GLOBAL HOLOMORPHIC INTERSECTION FORMULA AND LOCAL EULER OBSTRUCTIONS FOR MIXED RESULTANTS

XIA LIAO

Abstract. In this short paper we are going to accomplish multiple goals. We will prove a global version of the holomorphic intersection formula due to Dubson, Lê, Ginsburg, Sabbah and Schürmann, derive the local Euler obstruction formula for discriminants as a special case of the holomorphic intersection formula, and finally give a local Euler obstruction formula for mixed resultants.

1. introduction

The study of discriminants for a single polynomial and the resultants for several polynomials has a long history in mathematics. Traditionally, discriminants and resultants appeared in elimination theory founded by the 19th century mathematician Cayley and Sylverster etc. as the polynomial conditions for one polynomial equation bearing multiple roots or several polynomial equations bearing common roots. Into the modern period, the word discriminant was endowed geometric meanings as the research was trending towards exploring the interplay between algebra and geometry. So the discriminant could mean either the dual variety of a projective variety (see, for example [6]), or the special locus in the base of a versal deformation of singularities where the degeneracy of the family occurs (see, for example [17]). Though the topic of the present paper is closely related to the discriminant in the first sense, it is nevertheless true that in no matter what sense, the "primitive invariants of the discriminant yield very subtle invariants of the family" (cited and rephrased from the Mathscinet review article for [17]). For dual varieties $X^\vee$, the primitive invariants are multiplicities and local Euler obstructions, and the subtle invariant of the family is the counting of the singularity types appearing in the hyperplane sections of the original variety X. This philosophy is thoroughly discussed in [4] [1], together with many applications in enumerative geometry. For further development of this circle of ideas in Thom polynomial theory, see [11].

The guiding question in our mind when writing up this paper is how primitive invariants of the resultants (or even the Chow varieties) reflect the subtle geometry of the objects which resultants parametrise. According to Cayley's trick (see §2), resultants can also be regarded as discriminants, so it seems, at least in principle, that a similar argument following the path paved in [1] would render a formula for local Euler obstructions for mixed resultants. This is indeed what happened, except that the argument in [1] is tailored to treat nonsingular original varieties while we aim to include resultants of singular projective variety in our study as well. The key technical result to handle the Euler obstruction of the dual of a singular variety is Theorem 3.3, which was proved in [5] based on the theory of topological Radon transformation. Combining Cayley's trick, Theorem 3.3 and the method employed by Aluffi in [3] calculating Milnor classes of hypersurfaces in projective bundles, we can derive one of our main results Theorem 4.1. Part of this theorem (in the range $s \leq m-1$ and $L_1 = \ldots = L_s$, see the notations introduced in §2) is equivalent to Theorem 3.2 in [5], and part of it (in the range $s \geq m$) is genuinely new, asserting that for example (see

Corollary 4.2 (c))

$$Eu_{R_m}(f_1, \ldots, f_m) = \chi(X_f)$$

where $(f_1, \ldots, f_m)$ is a point on the mixed resultant R_m, X is an $(m-1)$-dimensional smooth complex projective variety and $\chi(X_f)$ is the topological Euler characteristic of the space $X_f = X \cap \{f_1 = \ldots = f_m = 0\}$.

In our endeavour to understand the existing proofs of Theorem 3.3 in the literature, we realise that this theorem can be interpreted as a special case of the holomorphic intersection formula for conic Lagrangian cycles. The intersection formula was due to Dubson, Lê, Ginsburg, Sabbah and Schürmann, originated first in holonomic D-module theory, gradually became independent from the D-module context, and whose true nature is now understood as being subject to stratified Morse theory. The formula concerns with the intersection of the graph of the differential of a holomorphic function with the conormal space of a complex analytic variety, but in this form it does not cater to our need because there is no non-constant holomorphic function on $\mathbb{P}^n$. An important feature of the present paper is that, we formulate and prove a global holomorphic intersection formula and an algebraic formula for the Milnor classes of pairs (Theorem 3.8 and Theorem 3.12) where we consider instead the intersection of the first jet of a section of a holomorphic line bundle with the space of conormal jets!

Theorem 3.8 and 3.12 generalise a few well-known theorems of Aluffi [2], Pragacz and Parusiński [12] about characteristic classes of singular hypersurfaces inside a smooth ambient space. With results proved in the present paper, we are able to express the Milnor class of the pair $\mathcal{M}(Z \cap \{f=0\} \subset Z; Eu_Z)$ in terms of the Segre class of the singular subscheme $Sing(Z,f)$, where Z is an irreducible complex analytic variety and f is a global section of a holomorphic line bundle over Z. Theorem 3.8 also allows us to give another proof of the major conclusion in [5], which is now a part of Corollary 4.3 in our paper.

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2. Cayley's trick for mixed resultants

Let X be an irreducible projective variety of dimension $m-1$ over a field k and let $L_1, \ldots, L_s$ be very ample line bundles on X. Fix the following notations:

$$\begin{aligned} V_i &= H^0(X, L_i) \\ V &= V_1 \oplus \ldots \oplus V_s \\ \tilde{X} &= \mathbf{P}(L_1 \oplus \ldots \oplus L_s). \end{aligned}$$

There are natural closed embeddings $X \subset \mathbf{P}(V_i^\vee)$ for every $i = 1, \ldots s$ and a natural closed embedding $\tilde{X} \subset \mathbf{P}(V^\vee)$.

Let $f_i \in V_i$ be s global sections. Classically, it was interesting to study the algebraic conditions which enforce the sections $f_1, \ldots f_s$ to intersect X non-transversally somewhere on X. More precisely, there are two cases.

(1) $s \le m-1$. Intersecting non-transversally is the statement that there exists $x \in reg(X)$ with $f_1(x) = \ldots = f_s(x) = 0$ and the system of equations $df_1(x) = \ldots = df_s(x) = 0$ determines a linear subspace of T_xX of codimension strictly less than s.

(2) $s \geq m$. Intersecting non-transversally is the statement of the existence of $x \in reg(X)$ with $f_1(x) = \ldots = f_s(x) = 0$.

