Tutorial 4 – Coding Lab 2:

Quantum Information Qiskit package, Density matrices & Superdense Coding

Week 6 : 04/11/2020

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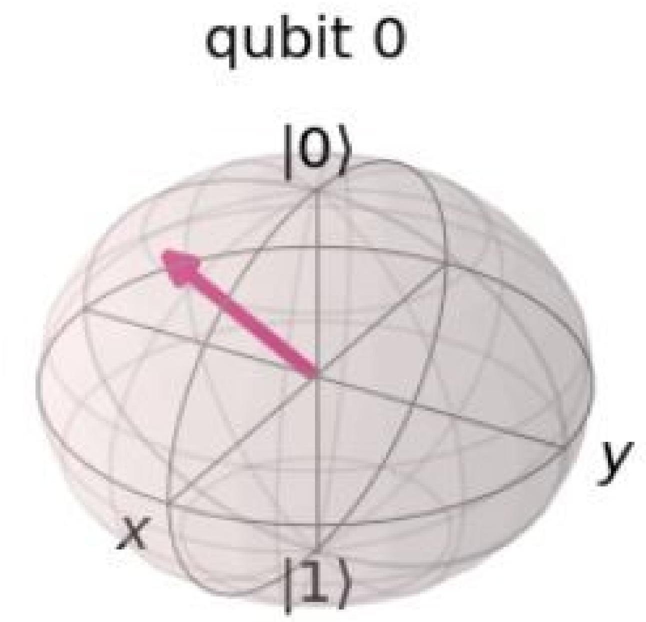
Quantum information module:

qiskit.quantum_info.<something>....

1. Random statevector

```
from qiskit.quantum_info import random_statevector
our_random_statevector =random_statevector(2, seed=None)
plot_bloch_multivector(our_random_statevector)
```




```
print(our_random_statevector)
```

```
Statevector([-0.42718534-0.64098579j, -0.37066602+0.51889942j],
            dims=(2,))
```

2 Minute Task:

Generate the Bloch sphere of the statevector $|r\rangle$

2. Pauli Strings, N-qubit Operators and evolution of statevectors

```
from qiskit.quantum_info.operators import Operator, Pauli
```

Some of the ways you can specify Single or Multiqubit operators:

1. Initiating Pauli Strings ex: XX + ZZ + II

2. Input your own NumPy array for the matrix of the operator

3. Translating a gate into an operator

AGAIN:

In 1994, David P. DiVincenzo

“proved that quantum gates operating on just two bits at a time are sufficient to construct a general quantum circuit”

Paper : https://arxiv.org/abs/cond-mat/9407022v1

3. Density Matrices

Let's find the density matrix of a Singlet:

```
Singlet_statevector_array = np.array([0,-1/np.sqrt(2),1/np.sqrt(2),0])
```

```
Singlet_statevector = Statevector(Singlet_statevector_array)
Singlet_statevector
```

```
Statevector([ 0.        +0.j, -0.70710678+0.j,  0.70710678+0.j,
              0.        +0.j],
            dims=(2, 2))
```

```
My_Singlet_density_matrix = DensityMatrix(Singlet_statevector)
My_Singlet_density_matrix
```

```
DensityMatrix([[ 0. +0.j,  0. -0.j,  0. +0.j,  0. +0.j],
               [ 0. +0.j,  0.5+0.j, -0.5+0.j,  0. +0.j],
               [ 0. +0.j, -0.5-0.j,  0.5+0.j,  0. +0.j],
               [ 0. +0.j,  0. -0.j,  0. +0.j,  0. +0.j]],
              dims=(2, 2))
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let’s Find the Purity of the state:
We can use the statevector or the density matrix

$$\rho^{Singlet} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Purity = tr(\rho^2)$$

For Pure states Purity = 1

Let’s Find the Purity of the state:
We can use the statevector or the density matrix

```
## finding purity given the statevector
Singlet_statevector.purity()
```

```
0.9999999999999996
```

```
## finding purity given the density matrix:
```

```
My_Singlet_density_matrix.purity()
```

```
(0.9999999999999996+0j)
```

Let's Find the reduced density matrix of the Singlet by tracing out one of its subsystems:

2.3.3: Finding reduced density matrices by tracing out subsystems:

