Homework 7

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A ECON 8010 Homework Assignment

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PS4.4

Problem

"Centipede with a possibly generous player" (see lecture notes) has a unique sequential equilibrium strategy profile in which both players sometimes choose to continue. Does this game possess any other weak sequential equilibrium profiles?

Solution

As found in the lecture, the unique equilibrium in which both players choose to continue is

$$\mu_1(y) = \frac{4}{5}; \sigma_2(c) = \frac{4}{19}; \sigma_1(c') = \frac{1}{5}; \sigma_1(c) = 1$$

There would be no other weak equilibria as there are no other rational profiles. For instance, if player 1 chooses to stop at s', player 2 would stop at s, thus causing the node not to be reached. If player 1 were to always play c', the probability does not align with beliefs. Thus, there can be no other profiles which are Bayesian rational.

Figure 5

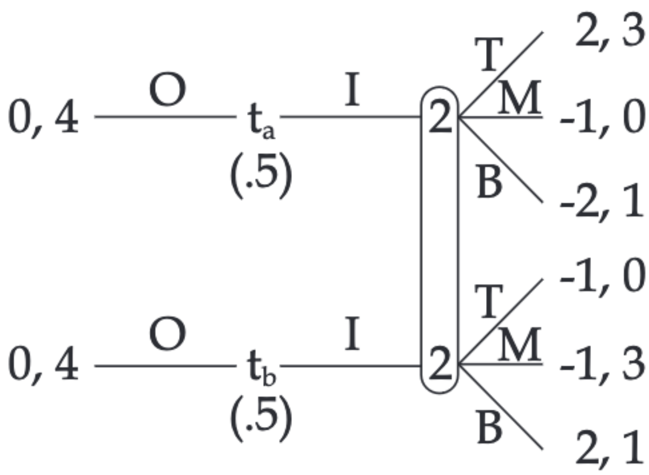

Figure 5

PS4.6

Problem

Compute all sequential equilirbia of the game in figure 5.

Solution

In all cases, B will not be played as there is a better move for each case a or b. Thus it becomes a game with two brances. For Case 1:

$$\sigma_1(O|t_b) = 1$$
$$\sigma_1(I|t_b) = 0$$
$$\mu_2(t_b|I) = 0$$
$$\sigma_2(T|I) = 1$$
$$\sigma_1(I|t_1) = 1$$

For Case 2:

$$\sigma_1(I|t_a) = 0$$
$$0 \geq -\sigma_2(M|I) + 2(1 - \sigma_2(M|I))$$
$$\sigma_2(M|I) \geq \frac{2}{3}$$
$$\text{if } \sigma_2(M|I) \in [\frac{2}{3}, 1) : \mu_2(t_a|I) = \frac{1}{2}$$

Because player 1 will not play I if type b, player 2 will always play M or $\sigma_2(M|I) = 1$. Equilibrium:

$$\sigma_2(M|I) = 1$$
$$\sigma_1(I|t_b) = 0$$
$$\mu_2(t_a|I) \geq \frac{1}{2}$$

Figure 6

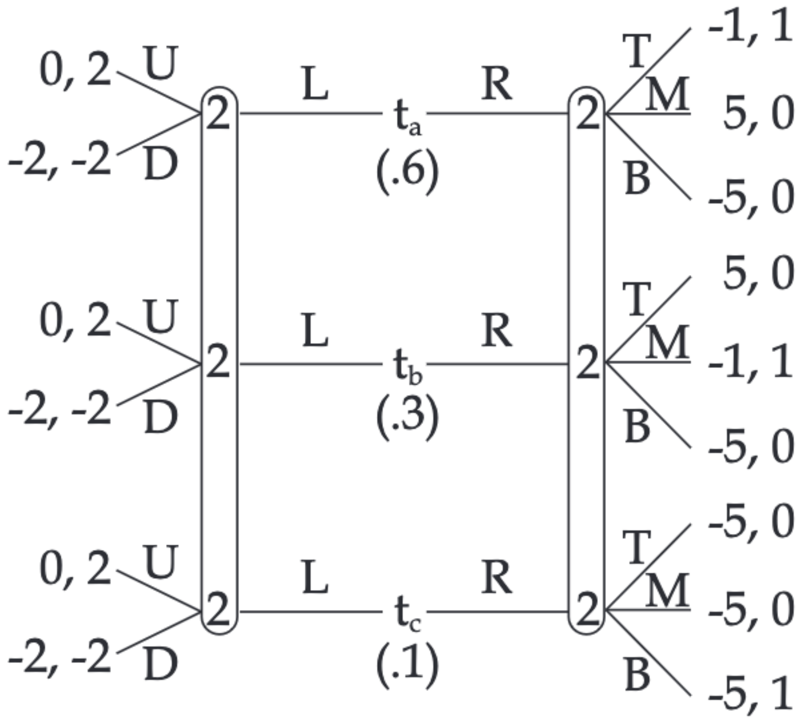

Figure 6

PS4.7

Problem

Compute all sequential equilibria of the game in figure 6.

Solution

Case 1:

$$\sigma_1(L|t_a) = \sigma_1(L|t_b) = \sigma_1(L|t_c) = 1$$
$$\mu_2(t_a|L) = \mu_2(t_a|R) = 0.6$$
$$\mu_2(t_b|L) = \mu_2(t_b|R) = 0.3$$
$$\mu_2(t_c|L) = \mu_2(t_c|R) = 0.1$$
$$\sigma_2(U|L) = 1$$
$$t_a :$$
$$0 > -(\sigma_2(T|R)) + 5(1 - \sigma_2(T|R))$$
$$0 > -6\sigma_2(T|R) + 5$$
$$\sigma_2(T|R) \geq \frac{5}{6}$$
$$t_b :$$
$$0 > 5(\sigma_2(T|R)) - (1 - \sigma_2(T|R))$$
$$0 > 6\sigma_2(T|R) - 1$$
$$\sigma_2(T|R) \leq \frac{1}{6}$$
$$\therefore \sigma_2(T|R) = \{\alpha|\alpha \in [\frac{1}{6}, \frac{5}{6}]\}$$

Case 2:

$$\sigma_1(L|t_a) = \sigma_1(L|t_b) = 0; \sigma_1(L|t_c) = 1$$
$$\sigma_1(R|t_a) = \sigma_1(R|t_b) = 1$$
$$\sigma_2(T|R) = \frac{1}{3}$$
$$\frac{\sigma_1(R|t_a)}{\sigma_1(R|t_a) + \sigma_1(R|t_b)} = \frac{2}{3} = \mu_2(t_a|R)$$
$$\frac{\sigma_1(R|t_b)}{\sigma_1(R|t_a) + \sigma_1(R|t_b)} = \frac{1}{3} = \mu_2(t_b|R)$$
$$\mu_2(t_c|L) = 1$$
$$\sigma_2(U|L) = 1$$

Case 3:

$$\sigma_1(R|t_a) = \frac{1}{10}; \sigma_1(R|t_b) = 1; \sigma_1(R|t_c) = 0$$
$$\mu_2(t_a|R) = \frac{1}{6}$$
$$\mu_2(t_b|R) = \frac{5}{6}$$
$$\mu_2(t_a|L) = \mu_2(t_b|L) = \frac{1}{2}$$
$$\sigma_2(T|R) = \frac{5}{6}$$
$$\sigma_2(U|L) = 1$$

Case 4:

$$\sigma_1(R|t_a) = 1; \sigma_1(R|t_b) = \frac{2}{5}; \sigma_1(R|t_c) = 0$$
$$\mu_2(t_a|R) = \frac{5}{6}$$
$$\mu_2(t_b|R) = \frac{1}{6}$$
$$\mu_2(t_b|L) = \frac{6}{11}$$
$$\mu_2(t_c|L) = \frac{5}{11}$$
$$\sigma_2(T|R) = \frac{1}{6}$$
$$\sigma_2(U|L) = 1$$

Question 4

Problem

Consider the following extensive form game of imperfect information played by three venture capital firms $i \in \{1, 2, 3\}$. The game takes place in five periods $t \in \{0, 1, 2, 3, 4\}$ and proceeds as follows.

