Tutorial 6 : problem Solving 4

Reviewing chapter 1 and 2 from Nielsen and Chuang for the midterm

Week 8 : 21/11/2020

TA: Hisham Ashraf Amer

Email: s-hisham.amer@zewailcity.edu.eg

QUESTION 1:

locate the state

$$|\psi\rangle = (-0.707 + i0.707)|1\rangle$$

on the Bloch sphere and sketch it

Ans 1:

Given $b = (-0.707 + i0.707)$

ALWAYS TURN a and b into polar form: $re^{i\phi}$

Here $a = 0$

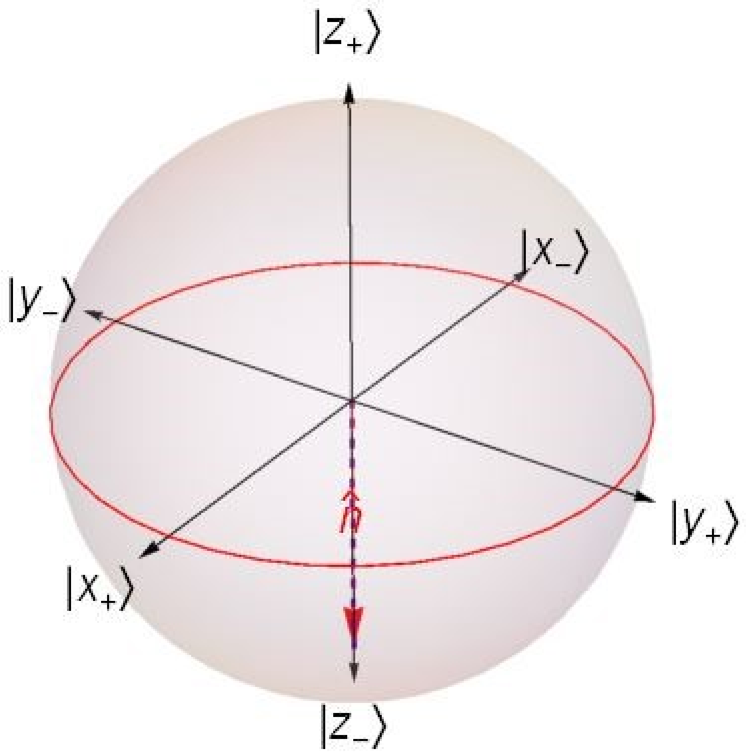
For b:

$r_b = 1$

$\phi = arctan\,1 = \frac{\pi}{4}$

So $b = 1e^{\frac{\pi}{4}}$

$$|\psi\rangle = \begin{pmatrix} 0 \\ 1e^{\frac{\pi}{4}} \end{pmatrix} = e^{\frac{\pi}{4}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



QUESTION 2:

Find 3 different orthonormal bases for a general 2 qubit Hilbert space, i.e. $\in C^2$

And mention if the states are entangled or not in each case.

NB: Each basis element in every set must be different from all the other basis elements from the other sets and any two that are similar up to a phase should be considered similar, for example (1,0,-1) and (-1,0,1) are considered the same up to a phase and will not be counted as different

Ans 2:

$$\text{Standard basis} \begin{bmatrix}1\\0\\0\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\\0\end{bmatrix};\begin{bmatrix}0\\0\\1\\0\end{bmatrix};\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\left|\psi_1\right\rangle = |00\rangle;\ \left|\psi_2\right\rangle = |01\rangle; \left|\psi_3\right\rangle\ = |10\rangle; \left|\psi_4\right\rangle\ = |11\rangle$$

$$\text{Bell basis } \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix};\begin{bmatrix}1\\0\\0\\-1\end{bmatrix};\begin{bmatrix}0\\1\\1\\0\end{bmatrix};\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right)$$

