Linear system with one control input

Step 1

Consider the following to a second order ODE:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x$$

The eigenvalues are $\lambda_1 = -10$ and $\lambda_1 = 1$, so unstable.

Step 2

Now let's consider the same system, but with control input:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Let us chose that $u = -2x_1$, in other words:

$$u = \begin{pmatrix} -2 & 0 \end{pmatrix} x$$

Then we can re-write this as:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} 1 & 0 \\ -5 & -10 \end{pmatrix} x + \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} x$$
$$\dot{x} = \begin{pmatrix} -1 & 0 \\ -5 & -10 \end{pmatrix} x$$

Now the eigenvalues are $\lambda_1 = -10$ and $\lambda_1 = -1$, so stable.

```
import numpy as np
from numpy.linalg import eig

A = np.array([[1, 0], [-5, -10]]) # state matrix
e, v = eig(A)
print("eigenvalues of A:", e)

A = np.array([[-1, 0], [-5, -10]]) # state matrix
e, v = eig(A)
print("eigenvalues of A:", e)
```

```
eigenvalues of A: [-10.   1.]
```

```
eigenvalues of A: [-10.  -1.]
```

Pole placement

There is a technique for finding suitable K matrix that would produced desired eigenvalues of the $A - BK$ system. It is called pole placement.

Watch the intoduction to pole placement for self-study: link. Notice the difference between the approach to "steady state" control design show there, and in the lecture.

```
import numpy as np
from numpy.linalg import eig
from scipy.integrate import odeint
from scipy.signal import place_poles

n = 2
A = np.array([[1, 0], [-5, -10]])
B = np.array([[1], [0]])

# x_dot from state space
def StateSpace(x, t):
    return A.dot(x)# + B*np.sin(t)

time = np.linspace(0, 1, 1000)
x0 = np.random.rand(n)  # initial state

solution = {"SS": odeint(StateSpace, x0, time)}

#desired eigenvalues
poles = np.array([-1, -2])
place_obj = place_poles(A, B, poles)

#found control gains
K = place_obj.gain_matrix;
print("K:", K)

#test that eigenvalues of the closed loop system are what they are supposed to be
e, v = eig((A - B.dot(K)))
print("eigenvalues of A - B*K:", e)
```

```
K: [[ -6.  -14.4]]
eigenvalues of A - B*K: [-1. -2.]
```

```
#desired eigenvalues
poles = np.array([-100, -200])
place_obj = place_poles(A, B, poles)

#found control gains
K = place_obj.gain_matrix;
```

```
print("K:", K)

#test that eigenvalues of the closed loop system are what they are supposed to be
e, v = eig((A - B.dot(K)))
print("eigenvalues of A - B*K:", e)

#notice how different the gain matrix is!
```

```
K: [[  291. -3420.]]
eigenvalues of A - B*K: [-200. -100.]
```

Task 1.1 Make the following systems stable, proposing appropriate control

$$\dot{x} = \begin{pmatrix} 10 & 0 \\ -5 & 10 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 0 & -8 \\ 1 & 30 \end{pmatrix} x + \begin{pmatrix} -2 \\ 1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 2 & 2 \\ -6 & 10 \end{pmatrix} x + \begin{pmatrix} 0 \\ 5 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 5 & -5 \\ 6 & 15 \end{pmatrix} x + \begin{pmatrix} -10 \\ 10 \end{pmatrix} u$$

Task 1.2 Make the following systems stable, proposing appropriate control

$$\dot{x} = \begin{pmatrix} 10 & 0 \\ -5 & 10 \end{pmatrix} x + \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 0 & -8 \\ 1 & 30 \end{pmatrix} x + \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 2 & 2 \\ -6 & 10 \end{pmatrix} x + \begin{pmatrix} 0 & -1 \\ 5 & -1 \end{pmatrix} u$$
$$\dot{x} = \begin{pmatrix} 5 & -5 \\ 6 & 15 \end{pmatrix} x + \begin{pmatrix} -10 & 3 \\ 10 & 3 \end{pmatrix} u$$

