Homework 2

Tate Mason

ECON - 8050

Problem 1: Costs of Business Cycle

Let utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^t U(c_t)$$

where the utility function is CRRA:

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

The consumption process is

$$c_t = c_{t-1}^{\alpha}\varepsilon_t \exp(\mu)$$

where

$$\mu = \frac{-\sigma_\varepsilon^2(1-\alpha)}{2(1-\alpha^2)}, \quad \log\varepsilon_t \sim N(0,\sigma_\varepsilon^2) \text{ and i.i.d.}$$

Thus, the log of consumption follows an AR(1) process:

$$\log c_t = \mu + \alpha \log c_{t-1} + \log\varepsilon_t$$

Part A

Find the unconditional mean of c_t, $E(c_t)$. (Hint: recall the properties of the lognormal distribution).

Part B

Define lifetime utility before any uncertainty is realized as:

$$V_0 = E_{-1}\sum_{t=0}^{\infty}\beta^t U(c_t)$$

Assume c_0 is drawn from the invariant (unconditional) distribution of c. Now define:

$$V(\lambda) = E_{-1}\sum_{t=0}^{\infty}\beta^t U[c_t(1+\lambda)]$$

This is lifetime utility when every period consumption is increased by $(1+\lambda)$. Express $V(\lambda)$ as a function of $\mu, \sigma_\varepsilon^2, \alpha, \gamma, \beta$.

Part C

Denote V_0 as the lifetime utility when c_t is deterministic and equal to its unconditional mean found in part A). Find the compensation λ such that $V(\lambda) = V_0$. Find how much compensation the consumer has to be given in order to be indifferent between the stochastic and deterministic cases, Provide economic intuition.

Part D

Denote the interest rate as r. Find consumption c_t.

Problem 2: Non-Expected Utility Framework

This problem follows the Kreps and Porteus (1978), Epstein and Zin (1991), and Weil (1990) frameworks.

Let remaining lifetime utility at time t, once c_t is known, be given by v_t, satisfying:

$$v_t = \left[(1-\beta)c_t^\rho + \beta(E_t v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{1}$$

where $1-\alpha$ represents risk aversion and $1-\rho$ represents the inverse of the intertemporal elasticity of substitution. In standard expected utility, $\alpha = \rho$.

Denote pre-realization lifetime utility at time t as U_t, where:

$$U_t = (E_t v_t^\alpha)^{\frac{1}{\alpha}}$$

Part A

Prove that multiplying c_t by λ for all $t = 0, 1, \ldots, \infty$ is equivalent to multiplying v_t by λ. (Hint: start by assuming this holds, substitute into equation (1), and show v_t scales linearly.)

Part B

Suppose for all t, we replace c_t with a deterministic constant $\bar{c} = E[c_t]$. Compare welfare in this case with uncertain c_t. Specifically, find η such that multiplying c_t by $(1+\eta)$ makes ex-ante welfare U_0 equal to that in the deterministic case. Express η in terms of U_0 and $\bar{c}$.

Part C

Suppose consumption follows one of two sequences: with probability $\frac{1}{2}$, $c_t = c_l$ for all t, and with probability $\frac{1}{2}$, $c_t = c_h$ for all t. The sequence is revealed at $t = 0$. Find η and analyze its dependence on ρ and α.

Part D

Now assume c_t is i.i.d., where each period $c_t = c_l$ with probability $\frac{1}{2}$ and c_h with probability $\frac{1}{2}$.

1. Derive an implicit equation for U_0.
2. Analyze whether η depends on α and ρ.

Part E

Solve for U_0 numerically using Matlab with given parameters: $\beta = 0.95$, $c_l = e^{0.98}$, $c_h = e^{1.02}$. Compute η for:

- $\alpha = 1, 0.5, -1$
- $\rho = 1, 0.5, -1$

Report results in a table and provide economic intuition. (Hint: Use an iterative approach to solve $U_0 = f(U_0)$ until convergence with tolerance 10^{-8}.)

1 Solution 1

(A)

Given c_t has a lognormal distribution by ϵ_{t+1}, take an arbitrary mean m and variance v. In this case, $\mathbb{E}[c_t] = \exp(m + \frac{v}{2})$. From there we can solve for the mean of unconditional expected consumption in time t.

$$\mathbb{E}[c_t] = \exp(m + \frac{v}{2})$$
$$m = \mu + \alpha m + 0$$
$$m(1 - \alpha) = \mu$$
$$\therefore\ m = \frac{\mu}{1 - \alpha} \text{ in expectation}$$