

# Assignment 2

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Due: February 6th, 11:59pm

## Instructions

- Show all your work and circle your final answers.
- Submit as a single document.

## Question 1 (10 points)

There is an island with only two consumers, Tom and Christina. There are two goods, apples ( $x$ ) and bananas ( $y$ ), available on the island. The consumers' utility function over bundles is given by:

$$\begin{aligned}\text{Tom: } u(x, y) &= x^\alpha y^{1-\alpha}, \\ \text{Christina: } u(x, y) &= x^\beta y^{1-\beta}.\end{aligned}$$

Tom has an endowment of  $\omega_{A,T} > 0$  apples and  $\omega_{B,T} > 0$  bananas, while Christina has an endowment of  $\omega_{A,C} > 0$  apples and  $\omega_{B,C} > 0$  bananas. Without loss of generality, the price of bananas is normalized to 1.

- (a) What is the Walrasian equilibrium of this exchange economy?
- (b) In equilibrium, what share of wealth does Tom spend on apples? What share of wealth does Christina spend on apples?

## Question 2 (10 points)

There is an island with only two consumers, Bob and Alice. There are two goods, apples ( $x$ ) and bananas ( $y$ ), available on the island. The consumers' utility function over bundles is given by:

$$\begin{aligned}\text{Bob: } u(x, y) &= x + y, \\ \text{Alice: } u(x, y) &= \min\{x, y\}.\end{aligned}$$

Bob's endowment is 1 apple and 0 bananas. Alice's endowment is 2 apples and 1 banana.

1. Show mathematically that no Walrasian equilibrium exists in this economy.

## Question 3 (15 points)

- (a) Prove the First Welfare Theorem.

- (b) Prove an equivalent characterization of Pareto-efficient allocations. Given any set of weights  $\mu_1, \dots, \mu_N \geq 0$  such that  $\sum \mu_i = 1$ , consider the solution  $x^*$  to the following problem:

$$\max_{x_1, \dots, x_N} \sum_{i=1}^N \mu_i u(x^i) \quad \text{subject to} \quad \sum_{i=1}^N x_i^j \leq \sum_{i=1}^N e_i^j \quad \forall j \in \{1, \dots, M\}.$$

Prove that any solution  $x^*$  is a Pareto-efficient allocation.

- (c) Provide an interpretation, in words, of what you showed in (b).

### Question 4 (20 points)

Consider a second-price auction with  $N$  bidders. Each bidder has an independent private value  $v_i$  drawn from a Uniform Distribution on  $[0, 1]$ .

- What is the expected revenue generated when all bidders bid truthfully? Provide a closed-form solution and show your work.
- Prove that all bidders bidding truthfully is an equilibrium of the second-price auction game. What is another equilibrium? Prove your example is an equilibrium.
- If the auctioneer sets a reserve price  $r$ , is it still a weakly dominant strategy to bid truthfully?
- Suppose  $N = 3$ . What is the optimal reserve price  $r$  the auctioneer should set? How much smaller/greater is the revenue compared to the auction with no reserve price?
- A student at another university thought revenue equivalence implied that the expected revenue should be the same. Explain why their reasoning is incorrect.

### Question 5 (15 points)

Suppose there are  $N$  bidders competing for a single object in an all-pay auction. Each bidder has an i.i.d. value  $v_i$  for the object drawn from some continuous distribution  $F$  with support  $[0, M]$ .

- Show that there is a symmetric equilibrium in increasing strategies.
- What is the expected revenue generated by this auction in the equilibrium from (a)? Explain your answer.

### Solution 1:

(a)

Given prices  $\{p_x, 1\}$  and allocations  $\{\omega_{A,T}, \omega_{A,C}, \omega_{B,T}, \omega_{B,C}\}$ , Tom solves:

$$\begin{aligned} u_T &= x^\alpha y^{1-\alpha} \\ \text{s.t. } p_x x + y &\leq p_x \omega_{A,T} + \omega_{B,T} \end{aligned}$$

and Cristina solves:

$$\begin{aligned} u_C &= x^\beta y^{1-\beta} \\ \text{s.t. } p_x x + y &\leq p_x \omega_{A,C} + \omega_{B,C} \end{aligned}$$