Particle Physics

Review for Particle Physics

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Contents

Chapter 1

Analytical Mechanics

1.1 Functional Derivative

1.1.1 Definition

Consider a quantity I defined as follows:

$$I := \int_A^B dx\, F(x) \tag{1.1}$$

Notice that I is not really a function of x, but if you had to say, it is more a "function" of F - may be $F(x) = e^x$, or $F(x) = ax^2 + bx + c$, or, etc. So, to denote the dependence of I on the function F, we write

$$I[f] := \int_A^B dx\, F(x) \tag{1.2}$$

This is called a functional. Now, imagine that F is a function of f, for example, $F[f] = f(x)^2$. By chain rule, a small change in F, denoted as δF, can be expressed as:

$$\delta F[f] = F[f + \delta f] - F[f] \tag{1.3}$$

$$= \frac{\partial F}{\partial f}\,\delta f \tag{1.4}$$

so, similarly,

$$\delta I[F] = I[F + \delta F] - I[F] \tag{1.5}$$

$$= \int_A^B dx\ \delta F[f] \tag{1.6}$$

$$= \int_A^B dx\, \frac{\partial F}{\partial f}\,\delta f \tag{1.7}$$

Then, the functional derivative of I with respect to f, $\frac{\delta I}{\delta f}$, is defined as follows:

Definition 1.1.1: Functional Derivative

If a function $X(x)$ exists, such that

$$\delta I = \int_A^B dx\, X(x)\, \delta f(x), \tag{1.8}$$

we say that $X(x)$ is the functional derivative of I with respect to f, and denote it as

$$\frac{\delta I}{\delta f} := X(x) \iff \delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\, \delta f. \tag{1.9}$$

1.1.2 Two function case

Consider a case where I is the functional of F, which is also a functional of f and g:

$$I[F[f,g]] = \int_B^A dx\, F[f(x), g(x)] \tag{1.10}$$

Or more generally, if a function $D(x)$ satisfies the following Now, let us add some small change of f, δf:

$$I[F[f+\delta f, g]] = \int_B^A dx\, F[f(x)+\delta f(x), g(x)] \tag{1.11}$$

$$= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f \tag{1.12}$$

and similarly, by adding δg,

$$I[F[f, g+\delta g]] = \int_B^A dx\, F[f(x), g(x)+\delta g(x)] \tag{1.13}$$

$$= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial g}\,\delta g \tag{1.14}$$

Combining these two, we have

$$I[F[f+\delta f, g+\delta g]] = \int_B^A dx\, F[f(x)+\delta f(x), g(x)+\delta g(x)] \tag{1.15}$$

$$= \int_B^A dx\, F[f,g] + \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.16}$$

$$\implies \quad I[F[f+\delta f, g+\delta g]] - I[F[f,g]] = \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.17}$$

In this case, we just like our normal derivative, we should denote the LHS as:

$$\delta I := \int_B^A dx\, \frac{\partial F}{\partial f}\,\delta f + \frac{\partial F}{\partial g}\,\delta g \tag{1.18}$$

or alternatively,

$$\delta I := \int_B^A dx\, \frac{\delta I}{\delta f}\,\delta f + \frac{\delta I}{\delta g}\,\delta g \tag{1.19}$$

1.1.3 Euler-Lagrange Equation

For the two function case, especially consider that $g = \frac{df}{dx}$, and let us see what happens. Specifically, let us set that $\delta f(A) = \delta f(B) = 0$. Then, we have:

$$\frac{\delta I}{\delta g}\,\delta g = \frac{\delta I}{\delta \frac{df}{dx}}\,\delta \frac{df}{dx} \tag{1.20}$$

we can change the order of the derivative:

$$= \frac{\delta I}{\delta \frac{df}{dx}}\frac{d\,\delta f}{dx} \tag{1.21}$$

from the differentiation of a product, we have

$$= \frac{d}{dx}\left(\frac{\delta I}{\delta \frac{df}{dx}}\,\delta f\right) - \frac{d}{dx}\frac{\delta I}{\delta \frac{df}{dx}}\,\delta f \tag{1.22}$$

