Homework 3

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Collaboration to varying degrees with Timothy Duhon, Josephine Hughes, Abdul Khan, Kasra Lak, Rachel Lobo, Mingzhou Wang, Wenyi Wang

An ECON - 8040 Homework Assignment

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Question 1

Problem

Consider the following two period planning problem

$$w(\bar{k}_1) = \max_{c_t, k_{t+1} \geq 0} \frac{c_1^{1-\sigma}}{1-\sigma} + \beta \frac{c_2^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_1 + k_2 = k_1^{\alpha} + (1-\delta)k_1$$
$$c_2 = k_2^{\alpha} + (1-\delta)k_2$$
$$k_1 = \bar{k}_1$$

The first order conditions for this problem is

$$c_1^{-\sigma} = \beta c_2^{-\sigma}(1 - \delta + \alpha k_2^{\alpha-1}).$$

Use the following parameters

β	σ	α	δ
0.95	2	0.4	0.1

Define

$$k_{ss} = \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

(a) Assume $\bar{k}_1 = k_{ss}$. Solve allocation of consumption and capital stock c_1, c_2, k_2. Note, you need to solve the following system of equations

$$c_1 + k_2 = k_1^{\alpha} + (1-\delta)k_1$$
$$c_2 = k_2^{\alpha} + (1-\delta)k_2$$
$$c_1^{-\sigma} = c_2^{-\sigma}\beta(1 - \delta + \alpha k_2^{\alpha-1})$$

q using the Newton method.

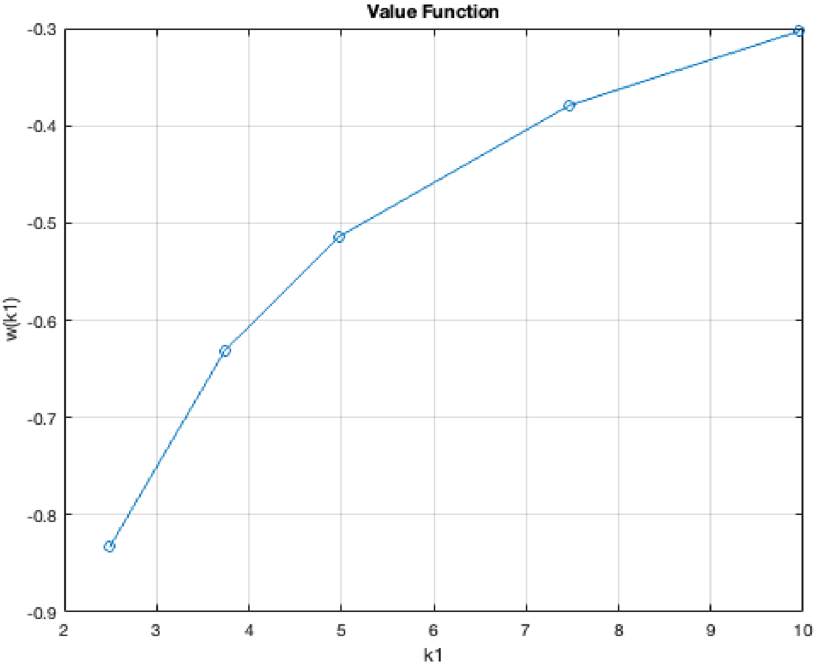
(b) Now, make the following grid $\mathcal{K} = \{\frac{1}{2}k_{ss}, \frac{3}{4}k_{ss}, k_{ss}, \frac{3}{2}k_{ss}, 2k_{ss}\}$ for $\bar{k}_0$. Solve allocations c_1, c_2, k_2 for all points on the grid. Using your answers, find value of $w(\bar{k}_1)$ for every point on the grid and plot $w(\bar{k}_1)$.

Solution

(a) Consumption in period 1 (c_1) is 3.7364

Consumption in period 2 (c_2) is 3.8593

Capital in period 2 (k_2) is 2.6478



(b)

k_1	c_1	c_2	k_2	$w(k_1)$
2.4907	2.2577	2.4341	1.4245	−0.8332
3.7361	3.0159	3.1670	2.0408	−0.6315
4.9815	3.7364	3.8593	2.6478	−0.5138
7.4722	5.1141	5.1759	3.8465	−0.3791
9.9630	6.4416	6.4387	5.0333	−0.3028

Question 2

Problem

Consider the following infinite horizon planning problem

$$\max_{c_t, k_{t+1} \geq 0} \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{C_t}{N_t}\right)^{1-\sigma}}{1-\sigma}$$

s.t.

$$\begin{aligned} C_t + K_{t+1} = AK_t^{\alpha}\left((1+\gamma)^t N_t\right)^{1-\alpha} + (1-\delta)K_t \\ N_{t+1} = (1+\eta)N_t \\ K_0 \quad \text{is given.} \end{aligned}$$

Where N_t is population, η is population growth rate, and γ is rate of growth of technology. A if TFP.

(a) Write this problem such that all variables are stationary.

(b) Write the problem in (a) recursively (as a Bellman). Write down the formula that determines steady state capita (per efficient units of labor).

(c) Assume $\sigma = 2, \eta = 0.01, \gamma = 0.02$. Choose parameters β, α, δ such that in the long run

$$\frac{K_{t+1} - (1-\delta)K_t}{Y_t} = 0.21,$$
$$\frac{K_t}{Y_t} = 3,$$
$$\text{Capital share of income} = \frac{1}{3}.$$

Choose A such that steady state capital (per efficient units of labor) is normalized to 1.