```
from qiskit.aqua.utils import get_subsystem_density_matrix
```

```
Singlet_reduced_density_matrix_array = get_subsystem_density_matrix(Singlet_statevector_array,[1])
Singlet_reduced_density_matrix_array
```

```
array([[0.5+0.j, 0. +0.j],
       [0. +0.j, 0.5+0.j]])
```

```
Singlet_reduced_density_matrix = DensityMatrix(Singlet_reduced_density_matrix_array)
Singlet_reduced_density_matrix
```

```
DensityMatrix([[0.5+0.j, 0. +0.j],
               [0. +0.j, 0.5+0.j]],
              dims=(2,))
```

```
Singlet_reduced_density_matrix.purity()
```

```
(0.49999999999999998+0j)
```

This shows that the reduced density matrix of The singlet is MAXIMALLY MIXED with a Purity = l/2

We illustrated how the reduced density matrix of a maximally entangled state, is a maximally mixed state

$$\text{Reduced } \rho^{Singlet} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

Purity of reduced density matrix = 1/2

4-Qubit GHZ State:

$$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

Simulated density matrix

$$\hat{\rho}^{(GHZ)} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$\rho^{(/A)} = \underbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}}$$

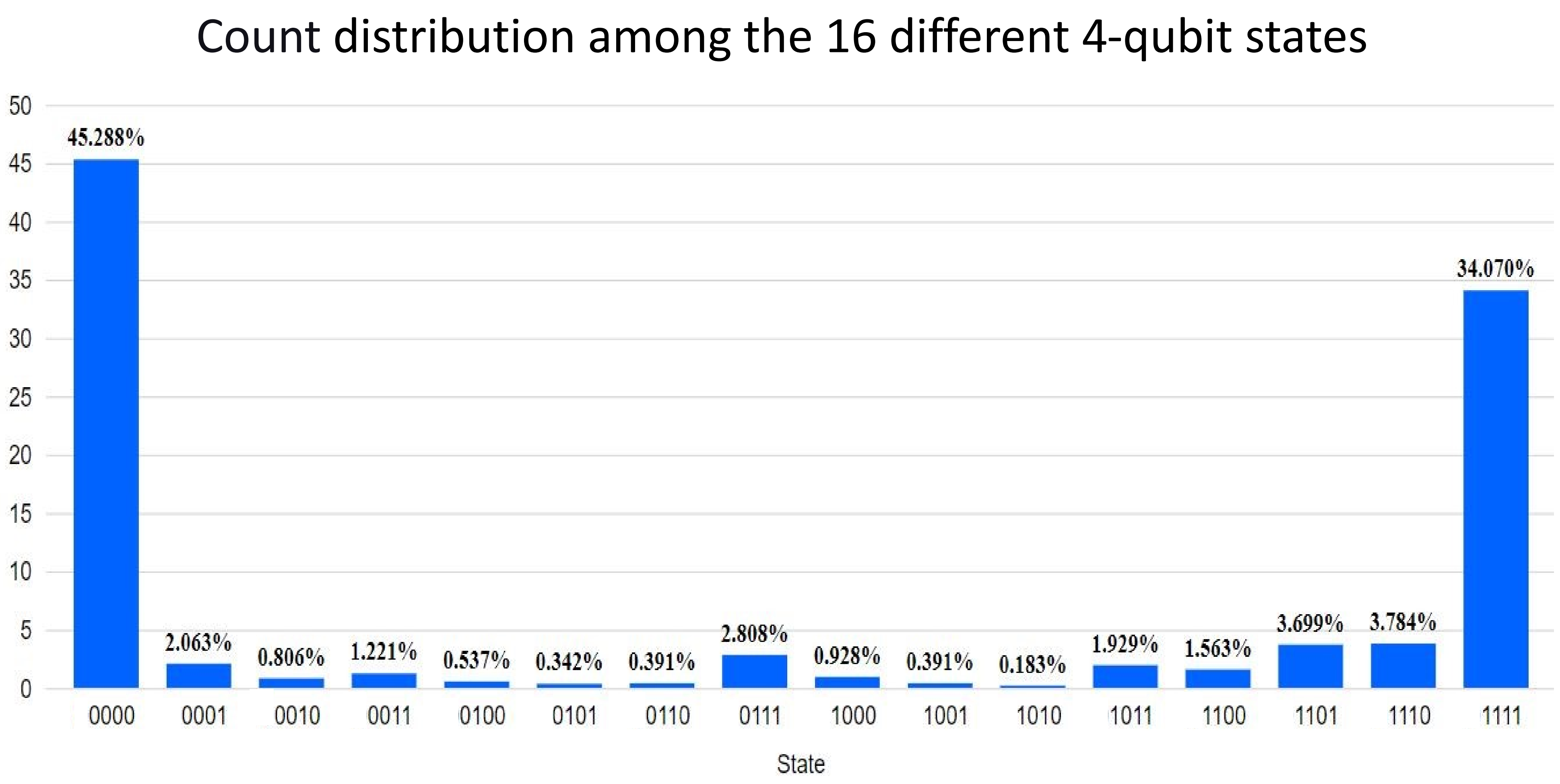
Reduced density matrix of partition A

$$\mathbf{(0.5\,,0.5)}$$

$$\rho^{(/B)} = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}}$$

Reduced density matrix of partition B

$$\mathbf{(0.5\,,0.5)}$$



Count distribution among the 16 different 4-qubit states
50
45
40
35
30
25
20
15
10
5
0
45.288%
2.063%
0.806%
1.221%
0.537%
0.342%
0.391%
2.808%
0.928%
0.391%
0.183%
1.929%
1.563%
3.699%
3.784%
34.070%
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
State

$$
\left(\begin{array}{cccccccccccccccc}
0.45288 & 0.0966588 & 0.060417 & 0.0743617 & 0.049315 & 0.0393554 & 0.0420804 & 0.112769 & 0.0648284 & 0.0420804 & 0.0287884 & 0.0934669 & 0.0841339 & 0.12943 & 0.130908 & 0.392806 \\
0.0966588 & 0.02063 & 0.0128949 & 0.0158711 & 0.0105254 & 0.00839968 & 0.00898127 & 0.0240685 & 0.0138364 & 0.00898127 & 0.00614434 & 0.0199488 & 0.0179568 & 0.0276243 & 0.0279399 & 0.083837 \\