- At $t = 0$, Nature chooses a state of the world $\theta \in \{H, L\}$ according to the probability distribution ρ_0, where
$$\rho_0(H) = \frac{1}{2}, \quad \rho_0(L) = \frac{1}{2}.$$
This state of the world corresponds to whether investing in a certain startup will yield a high payoff (state H) or a low payoff (state L). It is not observed by any of the firms.

- At $t = 1$, Nature chooses a signal $r_1 \in \{h, \ell\}$ according to the probability distribution $\rho(\cdot|\theta)$, where
$$\rho(h|H) = \frac{3}{4}, \quad \rho(\ell|H) = \frac{1}{4}, \quad \rho(h|L) = \frac{1}{4}, \quad \rho(\ell|L) = \frac{3}{4}.$$

This signal is observed by firm 1 (only). Then, firm 1 chooses to either invest (Y_1) or not invest (N_1). This action is observed by firms 2 and 3.

- At $t = 2$, Nature chooses another signal $r_2 \in \{h, \ell\}$ according to $\rho(\cdot|\theta)$. This signal is observed by firm 2 (only). Then, firm 2 chooses to either invest (Y_2) or not invest (N_2). This action is observed by firm 3.
- At $t = 3$, Nature chooses yet another signal $r_3 \in \{h, \ell\}$ according to $\rho(\cdot|\theta)$. This signal is observed by firm 3 (only). Then, firm 3 chooses to either invest (Y_3) or not invest (N_3).
- At $t = 4$, the game ends. Each firm i who chose to invest (i.e., chose action Y_i) receives a payoff of 1 if $\theta = H$ and $-\frac{9}{10}$ if $\theta = L$. Each firm who chose not to invest (i.e., chose action N_i) receives a payoff of zero, regardless of the state of the world.

Note that firms' beliefs are only relevant insofar as they describe the probability placed on each state of the world. Thus, instead of writing beliefs as the probabilities of each decision node (e.g., $\mu_3(H, r_1 = \ell, N_1, r_2 = h, Y_2, r_3 = \ell)$), we can simply write them as the probability of θ given the current information set (e.g., $\mu_3(H|N_1, Y_2, r_3 = \ell)$).

a) What are the unique Bayesian beliefs $\mu_1(H|r_1 = h)$, $\mu_1(H|r_1 = \ell)$ for player 1 after $r_1 \in \{h, \ell\}$?

b) In any weak sequential equilibrium, what action will firm 1 take after signal h? After ℓ?

c) What are the unique Bayesian beliefs

$$\mu_2(H|Y_1, r_2 = h), \quad \mu_2(H|Y_1, r_2 = \ell), \quad \mu_2(H|N_1, r_2 = h), \quad \mu_2(H|N_1, r_2 = \ell)?$$

d) In any weak sequential equilibrium, what action will firm 2 take after $(Y_1, r_2 = h)$? After $(Y_1, r_2 = \ell)$? After $(N_1, r_2 = h)$? After $(N_1, r_2 = \ell)$?

e) Solve for all weak sequential equilibria. (You only need to solve for beliefs about the state, not about individual decision nodes that follow the same state.)

f) In any weak sequential equilibrium, what is the probability of a history (s_1, s_2) occurring after which

   (a) firm 3 invests (i.e., plays Y_3), no matter what signal r_3 she receives?

   (b) firm 3 does not invest (i.e., plays N_3), no matter what signal r_3 she receives?

   (c) firm 3 plays the "correct" action (i.e., Y_3 if the state is H or N_3 if the state is L), no matter what signal r_3 she receives?

