You Cannot Press Out the Black Hole

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Abstract. *It is shown that a ball-shaped black hole region homeomorphic with D^n cannot be pressed out, along whichever axis penetrating the black hole region, into a black ring with a doughnut-shaped black hole region homeomorphic with $S^1 \times D^{n-1}$. A more general prohibition law for the change of the topology of black holes, including a version of no-bifurcation theorems for black holes, is given.*

1. *Introduction* We would like to discuss here the dynamical aspects of black hole space-times. The analysis of such nonstationary space-times is often difficult due to the lack of the geometrical symmetry of the space-time. However, several results have been known, which are kinematical, in the sense that they are deduced from causal and topological structures of space-times, but not sensitive to the details of the Einstein equation.

Among these is on the topology of black hole horizons. It is known that an apparent horizon in a 4-dimensional space-time must be diffeomorphic with the 2-sphere [1, 2, 3]. In particular, the stationary black hole event horizon must be the topological 2-sphere. This follows from the fact that the event horizon coincides with the apparent horizon in every stationary black hole space-time. This theorem for the topology of black holes does not exclude an event horizon with nonspherical topology. In fact, for black holes formed by gravitational collapses, the spatial section of the event horizon can in general be homeomorphic with the torus, or even with the closed orientable surface of arbitrarily high genus.

An attractive point of view for the topology of the event horizons is proposed by Siino [4]. His observation is that the topological information of the event horizon is encoded in the crease set of the event horizon, which is the set consisting of past end points of all the null geodesic generators of the event horizon. Actually, the crease set of the event horizon is quite relevant to the change of the topology of the black hole. In particular, the spatial section of the event horizon changes its topology only when the spatial hypersurface, in which it is embedded, intersects the crease set of the event horizon. On the other hand, Ida [5] has shown that the crease set has the same homotopy type with that of the world hypersurface of the event horizon.

Another well-known result, which we would like to mention, is the no-bifurcation theorem for black holes [1, 6], stating that a black hole cannot bifurcate into two or more black holes. Ida and Siino [7] observe that there is a more general form of the prohibition law for the change of the topology of black holes than the no-bifurcation law, that is, there are many other forbidden processes in the black hole dynamics. A typical example of the prohibition law concerns the formation of a black ring, by which we mean a black hole with a horizon homeomorphic with $S^1 \times S^k$ $(k \geq 1)$, from a spherical black hole. There are essentially two kinds of such processes. Let us consider a black hole in an n-dimensional space. Let the black hole region be a topological n-disk. Then, the black hole horizon is its boundary $(n-1)$-sphere. The black hole region will undergo deformation with the time development of the space-time. Let a pair of horn-shaped black hole regions grow from the body of the black hole, and then let tips of the horns merge. Then, we will obtain a black ring, which has a doughnut-shaped black hole region homeomorphic with $S^1 \times D^{n-1}$. Another way to get a black ring is by pressing out the ball-shaped black hole region along whichever axis penetrating the black hole region. It is argued in Ref. [7] that the latter "pressing-out process", which corresponds to the "white $(n-2)$-handle attachment" in the terminology of Ref. [7], cannot be realized in any space-time. However, the reasoning in Ref. [7] requires a technical assumption such that the time function induced on a slightly deformed event horizon is a Morse function on it, which is not fully satisfactory.

In the rest of the paper, we show, under milder conditions, that a spherical black hole cannot be pressed out into a black ring.

2. *Preliminaries* We mean by a space-time M a Hausdorff, orientable, time-orientable, Lorentzian manifold[1] with $\dim M \geq 4$. Note that the existence of a Lorentzian metric guarantees the paracompactness. Here, we consider weakly asymptotically simple and strongly future asymptotically predictable space-times [6]. The former condition is one of the standard conditions for the space-time to be asymptotically flat, and the latter requires that there is a partial Cauchy surface S such that $\mathscr{I}^+ \subset \overline{D^+(S)}$ and $\overline{J^-(\mathscr{I}^+)} \cap J^+(S) \subset D^+(S)$ hold.