Clearly, those $f_1, \ldots, f_s$ satisfying the condition in either (1) or (2) is a $(k^*)^s$-invariant closed subvariety of V. In this paper, we treat them only as k^*-invariant subsets of V for the diagonal k^*-action.

Definition 2.1. *The closures of the sets defined by either condition (1) or (2) in* $\mathbf{P}(V)$ *are called the mixed resultants of* X *associated to* $L_1, \ldots, L_s$*, denoted by* $R(X, L_1, \ldots, L_s)$*. If no ambiguity arises, we call them mixed resultants for short, and denote them by the simple notation* R_s*.*

Remark 2.2. *When* $s \geq m$*,* R_s *can be defined as the projectivisation of the set of* $f = (f_1, \ldots, f_s)$ *such that the system of equations* $f_1(x) = \ldots = f_s(x) = 0$ *has a common solution on* X*.*

Remark 2.3. *The terminology we choose to use here does not agree completely with the classical ones. Classically, it is more common to consider the case* $L_1 = \ldots = L_s$*, so that the sets in (1) and (2) are* $GL(s)$*-invariant in* V*.*

- *The corresponding sets in the Grassmanian* $G(s, V)$ *defined by the conditions in (1) are called higher associated hypersurfaces* [6] *or coisotropic varieties* [7] *or* k*-dual varieties* [5] *. The name associated hypersurface is chosen based on a wrong assertion (*[6] *chapter 3 proposition 2.11) which claims that conditions in (1) define hypersurfaces in the the Grassmanian. This mistake is fixed in* [7] *where the precise condition under which the coisotropic variety is a hypersurface is given.*
- *The case that* $L_1, \ldots L_s$ *are possibly not the same is discussed in* [6]*, but only in the situation* $s = m$*. It is shown in this restricted case the corresponding variety in* $P(V)$ *(or in the Grassmanian if* $L_1 = \ldots = L_s$*) is always a hypersurface (*[6] *Chapter 3 Proposition 3.1). The variety in* $P(V)$ *is called the mixed resultant associated to* $L_1, \ldots, L_s$ *loc.cit., and this is where our terminology comes from.*

Recall that $\tilde{X} \subset \mathbf{P}(V^\vee)$ and $R(X, L_1, \ldots, L_s) \subset \mathbf{P}(V)$. Cayley's trick is the simple observation that $\tilde{X}$ and $R(X, L_1, \ldots, L_s)$ are related by projective duality. The $s = m$ case is discussed in [6] Chapter 3 Proposition 3.4; the $s \leq m - 1, L_1 = \ldots = L_s$ case is discussed in [7]. Technically there is no essential difference between the two cases, and this is why we choose to use the uniform name R_s in both cases.

Proposition 2.4. *(Cayley's trick)* $\tilde{X}$ *is the projective dual of* $R(X, L_1, \ldots, L_s)$*.*

Proof. The argument in [6] can be easily modified to fit this slightly more general situation. For the convenience of the reader we offer a proof using only basic linear algebra.

Let $y \in \tilde{X}$ be a nonsingular point in the projective bundle lying over $x \in X$ and let $f = (f_1, \ldots, f_s) \in V$ be an equation of a hypersurface H in $\mathbf{P}(V^\vee)$ passing through y. Choose a local section through y of the projective bundle $\pi : \tilde{X} \to X$ and denote it by Y. Then H is tangent to $\tilde{X}$ at y if and only if (a) H is tangent to the fibre of the projection π at y and (b) H is tangent to Y at y.

Algebraically, the closed embeddings $X \subset \mathbf{P}(V_i^\vee)$ associate 1-dimensional linear subspaces $l_i \subset V_i^\vee$ to $x \in X$. The point $y \in \tilde{X}$ corresponds to a 1-dimensional vector subspace $l \subset l_1 \oplus \ldots \oplus l_s$. At the level of tangent vectors, there is a diagonal injection $T_x X \to \bigoplus_i Hom(l_i, V_i^\vee / l_i)$. The tangent space $T_y Y$ can be identified with the image of the composition $T_x X \to \bigoplus_i Hom(l_i, V_i^\vee / l_i) \to Hom(l, V^\vee / \bigoplus_i l_i) \to Hom(l, V^\vee / l)$ where the last map involves a choice of lifting vectors from $V^\vee / \bigoplus_i l_i$ to $V^\vee / l$. The ambiguity lies in

$\bigoplus_i l_i/l$, which is exactly the space of infinitesimal section of π through y. Since our goal is to decide when the hyperplane H is tangent to $\tilde{X}$ at y and this is independent from the choice of Y, we will see immediately and algebraically this choice is inessential.

Unraveling the tangency condition, we have

(a) H is tangent to the fibre of π at $y \iff H$ is tangent to the projective subspace $\mathbf{P}(l_1 \oplus \ldots \oplus l_s) \subset \mathbf{P}(V^\vee) \iff f_i(l_i) = 0$ for every i. The last condition means that the hypersurfaces defined by $f_i = 0$ in $\mathbf{P}(V_i^\vee)$ pass through $x \in X$ via the embeddings $X \subset \mathbf{P}(V_i^\vee)$.

