Tutorial 5 : problem Solving 3

Sample questions for the 2nd quiz, superdense coding, density matrices and cloning

Week 7 : 11/11/2020

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Q/1:

Given :

For Alice’s Observables:

$$Q = Z_1 \qquad R = X_1$$

For Alice’s Observables:

$$S = \frac{-Z_2 - X_2}{\sqrt{2}} \qquad \& \qquad T = \frac{Z_2 - X_2}{\sqrt{2}}$$

Show that for the Singlet state

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$

Q/1 Ans

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$RT = R \otimes T = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

$$E(RT) = \langle RT \rangle = \langle \psi | R \otimes T | \psi \rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

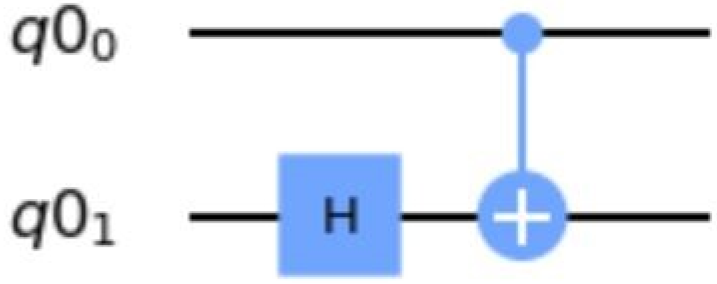
$$E(RT) = E(QS) = E(RS) = -E(QT) = \frac{1}{\sqrt{2}}$$

So

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$

Q/2:

Given this circuit, what is the resulting state and is it entangled:



Q/2 Ans

$$\left.\begin{array}{r} 1^{st} qubit\text{:} \quad |0\rangle \\[2ex] 2^{nd} qubit\text{:} \quad \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \end{array}\right\} \quad |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \rightarrow \quad \frac{|00\rangle + |01\rangle}{\sqrt{2}}$$

$$\text{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle\right) = \frac{1}{\sqrt{2}}\text{CNOT}(|00\rangle) + \frac{1}{\sqrt{2}}\text{CNOT}(|01\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$= |0\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

PRODUCT STATE

Q/3:

Measurement in NON-STANDARD basis:

Describe how we can perform a measurement in the X-basis (remember unmodified circuits can only measure along a Z-basis, so we will need to add some gates before measurement to change that).

Hint: (you need to use gates prior to your measurement along Z)

Q/3 Ans

An x-basis measurement = a Z-basis measurement following an X-basis $\longrightarrow$ Z-basis transformation

So how do we get the unitary for an X-basis $\longrightarrow$ Z-basis transformation?

We simply find the Unitary that takes the standard basis (Z-basis) to the X-basis, so performs Z-basis $\longrightarrow$ X-basis, and call it U.

So $U^\dagger$ will take us in the opposite direction from X-basis $\longrightarrow$ Z-basis

To find U we must find the U which satisfies:

$$U(|0\rangle \; and \; |1\rangle) \longrightarrow |+\rangle \; and \; |-\rangle$$

Q/3 Ans

$$U(|0\rangle \; and \; |1\rangle) \longrightarrow |+\rangle \; and \; |-\rangle$$

We already know a Unitary which does this:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle; \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

So the H-gate takes the standard basis $|0\rangle \; and \; |1\rangle$ and maps them to $|+\rangle \; and \; |-\rangle$, and it is our unitary U, and in this case it is only 1-gate.

We also Know $H^{\dagger} = H$, so to perform a measurement in the X-basis we apply the H-Gate to the single qubit before measuring it.

Q/3 Ans

To make this even clearer:

Consider we begin with the state $|+\rangle$, and we want to measure in the $|+/-\rangle$ basis:

First we apply the H-Gate before measurement:

$$H\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle$$

This brings us to the $|0\rangle$ state, so if we measure along the Z-basis now we get :
State $|0\rangle$ with 100 percent probability, this means we were originally in the $|+\rangle$ state.
If we had gotten $|1\rangle$, that would mean we were originally in the $|-\rangle$ state.

So we have basically indirectly inferred the status of our state in the the $|+\rangle$ and $|-\rangle$ measurement basis, from making measurements in the $|0\rangle$ and $|1\rangle$.

Q/4 Superdense Coding

How can Alice use superdense coding to send the 01 message to Bob. Draw the circuit, and a step by step guide of what happens to both her and Bob's qubits during the entire process up till measurement.

NB: Please make sure you consider what Bob has to do, on receiving both qubits, to measure the bipartite system in the Bell-basis.

Q/4 Ans

Alice and Bob just need to communicate one of the 4 Bell states, each with it's own 2-bit message, making that 4 possible messages, 00, 01, 10, 11.

