Homework 4

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An ECON - 8040 Homework Assignment

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Question 1

Problem

Consider the following planning problem

$$w(k_0) = \max_{k_{t+1}, c_t \geq 0} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + k_{t+1} \leq z k_t + (1-\delta) k_t$$
$$k_0 \text{ given}$$

(a) Write this problem as a dynamic programming problem.

(b) Solve the Bellman equation you wrote using guess-and-verify method (hint: try $v(k) = A\frac{k_t^{1-\sigma}}{1-\sigma}$).

(c) What is the growth rate of k_t and c_t in this economy? (use you answer in part (b) to answer this). Under what condition does the economy grow?

Solution

(a) The Bellman equation can be written as such

$$v(k_0) = \max_{k_1 \geq 0} \{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta v(k_1)\}$$

s.t.

$$c_t + k_1 \leq z k_0 + (1-\delta) k_0$$

(b) Solving the Bellman equation.

First, let's plug in $v(k_1) = A\frac{k_1^{1-\sigma}}{1-\sigma}$

$$\max_{k_1 \geq 0} \{\frac{(zk_0 + (1-\delta)k_0 - k_1)^{1-\sigma}}{1-\sigma} + \beta A \frac{k_1^{1-\sigma}}{1-\sigma}\}$$

Then, find the FOC w.r.t. k_1

$$-(zk_0 + (1-\delta)k_0 - k_1)^{-\sigma} + \beta A k_1^{-\sigma}$$
$$\beta A k_1^{-\sigma} = (zk_0 + (1-\delta)k_0 - k_1)^{-\sigma}$$

Now, solve for k_1

$$(\beta A)^{\frac{1}{-\sigma}} k_1 = zk_0 + (1-\delta)k_0 - k_1$$
$$k_1((\beta A)^{\frac{1}{-\sigma}} + 1) = zk_0 + (1-\delta)k_0$$
$$\boxed{k_1 = \frac{zk_0 + (1-\delta)k_0}{1 + (\beta A)^{\frac{1}{-\sigma}}}}$$

A quick simplification:

$$R \equiv z + (1-\delta)$$

Plug back into maximization and solve for A

$$A\frac{k_0^{1-\sigma}}{1-\sigma} = \frac{(Rk_0 - k_1)^{1-\sigma}}{1-\sigma} + \beta A \frac{k_1^{1-\sigma}}{1-\sigma}$$
$$\boxed{A = (\frac{R}{(1+\beta A)^{\frac{1}{\sigma}}})^{1-\sigma} + (\frac{\beta A}{1 + (\beta A)^{\frac{1}{\sigma}}})^{1-\sigma}}$$

0.0.1 (c) Growth rate of capital:

$$g(k) = \frac{k_1}{k_0}$$
$$k_1 = k_0 \times \frac{R}{1+(\beta A)^{\frac{1}{\sigma}}}$$
$$\boxed{g(k) = \frac{R}{1+(\beta A)^{\frac{1}{\sigma}}}}$$

Because consumption only relies on capital in this equation, it is reasonable to assume that $g(c) = g(k)$. This would imply growth occurs when $R > 1 + (\beta A)^{\frac{1}{\sigma}}$.

Question 2

Problem

Let n_t denote hours worked. Consider the following planning problem (n_t is hours worked)

$$w(k_0) = \max \sum_{t=0}^{\infty} \beta^t [\theta \log c_t + (1-\theta)\log(1-n_t)]$$

s.t.

$$c_t + k_{t+1} \leq z k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$$
$$k_{t+1}, c_t \geq 0$$
$$1 \geq n_t \geq 0$$
$$k_0 \text{ given}$$

We want to transform this problem to a formulation that we can solve using dynamic programming (Note that if $\alpha = 1$ this problem is a special case of question 1. Also, if $\theta = 1$ and $\delta = 1$ this transforms to the example we solved in class). To do this, we break it down to few steps. Consider the problem of choosing optimal hours worked, given current capital k and future capital k'. Write down the optimality condition for n in the following

$$F(k,k') = \max_n \theta \log(z k^{\alpha} n^{1-\alpha} + (1-\delta)k - k') + (1-\theta)\log(1-n)$$

you won't be able to find a closed form solution for n (yet). Call the optimal solution to the above problem $n(k,k')$.

(a) Write the planning problem as a dynamic programming problem (hint: utilize the function $F(k,k')$ above which is equivalent to period utility function for optimally chosen c and n, taking k and k' as given).