0.060417 & 0.0128949 & 0.00806 & 0.00992031 & 0.00657892 & 0.00525026 & 0.00561379 & 0.0150441 & 0.00864851 & 0.00561379 & 0.00384055 & 0.0124691 & 0.011224 & 0.0172667 & 0.017464 & 0.0524027 \\
0.0743617 & 0.0158711 & 0.00992031 & 0.01221 & 0.00809739 & 0.00646206 & 0.00690949 & 0.0185164 & 0.0106447 & 0.00690949 & 0.00472698 & 0.015347 & 0.0138146 & 0.021252 & 0.0214948 & 0.0644977 \\
0.049315 & 0.0105254 & 0.00657892 & 0.00809739 & 0.00537 & 0.00428549 & 0.00458222 & 0.0122796 & 0.00705929 & 0.00458222 & 0.00313482 & 0.0101778 & 0.0091615 & 0.0140938 & 0.0142549 & 0.0427733 \\
0.0393554 & 0.00839968 & 0.00525026 & 0.00646206 & 0.00428549 & 0.00342 & 0.0036568 & 0.00979967 & 0.00563361 & 0.0036568 & 0.00250172 & 0.0081223 & 0.00731127 & 0.0112475 & 0.011376 & 0.0341349 \\
0.0420804 & 0.00898127 & 0.00561379 & 0.00690949 & 0.00458222 & 0.0036568 & 0.00391 & 0.0104782 & 0.00602369 & 0.00391 & 0.00267494 & 0.00868469 & 0.0078175 & 0.0120263 & 0.0121637 & 0.0364985 \\
0.112769 & 0.0240685 & 0.0150441 & 0.0185164 & 0.0122796 & 0.00979967 & 0.0104782 & 0.02808 & 0.0161426 & 0.0104782 & 0.00716843 & 0.0232737 & 0.0209497 & 0.0322286 & 0.0325967 & 0.0978103 \\
0.0648284 & 0.0138364 & 0.00864851 & 0.0106447 & 0.00705929 & 0.00563361 & 0.00602369 & 0.0161426 & 0.00928 & 0.00602369 & 0.00412097 & 0.0133795 & 0.0120435 & 0.0185275 & 0.0187391 & 0.056229 \\
0.0420804 & 0.00898127 & 0.00561379 & 0.00690949 & 0.00458222 & 0.0036568 & 0.00391 & 0.0104782 & 0.00602369 & 0.00391 & 0.00267494 & 0.00868469 & 0.0078175 & 0.0120263 & 0.0121637 & 0.0364985 \\
0.0287884 & 0.00614434 & 0.00384055 & 0.00472698 & 0.00313482 & 0.00250172 & 0.00267494 & 0.00716843 & 0.00412097 & 0.00267494 & 0.00183 & 0.00594144 & 0.00534817 & 0.0082275 & 0.00832149 & 0.0249696 \\
0.0934669 & 0.0199488 & 0.0124691 & 0.015347 & 0.0101778 & 0.0081223 & 0.00868469 & 0.0232737 & 0.0133795 & 0.00868469 & 0.00594144 & 0.01929 & 0.0173638 & 0.0267121 & 0.0270173 & 0.0810685 \\
0.0841339 & 0.0179568 & 0.011224 & 0.0138146 & 0.0091615 & 0.00731127 & 0.0078175 & 0.0209497 & 0.0120435 & 0.0078175 & 0.00534817 & 0.0173638 & 0.01563 & 0.0240448 & 0.0243195 & 0.0729736 \\