(b) H is tangent to Y at $y \iff$ the composition $T_xX \to Hom(l, V^\vee/l) \to Hom(l,k)$ is 0 where the second arrow is the composition with f (by assumption $f(l) = 0$ since H passes through y). In case condition (a) also holds, the choice made for lifting $Hom(l, V^\vee/\bigoplus_i l_i)$ to $Hom(l, V^\vee/l)$ is irrelevant, and the composition is the same as $T_xX \to \bigoplus_i Hom(l_i, V_i^\vee/l_i) \xrightarrow{(\oplus_i f)\circ} Hom(l,k)$. Note that $Hom(T_xX, Hom(l,k)) \cong Hom(l, T_x^*X)$. In geometric language this means that these $f_1, \ldots, f_s$ give rise to a morphism $\mathscr{O}(-1) \to T_x^*X$ vanishing at l, where $\mathscr{O}(-1)$ denotes the tautological subbundle of $\mathbf{P}(l_1 \oplus \ldots \oplus l_s)$. In more plain language, we can choose generators $v_1, \ldots, v_s$ for $l_1, \ldots, l_s$. Every vector $v \in l_1 \oplus \ldots \oplus l_s$ has a unique expression $v = a_1v_1 + \ldots + a_sv_s$ for some $a_1, \ldots, a_s \in k$. Then $f_1, \ldots, f_s$ give rise to a map $k^s \to T_x^*X$ (which depends also on the choice of $v_1, \ldots, v_s$ and the projective embeddings $X \subset \mathbf{P}(V_i^\vee)$). From any nonzero vector $(a_1, \ldots, a_s)$ in the kernel of this map we can create a nonzero vector $v = a_1v_1 + \ldots + a_sv_s$ which generate a 1-dimensional vector subspace of $l_1 \oplus \ldots \oplus l_s$ (hence a point in $\tilde{X}$) where the tangency of H to $\tilde{X}$ occurs.

To summarise, H is tangent to $\tilde{X}$ at some nonsingular $y \in \tilde{X}$ if and only if $f_i(x) = 0$ for all i and the kernel of the map $k^s \to T_x^*X$ is nontrivial.

Now let us go back to the original two cases $s \leq m-1$ and $s \geq m$. If $s \geq m$, we see that the map $k^s \to T_x^*X$ automatically has nontrivial kernels, so that H is tangent to $\tilde{X}$ at some nonsingular $y \in \tilde{X}$ if and only if $f_i(x) = 0$ for all i. If $s \leq m-1$, the map $k^s \to T_x^*X$ has a nontrivial kernel if and only if the image of $k^s \to T_x^*X$ has dimension less than s, which is equivalent to the codimension condition in (1). □

3. The local Euler obstruction for ordinary discriminants

From now on, varieties will be defined over an algebraically closed field of characteristic 0. First, recall the definition of the Milnor class of a pair from [15] definition 5.2.

Definition 3.1. *For a regular embedding $i : Y \to X$, and a constructible function $\alpha \in CF(X)$, the difference*

$$\mathcal{M}(Y \subset X; \alpha) = c(N_YX)^{-1} \cap i^*c_*(\alpha) - c_*(i^*\alpha) \in H_*(X)$$

is called the Milnor class of the pair $Y \subset X$ relative to the constructible function α.

The notations require some explanation. The homology functor H_* in the definition depends on the category we are working with. We will use Chow homology when working with algebraic varieties, and Borel-Moore homology when working with complex analytic varieties. Next, c_* is the Chern class transformation of Schwartz and MacPherson [9], which is a natural transformation from constructible functions to homology. The readers may consult the lecture note [15] for its basics. We only mention that in this theory the local Euler obstruction is playing a prominent role. If X is an irreducible (algebraic or complex analytic) subvariety of a (algebraic or complex analytic) manifold M, the dual local obstruction $Eu_X^\vee$ is the constructible function on X associated with the conormal space $T^*_X M$ which is by definition the closure of $T^*_{X_{reg}} M$ in T^*M, and it is related to the local Euler obstruction Eu_X by $Eu_X = (-1)^{\dim X} Eu^\vee_X$. Moreover, if X is also smooth, then $Eu_X = 1_X$ the constant function with value 1 on X. This point will be used when we compute the local Euler obstruction for a fibre bundle with smooth fibres. Finally, note that the pull-back i^* is well-defined for both constructible functions and homology classes so the expression in the definition makes sense.

For our purpose, we need a result which states that the degree of the Milnor class can be computed using "a generic slice". This result is well-known among experts but we are not able to locate a precise reference. So we state it here and give a quick proof.

Proposition 3.2. *Let X be a projective variety in $(\mathbb{P}^n)^\vee$, let H be a hyperplane such that the inclusion $i : X \cap H \to X$ is a regular embedding of codimension 1 and let α be a constructible function on X. Next, let H' be a generic hyperplane in the sense that H' is transversal to a fixed Whitney stratification of X adapted to α. Then*

$$\int \mathcal{M}(X \cap H \subset X; \alpha) = \chi(X \cap H'; \alpha) - \chi(X \cap H; \alpha).$$

Proof. The functoriality of c_* implies that

$$\int c_*(i^*\alpha) = \chi(X \cap H; \alpha).$$

Let $i' : X \cap H' \to X$ be the inclusion which is necessarily a regular embedding of codimension 1. Also denote the line bundle $\mathscr{O}(1)$ by L, so that the normal bundle to $X \cap H$ $(X \cap H')$ in X is isomorphic to $L|_{X\cap H}$ $(L|_{X\cap H'})$. We have

$$\begin{aligned}\int c(L|_{X\cap H})^{-1} \cap i^* c_*(\alpha) &= \int c(L)^{-1} c_1(L) \cap c_*(\alpha)\\ &= \int c(L|_{X\cap H'})^{-1} \cap i'^* c_*(\alpha)\\ &= \int c_*(i'^*\alpha)\\ &= \chi(X \cap H'; \alpha)\end{aligned}$$

where the first two equalities follow from the self-intersection formula and the third equality follows from the Verdier-Riemann-Roch formula ([16] Corollary 2.7). □

Our study of the local Euler obstruction for mixed resultants depends on the following theorem.

Theorem 3.3. *(strong version) Let X be an irreducible projective variety in $(\mathbb{P}^n)^\vee$ and let $X^\vee$ its dual in $\mathbb{P}^n$. Let $H \in X^\vee$ be any point on the dual variety and assume $X \cap H$ is a hypersurface in X (rather than X itself), then*

$$Eu_{X^\vee}(H) = (-1)^{\dim X - 1} \int \mathcal{M}(X \cap H \subset X; Eu_X).$$

In [5], Ernström developed the theory of topological Radon transformation. Using this theory he computed the local Euler obstructions for the k-dual varieties (a synonym for our R_s when $s \leq m-1$, see Remark 2.3). The projective duality is a special case of the point-k-plane incidence correspondence studied by him, and in this very case [5] Theorem 3.2 in the guise of topological Radon transformation is equivalent to Theorem 3.3 here.

