

Homework 2

Tate Mason

ECON - 8050

Problem 1: Costs of Business Cycle

Let utility be given by:

$$E_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

where the utility function is CRRA:

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

The consumption process is

$$c_t = c_{t-1}^\alpha \varepsilon_t \exp(\mu)$$

where

$$\mu = \frac{-\sigma_\varepsilon^2(1-\alpha)}{2(1-\alpha^2)}, \quad \log \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \text{ and i.i.d.}$$

Thus, the log of consumption follows an AR(1) process:

$$\log c_t = \mu + \alpha \log c_{t-1} + \log \varepsilon_t$$

Part A

Find the unconditional mean of c_t , $E(c_t)$. (Hint: recall the properties of the lognormal distribution).

Part B

Define lifetime utility before any uncertainty is realized as:

$$V_0 = E_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Assume c_0 is drawn from the invariant (unconditional) distribution of c . Now define:

$$V(\lambda) = E_{-1} \sum_{t=0}^{\infty} \beta^t U[c_t(1+\lambda)]$$

This is lifetime utility when every period consumption is increased by $(1+\lambda)$. Express $V(\lambda)$ as a function of $\mu, \sigma_\varepsilon^2, \alpha, \gamma, \beta$.

Part C

Denote V_0 as the lifetime utility when c_t is deterministic and equal to its unconditional mean found in part A). Find the compensation λ such that $V(\lambda) = V_0$. Find how much compensation the consumer has to be given in order to be indifferent between the stochastic and deterministic cases, Provide economic intuition.

Part D

Denote the interest rate as r . Find consumption c_t .

Problem 2: Non-Expected Utility Framework

This problem follows the Kreps and Porteus (1978), Epstein and Zin (1991), and Weil (1990) frameworks.

Let remaining lifetime utility at time t , once c_t is known, be given by v_t , satisfying:

$$v_t = \left[(1 - \beta)c_t^\rho + \beta(E_t v_{t+1}^\alpha)^\frac{\rho}{\alpha} \right]^\frac{1}{\rho} \quad (1)$$

where $1 - \alpha$ represents risk aversion and $1 - \rho$ represents the inverse of the intertemporal elasticity of substitution. In standard expected utility, $\alpha = \rho$.

Denote pre-realization lifetime utility at time t as U_t , where:

$$U_t = (E_t v_t^\alpha)^\frac{1}{\alpha}$$

Part A

Prove that multiplying c_t by λ for all $t = 0, 1, \dots, \infty$ is equivalent to multiplying v_t by λ . (Hint: start by assuming this holds, substitute into equation (1), and show v_t scales linearly.)

Part B

Suppose for all t , we replace c_t with a deterministic constant $\bar{c} = E[c_t]$. Compare welfare in this case with uncertain c_t . Specifically, find η such that multiplying c_t by $(1 + \eta)$ makes ex-ante welfare U_0 equal to that in the deterministic case. Express η in terms of U_0 and \bar{c} .

Part C

Suppose consumption follows one of two sequences: with probability $\frac{1}{2}$, $c_t = c_l$ for all t , and with probability $\frac{1}{2}$, $c_t = c_h$ for all t . The sequence is revealed at $t = 0$. Find η and analyze its dependence on ρ and α .

Part D

Now assume c_t is i.i.d., where each period $c_t = c_l$ with probability $\frac{1}{2}$ and c_h with probability $\frac{1}{2}$.

1. Derive an implicit equation for U_0 .
2. Analyze whether η depends on α and ρ .

Part E

Solve for U_0 numerically using Matlab with given parameters: $\beta = 0.95$, $c_l = e^{0.98}$, $c_h = e^{1.02}$. Compute η for:

- $\alpha = 1, 0.5, -1$
- $\rho = 1, 0.5, -1$

Report results in a table and provide economic intuition. (Hint: Use an iterative approach to solve $U_0 = f(U_0)$ until convergence with tolerance 10^{-8} .)

1 Solution 1

(A)

If $c_t \sim N(m, v)$ for some mean m and variance v , $\log c_t$ has a log normal distribution such that $\mathbb{E}[c_t] = \exp[m + \frac{v}{2}]$. Due to c_t exhibiting an AR(1) process, we can say that $m = \mu + \alpha m + 0 \rightarrow m(1 - \alpha) = \mu \rightarrow m = \frac{\mu}{1 - \alpha}$. The unconditional variance can be found by the following $v = \alpha^2 v + \sigma_\epsilon^2 \rightarrow v = \frac{\sigma_\epsilon^2}{(1 - \alpha^2)}$. Thus, $\mathbb{E}[c_t] = \exp[\frac{\mu}{1 - \alpha} + \frac{\sigma_\epsilon^2}{2(1 - \alpha^2)}]$ Subbing in the given μ , $\exp[-\frac{\sigma_\epsilon^2 \mu}{2(1 - \alpha^2)(1 - \alpha)} + \frac{\sigma_\epsilon^2}{2(1 - \alpha^2)}] \rightarrow \exp(0) = 1 = \mathbb{E}[c_t]$

