TD 2

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Problem 1.[1] *Let S be a non-empty subset of $\mathbb{R}$ and M an upper bound for S. Prove that $M = \sup S$ if and only if, for all $\varepsilon > 0$, there exists some $x \in S$ such that $M - \varepsilon \leqslant x \leqslant M$.*

The following exercises are all from Section 2.5 of the book.

Problem 2.[2] *We say that $\lim_{n\to\infty} a_n = +\infty$ if, for every real number $R \in \mathbb{R}$, there exists some integer $N \in \mathbb{N}$ such that $a_n > R$ for all $n \geqslant N$. Show that a divergent monotone-increasing sequence converges to infinity in this sense.*

Problem 3. *Let $x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{\ldots + \sqrt{n}}}}}$ for $n \in \mathbb{N} \setminus \{0\}$.*

(a) *Show that $x_n < x_{n+1}$ for all $n \in \mathbb{N} \setminus \{0\}$.*

(b) *Show that $x_{n+1}^2 \leqslant 1 + \sqrt{2} x_n$ for all $n \in \mathbb{N} \setminus \{0\}$.*

(c) *Hence show that $(x_n)_{n\in\mathbb{N}\setminus\{0\}}$ is bounded above by 2. Deduce that its limit as $n \to \infty$ exists.*

Problem 4.

(a) *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Define a sequence $(b_n)_{n\in\mathbb{N}}$ by*

$$b_n = \sup\{a_k \mid k \geqslant n\}.$$

Prove that (b_n) converges. (This limit is called the limit superior *of (a_n), often written as* $\limsup$.*)*

(b) *Without redoing the proof of the previous part of this exercise, show that the* limit inferior *always converges as well, where $\liminf a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geqslant n\}$.*

[1] This proof is slightly wordy and over-detailed, but this is arguably a better fault than being vague and imprecise. Think of this proof as a supremum of the set of proofs ordered by 'amount of detail necessary for a good proof'.

[2] This problem is sort of a converse to the *Bolzano–Weierstrass theorem* (a fundamental result in real analysis), and also related to (one of) the *monotone convergence theorem(s)*.