Tutorial 1 : problem Solving 1

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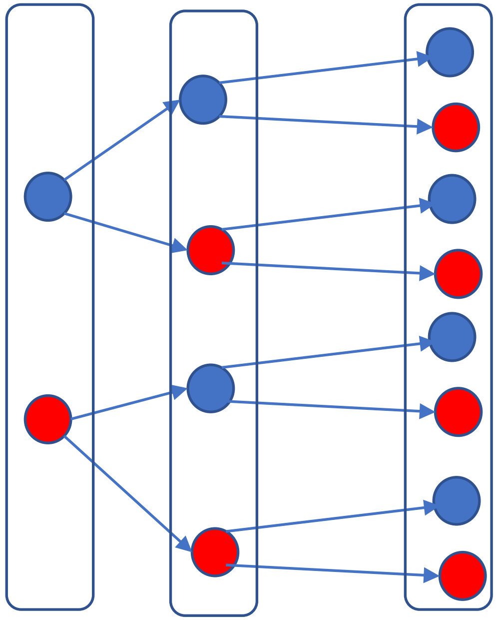
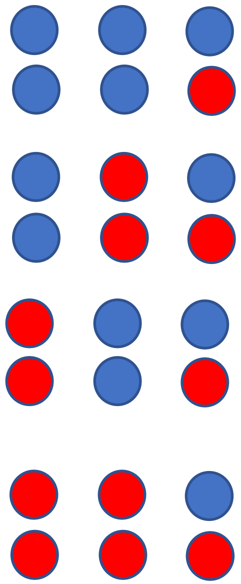
Number of basis states

Single Qubit $\longrightarrow$ 2 $\qquad |0\rangle$ *or* $|1\rangle$

A 2 qubit system $\longrightarrow |0\rangle \otimes |0\rangle$





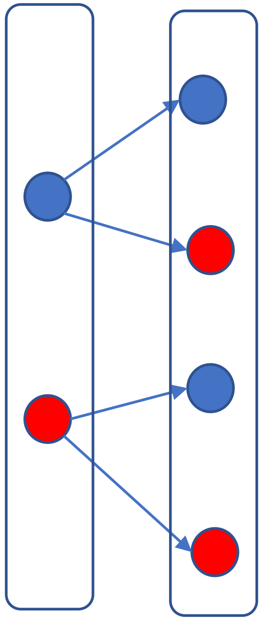
$|0\rangle \otimes |0\rangle \otimes |0\rangle \longrightarrow |000\rangle$

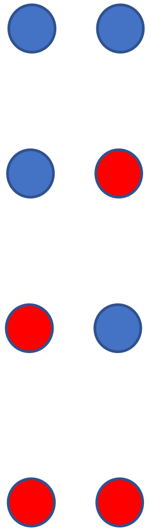
$|0\rangle \otimes |1\rangle \otimes |1\rangle \longrightarrow |011\rangle$

A 2 qubit system in terms of linear Algebra:

Examples:

$$|0\rangle\otimes|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1\\0\end{bmatrix}\\0\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad ; \quad |0\rangle\otimes|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}0\\1\end{bmatrix}\\0\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$





$$\left.\begin{aligned}
&\longrightarrow |0\rangle\otimes|0\rangle \longrightarrow |00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\
&\longrightarrow |0\rangle\otimes|1\rangle \longrightarrow |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\\
&\longrightarrow |1\rangle\otimes|0\rangle \longrightarrow |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\\
&\longrightarrow |1\rangle\otimes|1\rangle \longrightarrow |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}
\end{aligned}\right\}$$

Orthonormal basis spanning the entire 2 qubit state space

QUESTION 1: if $\left|\psi^{AB}\right\rangle = \begin{bmatrix} 1/\sqrt{4} \\ 0 \\ \sqrt{3}/\sqrt{4} \\ 0 \end{bmatrix}$ expand it in terms of the **standard basis,** first by using dirac notation then using explicit vector algebra:

The most general 2 qubit state is $\left|\psi^{AB}\right\rangle = \sum_{i,j} C_{ij}|i,j\rangle \; = C_{00}|0,0\rangle + C_{01}|0,1\rangle + C_{10}|1,0\rangle + C_{11}|1,1\rangle$

Recall in real space when we defined the dot product:

$$A \cdot B = A^T B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots . a_n b_n$$

This is related to the concept of "overlap/projection" between vectors and by extension "length"

Where length of $\bar{A} = \sqrt{\bar{A} \cdot \bar{A}}$

And the projection of $\bar{A}$ along basis $\hat{e}_1$ for example $\hat{e}_x \;\; is \;\; \bar{\mathrm{A}} \cdot \hat{e}_1$ Unit vector along the x but we are dealing here with a complex space, and the more general operation is the "inner product"

$$\begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \langle A|B\rangle$$

Dagger = conjugate transpose

$$\langle A| = \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^\dagger$$

$$|\psi^{AB}\rangle = \sum_{i=0,j=0}^{1} C_{ij}|i,j\rangle = C_{00}|0,0\rangle + C_{01}|0,1\rangle + C_{10}|1,0\rangle + C_{11}|1,1\rangle$$

$C_{ij} = \langle i,j|\psi^{AB}\rangle$ so for example $C_{01} = \langle 01|\psi^{AB}\rangle$

$$C_{01} = \langle 01|\psi^{AB}\rangle = \langle 01|\left(\sqrt{\frac{1}{4}}|0,0\rangle + \cancel{0\,|0,1\rangle} + \sqrt{\frac{3}{4}}|1,0\rangle + \cancel{0\,|1,1\rangle}\right)$$

$C_{01} = \sqrt{\frac{1}{4}}\langle 01|00\rangle + \sqrt{\frac{3}{4}}\langle 01|10\rangle = 0$ Remember orthogonality : $\langle \hat{e}_i|\hat{e}_j\rangle = \delta_{i,j}$

$$C_{00} = \langle 00|\psi^{AB}\rangle = \langle 00|\sqrt{\frac{1}{4}}|00\rangle\rangle = \sqrt{\frac{1}{4}}$$

$$C_{10} = \langle 10|\psi^{AB}\rangle = \langle 10|\sqrt{\frac{3}{4}}|10\rangle\rangle = \sqrt{\frac{3}{4}}$$

$$C_{11} = \langle 11|\psi^{AB}\rangle = \langle 11|\,0|11\rangle\rangle = 0$$

To do C_{10} explicitly:

$$C_{10} = \langle 10|\psi^{AB}\rangle = \langle 10|\left(\sqrt{\frac{1}{4}}|0,0\rangle + 0\,|0,1\rangle + \sqrt{\frac{3}{4}}|1,0\rangle + 0\,|1,1\rangle\right)$$

$$= \sqrt{\frac{1}{4}}\,\langle 10|00\rangle + \sqrt{\frac{3}{4}}\,\langle 10|10\rangle$$

$$= \sqrt{\frac{1}{4}}\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sqrt{\frac{3}{4}}\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{\frac{3}{4}}$$

Projection Operators:

$$we\ showed\ that\ Generally\ \left|\psi^{A_1,A_2,\ldots.A_n}\right\rangle = \sum_{a_1,a_2,\ldots.a_n} \left|a_1, a_2, \ldots a_n\right\rangle\ C_{a_1 a_2 \ldots a_n}$$

and since $C_{a_1,a_2,\ldots.a_n} = \langle a_1, a_2, \ldots.a_n|\psi^{A_1,A_2,\ldots.A_n}\rangle$

so

$$\left|\psi^{A_1,A_2,\ldots.A_n}\right\rangle = \sum_{a_1,a_2,\ldots.a_n} |a_1, a_2, \ldots a_n\rangle\langle a_1, a_2, \ldots.a_n|\ \psi^{A_1,A_2,\ldots.A_n}\rangle$$

$\mathbb{P}_{a_1,a_2,\ldots.a_n} = |a_1, a_2, \ldots a_n\rangle\langle a_1, a_2, \ldots.a_n| =$ projection operator on state $|a_1, a_2, \ldots a_n\rangle$

