Twisting cochains and Chern classes in complex geometry

An approach using derived analytic geometry

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Contents

Having all the answers just means you've been asking boring questions.
—*Joey Comeau and Emily Horne.*

Part I

Introduction

0.1 History and comparisons

what do we do that others don't?

PROOF READ ALL OF THIS PART basically want a section-by-section description

Part I

[Gre80] and [Dup76] work for smooths things; we 'skip' the smooth stuff and go straight to holomorphic forms with smooth parameters (whence another version of dupont's fibre integration lemma)

In fact, this is where the power of the simplicial construction is hiding: in the smooth case we can always construct global connections, whereas we cannot always do so in the holomorphic case. By working with forms over $|\Delta_p| \times X_{\mathscr{U}}^p$ we are mixing in 'just enough' smooth data to be able to construct global connections. This is explained in Section 4.

0.2 Part II

!

Part III

TODO

historical overview (nothing new here!)

Sections 7 and 8

!!!!!!!!!!!!!!!!This section is largely just notes on, and summaries of, important papers in the literature of twisting cochains.

No historical account is ever complete. In particular, we don't really mention the 'first' reference to twisting cochains: [Bro59]; we also don't follow what happened to the subject when it branched off into differential homological algebra (namely [Moo70]), even though this also predates all of the material that we *do* cover.* Our focus is really split into two parts: firstly, the development of twisting cochains by Toledo and Tong (and O'Brian) in [TT76; TT78; OTT81] using Čech cohomology and explicit methods; secondly, the development of twisted complexes (from [BK91]†) and the application of the language of DG-categories in [BHW15; Wei16a; Wei16b]. We discuss only briefly some of the generalisations to A_∞-categories, such as [Fao15].

Here we give the second of our two historical overviews on twisting cochains. This is much shorter than the first, and only briefly describes the modern viewpoint of twisting cochains as twisted complexes.!!!!!!!!!!!

Part IV

TODO

reviewing green's construction; adapting it to our framework

*For a good summary of the subject from a differential and lie-algebraic viewpoint, see [Sta09].

†In some papers this is cited as *Framed triangulated categories* instead of *Enhanced triangulated categories*, but this is just an artefact of translation from the original paper (which is in Russian).

Part II
Vector bundles and Hodge cohomology

1 The story

(i) twisting cochains resolve coherent sheaves

  (a) Green's resolution is actually a strict resolution: we use the twisting cochains (up to homotopy) to construct true simplicial vector bundles; we get a concrete way of computing things **question**: is this resolution cofibrant?

  (b) after this, we forget completely about twisting cochains

(ii) Green's construction (plus Dupont's fibre integration) gives us classes in Hodge cohomology for simplicial vector bundles

(iii) Julien's thesis/paper tells us what we need to show these classes satisfy to know that they are Chern classes (since he's already proved that Hodge cohomology satisfies nice enough properties)

  (a) *agrees on line bundles:* this is a straightforward calculation

  (b) *functorial under pullbacks:* N.B. for coherent sheaves, we get that the Chern class of the derived pullback is the pullback of the Chern class, but using Green's resolution we can work with the usual pullback to calculate the derived one, i.e. we just need to show that the Chern class of simplicial vector bundles is functorial under the true pullback

  (c) *Whitney sum for short exact sequences:* using Lemma 1.0.1, it suffices to show this for split exact sequences of coherent sheaves, but this is in Green

  (d) *Riemann-Roch for closed immersions:* apparently this follows from the other three properties by "deformations to the normal cone" or some such algebraic geometry magic

Lemma 1.0.1
If Chern classes are additive on every split exact sequence of coherent sheaves then they are additive on every short exact sequence of coherent sheaves. ⌟

Proof. Let $0 \to \mathscr{F} \xrightarrow{\iota} \mathscr{G} \xrightarrow{\pi} \mathscr{H} \to 0$ be a short exact sequence of coherent sheaves on X, and $t \in \mathbb{C}$ (which can be thought of as $t \in \Gamma(\mathbb{C}, \mathbb{P}^1)$). Write $p\colon X \times \mathbb{P}^1 \to X$ to mean the projection map. Define

$$\mathscr{N} = \operatorname{Ker}\left(p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \xrightarrow{\pi(1)-t\cdot\mathrm{id}} p^*\mathscr{H}(1)\right)$$

where $(\pi(1) - t \cdot \mathrm{id})\colon (g \otimes y, h) \mapsto \pi(g) \otimes y - h \otimes t$. We claim that this gives a short exact sequence

$$0 \to p^*\mathscr{F}(1) \to \mathscr{N} \to p^*\mathscr{H} \to 0 \tag{1.0.2}$$

of sheaves over $X \times \mathbb{P}^1$. where the maps are the 'obvious' ones: $p^*\mathscr{F}(1) \to \mathscr{N}$ is the map $\iota(1)\colon p^*\mathscr{F}(1) \to p^*\mathscr{G}(1)$ included into $p^*\mathscr{G}(1) \oplus p^*\mathscr{H}$ (which we prove lands in $\mathscr{N}$ below); and $\mathscr{N} \to p^*\mathscr{H}$ is the projection $p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \to p^*\mathscr{H}$ restricted to $\mathscr{N}$

- To prove surjectivity, let $h \in \Gamma(U, p^*\mathscr{H})$. Then $h \otimes t \in \Gamma(U, p^*\mathscr{H}(1))$. But $\pi\colon \mathscr{G} \to \mathscr{H}$ is surjective, and thus so too is the induced map $\pi(1)\colon p^*\mathscr{G}(1) \to p^*\mathscr{H}(1)$, hence there exists $g \otimes y \in \Gamma(U, p^*\mathscr{G}(1))$ such that $\pi(g \otimes y) = h \otimes t$. Thus $(g \otimes y, h) \in \mathscr{N}$ maps to h.

- To prove injectivity (and that this map is indeed well defined), we use the fact that tensoring with $\mathscr{O}(1)$ is exact, and so, in particular, $\iota(1)\colon p^*\mathscr{F}(1) \to p^*\mathscr{G}(1)$ is injective. The inclusion into the direct sum $p^*\mathscr{G}(1) \oplus p^*\mathscr{H}$ is injective by the definition of a direct sum, so all that remains to show is that the image of this composite map is contained inside $\mathscr{N}$. Let $f \otimes x \in \Gamma(U, p^*\mathscr{F}(1))$. Then this maps to $(\iota(f) \otimes x, 0) \in p^*\mathscr{G}(1) \oplus p^*\mathscr{H}$, but this is clearly in the kernel of $\pi(1) - t \cdot \mathrm{id}$ since $\pi\iota(f) = 0$.

- To prove exactness, it suffices to show that $\operatorname{Ker}(\mathscr{N} \to p^*\mathscr{F}) \cong p^*\mathscr{F}(1)$. But

$$\begin{aligned}
\operatorname{Ker}(\mathscr{N} \to p^*\mathscr{F}) &= \{(g \otimes y, h) \in \mathscr{N} \mid h = 0\} \\
&= \{(g \otimes y, h) \in p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \mid h = 0 \text{ and } \pi(g) \otimes y - h \otimes t = 0\} \\
&= \{(g \otimes y, h) \in p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \mid \pi(g) \otimes y = 0\} \\
&= \{(g \otimes y, h) \in p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \mid (g \otimes y) \in \operatorname{Im}\iota(1)\} \\
&\cong p^*\mathscr{F}(1).
\end{aligned}$$

Now we claim that the short exact sequence (1.0.2) is split for $t = 0$, and has $\mathscr{N} \cong p^*\mathscr{G}$ for $t \neq 0$. Formally, we do this by looking at the pullback of the map $X \times \{t\} \to X \times \mathbb{P}^1$, but for the moment we just 'pick a value for t'.

- $t = 0$. By definition,

$$\begin{aligned}
\mathscr{N} &= \operatorname{Ker}\left(p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \xrightarrow{\pi(1)-t\cdot\mathrm{id}} p^*\mathscr{H}(1)\right) \\
&= \operatorname{Ker}\left(p^*\mathscr{G}(1) \oplus p^*\mathscr{H} \xrightarrow{\pi(1)} p^*\mathscr{H}(1)\right)
\end{aligned}$$

$$
\begin{aligned}
&\cong \operatorname{Ker}\left(p^*\mathscr{G}(1) \xrightarrow{\pi(1)} p^*\mathscr{H}(1)\right) \oplus p^*\mathscr{H} \\
&\cong p^*\mathscr{F}(1) \oplus p^*\mathscr{H}.
\end{aligned}
$$

- $t \neq 0$. Define the injective morphism $\varphi\colon p^*\mathscr{G} \to \mathscr{N}$ of coherent sheaves by $\varphi(g) = (g \otimes t, \pi(g))$. To see that this is also surjective, let $(g \otimes y, h) \in \mathscr{N}$. If $y = 0$ then we must have $h = 0$, and so $(g \otimes y, h) = (0,0) = \varphi(0 \otimes 0)$. If $y \neq 0$ then $\pi(g) \otimes y - h \otimes t = 0$, with $y, t \neq 0$, whence $\pi(g) = \frac{y}{t}h$. Then $(g \otimes y, h) = (\frac{t}{y} g \otimes t, \pi(\frac{t}{y} g)) = \varphi(\frac{t}{y} g)$.

As one final ingredient, note that any coherent sheaf on X pulled back to a sheaf on $X \times \mathbb{P}^1$ is flat over $\mathbb{P}^1$, and so $\mathscr{N}$ is flat over $\mathbb{P}^1$, since both $\mathscr{F}(1)$ and $\mathscr{H}$ are. Thus, for $\tau_t\colon X \times \{t\} \to X \times \mathbb{P}^1$ given by a choice of $t \in \mathbb{C}$, the derived pullback $\mathbb{L}\tau_t^*\mathscr{N}$ agrees with the usual pullback $\tau_t^*\mathscr{N}$.

Now we use the $\mathbb{P}^1$-homotopy invariance of de Rham cohomology: the induced map

$$
\tau_t^*\colon H^\bullet\left(X \times \mathbb{P}^1, \Omega^\bullet_{X \times \mathbb{P}^1}\right) \to H^\bullet\left(X \times \{t\}, \Omega^\bullet_{X \times \{t\}}\right)
$$

doesn't depend on the choice of t. Since $X \times \{t\}$ is (canonically) homotopic to X, we can identify $(p\tau_t)^*$ with the identity on $H^\bullet(X, \Omega^\bullet_X)$. Since Equation (1.0.2) splits for $t = 0$, by our hypothesis, flatness, and the fact that Green's construction is functorial under derived pullback, we know that

$$
\begin{aligned}
\tau_0^* c(\mathscr{N}) = c(\mathbb{L}\tau_0^*\mathscr{N}) = c(\tau_0^*\mathscr{N}) &= c(\tau_0^* p^*\mathscr{F}(1) \oplus \tau_0^* p^*\mathscr{H}) \\
&= c(\mathscr{F}) \wedge c(\mathscr{H}).
\end{aligned}
$$

But we also know that $\mathscr{N} \cong p^*\mathscr{G}$ for $t \neq 0$, and so

$$
c(\mathscr{G}) = (p\tau_t)^* c(\mathscr{G}) = \tau_t^* c(\mathscr{N}).
$$

So, finally, the t-invariance of τ^* tells us that

$$
c(\mathscr{G}) = c(\mathscr{F}) \wedge c(\mathscr{H}). \qquad \square
$$

2 Preliminaries

Throughout, let $\mathcal{E}$ be a vector bundle of rank r on X, where $(X,\mathcal{O}_X)$ is a (paracompact) complex-analytic manifold, with a 'sufficiently nice'* open cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$. For all $\alpha \in I$, let ∇_α be some flat† connection over U_α.

The isomorphisms $\varphi_\alpha\colon \mathcal{E}|_{U_\alpha} \xrightarrow{\sim} (\mathcal{O}_X|_{U_\alpha})^{\mathrm{r}}$ define

$$M_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\colon (\mathcal{O}_X|_{U_\beta})^{\mathrm{r}} \xrightarrow{\sim} (\mathcal{O}_X|_{U_\alpha})^{\mathrm{r}}. \tag{2.0.1}$$

So picking a basis of ∇_α-flat sections $\{s_1^\alpha,\ldots,s_{\mathrm{r}}^\alpha\}$ over U_α means that the $M_{\alpha\beta}$ can be realised as explicit $(\mathrm{r}\times\mathrm{r})$-matrices that describe the base change from the trivialisation over U_α to the trivialisation over U_β:

$$s_k^\alpha = \sum_\ell (M_{\alpha\beta})_k^\ell s_\ell^\beta \tag{2.0.2}$$

2.1 The Atiyah class

Definition 2.1.1 [Atiyah exact sequence]
The *Atiyah exact sequence of* $\mathcal{E}$ is the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{E}\otimes\Omega_X^1 \to J^1(\mathcal{E}) \to \mathcal{E} \to 0$$

where $J^1(\mathcal{E}) = (\mathcal{E}\otimes\Omega_X^1)\oplus\mathcal{E}$ as a $\mathbb{C}_X$-module‡ but we define the $\mathcal{O}_X$-action by

$$f(s\otimes\omega, t) = (fs\otimes\omega + t\otimes \mathrm{d}f, ft). \qquad \lrcorner$$

By the above definition, a holomorphic connection on $\mathcal{E}$ is exactly a splitting of the Atiyah exact sequence of $\mathcal{E}$.

Note 2.1.2
Recall that we can extend any connection $\nabla\colon \mathcal{F} \to \mathcal{F}\otimes\Omega^1$ to a map $\nabla\colon \mathcal{F}\otimes\Omega^r \to \mathcal{F}\otimes\Omega^{r+1}$ by enforcing the Leibnitz rule:

$$\nabla(s\otimes\omega) = \nabla(s)\wedge\omega + s\otimes \mathrm{d}\omega. \qquad \lrcorner$$

*That is, locally finite and Stein, and such that $\mathcal{E}$ is free over each U_α

†These connections aren't required to be flat in order for Lemma 2.1.4 to hold, but we make this assumption now anyway. For example, we could simply pick the trivial connection d over each U_α.

‡Where $\mathbb{C}_X$ is the constant sheaf.

Definition 2.1.3 [Atiyah class]
The *Atiyah class* $\mathrm{at}_{\mathcal{E}}$ *of* $\mathcal{E}$ is the class of $J^1(\mathcal{E})$ in the first Ext group:

$$\mathrm{at}_{\mathcal{E}} = [J^1(\mathcal{E})] \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)$$ ⌟

Lemma 2.1.4
The Atiyah class of $\mathcal{E}$ *is represented by the cocycle**

$$\{\nabla_\beta - \nabla_\alpha\}_{\alpha,\beta \in I} \in \check{\mathscr{C}}^1_{\mathscr{U}}\Big(\mathscr{H}\!om(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)\Big).$$ ⌟

Proof. First, recall that the difference of any two connections is simply an $\mathcal{O}_X$-linear map. Secondly, note that we do indeed have a cocycle:

$$(\nabla_\beta - \nabla_\alpha) + (\nabla_\gamma - \nabla_\beta) = \nabla_\gamma - \nabla_\alpha.$$

Thus $\{\nabla_\beta - \nabla_\alpha\}_{\alpha,\beta} \in \check{\mathscr{C}}^1\Big(X, \mathscr{H}\!om(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)\Big)$. Then we use the isomorphisms

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X) \cong \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X[1]) \cong H^1\Big(X, \mathscr{H}\!om(\mathcal{E}, \mathcal{E} \otimes \Omega^1_X)\Big).$$

Finally, we have to prove that this class defined in homology agrees with that defined in our definition of the Atiyah class of $\mathcal{E}$. This fact is true in more generality, and we prove it as so.

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence in some abelian category $\mathscr{C}$. The definition of $[\mathcal{B}] \in \mathrm{Ext}^1_{\mathscr{C}}(\mathcal{C}, \mathcal{A})$ is as the class in $\mathrm{Hom}_{\mathcal{D}(\mathscr{C})}(\mathcal{C}, \mathcal{A}[1])$ of a canonical morphism $\mathcal{C} \to \mathcal{A}[1]$ constructed using $\mathcal{B}$ as follows:

(i) take the quasi-isomorphism $(\mathcal{A} \to \mathcal{B}) \xrightarrow{\sim} \mathcal{C}$, where $\mathcal{B}$ is in degree 0;

(ii) invert it to get a map $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \to \mathcal{B})$ such that the composite $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \to \mathcal{B}) \to \mathcal{B} \to \mathcal{C}$ is the identity;

(iii) compose with the identity map $(\mathcal{A} \to \mathcal{B}) \to \mathcal{A}[1]$.

