Assignment 2

Tate Mason

Due: February 6th, 11:59pm

Instructions

- Show all your work and circle your final answers.
- Submit as a single document.

Question 1 (10 points)

There is an island with only two consumers, Tom and Christina. There are two goods, apples (x) and bananas (y), available on the island. The consumers' utility function over bundles is given by:

$$\text{Tom: } u(x,y) = x^{\alpha} y^{1-\alpha},$$
$$\text{Christina: } u(x,y) = x^{\beta} y^{1-\beta}.$$

Tom has an endowment of $\omega_{A,T} > 0$ apples and $\omega_{B,T} > 0$ bananas, while Christina has an endowment of $\omega_{A,C} > 0$ apples and $\omega_{B,C} > 0$ bananas. Without loss of generality, the price of bananas is normalized to 1.

(a) What is the Walrasian equilibrium of this exchange economy?

(b) In equilibrium, what share of wealth does Tom spend on apples? What share of wealth does Christina spend on apples?

Question 2 (10 points)

There is an island with only two consumers, Bob and Alice. There are two goods, apples (x) and bananas (y), available on the island. The consumers' utility function over bundles is given by:

$$\text{Bob: } u(x,y) = x + y,$$
$$\text{Alice: } u(x,y) = \min\{x, y\}.$$

Bob's endowment is 1 apple and 0 bananas. Alice's endowment is 2 apples and 1 banana.

1. Show mathematically that no Walrasian equilibrium exists in this economy.

Question 3 (15 points)

(a) Prove the First Welfare Theorem.

(b) Prove an equivalent characterization of Pareto-efficient allocations. Given any set of weights $\mu_1, \dots, \mu_N \geq 0$ such that $\sum \mu_i = 1$, consider the solution x^* to the following problem:

$$\max_{x_1,\dots,x_N} \sum_{i=1}^{N} \mu_i u(x^i) \quad \text{subject to} \quad \sum_{i=1}^{N} x_i^j \leq \sum_{i=1}^{N} e_i^j \quad \forall j \in \{1, \dots, M\}.$$

Prove that any solution x^* is a Pareto-efficient allocation.

(c) Provide an interpretation, in words, of what you showed in (b).

Question 4 (20 points)

Consider a second-price auction with N bidders. Each bidder has an independent private value v_i drawn from a Uniform Distribution on $[0, 1]$.

(a) What is the expected revenue generated when all bidders bid truthfully? Provide a closed-form solution and show your work.

(b) Prove that all bidders bidding truthfully is an equilibrium of the second-price auction game. What is another equilibrium? Prove your example is an equilibrium.

(c) If the auctioneer sets a reserve price r, is it still a weakly dominant strategy to bid truthfully?

(d) Suppose $N = 3$. What is the optimal reserve price r the auctioneer should set? How much smaller/greater is the revenue compared to the auction with no reserve price?

(e) A student at another university thought revenue equivalence implied that the expected revenue should be the same. Explain why their reasoning is incorrect.

Question 5 (15 points)

Suppose there are N bidders competing for a single object in an all-pay auction. Each bidder has an i.i.d. value v_i for the object drawn from some continuous distribution F with support $[0, M]$.

(a) Show that there is a symmetric equilibrium in increasing strategies.

(b) What is the expected revenue generated by this auction in the equilibrium from (a)? Explain your answer.

Solution 1:

(a)

Given prices $\{p_x, 1\}$ and allocations $\{x, y, \omega_{A,T}, \omega_{A,C}, \omega_{B,T}, \omega_{B,C}\}$, Tom solves:

$$u_T = x^\alpha y^{1-\alpha}$$
$$\text{s.t.} p_x x + y = p_x \omega_{A,T} + \omega_{B,T}$$

and Cristina solves:

$$u_C = x^\beta y^{1-\beta}$$
$$\text{s.t.} p_x x + y = p_x \omega_{A,C} + \omega_{B,C}$$