שאלה 1

א. $y_i = 1 + \beta_1 e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}} + \varepsilon_i$, $E(\varepsilon_i)=0$ $Var(\varepsilon_i)=\sigma^2$ $\forall i$, $\varepsilon_i, \varepsilon_j$ בלתי תלויים

$$\frac{\partial y_i}{\partial \beta_1} = e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}} + \frac{\beta_1 x_{1i}}{\beta_2 x_{2i}} e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}$$

$$\frac{\partial y_i}{\partial \beta_2} = \beta_1 e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}} \cdot \frac{-x_{2i}}{(\beta_2 x_{2i})^2} = \frac{-\beta_1^2 x_{1i}}{\beta_2^2 x_{2i}} \cdot e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}$$

$$W_{n\times 2} = \begin{pmatrix} \frac{\partial y_1}{\partial \beta_1}(x_{11}, x_{21}) & \frac{\partial y_1}{\partial \beta_2}(x_{11}, x_{21}) \\ \vdots & \vdots \\ \frac{\partial y_1}{\partial \beta_1}(x_{1n}, x_{2n}) & \frac{\partial y_1}{\partial \beta_2}(x_{1n}, x_{2n}) \end{pmatrix}_{n\times 2} = \begin{pmatrix} e^{\frac{\beta_1 x_{11}}{\beta_2 x_{21}}}\left(1 + \frac{\beta_1 x_{11}}{\beta_2 x_{21}}\right) & \frac{-\beta_1^2 x_{11}}{\beta_2^2 x_{21}} \cdot e^{\frac{\beta_1 x_{11}}{\beta_2 x_{21}}} \\ \vdots & \vdots \\ e^{\frac{\beta_1 x_{1n}}{\beta_2 x_{2n}}}\left(1 + \frac{\beta_1 x_{1n}}{\beta_2 x_{2n}}\right) & \frac{-\beta_1^2 x_{1n}}{\beta_2^2 x_{2n}} \cdot e^{\frac{\beta_1 x_{1n}}{\beta_2 x_{2n}}} \end{pmatrix}_{n\times 2}$$

$$e_i^{(j)} = y_i - f(\beta_j, x_i) = y_i - \left(1 + \beta_1^{(j)} e^{\frac{\beta_1^{(j)} x_{1i}}{\beta_2^{(j)} x_{2i}}}\right)$$

$$\beta^{(j+1)} = \beta^{(j)} + [W^{(j)T} \cdot W^{(j)}]^{-1} \cdot W^{(j)T} \cdot e^{(j)}$$

$$S(\beta_1,\beta_2) = \sum_{i=1}^{n} \left(y_i - F(x_{1i}, x_{2i})\right)^2 = \sum_{i=1}^{n} \left(y_i - 1 - \beta_1 \cdot e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}\right)^2$$

$$\frac{\partial S}{\partial \beta_1} = -2 \cdot \sum_{i=1}^{n} \left(y_i - 1 - \beta_1 e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}\right) \cdot e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}} \left(1 + \frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}\right)$$

$$\frac{\partial S}{\partial \beta_2} = 2 \cdot \sum_{i=1}^{n} \left(y_i - 1 - \beta_1 e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}\right) \frac{\beta_1^2 x_{1i}}{\beta_2^2 x_{2i}} \cdot e^{\frac{\beta_1 x_{1i}}{\beta_2 x_{2i}}}$$

שאלה 2

א. רגרסיית פואסון. Y_i - מס' הפעמים שהתרחש ; תלויה במשתנה מסביר

לכל תצפית i יש P משתנים מסבירים X_i. $\quad Y_i | X_i \sim Pois(\lambda_i)$

נניח כי יש שני פרמטרים: $\quad \ln(\lambda_i) = \beta_0 + \beta_1 X_i \Rightarrow \lambda_i = e^{\beta_0 + \beta_1 X_i}$

$$\Rightarrow Y_i | X_i \sim Pois\left(e^{\beta_0 + \beta_1 X_i}\right)$$

$$f_{Y_i|X_i}(y) = \frac{e^{-\lambda_i} \lambda_i^{Y_i}}{Y_i!} = \frac{e^{-e^{\beta_0 + \beta_1 X_i}} \cdot e^{Y_i(\beta_0 + \beta_1 X_i)}}{Y_i!}$$

$$L(y, \beta_0, \beta_1) = \prod_{i=1}^{n} f_{Y_i|X_i}(y) = \prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{Y_i}}{Y_i!} = \prod_{i=1}^{n} \frac{e^{-e^{\beta_0 + \beta_1 X_i}} \cdot e^{Y_i(\beta_0 + \beta_1 X_i)}}{Y_i!} =$$

$$= \frac{e^{-\sum_{i=1}^{n} e^{\beta_0 + \beta_1 X_i}} \cdot e^{\sum Y_i(\beta_0 + \beta_1 X_i)}}{\prod_{i=1}^{n} Y_i!}$$

