Assignment #3

Due Date: February 27th, 11:59pm

Please Show Your Work and Circle Your Final Answer
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Question 1: Optimal Auctions (25 pts)

In this problem, you will compute the auction that maximizes the auctioneer's expected revenue.

There is a seller looking to sell a single object. There are N bidders (indexed by $i \in \{1, \ldots, N\}$) for this object. Each bidder i has an i.i.d value v_i drawn from a distribution F with support on $[0, M]$. We assume that the density function f is continuous, where $F(a) = \int_0^a f(x)dx$. Furthermore, assume that $\frac{f(x)}{1-F(x)}$ is non-decreasing.

(a) Write down the seller's optimization problem. *Hint: use the revelation principle we discussed in class.*

(b) Let's take a look at the constraints representing the bidders' incentive to report their true value to the auctioneer. Take a given bidder i and the corresponding incentive constraints. Write these constraints as the solution to a maximization problem.

(c) Reformulate the constraints using the envelope theorem. That is, rewrite the incentive constraints in terms of the value that bidder i must receive. After applying the envelope theorem, rearrange the equation to get an expression for $t(v_i)$.

(d) Solve the optimization problem for the seller. What is the allocation and transfer rule that maximize expected revenue?

Hint: Go back to the seller's optimization problem in (a). Replace the incentive constraints with the expression in (c).

(e) Interpret your answer in (d). What kind of auction is this?

Question 2 (15pts)

A seller is selling a single object. There are N bidders (indexed by $i \in \{1, \ldots, N\}$). Each bidder i has an i.i.d value v_i drawn from a distribution F with support $[0, M]$. We assume that the density function f is continuous, where $F(a) = \int_0^a f(x)dx$. Furthermore, assume $\frac{f(x)}{1-F(x)}$ is non-decreasing.

Let $OPT(N)$ be the expected revenue from the optimal auction with N bidders (see 4d). Let $\mathcal{S}(N+1)$ denote the expected revenue from a second-price auction with $N+1$ bidders.

(a) Prove that $\mathcal{S}(N+1) \geq OPT(N)$.

(b) Interpret the result in (a). What does it mean? What is the takeaway?

Question 3: Correlated Values (10pts)

In the previous problems, we assume independent, private values. To see why the independence assumption is important from the sellers' perspective when it comes to choosing which auction to use to maximize revenue, consider the following environment.

There are two bidders. Each has a *private* value for a given object. Bidder i's value is $v_i \in \{1, 2\}$. The values are correlated. With probability $\frac{1}{3}$, both bidders have a value of 1 for the object. With probability $\frac{1}{3}$, both bidders have a value of 2 for the object. With probability $\frac{1}{6}$, the first bidder has a value 2 for the object and the second bidder has a value of 1. With probability $\frac{1}{6}$, the first bidder has a value of 1 for the object, and the second bidder has a value of 2.

(a) Suppose the auctioneer runs a second price auction with random tie-breaking. Is it still a weakly dominant strategy to bid truthfully? What is the expected revenue generated? What is the ex-ante expected surplus for a bidder?

(b) Think about the mathematical representation of a second-price auction: an allocation rule and transfer rule mapping bids to probabilities each bidder receives the object and payments. Come up with an allocation and transfer rule that extracts full surplus from the bidders.

Hint: keep the allocation rule the exact same as in a second-price auction. Just change the transfer rule.

Question 4 (25pts)

This question involves the Principal-Agent problem we examined in Lecture.

The principal is a seller, and the agent is a buyer. The Principal can sell the agent any quantity $x \geq 0$ of a good in exchange for payment. The principal's payoff from selling x units of the good for a total price t, is $t - c(x)$, where c is a strictly convex cost function. The agent's payoff from purchasing x units of the good for total price t is $v(x, \theta) - t$, where θ is the agents' private marginal value for the good.

The buyer's private value θ is drawn from a distribution F on $[0, M]$. Assume that the density function f is continuous and $\frac{f(x)}{1-F(x)}$ is non-decreasing.

The principal (seller) has full commitment power, so is free to choose any mechanism she likes subject to the individual rationality constraint for the buyer.

(a) Write down the seller's optimization problem (e.g. objective, IC, and IR constraints). Explain why it suffices to just look at direct mechanisms.

(b) Rewrite the constraints using the ICFOC and monotonicity condition.

(c) Work out the optimal mechanism (e.g. menu/contract) that the seller will offer.

Hint: For (a)-(c), approach the problem in the exact same way we did in lecture.

(d) Under what conditions on the utility function $v(x, \theta)$ can you guarantee that the marginal markup $t'(x) - c'(x)$ is decreasing? *Hint: Recognize that $t(\theta)$ is the total payment by a type θ buyer purchasing $x(\theta)$ units. Given the optimal mechanism, find an expression for $t'(x)$ where x is the total units purchased. Use the fact that optimality of reporting truthfully also means optimal to purchase $x(\theta)$ units from the "menu".*

(e) Show that if marginal cost is constant, the optimal payment scheme involves quantity discounting.

(f) Suppose $v(x, \theta) = \theta \cdot \gamma(x)$, where γ is strictly increasing and concave. Assume the seller's cost function exhibits constant marginal cost and $F(a) = 1 - a^{-\delta}$ for some $\delta > 0$ (power-law distribution). Show that the optimal payment scheme is a two-part tariff.