Homework 2

Tate Mason

An ECON - 8070 Homework Assignment

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1 Question 6.8

1.1 Solution

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\frac{1}{n}\sum_{i=1}^{n} X_i)^2$$
$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X_n})^2$$
$$\Rightarrow \tilde{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \Rightarrow E[\tilde{\sigma}^2] = E[(X_i - \mu)^2] = \sigma^2$$

this shows that $\tilde{\sigma}^2$ is unbiased.

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 - (\bar{X_n} - \mu)^2$$
$$\hat{\sigma}^2 = \tilde{\sigma}^2 - (\bar{X_n} - \mu)^2$$

2 Question 6.14

2.1 Solution

$$E[\bar{X_n}] = \frac{1}{n}\sum_{i=1}^{n} \mu_i$$
$$\text{var}[\bar{X_n}] = \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2$$

3 Question 7.7

3.1 Solutions

(a)

$$E[\bar{X}_n^*] = \text{var}(\frac{1}{n}\sum_{i=1}^{n} w_i X_i) = \frac{1}{n^2}\sum_{n=1}^{i} w_i^2 \text{var}(X_i)$$
$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} w_i \mu$$
$$\Rightarrow \frac{\mu}{n}\sum_{i=1}^{n} w_i = \mu$$

because $\sum_{i=1}^{n} w_i = 1$.

(b)

$$\text{var}(\bar{X_n}^*) = \text{var}(\frac{1}{n}\sum_{i=1}^{n} w_i X_i) = \frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \text{var}(X_i)$$
$$X_i \text{ are i.i.d, therefore } = \frac{\sigma^2}{n^2}\sum_{i=1}^{n} w_i^2$$

(c) $\bar{X_n}^* - \mu \xrightarrow{p} 0$ if $\text{var}(\bar{X_n}^* - \mu) \to 0$ as $n \to \infty$

$$\text{var}(\bar{X_n}^* - \mu) = \text{var}(\bar{X_n}^*) = (\frac{\sigma}{n})^2 \sum_{i=1}^{n} w_i^2$$

if $\frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \to 0$ as $n \to \infty$, $\text{var}(\bar{X_i}^* - \mu) \to 0$. Therefore, $\bar{X_n}^* - \mu \xrightarrow{p} 0$

(d)

$$\frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \leq \frac{1}{n_2}(\max_{i\leq n} w_i)^2 = \frac{1}{n^2} \cdot n \cdot (\ i \leq n w_i)^2 = \frac{1}{n} \cdot (\max_{i\leq n} w_i)^2$$

if $\max_{i\leq n} w_i \to 0$ as $n \to \infty$, $\frac{1}{n}(\ i \leq n w_i)^2 \to 0$ as $n \to \infty$. Therefore, $\frac{1}{n^2}\sum_{i=1}^{n} w_i^2 \to 0$ as $n \to \infty$.

4 Question 7.8

4.1 Solution

Argument will be shown for $\bar{X}_{1n}$ but is identical to that of $\bar{X}_{2n}$

Proof. $E[\bar{X}_{1n}] = \mu$, $(\bar{X}_{1n}) = \frac{2\sigma^2}{n}$, $P(|\bar{X}_{1n} - \mu| > \epsilon) \leq \frac{\text{var}(\bar{X}_{1n})}{\epsilon^2} = (\frac{2\sigma^2}{n\epsilon^2})$. As $n \to \infty$, $(\frac{2\sigma^2}{n\epsilon^2}) \to 0$ for any fixed $\epsilon > 0$. Therefore, $P(|\bar{X}_{1n} - \mu| > \epsilon) \to 0$ as $n \to \infty$, meaning $\bar{X}_{1n}$ is consistent for μ. The same process would be applied for $\bar{X}_{2n}$, implying that $\bar{X}_{1n}, \bar{X}_{2n}$ are consistent for $E[X] = \mu$. □

5 Question 8.1

5.1 Solution

(a)

$$E[X] = 1 \cdot p(X = 1) + 0 \cdot p(X = 0) = p + 0 \cdot (1 - p) = p$$

(b)

$$\hat{p} = (\tfrac{1}{n}) \textstyle\sum_{i=1}^{n} X_i$$

(c)

$$\text{var}[\hat{p}] = \text{var}[\tfrac{1}{n} \textstyle\sum_{i=1}^{n} X_i] = \tfrac{1}{n^2} \textstyle\sum_{i=1}^{n} \text{var}[X_i]$$

because X_i are i.i.d, $= \frac{1}{n}\text{var}[X]$.

$$\text{var}[X] = p(1-p) \Rightarrow \text{var} = \tfrac{p(1-p)}{n}$$

(d) As $n \to \infty$, $\sqrt{n(\hat{p} - p)} \sim N(0, \sigma^2)$ such that $\sigma^2 = p(1-p) = \text{var}[X] \therefore$ $n \to \infty$, $\sqrt{n(\hat{p} - p)} \sim N(0, p(1-p))$.

6 Question 8.7 - (a) & (c)

6.1 Solution

(a) Let $g(\theta) = \theta^2 g'(\theta) = 2\theta$. Then, by the Delta method:

$$\sqrt{n(\hat{\theta}^2 - \theta^2)} \xrightarrow{d} N(0, 4\theta^2 v^2)$$
$$\therefore \hat{\theta}^2 \xrightarrow{d} N(\theta^2, \tfrac{4\theta^2 v^2}{n})$$

(c) Let $g(\theta) = \theta^k g'(\theta) = k\theta^{k-1}$. Then, by the delta method:

$$\sqrt{n(\hat{\theta}^k - \theta^k)} \xrightarrow{d} N(\theta^k, \tfrac{k^2\theta^{2k-2}v^2}{n})$$
$$\therefore \hat{\theta}^k \xrightarrow{d} N(\theta^k, \tfrac{k^2\theta^{2k-2}v^2}{n})$$