Homework 2

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ECON - 8050

Problem 1: Costs of Business Cycle

Let utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^t U(c_t)$$

where the utility function is CRRA:

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

The consumption process is

$$c_t = c_{t-1}^{\alpha}\varepsilon_t \exp(\mu)$$

where

$$\mu = \frac{-\sigma_\varepsilon^2(1-\alpha)}{2(1-\alpha^2)}, \quad \log\varepsilon_t \sim N(0,\sigma_\varepsilon^2) \text{ and i.i.d.}$$

Thus, the log of consumption follows an AR(1) process:

$$\log c_t = \mu + \alpha\log c_{t-1} + \log\varepsilon_t$$

Part A

Find the unconditional mean of c_t, $E(c_t)$. (Hint: recall the properties of the lognormal distribution).

Part B

Define lifetime utility before any uncertainty is realized as:

$$V_0 = E_{-1}\sum_{t=0}^{\infty}\beta^t U(c_t)$$

Assume c_0 is drawn from the invariant (unconditional) distribution of c. Now define:

$$V(\lambda) = E_{-1}\sum_{t=0}^{\infty}\beta^t U[c_t(1+\lambda)]$$

This is lifetime utility when every period consumption is increased by $(1+\lambda)$. Express $V(\lambda)$ as a function of $\mu, \sigma_\varepsilon^2, \alpha, \gamma, \beta$.

Part C

Denote V_0 as the lifetime utility when c_t is deterministic and equal to its unconditional mean found in part A). Find the compensation λ such that $V(\lambda) = V_0$. Find how much compensation the consumer has to be given in order to be indifferent between the stochastic and deterministic cases, Provide economic intuition.

Part D

Denote the interest rate as r. Find consumption c_t.

Problem 2: Non-Expected Utility Framework

This problem follows the Kreps and Porteus (1978), Epstein and Zin (1991), and Weil (1990) frameworks.

Let remaining lifetime utility at time t, once c_t is known, be given by v_t, satisfying:

$$v_t = \left[(1-\beta)c_t^\rho + \beta(E_t v_{t+1}^\alpha)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \tag{1}$$

where $1-\alpha$ represents risk aversion and $1-\rho$ represents the inverse of the intertemporal elasticity of substitution. In standard expected utility, $\alpha = \rho$.

Denote pre-realization lifetime utility at time t as U_t, where:

$$U_t = (E_t v_t^\alpha)^{\frac{1}{\alpha}}$$

Part A

Prove that multiplying c_t by λ for all $t = 0, 1, \ldots, \infty$ is equivalent to multiplying v_t by λ. (Hint: start by assuming this holds, substitute into equation (1), and show v_t scales linearly.)

Part B

Suppose for all t, we replace c_t with a deterministic constant $\bar{c} = E[c_t]$. Compare welfare in this case with uncertain c_t. Specifically, find η such that multiplying c_t by $(1+\eta)$ makes ex-ante welfare U_0 equal to that in the deterministic case. Express η in terms of U_0 and $\bar{c}$.

Part C

Suppose consumption follows one of two sequences: with probability $\frac{1}{2}$, $c_t = c_l$ for all t, and with probability $\frac{1}{2}$, $c_t = c_h$ for all t. The sequence is revealed at $t = 0$. Find η and analyze its dependence on ρ and α.

Part D

Now assume c_t is i.i.d., where each period $c_t = c_l$ with probability $\frac{1}{2}$ and c_h with probability $\frac{1}{2}$.

1. Derive an implicit equation for U_0.
2. Analyze whether η depends on α and ρ.

Part E

Solve for U_0 numerically using Matlab with given parameters: $\beta = 0.95$, $c_l = e^{0.98}$, $c_h = e^{1.02}$. Compute η for:

- $\alpha = 1, 0.5, -1$
- $\rho = 1, 0.5, -1$

Report results in a table and provide economic intuition. (Hint: Use an iterative approach to solve $U_0 = f(U_0)$ until convergence with tolerance 10^{-8}.)