Task 1.3 Give example of an unstable system that can't be stabilized...

of the form $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{2\times2}$

- where $B \in \mathbb{R}^{2\times1}$
- where $B \in \mathbb{R}^{2\times2}$
- where $B \in \mathbb{R}^{2\times3}$

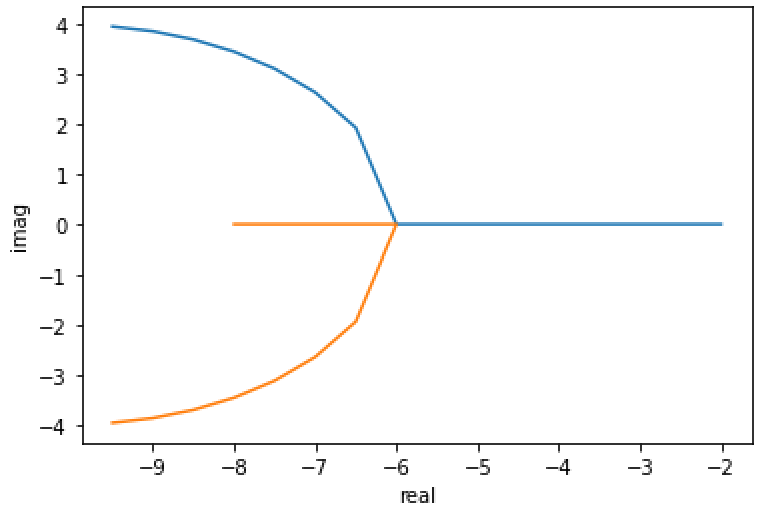
Root locus

Consider the following question: given system $\dot{x} = Ax + Bu$ and control $u = -Kx$, how does the change in K changes the eigenvalues of theresulting matrix $(A - BK)$?

Root locus method is drawing the graph of eigenvalues of the matrix $(A - BK)$ for a given change of matrix K. We only vary a single component of K, so the result is a line.

```
import matplotlib.pyplot as plt

A = np.array([[1, -7], [2, -10]])
B = np.array([[1], [0]])
K0 = np.array([[1, 1]]);

k_min = 1;
k_max = 10;
k_step = 0.1;

Count = np.floor((k_max-k_min)/k_step)
Count = Count.astype(int)

k_range = np.linspace(k_min, k_max, Count)

E = np.zeros((Count, 4))

for i in range(Count):
    K0[0, 0] = k_range[i]

    ei, v = eig((A - B.dot(K0)))

    E[i, 0] = np.real(ei[0])
    E[i, 1] = np.imag(ei[0])
    E[i, 2] = np.real(ei[1])
    E[i, 3] = np.imag(ei[1])

    #print("eigenvalues of A - B*K:", ei)


plt.plot(E[:, 0], E[:, 1])
plt.plot(E[:, 2], E[:, 3])
plt.xlabel('real')
```

```
plt.ylabel('imag')
plt.show()
```



Task 2.1 Plot root locus

- For a system with A with imaginary eigenvalues
- For a system with A with real eigenvalues
- For a system where real parts of eigenvalues of $(A - BK)$ are all positive
- For a system where real parts of eigenvalues of $(A - BK)$ are all negative

Reaction to inputs

Task 3 Step functions

Task 3.1 Simulate one of the given systems with a step function as an imput.

Task 3.2 Linear combination of solutions

Simulate one of the given systems with two different step functions f_1, f_2 as an imput, and as a sum of those $f_1 + f_2$ as an imput. Compare the sum of the solutions for the f_1, f_2 with the solution for $f_1 + f_2$.

$$f_1 = \begin{cases} 1, & t \geq t_1 \\ 0, & t < t_1 \end{cases}$$
$$f_2 = \begin{cases} 1, & t \geq t_2 \\ 0, & t < t_2 \end{cases}$$

Task 4 Sinusoidal inputs

Simulate one of the prevuiously given function for a sinusoidal input $u = sin(wt)$.