If the space-time M is strongly future asymptotically predictable from a partial Cauchy surface S, there is a homeomorphism

$$f : [0,\infty) \times S \to D^+(S); (\tau, x) \mapsto S_\tau(x),$$

such that the conditions

(1) for each $\tau \in [0,\infty)$, $S_\tau = f(\tau, S)$ is a partial Cauchy surface homeomorphic with S,
(2) $S_0 = S$,
(3) for $0 < \tau < \tau'$, $S_{\tau'} \subset I^+(S_\tau)$,
(4) for $0 < \tau < \tau'$ and for each $p \in S_\tau$, every future inextendible causal curve starting from p intersects $S_{\tau'}$ or $\mathscr{I}^+$,

hold (See Ref. [6] Prop. 9.2.3.).

The function $t = p_1 \circ f^{-1} : D^+(S) \to [0,\infty)$ can be regarded as a time function on $D^+(S)$, where p_1 denotes the projection onto the first component. Thus, due to the existence of the global time coordinate t, the black hole region at time: $t = \tau$ can be naturally defined.

Definition 1 (Black hole region)**.** For each $\tau \in [0,\infty)$, the black hole region B_τ at time $t = \tau$ is defined by

$$B_\tau = S_\tau \cap (M \setminus \overline{J^-(\mathscr{I}^+)}).$$

In the following, we assume that each partial Cauchy surface S_τ is a smooth hypersurface in the space-time M, and the time function t is a smooth function on $D^+(S)$.

3. *"Black holes are not pressed out" theorem* In order to formulate whether or not the black hole region is pressed out, we focus on the topological property of simple closed curves in the black hole region.

[1]By Lorentzian manifold, we mean a differentiable manifold endowed with a nondegenerate, smooth metric tensor field with a signature of $(-,+,+,\cdots,+)$.

At some initial time, let a simple closed curve L in the black hole region have the property that it bounds a 2-disk in the black hole region. If L lost this property after a while, we could say that the black hole region is pressed out. Conversely, there would be such a simple closed curve with this property, whenever the black hole region is pressed out.

We formulate this by introducing a couple of definitions.

Definition 2 (B-contractibility)**.** A simple closed curve $L : S^1 \to B_\tau$, which is a smooth embedding in B_τ, is said to be B-contractible in B_τ, if it is contractible to a point within B_τ, that is if there is a homotopy $f : S^1 \times [0,1] \to B_\tau$ such that the conditions:

(1) for each $\theta \in S^1$, $f(\theta, 0) = L(\theta)$ holds,
(2) $f(\,\cdot, 1) : S^1 \to B_\tau$ is a constant map,

are satisfied.

Definition 3 (Descendent of a simple closed curve)**.** Let $L : S^1 \to S_{\tau_i}$ ($\tau_i \geq 0$) be a simple closed curve smoothly embedded in S_{τ_i}. For $\tau_f > \tau_i$, a simple closed curve $L' : S^1 \to S_{\tau_f}$ is called a descendent of L, if there is a compact Lorentzian 2-submanifold N smoothly embedded in $D^+(S_{\tau_i})$, whose boundary is the disjoint union $L(S^1) \sqcup L'(S^1)$.

In this terminology, we state our main theorem claiming that any black hole cannot be pressed out.

Theorem 4. *Every descendent of a B-contractible simple closed curve is B-contractible.*

In order to prove this theorem, we need the following lemma asserting that a nonsingular timelike vector field on a closed Lorentzian submanifold of the space-time M can be globally extended to that on M.

Lemma 5. *For any closed Lorentzian submanifold N smoothly embedded in the space-time M, there exists a smooth, future-directed, timelike vector field in M, which is tangent to N, everywhere on N.*

Proof. For any point $p \in N$, there is an open neighborhood U_p of p such that $\overline{U}_p$ is compact and that there is a nonsingular future-directed timelike vector field Y_p on U_p, which is tangent to N everywhere on $N \cap U_p$, for the space-time M is a time-oriented Lorentzian manifold. Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover of $M \setminus N$, such that each $\overline{V}_\alpha$ is compact and that on each V_α, there is a nonsingular future-directed timelike vector field Y_α. Since $\{U_p\}_{p\in N}$, $\{V_\alpha\}_{\alpha\in A}$ is an open cover of M, and M is a paracompact manifold, there is a locally finite refinement $\{W_j\}_{j\in J}$ of $\{U_p\}_{p\in N}$, $\{V_\alpha\}_{\alpha\in A}$. Then, for each $j \in J$, $\overline{W}_j$ is compact.

For each $j \in J$ such that $W_j \cap N \neq \emptyset$ holds, we can choose a point $p \in W_j \cap N$ such that W_j is contained in U_p. Then, define a local vector field X_j on W_j as the restriction of Y_p on W_j.