A weak version of this theorem appears more frequently in the literature.

Theorem 3.4. *(weak version) Let X be a nonsingular projective variety in $(\mathbb{P}^n)^\vee$ and let $X^\vee$ its dual in $\mathbb{P}^n$. Assume $X^\vee$ is a hypersurface. Let $H \in X^\vee$ be any point on the dual variety, then*

$$Eu_{X^\vee}(H) = (-1)^{\dim X - 1} \int \mathcal{M}(X \cap H)$$

where $\mathcal{M}(X \cap H) := \mathcal{M}(X \cap H \subset X; 1_X)$ for a nonsingular X is called the Milnor class of $X \cap H$. (We caution the reader that there are different sign conventions in the literature for the Milnor class.)

There are various proofs for the weak formula. For example, Aluffi showed in [1] that $Eu_{X^\vee}(H)$ equals the degree of the μ-class of the singular subscheme of $X \cap H$. Combining with [2] proposition IV.1 which states that the degree of the μ-class equals the degree of the signed difference of the virtual Chern class and the Chern-Schwartz-MacPherson of $X \cap H$ proves the weak formula. However, this approach relies essentially on the smoothness of X. [5] states the weak formula as a corollary of the topological Radon transformation formula, and it also contains a short summary of other forms of this formula in the work of Dimca, Pragacz and Parusiński.

In the rest of this section I will explain how to derive the strong formula from the holomorphic intersection formula for conic Lagrangian varieties. This approach appears to be more conceptual than the existing proofs in the literature.

First, let us recall the holomorphic intersection formula.

Theorem 3.5. *Let M be a complex manifold, let $Z \subset M$ be a reduced irreducible analytic subset and let $f : M \to \mathbb{C}$ be a holomorphic function. Suppose the intersection $|T^*_Z M \cap df(M)|$ is projected onto a compact subset K of $Z \cap \{f = 0\}$, then*

$$\sharp([T^*_Z M] \cap [df(M)]) = \int_K Eu^\vee_Z + Eu^\vee_{f|Z} = (-1)^{\dim Z - 1} \int_K \phi_f(Eu_Z)$$

where ϕ_f is the vanishing cycle functor.

Remark 3.6. *The second equality is the easier part. In fact,*

$$(-1)^{\dim Z - 1} \phi_f(Eu_Z) = -\phi_f Eu^\vee_Z = Eu^\vee_Z - \psi_f Eu^\vee_Z = Eu^\vee_Z + Eu^\vee_{f|Z}$$

as constructible functions on $Z \cap \{f = 0\}$. (see [13] *Theorem 4.3)*

The holomorphic intersection formula is the work of many mathematicians. The version we are referring to is taken from [13]. The reader is encouraged to see [14] for a thorough discussion of the intersection formula, including historical remarks and a beautiful generalisation in real geometry.

To facilitate our discussion below, we need to recall briefly the proof of Theorem 3.5 given in [13]. First, we can embed M into $M \times \mathbb{C}$ by the graph of f, so that we are in the following situation.

$$\begin{tikzcd} Z \arrow[r] \arrow[drr, "f"'] & M \arrow[r, "\text{graph}"] & M \times \mathbb{C} \arrow[d] \\ & & \mathbb{C}, \end{tikzcd}$$

Notation 3.7. *The usual notation for the normal cone sometimes will be too cumbersome, so we will use the notation $C(A, B)$ below to denote the normal cone to $A \cap B$ in B whenever A, B are subschemes of an ambient space M. If A has a normal bundle $N_A M$, then $C(A, B)$ is a subcone of $N_A M\big|_{A \cap B}$.*

Using stratification theory, Sabbah proved that

$$[C\Big(M \times T^*\mathbb{C}, T^*_Z(M\times\mathbb{C})\Big)] = [T^*_f M] + \sum_i m_i [T^*_{Z_i} M \times T^*_0\mathbb{C}] \tag{1}$$

where

- This is an equality of cycles (as opposed to homology classes) in $T^*(M\times\mathbb{C})$.
- $T^*_f M$ is the conormal space relative to the fibre of f in M.
- Those Z_i are compact analytic varieties contained in $f^{-1}(0)$.
- $T^*_f M$ computes $Eu^\vee_{f|Z} = -\psi_f(Eu^\vee_Z)$. The cone $C\Big(M\times T^*\mathbb{C}, T^*_Z(M\times\mathbb{C})\Big)$ computes $Eu^\vee_Z$ as $T^*_Z(M\times\mathbb{C})$ does.

The properties listed above suffice to prove the holomorphic intersection formula, modulo standard technics from intersection theory. In fact, Sabbah implicitly proved the stronger statement that $\sum_i m_i[T^*_{Z_i}M]$ is the characteristic cycle of $(-1)^{\dim Z-1}\phi_f(Eu_Z)$ loc.cit.

Since our aim is to work over $(\mathbb{P}^n)^\vee$, and there is no non-constant global holomorphic functions, the theorem cited above is not sufficient for our purpose. To state a global analogue of the holomorphic intersection formula, let us consider the following situation.

