

# Homework 4

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## 1 Question 1

### 1.1 Problem

Consider the following planning problem

$$w(k_0) = \max_{k_{t+1}, c_t \geq 0} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &\leq zk_t + (1 - \delta)k_t \\ k_0 &\text{ given} \end{aligned}$$

- Write this problem as a dynamic programming problem.
- Solve the Bellman equation you wrote using guess-and-verify method (hint: try  $v(k) = A \frac{k_t^{1-\sigma}}{1-\sigma}$ ).
- What is the growth rate of  $k_t$  and  $c_t$  in this economy? (use you answer in part (b) to answer this). Under what condition does the economy grow?

## 2 Question 2

### 2.1 Problem

Let  $n_t$  denote hours worked. Consider the following planning problem ( $n_t$  is hours worked)

$$w(k_0) = \max \sum_{t=0}^{\infty} \beta^t [\theta \log c_t + (1 - \theta) \log(1 - n_t)]$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &\leq zk_t^\alpha n_t^{1-\alpha} + (1 - \delta)k_t \\ k_{t+1}, c_t &\geq 0 \\ 1 &\geq n_t \geq 0 \\ k_0 &\text{ given} \end{aligned}$$

We want to transform this problem to a formulation that we can solve using dynamic programming (Note that if  $\alpha = 1$  this problem is a special case of question 1. Also, if  $\theta = 1$  and  $\delta = 1$  this transforms to the example we solved in class). To do this, we break it down to few steps. Consider the problem of choosing optimal hours worked, given current capital  $k$  and future capital  $k'$ . Write down the optimality condition for  $n$  in the following

$$F(k, k') = \max_n \theta \log(zk^\alpha n^{1-\alpha} + (1 - \delta)k - k') + (1 - \theta) \log(1 - n)$$

you won't be able to find a closed form solution for  $n$  (yet). Call the optimal solution to the above problem  $n(k, k')$ .

- Write the planning problem as a dynamic programming problem (hint: utilize the function  $F(k, k')$  above which is equivalent to period utility function for optimally chosen  $c$  and  $n$ , taking  $k$  and  $k'$  as given).

- From now on, assume  $\delta = 1$ . Now guess the value function  $V(k) = A + B \log k$ . Derive optimal condition for  $k'$  (just like we did for example with  $\theta = 1$  in class). Solve for  $k'$  as function of parameters of  $B$  and  $n$ .

- (c) Rewrite the optimality condition in (a), replace for  $k'$  from (c) and solve for optimal  $n$  (this won't depend on  $k$ , only on parameters and  $B$ ). Remember to impose  $\delta = 1$ .
- (d) Now, replace optimal  $k'$  and  $n$  in the Bellman equation and solve for coefficient  $B$ .
- (e) Write the optimal policy functions for  $n, k'$ , and  $c$  as a function of  $k$ .

### 3 Question 3

#### 3.1 Problem

Consider an economy with two types of capital. Let  $k_t$  denote physical capital and  $h_t$  denote human capital. Output is produced using physical and human capital

$$y_t = z k_t^\alpha h_t^{1-\alpha}.$$

The final good can be consumed, or can be invested in making physical or human capital. Assume both types of capital depreciate at rate  $\delta$ . Assume households have log preferences over consumption (and labor is inelastically supplied).

- (a) Write the planning problem that maximizes the welfare of the representative household for an initial stock of physical and human capital  $(k_0, h_0)$ . Denote the value to the planner as  $w(k_0, h_0)$ .
- (b) Write the planning problem recursively. This means write a Bellman equation that corresponds to the planning problem in part (a).
- (c) Assume full depreciation  $\delta = 1$ . Solve the Bellman equation using guess and verify. Find the optimal policy functions for future physical and human capital as function of current physical and human capital.

### 4 Question 4

#### 4.1 Problem

Consider the following planning problem

$$w(\bar{k}_0) = \max_{\{(c_t, k_{t+1}, x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \theta_t \log c_t$$

s.t.

$$\begin{aligned} c_t + k_{t+1} &= k_t \\ k_0 &= \bar{k}_0 \end{aligned}$$

and

$$\theta_t = \begin{cases} \theta_H & t \text{ is even} \\ \theta_L & t \text{ is odd} \end{cases}$$

- (a) Write this problem only in terms of a sequence of capital  $\{k_{t+1}\}_{t=0}^{\infty}$ .
- (b) Write the problem in part (a) as a Bellman equation(s). (hint: Note that we have an additional (exogenous) state variable  $\theta_t$ . So you need to write two equations, one for  $v(k, \theta_L)$  and one for  $v(k, \theta_H)$ . Also, we know how  $\theta_t$  evolves.)

- (c) Solve the Bellman equation in part (b) using guess and verify method.
- (d) Write the formula for optimal policy functions  $g(k, \theta_L)$  and  $g(k, \theta_H)$ .
- (e) Start from  $k_0 = 1$ . Describe how you will simulate the optimal path of capital stock  $\{k_{t+1}\}_0^\infty$ . Write down  $k_t$  for  $k = 1, 2, 3$ .