TD 5 solution

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May 4, 2019

Problem 1. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [b,c] \to \mathbb{R}$ be continuous functions that agree on the overlap (i.e. such that $f(b) = g(b)$. Show that $h\colon [a,c] \to \mathbb{R}$, defined by*

$$h(x) = \begin{cases} f(x) & x \in [a,b] \\ g(x) & x \in [b,c] \end{cases}$$

is continuous.

Solution 1.
We know that h is continuous at every point in $[a,b) \cup (b,c]$; it just remains to show that h is continuous at b. Let $\varepsilon > 0$. Since f is continuous, there exists some $\delta_1 > 0$ such that

$$|b-x| < \delta_1 \implies |f(b) - f(x)| < \varepsilon.$$

Similarly, by continuity of g, there exists some $\delta_2 > 0$ such that

$$|b-x| < \delta_2 \implies |g(b) - g(x)| < \varepsilon.$$

Then, for $\delta = \min\{\delta_1, \delta_2\}$, we have that

$$|h(b) - h(x)| = \begin{cases} |f(b) - f(x)| & x \leqslant b \\ |g(b) - g(x)| & x > b \end{cases}$$

whence, in both cases,

$$|b-x| < \delta \implies |h(b) - h(x)| < \varepsilon.$$

Problem 2. *Assume that the temperature $T(x)$ at a point x on a sphere of radius 1 is continuous in space, i.e. a continuous function $T\colon S^2 \to \mathbb{R}$. Show that there is a point $y \in S^2$ on the surface such that $T(y) = T(-y)$. Hint: consider $f(x) = T(x) - T(-x)$ and compare $f(x)$ with $f(-x)$.*

Solution 2. We see that $f(-x) = T(-x) - T(x) = -f(x)$. Looking at what the question is asking, we see that we wish to find some $x_0 \in S^2$ such that $f(x_0) = 0$. But if $f(x_0) \neq 0$ then either $f(x_0) > 0$ or $f(x_0) < 0$. Without loss of generality, assume that $f(x_0) > 0$. Then $f(-x_0) = -f(x_0) < 0$, whence, by your favourite version of the Intermediate Value Theorem, there exists some x_0' 'in between' x_0 and $-x_0$ (for example, restricting f to a function on $[a,b] \subset \mathbb{R}$ by taking a line joining x_0 and $-x_0$) such that $f(x_0') = 0$.

Problem 3. *Let $f\colon \overline{B}(0;1) \to \mathbb{R}$ be a continuous function, where $\overline{B}(0;1) \subset \mathbb{R}^2$ is the closed ball of radius 1, centred at $(0,0)$. Show that f cannot be injective.*

Solution 3. Note that $\overline{B}(0;1)$ and $\mathbb{R}$ are not homeomorphic, since, for example, if we remove a point from the former then the resulting space remains connected, which is not true for the latter. Further, $\overline{B}(0;1)$ is compact, and $\mathbb{R}$ is Hausdorff.[1] This means that we can obtain a proof by contradiction using the following corollary.

Theorem. Let $f\colon X \to Y$ be a continuous *bijection* between topological spaces, with X *compact* and Y *Hausdorff*. Then f is a homeomorphism.

Proof. See, for example, `https://proofwiki.org/wiki/Continuous_Bijection_from_Compact_to_Hausdorff_is_Homeomorphism`.

Corollary. Let $f\colon X \to Y$ be a continuous *injection* between topological spaces, with X *compact* and Y *Hausdorff*. Then f is a homeomorphism from X to $f(X)$.

[1] That is, for every $x \neq y \in \mathbb{R}$, there exist **disjoint** open subsets $U, V \subset \mathbb{R}$ such that $x \in U$ and $y \in V$.