ב. $$\ell(y, \beta_0, \beta_1) = \ln(L(y, \beta_0, \beta_1)) = -\sum_{i=1}^{n} e^{\beta_0 + \beta_1 X_i} + \sum_{i=1}^{n} Y_i(\beta_0 + \beta_1 X_i) - \sum_{i=1}^{n} \ln(Y_i!)$$

$$S(\beta_0, \beta_1) = \begin{pmatrix} \frac{\partial \ell(y, \beta_0, \beta_1)}{\partial \beta_0} \\ \frac{\partial \ell(y, \beta_0, \beta_1)}{\partial \beta_1} \end{pmatrix} = \begin{pmatrix} -\sum_{i=1}^{n} e^{\beta_0 + \beta_1 X_i} + \sum_{i=1}^{n} Y_i \\ -\sum_{i=1}^{n} X_i \cdot e^{\beta_0 + \beta_1 X_i} + \sum_{i=1}^{n} X_i Y_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

אין פתרון של נוסח סגור ל β_0 ו- β_1.

5. נגזור את השנייה בהתאם:

$$\frac{\partial^2 \ell(y,\beta_0,\beta_1)}{\partial^2 \beta_0} = \frac{\partial\left(-\sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i} + \sum_{i=1}^{n} y_i\right)}{\partial \beta_0} = -\sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i}$$

$$\frac{\partial^2 \ell(y,\beta_0,\beta_1)}{\partial \beta_0 \partial \beta_1} = \frac{\partial\left(-\sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i} + \sum_{i=1}^{n} y_i\right)}{\partial \beta_1} = -\sum_{i=1}^{n} x_i e^{\beta_0+\beta_1 x_i} = \frac{\partial^2 \ell(y,\beta_0,\beta_1)}{\partial \beta_1 \partial \beta_0}$$

$$\frac{\partial^2 \ell(y,\beta_0,\beta_1)}{\partial^2 \beta_1} = \frac{\partial\left(-\sum_{i=1}^{n} x_i \cdot e^{\beta_0+\beta_1 x_i} + \sum_{i=1}^{n} x_i y_i\right)}{\partial \beta_1} = -\sum x_i^2 e^{\beta_0+\beta_1 x_i}$$

$$H(\beta_0,\beta_1) = \begin{pmatrix} -\sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i} & -\sum_{i=1}^{n} x_i e^{\beta_0+\beta_1 x_i} \\ -\sum_{i=1}^{n} x_i e^{\beta_0+\beta_1 x_i} & -\sum x_i^2 e^{\beta_0+\beta_1 x_i} \end{pmatrix}$$

לפי אלגוריתם N-R הערכים באיטרציה הבאה (j+1) נתון על ידי המשוואה:

$$\begin{pmatrix} \beta_0^{(j+1)} \\ \beta_1^{(j+1)} \end{pmatrix} = \begin{pmatrix} \beta_0^{(j)} \\ \beta_1^{(j)} \end{pmatrix} - H(\beta_0^{(j)},\beta_1^{(j)})^{-1} \cdot S(\beta_0^{(j)},\beta_1^{(j)})^{T} =$$

$$= \begin{pmatrix} \beta_0^{(j)} \\ \beta_1^{(j)} \end{pmatrix}_{2\times1} - \begin{pmatrix} -\sum_{i=1}^{n} e^{\beta_0^j+\beta_1^j x_i} & -\sum_{i=1}^{n} x_i e^{\beta_0^j+\beta_1^j x_i} \\ -\sum_{i=1}^{n} x_i e^{\beta_0^j+\beta_1^j x_i} & -\sum x_i^2 e^{\beta_0^j+\beta_1^j x_i} \end{pmatrix}^{-1}_{2\times2} \cdot \begin{pmatrix} -\sum_{i=1}^{n} e^{\beta_0^j+\beta_1^j x_i} + \sum_{i=1}^{n} y_i \\ -\sum_{i=1}^{n} x_i \cdot e^{\beta_0^j+\beta_1^j x_i} + \sum_{i=1}^{n} x_i y_i \end{pmatrix}_{2\times1} =$$

