the path:

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q_i, \dot{q}_i, t) = \delta \int_{t_i}^{t_f} dt\, \sum_i p_i \cdot \dot{q}_i - H(q_i, \dot{q}_i, t) = 0 \tag{1.111}$$

In the **Variational Principle**, we postulate that the motion of a particle is such that the action is stationary, yielding the Euler-Lagrange equation or Hamilton's equations:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{1.112}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.113}$$

Chapter 2

Quantum Mechanics

2.1 Historical Background

2.1.1 de Broglie Hypothesis

In 1905, Albert Einstein found the photoelectric effect, suggesting that light has a particle like property, such as discrete energy [4]. Following this work, Arthur Compton in 1923 discovered the Compton effect, which is the scattering of X-rays by electrons, further supporting the particle nature of light and confirming the discrete momentum of photons [5]. Specifically, energy and momentum are each related to the frequency and wavelength of light, respectively, as follows:

Proposition 2.1.1 Planck-Einstein relation

For a photon with frequency ν and wave length λ, the energy E and momentum p are given by

$$E = h\nu, \quad p = \frac{h}{\lambda} = \frac{h\nu}{c} = \frac{E}{c}, \tag{2.1}$$

In 1925, Louis de Brogile posulated that other particles (or any matter) also have a wave-like property, and the energy-frequency/ momentum-wavelength relation is given by the Planck-Einstein relation Eq. (2.1) [6].

2.1.2 Bohr Model

In 1913, Niels Bohr proposed a model of the hydrogen atom, which describes the electron as a particle orbiting the nucleus in discrete energy levels [7]. The implication of this model is that electrons have a fixed, discrete(quantized) angular momentum $\vec{L} := \vec{r} \times \vec{p}$. For a particle orbiting in a circular orbit, the angular momentum is given by

$$\vec{L} = rp = \frac{hr}{\lambda} \tag{2.2}$$

For the electron wave to be continous around the orbit of radius r, the wavelength must be an integer multiple of the circumference of the orbit:

$$\lambda = \frac{2\pi r}{n}, \quad n = 1, 2, 3, \ldots \implies \vec{L} = n\frac{h}{2\pi} := n\hbar \tag{2.3}$$

2.2 Canonical Quantization

2.3 Bra-ket Notation and Operator Formalism

2.4 Schrödinger Equation

2.5 Harmonic Oscillator

Chapter 3

Physics of Fields

3.1 1D String, Revisited

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