QUESTION 2:

Prove that the vectors $|+\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ **and** $|-\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ $\boldsymbol{span\ the\ entire\ single\ qubit\ state\ space,}$

then expand $\left|\psi^{A_1}\right\rangle = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ **in terms of this non–standard basis:**

Check orthogonality : do $\langle \hat{e}_i | \hat{e}_j \rangle = \delta_{i,j}$? $\qquad \langle + | + \rangle = \langle - | - \rangle = 1\,;\ \ \langle + | - \rangle = \langle - | + \rangle = 0$

Since this is a 2-D space, we have 2 projection operators and so

$$\left|\psi^{A_1,A_2,\ldots.A_n}\right\rangle = \sum_{a_1,a_2,\ldots.a_n=0} |a_1, a_2, \ldots a_n\rangle\langle a_1, a_2, \ldots.\, a_n|\ \psi^{A_1,A_2,\ldots.A_n}\rangle \longrightarrow \left|\psi^{A_1}\right\rangle = \sum_{a_1} |a_1\rangle\langle a_1|\ \psi^{A_1}\rangle$$

$$= \underbrace{|+\rangle\langle +|}_{\text{Projection operator } \mathbb{P}_+}\ \psi^{A_1}\rangle + \underbrace{|-\rangle\langle -|}_{\text{Projection operator } \mathbb{P}_-}\ \psi^{A_1}\rangle$$

$$= |+\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} + |-\rangle \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1+\sqrt{3}}{2\sqrt{2}}|+\rangle + \frac{1-\sqrt{3}}{2\sqrt{2}}|-\rangle$$

QUESTION 3:

Exercise 2.2: (Matrix representations: example) Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Since this is a 2-D space, we have 2 projection operators and so

Input basis

output basis

$$\begin{array}{c|c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix} \end{array} = \begin{bmatrix} \langle 0|1\rangle & \langle 0|0\rangle \\ \langle 1|1\rangle & \langle 1|0\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

output basis

$$\begin{array}{c|c} \begin{array}{c} |1\rangle \\ |0\rangle \end{array} & \begin{bmatrix} \langle 1|A|0\rangle & \langle 1|A|1\rangle \\ \langle 0|A|0\rangle & \langle 0|A|1\rangle \end{bmatrix} \end{array} = \begin{bmatrix} \langle 1|1\rangle & \langle 1|0\rangle \\ \langle 0|1\rangle & \langle 0|0\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

QUESTION 4: Parametrize the most general single qubit state such that it maps to the Bloch sphere,

then *locate the state* $\left|\psi^{A_1}\right\rangle = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$ *on the sphere* and draw it

curious question: How many parameters does a typical c^2 vector have, how many does a single qubit have, why are they not the same

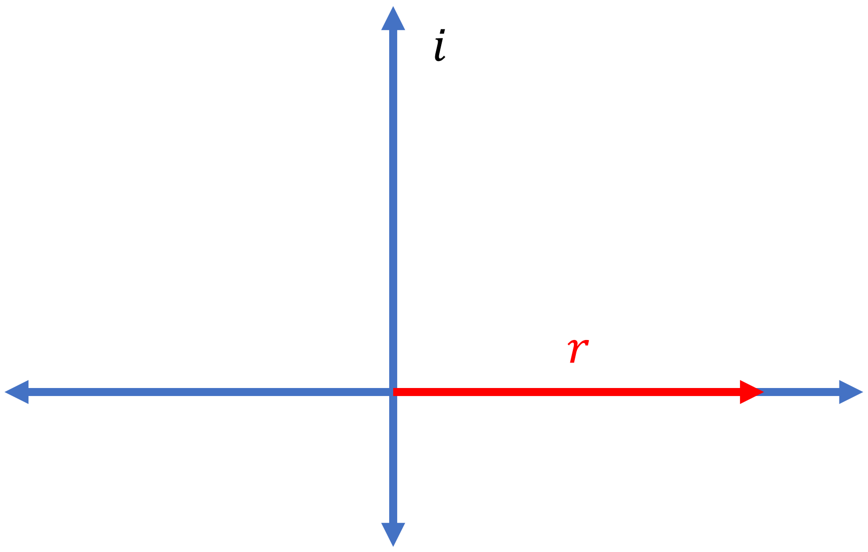
Remember:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \frac{(i\theta)^8}{8!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \cdots$$