$$\left|\beta_{00}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle);\ \left|\beta_{10}\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle);\ \left|\beta_{01}\right\rangle\ = \frac{1}{\sqrt{2}}\left(|01 + |10\rangle\rangle\right);\ \left|\beta_{11}\right\rangle\ = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\text{Extra basis } \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix};\begin{bmatrix}1\\-1\\0\\0\end{bmatrix};\begin{bmatrix}0\\0\\1\\1\end{bmatrix};\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right)$$

$$\left|\phi_1\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle);\ \left|\phi_2\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle);\ \left|\phi_3\right\rangle\ = \frac{1}{\sqrt{2}}\left(|10 + |11\rangle\rangle\right);\ \left|\phi_4\right\rangle\ = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

$$\left(|0\rangle \otimes \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}\right) \ldots\ etc \ldots \ldots .. product\ states\ while\ \beta_{ij}\ are\ entangled$$

QUESTION 3:

Show that the Bell states are orthogonal

Ans 3:

Show that the Bell states are orthogonal

$$a|\psi_1\rangle + b|\psi_2\rangle + c|\psi_3\rangle + d|\psi_4\rangle = 0$$

$\therefore \frac{1}{\sqrt{2}}\begin{bmatrix} a+b \\ c+d \\ c-d \\ a-b \end{bmatrix}$ =0 … We only have the trivial solution

QUESTION 4:

Give an example of a correlated state and an example of an anti-correlated state

Ans 4:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad co$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \; anti$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad co$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \; anti$$

QUESTION 5:

True or false: Given the state

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

This means the state of the first qubit is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

Ans 5:

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

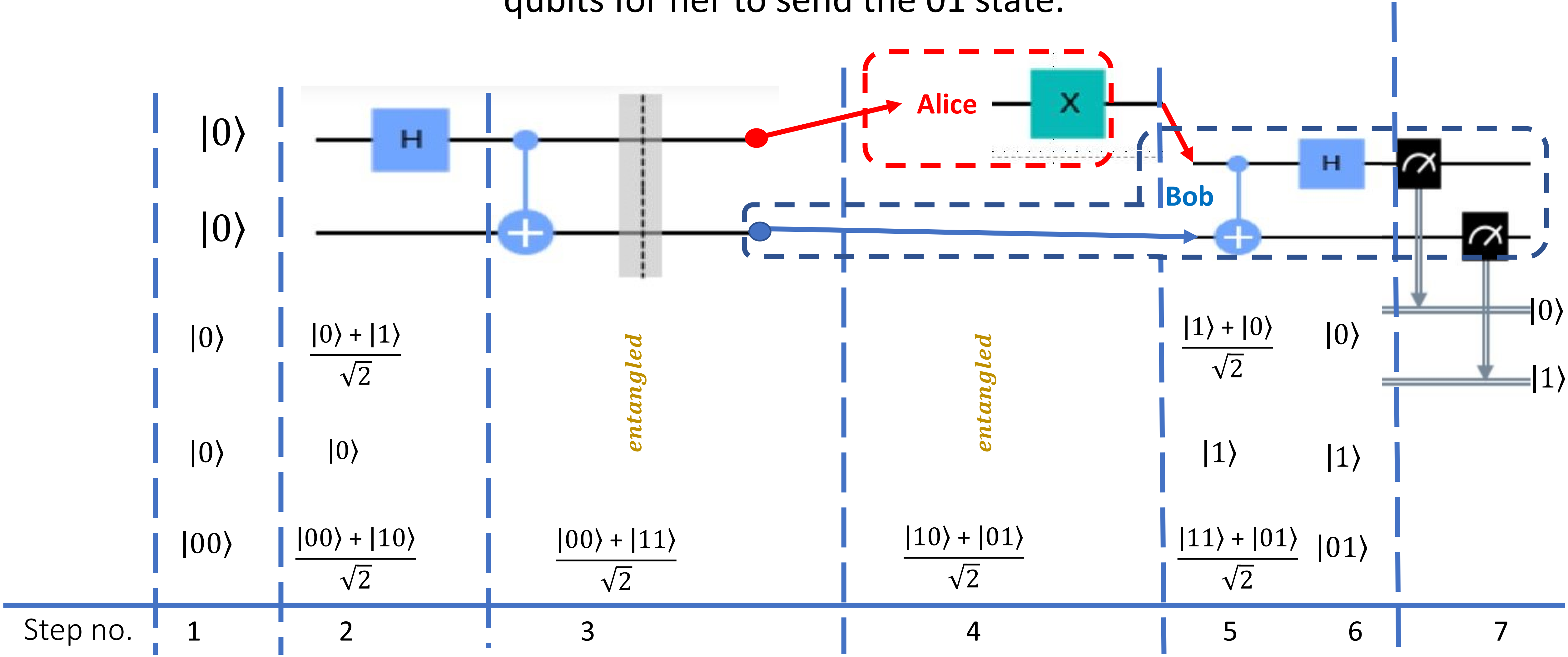
False

this is an entangled state, if I were to say that about the first qubit, that would mean that someone could claim the second qubit was $\frac{|1\rangle+|0\rangle}{\sqrt{2}}$ making the combined state

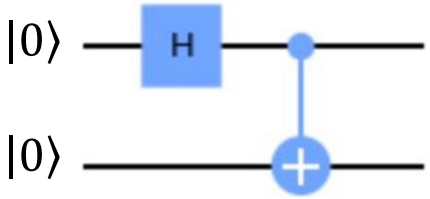
$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|1\rangle + |0\rangle}{\sqrt{2}} = \frac{1}{2}(\,|00\rangle + |01\rangle + |10\rangle + |11\rangle) \neq |\beta_{01}\rangle$$

Superdense Coding

How can Alice code for 01, draw a step by step guide of what happens to both her and Bob's qubits for her to send the 01 state.



First: we initialize an entangled state using this circuit:



After applying the Hadamard gate to the 1^{st} qubit the bipartite state gives

$$1^{st} qubit: \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$2^{nd} qubit: \ |0\rangle$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \longrightarrow \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

After applying the CNOT-Gate

$$\mathrm{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}\mathrm{CNOT}(|00\rangle) + \frac{1}{\sqrt{2}}\mathrm{CNOT}(|10\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

The ENTANGLED Bell-State (β_{00})

Second: Alice gets the first qubit and applies some gates to it based on what message she intends on sending, and Bob gets the second qubit and does nothing but wait for Alice's qubit so that he may start to decode the message:

To convey the 01 message, Alice must turn the Bell β_{00} state into the β_{01} Bell state

To do that she applies the X-Gate to her qubit, yielding the overall bipartite state:

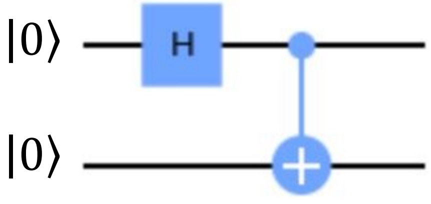
$$\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \beta_{01}$$

When she sends her qubit to Bob, he will have this bell-state with him

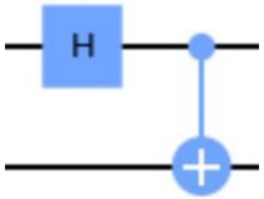
Now after getting a hold of Alice’s qubit, Bob needs to measure the bipartite state in the Bell-basis to determine which of the 4-Bell states Alice’s measurements have placed the two qubits into:

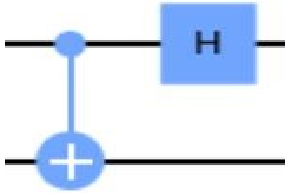
i.e. Bob needs to do a measurement in a non-standard basis, so we need to apply a unitary before measurement.