In the case of locally-free sheaves on X we can realise the quasi-isomorphism $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \to \mathcal{B})$ as $\mathcal{C} \xrightarrow{\sim} \check{\mathscr{C}}^\bullet_{\mathscr{U}}(\mathcal{A} \to \mathcal{B})$, using the Čech complex of a complex:

$$\check{\mathscr{C}}^\bullet_{\mathscr{U}}(\mathcal{A} \to \mathcal{B}) = \check{\mathscr{C}}^0_{\mathscr{U}}(\mathcal{A}) \xrightarrow{(\check{\delta}, f)} \check{\mathscr{C}}^1_{\mathscr{U}}(\mathcal{A}) \oplus \check{\mathscr{C}}^0_{\mathscr{U}}(\mathcal{B}) \xrightarrow{(\check{\delta}, -f, \check{\delta})} \check{\mathscr{C}}^2_{\mathscr{U}}(\mathcal{A}) \oplus \check{\mathscr{C}}^1_{\mathscr{U}}(\mathcal{B}) \xrightarrow{(\check{\delta}, f, \check{\delta})} \ldots$$

where $\check{\mathscr{C}}^0_{\mathscr{U}}(\mathcal{A})$ is in degree -1. If we have local sections $\sigma_\alpha \colon \mathcal{C}|_{U_\alpha} \to \mathcal{B}|_{U_\alpha}$ then $\sigma_\beta - \sigma_\alpha$ lies in the kernel $\mathrm{Ker}(\mathcal{B}|_{U_{\alpha\beta}} \to \mathcal{C}|_{U_{\alpha\beta}})$, and so we can lift this difference to $\mathcal{A}$, giving us the map

$$(\{\sigma_\alpha\}_\alpha, \{\sigma_\beta - \sigma_\alpha\}_{\alpha,\beta}) \colon \mathcal{C} \to \check{\mathscr{C}}^0_{\mathscr{U}}(\mathcal{B}) \oplus \check{\mathscr{C}}^1_{\mathscr{U}}(\mathcal{A}).$$

*We omit the restriction from our notation: really we mean $\nabla_\alpha|_{U_{\alpha\beta}} - \nabla_\beta|_{U_{\alpha\beta}}$

This map we have constructed is exactly $[B]$. More precisely,

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{C},\mathcal{A}) & \cong & \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{C},\mathcal{A}[1]) & \cong & H^1\Big(X,\mathscr{H}\!om(\mathcal{C},\mathcal{A})\Big) \\ [B] & \leftrightarrow & \mathcal{C}\xrightarrow{\sim}(A\to B)\to A[1] & \leftrightarrow & [\{\sigma_\alpha-\sigma_\beta\}_{\alpha,\beta}]. \end{array}$$

□

Note 2.1.5
When $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are sheaves of $\mathcal{O}_X$-modules, with $\mathcal{H}$ locally free, we have the isomorphism

$$\mathscr{H}\!om(\mathcal{F},\mathcal{G}\otimes\mathcal{H})\cong\mathscr{H}\!om(\mathcal{F},\mathcal{G})\otimes\mathcal{H}.$$

This means that, taking the trivialisation over U_α, we can consider $\omega_{\alpha\beta}=(\nabla_\beta-\nabla_\alpha)$ as an $\mathfrak{r}\times\mathfrak{r}$-matrix of 1-forms on X, where $\mathfrak{r}=\operatorname{rank}\mathcal{E}$, since

$$H^1\Big(X,\mathscr{H}\!om(\mathcal{E},\mathcal{E}\otimes\Omega^1_X)\Big)\cong H^1\Big(X,\Omega^1_X\Big(\mathscr{E}\!nd(\mathcal{E})\Big)\Big)$$

where $\Omega^r_X(\mathcal{F})=\Omega^r_X\otimes\mathcal{F}$ is the collection of r-forms on X with values in $\mathcal{F}$. We calculate $\omega_{\alpha\beta}$ explicitly in Section 3.1. ⌟

Definition 2.1.6
We write $\omega_{\alpha\beta}$ to mean $(\nabla_\beta-\nabla_\alpha)$ considered as an $\mathfrak{r}\times\mathfrak{r}$-matrix of 1-forms on X, as in Note 2.1.5. We calculate $\omega_{\alpha\beta}$ explicitly in Equation (3.1.1). ⌟

Recall that for sheaves $\mathscr{F},\mathscr{G}$ of $\mathcal{O}_X$-modules we have the cup product

$$\smile\colon H^m(X,\mathscr{F})\otimes H^n(X,\mathscr{G})\to H^{m+n}(X,\mathscr{F}\otimes\mathscr{G})$$

which is given in Čech cohomology by the tensor product: $(a\smile b)_{ijk}=(a)_{ij}\otimes(b)_{jk}$.

Example 2.1.7 [Formal construction of the second Atiyah class]
Take

$$(\mathrm{at}_\mathcal{E}\otimes\mathrm{id}_{\Omega^1_X})\smile(\mathrm{at}_\mathcal{E})\in H^2\Big(X,\mathscr{H}\!om(\mathcal{E}\otimes\Omega^1_X,\mathcal{E}\otimes\Omega^1_X\otimes\Omega^1_X)\otimes\mathscr{H}\!om(\mathcal{E},\mathcal{E}\otimes\Omega^1_X)\Big)$$

and apply the composition map

$$H^m\Big(X,\mathscr{H}\!om(\mathcal{G},\mathcal{H})\otimes\mathscr{H}\!om(\mathcal{F},\mathcal{G})\Big)\to H^m\Big(X,\mathscr{H}\!om(\mathcal{F},\mathcal{H})\Big)$$

to obtain

$$(\mathrm{at}_\mathcal{E}\otimes\mathrm{id}_{\Omega^1_X})\smile(\mathrm{at}_\mathcal{E})\in H^2\Big(X,\mathscr{H}\!om(\mathcal{E},\mathcal{E}\otimes\Omega^1_X\otimes\Omega^1_X)\Big)\cong H^2\Big(X,\mathscr{E}\!nd(\mathcal{E})\otimes\Omega^1_X\otimes\Omega^1_X\Big).$$

Finally, apply the wedge product to get

$$(\mathrm{at}_\mathcal{E}\otimes\mathrm{id}_{\Omega^1_X})\wedge(\mathrm{at}_\mathcal{E})\in H^2\Big(X,\mathscr{E}\!nd(\mathcal{E})\otimes\Omega^2_X\Big).$$

⌟

THIS CONSTRUCTION GIVES US THE EXPONENTIAL CHERN CLASSES. TO OBTAIN THE USUAL ONES, YOU NEED TO WEDGE THE ENDOMORPHISM PART, NOT THE FORM PART.

Definition 2.1.8 [Higher Atiyah classes]
The p*-th Atiyah class* is the class

$$\mathrm{at}^p_{\mathcal{E}} = (\mathrm{at}_{\mathcal{E}} \otimes \mathrm{id}^{\otimes(p-1)}_{\Omega^1_X}) \wedge \ldots \wedge (\mathrm{at}_{\mathcal{E}} \otimes \mathrm{id}_{\Omega^1_X}) \smile (\mathrm{at}_{\mathcal{E}}) \in H^p\Big(X, \mathscr{E}nd(\mathcal{E}) \otimes \Omega^p_X)\Big)$$

where we apply composition and the wedge product as in Example 2.1.7. ⌟

Example 2.1.9 [Local expression for the second Atiyah class]
We can find an explicit representative for the second Atiyah class by using Note 2.1.5:

$$\begin{aligned}\Big((\mathrm{at}_{\mathcal{E}} \otimes \mathrm{id}_{\Omega^1_X}) \smile (\mathrm{at}_{\mathcal{E}})\Big)_{\alpha\beta\gamma} &= (\mathrm{at}_{\mathcal{E}} \otimes \mathrm{id}_{\Omega^1_X})_{\alpha\beta} \otimes (\mathrm{at}_{\mathcal{E}})_{\beta\gamma} \\ &\leftrightarrow (\omega_{\alpha\beta} \otimes \mathrm{id}_{\Omega^1_{U_{\alpha\beta}}}) \otimes \omega_{\beta\gamma} \in \mathcal{M}_{\mathrm{r}}(\Omega^1_{U_{\alpha\beta}} \otimes \Omega^1_{U_{\alpha\beta}}) \otimes \mathcal{M}_{\mathrm{r}}(\Omega^1_{U_{\beta\gamma}})\end{aligned}$$

where $\mathcal{M}_{\mathrm{r}}(A)$ is the collection of A-valued $(\mathrm{r} \times \mathrm{r})$-matrices. Now, before composing these two matrices, as described in Example 2.1.7 and Definition 2.1.8, we first have to account for the change of trivialisation $U_{\beta\gamma} \to U_{\alpha\beta}$. That is, after applying composition and the wedge product, we have

$$(\mathrm{at}^2_{\mathcal{E}})_{\alpha\beta\gamma} = [\omega_{\alpha\beta} \wedge M_{\alpha\beta}\omega_{\beta\gamma}M^{-1}_{\alpha\beta}].$$ ⌟

Example 2.1.10 [Local expressions for higher Atiyah classes]
We know that $\mathrm{at}^3_{\mathcal{E}}$ is represented locally by $\omega_{\alpha\beta}\omega_{\beta\gamma}\omega_{\gamma\delta}$, but where $\omega_{\beta\gamma}$ and $\omega_{\gamma\delta}$ undergo a base change to become $\Omega^1_{U_{\alpha\beta}}$-valued. But then, do we

(i) base change $\omega_{\gamma\delta}$ to an $\Omega^1_{U_{\beta\gamma}}$-valued form,

(ii) compose with $\omega_{\beta\gamma}$,

(iii) then base change this composition to an $\Omega^1_{U_{\alpha\beta}}$-valued form;

or do we instead

(i) base change both $\omega_{\gamma\delta}$ and $\omega_{\beta\gamma}$ to $\Omega^1_{U_{\alpha\beta}}$-valued forms,

(ii) then perform the triple composition?

That is:

$$\omega_{\alpha\beta} \wedge M_{\alpha\beta}(\omega_{\beta\gamma} \wedge M_{\beta\gamma}\omega_{\gamma\delta}M^{-1}_{\beta\gamma})M^{-1}_{\alpha\beta} \overset{?}{=} \omega_{\alpha\beta} \wedge M_{\alpha\beta}\omega_{\beta\gamma}M^{-1}_{\alpha\beta} \wedge M_{\alpha\gamma}\omega_{\gamma\delta}M^{-1}_{\alpha\gamma}.$$

The happy answer is that these two constructions are in fact equal, thanks to the cocycle condition on the $M_{\alpha\beta}$ and some form of associativity*, and so we can use whichever one we so please. ⌟

*That is, $A \wedge MB = AM \wedge B$, where M is a matrix of 0-forms.

2.2 Simplicial forms

Note 2.2.1
For us, the p-simplex Δ_p is the ordered set $[p]=[0,1,\ldots,p]$, and its geometric realisation $|\Delta_p|$ is the *smooth* space $\left\{(t_0,\ldots,t_p) \mid \sum_i t_i = 1\right\} \subset \mathbb{R}^{n+1}$. We write $f_i\colon \Delta_{p-1} \to \Delta_p$ to mean the i-th face map. ⌟

Definition 2.2.2 [Nerve of a cover]
Given some topological space Y with a cover $\mathscr{V} = \{V_\beta\}_{\beta\in J}$ we define the ***nerve*** $Y^\bullet_{\mathscr{V}}$ to be the simplicial space given by

$$Y^p_{\mathscr{V}} = \bigsqcup_{\beta_0,\ldots,\beta_p \in J} \{V_{\beta_0\cdots\beta_p} \mid V_{\beta_0\cdots\beta_p} \neq \varnothing\}$$

and with face maps acting by

$$Y^p_{\mathscr{V}} f_i\colon V_{\beta_0\cdots\beta_p} \mapsto V_{\beta_0\cdots\hat{\beta}_i\cdots\beta_p}$$

and degeneracy maps

$$Y^p_{\mathscr{V}} g_i\colon V_{\beta_0\cdots\beta_p} \mapsto V_{\beta_0\cdots\beta_i\beta_i\cdots\beta_p}.$$

⌟

Lemma 2.2.3
We have the local *decomposition*

$$\Omega^r_{Y\times Z} \cong \bigoplus_{i+j=r} \pi_Y^*\Omega^i_Y \otimes_{\mathcal{O}_{Y\times Z}} \pi_Z^*\Omega^j_Z$$

where π_Y and π_Z are the projection maps. ⌟

Proof. Taking small enough open sets $U\times V \subset X\times Y$, so that we have local coordinates over U and over V, gives us the result almost immediately. □

Definition 2.2.4 [Forms of type (i,j)]
We write $\Omega^{i,j}_{Y\times Z} = \Omega^i_Y \otimes_{\mathcal{O}_{Y\times Z}} \Omega^j_Z$ and call its elements* ***forms of type*** (i,j). ⌟

Definition 2.2.5 [Simplicial differential forms]
A ***simplicial differential r-form*** on a simplicial space $Y^\bullet$ is a collection $\omega = \left\{\omega_p \in \Omega^r_{|\Delta_p|\times Y^p}\right\}_{p\in\mathbb{N}}$ of forms that are smooth on $|\Delta_p|$ and holomorphic on Y^p satisfying the condition

$$(\mathrm{id}\times Y^\bullet f_i)^*\omega_{p-1} = (|f_i|\times\mathrm{id})^*\omega_p \in \Omega^r_{|\Delta_{p-1}|\times Y^p}$$

*Whenever there is no chance of confusion with the degrees coming from the Dolbeault bicomplex.

for all p. We write $\widetilde{\Omega}^r_{Y^\bullet}$ to mean the collection of all simplicial differential r-forms on $Y^\bullet$. Using Definition 2.2.4 we refer to simplicial forms *of type* (i,j) *on* $|\Delta_p \times Y^p|$ to mean a form of degree i on the p-simplex and of degree j on Y^p. ⌟

The condition in the definition of simplicial differential forms is rather natural when compared to the fat realisation of of a simplicial space. This point of view is covered in [Dup76].

Lemma 2.2.6 [Fibre integration]
There is a quasi-isomorphism

$$\int_{|\Delta|} : \widetilde{\Omega}^r_{Y^\bullet} \xrightarrow{\sim} \bigoplus_{p=0}^{r} \Omega^{r-p}_{Y^p}$$

*induced** *by* fibre integration

$$\int_{|\Delta_p|} : \Omega^r_{Y^\bullet} \xrightarrow{\sim} \Omega^{r-p}_{Y^p}$$

which is given by integrating the type $(p, r-p)$ *part of a simplicial form over the geometric realisation of the* p*-simplex with its canonical orientation (see Section A).* ⌟

Proof. The classical proof is [Dup76, Theorem 2.3], and the fact that the morphism is given by integrating over the simplices is mentioned in [Dup76, §2, Remark 1]. However, this proof is for the smooth case. Although the proof for the holomorphic case is almost identical, we give it in Section A (for the case $Y^\bullet = X^\bullet_{\mathscr{U}}$) so as to not miss the myriad of confusions that can arise from choices of orientation and signs. □

Example 2.2.7 [Simplicial differential forms on the nerve]
Taking $Y^\bullet = X^\bullet_{\mathscr{U}}$ gives

$$\int_{|\Delta|} : \widetilde{\Omega}^r_{X^\bullet_{\mathscr{U}}} \xrightarrow{\sim} \bigoplus_{p=0}^{r} \Omega^{r-p}_{X^p_{\mathscr{U}}} \cong \operatorname{Tot}^r \check{\mathscr{C}}^\bullet_{\mathscr{U}}(\Omega^\bullet_X).$$

It is interesting to note that the conditions we impose on $\mathscr{U}$ are only to ensure that this quasi-isomorphism *calculates* de-Rham cohomology; the actual quasi-isomorphism in Lemma 2.2.6 does *not* depend on the choice of cover. ⌟

*See Section A.

3 Manual construction

It is a classical fact* that $H^r\operatorname{Tot}^\bullet\check{\mathscr{C}}^\circ_{\mathscr{U}}(\Omega^\circ_X)\cong H^r(X,\mathbb{C})$. Say (as will be the case with the Atiyah class) we are given some $\mathfrak{a}_0\in\check{\mathscr{C}}^p_{\mathscr{U}}(\Omega^p_X)$, with $\check{\delta}\mathfrak{a}_0=0$ but $\mathrm{d}\mathfrak{a}_0\neq 0$. Define $\mathfrak{a}_p=0\in\check{\mathscr{C}}^0_{\mathscr{U}}(\Omega^{2p}_X)$. If we can find $\mathfrak{a}_i\in\check{\mathscr{C}}^{p-i}_{\mathscr{U}}(\Omega^{p+i}_X)$ for $i=1,\ldots,(p-1)$ such that $\check{\delta}\mathfrak{a}_i=\mathrm{d}\mathfrak{a}_{i-1}$ then

$$(0,\pm\mathfrak{a}_{p-1},\ldots,\pm\mathfrak{a}_1,\mathfrak{a}_0,0,\ldots,0)\in\operatorname{Tot}^{2p}\check{\mathscr{C}}^\bullet_{\mathscr{U}}(\Omega^\bullet_X)$$

is $(\check{\delta}\pm\mathrm{d})$-closed†, and thus represents a cohomology class in $H^{2p}\operatorname{Tot}^\bullet\check{\mathscr{C}}^\circ_{\mathscr{U}}(\Omega^\circ_X)$, and thus a cohomology class in $H^{2p}(X,\mathbb{C})$.

