1] elementary forms

Let $\{i_0 \ldots\ldots i_k\} = I$. Define

$$dt_I = dt_{i_0} \wedge \ldots\ldots \wedge dt_{i_k}$$

$$\beta_I = \sum_{a=0}^{k} (-1)^a t_{i_a} dt_{i_0} \wedge \ldots \wedge \widehat{dt_{i_a}} \wedge \ldots \wedge dt_{i_k}$$

β_I is called an elementary form.

It satisfies $d\beta_I = dt_I \times (k+1)$

2] Face maps

Given p, $I = \{i_0 < i_1 < \ldots < i_k\}$ where $i_k \leq p$ there is an associated face map $f_{I,p} : [k] \to [p]$ obtained by mapping $0, 1, \ldots, k$ to $i_0, \ldots, i_k$

(note $f_{I,p}$ is called μ_I in Dupont's paper.
This notation SUCKS because it depends on p)

3] The two complexes

1^{st} one $\quad A^*(\Delta \times X)$ simplicial forms

2^{nd} one $\quad A^*(X)$ naïve simplicial DR cx

We saw that fiber int. is a morphism

$$\int_\Delta : A^*(\Delta \times X) \to A^*(X)$$

Now we define a morphism

$$\mathcal{E} : A^*(X) \to A^*(\Delta \times X)$$

as follows:

If $\omega \in A^{\ell}(X_k)$ then

$$\begin{cases} \mathcal{E}(\omega)^{(p)} = 0 \quad \text{if } p < k \\ \mathcal{E}(\omega) = k! \displaystyle\sum_{\substack{|I| = k+1 \\ i_k \leq p}} \beta_I \wedge X(f_{I,p})^* \omega \end{cases}$$

Recall that $f_{I,p} : [k] \to [p]$

$$X(f_{I,p}) : X_p \to X_k$$

<u>Goal 1</u> This guy is a simplicial form

Fix p, let $0 \leq i \leq p$ and let's restrict $\mathcal{E}(\omega)$ to $\{i^{\text{th}} \text{ face of } \Delta_p\} \times X_p$

If you set $t_i = 0$, then

$$\beta_{I \mid \{t_i = 0\}} = \begin{cases} \beta_I & \text{if } i \notin I \\ 0 & \text{if } i \in I \text{ (because } t_i \text{ or } dt_i \text{ appears)} \end{cases}$$

Hence you get $k! \displaystyle\sum_{\substack{|I| = k+1 \\ i_k \leq p \quad i \notin I}} \beta_I \wedge X(f_{I,p})^* \omega$

Here we use implicitly the coordinates

$t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_p$ on Δ_{p-1}.

If we re-label these coordinates as

$t_0, \ldots, t_{p-1}$, this is

$$k! \sum_{\substack{|J|=k+1 \\ j_k \leq p-1}} \beta_J \wedge X(f_{\tilde{J},p})^* \omega$$

where $\tilde{J}$ is obtained as follows: you put the same elements as J if they are $< i$, and you add 1 to all elements larger than or equal to i.

<u>Now</u> you can convince yourself than

$f_{\tilde{J},p}$ is $f_i \circ f_{J,p-1}$ where f_i is the i^{th} face map $[p-1] \to [p]$. Thus you get

$$k! \sum_{\substack{|J|=k+1 \\ j_k \leq p-1}} \beta_J \wedge \left(X(f_{J,p-1}) \circ X(f_i)\right)^* \omega$$

$$= X(f_i)^* \omega^{(p-1)}$$

end of goal 1

$\downarrow$

$\blacksquare$

Goal 2 This is a morphism of complexes.

Let's go *sigh*

$$d\,\mathcal{E}^{(p)}(\omega) = k! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} \left\{ d\beta_I \wedge X(f_{I,p})^* \omega + (-1)^k \beta_I \wedge X(f_{J,p})^* d\omega \right\}$$

- Second term is $(-1)^k \mathcal{E}^{(p)}(d\omega)$ which is great.

- First term is $k! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} (k+1)\, dt_I \wedge X(f_{I,p})^* \omega$

$$= (k+1)! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} dt_I \wedge X(f_{I,p})^* \omega$$

Now we compute

$\mathcal{E}^{(p)}(\delta\omega)$. It is

$$(k+1)! \sum_{\substack{|J|=k+1 \\ j_{k+1} \leq p}} \beta_J \wedge X(f_{J,p})^* \delta\omega$$

and look at interesting cancellations

$$\delta\omega = \sum_{i=0}^{k+1} (-1)^i X(f_i)^* \omega$$

$$\left(f_i : [k] \to [k+1] \ i^{th} \text{ face map} \right)$$

$$\left(X_p \xrightarrow{X(f_{J,p})} X_{k+1} \xrightarrow{X(f_i)} X_k \right)$$

so get

$$(k+1)! \sum_{\substack{|J|=k+1 \\ j_{k+1} \leq p}} \sum_{i=0}^{k+1} \beta_J \wedge X(\underbrace{f_{J,p} \circ f_i}_{\text{investigate this face map.}})^* \omega \times (-1)^i$$

$$\{0, \dots, k\} \to \{0, \dots, \overset{\overset{i}{\wedge}}{}, k+1\} \to \{j_0, \dots, \hat{j}_i, \dots, j_{k+1}\}$$

we now group the terms corresponding to the same face map $f_{J,p+1} \circ f_i$.

Given $L = \ell_0 < \ell_1 < \dots < \ell_k \leq p$ we can

add
$$\begin{cases} \ell' \in]0, \ell_0[& i=0 \\ \ell' \in]\ell_0, \ell_1[& i=1 \\ \ell' \in]\ell_1, \ell_2[& i=2 \\ \vdots & \\ \ell' \in]\ell_k, p[& i=k+1 \end{cases}$$

and take $J = L \cup \{\ell'\}$ (reordered)

Now we sum all the corresponding β_J,

i.e. $$\sum_{\ell' \notin L} (-1)^{\text{position of } \ell' \text{ in } L} \times \beta_{L \cup \{\ell'\}}$$

$$= \sum_{\ell' \notin L} (-1)^{\nu(L,\ell')} \times (\text{smtg charming})$$

$$= \sum_{\ell'=0}^{\ell_1 - 1} \beta_{\ell' \ell_0 \ell_1 \dots \ell_{k+1}}$$

$$- \sum_{\ell'=\ell_0+1}^{\ell_1 - 1} \beta_{\ell_0 \ell' \ell_1 \dots \ell_{k+1}}$$

$+ \dots\dots$

which is after thinking a little bit

$$\sum_{\substack{\ell'=0 \\ \ell' \notin L}}^{p} \Big(t_{\ell'}\, dt_{\ell_0} \wedge dt_{\ell_1} \wedge \dots \wedge dt_{\ell_{k+1}}$$

$$- t_{\ell_0}\, dt_{\ell'} \wedge dt_{\ell_1} \wedge \dots\dots \wedge dt_{\ell_{k+1}}$$

$$+ \dots\dots \Big)$$

All terms except the first one vanish because $\sum dt_{\ell'} = 0$. The first one is dt_L.

Hence we get

$$\sum_{\substack{|L|=k+1 \\ \ell_k \leq p}} dt_L \wedge X(f_{L,p+1})$$

Goal 2 ✓

↓

$\blacksquare$