How does the choice of w affects the result?

(not graded): Watch video on "frequency responce" and find how you could use the proposed method to analyse the effect of w in your problem.

Now, let us see how to plot ferquency responce in a plot, via scipy library:

```
from scipy.signal import ss2tf
from scipy.signal import freqz


A = np.array([[1, -7], [2, -10]])
B = np.array([[1], [0]])
C = np.eye(2)
D = np.zeros((2, 1))

num, den = ss2tf(A, B, C, D)

print("num:", num)
print("den:", den)

w1, h1 = freqz(num[0, :], den)
w2, h2 = freqz(num[1, :], den)

plt.subplot(211)
plt.plot(w1, 20 * np.log10(abs(h1)), 'b')
plt.ylabel('Amplitude [dB]', color='b')
plt.xlabel('Frequency [rad/sample]')

plt.subplot(212)
plt.plot(w2, 20 * np.log10(abs(h2)), 'b')
plt.ylabel('Amplitude [dB]', color='b')
plt.xlabel('Frequency [rad/sample]')
```

```
num: [[ 0.  1. 10.]
 [ 0.  0.  2.]]
den: [1. 9. 4.]
Text(0.5, 0, 'Frequency [rad/sample]')
```

Task 4.1 Make frequency diagrams for 2 of the systems you studied in the tasks 1.1 and 1.2

5. Point-to-point control

Given system:

$$\dot{x} = \begin{pmatrix} 10 & 5 \\ -5 & -10 \end{pmatrix} x + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u$$

let us drive it towards the point $x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We define our control as:

$$u = -K(x - x^*) + u^*$$

Step 1 - Feed-forward design

We know that $\dot{x}^* = 0$ and that at the node our dynamics obtains the form:

$$0 = \begin{pmatrix} 10 & 5 \\ -5 & -10 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u^*$$

In other words, $u^* = 5$.

Step 2

We define control error as $e = x - x^*$.

Now we write error dynamics:

$$\dot{x} - \dot{x}^* = \begin{pmatrix} 10 & 5 \\ -5 & -10 \end{pmatrix} x - \begin{pmatrix} 10 & 5 \\ -5 & -10 \end{pmatrix} x^* + \begin{pmatrix} -1 \\ 2 \end{pmatrix} (-K(x - x^*) + u^*) - \begin{pmatrix} -1 \\ 2 \end{pmatrix} u$$

$$\dot{e} = \begin{pmatrix} 10 & 5 \\ -5 & -10 \end{pmatrix} e - \begin{pmatrix} -1 \\ 2 \end{pmatrix} Ke$$