Note 3.0.1
Although we have the isomorphism $H^r\operatorname{Tot}^\bullet\check{\mathscr{C}}^\circ_{\mathscr{U}}(\Omega^\circ_X)\cong H^r(X,\mathbb{C})$, we don't necessarily have an easy way of computing explicitly what a closed class in the Čech-de Rham complex maps to, ***unless*** it has a non-zero degree-$(0,r)$ part, in which case it maps exactly to this. There is an important difference in our approach to closed de-Rham classes and Deligne classes: what we construct in Part IV is 'clearly' the Chern class, in particular because it has a non-zero degree-$(0,r)$‡ part that 'is' the Chern class; but what we construct here has no such term, and so we need to show that this lift of the Atiyah class (which 'is' the Chern class) really ***still is*** the Chern class, and this is the purpose of Section 5. ⌟

3.1 The first Atiyah class

We wish to calculate $(\nabla_\beta-\nabla_\alpha)$ in the basis $\{s^\alpha_1,\ldots,s^\alpha_{\mathrm{r}}\}$ and express it as some endomorphism-valued 1-form $\omega_{\alpha\beta}$, so that $\mathrm{at}_{\mathcal{E}}=[\{\omega_{\alpha\beta}\}_{\alpha,\beta}]$.

$$\begin{aligned}(\nabla_\beta-\nabla_\alpha)(s^\alpha_k)=\nabla_\beta s^\alpha_k&=\nabla_\beta\sum\nolimits_\ell(M_{\alpha\beta})^\ell_k s^\beta_\ell\\&=\sum\nolimits_\ell\Big(\nabla_\beta(s^\beta_\ell)(M_{\alpha\beta})^\ell_k+s^\beta_\ell\otimes\mathrm{d}(M_{\alpha\beta})^\ell_k\Big)\\&=\sum\nolimits_\ell\Big(s^\beta_\ell\otimes\mathrm{d}(M_{\alpha\beta})^\ell_k\Big)\\&=\sum\nolimits_\ell\Big(\Big(\sum\nolimits_m(M^{-1}_{\alpha\beta})^m_\ell s^\alpha_m\Big)\otimes\mathrm{d}(M_{\alpha\beta})^\ell_k\Big)\end{aligned}$$

*Since $\mathscr{U}$ is Stein (and any finite intersection of Stein open sets is also Stein) and $\mathcal{E}$ is coherent, Cartan's Theorem B tell us that $\check{\mathbb{H}}^k(\mathscr{U},\Omega^\bullet_X)\cong\check{\mathbb{H}}^k(X,\Omega^\bullet_X)$; since X is paracompact we know that $\check{\mathbb{H}}^k(X,\Omega^\bullet_X)\cong\mathbb{H}^k(X,\Omega^\bullet_X)$; and [Voi02, Theorem 8.1] tells us that $\mathbb{H}^k(X,\Omega^\bullet_X)\cong H^k(X,\mathbb{C})$.

†The signs depend on the Čech degree: the total differential, acting on an ℓ-cocycle of k-forms, is $(-1)^\ell\mathrm{d}+\check{\delta}$.

‡There is a slight change in the degree-labelling of Deligne cohomology, but this is just a technicality.

$$
\begin{aligned}
&= \sum\nolimits_{\ell,m} s_m^\alpha \otimes (M_{\alpha\beta}^{-1})_\ell^m \mathrm{d}(M_{\alpha\beta})_k^\ell \\
&= \sum\nolimits_m \left(s_m^\alpha \otimes M_{\alpha\beta}^{-1} \mathrm{d}(M_{\alpha\beta}) \right)_k^m .
\end{aligned}
$$

This means that $(\nabla_\beta - \nabla_\alpha)$ is given by the matrix of 1-forms

$$
\omega_{\alpha\beta} = M_{\alpha\beta}^{-1} \mathrm{d} M_{\alpha\beta}. \tag{3.1.1}
$$

Lemma 3.1.2
The element $\mathrm{tr}(\mathrm{at}_{\mathcal{E}})$ *is* d*-closed. More precisely,* $\{\mathrm{d}\,\mathrm{tr}(\omega_{\alpha\beta})\}_{\alpha,\beta} \in \check{\mathscr{C}}_{\mathscr{U}}^1(\Omega_X^2)$ *is zero.* ⌟

Proof. Note the following two facts: if $\varphi(m) = m^{-1}$ then $d\varphi(m)(h) = -m^{-1}hm^{-1}$; and tr and d commute. Thus

$$
\begin{aligned}
\mathrm{d}\,\mathrm{tr}(\omega_{\alpha\beta}) &= \mathrm{tr}(\mathrm{d}\omega_{\alpha\beta}) \\
&= \mathrm{tr}(\mathrm{d}M_{\alpha\beta}^{-1}\mathrm{d}M_{\alpha\beta} - M_{\alpha\beta}^{-1}\mathrm{d}^2 M_{\alpha\beta}) \\
&= \mathrm{tr}(\mathrm{d}M_{\alpha\beta}^{-1}\mathrm{d}M_{\alpha\beta}) \\
&= \mathrm{tr}(-M_{\alpha\beta}^{-1}\mathrm{d}M_{\alpha\beta}M_{\alpha\beta}^{-1}\mathrm{d}M_{\alpha\beta}) \\
&= -\mathrm{tr}(\omega_{\alpha\beta}^2).
\end{aligned}
$$

But now, since $\omega_{\alpha\beta}$ is a matrix of 1-forms, which are skew-symmetric, we see that

$$
-\mathrm{tr}(\omega_{\alpha\beta}^2) = -\sum\nolimits_{a,b} (\omega_{\alpha\beta})_b^a \wedge (\omega_{\alpha\beta})_a^b = 0.
$$
□

Lemma 3.1.3
The element $\mathrm{tr}(\mathrm{at}_{\mathcal{E}})$ *is* $\check{\delta}$*-closed. More precisely,* $\{\check{\delta}\,\mathrm{tr}(\omega_{\alpha\beta})\}_{\alpha,\beta} \in \check{\mathscr{C}}_{\mathscr{U}}^2(\Omega_X^1)$ *is zero.* ⌟

Proof.
$$
\check{\delta}\,\mathrm{tr}(\omega_{\alpha\beta}) = \mathrm{tr}(\omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta}) = 0
$$
where the second equality is shown in Section 3.2. □

This calculation can be summarised in the following diagram:

$$
\begin{tikzcd}
0 \\
\mathrm{tr}(\omega_{\alpha\beta}) \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & 0
\end{tikzcd} \tag{3.1.4}
$$

3.2 The second Atiyah class

By Example 2.1.9 we know that $\mathrm{at}^2_{\mathcal{E}} = [\{\omega_{\alpha\beta}M_{\alpha\beta}\omega_{\beta\gamma}M^{-1}_{\alpha\beta}\}_{i,j}]$. We introduce the following notation: $A = \omega_{\alpha\beta}$, $B = \omega_{\alpha\gamma}$, $M = M_{\alpha\beta}$, $X = M\omega_{\beta\gamma}M^{-1}$. Thus $\mathrm{at}^2_{\mathcal{E}} = AX$. Now, $\omega_{\alpha\beta} = M^{-1}_{\alpha\beta}\mathrm{d}M_{\alpha\beta}$, whence $\mathrm{d}A = -A^2$, and similarly for B and X. Further, by differentiating the cocycle condition $M_{\alpha\beta}M_{\beta\gamma} = M_{\alpha\gamma}$ and right-multiplying by $M^{-1}_{\alpha\gamma}$, we see that* $A + X = B$. Hence

$$\mathrm{at}^2_{\mathcal{E}} = A(B - A).$$

Using the fact that $\mathrm{d}A = A^2$ we see that $\mathrm{d}A^2 = 0$, whence

$$\mathrm{d}\,\mathrm{tr}(\mathrm{at}^2_{\mathcal{E}}) = \mathrm{d}\,\mathrm{tr}(AB - A^2) = -\,\mathrm{tr}(A^2B - AB^2).$$

We want $f \in \check{\mathscr{C}}^1_{\mathscr{U}}(\Omega^3_X)$ such that $\delta f = -\mathrm{d}\,\mathrm{tr}(\mathrm{at}^2_{\mathcal{E}})$ and $\mathrm{d}f = 0$. It is clear that we need, at least, f to be (the trace of) a polynomial of homogeneous degree 3 in the one variable $A = \omega_{\alpha\beta}$. But then $f(A) = \mathrm{tr}(A^3)$ is, up to a scalar multiple, our only option. We set $f(A) = \frac{1}{3}\,\mathrm{tr}(A^3)$ and compute its Čech coboundary:

$$\begin{aligned}(\delta f)_{\alpha\beta\gamma} &= f(\omega_{\beta\gamma}) - f(\omega_{\alpha\gamma}) + f(\omega_{\alpha\beta})\\ &= f(B-A) - f(B) + f(A)\\ &= \frac{1}{3}\,\mathrm{tr}\Big((B-A)^3 - B^3 + A^3\Big)\\ &= \frac{1}{3}\,\mathrm{tr}(-AB^2 - BAB - B^2A + A^2B + ABA + BA^2)\end{aligned}$$

Lemma 3.2.1
Let $x_1,\ldots,x_n$ be square matrices of 1-forms. Then

$$\mathrm{tr}(x_1x_2\cdots x_n) = (-1)^{n-1}\,\mathrm{tr}(x_2x_3\cdots x_nx_1).$$ ⌟

Proof. This is just using the anti-commutativity of the wedge product:

$$\begin{aligned}\mathrm{tr}(x_1x_2\cdots x_n) &= \sum_{a_i}(x_1)^{a_1}_{a_2}\wedge(x_2)^{a_2}_{a_3}\wedge\cdots\wedge(x_n)^{a_n}_{a_1}\\ &= -\sum_{a_i}(x_2)^{a_2}_{a_3}\wedge(x_1)^{a_1}_{a_2}\wedge\cdots\wedge(x_n)^{a_n}_{a_1}\end{aligned}$$

*As we already said, $M\omega_{\beta\gamma}M^{-1}$ is the natural way of thinking of X as being a map *into* something lying over U_i, so this equation should be read as a cocycle condition over U_i by thinking of it as $\omega_{\alpha\beta}+\tilde{\omega}_{\beta\gamma} = \omega_{\alpha\gamma}$, where the tilde corresponds to a base change (see Note 3.3.6 for a 'better' statement). Note that this is also the result we expect, since $\omega_{\alpha\beta}$ corresponds to $\nabla_\beta - \nabla_\alpha$, and this clearly satisfies the additive cocycle condition, as already shown.

$$
\begin{aligned}
&= \ldots = (-1)^{(n-1)} \sum_{a_i} (x_2)^{a_2}_{a_3} \wedge (x_3)^{a_3}_{a_4} \wedge \cdots \wedge (x_n)^{a_n}_{a_1} \wedge (x_1)^{a_1}_{a_2} \\
&= (-1)^{n-1} \operatorname{tr}(x_2 x_3 \cdots x_n x_1). \qquad \square
\end{aligned}
$$

Using Lemma 3.2.1 we see that

$$
\begin{aligned}
(\delta f)_{\alpha\beta\gamma} &= \frac{1}{3} \operatorname{tr}(-AB^2 - AB^2 - AB^2 + A^2B + A^2B + A^2B) \\
&= \operatorname{tr}(A^2B - AB^2) \\
&= -\mathrm{d} \operatorname{tr}(\mathrm{at}^2_{\mathcal{E}}).
\end{aligned}
$$

Now we just have to worry about whether or not $\mathrm{d}f$ is zero. But

$$
\begin{aligned}
\mathrm{d}f(A) &= \mathrm{d}\frac{1}{3} \operatorname{tr}(A^3) \\
&= \frac{1}{3} \operatorname{tr}(\mathrm{d}AA^2 - A\mathrm{d}A^2) \\
&= -\frac{1}{3} \operatorname{tr}(A^4),
\end{aligned}
$$

and we know* that $\operatorname{tr}(A^{2n}) = 0$ for any integer n, so we are done.

Finally, note that†

$$
\begin{aligned}
\check{\delta} \operatorname{tr}\Big(\omega_{\alpha\beta}(\omega_{\alpha\gamma} - \omega_{\alpha\beta})\Big) &= \check{\delta} \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\
&= \operatorname{tr}(\omega_{\beta\gamma}\omega_{\beta\delta}) - \operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\delta}) + \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\delta}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\
&= \operatorname{tr}\Big((\omega_{\alpha\gamma} - \omega_{\alpha\beta})(\omega_{\alpha\delta} - \omega_{\alpha\beta})\Big) - \operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\delta}) + \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\delta}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\
&= -\operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\beta}) + \operatorname{tr}(\omega^2_{\alpha\beta}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\
&= \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) = 0.
\end{aligned}
$$

*By exactly the same argument for showing that $\operatorname{tr}(A^2) = 0$ where A is a matrix of 1-forms: we can cyclically permute to obtain that $\operatorname{tr}(A^{2n}) = \operatorname{tr}(A^{2n-1}A) = -\operatorname{tr}(AA^{2n-1})$, whence $\operatorname{tr}(A^{2n}) = 0$.

†But really this is superfluous: $\omega_{\alpha\beta}$ is a 1-cocycle by definition.

This calculation can be summarised by the following commutative diagram:

$$\begin{tikzcd}
0 & & \\
-\frac{1}{3}\operatorname{tr}(A^3) \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & \operatorname{tr}(A(B-A)B) & \\
& \underbrace{\operatorname{tr}\big(A(B-A)\big)}_{\operatorname{tr}\operatorname{at}^2_{\mathcal{E}}} \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & 0
\end{tikzcd} \tag{3.2.2}$$

Tahis gives us the closed element

$$\operatorname{tr}\operatorname{at}^2_{\mathcal{E}} \mapsto \left(0, \frac{1}{3}\operatorname{tr}(A^3), \operatorname{tr}\big(A(B-A)\big), 0, 0\right) \in \operatorname{Tot}^4 \check{\mathscr{C}}^{\bullet}_{\mathscr{U}}(\Omega^{\bullet}_X) \tag{3.2.3}$$

3.3 The third Atiyah class

We now extend the notation from Section 3.2: write

$$\begin{aligned}
A &= \omega_{\alpha\beta} & M &= M_{\alpha\beta} & X &= M\omega_{\beta\gamma}M^{-1}\\
B &= \omega_{\alpha\gamma} & N &= M_{\alpha\gamma} & Y &= N\omega_{\gamma\delta}N^{-1}\\
C &= \omega_{\alpha\delta} & & & &
\end{aligned}$$

so that $\operatorname{at}^3_{\mathcal{E}} = AXY = A(B-A)(C-B)$.

The first step is simple:

$$\mathrm{d}\operatorname{tr}\operatorname{at}^3_{\mathcal{E}} = -\operatorname{tr}\big(A(B-A)(C-B)C\big) \in \check{\mathscr{C}}^3_{\mathscr{U}}(\Omega^4_X).$$

Now, trying to find some $\varphi \in \check{\mathscr{C}}^2_{\mathscr{U}}(\Omega^4_X)$ such that $\check{\delta}\varphi = \mathrm{d}\operatorname{tr}\operatorname{at}^3_{\mathcal{E}}$ is slightly harder. The most naïve approach is to find all the monomials in $\check{\mathscr{C}}^2_{\mathscr{U}}(\Omega^4_X)$, apply the Čech differential, and equate coefficients. Using the fact that we can cyclically permute under the trace, finding all the monomials is the same as finding all degree 2 monomials in non-commutative variables X and Y, modulo equivalence under cyclic permutation, and there are just four of these: X^2Y^2, $(XY)^2$, X^3Y, and XY^3. Thus, we find that

$$\varphi = -\frac{1}{4}\operatorname{tr}\left(\big(A(B-A)\big)^2\right) + \frac{1}{2}\operatorname{tr}(A^2(B-A)^2) - \frac{1}{2}\operatorname{tr}\big(A^3(B-A)\big) - \frac{1}{2}\operatorname{tr}\big(A(B-A)^3\big). \tag{3.3.1}$$

Noting that $\mathrm{d}\varphi$ can be factored as $\frac{1}{10}\operatorname{tr}\Big((B-A)^5 - B^5 + A^5\Big)$ we see that

$$\mathrm{d}\varphi = \check{\delta}\frac{1}{10}\operatorname{tr}(A^5). \tag{3.3.2}$$

This calculation can be summarised in the following commutative diagram:

$$\begin{tikzcd}
0 & & & \\
\frac{1}{10}\operatorname{tr}(A^5) \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & \frac{1}{10}\operatorname{tr}\Big((B-A)^5 - B^5 + A^5\Big) & & \\
& \begin{array}{c} -\frac{1}{4}\operatorname{tr}\Big(\big(A(B-A)\big)^2\Big) + \frac{1}{2}\operatorname{tr}(A^2(B-A)^2) \\ -\frac{1}{2}\operatorname{tr}\Big(A^3(B-A)\Big) - \frac{1}{2}\operatorname{tr}\Big(A(B-A)^3\Big) \end{array} \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & -\operatorname{tr}\Big(A(B-A)(C-B)C\Big) & \\
& & \underbrace{\operatorname{tr}\Big(A(B-A)(C-B)\Big)}_{\mathrm{at}^3_{\mathcal{E}}} \arrow[u, "\mathrm{d}"] \arrow[r, "\check{\delta}"] & 0
\end{tikzcd} \tag{3.3.3}$$

Taking the signs of the total differential into account, this gives us the closed element

$$\operatorname{tr}\mathrm{at}^3_{\mathcal{E}} \mapsto \Big(0, -\frac{1}{10}\operatorname{tr}(A^5), \rho(A,X), \operatorname{tr}\Big(AXY\Big), 0, 0, 0\Big) \in \operatorname{Tot}^6 \check{\mathscr{C}}^\bullet_{\mathscr{U}}(\Omega^\bullet_X) \tag{3.3.4}$$

where

$$\rho(A,X) = -\frac{1}{4}\operatorname{tr}\Big(AXAX\Big) + \frac{1}{2}\operatorname{tr}\Big(A^2X^2\Big) - \frac{1}{2}\operatorname{tr}\Big(A^3X\Big) - \frac{1}{2}\operatorname{tr}\Big(AX^3\Big).$$

Definition 3.3.5

To avoid having to write tr everywhere, we use the notation $\mathrel{\mathring{=}}$ to mean ‘equal up to a cyclic permutation and corresponding sign’, i.e.

$$x_1x_2\cdots x_n \mathrel{\mathring{=}} (-1)^{n+1}x_2x_3\cdots x_nx_1 \mathrel{\mathring{=}} (-1)^{m(n+1)}x_mx_{m+1}\cdots x_nx_1x_2\cdots x_{m-1}. \qquad \lrcorner$$

Note 3.3.6
Using the above notation we have the following relations:

(i) $\omega_{\alpha\beta}^2 \mathrel{\mathring{=}} 0$;

(ii) $M\omega_{\alpha\beta}M^{-1} \mathrel{\mathring{=}} \omega_{\alpha\beta}$ for all invertible matrices M.