Let M be an m-dimensional complex manifold, let L be a holomorphic line bundle on M and let $f\in H^0(M,L)$ be a global section. Denote by P^1L the bundle of principal parts of L. Recall that there is a short exact sequence of vector bundles on M

$$0 \ \to T^*M\otimes L \to P^1L \to L \to 0. \tag{2}$$

Let Z be a reduced and irreducible analytic closed subset of M and $Z\neq M$. Suppose the equation $f=0$ defines a hypersurface in Z. Define the space of the conormal jets JT^*_ZM as the closure of the set

$$\{\Big(j(g)\Big)(x)\in P^1L(x) \mid x\in reg(Z), g(x)=0, j(g) \text{ annihilates the tangent space } T_xX\},$$

so in particular JT^*_ZM is a $\mathbb{C}^*$-invariant analytic subset of $T^*M\otimes L$ of dimension m. Let $\pi:\mathbf{P}(JT^*_ZM)\to Z$ be the projection, let ξ be the tautological line bundle over $\mathbf{P}(JT^*_ZM)$, and let $\zeta_1=\pi^*(P^1L)/\xi$, $\zeta_2=\pi^*(T^*M\otimes L)/\xi$ be the quotient bundles of rank m and $m-1$ respectively. The section $j(f)\in H^0(M,P^1L)$ induces a section $\overline{j(f)}\in H^0(\mathbf{P}(JT^*_ZM),\zeta_1)$. Consider the following Cartesian diagram

$$\begin{array}{ccc} Sing(Z,f) & \longrightarrow & \mathbf{P}(JT^*_ZM) \\ \big\downarrow & & \big\downarrow{\scriptstyle \overline{j(f)}} \\ \mathbf{P}(JT^*_ZM) & \longrightarrow & \zeta_1 \end{array}$$

where the bottom arrow is the zero section embedding, and $Sing(Z,f)$ is the zero scheme of $\overline{j(f)}$. Intuitively, according to stratified Morse theory, this space contains the information about $\phi_f(Eu_Z)$ along $Z\cap\{f=0\}$.

Theorem 3.8. *Suppose $\pi(Sing(Z,f))$ is compact (In particular, this is true if Z is compact), then*

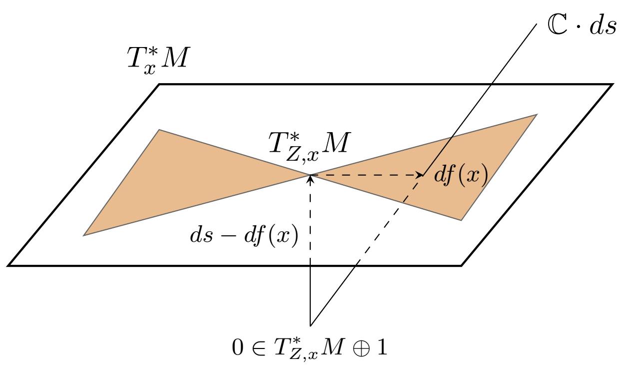
$$\int c(T^*M\otimes L)\cap s\Big(Sing(Z,f),\mathbf{P}(JT^*_ZM)\Big) = (-1)^{\dim Z-1}\int \mathcal{M}\Big(Z\cap\{f=0\}\subset Z; Eu_Z\Big)$$

To prove this theorem, we first reformulate Sabbah's key equation (1) without using the graph embedding. So let's return to the local situation briefly. In the discussion below, we will denote the graph of $f: M\to\mathbb{C}$ by $\tilde{M}$.

Note that there is a canonical decomposition $T^*(M\times\mathbb{C})\big|_{\tilde M}\cong T^*M\oplus T^*_{\tilde M}(M\times\mathbb{C})$ where the T^*M component is the restriction to $\tilde M$ of $pr_1^*(T^*M)$ and pr_1 is the natural projection $M\times\mathbb{C}\to M$. If s is the coordinate on $\mathbb{C}$, then T^*M consists of those covectors having no ds part. The following observations are simple.

- $ds-df$ is a non-vaninshing section of $T^*_{\tilde M}(M\times\mathbb{C})$ so that $T^*_{\tilde M}(M\times\mathbb{C})$ is a trivial line bundle.
- α is a covector in $T^*_Z M$ if and only if $\alpha+ds-df$ is a covector in $T^*_Z(M\times\mathbb{C})$.

In other words, we can make the identification $T^*(M\times\mathbb{C})\big|_{\tilde M}\cong T^*M\oplus 1$ and $T^*_Z(M\times\mathbb{C})\cong T^*_Z(M)\oplus 1$ under this identification.



Let $x\in Z$ be any point and denote the fibre of the $T^*_Z M$ over x by $T^*_{Z,x}M$. The figure indicates that the normal directions to $\mathbb{C}\cdot ds$ in $T^*_{Z,x}(M\times\mathbb{C})$ are equal to the normal directions to $df(x)$ in $T^*_{Z,x}M$, modulo the $\mathbb{C}^*$-action along the ds direction. Letting x move on Z, we conclude that the projectivisation of $C(M\times T^*\mathbb{C}, T^*_Z(M\times\mathbb{C}))$ along the ds-direction equals $C(df(M), T^*_Z M)$. Combining with (1), we obtain

Proposition 3.9.
$$[C(df(M),T^*_Z M)]=\sum_i m_i[T^*_{Z_i}M]$$
which is the characteristic cycle of $(-1)^{\dim Z-1}\phi_f(Eu_Z)$.

Proof. This is a direct consequence of equation (1). One notes that the 1-dimensional space $T^*_0\mathbb{C}$ becomes a point after projectivisation along the ds-direction, and the component $T^*_f M$ is killed by the projectivisation because it is completely contained in the 0-section of the ds-direction. ☐

Let's now convert back to the global situation, so that $f\in H^0(M,L)$ in the sequel.

Corollary 3.10.
$$[C\Big((j(f))(M), JT^*_Z M\Big)]=\sum_i m_i[JT^*_{Z_i}M]$$
where $Z_i\subset\{f=0\}$ *for all* i *and* $\sum_i m_i[T^*_{Z_i}M]$ *when restricted to a small neighbourhood* U *of any* $x\in Z$ *is the characteristic cycle of* $(-1)^{\dim Z-1}\phi_{f|U}(Eu_{Z\cap U})$. *In the last statement* $f|U$ *is regarded as a local holomorphic function on* U.