(d) Use a discretized grid for current stock of capital that has 100 nodes. For minimum and maximum capital, use $0.1k_{ss}$ and $2k_{ss}$. Solve the Bellman equation in part (b) using value function iteration.

(e) Start from $k_0 = 0.5k_{ss}$. Simulate the path of capital, consumption and output for 50 periods.

Solution

(a)

$$c_t = \frac{C_t}{(1+\eta)(1+\gamma)N_0}$$
$$k_t = \frac{K_t}{(1+\eta)(1+\gamma)N_0}$$

With this in mind, our new formula can be written as such

$$\max_{c_t, k_{t+1} \geq 0} \sum_{t+0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

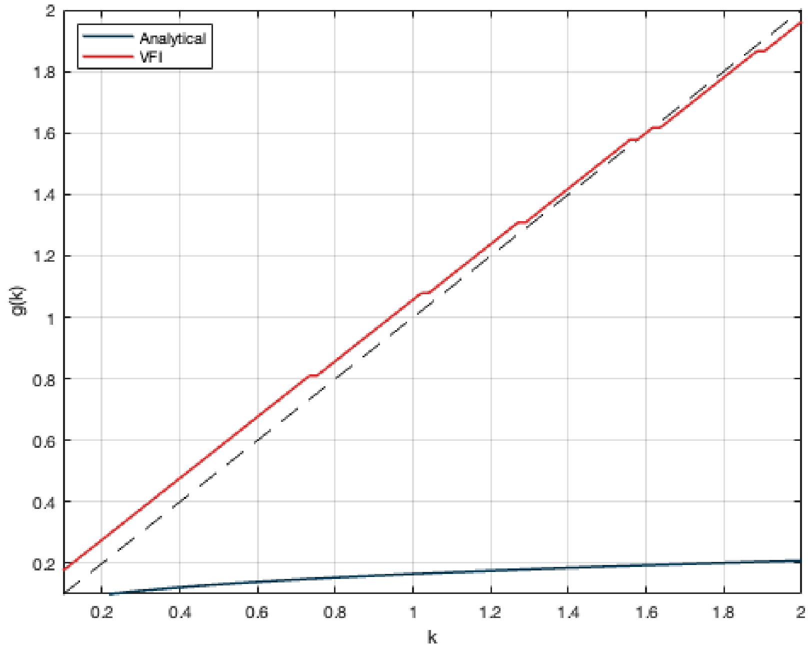
$$c_t = Ak_t^\alpha + (1-\delta) - (1+\gamma)(1+\eta)k_{t+1}$$

(b)

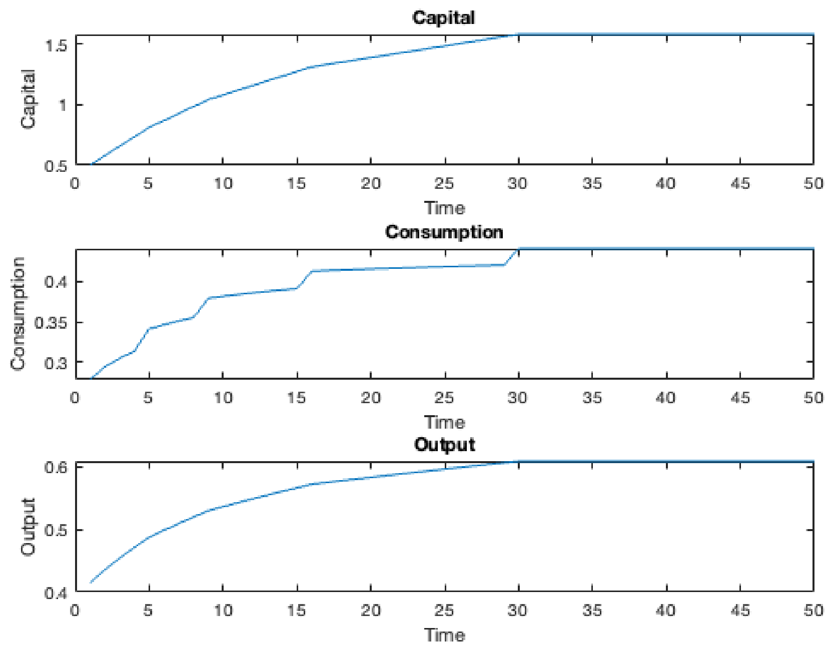
$$\max_{k_{t+1}} \left(\frac{(Ak_t^\alpha + (1-\delta) - (1+\gamma)(1+\eta)k_{t+1})^{1-\sigma}}{1-\sigma} + \beta\left(\frac{k_{t+1}^{1-\sigma}}{1-\sigma}\right)\right)$$
$$A\alpha k_{ss}^{\alpha-1} = \frac{(1+\gamma)(1+\eta)}{\beta} - (1-\delta)$$

(c)

$$\alpha = \frac{1}{3}$$
$$\delta \approx 0.0767$$
$$\beta \approx 0.9804$$
$$A \approx 0.5226$$



(d)



(e)

Note

I am fairly certain I am off in my calculations in the above parameters. Going to be tinkering over the weekend with it because I want to get good at these problems, but wanted to at least submit what I had. The VFI graph especially looks bad. The analytical and VFI are both concave, but not near each other like they are in your example code. Again, will keep working on this.