Thus $\omega_{\alpha\beta} + \omega_{\beta\gamma} \mathrel{\mathring{=}} \omega_{\alpha\gamma}$. ⌟

3.4 The fourth Atiyah class

There are clearly patterns that we can spot looking at Equations (3.2.2) and (3.3.3), such as the fact that the Čech 1-cocycle seems to be (some constant multiple of) $\operatorname{tr}(A^{2k-1})$. But beyond this, trying to work in full generality with the k-th Atiyah class is difficuly, no small part thanks to the large number of terms. It seems believable that the even Atiyah classes and the odd Atiyah classes would follow different patterns, but unfortunately we don't have many explicit cases to study: $k = 0, 1$ are both trivial, and $k \geqslant 4$ is too large for us be able to spot any patterns. For completeness we give the lift of the fourth Atiyah class below, which was calculated using a Haskell program written by the author*, but after this we will look for a different method of finding a lift, using simplicial data.

The element in the total complex is

$$\left(0, -\frac{1}{35}\operatorname{tr}\mathrm{at}_{\mathcal{E}}^{4,(1,7)}, \frac{1}{5}\operatorname{tr}\mathrm{at}_{\mathcal{E}}^{4,(2,6)}, \frac{1}{5}\operatorname{tr}\mathrm{at}_{\mathcal{E}}^{4,(3,5)}, \operatorname{tr}\mathrm{at}_{\mathcal{E}}^{4,(4,4)}, 0, 0, 0, 0\right) \tag{3.4.1}$$

where

$$\begin{aligned}
\mathrm{at}_{\mathcal{E}}^{4,(4,4)} &= A(B-A)(C-B)(D-C) \\
\mathrm{at}_{\mathcal{E}}^{4,(3,5)} &= \frac{13}{5}A^5 + 13A^4(B-A) + 5A^3(B-A)^2 + 5A^3(B-A)(C-A) \\
&\quad + 3A^3(C-A)(B-A) + 4A^2(B-A)A(B-A) + 4A^2(B-A)A(C-A) \\
&\quad + 3A^2(B-A)^3 - A^2(B-A)^2(C-A) + 5A^2(B-A)(C-A)^2 \\
&\quad + 5A^2(C-A)A(B-A) + 2A^2(C-A)(B-A)^2 + A^2(C-A)(B-A)(C-A) \\
&\quad + 3A^2(C-A)^2(B-A) - A(B-A)A(C-A)(B-A) + 5A(B-A)A(C-A)^2 \\
&\quad - 5A(B-A)^2(C-A)(B-A) + 5A(B-A)(C-A)A(C-A) + 5A(B-A)(C-A)^3 \\
&\quad + 4\big(A(C-A)\big)^2(B-A) - 2A(C-A)(B-A)^3 + 4A(C-A)(B-A)^2(C-A)
\end{aligned}$$

*It just uses the naïve approach from Section 3.3 of repeatedly calculating all of the non-commutative monomials, modulo cyclic permutations, applying the Čech derivative, and then equating coefficients.

$$
\begin{aligned}
&+A\Big((C-A)(B-A)\Big)^2+2A(C-A)^2(B-A)^2+A(C-A)^2(B-A)(C-A)\\
&+3A(C-A)^3(B-A)\\
\mathrm{at}_{\mathcal{E}}^{4,(2,6)}&=5A^5(B-A)-4A^4(B-A)^2+A^3(B-A)A(B-A)+A^3(B-A)^3\\
&-5A^2(B-A)A(B-A)^2-4A^2(B-A)^2A(B-A)-4A^2(B-A)^4\\
&+\frac{1}{3}\Big(A(B-A)\Big)^3+A(B-A)A(B-A)^3+A(B-A)^5\\
\mathrm{at}_{\mathcal{E}}^{4,(1,7)}&=A^7
\end{aligned}
$$

4 Simplicial construction

4.1 The global simplicial connection

Write $\pi\colon|\Delta_p|\times X_{\mathscr{U}}^p\to X_{\mathscr{U}}^p$ to mean the projection map; write $\mathcal{E}^p$ to mean $\mathcal{E}$ as a sheaf on $X_{\mathscr{U}}^p$; and write $Y^p=|\Delta_p|\times X_{\mathscr{U}}^p$. Define $\overline{\mathcal{E}^p}=\pi^*\mathcal{E}^p=\mathcal{E}^p\otimes_{\mathcal{O}_{X_{\mathscr{U}}^p}}\mathcal{O}_{Y^p}$.

Note 4.1.1
The map $\mathcal{O}_{|\Delta_p|}\to\mathcal{O}_{Y^p}$ gives an $\mathcal{O}_{|\Delta_p|}$-action on $\mathcal{E}^p$. Further, we can use π to pull back sections, and so extend the ∇_α to connections on $\overline{\mathcal{E}^p}$. ⌟

Definition 4.1.2 [Global simplicial connection]
Using Note 4.1.1 we can define

$$\nabla_\mu^{(p)}=\sum_{i=0}^p t_i\nabla_{\alpha_i}\colon\overline{\mathcal{E}^p}\to\overline{\mathcal{E}^p}\otimes\Omega^1_{|\Delta_p|\times X_{\mathscr{U}}^p}$$

which acts on a section $s\otimes\varphi$ of $\overline{\mathcal{E}^p}$ over $U_{\alpha_0\ldots\alpha_p}$ by

$$
\begin{aligned}
\nabla_\mu^{(p)}(s\otimes\varphi)=\sum_{i=0}^p t_i\nabla_{\alpha_i}(s\otimes\varphi)&=\sum_i t_i\nabla_{\alpha_i}(\varphi s\otimes 1)\\
&=\sum_i t_i\Big(\varphi\pi^*\Big(\nabla_{\alpha_i}(s)\Big)+s\otimes 1\otimes \mathrm{d}\varphi\Big)\\
&=\sum_i \pi^*\Big(t_i\varphi\otimes\nabla_{\alpha_i}(s)\Big)+s\otimes 1\otimes t_i\mathrm{d}\varphi.
\end{aligned}
$$
⌟

Note that we have an alternative expression for $\nabla_\mu^{(p)}$ given by

$$\nabla_\mu^{(p)} = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i(\nabla_{\alpha_i} - \nabla_{\alpha_0}) = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i \omega_{\alpha_0\alpha_i}. \tag{4.1.3}$$

So, recalling Equation (3.1.1), and that the $s_{\alpha_i}^\ell$ are ∇_{α_i}-flat, we see that the p-th simplicial-level curvature acts by

$$\begin{aligned}
\kappa\Big(\nabla_\mu^{(p)}\Big)(s_{\alpha_0}^\ell) &= \Big(\nabla_{\alpha_0} + \sum_{i=1}^{p} t_i\omega_{\alpha_0\alpha_i}\Big)^2 (s_{\alpha_0}^\ell) \\
&= \nabla_{\alpha_0}^2(s_{\alpha_0}^\ell) + \sum_{i=1}^{p}\Big[\Big(\nabla_{\alpha_0}(s_{\alpha_0}^\ell)\wedge t_i\omega_{\alpha_0\alpha_i}\Big)\otimes + s_{\alpha_0}^\ell \otimes \mathrm{d}(t_i\omega_{\alpha_0\alpha_i})\Big] \\
&\quad + \sum_{i=1}^{p}\Big(t_i\omega_{\alpha_0\alpha_i}\wedge\nabla_{\alpha_0}(s_{\alpha_0}^\ell)\Big) + \sum_{i,j=1}^{p} s_{\alpha_0}^\ell \otimes (t_j t_i \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i}) \\
&= \sum_{i=1}^{p} s_{\alpha_0}^\ell \otimes \mathrm{d}(t_i\omega_{\alpha_0\alpha_i}) + \sum_{i,j=1}^{p} s_{\alpha_0}^\ell \otimes (t_j t_i \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i})
\end{aligned}$$

which is simply* $\mathrm{d}\overline{\omega}_{(p)} + \overline{\omega}_{(p)}^2$ on $U_{\alpha_0\cdots\alpha_p}$ in the U_{α_0} trivialisation, where $\overline{\omega}_{(p)}$ is the matrix of $(0,1)$-forms given by $\sum_{i=1}^{p} t_i\omega_{\alpha_0\alpha_i}$. Then

$$\mathrm{d}\overline{\omega}_{(p)} + \overline{\omega}_{(p)}^2 \in \Gamma\Big(X_{\mathscr{U}}^p, \Omega^2_{|\Delta_p|\times X_{\mathscr{U}}^p}(\mathscr{End}(\overline{\mathcal{E}^p}))\Big).$$

Note 4.1.4
Note that $\mathscr{End}(\overline{\mathcal{E}^p})$-valued forms on $|\Delta_p|\times X_{\mathscr{U}}^p$ are (locally) ‘usual’ forms on $|\Delta_p|$ times the identity matrix†, tensored with $(\mathfrak{r}\times\mathfrak{r})$-matrix-valued forms on $X_{\mathscr{U}}^p$. ⌟

Definition 4.1.5 [Simplicial Atiyah class]
We define the *k-th simplicial Atiyah class* as

$$\underline{\mathrm{at}}_{\mathcal{E}}^k = \Big\{\epsilon_k\Big(\mathrm{d}\overline{\omega}_{(p)} + \overline{\omega}_{(p)}^2\Big)^k \in \Gamma\Big(X_{\mathscr{U}}^p, \Omega^{2k}_{|\Delta_p|\times X_{\mathscr{U}}^p}\Big(\mathscr{End}(\overline{\mathcal{E}^p})\Big)\Big)\Big\}_{p\in\mathbb{N}}$$

where $\overline{\omega}_{(p)} = \sum_{i=1}^{p} t_i\omega_{\alpha_0\alpha_i}$ and $\epsilon_k = (-1)^{k(k-1)/2}$. ⌟

*Recall (Section A) that the de Rham differential on a form of type (i,j) is given by $\mathrm{d}_t + (-1)^i\mathrm{d}_X$, where d_t is the differential in the t_i. Hence, since $\overline{\omega}_{(p)}$ is a (matrix of) $(0,1)$ forms, the differential is simply ‘the usual one’, i.e. we use the product rule (which is the same as applying $\mathrm{d}_t + \mathrm{d}_X$)

†That is, we write $\mathrm{d}t_i$ when really we mean $\mathrm{d}t\cdot I_{\mathfrak{r}}$.

Note 4.1.6

The sign ϵ_k in Definition 4.1.5 comes from the fact that we want the wedge product of simplicial forms to respect fibre integration in some sense. Here, we want the sign of the (k,k)-term of $\operatorname{tr}\int_{|\Delta|}\underline{at}_{\mathcal{E}}^k$ to agree with the sign of the Čech k-cocycle of k-forms $\mathrm{at}_{\mathcal{E}}^k$. As shown in the proof of Theorem 4.5.1, the (k,k)-term involves changing the sign T_{k-1} times, where T_{k-1} is the $(k-1)$-th triangle number. ⌟

Theorem 4.1.7

The k-th simplicial Atiyah class defines a simplicial form $\underline{\mathrm{at}}_{\mathcal{E}}^k(U_\alpha)$ over every $U_\alpha \in \mathscr{U}$. That is, we can write

$$\underline{\mathrm{at}}_{\mathcal{E}}^k \in \Gamma\left(X_{\mathscr{U}}^\bullet, \widetilde{\Omega}_{X_{\mathscr{U}}^\bullet}^{2k}\left(\mathscr{E}nd(\overline{\mathcal{E}^\bullet})\right)\right).$$ ⌟

Proof. Since d and the wedge product both commute with pullbacks, we see that both $\mathrm{d}\mu$ and $\mu\wedge\mu$ are simplicial forms if μ is (and clearly $-\mu$ is also a simplicial form). Thus it only remains to show that $\overline{\omega}_{(p)} = \sum_{i=1}^p t_i\omega_{\alpha_0\alpha_i}$ is a simplicial form, by showing that, for all face maps $f_j\colon \Delta_{p-1}\to\Delta_p$, we have the equality

$$(\mathrm{id}\times|f_j|)^*\overline{\omega}_{(p)} = (X_{\mathscr{U}}^\bullet(f_j)\times\mathrm{id})^*\overline{\omega}_{(p-1)} \in \Omega^1_{|\Delta_{p-1}|\times X_{\mathscr{U}}^p}\left(\mathscr{E}nd(\overline{\mathcal{E}^p})\right).$$

But this is just asking that $\overline{\omega}_{(p)} = \overline{\omega}_{(p-1)}$ on the face $\{t_j=0\}$ of $|\Delta_p|\times U_{\alpha_0\cdots\alpha_p}$ for any $U_{\alpha_0\cdots\alpha_p}\in X_{\mathscr{U}}^p$, and there both sides are equal to $\sum_{i=1,i\neq j}^p t_i\omega_{\alpha_0\alpha_i}$. □

Note 4.1.8

There is no reason to expect the simplicial Atiyah class to be a cocycle *before* applying fibre integration. ⌟

Lemma 4.1.9

The trace of the k-th simplicial Atiyah class is d*-closed.* ⌟

Proof. It suffices (thanks to the product rule) to show that the trace of the first simplicial Atiayh class is d-closed, and

$$\begin{aligned}
\mathrm{d}\left(\mathrm{d}\overline{\omega}_{(p)} + \overline{\omega}_{(p)}^2\right) &= \mathrm{d}^2\overline{\omega}_{(p)} + \mathrm{d}\overline{\omega}_{(p)}^2\\
&= \mathrm{d}\left(\sum_{i,j=1}^p t_it_j\omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j}\right)\\
&= \sum_{i,j=1}^p\left[\mathrm{d}_{|\Delta|}(t_it_j)\otimes\omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j} + t_it_j\mathrm{d}_X(\omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{p} \Big[t_j \mathrm{d}_{|\Delta|} t_i \otimes \omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j} + t_i \mathrm{d}_{|\Delta|} t_j \otimes \omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j} \\
&\qquad + t_i t_j \mathrm{d}_X(\omega_{\alpha_0\alpha_i})\omega_{\alpha_0\alpha_j} - t_i t_j \omega_{\alpha_0\alpha_i} \mathrm{d}_X(\omega_{\alpha_0\alpha_j}) \Big] \\
&\mathrel{\mathring{=}} \sum_{i,j=1}^{p} \Big[t_j \mathrm{d}_{|\Delta|} t_i \otimes \omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j} - t_i \mathrm{d}_{|\Delta|} t_j \otimes \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i} \\
&\qquad - t_i t_j \omega_{\alpha_0\alpha_i}^2 \omega_{\alpha_0\alpha_j} + t_i t_j \omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j}^2 \Big] \\
&\mathrel{\mathring{=}} \sum_{i,j=1}^{p} \Big[- t_i t_j \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i}^2 + t_i t_j \omega_{\alpha_0\alpha_i}\omega_{\alpha_0\alpha_j}^2 \Big] = 0. \qquad \square
\end{aligned}
$$

4.2 The first simplicial Atiyah class

Working over $U_{\alpha_0\cdots\alpha_p}$ gives us an expression* for the first simplicial Atiyah class:

$$
\underline{\mathrm{at}}_{\mathcal{E}}^1(U_{\alpha_0\cdots\alpha_p}) = \left\{ \sum_{i=1}^{p} \mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i} - \sum_{i=1}^{p} t_i \omega_{\alpha_0\alpha_i}^2 + \sum_{i,j=1}^{p} t_j \omega_{\alpha_0\alpha_j} t_i \omega_{\alpha_0\alpha_i} \right\}_{p\in\mathbb{N}}. \tag{4.2.1}
$$

Equation (A.0.1) tells us that the fibre integral of $\underline{\mathrm{at}}_{\mathcal{E}}^1$ depends only on the $(2,0)$, $(1,1)$, and $(0,2)$ parts. Looking at Equation (4.2.1) we see that there is no $(2,0)$ part, and so

$$
\begin{aligned}
\int_{|\Delta|} \underline{\mathrm{at}}_{\mathcal{E}}^1(U_{\alpha_0\cdots\alpha_p}) &= \int_{|\Delta_1|} \sum_{i=1}^{p=1} \mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i} - \int_{|\Delta_0|} \sum_{i=1}^{p=0} t_i \omega_{\alpha_0\alpha_i}^2 + \int_{|\Delta_0|} \sum_{i,j=1}^{p=0} t_j \omega_{\alpha_0\alpha_j} t_i \omega_{\alpha_0\alpha_i} \\
&= \int_{|\Delta_1|} \mathrm{d}t_1 \otimes \omega_{\alpha_0\alpha_1}.
\end{aligned}
$$

Note 4.2.2
Although the $(0,2k)$ part of $\underline{\mathrm{at}}_{\mathcal{E}}^k$ is generally non-zero, when we fibre integrate we only look at it on the 0-simplex, and there it *is* zero (since both sums disappear). ⌟

Continuing our calculation gives the same result as in Section 3.1:

$$
\mathrm{tr} \int_{|\Delta|} \underline{\mathrm{at}}_{\mathcal{E}}^1(U_{\alpha_0\cdots\alpha_p}) = \mathrm{tr} \int_0^1 \omega_{\alpha_0\alpha_1} \mathrm{d}t_1 = \underbrace{\mathrm{tr}(\omega_{\alpha_0\alpha_1})}_{p=1}. \tag{4.2.3}
$$

*Recall that $\mathrm{d}\omega_{\alpha\beta} = -\omega_{\alpha\beta}^2$.