Remark 3.11. $\phi_{f|U}$ *is a priori only an analytically constructible function on* U, *but these glue to a global analytically constructible function because its projectivised characteristic cycle is given by an analytic cycle in* $\mathbf{P}(T^*M\otimes L)\cong\mathbf{P}(T^*M)$. *In case* Z *and* f *are* *algebraic, the Z_is' are also algebraic, and the glueing process will give us a global algebraically constructible function. By a slight abuse of notation, we denote this constructible function by $\phi_f(Eu_Z)$.*

Proof. The naive argument goes as follows. $C\Big((j(f))(M), JT^*_Z M\Big)$ is a globally defined cone over $Z \cap \{f = 0\}$. Examining the cone locally and use proposition 3.9, one will derive all the statements in the corollary. However, there is a small gap in this argument. Since $j(f)$ is a section of P^1L, the normal cone $C\Big((j(f))(M), JT^*_Z M\Big)$ is in principle just a subcone of P^1L rather than $T^*M \otimes L$. To remove the flaw, we argue that locally $j(f)$ is given by the map

$$\begin{aligned} \mathbb{C}^m \supset U &\to \mathbb{C}^{m+1} \\ (x_1, \ldots, x_m) &\mapsto \Big(h(x_1, \ldots, x_m), \frac{\partial h}{\partial x_1}(x_1, \ldots, x_m), \ldots, \frac{\partial h}{\partial x_m}(x_1, \ldots, x_m)\Big) \end{aligned}$$

where h is a local equation of f, $m = \dim M$ and $x_1, \ldots, x_m$ are local analytic coordinates on U. Let $(y_0, \ldots, y_m)$ be the coordinates on $\mathbb{C}^{m+1}$, and let $\mathscr{R}$ be the ring of $T^*_{Z\cap U}U$ which can be regarded as a quotient ring of $\mathscr{O}_{U\times\mathbb{C}^m}$. The intersection $j(f)(U) \cap T^*_{Z\cap U}U$ is defined by the ideal $(-h, y_1 - \frac{\partial h}{\partial x_1}, \ldots, y_m - \frac{\partial h}{\partial x_m})\mathscr{R}$. So to prove the corollary, it suffices to prove that h is integral over $(y_1 - \frac{\partial h}{\partial x_1}, \ldots, y_m - \frac{\partial h}{\partial x_m})\mathscr{R}$. Note that if $Z = M$ then $T^*_Z M$ is the zero section of T^*M so that $y_1 = \ldots = y_m = 0$ in $\mathscr{R}$, and it is a well-known result that h is integral over $(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_m})$ in $\mathscr{O}_U$. The proof of the special case can be easily adapted to take care of the general case. Indeed, let $(z_0, \alpha_0) \in T^*_{Z\cap U}U$ and let

$$\begin{aligned} \rho : (\mathbb{C}, 0) &\to (T^*_{U\cap Z}(U), (z_0, \alpha_0)) \\ t &\mapsto (x_1(t), \ldots, x_m(t), \alpha_1(t), \ldots, \alpha_m(t)) \end{aligned}$$

be any analytic map germ with $(z_0, \alpha_0) = (x_1(0), \ldots, x_m(0), \alpha_1(0), \ldots, \alpha_m(0))$. We have

$$\begin{aligned} \frac{d\rho^*(h)}{dt} &= \frac{\partial h}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial h}{\partial x_m}\frac{dx_m}{dt} \\ &= \sum_i (\frac{\partial h}{\partial x_i} - \alpha_i)\frac{dx_i}{dt} + \sum_i \alpha_i \frac{dx_i}{dt} \\ &= \sum_i (\frac{\partial h}{\partial x_i} - \alpha_i)\frac{dx_i}{dt} \end{aligned}$$

where the last equation follows from the Whitney A condition. Indeed, fix a Whitney stratification of Z. We can assume that $(x_1(t), \ldots, x_m(t))$ remains staying inside one stratum for $0 < |t| \ll 1$. Since $(x_1(t), \ldots, x_m(t), \alpha_1(t), \ldots, \alpha_m(t))$ is a limiting conormal vector from the largest stratum Z_{reg}, the Whitney A condition guarantees that $(\alpha_1(t), \ldots, \alpha_m(t))$ annihilates $(\frac{dx_1}{dt}(t), \ldots \frac{dx_m}{dt}(t))$. So $\sum_i \alpha_i \frac{dx_i}{dt}$ is identically 0 in a small punctured neighbourhood of $0 \in \mathbb{C}$, which implies that $\sum_i \alpha_i \frac{dx_i}{dt} = 0$ in $(\mathbb{C}, 0)$ by continuity. Therefore, $\operatorname{ord}_t(\rho^*(h)) = \operatorname{ord}_t(t\frac{d\rho^*(h)}{dt}) \geq \min_i \operatorname{ord}_t((\frac{\partial h}{\partial x_i}(t) - \alpha_i(t))x_i(t))$, and we obtain h is integral over $((y_1 - \frac{\partial h}{\partial x_1})x_1, \ldots, (y_m - \frac{\partial h}{\partial x_m})x_m)\mathscr{R}$ by the arcwise criterion for integral dependence (see [17] §2). Since the last ideal is contained in $(y_1 - \frac{\partial h}{\partial x_1}, \ldots, y_m - \frac{\partial h}{\partial x_m})\mathscr{R}$, we conclude that h is also integral over $(y_1 - \frac{\partial h}{\partial x_1}, \ldots, y_m - \frac{\partial h}{\partial x_m})\mathscr{R}$. □

Since the characteristic cycle of $\phi_f(Eu_Z)$ is known now, its Chern class transformation follows from formal machinery. Let ξ' and ζ' be the tautological line bundle and quotient bundle on $P(T^*_Z M)$. Via the identification $P(JT^*_Z M) \cong P(T^*_Z M)$, we have $\xi \cong \xi' \otimes L$ and $\zeta_2 \cong \zeta' \otimes L$.

Theorem 3.12.

$$(-1)^{\dim Z-1} c_*(\phi_f(Eu_Z)) = (-1)^{m-1}\pi_*\Big(c(\zeta'^\vee) \cap [\mathbf{P}\Big(C\big((j(f))(M), JT^*_Z M\big)\Big)]\Big)$$

$$\mathcal{M}(Z \cap \{f=0\} \subset Z; Eu_Z) = \frac{1}{c(L)} \cap c_*(\phi_f(Eu_Z))$$

Proof. The procedure of associating a Lagrangian cycle and a homology class to a constructible function is standard nowadays. See for example [15]. The relation between the Milnor class and the Chern class transformation of the vanishing cycle constructible function is found in [15] Corollary 5.4, cf. also [12] Theorem 0.2 and Lemma 4.1. □

Theorem 3.8 follows from taking the degree of the second equation.