0.12943 & 0.0276243 & 0.0172667 & 0.021252 & 0.0140938 & 0.0112475 & 0.0120263 & 0.0322286 & 0.0185275 & 0.0120263 & 0.0082275 & 0.0267121 & 0.0240448 & 0.03699 & 0.0374126 & 0.112261 \\
0.130908 & 0.0279399 & 0.017464 & 0.0214948 & 0.0142549 & 0.011376 & 0.0121637 & 0.0325967 & 0.0187391 & 0.0121637 & 0.00832149 & 0.0270173 & 0.0243195 & 0.0374126 & 0.03784 & 0.113543 \\
0.392806 & 0.083837 & 0.0524027 & 0.0644977 & 0.0427733 & 0.0341349 & 0.0364985 & 0.0978103 & 0.056229 & 0.0364985 & 0.0249696 & 0.0810685 & 0.0729736 & 0.112261 & 0.113543 & 0.3407
\end{array}\right)
$$

$$
\begin{pmatrix} 0.5348 & 0.270896 \\ 0.270896 & 0.46523 \end{pmatrix}
$$

Reduced density matrix of partition A – 1^{st} Qubit

$$
\begin{pmatrix}
0.47351 & 0.0762881 & 0.0577146 & 0.0661489 & 0.0738097 & 0.0487371 & 0.111758 & 0.214745 \\
0.0762881 & 0.02027 & 0.013041 & 0.0241302 & 0.015558 & 0.0191876 & 0.032476 & 0.0819616 \\
0.0577146 & 0.013041 & 0.00879 & 0.0143819 & 0.0107161 & 0.0112571 & 0.020409 & 0.0483898 \\
0.0661489 & 0.0241302 & 0.0143819 & 0.03199 & 0.0165019 & 0.0259486 & 0.0400461 & 0.109974 \\
0.0738097 & 0.015558 & 0.0107161 & 0.0165019 & 0.01319 & 0.0128057 & 0.0240698 & 0.0552376 \\
0.0487371 & 0.0191876 & 0.0112571 & 0.0259486 & 0.0128057 & 0.02112 & 0.0320603 & 0.08939 \\
0.111758 & 0.032476 & 0.020409 & 0.0400461 & 0.0240698 & 0.0320603 & 0.05262 & 0.13658 \\
0.214745 & 0.0819616 & 0.0483898 & 0.109974 & 0.0552376 & 0.08939 & 0.13658 & 0.37854
\end{pmatrix}
$$

Reduced density matrix of partition B – $2^{nd}, 3^{rd}, 4^{th}$ Qubits

$$\begin{pmatrix}
0.45288 & 0.0966588 & 0.060417 & 0.0743617 & 0.049315 & 0.0393554 & 0.0420804 & 0.112769 & 0.0648284 & 0.0420804 & 0.0287884 & 0.0934669 & 0.0841339 & 0.12943 & 0.130908 & 0.392806 \\
0.0966588 & 0.02063 & 0.0128949 & 0.0158711 & 0.0105254 & 0.00839968 & 0.00898127 & 0.0240685 & 0.0138364 & 0.00898127 & 0.00614434 & 0.0199488 & 0.0179568 & 0.0276243 & 0.0279399 & 0.083837 \\
0.060417 & 0.0128949 & 0.00806 & 0.00992031 & 0.00657892 & 0.00525026 & 0.00561379 & 0.0150441 & 0.00864851 & 0.00561379 & 0.00384055 & 0.0124691 & 0.011224 & 0.0172667 & 0.017464 & 0.0524027 \\
0.0743617 & 0.0158711 & 0.00992031 & 0.01221 & 0.00809739 & 0.00646206 & 0.00690949 & 0.0185164 & 0.0106447 & 0.00690949 & 0.00472698 & 0.015347 & 0.0138146 & 0.021252 & 0.0214948 & 0.0644977 \\