1 Solution 1

(A)

If $c_t \sim N(m, v)$ for some mean m and variance v, $\log c_t$ has a log normal distribution such that $\mathbb{E}[c_t] = \exp[m + \frac{v}{2}]$. Due to c_t exhibiting an AR(1) process, we can say that $m = \mu + \alpha m + 0 \to m(1-\alpha) = \mu \to m = \frac{\mu}{1-\alpha}$. The unconditional variance can be found by the following $v = \alpha^2 v + \sigma_\epsilon^2 \to v = \frac{\sigma_\epsilon^2}{(1-\alpha^2)}$. Thus, $\mathbb{E}[c_t] = \exp[\frac{\mu}{1-\alpha} + \frac{\sigma_\epsilon^2}{2(1-\alpha^2)}]$ Subbing in the given μ, $\exp[-\frac{\sigma_\epsilon^2 \mu}{2(1-\alpha^2)(1-\alpha)} + \frac{\sigma_\epsilon^2}{2(1-\alpha^2)}] \to \exp(0) = 1 = \mathbb{E}[c_t]$

(B)

$$V_0 = \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$V(\lambda) = \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t U(c_t(1+\lambda))$$
$$U(c_t(1+\lambda)) = (1+\lambda)^{1-\gamma} \frac{c_t^{1-\gamma}}{1-\gamma}$$
$$V(\lambda) = (1+\lambda)^{1-\gamma} \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} = (1+\lambda)^{1-\gamma} V_0$$

Now, we apply the same distribution as in (A) to $c_t^{1-\gamma}$.

$$\mathbb{E}[c_t^{1-\gamma}] = \exp[(1-\gamma)m + \frac{(1-\gamma)v}{2}]$$
$$\mathbb{E}[c_t^{1-\gamma}] = \exp[\frac{(1-\gamma)\mu}{1-\alpha} + \frac{(1-\gamma)^2\sigma_\epsilon^2}{2(1-\alpha^2)}]$$
$$\therefore\ V_0 = \frac{1}{1-\gamma} \sum_{t=0}^{\infty} \beta^t \mathbb{E}[c_t^{1-\gamma}]$$
$$= \frac{1}{1-\gamma} \frac{1}{1-\beta} \exp[\frac{(1-\gamma)\mu}{1-\alpha} + \frac{(1-\gamma)^2\sigma_\epsilon^2}{2(1-\alpha^2)}]$$
$$V(\lambda) = (1+\lambda)^{1-\gamma} \frac{1}{1-\gamma} \frac{1}{1-\beta} \exp[\frac{(1-\gamma)\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}]$$

(C)

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(1) = \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma} = \frac{1}{(1-\beta)(1-\gamma)}$$

Indifference implies that $V_0 = V(\lambda)$. Therefore:

$$
\begin{aligned}
(1+\lambda)^{1-\gamma} \exp[\frac{(1-\gamma)\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}] = 1 \\
\Rightarrow (1-\gamma)\log(1+\gamma) + \frac{(1-\gamma)\sigma_\epsilon^2}{(1-\alpha)} + \frac{(1-\gamma)^2\sigma_\epsilon^2}{2(1-\alpha^2)} = 0 \\
\log(1+\lambda) = -\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)} \\
1+\lambda = \exp[-\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}] \\
\lambda = \exp[-\frac{\mu}{1-\alpha} + \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}] - 1 \\
\lambda = \exp[\frac{\sigma_\epsilon^2}{2(1-\alpha^2)} - \frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\alpha^2)}] - 1 \\
\lambda = \exp[\frac{\sigma_\epsilon^2}{2(1-\alpha^2)}(1-1+\gamma)] - 1 \\
\lambda = \exp[\frac{\gamma\sigma_\epsilon^2}{2(1-\alpha^2)}] - 1
\end{aligned}
$$

(D)

$$
\begin{aligned}
c_t^{-\gamma} = \beta(1+r)\mathbb{E}_t(c_{t+1}^{-\gamma}) \\
1 = \beta(1+r)\mathbb{E}_t(\frac{c_t^\alpha \epsilon_{t+1}\exp[\mu]}{c_{t-1}\epsilon_t \exp[\mu]})^{-\gamma} \\
\beta(1+r)\mathbb{E}_t(\alpha \log\frac{c_{t+1}}{c_t} + \log(\frac{\epsilon_{t+1}}{\epsilon_t}) + 1) = 1 \\
\beta(1+r)\mathbb{E}_t(\alpha\Delta\log c_{t+1} + 1) \text{ s.t. } \Delta\log c_{t+1} \sim N(\mathbb{E}_t\Delta\log c_{t+1}, v_t\Delta\log c_{t+1}) \\
\mathbb{E}_t(-\gamma\alpha\Delta\log c_{t+1} + 1) = (-\gamma\alpha\Delta\log c_{t+1} + \frac{1}{2}(\gamma\alpha)^2 v_t\Delta\log c_{t+1}) \\
\mathbb{E}_t\Delta\log c_{t+1} = \frac{\log\beta(1+r)}{\gamma\alpha} + \frac{1}{2}\gamma\alpha v_t\Delta\log c_{t+1}
\end{aligned}
$$