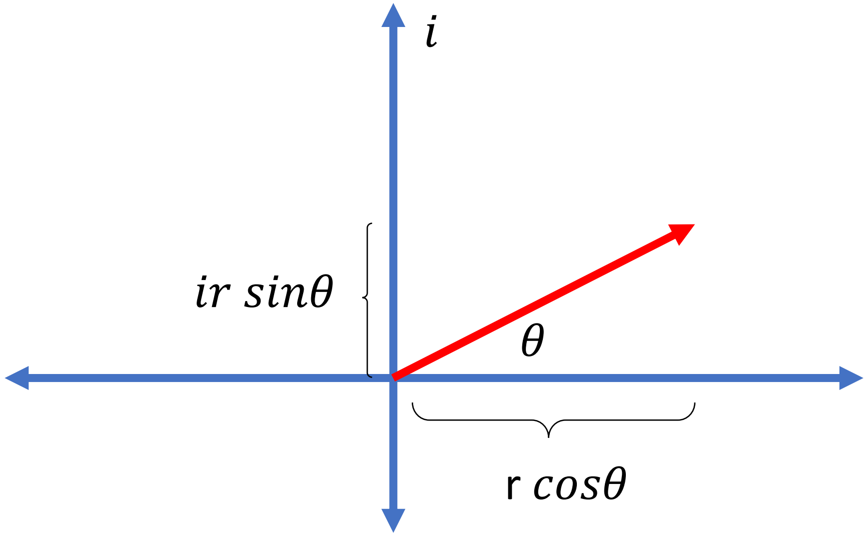
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$



real axis

Multiplying by $e^{i\theta}$ **rotates**
a vector in complex space



So the most general Complex number $Z = re^{i\theta}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \binom{\alpha}{\beta} \in C^2$$ since α and β are complex

C^2 is the Complex equivalent of R^2 a coordinate in C^2 has the form $(\underbrace{a + bi}_{\alpha}, \underbrace{c + di}_{\beta})$

We have 4 parameters …. a, b, c & d

The Qubit has only 2, although it $\in C^2$ HOW ?

$$|\psi\rangle = r_\alpha e^{i\phi_1}|0\rangle + r_\beta e^{i\phi_2}|1\rangle = \underbrace{e^{i\phi_1}}_{Global\ phase}(r_\alpha|0\rangle + r_\beta e^{i(\phi_2 - \phi_1)}|1\rangle)$$

We can ignore it due to Global phase invariance

If we define $\phi = \phi_2 - \phi_1$ we now have 3 parameters r_α, r_β, ϕ instead of 4

Define $|\psi\rangle' = \left(r_\alpha|0\rangle + r_\beta e^{i(\phi_2-\phi_1)}|1\rangle\right)$

So $|\psi\rangle' = |\psi\rangle$

up to a global phase (i.e. the phase $e^{i\phi_1}$)

$given\ \langle\psi|\psi\rangle = 1$ **normalization condition which preserves probability**

$$(\, r_\alpha^*\langle 0| + r_\beta^* e^{-i(\phi)}\langle 1|\,)\ \ (\, r_\alpha|0\rangle + r_\beta e^{i(\phi)}|1\rangle\,)$$

$$(\, r_\alpha^* r_\alpha\langle 0||0\rangle + \ + r_\beta^* r_\beta e^{-i(\phi)} e^{i(\phi)}\langle 1||1\rangle) + \ (\, r_\alpha r_\beta^* e^{-i(\phi)}\langle 1||0\rangle + r_\alpha^* r_\beta e^{i(\phi)}\langle 0||1\rangle\,)$$

$$|r_\alpha|^2 + \left|r_\beta e^{i(\phi)}\right|^2 = 1$$

so $|\mathrm{r}_\alpha|^2 + \left|\mathrm{r}_\beta\right|^2 = 1$ since phases squared cancel under complex squaring