(b) From now on, assume $\delta = 1$. Now guess the value function $V(k) = A + B\log k$. Derive optimal condition for k' (just like we did for example with $\theta = 1$ in class). Solve for k' as function of parameters of B and n.

(c) Rewrite the optimality condition in (a), replace for k' from (c) and solve for optimal n (this won't depend on k, only on parameters and B). Remember to impose $\delta = 1$.

(d) Now, replace optimal k' and n in the Bellman equation and solve for coefficient B.

(e) Write the optimal policy functions for n, k', and c as a function of k.

Solution

(a) The Bellman equation is as follows

$$v(k_0) = \max_{k} \{F(k, k') + \beta v(k')\}$$

s.t.

$$c_t + k' \leq z k^\alpha n_t 1 - \alpha + (1-\delta)k$$

(b) Now, let's find optimal k'

$$\theta \log(zk^\alpha n^{1-\alpha} + (1-\delta)k - k') + (1-\theta)\log(1-n) + \beta(A + B\log(k'))$$
$$-\tfrac{\theta}{zk^\alpha n^{1-\alpha} - k'} + \tfrac{\beta B}{k'} = 0$$
$$\tfrac{\theta}{zk^\alpha n^{1-\alpha} - k'} = \tfrac{\beta B}{k'}$$
$$\boxed{k' = \frac{(\beta B(zk^\alpha n^{1-\alpha}))}{\theta + \beta B}}$$

(c) Solving for optimal n

$$n(k, k') = \theta \log(zk^\alpha n^{1-\alpha} - \tfrac{(\beta B(zk^\alpha n^{1-\alpha}))}{\theta + \beta B}) + (1-\theta)\log(1-n)$$

First order condition with respect to n

$$\tfrac{\theta(1-\alpha)(zk^\alpha n^{1-\alpha})}{zk^\alpha n^{1-\alpha}} = \tfrac{1-\theta}{1-n}$$
$$\tfrac{\theta(1-\alpha)}{\theta+\beta B} = \tfrac{1-\theta}{1-n}$$
$$\boxed{n = \frac{\theta(1-\alpha) + \beta B}{\theta + \beta B}}$$

(d) Solving for B

$$A + B\log k = \theta \log(\tfrac{\theta z k^\alpha n^{1-\alpha}}{\theta + \beta B}) + (1-\theta)\log(1-n) + \beta A + \beta B \log(\tfrac{\beta B z k^\alpha n^{1-\alpha}}{\theta+\beta B})$$
$$= \theta\alpha\log(k) + \theta\log(z) + \theta(1-\alpha)\log(n) - \theta\log(\theta+\beta B) + (1-\theta)\log(1-n) + \beta A +$$
$$\beta B\alpha\log(k) + \beta B\log(z) + (1-\alpha)\beta B\log(n) - \beta B\log(\theta + \beta B)$$

After many steps of simplification and grouping, we arrive at

$$\boxed{B = \frac{\theta\alpha}{1-\alpha\beta}}$$

(e) Optimal Policy Functions

$$g(n) = \tfrac{\theta(1-\alpha)+\beta B}{\theta+\beta B}$$
$$\boxed{g(n) = \frac{\theta(1-\alpha) + \beta\frac{\theta\alpha}{1-\alpha\beta}}{\theta + \beta\frac{\theta\alpha}{1-\alpha\beta}}}$$
$$g(k) = \tfrac{\beta B z k^\alpha n^{1-\alpha}}{\theta+\beta B}$$
$$\boxed{g(k) = \frac{\beta\frac{\theta\alpha}{1-\alpha\beta} z k^\alpha n^{1-\alpha}}{\theta + \beta\frac{\theta\alpha}{10-\alpha\beta}}}$$
$$\boxed{g(c) = zk^\alpha(\frac{\theta(1-\alpha) + \beta\frac{\theta\alpha}{1-\alpha\beta}}{\theta + \beta\frac{\theta\alpha}{1-\alpha\beta}})^{1-\alpha} - \frac{\beta\frac{\theta\alpha}{1-\alpha\beta} z k^\alpha n^{1-\alpha}}{\theta + \beta\frac{\theta\alpha}{10-\alpha\beta}}}$$

Problem 3

Problem

Consider an economy with two types of capital. Let k_t denote physical capital and h_t denote human capital. Output is produced using physical and human capital

$$y_t = zk_t^{\alpha}h_t^{1-\alpha}.$$

The final good can be consumed, or can be invested in making physical or human capital. Assume both types of capital depreciate at rate δ. Assume households have log preferences over consumption (and labor is inelastically supplied).