0.049315 & 0.0105254 & 0.00657892 & 0.00809739 & 0.00537 & 0.00428549 & 0.00458222 & 0.0122796 & 0.00705929 & 0.00458222 & 0.00313482 & 0.0101778 & 0.0091615 & 0.0140938 & 0.0142549 & 0.0427733 \\
0.0393554 & 0.00839968 & 0.00525026 & 0.00646206 & 0.00428549 & 0.00342 & 0.0036568 & 0.00979967 & 0.00563361 & 0.0036568 & 0.00250172 & 0.0081223 & 0.00731127 & 0.0112475 & 0.011376 & 0.0341349 \\
0.0420804 & 0.00898127 & 0.00561379 & 0.00690949 & 0.00458222 & 0.0036568 & 0.00391 & 0.0104782 & 0.00602369 & 0.00391 & 0.00267494 & 0.00868469 & 0.0078175 & 0.0120263 & 0.0121637 & 0.0364985 \\
0.112769 & 0.0240685 & 0.0150441 & 0.0185164 & 0.0122796 & 0.00979967 & 0.0104782 & 0.02808 & 0.0161426 & 0.0104782 & 0.00716843 & 0.0232737 & 0.0209497 & 0.0322286 & 0.0325967 & 0.0978103 \\
0.0648284 & 0.0138364 & 0.00864851 & 0.0106447 & 0.00705929 & 0.00563361 & 0.00602369 & 0.0161426 & 0.00928 & 0.00602369 & 0.00412097 & 0.0133795 & 0.0120435 & 0.0185275 & 0.0187391 & 0.056229 \\
0.0420804 & 0.00898127 & 0.00561379 & 0.00690949 & 0.00458222 & 0.0036568 & 0.00391 & 0.0104782 & 0.00602369 & 0.00391 & 0.00267494 & 0.00868469 & 0.0078175 & 0.0120263 & 0.0121637 & 0.0364985 \\
0.0287884 & 0.00614434 & 0.00384055 & 0.00472698 & 0.00313482 & 0.00250172 & 0.00267494 & 0.00716843 & 0.00412097 & 0.00267494 & 0.00183 & 0.00594144 & 0.00534817 & 0.0082275 & 0.00832149 & 0.0249696 \\
0.0934669 & 0.0199488 & 0.0124691 & 0.015347 & 0.0101778 & 0.0081223 & 0.00868469 & 0.0232737 & 0.0133795 & 0.00868469 & 0.00594144 & 0.01929 & 0.0173638 & 0.0267121 & 0.0270173 & 0.0810685 \\
0.0841339 & 0.0179568 & 0.011224 & 0.0138146 & 0.0091615 & 0.00731127 & 0.0078175 & 0.0209497 & 0.0120435 & 0.0078175 & 0.00534817 & 0.0173638 & 0.01563 & 0.0240448 & 0.0243195 & 0.0729736 \\
0.12943 & 0.0276243 & 0.0172667 & 0.021252 & 0.0140938 & 0.0112475 & 0.0120263 & 0.0322286 & 0.0185275 & 0.0120263 & 0.0082275 & 0.0267121 & 0.0240448 & 0.03699 & 0.0374126 & 0.112261 \\
0.130908 & 0.0279399 & 0.017464 & 0.0214948 & 0.0142549 & 0.011376 & 0.0121637 & 0.0325967 & 0.0187391 & 0.0121637 & 0.00832149 & 0.0270173 & 0.0243195 & 0.0374126 & 0.03784 & 0.113543 \\