Step 3 - feedback design

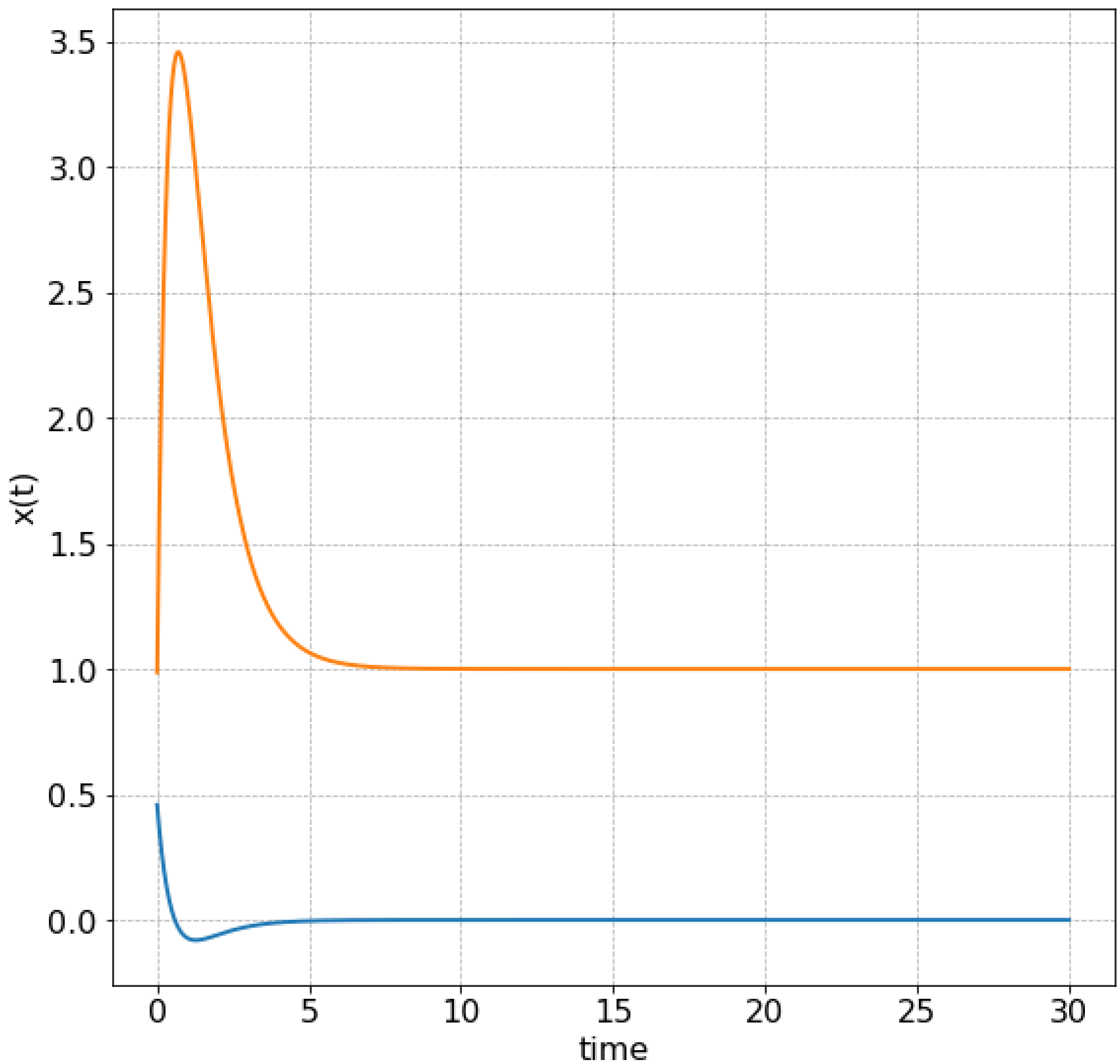
```
A = np.array([[10, 5], [-5, -10]])
B = np.array([[-1], [2]])

#desired eigenvalues
poles = np.array([-1, -2])
place_obj = place_poles(A, B, poles)

#found control gains
K = place_obj.gain_matrix;
print("K:", K)

#test that eigenvalues of the closed loop system are what they are supposed to be
e, v = eig((A - B.dot(K)))
print("eigenvalues of A - B*K:", e)
```

```
K: [[-13.26666667  -5.13333333]]
eigenvalues of A - B*K: [-2. -1.]
```

Simulate forward wirth the found control

```
x_desired = np.array([0, 1])
u_desired = np.array([5])


def StateSpace(x, t):
    u = -K.dot(x - x_desired) + u_desired
    return A.dot(x) + B.dot(u)

time = np.linspace(0, 30, 30000)
x0 = np.random.rand(n)  # initial state

solution = {"solution_1": odeint(StateSpace, x0, time)}


plt.rcParams['figure.figsize'] = [20, 20]

# plt.rcParams["font.family"] = "Old Times American"
plt.rcParams["font.size"] = 16
plt.rcParams["font.weight"] = 'normal'


# plt.subplot(221)
plt.plot(time, solution["solution_1"], linewidth=2)
plt.xlabel('time')
plt.ylabel('x(t)')
```

```
plt.grid(color='k', linestyle='--', linewidth=0.7, alpha=0.3)
# plt.title('autonomous')
```



Task 5.1 Design point-to-point control and simulate two systems:

- where $B \in \mathbb{R}^{2\times 1}$
- where $B \in \mathbb{R}^{2\times 2}$

6. Discrete systems

Let's consider discrete system: $x_{i+1} = Ax_i + Bu_i$

Task 6.1

Find which of the followig systems is stable:

$$x_{i+1} = \begin{pmatrix} 0.5 & 0.1 \\ -0.05 & 0.2 \end{pmatrix} x_i$$

$$x_{i+1} = \begin{pmatrix} 1 & -2 \\ 0 & 0.3 \end{pmatrix} x_i$$

$$x_{i+1} = \begin{pmatrix} -5 & 0 \\ -0.1 & 1 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} u_i, \quad u_i = \begin{pmatrix} 0 & 0.2 \end{pmatrix} x_i$$

$$x_{i+1} = \begin{pmatrix} -2.2 & -3 \\ 0 & 0.5 \end{pmatrix} x_i + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u_i, \quad u_i = 10$$

Task 6.2

Propose control that makes the following systems stable:

$$x_{i+1} = \begin{pmatrix} 1 & 1 \\ -0.4 & 0.1 \end{pmatrix} x_i + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} u_i$$

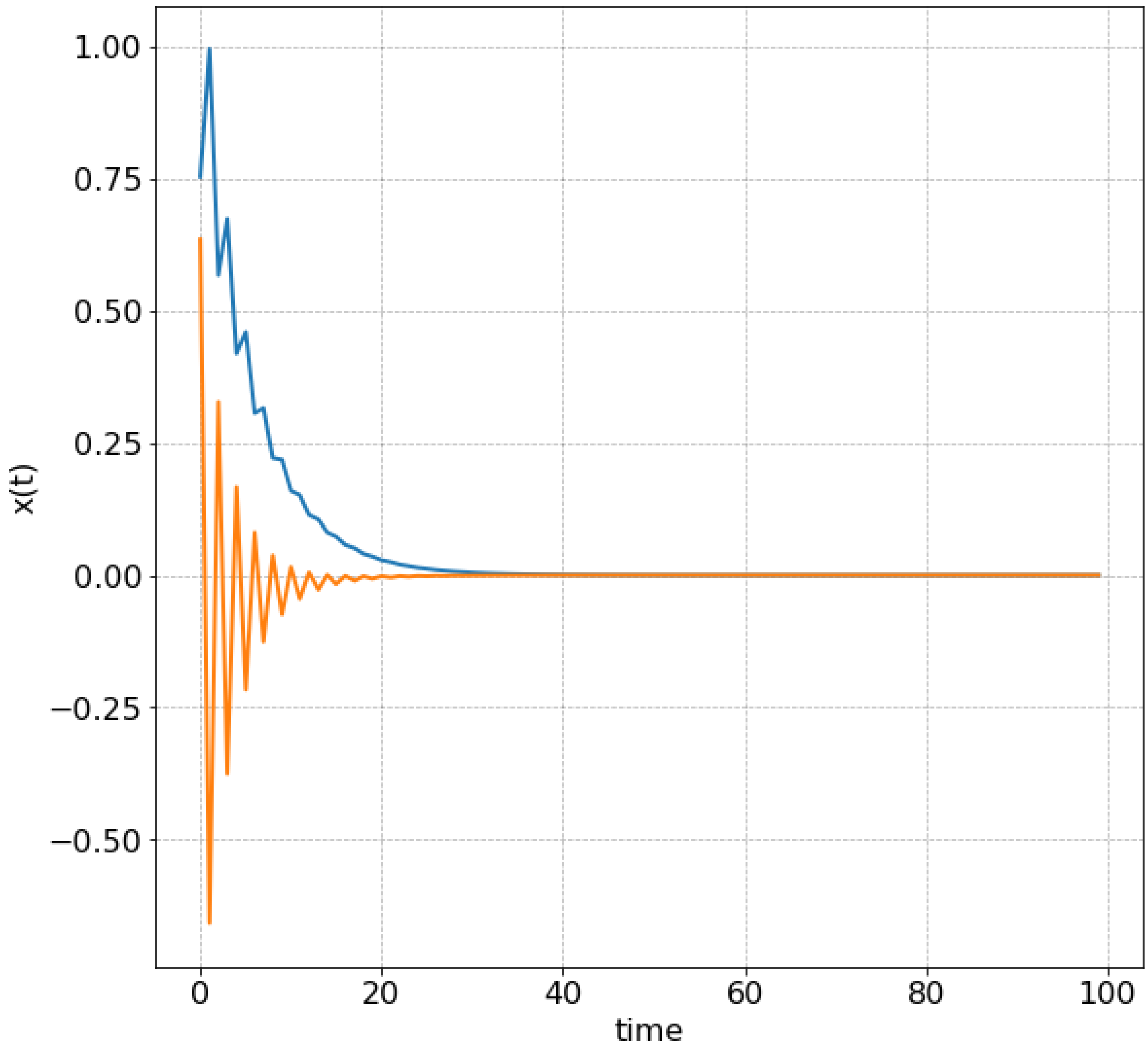
$$x_{i+1} = \begin{pmatrix} 0.8 & -0.3 \end{pmatrix} x_i + \begin{pmatrix} -1 \end{pmatrix} u_i$$