Note 4.2.4
We indicate that the result in Equation (4.2.3) 'lives in' $p = 1$ to mean that the result is a Čech 1-cocycle. Generally, as in the manual construction, the (fibre integral of the) k-th simplicial Atiyah class will have terms in $\check{\mathscr{C}}^{k-i}_{\mathscr{U}}(\Omega^{k+i}_X)$ for $i = 0, \ldots, k-1$. This is just notational laziness: we write sums to mean direct sums. ⌟

4.3 The second simplicial Atiyah class

We know that

$$\underline{\mathrm{at}}^2_{\mathcal{E}} = \left\{ \left(\sum_{i=1}^{p} \mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i} - \sum_{i=1}^{p} t_i \omega^2_{\alpha_0\alpha_i} + \sum_{i,j=1}^{p} t_j \omega_{\alpha_0\alpha_j} t_i \omega_{\alpha_0\alpha_i} \right)^2 \right\}_{p\in\mathbb{N}}$$

but also that the only parts that will be non-zero after fibre integration are the $(2,2)$ parts on the 2-simplex, and the $(1,3)$ parts on the 1-simplex. The only $(2,2)$ part comes from the $(\mathrm{d}\overline{\omega})^2$ term, and, recalling Section A and remembering the sign $\epsilon_2 = -1$, this is

$$\begin{aligned}
-\int_{|\Delta_2|} \sum_{i,j=1}^{p} (\mathrm{d}t_j \otimes \omega_{\alpha_0\alpha_j}) \wedge (\mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i}) &= \int_{|\Delta_2|} \sum_{i,j=1}^{p=2} \mathrm{d}t_j \mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i} \\
&= \int_{|\Delta_2|} \Big((\mathrm{d}t_1)^2 \otimes \omega^2_{\alpha_0\alpha_1} + \mathrm{d}t_1\mathrm{d}t_2 \otimes \omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2} \\
&\qquad + \mathrm{d}t_2\mathrm{d}t_1 \otimes \omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1} + (\mathrm{d}t_2)^2 \otimes \omega^2_{\alpha_0\alpha_2} \Big) \\
&= \int_{|\Delta_2|} \mathrm{d}t_1\mathrm{d}t_2 \otimes \Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2} - \omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1} \Big) \\
&= \int_0^1 \int_0^{1-t_2} \mathrm{d}t_1\mathrm{d}t_2 \otimes \Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2} - \omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1} \Big) \\
&= \frac{1}{2} \Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2} - \omega_{\alpha_0\alpha_2}\omega_{\alpha_0\alpha_1} \Big) \\
&\mathrel{\mathring{=}} \frac{1}{2} \cdot 2 \cdot \omega_{\alpha_0\alpha_1}\omega_{\alpha_0\alpha_2} \\
&\mathrel{\mathring{=}} \omega_{\alpha_0\alpha_1}(\omega_{\alpha_0\alpha_1} + \omega_{\alpha_1\alpha_2}) \\
&\mathrel{\mathring{=}} \omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2}.
\end{aligned}$$

So far, this agrees with the result found in Section 3.2:

$$\mathrm{tr} \int_{|\Delta|} \underline{\mathrm{at}}^2_{\mathcal{E}}(U_{\alpha_0\cdots\alpha_p}) = \underbrace{\mathrm{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2})}_{p=2} + \underbrace{\ ?\ }_{p=1} \tag{4.3.1}$$

Now, remembering again the sign $\epsilon_2 = -1$, we calculate the $(1,3)$ part on the 1-simplex:

$$\begin{aligned}
&\sum_{i,j=1}^{p} \mathrm{d}t_j \otimes \omega_{\alpha_0\alpha_j} \wedge t_i \omega^2_{\alpha_0\alpha_i} + \sum_{i,j=1}^{p} t_j \omega^2_{\alpha_0\alpha_j} \wedge (\mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i}) \\
&- \sum_{i,j,k=1}^{p} (\mathrm{d}t_k \otimes \omega_{\alpha_0\alpha_k}) \wedge t_j \omega_{\alpha_0\alpha_j} t_i \omega_{\alpha_0\alpha_i} - \sum_{i,j,k=1}^{p} t_k \omega_{\alpha_0\alpha_k} t_j \omega_{\alpha_0\alpha_j} \wedge (\mathrm{d}t_i \otimes \omega_{\alpha_0\alpha_i}) \\
&\xmapsto{\int_{|\Delta|}} \int_{|\Delta_1|} 2\omega^3_{\alpha_0\alpha_1}(t_1 \mathrm{d}t_1 - t_1^2 \mathrm{d}t_1) \\
&\qquad = \frac{1}{3}\omega^3_{\alpha_0\alpha_1}.
\end{aligned}$$

Thus, exactly as in Section 3.2, we see that

$$\operatorname{tr} \int_{|\Delta|} \underline{\mathrm{at}}^2_{\mathcal{E}}(U_{\alpha_0\cdots\alpha_p}) = \underbrace{\operatorname{tr}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2})}_{p=2} + \underbrace{\frac{1}{3}\operatorname{tr}(\omega^3_{\alpha_0\alpha_1})}_{p=1}. \tag{4.3.2}$$

4.4 The third simplicial Atiyah class

There is a subtlety in the calculations when we reach the third simplicial Atiyah class, due to our choice of conventions for Čech cocycles. We don't assume skew-symmetry of cocycles (i.e. that exchanging two indices changes sign), but skew-symmetrisation of cocycles is a quasi-isomorphism, and so doesn't change the cohomology class*. If we had worked with skew-symmetric Čech cocycles from the start then this calculation would appear in some sense more natural, but we didn't.

We write μ_i to mean $\omega_{\alpha_0\alpha_i}$. Then, as before, we know that the $(3,3)$ part of $\int_{|\Delta|}\underline{\mathrm{at}}^3_{\mathcal{E}}$ is

$$\begin{aligned}
-\int_{|\Delta_3|} \sum_{i,j,k=1}^{p} (\mathrm{d}t_k \otimes \mu_k) \wedge (\mathrm{d}t_j \otimes \mu_j) \wedge (\mathrm{d}t_i \otimes \mu_i) &= \int_{|\Delta_3|} \sum_{i,j,k=1}^{p=3} \mathrm{d}t_k \mathrm{d}t_j \mathrm{d}t_i \otimes \mu_k \mu_j \mu_i \\
&= \int_{|\Delta_3|} \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \otimes \mu_{\sigma(1)}\mu_{\sigma(2)}\mu_{\sigma(3)} \\
&= \frac{1}{6} \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \mu_{\sigma(1)}\mu_{\sigma(2)}\mu_{\sigma(3)}
\end{aligned}$$

*All this says is that you have (at least) two models of Čech cocycles that are quasi-isomorphic: cocycles with arbitrarily-ordered indices, and cocycles with arbitrarily-ordered but skew-symmetric indices.

$$
\begin{aligned}
&\mathrel{\mathring{=}} \frac{1}{2}(\mu_1\mu_2\mu_3 - \mu_1\mu_3\mu_2)\\
&\mathrel{\mathring{=}} \frac{1}{2}\Big(\omega_{\alpha_0\alpha_1}(\omega_{\alpha_0\alpha_1}+\omega_{\alpha_1\alpha_2})(\omega_{\alpha_0\alpha_1}+\omega_{\alpha_1\alpha_2}+\omega_{\alpha_2\alpha_3})\\
&\qquad - \omega_{\alpha_0\alpha_1}(\omega_{\alpha_0\alpha_1}+\omega_{\alpha_1\alpha_2}+\omega_{\alpha_2\alpha_3})(\omega_{\alpha_0\alpha_1}+\omega_{\alpha_1\alpha_2})\Big)\\
&\mathrel{\mathring{=}} \frac{1}{2}\Big(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2}\omega_{\alpha_2\alpha_3} - \omega_{\alpha_0\alpha_1}\omega_{\alpha_2\alpha_3}\omega_{\alpha_1\alpha_2}\Big).
\end{aligned}
$$

But note that both of the terms in this last expression skew-symmetrise to the same thing:

$$
\sum_{\sigma\in S_4}\operatorname{sgn}(\sigma)\,\omega_{\alpha_{\sigma(0)}\alpha_{\sigma(1)}}\omega_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}}\omega_{\alpha_{\sigma(2)}\alpha_{\sigma(3)}} = -\sum_{\sigma\in S_4}\operatorname{sgn}(\sigma)\,\omega_{\alpha_{\sigma(0)}\alpha_{\sigma(1)}}\omega_{\alpha_{\sigma(2)}\alpha_{\sigma(3)}}\omega_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}}.
$$

Thus, writing ς_p to mean the skew-symmetrisation of a Čech p-cocycle, we have shown that

$$
\varsigma_3\operatorname{tr}\int_{|\Delta_3|}\underline{\mathrm{at}}^3_{\mathcal{E}} = \varsigma_3\operatorname{tr}(\mathrm{at}^3_{\mathcal{E}})
$$

whence they both represent the same class in cohomology.

The $(2,4)$ term comes from*

$$
X^2Y + XYX + YX^2 - X^2Z - XZX - ZX^2 \tag{4.4.1}
$$

where

$$
X = \sum_{i=1}^{2}\mathrm{d}t_i\otimes\mu_i \qquad Y = \sum_{i=1}^{2}t_i\mu_i^2 \qquad Z = \sum_{i,j=1}^{2}t_it_j\mu_i\mu_j.
$$

Using that $\int_{|\Delta_2|}\mathrm{d}t_1\mathrm{d}t_2 = \int_0^1\int_0^{1-t_2}\mathrm{d}t_1\mathrm{d}t_2$ we can calculate the following:

(i) $\int_{|\Delta_2|}t_1\mathrm{d}t_1\mathrm{d}t_2 = \int_{|\Delta_2|}t_2\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{6}$;

(ii) $\int_{|\Delta_2|}t_1^2\mathrm{d}t_1\mathrm{d}t_2 = \int_{|\Delta_2|}t_2^2\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{12}$;

(iii) $\int_{|\Delta_2|}t_1t_2\mathrm{d}t_1\mathrm{d}t_2 = \frac{1}{24}$.

Then we can integrate Equation (4.4.1):

$$
\begin{aligned}
\int_{|\Delta_2|}X^2(Z-Y) + X(Z-Y)X + (Z-Y)X^2 &= \frac{1}{2}\mu_1^3\mu_2 + \frac{1}{2}\mu_1\mu_2^3 - \frac{1}{4}\mu_1\mu_2\mu_1\mu_2\\
&= \frac{1}{4}(\omega_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2})^2 + \frac{1}{2}(\omega^3_{\alpha_0\alpha_1}\omega_{\alpha_1\alpha_2} + \omega_{\alpha_0\alpha_1}\omega^3_{\alpha_1\alpha_2})
\end{aligned}
$$

*Not forgetting that $\epsilon_3 = -1$.

Comparing this with Equation (3.3.1), we have the same, save a missing $\frac{1}{2}\omega^2_{\alpha_0\alpha_1}\omega^2_{\alpha_1\alpha_2}$ term. But this skew-symmetrises to zero, since it is invariant under the permutation that swaps 0 and 2. Thus the $(2,4)$ terms agree in cohomology.

Finally, the $(1,5)$ term is

$$\begin{aligned} -\int_{|\Delta_1|}(3t_1^2+3t_1^4-6t_1^3)\mathrm{d}t_1\otimes\mu_1^5 &= -\int_0^1(3t_1^2+3t_1^4-6t_1^3)\mathrm{d}t_1\otimes\mu_1^5 \\ &= -\left(1+\frac{3}{5}-\frac{3}{2}\right)\mu_1^5 = -\frac{1}{10}\omega^5_{\alpha_0\alpha_1} \end{aligned}$$

which agrees exactly with Equation (3.3.2).

4.5 General results

It would be reassuring to see that the classes that we obtain through fibre integration really do agree with our manual construction. This is the content of the following theorem.

Theorem 4.5.1
The degree-(k,k) term in $\operatorname{tr}\int_{|\Delta|}\underline{\mathrm{at}}^k_{\mathcal{E}}$ is equivalent to $\operatorname{tr}(\mathrm{at}^k_{\mathcal{E}})$. That is,

$$\varsigma_k\left(\operatorname{tr}\int_{|\Delta|}\underline{\mathrm{at}}^k_{\mathcal{E}}\right)^{(k,k)} = \varsigma_k\operatorname{tr}(\mathrm{at}^k_{\mathcal{E}})$$

where ς_k denotes the skew-symmetrisation of a Čech k-cocycle. ⌟

Proof. **First we rewrite the left-hand side**. Generalising Section 4.4, we can write the term coming from fibre integration as

$$\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\mu_{\sigma(1)}\cdots\mu_{\sigma(k)} \overset{\varsigma_k}{\longmapsto} \frac{1}{k!(k+1)!}\sum_{\tau\in S_{k+1}}\sum_{\sigma\in S_k}\operatorname{sgn}(\tau\sigma)\omega_{\tau(0)\tau\sigma(1)}\cdots\omega_{\tau(0)\tau\sigma(k)}$$

where $S_k\leqslant S_{k+1}$ acts on $\{0,1,\ldots,k\}$ by fixing 0. But then, since $\omega(0)=0$, we can rewrite this as

$$\frac{1}{k!(k+1)!}\sum_{\tau\in S_{k+1}}\sum_{\sigma\in S_k}\operatorname{sgn}(\tau\sigma)\omega_{\tau\sigma(0)\tau\sigma(1)}\cdots\omega_{\tau\sigma(0)\tau\sigma(k)}.$$

Now we can use the fact that multiplication by an element of $S_k\leqslant S_{k+1}$ is an automorphism to do a change of variables. This gives us

$$\frac{1}{(k+1)!}\sum_{\eta\in S_{k+1}}\operatorname{sgn}(\eta)\,\omega_{\eta(0)\eta(1)}\cdots\omega_{\eta(0)\eta(k)}$$

which is trivially equal to

$$\frac{1}{(k+1)!}\sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta) \prod_{i=1}^{k} \left(\omega_{\eta(0)\eta(i)} - \omega_{\eta(0)\eta(0)}\right) \tag{4.5.2}$$

where we use that $\omega_{ii} = 0$, by definition.

Next we rewrite the right-hand side. The skew-symmetrisation is simply

$$\begin{aligned}
\varsigma_k \operatorname{tr}(\mathrm{at}_{\mathcal{E}}^k) &= \frac{1}{(k+1)!}\sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta)\, \omega_{\eta(0)\eta(1)}\omega_{\eta(1)\eta(2)}\cdots\omega_{\eta(k-1)\eta(k)} \\
&= \frac{1}{(k+1)!}\sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta)\, \omega_{\eta(0)\eta(1)}(\omega_{\eta(0)\eta(2)} - \omega_{\eta(0)\eta(1)})\cdots(\omega_{\eta(0)\eta(k)} - \omega_{\eta(0)\eta(k-1)}) \\
&= \frac{1}{(k+1)!}\sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta) \prod_{i=1}^{k} \left(\omega_{\eta(0)\eta(i)} - \omega_{\eta(0)\eta(i-1)}\right) \qquad (4.5.3)
\end{aligned}$$

again using that $\omega_{ii} = 0$.