To reach the Bell-state we used this circuit



So our unitary for transferring the standard basis to the Bell-basis is

$U =$ 

$so\ U^\dagger =$ 

Now we just need to apply the $U^\dagger$ to our circuit before measurement to measure in the Bell-basis

$$U^\dagger\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right) \longrightarrow H^A\left(CNOT\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)\right) = H^A\left(\frac{|11\rangle + |01\rangle}{\sqrt{2}}\right) = H^A\left(\frac{|1\rangle + |0\rangle}{\sqrt{2}}\right) \otimes |1\rangle = |01\rangle$$

QUESTION 6:

Nielsen and Chunag : Exercise 2.65

Express the states $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ in a basis in which they are not the same up to a relative phase shift

Ans 6:

$$\left|\psi^{A_1,A_2,\ldots.A_n}\right\rangle = \sum_{a_1,a_2,\ldots.a_n=0} |a_1,a_2,\ldots a_n\rangle\langle a_1,a_2,\ldots.a_n|\ \psi^{A_1,A_2,\ldots.A_n}\rangle$$

Since we have 2 Qubits:

$$\left|\psi^{A_1}\right\rangle = \sum_{a_1} |a_1\rangle\langle a_1|\ \psi^{A_1}\rangle \ = \underbrace{|+\rangle\langle+|}_{\text{Projection operator } \mathbb{P}_+}\ \psi^{A_1}\rangle + \underbrace{|-\rangle\langle-|}_{\text{Projection operator } \mathbb{P}_-}\ \psi^{A_1}\rangle$$

$$\psi^A = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} + |-\rangle\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}^x$$

$$\psi^B = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = |+\rangle\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} + |-\rangle\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix} = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}^x$$

Ans 6:

To get the relative phase compare the 2 states in our new basis:

$$\psi^A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^x ?= \psi^b = \begin{pmatrix} b_1 \\ b_2 e^{i\phi} \end{pmatrix}^x$$

$$originally: \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}^z = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} e^{i\pi} \end{pmatrix}^z$$

Here we have

$$\psi^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^x \ ; \ \psi^b \begin{pmatrix} 0 \\ 1 \end{pmatrix}^x$$

Is there a phase $e^{i\phi}$ to turn $1 \longrightarrow 0$ …… NO

So we have our solution in the X basis

QUESTION 7:

True or False

two non-orthogonal states can be reliably distinguished

Prove your answer

Ans 7:

False

Proof: by contradiction : we went over Box 2.3 (you need to know the rigorous proof)

Qualitatively:

non-orthogonality always implies that either state may be broken down into a part along the other state, and a part orthogonal to it. So given two non orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ $|\psi_2\rangle$ for e.g. can be broken down into $\alpha|\psi_1\rangle + \beta|\varphi\rangle$, where $|\varphi\rangle$ is orthogonal to $|\psi_1\rangle$ and α is never zero, meaning any measurement operators designed to measures $|\psi_1\rangle$ will always give a 1 when acting on $|\psi_1\rangle$ but also have a non zero probability of getting a measurement out of $|\psi_2\rangle$,

And the opposite is true, a measurement operator set up for measuring $|\psi_2\rangle$ will certainly give a reading with $|\psi_2\rangle$ but also has a non zero probability to give a zero reading with $|\psi_1\rangle$, since part of $|\psi_1\rangle$ is along $|\psi_2\rangle$, since they are not orthogonal

QUESTION 8:

Nielsen and Chunag : Exercise 2.66

Ans 8:

Nielsen and Chunag : Exercise 2.66

$$\langle X_1 Z_2 \rangle = \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) =$$

$$\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)\left(\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right) = 0$$

QUESTION 9:

Nielsen and Chunag : Exercise 2.57

(Cascaded measurements are single measurements)

Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_l\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ with the representation $N_{lm} \equiv M_m L_l$.