(B)

$$\begin{aligned} V_0 &= \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ V(\lambda) &= \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t(1 + \lambda)) \\ U(c_t(1 + \lambda)) &= (1 + \lambda)^{1 - \gamma} \frac{c_t^{1 - \gamma}}{1 - \gamma} \\ V(\lambda) &= (1 + \lambda)^{1 - \gamma} \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1 - \gamma}}{1 - \gamma} = (1 + \lambda)^{1 - \gamma} V_0 \end{aligned}$$

Now, we apply the same distribution as in (A) to $c_t^{1 - \gamma}$.

$$\begin{aligned} \mathbb{E}[c_t^{1 - \gamma}] &= \exp[(1 - \gamma)m + \frac{(1 - \gamma)v}{2}] \\ \mathbb{E}[c_t^{1 - \gamma}] &= \exp[\frac{(1 - \gamma)\mu}{1 - \alpha} + \frac{(1 - \gamma)^2 \sigma_\epsilon^2}{2(1 - \alpha^2)}] \\ \therefore V_0 &= \frac{1}{1 - \gamma} \sum_{t=0}^{\infty} \beta^t \mathbb{E}[c_t^{1 - \gamma}] \\ &= \frac{1}{1 - \gamma} \frac{1}{1 - \beta} \exp[\frac{(1 - \gamma)\mu}{1 - \alpha} + \frac{(1 - \gamma)^2 \sigma_\epsilon^2}{2(1 - \alpha^2)}] \\ V(\lambda) &= (1 + \lambda)^{1 - \gamma} \frac{1}{1 - \gamma} \frac{1}{1 - \beta} \exp[\frac{(1 - \gamma)\mu}{1 - \alpha} + \frac{(1 - \gamma)^2 \sigma_\epsilon^2}{2(1 - \alpha^2)}] \end{aligned}$$

(C)

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(1) = \sum_{t=0}^{\infty} \beta^t \frac{1}{1 - \gamma} = \frac{1}{(1 - \beta)(1 - \gamma)}$$

Indifference implies that $V_0 = V(\lambda)$. Therefore:

$$\begin{aligned}
(1 + \lambda)^{1-\gamma} \exp\left[\frac{(1-\gamma)\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}\right] &= 1 \\
\Rightarrow (1-\gamma) \log(1 + \lambda) + \frac{(1-\gamma)\sigma_\epsilon^2}{(1-\alpha)} + \frac{(1-\gamma)^2\sigma_\epsilon^2}{2(1-\alpha^2)} &= 0 \\
\log(1 + \lambda) &= -\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)} \\
1 + \lambda &= \exp\left[-\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}\right] \\
\lambda &= \exp\left[-\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}\right] - 1 \\
\lambda &= \exp\left[\frac{\sigma_\epsilon^2}{2(1-\alpha^2)} - \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}\right] - 1 \\
\lambda &= \exp\left[\frac{\sigma_\epsilon^2}{2(1-\alpha^2)}(1 - 1 + \gamma)\right] - 1 \\
\lambda &= \exp\left[\frac{\gamma\sigma_\epsilon^2}{2(1-\alpha^2)}\right] - 1
\end{aligned}$$

(D)

$$\begin{aligned}
c_t^{-\gamma} &= \beta(1+r)\mathbb{E}_t(c_{t+1}^{-\gamma}) \\
1 &= \beta(1+r)\mathbb{E}_t\left(\frac{c_t^\alpha \epsilon_{t+1} \exp[\mu]}{c_{t-1} \epsilon_t \exp[\mu]}\right)^{-\gamma} \\
\beta(1+r)\mathbb{E}_t\left(\alpha \log \frac{c_{t+1}}{c_t} + \log\left(\frac{\epsilon_{t+1}}{\epsilon_t}\right) + 1\right) &= 1 \\
\beta(1+r)\mathbb{E}_t(\alpha \Delta \log c_{t+1} + 1) \text{ s.t. } \Delta \log c_{t+1} &\sim N(\mathbb{E}_t \Delta \log c_{t+1}, v_t \Delta \log c_{t+1}) \\
\mathbb{E}_t(-\gamma \alpha \Delta \log c_{t+1} + 1) &= (-\gamma \alpha \Delta \log c_{t+1} + \frac{1}{2}(\gamma \alpha)^2 v_t \Delta \log c_{t+1}) \\
\mathbb{E}_t \Delta \log c_{t+1} &= \frac{\log \beta(1+r)}{\gamma \alpha} + \frac{1}{2} \gamma \alpha v_t \Delta \log c_{t+1}
\end{aligned}$$

Solution 2

(A)

Proof. $v_t = [(1-\beta)(\lambda c_t)^\rho + \beta(\mathbb{E}[\lambda v_{t+1}]^\alpha)^\frac{\rho}{\alpha}]^\frac{1}{\rho}$. This allows us to move the λ term out such that $v_t = [\lambda^\rho(1-\beta)(c_t^\rho) + \lambda^\rho \beta(\mathbb{E}[v_t]^\alpha)^\frac{\rho}{\alpha}]^\frac{1}{\rho}$. Finally, we can show that utility is linearly scaled by lambda such that $v_t = \lambda[(1-\beta)c_t^\rho + \beta(\mathbb{E}c_t^\alpha)^\frac{\rho}{\alpha}]^\frac{1}{\rho}$. \square

(B)