

# Stabilizing Control

## Control Theory, Lecture 4

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# CHANGING STABILITY

Here are two LTIs:

$$\dot{x} = 2x \tag{1}$$

$$\dot{x} = 2x + u \tag{2}$$

First one is unstable. Second one - we don't know until we know what  $u$  is.

If we pick  $u = 0$ , the result is an unstable equation. But we can also pick  $u$  such that the resulting dynamics is stable, such as  $u = -3x$ :

$$\dot{x} = 2x + u = 2x - 3x = -x \tag{3}$$

So, we can use *control input*  $u$  to change stability of the system!

## Definition

The problem of finding control law  $\mathbf{u}$  that make a certain solution  $\mathbf{x}^*$  of dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$  stable is called *stabilizing control problem*

This is true for both linear and non-linear systems. But for linear systems we can get a lot more details about this problem, if we restrict our choice of control law.

Consider an LTI system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (4)$$

and let us chose control as a linear function of the *state*  $\mathbf{x}$ :

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \quad (5)$$

Thus, we know how the system is going to look when the control is applied:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \quad (6)$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \quad (7)$$

Note that (7) is an autonomous system. We call this (a system that was not autonomous, but became one after substituting control law) a *closed loop* system.

Observing the system  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$  we obtained, we can notice that we already have the tools to analyse its stability. Namely:

### Stability condition for LTI closed-loop system

The real parts of the eigenvalues of the matrix  $(\mathbf{A} - \mathbf{BK})$  should be negative for asymptotic stability, or non-positive for stability in the sense of Lyapunov

So, all you need to do is to find such  $\mathbf{K}$  that  $(\mathbf{A} - \mathbf{BK})$  has eigenvalues with negative real parts, and you made a stable closed-loop system!

# AFFINE CONTROL

## Part 1

We don't have to limit ourselves to just this  $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$  and  $\mathbf{u} = -\mathbf{Kx}$  pair.

In fact, this pair mostly works for the simple case when the solution we want to stabilize is trivial  $\mathbf{x}^*(t) = 0$ .

# AFFINE CONTROL

## Part 2

Let us consider a slightly more complicated system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (8)$$

This is called *affine system*, because of the constant term  $\mathbf{c}$ .

What is the control that stabilizes this system? Let us propose an *affine control law*:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \quad (9)$$

where  $\mathbf{u}^*$  is a constant term.



Thus, from  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}$  and  $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$  we get the following closed-loop system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u}^* + \mathbf{c} \quad (10)$$

And as long as we can choose such  $\mathbf{u}^*$  that  $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$ , we will get back to the previously seen form  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ .

### Existence of the stabilizing control

Notice that same as it is possible that there exists no such  $\mathbf{K}$  that  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$  is stable, same there might exist no such  $\mathbf{u}^*$  that  $\mathbf{B}\mathbf{u}^* = -\mathbf{c}$

Let us now consider an arbitrary solution  $\mathbf{x}^* = \mathbf{x}^*(t)$  for the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (11)$$

and try to find a stabilizing control for it.

Our first step is to notice that, if  $\mathbf{x}^* = \mathbf{x}^*(t)$  is a solution, that means that it satisfies the ODE (11):

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \quad (12)$$

where  $\mathbf{u}^* = \mathbf{u}^*(t)$  is some control law, for which the solution  $\mathbf{x}^* = \mathbf{x}^*(t)$  is obtained.

If we are not given  $\mathbf{u}^* = \mathbf{u}^*(t)$ , we can compute it as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*) \quad (13)$$

where  $\mathbf{B}^+$  is a pseudo-inverse, and the solution to this least-squared problem will have to have no residual (since  $\mathbf{x}^* = \mathbf{x}^*(t)$  is a solution).

$$\|\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{B}\mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^*)\| = 0 \quad (14)$$

Now, let us introduce the concept of *control error*  $\mathbf{e}$ :

$$\mathbf{e} = \mathbf{x} - \mathbf{x}^* \quad (15)$$

### Control error and stability

If control error goes to zero asymptotically, every solution goes to  $\mathbf{x}^*$ .

