TD 1

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Mathematical formalism.

The objective of these exercises is for you to practise writing proofs: the exercises themselves are not supposed to be difficult, but you should write the solutions as best as you can.

Problem 1. *Prove that, for every odd integer $x \in \mathbb{Z}$, there exists some integer $y \in \mathbb{Z}$ such that $x^2 = 8y + 1$.*

Problem 2. *Let $c \neq 0$ be a real number. Show that the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = cx$ is injective.*

Problem 3. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be injective. Show that their composition $f \circ g \colon \mathbb{R} \to \mathbb{R}$ is also injective.*

Problem 4. *Fix a real number $x \neq 1$. Show, by induction, that, for every non-negative integer $n \in \mathbb{N}$, the following equality holds.*

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Dedekind cuts.

In the following exercises, $+$, $\cdot$, $<$, and $\leqslant$ all refer to operations on the *rational numbers*, and so we cannot use their properties (as we know them) on the *irrational* numbers: all we can use is their properties on $\mathbb{Q}$, as well as the properties of Dedekind cuts.

Problem 5. *Let x and y be Dedekind cuts, and define their* sum $x \oplus y$ *as*

$$x \oplus y = \{d + e \mid d \in x \text{ and } e \in y\}.$$

Prove that $x \oplus y$ is a Dedekind cut.

Problem 6. *Define $Z = \{q \in \mathbb{Q} \mid q < 0\}$ (where $0 \in \mathbb{Q}$), which we assume is a Dedekind cut. Show that, for any Dedekind cut x, we have that $x \oplus Z = x$.*

Recall that, in the lectures, we defined a total order $\preceq$ on Dedekind cuts by

$$x \preceq y \iff x \subseteq y.$$

We say that a Dedekind cut x is *non-negative* if $Z \preceq x$.

Problem 7. *Define the product $x \odot y$ of two non-negative Dedekind cuts x and y by*

$$x \odot y = Z \cup \{d \cdot e \mid d \in x \setminus Z \text{ and } e \in y \setminus Z\}.$$

Show that $x \odot y$ is a Dedekind cut.

Problem 8. *For a Dedekind cut x, we define its* negative $\ominus x$ *by*

$$\ominus x = \{-q \mid q \in \mathbb{Q} \setminus x \text{ and } q \text{ is not a minimum element of } \mathbb{Q} \setminus x\}.$$

(a) (Warm up.) *Show that $\ominus Z = Z$.*

(b) (Useful technical step.) *Show that $\mathbb{Q} \setminus x$ is* closed upwards, *i.e. if $q \in \mathbb{Q} \setminus x$ and $q' \in \mathbb{Q}$ is such that $q' > q$ then $q' \in \mathbb{Q} \setminus x$.*

(c) *Show that $\ominus x$ is indeed a Dedekind cut.*

(d) (Another technical step.) *Show that, for every $r < 0$, there exists some $d \in x$ such that $d - r \in \mathbb{Q} \setminus x$.*

(e) *Show that $x \oplus (\ominus x) = Z$.*