Proof of Theorem 3.8. Since $\mathbf{P}\Big(C\big((j(f))(M), JT^*_Z M\big)\Big)$ is purely $(m-1)$-dimensional, we need to pair it with the degree $(m-1)$ part of $\frac{1}{c(L)} \cdot c(\zeta'^\vee)$. We have

$$\frac{1}{c(L)} \cdot c(\zeta'^\vee) = \Big(1 - c_1(L) + c_1^2(L) - \dots\Big)\Big(1 - c_1(\zeta') + \dots + (-1)^{m-1}c_{m-1}(\zeta')\Big)$$

whose degree $m-1$ part is

$$(-1)^{m-1}\Big(c_{m-1}(\zeta') + c_1(L)c_{m-2}(\zeta') + \dots\Big) = (-1)^{m-1}c_{m-1}(\zeta' \otimes L) = (-1)^{m-1}c_{m-1}(\zeta_2).$$

Now Theorem 3.12 implies that

$$\begin{aligned}(-1)^{\dim Z-1}\int \mathcal{M}(Z \cap \{f=0\} \subset Z; Eu_Z) &= \int c_{m-1}(\zeta_2) \cap [\mathbf{P}\Big(C\big((j(f))(M), JT^*_Z M\big)\Big)] \\ &= \int c(\zeta_2) \cap [\mathbf{P}\Big(C\big((j(f))(M), JT^*_Z M\big)\Big)] \\ &= \int c(T^*M \otimes L) \cdot \frac{1}{c(\xi)} \cap [\mathbf{P}\Big(C\big((j(f))(M), JT^*_Z M\big)\Big)] \\ &= \int c(T^*M \otimes L) \cap s\Big(Sing(Z,f), \mathbf{P}(JT^*_Z M)\Big)\end{aligned}$$

□

Remark 3.13. *Theorem 3.8 and 3.12 generalise the CSM class formula for singular hypersurfaces inside smooth varieties* [2] [12]. *One can also formulate Theorem 3.12 using Aluffi's tensor notation in* [2]. *We leave it as an exercise to the interested reader.*

Remark 3.14. *A formula of similar type computing the Chern class transformation of the μ-constructible function in the context of a TB-generic holomorphic map between two complex manifolds is announced by Ohmoto* [10]. *We claimed an algebraic proof of Ohmoto's formula in* [8], *but the proof contains serious mistakes. So a genuine proof of Ohmoto's formula is still missing.*

Next, we will prove that over $\mathbb{C}$ Theorem 3.3 follows from Theorem 3.8.

Proof of Theorem 3.3 by the holomorphic intersection formula. Let X be an irreducible projective variety and let L be a very ample line bundle. To simplify notation, set $W = H^0(X, L)$, $P_1 = \mathbf{P}(W^\vee)$ and $P_2 = \mathbf{P}(W)$, so we have projective embeddings $X \subset P_1$ and $X^\vee \subset P_2$. By [6] Chapter 2 proposition 1.2, $H^0(P_1, P^1L) \cong W$, P^1L is identified with

$P_1 \times W$ the trivial bundle with fibre W, and the exact sequence (2) is identified with the twisted Euler sequence

$$0 \to T^*P_1 \otimes L \to P_1 \times W \to L \to 0.$$

Therefore $c(T^*P_1 \otimes L) = \frac{1}{c(L)}$.

Let $H \in X^\vee \subset P_2$ be a point in $X^\vee$ and let $f \in W$ be any of its equation. Note that $\mathbf{P}(JT^*_X P_1)$ is canonically embedded in $\mathbf{P}(P^1 L) = P_1 \times \mathbf{P}(H^0(P_1, P^1 L)) = P_1 \times P_2$, so we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{P}(T^*_X P_1) & \xrightarrow{\cong} & \mathbf{P}(JT^*_X P_1) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \text{jet projection}} \\ P_2 & \xrightarrow{\cong} & \mathbf{P}(H^0(P_1, P^1 L)). \end{array}$$

If we identify H with the 1-dimensional vector space generated by $j(f)$ in $\mathbf{P}(H^0(P_1, P^1 L))$ via the lower row, then we see immediately that $p^{-1}(H) = Sing(X, f)$ via the upper row identification. So Theorem 3.8 implies that

$$\int \frac{1}{c(L)} \cap s(p^{-1}(H), \mathbf{P}(T^*_X P_1)) = (-1)^{\dim X - 1} \int \mathcal{M}(X \cap H \subset X; Eu_X).$$

Finally, the biduality theorem implies that $\mathbf{P}(T^*_X P_1) = \mathbf{P}(T^*_{X^\vee} P_2)$. Consider the Euler sequence on P_2

$$0 \to T^*P_2 \to W^\vee \otimes \mathscr{O}_{P_2}(-1) \to \mathscr{O}_{P_2} \to 0.$$

This sequence implies that there is a natural embedding $\mathbf{P}(T^*P_2) \subset \mathbf{P}(W^\vee \otimes \mathscr{O}_{P_2}(-1)) \cong P_2 \times \mathbf{P}(W^\vee) = P_2 \times P_1$, and the tautological subbundle of $\mathbf{P}(T^*P_2)$ can be identified with $\mathscr{O}_{P_1}(-1) \otimes \mathscr{O}_{P_2}(-1) \cong L^\vee \otimes \mathscr{O}_{P_2}(-1)$.