Ans 9:

Let the State after first measurement be $|a\rangle$ so

$$|a\rangle = \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

Apply M_m; the new state after the second measurement will be :

$$|b\rangle = \frac{M_m|a\rangle}{\sqrt{\langle a|M_m^\dagger M_m|a\rangle}} = \frac{M_m \dfrac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}{\sqrt{\dfrac{(L_l|\psi\rangle)^\dagger}{\underbrace{\left(\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}\right)^*}_{\text{Real}}} M_m^\dagger M_m \dfrac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}} = \frac{M_m L_l|\psi\rangle}{\sqrt{(L_l|\psi\rangle)^\dagger\, M_m^\dagger M_m L_l|\psi\rangle}} = \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}}$$

so

$$M_m L_l|\psi\rangle = \frac{N_{lm}|\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}}$$

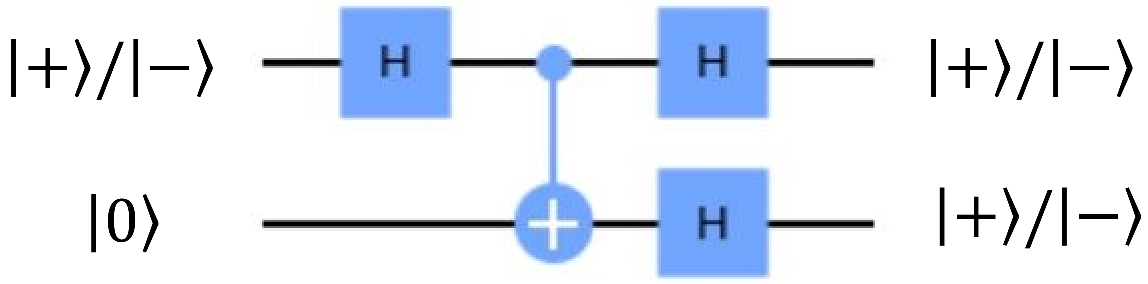
where $\{N_{lm}\}$ are the set of measurement operators defined by $N_{lm} = M_m L_l$

QUESTION 10:

Can two orthogonal states, say $|\phi\rangle$ and $|\psi\rangle$, be cloned, with data input either $|\phi\rangle$ or $|\psi\rangle$ and the target prepared in $|0\rangle$. Explain using detailed matrix equation

Ans 10:

Yes they can, as long as we know the possible input states, and they are orthogonal (i.e. distinguishable), we can find a unitary U that will take either of the states (we won't know who) and clone them given a $|0\rangle$ target



$$(H^1 \otimes H^2)CNOT\big((H|+\rangle) \otimes |0\rangle\big) = (H^1 \otimes H^2)CNOT|00\rangle = (H^1 \otimes H^2)|00\rangle = |+\,+\rangle$$

$$(H^1 \otimes H^2)CNOT\big((H|-\rangle) \otimes |0\rangle\big) = (H^1 \otimes H^2)CNOT|10\rangle = (H^1 \otimes H^2)|11\rangle = |-\,-\rangle$$

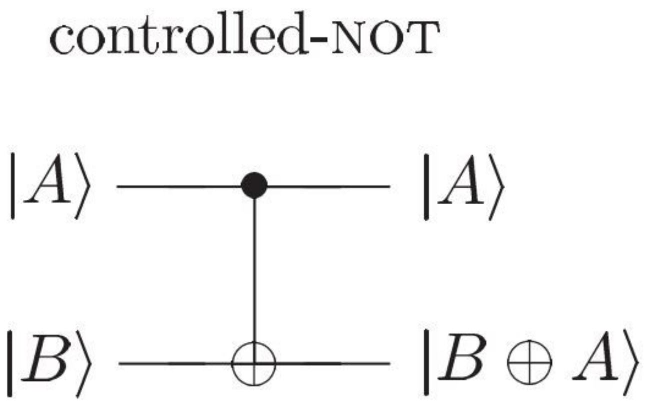
QUESTION 11:

