

# Discrete Dynamics

## Control Theory, Lecture 6

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The following dynamical system is called *discrete*:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \quad (1)$$

Note that those:

- have no derivatives in the equation;
- are easily simulated.

The affine control for this system can be given as:

$$\mathbf{u}_i = -\mathbf{K}\mathbf{x}_i + \mathbf{u}_i^* \quad (2)$$

# STABILITY OF THE DISCRETE DYNAMICS

## Part 1

Let us consider stability of the discrete dynamical system  
 $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$ .

We will attack the problem in the same way as before, first assuming that  $\mathbf{A} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}$ , where  $\mathbf{D}$  is a diagonal matrix with eigenvalues of  $\mathbf{A}$  on its diagonal:

$$\mathbf{x}_{i+1} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \quad (3)$$

Multiplying both sides of the equation by  $\mathbf{V}$  and defining  $\mathbf{z}_i = \mathbf{V}\mathbf{x}_i$ , we get:

$$\mathbf{V}\mathbf{x}_{i+1} = \mathbf{V}\mathbf{V}^{-1}\mathbf{D}\mathbf{V}\mathbf{x}_i \quad (4)$$

$$\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i \quad (5)$$

# STABILITY OF THE DISCRETE DYNAMICS

## Part 2

Now, considering  $\mathbf{z}_{i+1} = \mathbf{D}\mathbf{z}_i$  we can see that the norm of the state  $\mathbf{z}_i$  would not increase iff the norm of the elements of  $\mathbf{D}$  (which as eigenvalues of  $\mathbf{A}$ ) are smaller than 1.

### Stability criterion

In general, discrete systems  $\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$  are stable as long as the eigenvalues of  $\mathbf{A}$  are smaller than 1 by absolute value:  $|\lambda_i(\mathbf{A})| \leq 1, \forall i$ . This is true for complex eigenvalues as well.

# DISCRETIZATION

## Finite difference

Consider linear time-invariant autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

The time derivative  $\dot{\mathbf{x}}$  can be replaced with a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + \Delta t) - \mathbf{x}(t)) \quad (7)$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t + 0.5\Delta t) - \mathbf{x}(t - 0.5\Delta t)) \quad (8)$$

or

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t}(\mathbf{x}(t) - \mathbf{x}(t - \Delta t)) \quad (9)$$

# DISCRETIZATION

## Finite difference notation

We can introduce notation:

$$\begin{cases} \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{x}_1 = \mathbf{x}(\Delta t) \\ \mathbf{x}_2 = \mathbf{x}(2\Delta t) \\ \dots \\ \mathbf{x}_n = \mathbf{x}(n\Delta t) \end{cases} \quad (10)$$

We say that  $\mathbf{x}_i$  is the value of  $\mathbf{x}$  at the time step  $i$ . Then the finite difference can be written, for example, as follows:

$$\dot{\mathbf{x}} \approx \frac{1}{\Delta t} (\mathbf{x}_{i+1} - \mathbf{x}_i) \quad (11)$$

# DISCRETIZATION

## Finite difference in an autonomous LTI

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_i \quad (12)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}_i \quad (13)$$

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Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}_{i+1} - \mathbf{x}_i) = \mathbf{A}\mathbf{x}_{i+1} \quad (14)$$

Isolating  $\mathbf{x}_{i+1}$  on the left hand side, we get:

$$\mathbf{x}_{i+1} = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}_i \quad (15)$$



# DISCRETIZATION

## Zero order hold

Defining *discrete state space matrix*  $\bar{\mathbf{A}}$  and *discrete control matrix*  $\bar{\mathbf{B}}$  as follows:

$$\bar{\mathbf{A}} = \mathbf{A}\Delta t + \mathbf{I} \quad (16)$$

$$\bar{\mathbf{B}} = \mathbf{B}\Delta t \quad (17)$$

We get discrete dynamics:

$$\mathbf{x}_{i+1} = \bar{\mathbf{A}}\mathbf{x}_i + \bar{\mathbf{B}}\mathbf{u}_i \quad (18)$$

This way of defining discrete dynamics is called *zero order hold (ZOH)*.

# ZOH AND OTHER TYPES OF DISCRETIZATION

## Zero order hold vs First order hold

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

Graphically, zero order hold is this:



First order hold is this:

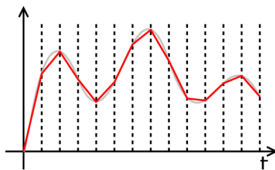


Figure 1: Different types of discretization

# ZOH AND OTHER TYPES OF DISCRETIZATION

## Exact discretization

Let the discrete state  $\mathbf{x}_i$  correspond to continuous state  $\mathbf{x}$  at the moment of time  $t_i$ . Then, we can say that the discretization is *exact* the following holds for any solution  $\mathbf{x}(t)$

$$\mathbf{x}_0 = \mathbf{x}(t_0) \rightarrow \mathbf{x}_i = \mathbf{x}(t_i), \quad \forall i \quad (19)$$

We can compute the exact discretization as follows:

$$\bar{\mathbf{A}} = e^{\mathbf{A}\Delta t} \quad (20)$$

$$\bar{\mathbf{B}} = \mathbf{B} \int_{t_0}^{t_0+\Delta t} e^{\mathbf{A}s} ds \quad (21)$$

- [Automatic Control 1 Discrete-time linear systems](#), Prof. Alberto Bemporad, University of Trento

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)

Check Moodle for additional links, videos, textbook suggestions.