0.392806 & 0.083837 & 0.0524027 & 0.0644977 & 0.0427733 & 0.0341349 & 0.0364985 & 0.0978103 & 0.056229 & 0.0364985 & 0.0249696 & 0.0810685 & 0.0729736 & 0.112261 & 0.113543 & 0.3407
\end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0.5348 & 0.270896 \\ 0.270896 & 0.46523 \end{pmatrix}}$$

Reduced density matrix of partition A – 1^{st} Qubit

$$\mathbf{(0.773135\,,0.226895)}$$

$$\underbrace{\begin{pmatrix}
0.47351 & 0.0762881 & 0.0577146 & 0.0661489 & 0.0738097 & 0.0487371 & 0.111758 & 0.214745 \\
0.0762881 & 0.02027 & 0.013041 & 0.0241302 & 0.015558 & 0.0191876 & 0.032476 & 0.0819616 \\
0.0577146 & 0.013041 & 0.00879 & 0.0143819 & 0.0107161 & 0.0112571 & 0.020409 & 0.0483898 \\
0.0661489 & 0.0241302 & 0.0143819 & 0.03199 & 0.0165019 & 0.0259486 & 0.0400461 & 0.109974 \\
0.0738097 & 0.015558 & 0.0107161 & 0.0165019 & 0.01319 & 0.0128057 & 0.0240698 & 0.0552376 \\
0.0487371 & 0.0191876 & 0.0112571 & 0.0259486 & 0.0128057 & 0.02112 & 0.0320603 & 0.08939 \\
0.111758 & 0.032476 & 0.020409 & 0.0400461 & 0.0240698 & 0.0320603 & 0.05262 & 0.13658 \\
0.214745 & 0.0819616 & 0.0483898 & 0.109974 & 0.0552376 & 0.08939 & 0.13658 & 0.37854
\end{pmatrix}}$$

Reduced density matrix of partition B – $2^{nd}, 3^{rd}, 4^{th}$Qubits

$$\mathbf{(0.773135\,,0.226895\,,0\,,0\,,0\,,0,0\,,0)}$$

Measurement in NON-STANDARD basis:

An x-basis measurement = a Z-basis measurement following an X-basis $\longrightarrow$ Z-basis transformation

So how do we get the unitary for an X-basis $\longrightarrow$ Z-basis transformation?

We simply find the Unitary that takes the standard basis (Z-basis) to the X-basis, so performs Z-basis $\longrightarrow$ X-basis, and call it U.

So $U^\dagger$ will take us in the opposite direction from X-basis $\longrightarrow$ Z-basis

To find U we must find the U which satisfies:

$$U(|0\rangle\ and\ |1\rangle) \longrightarrow |+\rangle\ and\ |-\rangle$$

$$U(|0\rangle \; and \; |1\rangle) \longrightarrow |+\rangle \; and \; |-\rangle$$

We already know a Unitary which does this:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle; \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