Now we prove equality. Since, *for each fixed* η, there are no relations satisfied between the $\omega_{\eta(0)\eta(i)}$, we have two polynomials in $(k+1)$ free non-commutative variables, homogeneous of degree k. To emphasise the fact the following argument is purely abstract, we write $x_i = \omega_{0i}$ and define an action of S_{k+1} on the x_i by $x_{\eta(i)} = \omega_{\eta(0)\eta(i)}$. Now, showing that Equations (4.5.2) and (4.5.3) are equal amounts to showing that

$$A := \sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta) \prod_{i=1}^{k} \left(x_{\eta(i)} - x_{\eta(0)}\right) = \sum_{\eta\in S_{k+1}} \operatorname{sgn}(\eta) \prod_{i=1}^{k} \left(x_{\eta(i)} - x_{\eta(i-1)}\right) =: B. \tag{4.5.4}$$

Write E to mean the $\mathbb{Z}$-linear span of degree-k monomials in the $(k+1)$ free non-commutative variables x_i, and let $\sigma_{p,q} \in S_{k+1}$ be the transposition that swaps p and q. Then $\sigma_{p,q}$ gives an involution on E, and thus $E \cong E_{p,q}(1) \oplus E_{p,q}(-1)$, where $E_{p,q}(\lambda)$ is the eigenspace corresponding to the eigenvalue λ.

Let $H_{p,q}$ be the $\mathbb{Z}$-linear subspace of E spanned by monomials that contain at least one x_p or x_q. This subspace is clearly stable* under $\sigma_{p,q}$, and so this space also splits as $H_{p,q} \cong H_{p,q}(1) \oplus H_{p,q}(-1)$. Further, we have the obvious inclusion $H_{p,q}(-1) \subseteq E_{p,q}(-1)$, but we see that if $X \in E_{p,q}(-1)$ then, in particular, X must contain at least one† x_p or x_q, so $X \in H_{p,q}$, whence $X \in H_{p,q}(-1)$. Thus $E_{p,q}(-1) = H_{p,q}(-1)$.

The intersection H of the $H_{p,q}$ over all (distinct) pairs $(p,q) \in \{0,\ldots,k\} \times \{0,\ldots,k\}$ is the $\mathbb{Z}$-linear span of all monomials containing all but one of the x_i (and, in particular,

*If a monomial X contains, say, one x_p, then $\sigma_{p,q}X$ contains one x_q.

†If not, then the action of $\sigma_{p,q}$ would be trivial and X would lie in $E_{p,q}(1)$.

containing k distinct x_i). But since $H_{p,q}(-1) = E_{p,q}(-1)$, we see that the intersection $E(-1)$ of all the $E_{p,q}(-1)$ is equal to $H(-1)$. Now both A and B are in $E_{p,q}(-1)$ for all p,q (since $\operatorname{sgn}(\omega_{p,q}) = -1$), and so $A, B \in E(-1) = H(-1)$. Since the coefficient of, for example, the $x_1 \cdots x_k$ term is equal (and *non-zero*) in both A and B (it is 1, because we have to take $\eta = \mathrm{id}$), it suffices to show that $H(-1)$ is one-dimensional to prove Equation (4.5.4).

So let $X, Y \in H(-1)$ be monomials. Then each one contains k distinct x_i, and so there exists some (unique) $\sigma \in S_{k+1}$ such that $\sigma X = \pm Y$. But, writing $\sigma = \sigma_{p_1,q_1} \cdots \sigma_{p_r,q_r}$, we know that $\sigma X = (-1)^r X$, whence $X = Y$, up to some sign (i.e. up to some scalar in $\mathbb{Z}$). $\square$

Part III
Chern classes

TODO! talk about Julien's thesis etc and why the following things are what we need to show

5 Axiomatic Chern classes

Throughout this section, let X be a paracompact complex-analytic manifold with a 'nice' (as in Section 2) cover $\mathscr{U}$.

5.1 Line bundles

Let $\mathcal{L}$ be a line bundle on X, defined by transition maps $\{g_{\alpha\beta}\}_{\alpha,\beta}$. The first Chern class is, classically,

$$c_1(\mathcal{L}) = \left\{\frac{\mathrm{d}g_{\alpha\beta}}{g_{\alpha\beta}}\right\}_{\alpha,\beta} \in \check{H}(\mathscr{U},\Omega_X^1). \tag{5.1.1}$$

Comparing this with Equation (4.2.3) we see that the trace of the first (simplicial) Atiyah class agrees with the first Chern class for line bundles.

5.2 Whitney sum formula

TODO: change this to the *total* Chern class, or Chern characters

Lemma 5.2.1 [Whitney sum formula]
should be for SESs *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles on X. Then, for all $k \in \mathbb{N}$, the Whitney sum formula holds:*

$$\underline{\mathrm{at}}^k_{\mathcal{E}_1\oplus\mathcal{E}_2} = \sum_{i+j=k} \underline{\mathrm{at}}^i_{\mathcal{E}_1}\,\underline{\mathrm{at}}^j_{\mathcal{E}_2}.$$ ⌟

Proof. **TODO!** □

5.3 Pullbacks

Lemma 5.3.1 [Functoriality under pullbacks]
Let $f\colon Y \to X$ be holomorphic (where Y is paracompact, and has a cover $\mathcal{V}$ satisfying the same properties as $\mathscr{U}$), and let $\mathcal{E}$ be a vector bundle on X. Then, for all $k \in \mathbb{N}$,

$$f^*(\underline{\mathrm{at}}_{\mathcal{E}}^k) = \underline{\mathrm{at}}_{(f^*\mathcal{E})}^k.$$ ⌟

Proof. **TODO! – need to talk about derived pullbacks (and flatness of multiplication of an SES by a scalar?)** □

5.4 The splitting principle

$\mathbb{P}^1$-homotopy invariance of cohomology theory

Theorem 5.4.1
Let $\mathcal{E}$ be a vector bundle on X. Then the trace of the k-th Atiyah class is the k-th Chern class:

$$\mathrm{tr} \int_{|\Delta|} \underline{\mathrm{at}}_{\mathcal{E}}^k \cong \mathrm{c}_k(\mathcal{E}) \in H^k(\Omega_X^{\geqslant k})???.$$ ⌟

Proof. **splitting principle TODO!** □

5.5 Simplicial Atiyah classes

6 Transgression

This will be used in Part IV.

Part IV
Vector bundles and Deligne cohomology

[Nie09; BM96; Gre80]

really give a good introduction to deligne cohomology in terms of differential cohomology and ∞-Chern-Weil theory, e.g. **WARNING: this is all *smooth*** `https://ncatlab.org/nlab/show/Deligne+cohomology#interpretation_in_terms_of_higher_parallel_transport` and `https://ncatlab.org/nlab/print/infinity-Chern-Weil+theory+introduction#ConnectionOnPrincipalBundle`

Part V
Twisting cochains

7 Holomorphic twisting cochains

Here we discuss and summarise (certain sections of) two seminal papers by Toledo and Tong. Each provides a slightly different viewpoint on why twisting cochains are interesting and useful, and [OTT81] is a very good demonstration of some of their applications (and also deals with the definitions in a slightly more abstract manner, using spectral sequences to discuss various properties).

7.1 Vector bundles and graded vector spaces

We start with [TT78] instead of the earlier [TT76] because, apart from a slight reversal of definitions*, this approach closely resembles what we wish to define in Section 9.

Let $V^\bullet = \{V_\alpha^\bullet\}$ be a collection of graded finite-dimensional $\mathbb{C}$-vector spaces† and write $L^\bullet(V) = \mathrm{End}^\bullet(V)$, where $\mathrm{End}^q(V)$ consists of degree-q endomorphisms. If we let X be some paracompact complex-analytic N-manifold then we define $\underline{V}^\bullet = \mathcal{O}_X \otimes_{\mathbb{C}} V^\bullet$ and $\underline{L}^\bullet(V) = \mathcal{O}_X \otimes_{\mathbb{C}} \mathrm{End}^\bullet(V)$. Then we let $\mathscr{U} = \{U_\alpha\}$ be a 'sufficiently-nice‡' open cover of X, where the α correspond to the α for $V^\bullet$ (i.e. there is a $V_\alpha^\bullet$ for each U_α).

Finally, for any complex of vector bundles $K^\bullet$, we define a Čech-style complex $\check{\mathscr{C}}_{\mathscr{U}}^\bullet(K^\circ)$ by letting an element of $\check{\mathscr{C}}_{\mathscr{U}}^p(K^q)$ be a Čech p-cochain c whose value $c_{\alpha_0\ldots\alpha_p}$ on the p-simplex $U_{\alpha_0\ldots\alpha_p}$ lies in $K^q(U_{\alpha_0})|_{\alpha_0\ldots\alpha_p}$. The Čech differential on this complex is actually a ***deleted*** Čech differential, written $\hat{\check{\delta}}$ instead of the usual $\check{\delta}$, i.e. it is the usual alternating sum of the cochain on $\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p$ but where we start from $i=1$, ***not*** $i=0$, and go up to $i=p+1$. This is natural, because we have defined our cochains to take values in $K(U_{\alpha_0})$, and something of the form $c_{\alpha_1\ldots\alpha_{p+1}}$ does ***not*** live here. This double complex also has a natural multiplication structure§ given by $(c^{p,s}\cdot d^{q,t})_{\alpha_0\ldots\alpha_{p+q}} = (-1)^{sq} c^{p,s}_{\alpha_0\ldots\alpha_p} d^{q,t}_{\alpha_p\ldots\alpha_{p+q}}$. This

*That is, here we start with some graded vector space and define a complex structure on it by defining a differential, whereas our situation later is that we ***already have*** a collection of complexes and wish to use the differential that comes with them. There is no real mathematical difference, but conceptually the approaches are slightly different.

†That is, $V_\alpha = \bigoplus_{i\in\mathbb{N}} V_{i,\alpha}$.

‡In particular, $\mathscr{U}$ is Stein, locally compact, and, if we are working with some locally-free sheaf $\mathcal{F}$ of rank r, assumed to be such that $\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_X^r$.

§In fact, it often has an algebra structure, but this isn't too important for us here.

structure plays nicely with all the differentials involved.

Now, a holomorphic vector bundle on X is defined by a holomorphic 1-cocycle with coefficients in $\mathrm{GL}(n,\mathbb{C})$, i.e. the transition maps on overlaps. We generalise this slightly by forgetting the grading on V and giving a 1-cochain $\{\mathrm{a}_{\alpha\beta}\}_{\alpha,\beta} \in \check{\mathscr{C}}^1_{\mathscr{U}}\left(\underline{L}(V)\right)$ satisfying the cocycle condition

$$\mathrm{a}_{\alpha\beta} \cdot \mathrm{a}_{\beta\gamma} = \mathrm{a}_{\alpha\gamma} \tag{7.1.1}$$

and *also* satisfying $\mathrm{a}_{\alpha\alpha} = \mathrm{id}$ (to ensure that our cochains take values in $\mathrm{GL}(V)$).

But note that Equation (7.1.1) is equivalent to

$$\hat{\delta}\mathrm{a} + \mathrm{a} \cdot \mathrm{a}, \tag{7.1.2}$$

and *this* is the equation that we are going to generalise. Since we forgot the grading on $V^\bullet$ we can think of a as $\mathrm{a}^{1,0}$, i.e. a 1-cocycle of degree 0. This leads to the following definition.

Definition 7.1.3 [Twisting cochain]
A *twisting cochain* over $V^\bullet$ is an element

$$\mathrm{a} = \sum_{k\geqslant 0} \mathrm{a}^{k,1-k}$$

of total degree 1 in the double complex*, where $\mathrm{a}^{k,1-k} \in \check{\mathscr{C}}^k_{\mathscr{U}}\left(\underline{L}^{1-k}(V)\right)$, such that

$$\hat{\delta}\mathrm{a} + \mathrm{a} \cdot \mathrm{a} = 0$$
$$\mathrm{a}^{1,0}_{\alpha\alpha} = \mathrm{id}.$$
⌟

Note 7.1.4
In the double complex, multiplication is *not* simply done component-wise. Instead, we take all possible combinations: $(\mathrm{a} \cdot \mathrm{b})^{k,k'} = \sum_{m+n=k,m'+n'=k'} \mathrm{a}^{m,m'} \cdot \mathrm{b}^{n,n'}$. ⌟

By looking at the equations in Definition 7.1.3 seperately, in each Čech degree, we can see why this is a 'good' generalisation.

- For $p = 0$ we simply get that $\mathrm{a}^{0,1}_\alpha \mathrm{a}^{0,1}_\alpha = 0$, that is, $\mathrm{a}^{0,1}_\alpha$ is an $\mathcal{O}_X$*-linear differential on the complex* $\underline{V}^\bullet|_{U_\alpha}$.
- For $p = 1$ we get that $\mathrm{a}^{0,1}_\alpha \mathrm{a}^{1,0}_{\alpha\beta} = \mathrm{a}^{1,0}_{\alpha\beta} \mathrm{a}^{0,1}_\beta$, that is, over $U_{\alpha\beta}$ the map
$$\mathrm{a}^{1,0}_{\alpha\beta} \colon (\underline{V}^\bullet|_{U_{\alpha\beta}}, \mathrm{a}^{0,1}_\beta) \to (\underline{V}^\bullet|_{U_{\alpha\beta}}, \mathrm{a}^{0,1}_\alpha)$$
is a *chain map of complexes.*

*We don't have any negative values of k because by our definition the Čech degree of any element must be at least zero.

- For $p = 2$, in the case of degenerate simplices (α, β, α) and (β, α, β), we see that $\mathrm{a}^{1,0}_{\alpha\beta}$ and $\mathrm{a}^{1,0}_{\beta\alpha}$ are ***chain homotopic inverses***, that is, the chain map $\mathrm{a}^{1,0}_{\alpha\beta}$ is a ***quasi-isomorphism***. For general simplices, we get that $\mathrm{a}^{1,0}_{\alpha\gamma}$ and $\mathrm{a}^{1,0}_{\alpha\beta}\mathrm{a}^{1,0}_{\beta\gamma}$ are chain homotopic.

- For $p \geqslant 3$ we get 'higher-order homotopy gluings'.*

In an ideal world we would hope to be able to glue the complexes $V^\bullet|_{U_\alpha}$ and $V^\bullet|_{U_\beta}$ in order to obtain some global object, but if this isn't possible then the next best thing to hope for is a twisting cochain, which lets us glue things 'up to homology'. To make this precise, the data of a twisting cochain a gives us

- $\mathcal{H}^\bullet_\alpha = H^\bullet(\underline{V}^\bullet|_{U_\alpha})$, which is a complex of locally-free sheaves;

- over each intersection $U_{\alpha\beta}$ we get an isomorphism $H(\mathrm{a}^{1,0}_{\alpha\beta})\colon \mathcal{H}^\bullet_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} \mathcal{H}^\bullet_\alpha|_{U_{\alpha\beta}}$ ***such that*** over $U_{\alpha\beta\gamma}$ we have the isomorphism $H(\mathrm{a}^{1,0}_{\alpha\gamma}) \cong H(\mathrm{a}^{1,0}_{\alpha\beta})H(\mathrm{a}^{1,0}_{\beta\gamma})$;

- some higher order gluing conditions.

The above data lets us define a complex of coherent sheaves $\mathcal{H}^\bullet(\mathrm{a})$. If we further have that $V^i = 0$ for $i > 0$, $\mathcal{H}^0(\mathrm{a}) \cong \mathcal{F}$ for some coherent sheaf $\mathcal{F}$, and $\mathcal{H}^i(\mathrm{a}) = 0$ otherwise, then we can think of the collection $\{(\underline{V}^\bullet|_{U_\alpha}, \mathrm{a}^{0,1}_\alpha)\}_\alpha$ as a ***local resolution of $\mathcal{F}$ by locally-free sheaves.***

Note 7.1.5

An important fact is that we can build a twisting cochain inductively by lifting identity maps and then constructing the higher $\mathrm{a}^{k,1-k}$ to satisfy the equation $\hat{\delta}\mathrm{a} + \mathrm{a} \cdot \mathrm{a} = 0$. This is covered in detail in [OTT81, p. 230] and [TT76, Lemma 8.13]. ⌟

Finally, we talk about the total complex structure that we get from a twisting cochain. It is not too hard (but is certainly tiring) to show that the conditions for a to be a twisting cochain are equivalent to the conditions for the morphism $D_\mathrm{a}\colon c \mapsto \hat{\delta}c + \mathrm{a} \cdot c$ to be a (degree-1) differential on $\mathrm{Tot}^\bullet \check{\mathscr{C}}^p_{\mathscr{U}}(\underline{V}^q)$, i.e. $D_\mathrm{a}^2 = 0$.