   (d) firm 3 plays the "incorrect" action (i.e., N_3 if the state is H or Y_3 if the state is L), no matter what signal r_3 she receives?

Solution

(a)

$$\mu_1(H|h) = \frac{\frac{3}{4}\cdot\frac{1}{2}}{(\frac{3}{4}\cdot\frac{1}{2}) + (\frac{1}{4}\cdot\frac{1}{2})}$$
$$\mu_1(H|h) = \frac{3}{4}$$
$$\mu_1(H|\ell) = \frac{\frac{1}{4}\cdot\frac{1}{2}}{(\frac{3}{4}\cdot\frac{1}{2}) + (\frac{1}{4}\cdot\frac{1}{2})}$$
$$\mu_1(H|\ell) = \frac{1}{4}$$

(b)

$$(H|h) = \mu_1(H|h) - (1 - \mu_1(H|h))(\frac{9}{10})$$
$$(H|h) = \frac{3}{4} - (\frac{9}{10}\cdot\frac{1}{4}) > 0$$
$$(H|h) = \frac{21}{40} > 0 \therefore \sigma_1(H|h) = Y_1$$
$$(H|\ell) = \mu_1(H|\ell) - (1 - \mu_1(H|\ell))(\frac{9}{10})$$
$$(H|\ell) = \frac{1}{4} - \frac{27}{40}$$
$$(H|\ell) = -\frac{17}{40} < 0 \therefore \sigma_1(H|\ell) = N_1$$

(c)

$$\mu_2(H|Y_1, r_2 = h) = \frac{(\frac{3}{4})^2\cdot\frac{1}{2}}{(\frac{3}{4}^2\cdot\frac{1}{2}) + (\frac{1}{4}^2\cdot\frac{1}{2})}$$
$$\mu_2(H|Y_1, r_2 = h) = \frac{9}{10}$$
$$\mu_2(H|N_1, r_2 = \ell) = \frac{(\frac{1}{4})^2\cdot\frac{1}{2}}{(\frac{1}{4}^2\frac{1}{2}) + (\frac{3}{4}^2\cdot\frac{1}{2})}$$
$$\mu_2(H|N_1, r = \ell) = \frac{1}{10}$$
$$\mu_2(H|Y_1, r = \ell) = \frac{1}{2}$$
$$\mu_2(H|N_1, r = h) = \frac{1}{2}$$

(d)

$$(\mu_2(H)) = \mu_2 + (1 - \mu_2(H))(-\frac{9}{10})$$
$$\mu_2 \geq \frac{9}{19}$$
$$\sigma_2(H|Y_1, h) = Y_2$$
$$\sigma_2(H|N_1, h) = Y_2$$
$$\sigma_2(H|Y_1, \ell) = Y_2$$
$$\sigma_2(H|N_1, \ell) = N_2$$

All but the case in which player one chooses not to invest when the state is low, player 2 will invest. In the outlier case, they will not invest, following the signal of the player before them.

(e)

$$\text{i)}$$
$$\mu_3(H|Y_1, Y_2, r_3 = h) = Y_3$$
$$\mu_3(H|Y_1, Y_2, r_3 = \ell) = Y_3$$
$$\text{(ii)}$$
$$\mu_3(H|N_1, Y_2, r_3 = h) = Y_3$$
$$\mu_3(H|N_1, Y_2, r_3 = \ell) = Y_3$$
$$\text{(iii)}$$
$$\mu_3(H|N_1, N_2, r_3 = h) = N_3$$
$$\mu_3(H|N_1, N_2, r_3 = \ell) = N_3$$

(f)

(a)

$$\frac{1}{2} + (\frac{1}{2})^2 = \frac{3}{4}$$

(b)

$$\frac{1}{2}^2 = \frac{1}{4}$$

(c)

$$(\frac{1}{2} \cdot \frac{3}{4}) + (\frac{1}{2} \cdot \frac{1}{4}) = \frac{1}{2}$$

(d)

$$1 - (f.c) = \frac{1}{2}$$