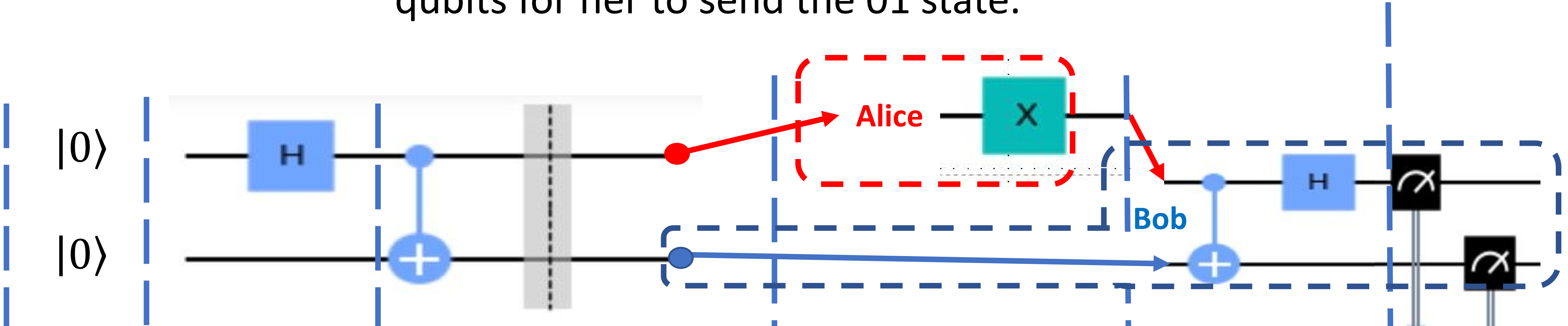
Alice needs to initiate this communication process through ONLY acting on one of 2 Qubits.
NB: The Bell states are sometimes named by the message they convey in superdense coding

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Q/4 Ans

Superdense Coding

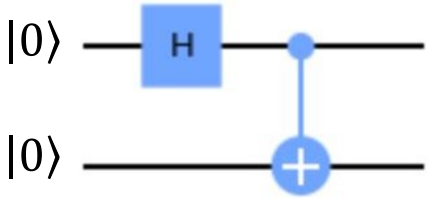
How can Alice code for 01, draw a step by step guide of what happens to both her and Bob’s qubits for her to send the 01 state.



Step no.	1	2	3	4	5	6	7
	$\vert 0\rangle$	$\frac{\vert 0\rangle + \vert 1\rangle}{\sqrt{2}}$	*entangled*	*entangled*	$\frac{\vert 1\rangle + \vert 0\rangle}{\sqrt{2}}$	$\vert 0\rangle$	$\vert 0\rangle$
	$\vert 0\rangle$	$\vert 0\rangle$			$\vert 1\rangle$	$\vert 1\rangle$	$\vert 1\rangle$
	$\vert 00\rangle$	$\frac{\vert 00\rangle + \vert 10\rangle}{\sqrt{2}}$	$\frac{\vert 00\rangle + \vert 11\rangle}{\sqrt{2}}$	$\frac{\vert 10\rangle + \vert 01\rangle}{\sqrt{2}}$	$\frac{\vert 11\rangle + \vert 01\rangle}{\sqrt{2}}$	$\vert 01\rangle$	

Q/4 Ans

First: we initialize an entangled state using this circuit:



After applying the Hadamard gate to the 1^{st} qubit the bipartite state gives $\longrightarrow$

$$\left.\begin{array}{l} 1^{st} qubit\text{:}\ \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \\[2ex] 2^{nd} qubit\text{:}\ \ |0\rangle \end{array}\right\} \ \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \longrightarrow \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

After applying the CNOT-Gate $\longrightarrow$

$$\mathrm{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}\mathrm{CNOT}(|00\rangle) + \frac{1}{\sqrt{2}}\mathrm{CNOT}(|10\rangle)$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

The ENTANGLED Bell-State (β_{00})

Q/4 Ans

Second: Alice gets the first qubit and applies some gates to it based on what message she intends on sending, and Bob gets the second qubit and does nothing but wait for Alice's qubit so that he may start to decode the message:

To convey the 01 message, Alice must turn the Bell β_{00} state into the β_{01} Bell state

To do that she applies the X-Gate to her qubit, yielding the overall bipartite state:

$$\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \beta_{01}$$

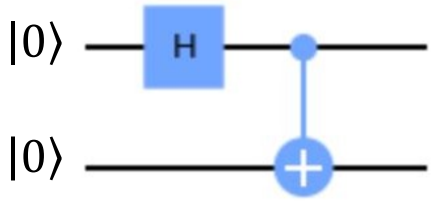
When she sends her qubit to Bob, he will have this bell-state with him

Q/4 Ans

Now after getting a hold of Alice’s qubit, Bob needs to measure the bipartite state in the Bell-basis to determine which of the 4-Bell states Alice’s measurements have placed the two qubits into:

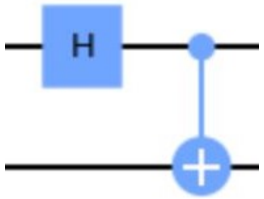
i.e. Bob needs to do a measurement in a non-standard basis, like we did in Q/2, so we need to apply a unitary before measurement.

To reach the Bell-state we used this circuit

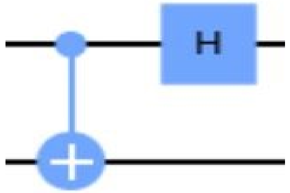


So our unitary for transferring the standard basis to the Bell-basis is

$U =$



$so\ U^\dagger =$



Now we just need to apply the $U^\dagger$ to our circuit before measurement to measure in the Bell-basis

$$U^\dagger\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right) \longrightarrow H^A\left(CNOT\left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)\right) = H^A\left(\frac{|11\rangle + |01\rangle}{\sqrt{2}}\right) = H^A\left(\frac{|1\rangle + |0\rangle}{\sqrt{2}}\right) \otimes |1\rangle = |01\rangle$$

Q/5: No cloning Theorem

Steeb/Hardy problem 2 – Cloning:

let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1 x_1^* + x_2 x_2^* = 1$$

be an arbitrary normalized vector in C^2 . Can we construct a 4 x 4 unitary matrix U such that

$$U\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}?$$

What does that Imply?

Q/5 Ans

No, prove by showing that the $RHS \neq LHS$:

RHS:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right) \otimes \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

LHS:

$$U\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = U\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$