Why would we want to consider a differential of this form? Well, conceptually, this looks like a first order perturbation of our deleted Čech differential: this is the point of view that we consider in Section 7.2. As a more concrete reason though, we can work through two specific examples.

(i) (Vector bundle: $\underline{V}$ ungraded and $\mathrm{a} = \mathrm{a}^{1,0}$). Returning to the example that we started with, we see that here

$$(D_\mathrm{a} c^p)_{\alpha_0 \ldots \alpha_{p+1}} = \mathrm{a}^{1,0}_{\alpha_0\alpha_1} c^p_{\alpha_1 \ldots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c^p_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}.$$

*Whatever this might mean, formally.

But recall that $\mathrm{a}^{1,0}_{\alpha_0\alpha_1}$ is a quasi-isomorphism, and so gives us a way of thinking of $c^p_{\alpha_1\ldots\alpha_{p+1}}$ as living in $\underline{L}(U_{\alpha_0})|_{\alpha_0\ldots\alpha_{p+1}}$, i.e. it fixes the problem we had with not being able to define the 'real' Čech differential on our complex and having to use the deleted one: D_{a} 'is' $\check{\delta}$.

(ii) (Complex of vector bundles: $\underline{V}^\bullet$ graded, $\mathrm{a} = \mathrm{a}^{0,1} + \mathrm{a}^{1,0}$). Here we consider the case when we have a ***real*** resolution (i.e. no higher-order gluing conditions) of some (coherent) sheaf by locally-free sheaves. Recalling that $\mathrm{a}^{0,1}$ is the differential (written d_V) on $V^\bullet$, we see that

$$D_{\mathrm{a}} c^{p,\bullet} = \check{\delta} c^{p,\bullet} + (-1)^p \mathrm{d}_V c^{p,\bullet}.$$

That is, we recover the usual total Čech bicomplex with differential $\check{\delta} \pm \mathrm{d}_V$.

Interestingly enough, we can actually start from this total differential and define twisting cochains from there. This is the approach that we study in Section 7.2.

7.2 Connections

Twisting cochains originally arose in a more general setting, for general bigraded algebras, in [TT76, §8], and were defined starting from the total differential D_{a} that we defined at the end of Section 7.1. The setting is as follows.

Let $A = \bigoplus_{p,q} A^{p,q}$ be a bigraded $\mathbb{C}$-algebra and $M = \bigoplus_{p,q} M^{p,q}$ a bigraded A-module (and thus also a bigraded $\mathbb{C}$-module). We assume that A and M are both zero for $p < 0$ and only non-zero for finitely-many q. Further, we take M to be a faithful A-module*: if $a \cdot m = 0$ for all m then $a = 0$.

Now take two $\mathbb{C}$-linear maps of bidegree $(1,1)$, say $D = D' + D''$ on A and $\nabla = \nabla' + \nabla''$ on M, where D' and ∇' are of bidegree $(1,0)$ and D'' and ∇'' are of bidegree $(0,1)$. Further, take both D and ∇ to be derivations:

$$D(a \cdot b) = (Da) \cdot b + (-1)^{|a|} a \cdot (Db)$$
$$\nabla(a \cdot m) = (Da) \cdot m + (-1)^{|a|} a \cdot (\nabla m)$$

where $|a| = p + q$ for $a \in A^{p,q}$. We assume also that D'' and ∇'' are differentials (i.e. square to zero), but we place ***no*** such assumption on D' and ∇'. In fact, in general, $(\nabla')^2$ will be a non-zero endomorphism of bidegree $(2,0)$, but we ***do*** assume that there exists some

*Another way of saying this is that we can identify A with a subalgebra of $\mathrm{End}_{\mathbb{C}}(M)$.

$k \in A^{2,0}$ such that $(\nabla')^2 m = k \cdot m$ for all m. Our final assumption is that the following anti-commutativity relations are satisfied:

$$\nabla'\nabla'' = -\nabla''\nabla$$
$$D'D'' = -D''D'.$$

A situation that will often arise when we have maps D and ∇ satisfying the properties above is that we want to modify ∇ by some element of A to obtain a true differential (we justify this statement shortly). Explicitly, we want to find some $\mathrm{a} = \sum_{i\geqslant 0} \mathrm{a}^{i,1-i}$ such that $\nabla_{\mathrm{a}}^2 = 0$, where we define $\nabla_{\mathrm{a}} m = \nabla m + a \cdot m$ satisfies . Note that

$$\nabla_{\mathrm{a}}^2 m = \nabla^2 m + \mathrm{a} \cdot \nabla m + \nabla(\mathrm{a} \cdot m) + \mathrm{a} \cdot \mathrm{a} \cdot m.$$

Using the derivation property and the definition of k we see that

$$\nabla_{\mathrm{a}}^2 m = (D\mathrm{a} + \mathrm{a} \cdot \mathrm{a} + k) \cdot m$$

for all m, and thus, since M is faithful, $\nabla_{\mathrm{a}}^2 = 0$ if and only if $D\mathrm{a} + \mathrm{a} \cdot \mathrm{a} + k = 0$. This leads to the following definition.

Definition 7.2.1 [Twisting cochain]
A *twisting cochain* for (M,∇) is an element $\mathrm{a} = \sum_{i\geqslant 0} \mathrm{a}^{i,1-i} \in A$ such that

$$D\mathrm{a} + \mathrm{a} \cdot \mathrm{a} = -k.$$

The *twisted complex* associated to a is the complex (M, ∇_{a}). ⌟

As for the justification for this definition, we look to differential geometry: if we think of ∇ as a connection then k corresponds to the curvature, and the definition $\nabla_{\mathrm{a}} = \nabla + \mathrm{a}$ looks like the local definition of a connection, i.e. it is just ∇ modified by an operator *of order zero*, and the fact that a is a twisting cochain tell us that this new connection ∇_{a} is *flat*. Also note that, when $k = \mathrm{a} = 0$, the twisted complex is simply our original bicomplex.

An important thing to note is that we have the following progression of relations:

(i) $\nabla^2 m = k \cdot m$ for all m;

(ii) $(D^2 a) \cdot m = \nabla^2(a \cdot m) - a \cdot \nabla^2 m = (k\dot{a} - a \cdot k) \cdot m$ for all m, by using the derivation assumption;

(iii) $D^2 a = k \cdot a - a \cdot k$, from the above, since M is faithful;

(iv) $k \cdot \nabla m = Dk \cdot m + k \cdot \nabla m$ for all m, by writing $\nabla^3 m$ in two ways (using the anti-commutativity relations and by using the derivation assumption);

(v) $Dk = 0$, from the above, since M is faithful.

Although we do not make use of these here, we state them because they are vital to the fact that, under sufficiently nice conditions*, we can always inductively construct a twisting cochain (see [TT76, Lemma 8.13]).

Finally, we briefly mention the ***Maurer-Cartan*** equations. If $(\mathfrak{g}, [-,-], \partial)$ is a dg-Lie algebra† with homological grading (i.e. ∂ is a differential of degree -1) then a ***Maurer-Cartan element*** is some $\mathrm{a} \in \mathfrak{g}$ of degree -1 such that

$$\partial \mathrm{a} + \frac{1}{2}[\mathrm{a}, \mathrm{a}] = 0.$$

! say at least a little bit more than this…

8 DG-categories

8.1 Definitions and motivation

One of the motivations behind the definition of ***twisted complexes over a dg-category*** found in [BK91] is the following: given a dg-category $\mathcal{A}$, what is the 'smallest' dg-category $\mathcal{A}'$ into which $\mathcal{A}$ embeds such that we can define a shift and functorial cones in $\mathcal{A}'$. It turns out that $\mathcal{A}'$ is exactly the category of twisted complexes over $\mathcal{A}$. Further, if $\mathcal{A}$ is ***pretriangulated***‡ then this embedding is a quasi-equivalence§ of dg-categories. This lets us 'pull back' the shift and cones from $\mathcal{A}'$ to ones in $\mathcal{A}$, and it turns out that the homotopy category of $\mathcal{A}$ is triangulated with this shift functor and these cones.

We refer the interested reader to the original paper for thorough definitions of this category and formalisations of the above suggestive motivations, and give only the definition of a twisted complex to see how it relates to what we have previously discussed.

Definition 8.1.1 [Twisted complex]
Let $\mathcal{A}$ be a dg-category. Then a ***twisted complex*** $\mathfrak{C}$ ***over*** $\mathcal{A}$ is a collection

$$\mathfrak{C} = \{(E_i \in \mathcal{A})_{i\in\mathbb{Z}},\ q_{ij}\colon E_i \to E_j\}$$

*Roughly speaking, we want A to be D''-acyclic.

†This can be phrased in terms of DG-categories, but is also equivalently a Lie algebra along with a differential that acts as a graded derivation with respect to the Lie bracket.

‡See [BK91, §1].

§That is, it induces an equivalence of the homotopy categories where only finitely-many of the E_i are non-zero, and the q_{ij} are morphisms of degree $i-j+1$ that satisfy

$$\mathrm{d}q_{ij} + \sum_s q_{sj} q_{is} = 0. \tag{8.1.2}$$

⌟

8.2 Twisted complexes and twisting cochains

The twisting cochains of Toledo and Tong, as defined in Section 7, are a specific example of the twisted complexes of Bondal and Kapranov, as we now show.

Using the notation from Section 7, we start with the data of a graded holomorphic vector bundle $\mathcal{E}_\alpha$ on every U_α and write $\mathcal{E} = \{\mathcal{E}_\alpha\}$. Let $B = \check{\mathscr{C}}^\bullet_{\mathscr{U}}(\mathcal{O}_X)$ and $E_0 = \check{\mathscr{C}}^\bullet_{\mathscr{U}}(\mathcal{E})$. Now, over each U_α we can consider the graded $\mathcal{O}_{U_\alpha}$-module $\mathcal{E}_\alpha$ as a complex $\mathcal{E}^\circ_\alpha$ with trivial differential, and so E_0 is a B-dg-module.

If we then define $E_i = 0$ for $i \neq 0$ we can ask what it means for $\{E_i\}_{i\in\mathbb{Z}}$ to have the structure of a twisted complex: we need to provide a degree-1 B-linear endomorphism $\mathrm{a} = q_{00}$ of E_0 such that $\mathrm{da}+\mathrm{a}\cdot\mathrm{a} = 0$. But the dg-algebra $\mathrm{End}_B(E_0)$ is exactly* $\check{\mathscr{C}}^\bullet_{\mathscr{U}}(\mathrm{End}(\mathcal{E})^\circ)$. Decomposing a into $\mathrm{a}^{k,1-k} \in \check{\mathscr{C}}^k_{\mathscr{U}}\left(\mathrm{End}^{1-k}(\mathcal{E})\right)$ we recover exactly the condition of Toledo and Tong. Thus any holomorphic twisting cochain gives us a twisted complex in the dg-category of B-dg-modules.

It is ***not*** the case, however, that by picking the 'right' dg-category $\mathcal{A}$ we recover the definition of holomorphic twisting cochains from that of twisted complexes. In particular, twisting cochains are twisted complexes with only ***one*** non-zero object. Further, this object E_0 is not an arbitrary B-module: it comes from $\check{\mathscr{C}}_{\mathscr{U}}(\mathcal{E})$, where $\mathcal{E}$ is graded.

8.3 DG-enhancement

The idea of holomorphic twisting cochains can be used to construct a dg-enhancement of the derived category of perfect complexes of $\mathcal{O}_X$-modules on X, as shown in [Wei16b]. There is only one difference in the definition when compared to that of Toledo and Tong: the identity condition on $\mathrm{a}^{1,0}_{\alpha\alpha}$ holds ***only up to chain homotopy.*** In a sense this is a more natural requirement, and it also guarantees that we can construct mapping cones in the category of twisting cochains: see [Wei16b, Remark 2.15].

Recall that a complex $S^\bullet$ of $\mathcal{O}_X$-modules on a locally ringed space $(X,\mathcal{O}_X)$ is ***perfect*** if, for any $x \in X$, there exists a bounded complex $\mathcal{E}^\bullet_{U_x}$ of vector bundles on some open neighbourhood U_x of x, together with a quasi-isomorphism $\mathcal{E}^\bullet_{U_x} \xrightarrow{\sim} S^\bullet|_{U_x}$. Let $\mathrm{Qcoh}_{\mathrm{perf}}(X)$

*[OTT81, §1] shows that we get a module structure of $\check{\mathscr{C}}_{\mathscr{U}} \bullet (\mathcal{E}^\circ)$ over the algebra $\check{\mathscr{C}}^\bullet_{\mathscr{U}}\left(\mathrm{End}^\circ(\mathcal{E})\right)$

be category of perfect complexes of quasi-coherent sheaves; $\mathrm{D}_{\mathrm{perf}}\big(\mathrm{QCoh}(X)\big)$ be the triangulated subcategory (of the derived category of complexes of quasi-coherent sheaves) consisting of perfect complexes; and $\mathrm{D}_{\mathrm{perf}}(X)$ be the triangulated subcategory (of the derived category of complexes of $\mathcal{O}_X$-modules) consisting of perfect complexes.

Note 8.3.1

When working in the complex-analytic setting we need to use *Fréchet quasi-coherent sheaves*: see [Wei16b, Remark 6.4]. ⌟

One of the main results of [Wei16b] is the construction of a sheafification functor

$$\mathcal{S}\colon \mathrm{Tw}_{\mathrm{perf}}(X) \to \mathrm{QCoh}_{\mathrm{perf}}(X)$$

that, under 'reasonable conditions', induces an equivalence of categories

$$\mathcal{S}\colon \operatorname{Ho}\mathrm{Tw}_{\mathrm{perf}}(X) \to \mathrm{D}_{\mathrm{perf}}\big(\mathrm{QCoh}(X)\big)$$

and (under further conditions) an equivalence

$$\mathcal{S}\colon \operatorname{Ho}\mathrm{Tw}_{\mathrm{perf}}(X) \to \mathrm{D}_{\mathrm{perf}}(X).$$

This gives us our desired dg-enhancement of the derived category of perfect complexes of $\mathcal{O}_X$-modules on X.

8.4 A-infinity categories

move this to the introduction?

Here we cheat: we refer the reader to two different sources that cover various aspects of the idea of twisted complexes in the setting of A_∞-categories.

(i) [Kel01]: This talks about twisted complexes in the sense of Bondal and Kapranov and shows how they are equivalently characterised by being the objects through which we factor the Yoneda embedding from an A_∞-category $\mathcal{A}$ to the category of $\mathcal{A}$-modules.

(ii) [Fao15]: There is a construction called the *simplicial nerve* for an A_∞-category that mirrors the classical nerve construction. Using the idea of pretriangulated dg-categories (that comes from the definition of twisted complexes) found in [BK91], this paper shows that the simplicial nerve of $\mathcal{A}$ is a stable ∞-category (in the sense of [Lur16]) whenever $\mathcal{A}$ is pretriangulated. It is interesting to compare this construction to that found in [Lur16, p. 1.3.1.6].

surely you know at least a little bit more than this now? come on mate

Part VI
Coherent sheaves

9 Green's resolution

9.1 Definitions

Note 9.1.1
We use almost the same definitions and notations as in Section 7, but with a few important differences. To avoid confusion, we redefine everything. ⌟

Let X be a paracompact complex-analytic N-manifold with sheaf of holomorphic functions $\mathcal{O}_X$, and let $\mathscr{U} = \{U_\alpha\}$ be a sufficiently-nice* open cover. Suppose that over each U_α we have a finite-length complex $(\mathcal{E}_\alpha^\bullet, \mathrm{d}_\alpha)$ of locally-free $\mathcal{O}_{U_\alpha}$-modules. Define

$$\operatorname{End}^q(\mathcal{E})|_{U_{\alpha_0\ldots\alpha_p}} = \left\{ \left(f^i \colon \mathcal{E}^i_{\alpha_p}|_{U_{\alpha_0\ldots\alpha_p}} \to \mathcal{E}^{i+q}_{\alpha_0}|_{U_{\alpha_0\ldots\alpha_p}} \right)_{i\in\mathbb{Z}} \middle| \mathrm{d}_{\alpha_p} \circ f^i = f^{i+1} \circ \mathrm{d}_{\alpha_0} \right\}.$$

There are two important differences here when compared to Section 7.1:

(i) the maps go from $\mathcal{E}_{\alpha_p}$ to $\mathcal{E}_{\alpha_0}$;

(ii) the maps are degree-q *chain maps*, i.e. they are 'true' maps of complexes and respect the differentials.

However, when using a twisting cochain to construct a total differential (as at the end of Section 7.1), we get the same result whether we use this definition or that in Section 7.1.