Simulation of descrete systems

Consider the system: $x_{i+1} = Ax_i$. Let us pick values for the matrix A and simulate it forward.

```
A = np.array([[0.9, 0.5], [-0.2, -0.8]])

e, v = eig((A))
print("eigenvalues of A:", e)

Count = 100
time = np.zeros((Count))
dt = 0.01

x0 = np.random.rand(n)  # initial state
solution = np.zeros((Count, 2))
solution[0, :] = x0

for i in range(0, Count-1):
    x = solution[i, :]
    x = A.dot(x)
    solution[i+1, :] = np.reshape(x, (1, 2))
    time[i] = dt*i

plt.subplot(221)
plt.plot(range(0, Count), solution, linewidth=2)
plt.xlabel('time')
plt.ylabel('x(t)')
plt.grid(color='k', linestyle='--', linewidth=0.7, alpha=0.3)
```

```
eigenvalues of A: [ 0.83898669 -0.73898669]
```



Task 6.3 Design point-to-point control and simulate two discrete systems:

- where $B \in \mathbb{R}^{2\times 1}$
- where $B \in \mathbb{R}^{2\times 2}$

7 Lyapunov equations

Solve Lyapunov equations for both discrete and continious systems to prove stability

```
from scipy.linalg import solve_continuous_lyapunov
from scipy.linalg import solve_discrete_lyapunov
```

```
Q = np.array([[-1, 0], [0, -1]])
```

```
A = np.array([[-10, 5], [-5, -10]])
e, v = eig(A)
print("eig(A)", e)

P = solve_continuous_lyapunov(A, Q)
print("P", P)
e, v = eig((A.transpose().dot(P) + P.dot(A)))
print("eig(A'P + P*A)", e)
print(" ")
print(" ")


A = np.array([[0.9, 0.5], [-0.2, -0.8]])
e, v = eig(A)
print("eig(A)", e)

P = solve_discrete_lyapunov(A, Q)
print("P", P)
print("(A'PA - P + Q ):")
print(((A.dot(P)).dot(A.transpose()) - P + Q))
```

```
eig(A) [-10.+5.j -10.-5.j]
P [[ 5.00000000e-02  7.34706413e-20]
 [-1.24900090e-18  5.00000000e-02]]
eig(A'P + P*A) [-1. -1.]


eig(A) [ 0.83898669 -0.73898669]
P [[-4.03347296  0.9268445 ]
 [ 0.9268445  -2.40207966]]
(A'PA - P + Q ):
[[0.00000000e+00 3.33066907e-16]
 [1.11022302e-16 4.44089210e-16]]
```

Task 7.1

Choose one of the continious and one of the discrete systems for which you designed control, and prove stability of the closed-loop version $(A - BK)$