TD 6

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March 18, 2019

Problem 1. *Let $C \subseteq \mathbb{R}^n$ be a compact subset and $f\colon C \to D$ a continuous injective function to some $D \subseteq \mathbb{R}^n$. Prove that the inverse function $f^{-1}\colon D \to C$ is also continuous. Hint: fix some $d \in D$, let $c = f^{-1}(d)$ and $\varepsilon > 0$; then show that there exists some $r > 0$ such that $B(d;r) \cap f(C \setminus B(c_0;\varepsilon)) = \varnothing$.*

Problem 2. *Let T be a right-angled triangle with sides of length 3, 4, and 5. Drop a perpendicular from the hypotenuse to the right-angle, splitting the triangle into two smaller triangles. Label the smaller triangle $T(0)$ and the larger triangle $T(1)$. Then divide each $T(i)$ in the same way, labelling the smaller part as $T(i0)$ and the larger as $T(i1)$. Repeat recursively.*

Now consider every point $x \in [0,1]$ in its binary expansion: $x = 0.i_1i_2i_1\ldots$, where $i_j \in \{0,1\}$. Define $f\colon [0,1] \to T$ by

$$f(0.i_1i_2i_3\ldots) = \bigcap_{n\geqslant 1} T(i_1i_2\ldots i_n).$$

(a) *Prove that $T(i_1\ldots i_n)$ has a diameter of at most $5(0.8)^n$.*

(b) *Prove that, if x has a finite binary expansion (i.e. that $x = 0.i_1\ldots i_n00000\ldots$, with $i_n = 1$), then the other binary representation (that ends in ones, namely $x = 0.i_1\ldots i_{n-1}011111\ldots$) maps to the same point under f.*

(c) *Using the above, show that f is well defined.*

(d) *Prove that f is continuous. Hint: if x and y agree for the first n digits, then how is $f(x)$ related to $f(y)$?*

(e) Prove that f is surjective.

Problem 3. *For $\alpha > 0$, we say that a function $f\colon [a,b] \to \mathbb{R}$ satisfies the* ***Lipschitz condition of order*** α *if there exists some constant $M > 0$ such that, for all $x, y \in [a,b]$,*

$$|f(y) - f(x)| \leqslant M|y-x|^\alpha.$$

Let $\mathsf{Lip}(\alpha)$ denote the set of all such functions.

(a) *Prove that, if $f \in \mathsf{Lip}(\alpha)$, then f is uniformly continuous.*

(b) *Prove that, if $f \in \mathsf{Lip}(\alpha)$ for $\alpha > 1$, then f is constant.*

(c) *Prove that $x^\alpha \in \mathsf{Lip}(\alpha)$ for all $\alpha \in (0,1)$.*