<u>שאלה ג</u>

$$X \sim Gompartz \Rightarrow f_X(x) = b\eta \cdot e^{\eta + bx - \eta e^{bx}} \cdot \mathbb{1}_{y \geq 0} \quad b, \eta > 0$$ א.

$$L(x,\eta,b) = \prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} b\eta \cdot e^{\eta + bx_i - \eta e^{bx_i}} = (b \cdot \eta)^n \cdot e^{\sum \eta + bx_i - \eta e^{bx_i}}$$

$$l(x,\eta,b) = \ln(L(x,\eta,b)) = n(\ln(b) + \ln(\eta)) + \sum(\eta + bx_i - \eta e^{bx_i}) =$$

$$= n(\ln(b) + \ln(\eta)) + n\eta + b\sum x_i - \eta \sum e^{bx_i} =$$

$$= n\ln(b) + n\ln(\eta) + n\eta + b\sum x_i - \eta \sum e^{bx_i}$$

$$S(\eta,b) = \begin{pmatrix} \frac{\partial l(x,\eta,b)}{\partial \eta} \\ \frac{\partial l(x,\eta,b)}{\partial b} \end{pmatrix} = \begin{pmatrix} \frac{n}{\eta} + n - \sum e^{bx_i} \\ \frac{n}{b} + \sum x_i - \eta \sum x_i e^{bx_i} \end{pmatrix}$$

$$\frac{\partial^2 l(x,\eta,b)}{\partial^2 \eta} = \frac{\partial \left(\frac{n}{\eta} + n - \sum e^{bx_i}\right)}{\partial \eta} = -\frac{n}{\eta^2}$$

$$\frac{\partial^2 l(x,\eta,b)}{\partial \eta \partial b} = \frac{\partial \left(\frac{n}{\eta} + n - \sum e^{bx_i}\right)}{\partial b} = -\sum x_i e^{bx_i}$$

$$\frac{\partial^2 l(x,\eta,b)}{\partial^2 b} = \frac{\partial \left(\frac{n}{b} + \sum x_i - \eta \sum x_i e^{bx_i}\right)}{\partial b} = \frac{-n}{b^2} - \eta \sum x_i^2 e^{bx_i}$$

$$\begin{pmatrix} \eta^{j+1} \\ b^{j+1} \end{pmatrix}_{2\times 1} = \begin{pmatrix} \eta^{j} \\ b^{j} \end{pmatrix}_{2\times 1} - \begin{pmatrix} -\frac{n}{\eta^2} & \sum x_i e^{bx_i} \\ \sum x_i e^{bx_i} & \frac{-n}{b^2} - \eta \sum x_i^2 e^{bx_i} \end{pmatrix}^{-1}_{2\times 2} \cdot \begin{pmatrix} \frac{n}{\eta} + n - \sum e^{bx_i} \\ \frac{n}{b} + \sum x_i - \eta \sum x_i e^{bx_i} \end{pmatrix}_{2\times 1}$$

<u>שאלה 6</u>

ב. (i)

$$f(x,y) = (1-x)^2 + 100(y-x^2)^2$$

$$\frac{\partial f(x,y)}{\partial x} = -2(1-x) - 400x(y-x^2)$$

$$\frac{\partial f(x,y)}{\partial y} = 200(y-x^2)$$

$$\begin{pmatrix} x^{j+1} \\ y^{j+1} \end{pmatrix} = \begin{pmatrix} x^{j} \\ y^{j} \end{pmatrix} - \gamma^j \begin{pmatrix} -2(1-x^j) - 400 \cdot x^j (y^j - x^{j^2}) \\ 200(y^j - x^{j^2}) \end{pmatrix}$$

(ii) נחשב הנגזרות השניות

$$\frac{\partial^2 f(x,y)}{\partial^2 x} = 2 - 400(y-x^2) + 800x^2 = 2 - 400y + 1200x^2$$

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = -400x = \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

$$\frac{\partial^2 f(x,y)}{\partial^2 y} = 200$$

$$\begin{pmatrix} x^{j+1} \\ y^{j+1} \end{pmatrix} = \begin{pmatrix} x^{j} \\ y^{j} \end{pmatrix} - \begin{pmatrix} 2-400y^j+400x^{j^2} & -400x^j \\ -400x^j & 200 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -2(1-x^j) - 400x^j(y^j - x^{j^2}) \\ 200(y^j - x^{j^2}) \end{pmatrix}$$