Computational Algorithm
SIM with life-cycle and Social Security

ECON 8050: Advanced Macroeconomics
Svetlana Pashchenko

1. Discretize all the state variables: $(k_1...k_n)$, $(z_1..z_m)$. To discretize z, use Tauchen-Hussey algorithm.

2. Guess r and b.

3. Compute $\frac{K}{N} = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha - 1}}$, compute implied $w = (1-\alpha)(\frac{K}{N})^{\alpha}$.

4. Solve the consumer's optimization problem using backward induction.

   (a) Start with the last year T and then solve for consumption, savings and the value function during retirement years:

   $$coptR(n, R{:}T),\ koptR(n, R{:}T),\ VR(n, R{:}T)$$

   The problem you are solving looks as follows:

   $$V_t^R(k_t) = \max_{c_t, k_{t+1}} \left\{u(c_t, 0) + \beta V_{t+1}^R(k_{t+1})\right\}$$
   $$st. \qquad c_t + k_{t+1} = k_t(1+r) + b$$

   (b) Solve for consumption, savings, labor supply and the value function during the working stage of a life-cycle:

   $$copt(n, m, 1{:}R-1),\ kopt(n, m, 1{:}R-1),$$
   $$lopt(n, m, 1{:}R-1),\ V(n, m, 1{:}R-1)$$

   The problem you are solving looks as follows:

   $$V_t(k_t, z_t) = \max_{c_t, k_{t+1}, l_t} \left\{ u(c_t, l_t) + \beta \sum_{z_{t+1}} P(z_{t+1}|z_t) V_{t+1}(k_{t+1}, z_{t+1}) \right\}$$
   $$st. \qquad c_t + k_{t+1} = k_t(1+r) + w \exp(z_t) \lambda_t l_t (1-\tau)$$

Note, the labor supply choice is a static one. You can express l_t as a function of k_{t+1} using FOC:

$$\frac{u_c}{u_l} = \frac{1}{w\exp(z)\lambda(1-\tau)}$$

Show that this can be transformed as:

$$l_t = \frac{\mu w\exp(z_t)\lambda_t(1-\tau) - (1-\mu)(k_t(1+r) - k_{t+1})}{w\exp(z_t)\lambda_t(1-\tau)}$$

Once you plug this in your objective function l_t will depend on k_{t+1} only.

5. Compute invariant distribution $\Gamma(k, z, t)$ and $\Gamma R(k, t)$ using non-stochastic simulations.

   Simulate forward, initialize $\Gamma(:, :, :) = 0$, $\Gamma R(:, :) = 0$.

   Start with $t = 1$. Since agents enter the model with zero assets, you have: $\Gamma(1, :, 1) = \Pi$ (where Π is invariant distribution of z).

   Loop over age and state variables $t = 1 : R-1, ik = 1 : n, iz = 1 : m$.

   Mass of the population with a particular combination of state variables: $masscur = \Gamma(ik, iz, t)$.

   Savings of this group: $kopt(ik, iz, t)$, you should locate the savings on $kgrid$ and find jlo and $jlo + 1$.

   Compute weight for interpolation: $w = \dfrac{kopt(ik, iz, t) - k_{jlo}}{k_{jlo+1} - k_{jlo}}$.

   Now, allocate this mass over state variables for age $t + 1$:

   Loop over tomorrow z: $izp = 1 : m$. For each izp:

   $$\Gamma(jlo, izp, t+1) = \Gamma(jlo, izp, t+1) + masscur * (1-w) * P(izp|iz)$$
   $$\Gamma(jlo+1, izp, t+1) = \Gamma(jlo+1, izp, t+1) + masscur * w * P(izp|iz)$$

   Note, for retirement ages, you do not need to loop over z. So be careful for $t = R - 1$.

Also note that you have to adjust for the population growth: every new cohort is $(1+n)$ times larger than the previous one. You can do it this way.

Create a vector $adj(1{:}T)$:

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g=1
for t=T:1
adj(t)=g
g=g*(1+n)
end
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Then you multiply $\Gamma(:,:,t)$ and $\Gamma R(:,t)$ by $adj(t)$ for each t.

Once you are done, normalize the total mass of people to 1.

6. Compute aggregate K and L implied by your decision rules and by the invariant distribution.

$$K = \sum_t \Gamma R(t) koptR(t) + \sum_t \sum_z \Gamma(z,t) kopt(z,t)$$
$$L = \sum_z \sum_t lopt(z,t)\lambda_t z_t \Gamma(z,t)$$

Then compute r_{new}.

7. Compute b_{new}:

$$b_{new} = \frac{\tau w \sum\limits_t \sum\limits_z \lambda_t z_t lopt(z,t)\Gamma(z,t)}{\sum\limits_t \Gamma R(t)}$$

(Note, $\sum\limits_t \Gamma R(t)$ is the measure of retirees).

8. Compare r and r_{new}, b and b_{new}. Update r and b: set $r = r_{new}$ and $b = b_{new}$. Go back to 2.