Next we set

$$\check{\mathscr{C}}^p_{\mathscr{U}}\left(\operatorname{End}^q(\mathcal{E})\right) = \prod_{\substack{(\alpha_0\ldots\alpha_p) \text{ s.t.} \\ U_{\alpha_0\ldots\alpha_p}\neq\varnothing}} \operatorname{End}^q(\mathcal{E})|_{U_{\alpha_0\ldots\alpha_p}}$$

and define our deleted Čech differential *almost* exactly as before:†

$$\hat{\delta} \colon \check{\mathscr{C}}^p_{\mathscr{U}}\left(\operatorname{End}^q(\mathcal{E})\right) \to \check{\mathscr{C}}^{p+1}_{\mathscr{U}}\left(\operatorname{End}^q(\mathcal{E})\right)$$
$$(\hat{\delta} c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i c_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}}|_{U_{\alpha_0\ldots\alpha_{p+1}}},$$

*Locally finite and Stein.

†This is really a natural modification to make, since we need to end up with a map from $\mathcal{E}_{\alpha_p}$ to $\mathcal{E}_{\alpha_0}$.

so the sum only goes from $i = 1$ to p, missing out both $i = 0$ *and* $i = p+1$. We can further make this bicomplex into an $\mathcal{O}_X$-module by defining the obvious multiplication:

$$(c^{p,q} \cdot d^{r,s})_{\alpha_0 \ldots \alpha_{p+r}} = (-1)^{qr} c^{p,q}_{\alpha_0 \ldots \alpha_p} d^{r,s}_{\alpha_p \ldots \alpha_{p+r}}.$$

Definition 9.1.2 [Holomorphic twisting cochain]
A *holomorphic twisting cochain* is an element $\mathfrak{a} = \sum_{k \geqslant 0} \mathfrak{a}^{k,1-k} \in \operatorname{Tot}^1 \check{\mathscr{C}}^\bullet_{\mathscr{U}}\big(\operatorname{End}^\bullet(\mathcal{E})\big)$ such that

(i) $\mathfrak{a}^{0,1}_\alpha = \mathrm{d}_\alpha$;

(ii) $\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0$. ⌟

Note 9.1.3
With Green's definition there is no requirement for $\mathfrak{a}^{1,0}$ to be the identity as of yet, but he imposes so in his definition of a holomorphic twisted resolution. ⌟

Definition 9.1.4 [Holomorphic twisted resolution]
Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. Then a *holomorphic twisted resolution of* $\mathcal{F}$ is a triple $(\mathscr{U}, \mathcal{E}^\bullet, \mathfrak{a})$ such that the following conditions are satisfied:

(i) $\mathscr{U} = \{U_\alpha\}$ is a locally-finite open Stein cover of X;

(ii) $\mathcal{E}^\bullet = (\mathcal{E}^\bullet_\alpha, \mathrm{d}_\alpha)$ is a collection* of local locally-free resolutions of $\mathcal{F}$ over each U_α of globally-bounded length;†

(iii) $\mathfrak{a}$ is a holomorphic twisting cochain *over* $\mathcal{F}$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{E}^\bullet \arrow[rr, "\mathfrak{a}^{1,0}"] \arrow[dr] & & \mathcal{E}^\bullet \arrow[dl] \\ & \mathcal{F} & \end{tikzcd}$$

(iv) on degenerate simplices of the form $\alpha = (\alpha_0 \ldots \alpha_p)$ with $\alpha_i = \alpha_{i+1}$ for some i, we have that $\mathfrak{a}^{1,0}_\alpha = \mathrm{id}$ and $\alpha^{k,1-k} = 0$ for $k > 1$. ⌟

This last condition is the identity condition that we've seen before (ensuring that we take values in some $\mathrm{GL}(n)$), since requiring $\mathfrak{a}^{k,1-k}$ to be zero for $k > 1$ just means that we want $\mathfrak{a}^{1,0}$ to be the identity 'on the nose'.

*There is a lot to gain from noting that $E^\bullet$ is more than just a collection: using the nerve of the cover we can realise it as a simplicial object of some sort.

†That is, each $\mathcal{E}^\bullet_\alpha$ is a resolution of $\mathcal{F}|_{U_\alpha}$ by locally-free $\mathcal{O}_{U_\alpha}$-modules. Further, there exists some $B \in \mathbb{N}$ such that every $\mathcal{E}^\bullet_\alpha$ is of length no more than B.

Note 9.1.5
[TT76, Lemma 8.13] and [TT78, Lemma 2.4] both show existence of a holomorphic twisting resolution when $\mathcal{F}$ is coherent (the latter shows a stronger result using the Hilbert syzygy theorem: we can ensure that our global-length bound B is no more than the dimension of X). ⌟

Definition 9.1.6 [Elementary sequence]
Given a ring R, we say that a sequence $0 \to M_r \to \ldots \to M_0 \to 0$ is *elementary* if it a sum of terms of the form $0 \to M \xrightarrow{\mathrm{id}} M \to 0$ for some R-modules M. ⌟

Theorem 9.1.7 [Green's simplicial resolution]
Let $\mathcal{S}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Let $(\mathscr{U}, \mathcal{E}^\bullet, \mathfrak{a})$ be a holomorphic twisted resolution of $\mathcal{S}$. Denote by $\mathcal{S}_\bullet$ the pullback of $\mathcal{S}$ to $X^\bullet_{\mathscr{U}}$. Then there exists a resolution of $\mathcal{S}_\bullet$ by simplicial sheaves of $\mathcal{O}_{X^\bullet_{\mathscr{U}}}$-modules:

$$0 \to \mathcal{F}^n_\bullet \to \ldots \to \mathcal{F}^0_\bullet \to \mathcal{S}_\bullet$$

where $n = \dim X$. Further, the $\mathcal{F}^i_\bullet$ all satisfy the following properties:

(i) $\mathcal{F}^i_\bullet$ *is locally free on each* $X^p_{\mathscr{U}}$;

(ii) $F^\bullet_0|_{U_\alpha} \cong \mathcal{E}^\bullet_\alpha$.

*Finally, for all simplices** $\gamma \leqslant \beta \leqslant \alpha$, *we have the following properties:*

(i) $\mathcal{F}^\bullet_\alpha \cong \mathcal{F}^\bullet_\beta \oplus E^\beta_\alpha$ *for some elementary sequence* E^β_α *in* $\mathcal{E}^\bullet_{\alpha_0}, \ldots, \mathcal{E}^\bullet_{\alpha_p}$;

(ii) $E^\gamma_\alpha \cong E^\gamma_\beta \oplus E^\beta_\alpha$;

(iii) over each U_α *there is the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^\bullet_\beta \arrow[r] & \mathcal{F}^\bullet_\alpha \arrow[r] & E^\beta_\alpha \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{F}^\bullet_\gamma \oplus E^\gamma_\beta \arrow[r] \arrow[u, "\wr"] & \mathcal{F}^\bullet_\gamma \oplus E^\gamma_\alpha \arrow[r] \arrow[u, "\wr"] & E^\beta_\alpha \arrow[r] \arrow[u, "\mathrm{id}"] & 0 \end{tikzcd}$$

(omitting the restriction notation), where the bottom map is induced by the natural inclusion $E^\gamma_\beta \to E^\gamma_\alpha$ *coming from* $E^\gamma_\alpha \cong E^\gamma_\beta \oplus E^\beta_\alpha$. ⌟

Proof. The explicit construction is given in [Gre80, §1]. □

*So $\alpha = (\alpha_0, \ldots, \alpha_p)$, and we write $\beta \leqslant \alpha$ to mean that β is a subsimplex of α.

Example 9.1.8 [Green's example]
Let $X = \mathbb{P}^1_{\mathbb{C}}$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the (Stein) cover $U_\alpha = X \setminus \{\infty\}$ and $U_\beta = X \setminus \{0\}$, and let $\mathcal{J} = \mathbb{I}(\{0\})$ be the sheaf of ideals corresponding to the subvariety $\{0\} \subset X$. Then $\mathcal{S} = \mathcal{O}_X/\mathcal{J}$ is a coherent sheaf.

The stalks of $\mathcal{S}$ are simple to understand: $\mathcal{S}_x = 0$ for $x \neq 0$, and $\mathcal{S}_0 = \mathbb{C}$. Thus, over U_α we have the resolution

$$\xi^\bullet_\alpha \colon 0 \to \mathcal{O}_X|_{U_\alpha} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X|_{U_\alpha} \to S|_{U_\alpha} \to 0,$$

and over U_β we have the resolution

$$\xi^\bullet_\beta \colon 0 \to 0 \to 0 \to S|_{U_\beta} \to 0.$$ ⌟

9.2 Gluing connections

TODO

Part VII

Appendices and references

Appendices

A The Dupont isomorphism (fibre integration)

Sign conventions

- We adopt the Koszul sign convention for the tensor product of two complexes:

$$a\otimes b = (-1)^{|a||b|} b\otimes a$$
$$(a\otimes b)\wedge(x\otimes y) = (-1)^{|b||x|}(a\wedge x)\otimes(b\wedge y).$$

- Write d_Y, $\mathrm{d}_{|\Delta_p|}$, and $\mathrm{d}_{|\Delta_p|\times Y^p}$ to mean the de Rham differential on Y, $|\Delta_p|$, and $|\Delta_p|\times Y^p$, respectively. Then the simplicial de Rham complex is isomorphic to the total complex of the de Rham bicomplex of simplicial forms grouped by type:

$$\left(\widetilde{\Omega}^\bullet_{Y^\circ}, \mathrm{d}_{|\Delta_\circ|\times Y^\circ}\right) \cong \left(\mathrm{Tot}^\bullet\,\Omega^{i,j}_{|\Delta_\circ|\times Y^\circ}, \mathrm{d}_{|\Delta_\circ|} + (-1)^i\mathrm{d}_Y\right).$$

  It's useful to note, however, that this total differential is simply 'the product rule': writing a type (i,j) form as $\tau\otimes\omega$ means that $\mathrm{d}(\tau\otimes\omega) = \mathrm{d}\tau\otimes\omega + (-1)^i\tau\otimes\mathrm{d}\omega$.

There is a subtlety in the above: $\mathrm{d}_{|\Delta_\circ|}+(-1)^i\mathrm{d}_Y$ is a differential on $\pi_1^{-1}\Omega^\bullet_{|\Delta_\circ|}\otimes_R\pi_2^{-1}\Omega^\bullet_{Y^\circ}$ where $\pi_1 = \pi^{-1}_{|\Delta_\circ|}$ and $\pi_2 = \pi_{Y^\circ}$ are the projection maps from the product, and

$$R = \pi_1^{-1}\mathcal{O}_{|\Delta_\circ|}\otimes\pi_2^{-1}\mathcal{O}_{Y^\circ}.$$

This differential extends to a differential on

$$\left(\pi_1^{-1}\Omega^\bullet_{|\Delta_\circ|}\otimes_R\pi_2^{-1}\Omega^\bullet_{Y^\circ}\right)\otimes_R\mathcal{O}_{|\Delta_\circ|\times Y^\circ}$$

which is exactly $\mathrm{d}_{|\Delta_\circ|\times Y^\circ}$.

Integrals of differential forms

- Since the integral of a k-form over an ℓ-manifold is only non-zero when $k = \ell$ we see that the fibre integral of some simplicial differential r-form $\omega = \{\omega_p^{i,j}\}_{p\in\mathbb{N},i+j=r}$ is determined entirely by the type $(p, r-p)$ parts on the p-simplices. That is,

$$\int_{|\Delta|} \omega = \int_{|\Delta_0|} \omega_0^{0,r} + \int_{|\Delta_1|} \omega_1^{1,r-1} + \ldots + \int_{|\Delta_r|} \omega_r^{r,0}. \tag{A.0.1}$$

- Given two simplicial forms ω and σ, we write $\omega \overset{\int_{|\Delta|}}{=} \sigma$ to mean that $\int_{|\Delta|} \omega = \int_{|\Delta|} \sigma$. We can therefore rewrite Equation (A.0.1) as

$$\omega \overset{\int_{|\Delta|}}{=} \sum_{p=0}^{r} \omega_p^{p,r-p}. \tag{A.0.2}$$

An important point is that we integrate terms of the form $\tau \otimes \omega$ and ***not*** $\omega \otimes \tau$. This makes a non-trivial difference, because

$$\int_{|\Delta_p|} \tau \otimes \omega = (-1)^{|\tau||\omega|} \int_{|\Delta_p|} \omega \otimes \tau$$

and so Equation (A.0.1) (with $r = 2k$ for some $k \in \mathbb{N}$) would have alternating signs if we swapped the order in the tensor product before integrating. However, it is only $\int_{|\Delta_p|} \tau \otimes \omega$ that gives a morphism of complexes with our choice of differentials.

Orientation

- The coordinates of the p-simplex can be written as $\{t_1, \ldots, t_p\}$ and $t_0 = 1 - \sum_{i=1}^{p} t_i$, with all t_i positively oriented. Recall that we pick the orientation to be such that $\int_{|\Delta_p|} \mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_p > 0$ and take the induced orientations on the boundaries. There is, however, a subtlety with the orientations of the boundaries.

- Suppose we have some oriented manifold M with boundary ∂M, and orientations on each connected component $(\partial M)_\alpha$ of the boundary (not necessarily the canonical ones). Define

$$\varepsilon_\alpha = \begin{cases} 1 & \text{if } (\partial M)_\alpha \text{ has the canonical orientation;} \\ -1 & \text{otherwise.} \end{cases}$$

Then, for any differential $(n-1)$-form ω on M,

$$\int_M \mathrm{d}\omega = \sum_\alpha \varepsilon_\alpha \int_{(\partial M)_\alpha} \omega.$$

Example A.0.3
Give $|\Delta_p|$ the orientation that makes $\mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_p$ positive, as above. Orientate the i-th face $(\partial|\Delta_p|)_i \simeq |\Delta_{p-1}|$ using the orientation on $|\Delta_{p-1}|$, so that $\mathrm{d}t_1 \wedge \ldots \wedge \mathrm{d}t_{p-1}$ is positive. Then $\varepsilon_i = (-1)^i$. ⌟

Fibre integration is a quasi-isomorphism

Proof of Lemma 2.2.6. We split this proof into steps, recalling Example 2.2.7.

(i) *The map* $\int_{|\Delta|}\colon \widetilde{\Omega}^r_{X^\bullet_{\mathscr{U}}} \to \bigoplus_{p=0}^r \Omega^{r-p}_{X^p_{\mathscr{U}}}$ *is a morphism of complexes.*

Let $\omega = \sum_{k\geqslant 0} \omega_k$ be a simplicial form of degree r, where $\omega_k \in \Omega^r_{|\Delta_k|\times X^k_{\mathscr{U}}}$. Then, recalling Example A.0.3,

$$\begin{aligned}
\int_{|\Delta_k|} \mathrm{d}\omega_k &= \int_{|\Delta_k|} \mathrm{d}_{|\Delta_\circ|}\omega_k + (-1)^k \int_{|\Delta_k|} \mathrm{d}_X \omega_k \\
&= \int_{\partial|\Delta_k|} \omega_k\Big|_{\partial|\Delta_k|} + (-1)^k \mathrm{d}_X\left(\int_{|\Delta_k|}\omega_k\right) \\
&= \sum_i (-1)^i \int_{|\Delta_{k-1}|} (f_i \times \mathrm{id}_{X^k_{\mathscr{U}}})^* \omega_k + (-1)^k \mathrm{d}_X\left(\int_{|\Delta_k|}\omega_k\right) \\
&= \sum_i (-1)^i \int_{|\Delta_{k-1}|} \big(\mathrm{id}_{|\Delta_k|} \times X(f_i)\big)^* \omega_{k-1} + (-1)^k \mathrm{d}_X\left(\int_{|\Delta_k|}\omega_k\right) \\
&= \check{\delta}\left(\int_{|\Delta_{k-1}|}\omega_{k-1}\right) + (-1)^k \mathrm{d}_X\left(\int_{|\Delta_k|}\omega_k\right).
\end{aligned}$$

So we have shown that fibre integration commutes with the differentials:

$$\begin{aligned}
\int_{|\Delta|} \mathrm{d}\omega_k &= \sum_{k\geqslant r}\left(\check{\delta}\left(\int_{|\Delta_{k-1}|}\omega_{k-1}\right) + (-1)^k \mathrm{d}_X\left(\int_{|\Delta_k|}\omega_k\right)\right) \\
&= \mathrm{d}\left(\int_{|\Delta|}\omega_k\right).
\end{aligned}$$

(ii) *There is a map* $\mathscr{E}\colon \bigoplus_{p=0}^{r} \Omega^{r-p}_{X^p_{\mathscr{U}}} \to \widetilde{\Omega}^r_{Y_\bullet}$.

Let $\omega \in \check{\mathscr{C}}^k_{\mathscr{U}}(\Omega^\ell_X)$.

TODO

(iii) *The map* $\mathscr{E}$ *is a morphism of complexes.*

TODO

(iv) *The maps* $\int_{|\Delta|}$ *and* $\mathscr{E}$ *are quasi-inverse.*

TODO

TODO □

Products

Lemma A.0.4
The morphism $\mathscr{E}\colon \bigoplus_{p=0}^{r} \Omega^{r-p}_{Y^p} \to \widetilde{\Omega}^r_{Y_\bullet}$ *commutes with products.* ⌟

Proof. **TODO** □

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