Because $\mathbf{P}(T^*_{X^\vee} P_2)$ is a subcone of $\mathbf{P}(T^*P_2)$, its tautological subbundle is also $L^\vee \otimes \mathscr{O}_{P_2}(-1)$. Using the Gonzalez-Sprinberg formula for the local Euler obstruction we obtain

$$Eu_{X^\vee}(H) = \int \frac{c(TP_2)}{c(L \otimes \mathscr{O}_{P_2}(1))} \cap s(p^{-1}(H), \mathbf{P}(T^*_{X^\vee} P_2)).$$

This expression is also equal to $\int \frac{1}{c(L)} \cap s(p^{-1}(H), \mathbf{P}(T^*_X P_1))$ because the restriction to $p^{-1}(H)$ of TP_2 and $\mathscr{O}_{P_2}(1)$ are both trivial. □

4. Local Euler obstructions for mixed resultants

We will continue to adopt the notations used in §2. Furthermore, let $p \in R_s$ be a point represented by the multi-section $f = (f_1, \ldots, f_s)$, and let $f' = (f'_1, \ldots, f'_s)$ be a generic multi-section. Let $X_f = X \cap \{f_1 = \ldots, = f_s = 0\}$ and let $X_{f'} = X \cap \{f'_1 = \ldots, = f'_s = 0\}$. Also assume $X_f \neq X$.

Theorem 4.1.

$$Eu_{R_s}(p) = (-1)^{\dim X + s}(\int_{X_{f'}} Eu_X - \int_{X_f} Eu_X).$$

This includes the following special cases.

Corollary 4.2. *(a) If $s = 1$, this is Theorem 3.3.*

(b) If X is smooth and X_f is a complete intersection of codimension s with $s \leq m - 1 = \dim X$, then $Eu_{R_s}(p) = \int \mathcal{M}(X_f)$ where $\mathcal{M}(X_f)$ is the Milnor class of the complete intersection X_f.

(c) If X is smooth and $s > m-1 = \dim X$, then $Eu_{R_s}(p) = (-1)^{\dim X+s+1}\chi(X_f)$ where $\chi(X_f)$ is the topological Euler characteristic of X_f.

Our work is greatly inspired by the work of Aluffi in [1] and [3]. In fact, in [3] he computed the CSM class and the Milnor class of a hypersurface in $\mathbf{P}(E)$ and their pushdown to the base where E is any vector bundle over a smooth base and the hypersurface is induced by any nonzero global section of E. Combining Aluffi's result with Cayley's trick yields part (b) and (c) of Corollary 4.2 immediately. The general case can be treated with essentially the same idea, and the additional technicality is covered by Theorem 3.3.

Proof of Theorem 4.1. Combine proposition 2.4, Proposition 3.2 and Theorem 3.3. Let H, H' be the hyperplanes in $\mathbf{P}(V^\vee)$ defined by $f = 0$ and $f' = 0$, we have

$$Eu_{R_s}(p) = (-1)^{\dim \tilde{X}-1}\Big(\int_{\tilde{X}\cap H'} Eu_{\tilde{X}} - \int_{\tilde{X}\cap H} Eu_{\tilde{X}}\Big).$$

Let $\pi : \tilde{X} \to X$ be the projection and let $y \in \tilde{X}$, $x = \pi(y) \in X$ be two points. We have $Eu_{\tilde{X}}(y) = Eu_X(x)$ since $\tilde{X}$ is a locally trivial fibre bundle over X with nonsingular fibres (by [9] page 426). To understand $\tilde{X} \cap H$, notice that $\tilde{X}$ is the union of the projective subspaces $\mathbf{P}(l_1 \oplus \ldots \oplus l_s)$ (see the notation used in the proof of proposition 2.4). As a point x is moving throughout the base X, the corresponding projective subspace $\mathbf{P}(l_1 \oplus \ldots \oplus l_s)$ is sweeping inside $\mathbf{P}(V^\vee)$ to form $\tilde{X}$. The intersection $\mathbf{P}(l_1 \oplus \ldots \oplus l_s) \cap H$ is either $\mathbf{P}(l_1 \oplus \ldots \oplus l_s)$ itself if $x \in X_f$ or a hyperplane in $\mathbf{P}(l_1 \oplus \ldots \oplus l_s)$ if $x \notin X_f$. Hence

$$\begin{aligned}
&\int_{\tilde{X}\cap H} Eu_{\tilde{X}}\\
=&\int_{\pi^{-1}(X\setminus X_f)} Eu_{\tilde{X}} + \int_{\pi^{-1}(X_f)} Eu_{\tilde{X}}\\
=&(s-1)\int_{X\setminus X_f} Eu_X + s\int_{X_f} Eu_X\\
=&(s-1)\int_X Eu_X + \int_{X_f} Eu_X
\end{aligned}$$

where we have used $\chi(\mathbb{P}^{s-1}) = s$ and $\chi(\mathbb{P}^{s-2}) = s-1$. Similarly

$$\int_{\tilde{X}\cap H'} Eu_{\tilde{X}} = (s-1)\int_X Eu_X + \int_{X_{f'}} Eu_X.$$

Taking the difference, we obtain the desired result. □

When $L_1 = \ldots = L_s$, the mixed resultant can be regarded as a subvariety in the Grassmanian $G(s, V)$, c.f. Remark 2.3. This is the usual resultant ($s > m-1$) or coisotropic variety ($s \le m-1$) in the literature and we denote it by GR_s. We also define the affine mixed resultant as the affine cone over R_s in V, and denote it by AR_s. Since GR_s is the quotient of an open subset of AR_s by $GL(s)$ and R_s is the quotient of an open subset of AR_s by k^*, we conclude by [9] page 426 that the local Euler obstructions of GR_s and R_s are essentially the same. The following corollary in the case $s \le m-1$ is proved in [5] again in the guise of the topological Radon transformation.

Corollary 4.3. *Let $f_1 \wedge \ldots \wedge f_s (\neq 0)$ represent a point in GR_s and let $f = (f_1, \ldots, f_s)$ represent a point in R_s. Then*

$$Eu_{GR_s}(f_1 \wedge \ldots \wedge f_s) = Eu_{R_s}(f) = (-1)^{\dim X+s}\Big(\int_{X_{f'}} Eu_X - \int_{X_f} Eu_X\Big).$$

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Department of Mathematical Sciences, Huaqiao University, Chenghua North Road 269, Quanzhou, Fujian, China

E-mail address: `xliao@hqu.edu.cn`