On the other hand, for each $j \in J$ such that $W_j \cap N = \emptyset$ holds, W_j is contained in V_α for some $\alpha \in A$, or otherwise in U_p for some $p \in N$. Then, define a local vector field X_j on W_j as the restriction of either Y_α or Y_p on W_j.

There is a partition of unity subordinate to the cover $\{W_j\}_{j\in J}$, that is, there is a collection of smooth functions $\{\rho_j\}_{j\in J}$ on M such that

(1) for each $j \in J$, $0 \leq \rho_j \leq 1$ on M,
(2) for each $j \in J$, $\mathrm{supp}(\rho_j) \subset W_j$,

(3) for every point $p \in M$, $\sum_j \rho_j(p) = 1$

hold.

Define the vector field X on M by

$$X = \sum_{j \in J} \rho_j X_j.$$

Since $\{W_j\}_{j \in J}$ is locally finite, every point in M has a neighborhood such that $\sum_j \rho_j X_j$ is a finite sum.

Clearly, X is a nonsingular, future-directed, timelike vector field on M, which is tangent to N, everywhere on N. This completes the proof of Lemma 5. □

Now, we are in a position to prove Theorem 4.

Proof of Theorem 4. Let $L : S^1 \to B_{\tau_i}$ be a B-contractible simple closed curve in the black hole region B_{τ_i}. Then, there is a homotopy $f : S^1 \times [0,1] \to B_{\tau_i}$ between L and the constant map: $S^1 \to \{p\}$, $p \in B_{\tau_i}$.

For $\tau_f > \tau_i$, let $B_{[\tau_i,\tau_f]}$ be the portion of the black hole region $M \setminus \overline{J^-(\mathscr{I}^+)}$ between B_{τ_i} and B_{τ_f}: $B_{[\tau_i,\tau_f]} = \{x \in M \setminus \overline{J^-(\mathscr{I}^+)}; t(x) \in [\tau_i, \tau_f]\}$. Let N be a smoothly embedded 2-dimensional compact Lorentzian submanifold of $B_{[\tau_i,\tau_f]}$, whose boundary consists of $L(S^1)$ and a simple closed curve $L'(S^1)$ in S_{τ_f}. N is properly embedded in $B_{[\tau_i,\tau_f]}$. By Lemma 5, there is a nonsingular smooth timelike vector field X on $B_{[\tau_i,\tau_f]}$, tangent to N everywhere on N. The vector field X gives a diffeomorphism $\varphi : B_{\tau_i} \times [0,1] \to B_{[\tau_i,\tau_f]}$, that is, for each $(x,r) \in B_{\tau_i} \times [0,1]$, $\varphi(x,r)$ is defined to be the point where the integral curve of X starting from x intersects $B_{(1-r)\tau_i + r\tau_f}$. Then, the descendent L' of L can be written as

$$L' : S^1 \to B_{\tau_f}; \theta \mapsto \varphi(L(\theta), 1).$$

Consider now the continuous map

$$f' : S^1 \times [0,1] \to B_{\tau_f}; (\theta, s) \mapsto \varphi(f(\theta, s), 1).$$

For $f'(\theta, 0) = L'(\theta)$ and $f'(\theta, 1) = \varphi(p, 1)$, the map f' gives a homotopy within closed curves in B_{τ_f}, between L' and the constant map onto $\varphi(p,1)$. Thus, the descendent L' of L is B-contractible. This completes the proof of Theorem 4. □

4. *Extension for B-contractible k-spheres* Theorem 4 can be easily extended for "B-contractible spheres" with general dimensions. We briefly state the extended theorem here. First, we need a few definitions.

Definition 6 (B-contractible k-sphere)**.** For $0 \le k \le \dim M - 2$, let $L : S^k \to B_\tau$ be a smooth embedding of a k-sphere into the black hole region at $t = \tau$. (The 0-sphere is a pair of points). Then, L is said to be B-contractible in B_τ, if it is contractible to a point within B_τ.

Accordingly, the notion of a descendent of an embedded k-sphere is defined as follows.

Definition 7 (Descendent of an embedded k-sphere)**.** For $0 \le k \le \dim M - 2$, let $L : S^k \to S_{\tau_i}$ $(\tau_i \ge 0)$ be a smooth embedding. For $\tau_f > \tau_i$, a smooth embedding $L' : S^k \to S_{\tau_f}$ is said to be a descendent of L, if there is a compact Lorentzian $(k+1)$-submanifold smoothly embedded in $D^+(S_{\tau_i})$, whose boundary is the disjoint union $L(S^k) \sqcup L'(S^k)$.