So the H-gate takes the standard basis $|0\rangle \; and \; |1\rangle$ and maps them to $|+\rangle \; and \; |-\rangle$, and it is our unitary U, and in this case it is only 1-gate.

We also Know $H^{\dagger} = H$, so to perform a measurement in the X-basis we apply the H-Gate to the single qubit before measuring it.

To make this even clearer:

Consider we begin with the state $|+\rangle$, and we want to measure in the $|+/-\rangle$ basis:

First we apply the H-Gate before measurement:

$$H\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle$$

This brings us to the $|0\rangle$ state, so if we measure along the Z-basis now we get :
State $|0\rangle$ with 100 percent probability, this means we were originally in the $|+\rangle$ state.
If we had gotten $|1\rangle$, that would mean we were originally in the $|-\rangle$ state.

So we have basically indirectly inferred the status of our state in the the $|+\rangle$ and $|-\rangle$ measurement basis, from making measurements in the $|0\rangle$ and $|1\rangle$.

Superdense Coding

How can Alice use superdense coding to send the 01 message to Bob. Draw the circuit, and a step by step guide of what happens to both her and Bob’s qubits during the entire process up till measurement.

NB: Please make sure you consider what Bob has to do, on receiving both qubits, to measure the bipartite system in the Bell-basis.

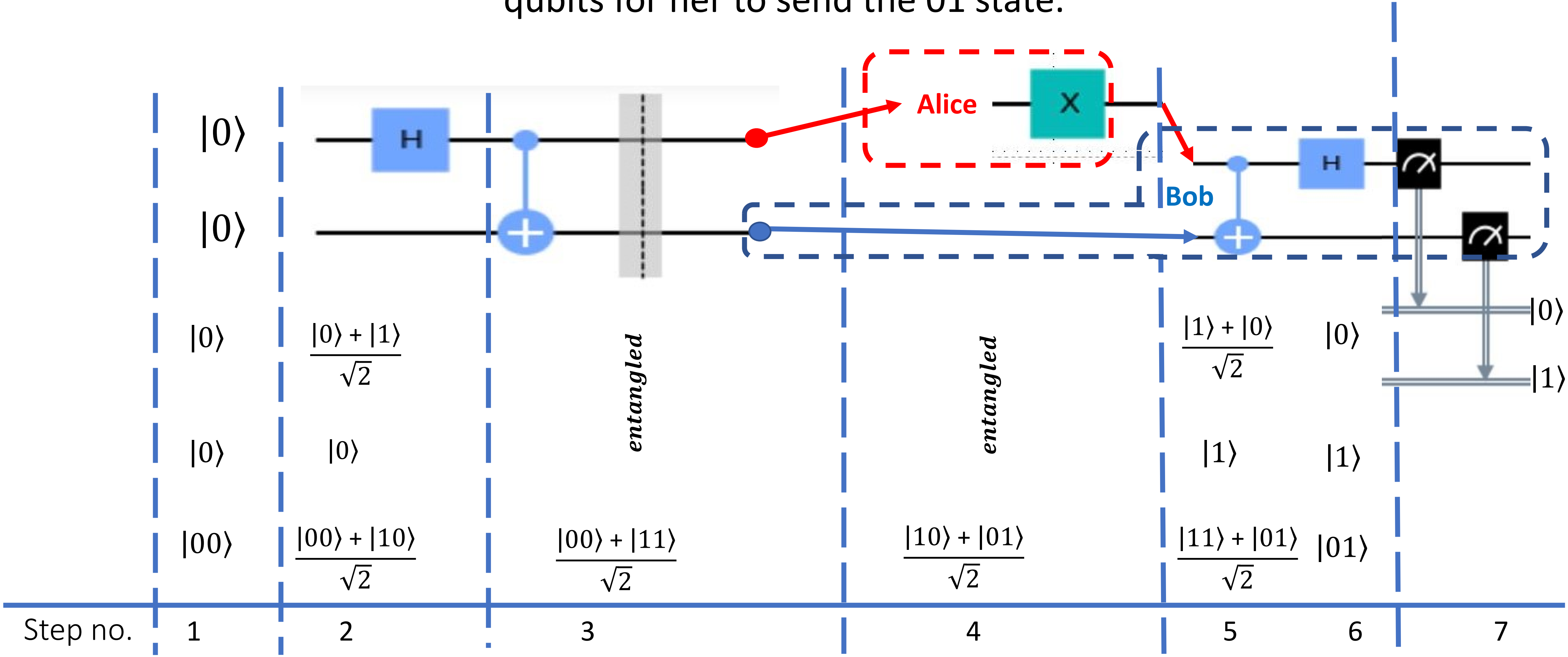
Alice and Bob just need to communicate one of the 4 Bell states, each with it's own 2-bit message, making that 4 possible messages, 00, 01, 10, 11.

Alice needs to initiate this communication process through ONLY acting on one of 2 Qubits.
NB: The Bell states are sometimes named by the message they convey in superdense coding

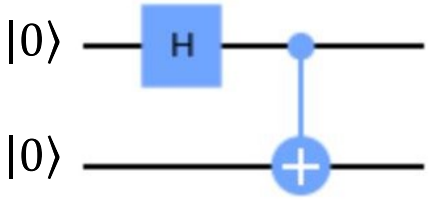
$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Superdense Coding

How can Alice code for 01, draw a step by step guide of what happens to both her and Bob's qubits for her to send the 01 state.



First: we initialize an entangled state using this circuit:



After applying the Hadamard gate to the 1^{st} qubit the bipartite state gives $\longrightarrow$

