

Homework 4

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1 Question 1

1.1 Problem

Consider the following planning problem

$$w(k_0) = \max_{k_{t+1}, c_t \geq 0} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &\leq zk_t + (1 - \delta)k_t \\ k_0 &\text{ given} \end{aligned}$$

- (a) Write this problem as a dynamic programming problem.
- (b) Solve the Bellman equation you wrote using guess-and-verify method (hint: try $v(k) = A \frac{k^{1-\sigma}}{1-\sigma}$).
- (c) What is the growth rate of k_t and c_t in this economy? (use you answer in part (b) to answer this). Under what condition does the economy grow?

2 Question 2

2.1 Problem

Let n_t denote hours worked. Consider the following planning problem (n_t is hours worked)

$$w(k_0) = \max \sum_{t=0}^{\infty} \beta^t [\theta \log c_t + (1 - \theta) \log(1 - n_t)]$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &\leq zk_t^\alpha n_t^{1-\alpha} + (1 - \delta)k_t \\ k_{t+1}, c_t &\geq 0 \\ 1 &\geq n_t \geq 0 \\ k_0 &\text{ given} \end{aligned}$$

We want to transform this problem to a formulation that we can solve using dynamic programming (Note that if $\alpha = 1$ this problem is a special case of question 1. Also, if $\theta = 1$ and $\delta = 1$ this transforms to the example we solved in class). To do this, we break it down to few steps. Consider the problem of choosing optimal hours worked, given current capital k and future capital k' . Write down the optimality condition for n in the following

$$F(k, k') = \max_n \theta \log(zk^\alpha n^{1-\alpha} + (1 - \delta)k - k') + (1 - \theta) \log(1 - n)$$

you won't be able to find a closed form solution for n (yet). Call the optimal solution to the above problem $n(k, k')$.

- (a) Write the planning problem as a dynamic programming problem (hint: utilize the function $F(k, k')$ above which is equivalent to period utility function for optimally chosen c and n , taking k and k' as given).

- (b) From now on, assume $\delta = 1$. Now guess the value function $V(k) = A + B \log k$. Derive optimal condition for k' (just like we did for example with $\theta = 1$ in class). Solve for k' as function of parameters of B and n .

- (c) Rewrite the optimality condition in (a), replace for k' from (c) and solve for optimal n (this won't depend on k , only on parameters and B). Remember to impose $\delta = 1$.
- (d) Now, replace optimal k' and n in the Bellman equation and solve for coefficient B .
- (e) Write the optimal policy functions for n, k' , and c as a function of k .

3 Question 3

3.1 Problem

Consider an economy with two types of capital. Let k_t denote physical capital and h_t denote human capital. Output is produced using physical and human capital

$$y_t = z k_t^\alpha h_t^{1-\alpha}.$$

The final good can be consumed, or can be invested in making physical or human capital. Assume both types of capital depreciate at rate δ . Assume households have log preferences over consumption (and labor is inelastically supplied).

- (a) Write the planning problem that maximizes the welfare of the representative household for an initial stock of physical and human capital (k_0, h_0) . Denote the value to the planner as $w(k_0, h_0)$.
- (b) Write the planning problem recursively. This means write a Bellman equation that corresponds to the planning problem in part (a).
- (c) Assume full depreciation $\delta = 1$. Solve the Bellman equation using guess and verify. Find the optimal policy functions for future physical and human capital as function of current physical and human capital.

4 Question 4

4.1 Problem

Consider the following planning problem

$$w(\bar{k}_0) = \max_{\{(c_t, k_{t+1}, x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c_t$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &= k_t \\ k_0 &= \bar{k}_0 \end{aligned}$$

and

$$\theta_t = \begin{cases} \theta_H & t \text{ is even} \\ \theta_L & t \text{ is odd} \end{cases}$$

- (a) Write this problem only in terms of a sequence of capital $\{k_{t+1}\}_{t=0}^{\infty}$.
- (b) Write the problem in part (a) as a Bellman equation(s). (hint: Note that we have an additional (exogenous) state variable θ_t . So you need to write two equations, one for $v(k, \theta_L)$ and one for $v(k, \theta_H)$. Also, we know how θ_t evolves.)

- (c) Solve the Bellman equation in part (b) using guess and verify method.
- (d) Write the formula for optimal policy functions $g(k, \theta_L)$ and $g(k, \theta_H)$.
- (e) Start from $k_0 = 1$. Describe how you will simulate the optimal path of capital stock $\{k_{t+1}\}_0^\infty$. Write down k_t for $k = 1, 2, 3$.