Now, subsituting this to the two function case, we get:

$$\delta I = \int_B^A dx\,\left[\left(\frac{\delta I}{\delta f} - \frac{d}{dx}\frac{\delta I}{\delta \frac{df}{dx}}\right)\delta f + \frac{d}{dx}\left(\frac{\delta I}{\delta \frac{df}{dx}}\,\delta f\right)\right] \tag{1.23}$$

the total derivative term is zero, since $\delta f(A) = \delta f(B) = 0$. Thus, we have

$$\delta I = \int_B^A dx\,\left(\frac{\delta I}{\delta f} - \frac{d}{dx}\frac{\delta I}{\delta \frac{df}{dx}}\right)\delta f \tag{1.24}$$

and since $I = \int_B^A dx\, F[f(x), g(x)]$, we can say that

$$\frac{\partial F}{\partial f} = \frac{\delta I}{\delta f}, \quad \frac{\partial F}{\partial g} = \frac{\delta I}{\delta g} \tag{1.25}$$

Then

$$\delta I = \int_B^A dx\,\left(\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial \frac{df}{dx}}\right)\delta f \tag{1.26}$$

And if we somehow want to find a minimum of I, we can set $\delta I = 0$:

$$\implies \quad \frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial \frac{df}{dx}} = 0 \tag{1.27}$$

This is called the Euler-Lagrange equation.

1.1.4 Important Property

In general, consider that the functional F is a function of $f_1, f_2, \ldots, f_n$:

$$F[f_1, f_2, \ldots, f_n] \implies \delta F = \sum_{j=1}^{n} \frac{\partial F}{\partial f_j} \delta f_j \tag{1.28}$$

$$\tag{1.29}$$

1.2 Lagrange Formalism

1.2.1 Quick Recap: Newtonian Mechanics

In Newtonian mechanics, the motion of a particle is described through a few important quantities: for a particle of (inertial) mass m, position $\vec{r}$, we have

$$\text{velocity} : \vec{v} = \frac{d\vec{r}}{dt} \tag{1.30}$$

$$\text{acceleration} : \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{1.31}$$

$$\text{momentum} : \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{1.32}$$

and the relations between these quantities, in the presence of external forces $\vec{F}_{\text{ext}}^{(i)}$ acting on the particle, are given by Newton's second law:

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = \sum_i \vec{F}_{\text{ext}}^{(i)} \tag{1.33}$$

The work done by such forces is given by

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}_{\text{ext}}^{(i)}, \quad \text{where } l \text{ is the path of the particle.} \tag{1.34}$$

This is the energy change of the particle through the motion:

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}_{\text{ext}}^{(i)} \tag{1.35}$$

$$= \int_{t_i}^{t_f} dt\, \vec{v} \cdot m\frac{d\vec{v}}{dt} \tag{1.36}$$

$$= \int_{t_i}^{t_f} dt\, \frac{m}{2}\frac{d}{dt}\vec{v}^2 \tag{1.37}$$

$$= \frac{m}{2}\vec{v}_f^2 - \frac{m}{2}\vec{v}_i^2 \tag{1.38}$$

meaning that $m\vec{v}^2/2$ is the energy due to the motion of the particle: the kinetic energy T:

$$T = \frac{m}{2}\vec{v}^2 \tag{1.39}$$

Now, often, the external force acting on the particle is due to a potential V:

$$\vec{F}_{\text{ext}} = -\nabla V \tag{1.40}$$

For example, for a 1D spring, the potential is given by

Example 1.2.1 (1D spring/ Harmonic potential)

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -kx \tag{1.41}$$

or the electrostatic potential:

Example 1.2.2 (Electrostatic potential)

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \implies F_{\text{ext}} = -\nabla V = -\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r} \tag{1.42}$$

1.2.2 Lagrangian and Euler-Lagrange Equation

Let us define quantities called the Lagrangian L and action S:

$$L := T - V, \qquad S := \int_{t_i}^{t_f} dt\, L \tag{1.43}$$

For one particle, the Lagrangian in general contains the position $q_i(t)$ and velocity $\dot{q}_i(t)$ and time t:

$$L = L(q_i, \dot{q}_i, t) \tag{1.44}$$

The action is then given by

$$S = \int_{t_i}^{t_f} dt\, L(q_i(t), \dot{q}_i(t), t) \tag{1.45}$$

1.3 Hamiltonian Formalism

Chapter 2

Quantum Mechanics