Tutorial 3 : problem Solving 2

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Good sources for questions:

1. Steeb, Hardy, 2004 Problems and Solutions in Quantum Computing.

2. Nielsen and Chuang, Quantum Computation and Quantum information 2011.

Q1

Problem 1. Can the *EPR-state* (*Einstein-Podolsky-Rosen state*)

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in the Hilbert space $\mathbf{C}^4$ be written as a product state?

Q1 Ans:

consider the general states for 2 individual qubits

$$|\psi^A\rangle = a|0\rangle + b|1\rangle \qquad |\psi^B\rangle = c|0\rangle + d|1\rangle$$

$$|\psi^{AB}\rangle = |\psi^A\rangle \otimes |\psi^B\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

$$= ac|0,0\rangle + ad|0,1\rangle + bc|1,0\rangle + bd|1,1\rangle$$

Solution : $ac = 0, \quad ad = \frac{1}{\sqrt{2}}, \quad bc = -\frac{1}{\sqrt{2}}, \quad bd = 0$

Another example:

$$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$ac = \frac{1}{2}, \qquad ad = \frac{1}{2}, \qquad bc = \frac{1}{2}, \qquad bd = \frac{1}{2}$$

Q2

Given the following 2 qubit density matrix of a NON-ENTANGLED bipartite state AB, find the reduced density matrix of subsystem A (the first qubit)

$$c = \rho^{AB} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}$$

Answer:

Since A and B are not entangled, so AB is a product state where $\rho^{AB} = \rho^{A} \otimes \rho^{B}$

$$\rho^{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad \rho^{B} = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

Q2: Easy method

$$\rho^A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad \rho^B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

$$\rho^{AB} = \rho^A \otimes \rho^B = \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}$$

To get the elements of ρ^A we just need to trace out subsystem B, i.e. $\rho^A = \mathrm{tr}_{\mathrm{B}}(\rho^{AB})$ to get $a_{11}, a_{12}, a_{21}\ and\ a_{22}$

Since ρ^B is a density matrix, we know that its trace is 1, so $b_{1,1} + b_{2,2} = 1$

Q2: Easy method

$$\rho^A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad \rho^B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

$$\rho^{AB} = \rho^A \otimes \rho^B = \begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}$$

We can look for the sum of 2 matrix components in ρ^{AB} where we can take the required matrix element a_{ij} as a common factor and leave behind the trace of ρ^B, which we know is 1. For example if we sum $c_{11} + c_{22}$ we get :

$$c_{11} + c_{22} = a_{11}(b_{1,1} + b_{2,2}) = a_{11}$$

Q2: Easy method

$$\begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}$$

Repeat for $a_{1,2}$; $a_{2,1}$ and a_{22}

$$c_{11} + c_{22} = a_{11}(b_{1,1} + b_{2,2}) = a_{11}$$
$$c_{13} + c_{24} = a_{12}(b_{1,1} + b_{2,2}) = a_{12}$$
$$c_{31} + c_{42} = a_{21}(b_{1,1} + b_{2,2}) = a_{21}$$
$$c_{33} + c_{44} = a_{22}(b_{1,1} + b_{2,2}) = a_{22}$$

Q2: general method

$$
\begin{array}{c c}
 & \begin{array}{cccc} \langle 00| & \langle 01| & \langle 10| & \langle 11| \end{array} \\
\begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} & \begin{bmatrix} a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{bmatrix}
\end{array}
$$

$$Matrix = a|0^A0^B\rangle\langle 0^A0^B| + b|0^A1^B\rangle\langle 0^A0^B| + c|1^A0^B\rangle\langle 0^A1^B| + d|1^A1^B\rangle\langle 1^A1^B|$$

$$= a(|0^A\rangle\langle 0^A| \otimes |0^B\rangle\langle 0^B|) + b(|0^A\rangle\langle 0^A| \otimes |1^B\rangle\langle 0^B|) + c(|1^A\rangle\langle 0^A| \otimes |0^B\rangle\langle 1^B)| + d(|1^A\rangle\langle 1^A| \otimes |1^B\rangle\langle 1^B|)$$

So

$$Tr_B(\rho^{AB}) = a|0^A\rangle\langle 0^A|\, {\color{red}Tr_B(\,|0^B\rangle\langle 0^B|)} + b(|0^A\rangle\langle 0^A|\, {\color{red}Tr_B(\,|1^B\rangle\langle 0^B|)}) + c|1^A\rangle\langle 0^A|\, {\color{red}Tr_B(|0^B\rangle\langle 1^B|)} + d|1^A\rangle\langle 1^A|{\color{red}Tr_B(|1^B\rangle\langle 1^B|)})$$

$$= a|0^A\rangle\langle 0^A| + d|1^A\rangle\langle 1^A| = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

Q3

Problem 9. Let $|\psi\rangle$ be a given state in the Hilbert space $\mathbf{C}^n$. Let X and Y be two $n \times n$ hermitian matrices. We define the *correlation* as

$$\langle\psi|XY|\psi\rangle - \langle\psi|X|\psi\rangle\langle\psi|Y|\psi\rangle .$$

Let $n = 4$ and

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$|\psi\rangle = \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) .$$

Find the correlation.

Q3

$$X|\psi\rangle = |\psi\rangle$$

$$(X|\psi\rangle)^\dagger = |\psi\rangle^\dagger X^\dagger = \langle\psi|X$$

So if $X|\psi\rangle = |\psi\rangle$ so $\langle\psi|X = \langle\psi|$

$$\langle\psi|XY|\psi\rangle - \langle\psi|X|\psi\rangle\langle\psi|Y|\psi\rangle = \langle\psi|Y|\psi\rangle - \langle\psi|Y|\psi\rangle = 0$$

Q4

IF the effect of matrix A on the orthonormal basis states of a bipartite binary system is

$$A|00\rangle = |00\rangle$$
$$A|01\rangle = |11\rangle$$
$$A|10\rangle = |10\rangle$$
$$A|11\rangle = |01\rangle$$

Find the matrix A:

(Show all the steps in detail)

NB: we did this together in the lab coding session

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Q5

Given the reduced density matrix of a bipartite system is

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

What can you infer regarding the entanglement of the original state?

Answer: It has 2 eigenvalues both equal to 0.5, so for a bipartite state, this means that AB was originally a maximally entangled state.

Q6:

Are density matrices Unique ?
Prove with an example.

Q6 Answer

Are density matrices Unique

They change with a change in basis:

Example:

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Is this a pure state ?

Yes, despite it having off-diagonal terms. We can find the basis

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \; ; \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$