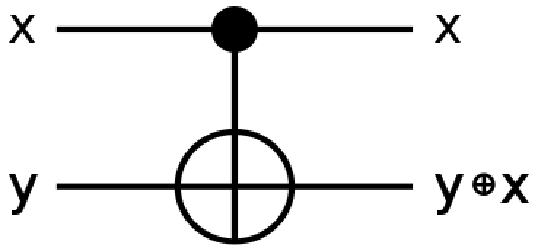
True or false:

1. In the teleportation protocol we covered, after Alice teleports her qubit she can still access her version of the teleported qubit. Explain

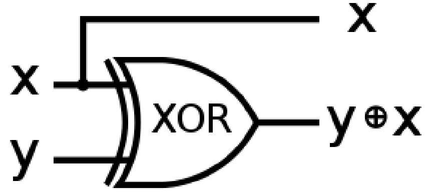
2. Problem 3 Chapter 9 : Teleportation Steeb and Hardy

Ans 11:





input		output	
x	y	x	y+x
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$



input		output	
x	y	x	y+x
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

$$so \quad CNOT = U_{XOR}$$

Ans 11:

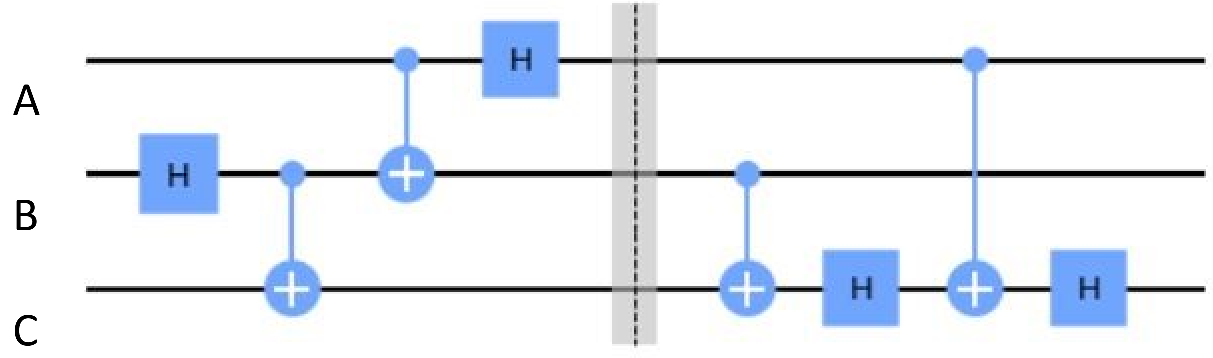
$$
I_4 \oplus U_{NOT} \oplus U_{NOT} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{array}
$$

$CNOT$ between the first and third qubit (target) in a 3-qubit system

We discussed the solution to part (i) in Problem 3 Chapter 9 : Teleportation Steeb and Hardy; you must be able to calculate these unitary operations.

Ans 11:

Teleportation circuit:



Define your Operators:

$$U_1 = I_2 \otimes U_H \otimes I_2, \qquad U_2 = I_2 \otimes U_{XOR}$$
$$U_3 = U_{XOR} \otimes I_2, \qquad U_4 = U_H \otimes I_2 \otimes I_2$$
$$U_5 = I_2 \otimes U_{XOR}, \qquad U_6 = I_2 \otimes I_2 \otimes U_H$$
$$U_7 = I_4 \oplus U_{NOT} \oplus U_{NOT} \quad , \qquad U_8 = I_2 \otimes I_2 \otimes U_H$$

Define initial state: $\mathrm{A} \otimes B \otimes C = |\psi\rangle \otimes |0\rangle \otimes |0\rangle \equiv (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes |0\rangle \equiv |\psi 00\rangle$

Run the circuit:

$$U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 |\psi 00\rangle$$

$$= \frac{a}{2}(|000\rangle + |100\rangle + |010\rangle + |110\rangle) + \frac{b}{2}(|011\rangle + |111\rangle + |001\rangle + |101\rangle)$$

$$= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |\psi\rangle$$