(a) Write the planning problem that maximizes the welfare of the representative household for an initial stock of physical and human capital (k_0, h_0). Denote the value to the planner as $w(k_0, h_0)$.

(b) Write the planning problem recursively. This means write a Bellman equation that corresponds to the planning problem in part (a).

(c) Assume full depreciation $\delta = 1$. Solve the Bellman equation using guess and verify. Find the optimal policy functions for future physical and human capital as function of current physical and human capital.

Solution

(a) Planning Problem

$$w(k_0, h_0) = \max_{k',h'} \textstyle\sum_{t=0}^{\infty} \beta^t (\log c_t)$$

s.t.

$$\begin{gathered} c_t + k_{t+1}h_{t+1} \leq y + (1-\delta)k_t + (1-\delta)h_t \\ y = zk^{\alpha}h^{1-\alpha} \\ h, k > 0 \end{gathered}$$

(b) Bellman Equation

$$\begin{aligned} v(k_0, h_0) &= \max_{k_{t+1},h_{t+1}} \{\log y_t + (1-\delta)(k_t + h_t) - k_{t+1} - h_{t+1} + \beta V(k_{t+1}, h_{t+1})\} \\ v(k_0, h_0) &= \max_{k_{t+1},h_{t+1}} \{\log(zk^{\alpha}h^{1-\alpha} + (1-\delta)(k_t + h_t) - k_{t+1} - h_{t+1}) + \beta V(k_{t+1}, h_{t+1})\} \end{aligned}$$

(c) Guess and Verify

With $\delta = 1$, the Bellman equation is as follows

$$v(k_0, h_0) = \max_{k_{t+1},h_{t+1}} \{\log(zk^{\alpha}h^{1-\alpha} - k_{t+1} - h_{t+1}) + \beta(A + B\log(k_{t+1}) + C\log(h_{t+1}))\}$$

Taking derivatives with respect to $k_t + 1$ and h_{t+1}

$$\boxed{\frac{\partial v}{\partial k_{t+1}} : \quad \frac{1}{zk^{\alpha}h^{1-\alpha} - h_{t+1} - k_{t+1}} = \frac{\beta B}{k_{t+1}}}$$

$$\boxed{\frac{\partial v}{\partial h_{t+1}} : \quad \frac{1}{zk^{\alpha}h^{1-\alpha} - k_{t+1} - h_{t+1}} = \frac{\beta C}{h_{t+1}}}$$

Simplifying and Solving

$$\boxed{h_{t+1} = \frac{\beta C(zk^{\alpha}h^{1-\alpha})}{1+\beta(B+C)}}$$
$$\boxed{k_{t+1} = \frac{\beta B(zk^{\alpha}h^{1-\alpha})}{1+\beta(B+C)}}$$

Plugging back into Bellman and simplifying logs

$$A + B\log k + C\log h =$$
$\log z + \alpha\log k + (1-\alpha)\log h - \log(1+\beta(B+C)) + \beta A + \beta B\log(\beta Bz) + \beta B(\alpha\log k + (1-\alpha)\log h) - \beta B(1+\beta(B+C)) + \beta C\log(\beta Cz) + \beta C(\alpha\log k + (1-\alpha)\log h)) - \beta C(1+\beta(B+C))$

After a bunch of algebra and grouping,

$$\boxed{B = \frac{\alpha}{1-\beta}}$$
$$\boxed{C = \frac{(1-\alpha)}{1-\beta}}$$

Next, solutions

$$\boxed{V(k,h) = A + \frac{\alpha}{1-\beta}\log k + \frac{1-\alpha}{1-\beta}\log h}$$
$$\boxed{k_{t+1} = \frac{\beta\alpha zk^{\alpha}h^{1-\alpha}}{1+\beta}}$$
$$\boxed{h_{t+1} = \frac{\beta(1-\alpha)zk^{\alpha}h^{1-\alpha}}{1+\beta}}$$

Question 4

Problem

Consider the following planning problem

$$w(\bar{k}_0) = \max_{\{(c_t,k_{t+1},x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t\theta_t\log c_t$$

s.t.

$$c_t + k_{t+1} = k_t$$
$$k_0 = \bar{k}_0$$

and

$$\theta_t = \begin{cases} \theta_H & t\ iseven \\ \theta_L & t\ isodd \end{cases}$$

(a) Write this problem only in terms of a sequence of capital $\{k_{t+1}\}_{t=0}^{\infty}$.