Then, the generalization of Theorem 4 is stated as follows.

Theorem 8. *Every descendent of a B-contractible k-sphere is B-contractible.*

We omit the proof, for it parallels to that of Theorem 4. Theorem 8 reduces, in the case of $k = 0$, to a version of "no-bifurcation theorems" implying that any black hole cannot bifurcate into several black holes.

Corollary 9 (No-bifurcation theorem for black holes)**.** *Let p and q be a pair of points belonging to the same connected component of the black hole region B_{τ_i}. For $\tau_f > \tau_i$, if p' belongs to $I^+(p) \cap B_{\tau_f}$ and q' belongs to $I^+(q) \cap B_{\tau_f}$, then p' and q' belong to the same connected component of the black hole region B_{τ_f}.*

Proof. This can be seen by noting that $L : S^0 \to B_\tau$ is B-contractible, if and only if $L(S^0)$ belongs to the same connected component of B_τ. □

5. *Concluding remarks* We have defined the notion of the "pressing-out process" for black hole event horizons in terms of the "B-contractiblity" of simple closed curves interpolated by a Lorentzian cobordism. Then, we have shown that these pressing-out processes are never realized (Theorem 4). One consequence derived from this is that an axisymmetric black hole cannot be converted into a black ring in any axisymmetric process. This can be seen as follows. Let $B_{\tau_i} \simeq D^n$ (homeomorphism) be a ball-shaped black hole region in an $(n+1)$-dimensional axisymmetric space-time M, and it is converted into a doughnut-shaped black hole region $B_{\tau_f} \simeq S^1 \times D^{n-1}$ after a while, such that each orbit of the $U(1)$-isometry on a point in B_{τ_f} is an incontractible circle in B_{τ_f}. There will be a timelike curve $\gamma : [0,1] \to M$ connecting a point $\gamma(0) \in B_{\tau_i}$ and a point $\gamma(1) \in B_{\tau_f}$. By the standard arguments of general position, we can assume that the timelike curve γ is smooth and it does not contain a fixed point of the $U(1)$-isometric action. Then, the orbit of $U(1)$-isometry on γ will be a Lorentzian 2-submanifold smoothly embedded in M, which interpolates a loop L in B_{τ_i} and a loop L' in B_{τ_f}. The loop L is a B-contracible simple closed curve in B_{τ_i}, for B_{τ_i} is simply connected, while the loop L' is not B-contractible, for it is an orbit of the $U(1)$-isometry on $\gamma(1) \in B_{\tau_f}$. This is impossible by Theorem 4. In this way, the formation of a black ring from a Kerr (or Myers-Perry) black hole must involve a non-axisymmetric process.

Theorem 8 controls more complicated dynamical evolutions of the topology of black holes in higher-dimensional space-times. It is known that the black hole no-hair property does not in general hold in higher-dimensional space-times. Rather, various types of black hole with nontrivial horizon topologies are allowed by the Einstein equation. These black holes are not necessarily stable in dynamics[8]. This implies that various dynamical processes involving the topology change of the black hole event horizon naturally take place. Our result here would serve as a good starting point to qualitatively understand such dynamically complicated processes.

Let us finally comment on the relationship between the previous work [7] and the present one. In Ref. [7], the dynamics of the topology of the event horizon is described in terms of a Morse function defined on an appropriately smoothed event horizon as a world hypersurface. Here, each topology change of the black hole region corresponds to a critical point of the Morse function and it is therefore classified using the Morse index λ of the critical point, where $\lambda = 0, \cdots, n$. Furthermore, each critical point is classified a priori into black or white according to the relative orientation between the black hole region and the external region around the critical point. Hence, each topology change is identified geometrically with black or white λ-handle attachment. In this way, any complicated topological dynamics of the black hole reduces to a comprehensible local dynamics. Then, it is argued in Ref. [7] that any change of the horizon topology corresponding to the white λ-handle attachment never occurs. This reasoning relies entirely on the existence of the Morse function on the event horizon, which is not guaranteed in general. The present work overcomes this drawback in terms of a global approach, which does not need such Morse functions. Theorem 8 for the B-contractible k-sphere is essentially the same as the statement that excludes the white $(n-k-1)$-handle attachment in the terminology of the local approach in Ref. [7].

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