Homework 3

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An ECON - 8070 Homework Assignment

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10.1

Problem

Let X be distributed Poisson: $\pi(k) = \frac{\exp(-\theta)\theta^k}{k!}$ for nonnegative integer k and $\theta > 0$.

(a) Find the log-likelihood function $l_n(\theta)$.

(b) Find the MLE $\hat{\theta}$ for θ

Solution

(a) We can obtain log likelihood by first finding the likelihood function

$$L(\theta) = \Pi_{i=1}^{n}[\tfrac{(\exp(-\theta)*\theta^{k_i})}{k_i!}]$$

Then, we take the natural log

$$l_n(\theta) = \log(L(\theta)) = \textstyle\sum_{i=1}^{n}[\log(\exp(-\theta)) + k_1\log(\theta) - \log(k_i!)]$$
$$= \textstyle\sum_{i=1}^{n}[-\theta + k_i\log(\theta) - \log(k_i!)]$$
$$\boxed{= -n\theta + \log(\theta)\sum_{i=1}^{n} k_i - \sum_{i=1}^{n}\log(k_i!)}$$

(b) Now we need to find $\hat{\theta} = \underset{\theta\in\Theta}{\arg\max}\, l_n(\theta)$. We can do this by taking the derivative of l_n w.r.t θ

$$\tfrac{dl_n(\theta)}{d\theta} = -n + \tfrac{1}{\theta}\textstyle\sum_{i=1}^{n} k_i = 0$$

Then, we solve for $\hat{\theta}$

$$\theta n = \textstyle\sum_{i=1}^{n} k_i$$
$$\boxed{\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} k_i}$$

The MLE is the sample mean.

10.6

Problem

Let X be Bernoulli $\pi(X|p) = p^x(1-p)^{1-x}$.

(a) Calculate the information for p by taking the variance of the score.

(b) Calculate the information for p by taking the expectation of (minus) the second derivative. Did you obtain the same answer?

Solution

(a) Let's start with log-likelihood:

$$l(p) = x\log(p) + (1-x)\log(p)$$

To find the score, we differentiate w.r.t. p

$$S(p) = \frac{x}{p} - \frac{(1-x)}{(1-p)}$$

Next, we will find the variance such that $[S(p)] = \mathbb{E}(S(p)^2)$

$$\mathbb{E}(s(p)^2) = \mathbb{E}(\frac{x^2}{p^2}) + \mathbb{E}(\frac{(1-x)^2}{(1-p)^2}) = \frac{1}{p} + \frac{1}{(1-p)}$$
$$\boxed{\therefore \mathcal{I} = \frac{1}{p(1-p)}}$$

(b) Second derivative

$$\frac{d^2 l(p)}{dp^2} = \frac{-x}{p^2} - \frac{(1-x)}{(1-p)^2}$$

Negative expectation

$$\begin{aligned}\mathcal{I}(p) &= -\mathbb{E}(\frac{d^2 l(p)}{dp^2} = \frac{-x}{p^2} - \frac{(1-x)}{(1-p)^2}) \\ &= \mathbb{E}(\frac{x}{p^2}) + \mathbb{E}(\frac{1-x}{(1-p)^2}) \\ &= \frac{p}{p^2} + \frac{(1-p)}{(1-p)^2} \\ &= \frac{1}{p} + \frac{1}{(1-p)} \\ &\boxed{= \frac{1}{p(1-p)}}\end{aligned}$$

11.3

Problem

A Bernoulli random variable X is

$$\mathbb{P}[X = 0] = 1 - p$$
$$\mathbb{P}[X = 1] = p$$

(a) Propose a moment estimator $\hat{p}$ for p.
(b) Find the variance of the asymptotic distribution of $\sqrt{n}(\hat{p} - p)$

Solution

(a) The first moment of a Bernoulli is the mean

$$\mathbb{E}[X] = p$$

The method of moments for p, here, is the sample mean

$$\boxed{hatp = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

(b) For a Bernoulli distribution $(X) = p(1-p)$. We can use central limit theorem to deduce $\sqrt{n(\hat{p} - p)} \to N(0, \sigma^2)$. We can put these together since $\sigma^2 = (X)$. So, the asymptotic distribution is $\sqrt{n(p\hat{p} - p)} \to N(0, p(1-p))$

11.4

Problem

Propse a moment estimator $\hat{\lambda}$ for the parameter λ of a Poisson distribution.

Solution

A Poisson distribution has distribution $[X] = \lambda$. As seen in the Bernoulli case, the moment estimator for λ is the sample mean

$$\boxed{\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

13.3

Problem

Take the exponential model with parameter λ. We want a test for $\mathbb{H}_0 : \lambda = 1$ against $\mathbb{H}_1 : \lambda \neq 1$.
(a) Develop a test based on the sample mean $\bar{X}_n$.

Solution

13.5

Problem

Take the model $XN(\mu, \sigma^2)$. Propose a test for $\mathbb{H}_0 : \mu = 1$ against $\mathbb{H}_1 : \mu \neq 1$.

Solution