We now have TWO parameters $r_{\beta\ or\ \alpha}$ & ϕ

From $|r_\alpha|^2 + \left|r_\beta e^{i(\phi)}\right|^2 = 1$ since $r_\beta e^{i(\phi)} = x + iy$

So $|\mathrm{r}_\alpha|^2 + |x + iy|^2 = 1$; $|r_\alpha|^2 + (x + iy)(x - iy) = 1$

$|\mathrm{r}_\alpha|^2 + |x|^2 + |y|^2 = 1$ maps to the surface of a sphere

$\mathrm{r}_\alpha = 1 * cos\theta$; $x = 1 * sin\theta cos\phi$; $y = 1 * sin\theta sin\phi$

Now from $|\psi\rangle' = \left(r_\alpha|0\rangle + r_\beta e^{i(\phi_2 - \phi_1)}|1\rangle\right) = r_\alpha|0\rangle + (x + iy)|1\rangle$

$$|\psi\rangle' = cos\theta'\,|0\rangle + (sin\theta' cos\phi + i sin\theta' sin\phi|1\rangle)$$

$$|\psi\rangle' = cos\theta'\,|0\rangle + sin\theta'(cos\phi + i sin\phi|1\rangle)$$

$$|\psi\rangle' = cos\theta'\,|0\rangle + sin\theta'(e^{i\phi})|1\rangle$$

What about the angles

What about the angles

A point on the opposite side of the sphere has angles

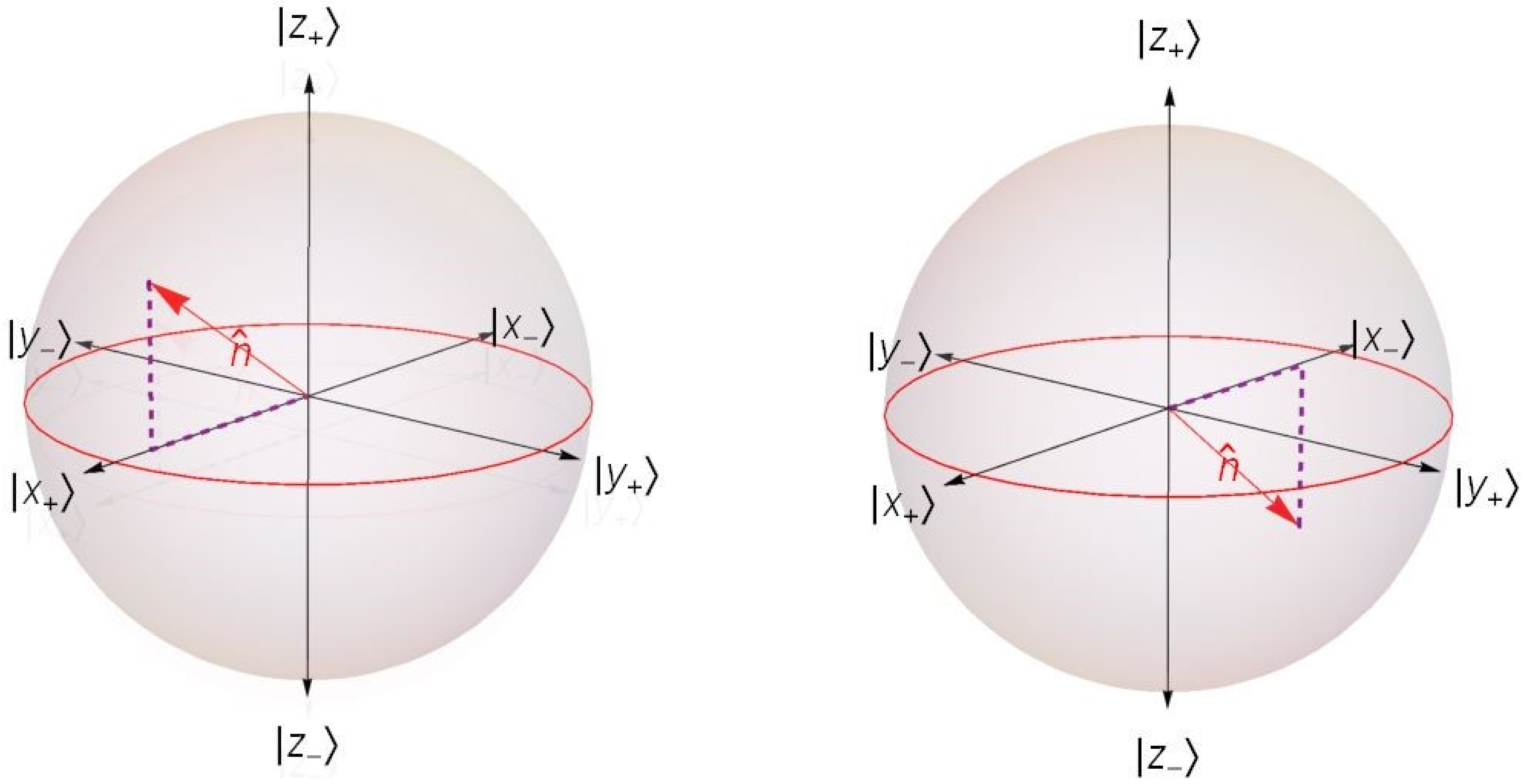
$\pi - \theta$ and $\phi + \pi$

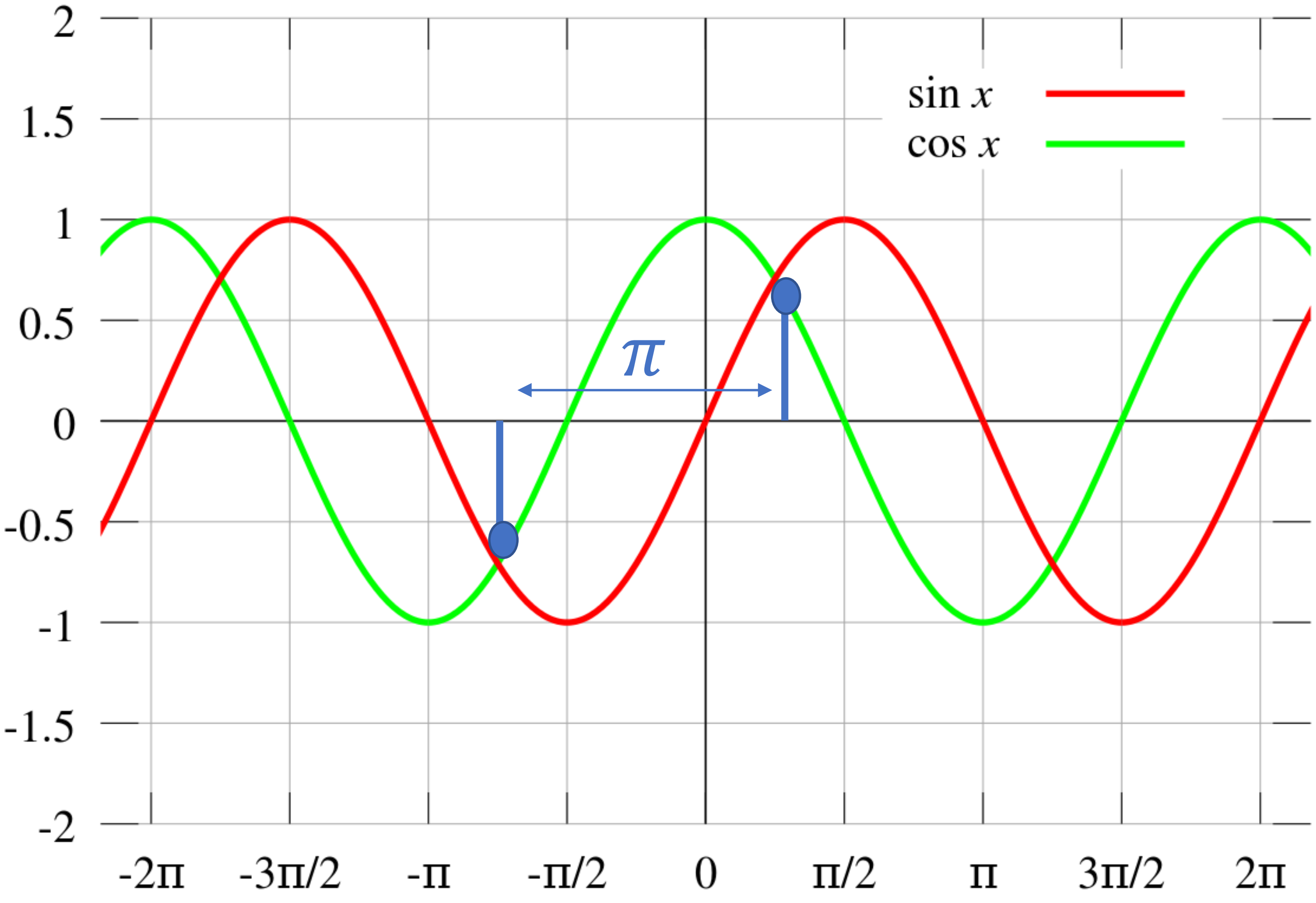
$$|\psi'\rangle = \cos(\pi - \theta')|0\rangle + e^{i(\phi+\pi)}\sin(\pi - \theta')|1\rangle$$
$$= -\cos(\theta')|0\rangle + e^{i\phi}e^{i\pi}\sin(\theta')|1\rangle$$
$$= -\cos(\theta')|0\rangle - e^{i\phi}\sin(\theta')|1\rangle$$
$$|\psi'\rangle = -|\psi\rangle$$