$$\left.\begin{array}{l} 1^{st} qubit\text{:}\ \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \\[2em] 2^{nd} qubit\text{:}\ \ |0\rangle \end{array}\right\} \ \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \longrightarrow \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

After applying the CNOT-Gate $\longrightarrow$

$$\text{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}\text{CNOT}(|00\rangle) + \frac{1}{\sqrt{2}}\text{CNOT}(|10\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

The ENTANGLED Bell-State (β_{00})

Second: Alice gets the first qubit and applies some gates to it based on what message she intends on sending, and Bob gets the second qubit and does nothing but wait for Alice's qubit so that he may start to decode the message:

To convey the 01 message, Alice must turn the Bell β_{00} state into the β_{01} Bell state

To do that she applies the X-Gate to her qubit, yielding the overall bipartite state:

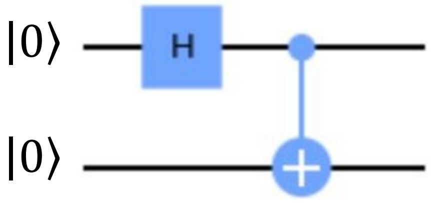
$$\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \beta_{01}$$

When she sends her qubit to Bob, he will have this bell-state with him

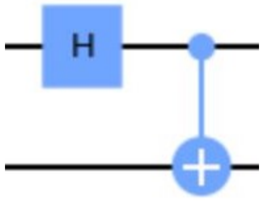
Now after getting a hold of Alice’s qubit, Bob needs to measure the bipartite state in the Bell-basis to determine which of the 4-Bell states Alice’s measurements have placed the two qubits into:

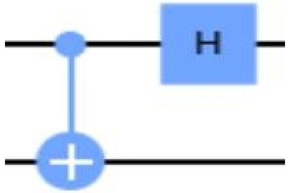
i.e. Bob needs to do a measurement in a non-standard basis, so we need to apply a unitary before measurement.

To reach the Bell-state we used this circuit



So our unitary for transferring the standard basis to the Bell-basis is

$U =$ 

$so\ U^\dagger =$ 

Now we just need to apply the $U^\dagger$ to our circuit before measurement to measure in the Bell-basis

$$U^\dagger\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right) \longrightarrow H^A\left(CNOT\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)\right) = H^A\left(\frac{|11\rangle + |01\rangle}{\sqrt{2}}\right) = H^A\left(\frac{|1\rangle + |0\rangle}{\sqrt{2}}\right) \otimes |1\rangle = |01\rangle$$

$Ungraded\ Assignment$:

Draw a circuit to code for the 11 message.