(b) Write the problem in part (a) as a Bellman equation(s). (hint: Note that we have an additional (exogenous) state variable θ_t. So you need to write two equations, one for $v(k,\theta_L)$ and one for $v(k,\theta_H)$. Also, we know how θ_t evolves.)

(c) Solve the Bellman equation in part (b) using guess and verify method.

(d) Write the formula for optimal policy functions $g(k, \theta_L)$ and $g(k, \theta_H)$.

(e) Start from $k_0 = 1$. Describe how you will simulate the optimal path of capital stock $\{k_{t+1}\}_0^\infty$. Write down k_t for $k = 1, 2, 3$.

Solution

(a) Planning Problem

$$w(\bar{k}_0) = \max_{\{k_t+1_{t=0}^\infty\}} \sum_{t=0}^\infty \beta^t \theta_t \log(k_t - k_{t+1})$$

s.t.

$$0 \le k_{t+1} \le k_t$$
$$k_0 = \bar{k}_0$$

(b) Bellman Equations

$$v(k, \theta_H) = \max_{k_{t+1}} \{\theta_H \log(k_t - k_{t+1}) + \beta V(k_{t+1}, \theta_L)\}$$
$$v(k, \theta_L) = \max_{k_{t+1}} \{\theta_L \log(k_t - k_{t+1}) + \beta V(k_{t+1}, \theta_H)\}$$

(c) Solving with Guess and Verify

Find the first order conditions for both Bellman Equations

$$v(k, \theta_H) = \max_{k_{t+1}} \{\theta_H \log(k_t - k_{t+1}) + \beta(A + B \log(k_{t+1}))\}$$
$$v(k, \theta_L) = \max_{k_{t+1}} \{\theta_L \log(k_t - k_{t+1}) + \beta(C + D \log(k_{t+1}))\}$$
$$\boxed{\frac{\theta_H}{(k_t - k_{t+1})} = \frac{\beta D}{k_{t+1}}}$$
$$\boxed{\frac{\theta_L}{(k_t - k_{t+1})} = \frac{\beta B}{k_{t+1}}}$$

Now, solving for k_{t+1} in both FOC's

$$\boxed{k_{t+1} = \frac{\beta D k}{\theta_H + \beta D} \text{ for } v(k, \theta_H)}$$
$$\boxed{k_{t+1} = \frac{\beta B k}{\theta_L + \beta B} \text{ for } v(k, \theta_L)}$$

Finally, solving for B and D

$$\boxed{D = \theta_L + \beta\theta_H}$$
$$\boxed{B = \theta_H + \beta\theta_L}$$

(d) Solving the Optimal policy function

Solve for k_{t+1} with the solved B and D.

$$g(k, \theta_H) = \tfrac{\beta(\theta_H + \beta\theta_L)k}{\theta_H + \beta(\theta_L + \beta\theta_H)}$$
$$g(k, \theta_L) = \tfrac{\beta(\theta_H + \beta\theta_L)k}{\theta_L + \beta(\theta_H + \beta\theta_L)}$$

(e) Simulation

Given $k_0 = 1$ and assuming k_1 is an odd day:

$$
\begin{gathered}
k_1 = g(1, \theta_L) = \frac{(\beta(\theta_H + \beta\theta_L))}{(\theta_L + \beta(\theta_H + \beta\theta_L))} \\
k_2 = g(k_1, \theta_H) = \frac{\beta(\theta_L + \beta\theta_H)\frac{(\beta(\theta_H + \beta\theta_L))}{(\theta_L + \beta(\theta_H + \beta\theta_L))}}{(\theta_H + \beta(\theta_L + \beta\theta_H))} = \beta^2 \\
k_3 = g(k_2, \theta_L) = \frac{\beta(\theta_H + \beta\theta_L)\beta^2}{(\theta_L + \beta(\theta_H + \beta\theta_L))} = \frac{\beta^3(\theta_H + \beta\theta_L)}{\theta_L + \beta(\theta_H + \beta\theta_L)}
\end{gathered}
$$