-1 *is a global phase ….* it's $e^{i\pi}$

so

we only need $\theta' = \frac{\theta}{2}$ *to span all the physically equivalent states*


$|z_+\rangle$
$|y_-\rangle$
$|x_-\rangle$
$\hat{n}$
$|x_+\rangle$
$|y_+\rangle$
$|z_-\rangle$
$|z_+\rangle$
$|y_-\rangle$
$|x_-\rangle$
$\hat{n}$
$|x_+\rangle$
$|y_+\rangle$
$|z_-\rangle$



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
sin x
cos x
π
-2π
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π

$$|\psi\rangle' = cos\theta' \, |0\rangle + sin\theta'(e^{i\phi})|1\rangle$$

$$|\psi\rangle' = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}(e^{i\phi})|1\rangle$$

$$|\psi\rangle' = \begin{pmatrix} \cos\frac{\theta}{2} \\ (e^{i\phi})\sin\frac{\theta}{2} \end{pmatrix}$$ **where theta is the angle in real 3D space**

$so\ the\ state\ |\psi^{A_1}\rangle = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$ *on the sphere* is parametrized by 2 angles θ and ϕ

$$|\psi^{A_1}\rangle = -\begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} = e^{i\pi}\begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix} \quad ; \quad |\psi^{A_1}\rangle' = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$\cos\frac{\theta}{2} = \frac{1}{2} \quad ; \quad \theta = \frac{2\pi}{3}$$

$$(e^{i\phi})\sin\frac{2\pi}{3(2)} = -\frac{\sqrt{3}}{2} \quad ; \ \phi = \pi$$

$$|\psi^{A_1}\rangle = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$|\psi^{A_1}\rangle' = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$|\psi\rangle' = \begin{pmatrix} \cos\frac{\theta}{2} \\ (e^{i\phi})\sin\frac{\theta}{2} \end{pmatrix}$$

$$\theta = \frac{2\pi}{3}$$

$$\phi = \pi$$