Remember that we have two simultaneous equations:

$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$  and  $\dot{\mathbf{x}}^* = \mathbf{Ax}^* + \mathbf{Bu}^*$ . We can now subtract one from the other to get:

$$\dot{\mathbf{x}} - \dot{\mathbf{x}}^* = \mathbf{Ax} - \mathbf{Ax}^* + \mathbf{Bu} - \mathbf{Bu}^* \quad (16)$$

in other words:

$$\dot{\mathbf{e}} = \mathbf{Ae} + \mathbf{Bv} \quad (17)$$

where  $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$

# ERROR DYNAMICS

## Part 5

We arrived at a new dynamical system  $\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v}$ , which is an LTI, and we are interested in stabilizing the solution  $\mathbf{e}^* = 0$ . We know how to do it with a linear control law:

$$\mathbf{v} = -\mathbf{K}\mathbf{e} \quad (18)$$

Now remember that  $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$  and  $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$ , this will become:

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^* \quad (19)$$

This control law  $\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^*$  can be thought of as consisting of two parts:

- Feedback control  $\mathbf{u}_{FB} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*)$ , which depends on the control error (which requires a feedback about the current state of your system)
- Feed-forward control  $\mathbf{u}_{FF} = \mathbf{u}^*$ , which depends only on the trajectory and the equations of dynamics of your system, but not on your current state

# AFFINE TRAJECTORY TRACKING

## Part 1

What we just did - stabilization of the arbitrary trajectory  $\mathbf{x}^* = \mathbf{x}^*(t)$  - is also called *trajectory tracking control*, or *trajectory stabilization*. The solution to stabilize is called *trajectory*.

Just for completeness, let's consider the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (20)$$

and stabilize trajectory  $\mathbf{x}^* = \mathbf{x}^*(t)$ .



# AFFINE TRAJECTORY TRACKING

## Part 2

We start by observing that, as before, our solution gives us equality:

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* + \mathbf{c} \quad (21)$$

and after introducing control error and subtracting (21) from the original dynamics (20), we get:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{v} \quad (22)$$

where  $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$ , which we already saw before. The only difference is that now  $\mathbf{u}^*$  is found as:

$$\mathbf{u}^* = \mathbf{B}^+(\dot{\mathbf{x}}^* - \mathbf{A}\mathbf{x}^* - \mathbf{c}) \quad (23)$$

# POINT-TO-POINT CONTROL

## Part 1

What if we want to move our system  $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$  from initial condition to some desired state  $\mathbf{x}^*$ . The difference is that we do not have a solution that we used before, only the desired *node* is given. This can be called *point-to-point control*

Let us start by giving the form of the control law:

$$\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^* \quad (24)$$

And thus we can re-write the dynamics as:

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BK}(\mathbf{x} - \mathbf{x}^*) + \mathbf{Bu}^* \quad (25)$$

# POINT-TO-POINT CONTROL

## Part 2

Let us consider how the system  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{BK}(\mathbf{x} - \mathbf{x}^*) + \mathbf{B}\mathbf{u}^*$  will behave at the point  $\mathbf{x}^*$ . We know that  $\dot{\mathbf{x}}^* = 0$ :

$$0 = \mathbf{A}\mathbf{x}^* - \mathbf{BK}(\mathbf{x}^* - \mathbf{x}^*) + \mathbf{B}\mathbf{u}^* \quad (26)$$

$$0 = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \quad (27)$$

which we saw before. It provides us solution to the forward dynamics:

$$\mathbf{u}^* = -\mathbf{B}^+ \mathbf{A}\mathbf{x}^* \quad (28)$$

Subtracting solution (27) from the original dynamics (25), we get familiar error dynamics  $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e}$ .

# PURE STATE FEEDBACK

## Part 1

Given  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$  and the desired state  $\mathbf{x}^*$  we can do a point-to-point control with the following *pure state feedback control*:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^* \quad (29)$$

We can re-write the dynamics as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{u}^* \quad (30)$$

As before, we know that at the node,  $\dot{\mathbf{x}}^* = 0$ :

$$0 = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \quad (31)$$

Thus we can solve for  $\mathbf{u}^*$

$$\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}^* \quad (32)$$

# PURE STATE FEEDBACK

## Part 2

The rest is the same. Error dynamics is  $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e}$ .

Note that when  $\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{u}^*$ , we got feed-forward control in the form:

$$\mathbf{u}^* = -\mathbf{B}^+(\mathbf{A} - \mathbf{BK})\mathbf{x}^*$$

But when we had  $\mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}^*) + \mathbf{u}^*$ , our feed-forward control was

$$\mathbf{u}^* = -\mathbf{B}^+\mathbf{A}\mathbf{x}^*$$

The difference has to do with how the two control methods behave at the node.

- Richard M. Murray Control and Dynamical Systems  
California Institute of Technology [Optimization-Based Control](#)
- [Dynamic Simulation in Python](#)

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)

Check Moodle for additional links, videos, textbook suggestions.