Solution 2

(A)

Proof. $v_t = [(1-\beta)(\lambda c_t)^\rho + \beta(\mathbb{E}[\lambda v_{t+1}]^\alpha)^{\frac{\rho}{\alpha}}]^{\frac{1}{\rho}}$ This allows us to move the λ term out such that $v_t = [\lambda^\rho(1-\beta)(c_t^\rho) + \lambda^\rho\beta(\mathbb{E}[v_t]^\alpha)^{\frac{\rho}{\alpha}}]^{\frac{1}{\rho}}$. Finally, we can show that utility is linearly scaled by lambda such that $v_t = \lambda[(1-\beta)c_t^\rho + \beta(\mathbb{E}c_t^\alpha)^{\frac{\rho}{\alpha}}]^{\frac{1}{\rho}}$. □

(B)

In the deterministic case, $v_t = v_{t+1} = U_0^d$. This allows us to write the uncertain case as $U_0^c(1+\eta) = U_0^d \Rightarrow \eta = \frac{U_0^d}{U_0^c} - 1 \Rightarrow \eta = \frac{\bar{c}}{U_0^c}$.

(C)

Using the probabilities $\Pi_L = 0.5,\ \Pi_H = 0.5$, we can derive value functions for the cases in which you get a low draw and a high draw.

$$\bar{c} = \Pi_L c_L + \Pi_H c_H$$
$$U_0 = (\mathbb{E}[v_0^\alpha])^{\frac{1}{\alpha}} = [\Pi_L c_L^\alpha + \Pi_H c_H^\alpha]^{\frac{1}{\alpha}}$$

Plugging into η formula derived above

$$\eta = \frac{\Pi_L c_L + \Pi_H c_H}{[\Pi_L c_L^\alpha + \Pi_H c_H^\alpha]^{\frac{1}{\alpha}}} - 1$$

This shows that η depends on the risk aversion parameter α. ρ does not appear as at period $t = 0$, consumption is constant and sees no uncertainty. The agent feels no disutility from pushing consumption to a future period t.

(D)

Let's start with the case in which $c_t = c_L$.

$$v_L = [(1-\beta)c_L^\rho + \beta(\mathbb{E}[v_{t+1}^\alpha])^{\frac{\rho}{\alpha}}]^{\frac{1}{\rho}}$$

Next, in the case in which $c_t = c_H$,

$$v_H = [(1-\beta)c_H^\rho + \beta(\mathbb{E}[v_{t+1}^\alpha])^{\frac{\rho}{\alpha}}]^{\frac{1}{\rho}}$$

Now, allow $R = \mathbb{E}[v_{t+1}^\alpha]^{\frac{1}{\alpha}} = [\Pi_L v_L^\alpha + \Pi_H v_H^\alpha]^{\frac{1}{\alpha}}$. From here, we can update the value functions in each state:

$$v_L = [(1-\beta)c_L^\rho + \beta R^\rho]^{\frac{1}{\rho}}$$
$$v_H = [(1-\beta)c_H^\rho + \beta R^\rho]^{\frac{1}{\rho}}$$

Now, we can use these functions to update U_0:

$$U_0 = (\mathbb{E}_0[v_0^\alpha])^{\frac{1}{\alpha}} - R$$
$$U_0 = \{\Pi_L[(1-\beta)c_L^\rho + \beta U_0]^{\frac{\alpha}{\rho}} + \Pi_H[(1-\beta)c_H^\rho + \beta U_0^\rho)]^{\frac{\alpha}{\rho}}\}$$

Here, in this case, $\alpha,\ \rho$ are present due to the fact that consumption can now vary in each period. This causes the agent to experience disutility ρ for creating a buffer stock as well as their risk aversion α.

(E)