Homework 6

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Due on Tuesday October 29, by 11:49pm

An ECON - 8040 Homework Assignment

November 4, 2024

Question 1

Problem

Consider the example of our endowment economy with two types of households. Assume utility function is

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \text{ for } \sigma > 0$$

Also assume endowments growth at constant rate, i.e.,

$$e_t^1 = \begin{cases} 2\gamma^t & \text{if t is even} \\ 0 & \text{if t is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 & \text{if t is even} \\ 2\gamma^t & \text{if t is odd} \end{cases}$$

Let $0 < \beta < 1$ be discount rate and assume $\beta\gamma^{1-\sigma} < 1$. (Note: for $\sigma = 1$, $u(c) = \log(c)$. Therefore the example we studied in class was a special case of this problem with $\gamma = 1$ and $\sigma = 1$)

Parts

(a)

Define (Arrow-Debreu) competitive equilibrium.

(b)

Drive household's Euler equation.

(c)

Write down a social planner problem for some Pareto weights and solve for Pareto efficient allocations (treat Pareto weights as parameters). hint: it is easier to solve everything in terms of the ratio of weights $\alpha \equiv \frac{\alpha_2}{\alpha_1}$.

(d)

Use Negishi Method to solve for competitive equilibrium allocations and prices.

(e)

Find Pareto weights that generate the following allocation $(c_t^1, c_t^2) = (\gamma^t, \gamma^t)$. Find transfers needed to implement this allocation in a competitive equilibrium.

(f)

How does growth rate of consumption depend on γ and σ? How do equilibrium prices depend on γ and σ?

(g)

Define a sequential market competitive equilibrium. Find interest rate in this equilibrium. How do equilibrium interest rate depend on γ and σ?

Solutions

(a)

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{{c_t^i}^{1-\sigma} - 1}{1-\sigma} \right)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$
$$\hat{c}_t^i \geq 0$$
$$0 < \beta < 1$$

Markets Clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = \hat{e}_t^1 + \hat{e}_t^2 \ \forall \ t$$

(b)

$$c_t^i : \frac{\beta^t}{{c_t^i}^{\sigma}} - \lambda p_t$$
$$c_{t+1}^i : \frac{\beta^{t+1}}{{c_{t+1}^i}^{\sigma}} - \lambda p_{t+1}$$
$$\beta \frac{{c_{t+1}^i}^{\sigma}}{{c_t^i}^{\sigma}} = \frac{p_t}{p_{t+1}}$$
$${c_{t+1}^i}^{\sigma} p_{t+1} = \beta {c_t^i}^{\sigma} p_t$$
$${c_{t+1}^i}^{\sigma} = \beta \frac{p_t}{p_{t+1}} {c_t^i}^{\sigma}$$
$${c_{t+1}^i}^{\sigma} = \beta \frac{p_t}{p_{t+1}} {c_t^i}^{\sigma}$$

(c)

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\alpha_1 \frac{{c_t^1}^{1-\sigma} - 1}{1-\sigma} + \alpha_2 \frac{{c_t^2}^{1-\sigma}}{1-\sigma}]$$

s.t.

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = 2\gamma^t; \frac{\mu}{2}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \alpha \sum_{t=0}^{\infty} \beta^t \frac{{c_t^i}^{1-\sigma} - 1}{1-\sigma} + \frac{\mu}{2}[\sum_{t=0}^{\infty} e_t^i - \sum_{t=0}^{\infty} c_t^i]$$
$$\mathcal{L}_c = \alpha\beta c^{-\sigma} - \frac{\mu}{2} = 0$$
$$c_t^1 : \alpha_1 \beta c_t^{1-\sigma} - \frac{\mu}{2}$$
$$c_t^2 : \alpha_2 \beta c_t^{2-\sigma} - \frac{\mu}{2}$$

Divide the two FOC's

$$\frac{\alpha_1 \beta c_t^{1-\sigma} - \frac{\mu}{2}}{\alpha_2 \beta c_t^{2-\sigma} - \frac{\mu}{2}}$$
$$1 = \frac{\alpha_1}{\alpha_2} (\frac{c_t^1}{c_t^2})^{-\sigma}$$
$$(\frac{\alpha_2}{\alpha_1})^{\sigma} = \frac{c_t^1}{c_t^2}$$
$$\alpha \equiv \frac{\alpha_2}{\alpha_1} \Rightarrow c_t^1 = c_t^2 \alpha^{\sigma}$$
$$\alpha^{\sigma} c_t^2 + c_t^2 = 2\gamma$$
$$c_t^2 = \frac{2\gamma}{1+\alpha^{\sigma}}$$
$$c_t^1 = \frac{2\gamma\alpha^{\sigma}}{1+\alpha^{\sigma}}$$

(d)

Solving for μ

$$c_t^1 : \frac{\mu_t}{2\alpha_1} = \frac{\beta^t}{c_t^{1\sigma}}$$
$$c_{t+1}^1 : \frac{\mu_{t+1}}{2\alpha_1} = \frac{\beta^{t+1}}{c_{t+1}^{1^{\sigma}}}$$
$$\frac{\mu_{t+1}}{mu_t} = \frac{\beta^{t+1}}{c_{t+1}^{\sigma}} \cdot \frac{c_t^{1\sigma}}{\beta_t} \to \frac{\mu_{t+1}}{\mu_t} = \beta(\frac{\frac{2\gamma^t \alpha^{\sigma}}{1+\alpha^{\sigma}}}{\frac{2\gamma^{t+1}\alpha^{\sigma}}{1+\alpha^{\sigma}}})$$
$$\frac{\mu_{t+1}}{\mu} = \beta\gamma^{\sigma} \to \mu_t = \beta^t \gamma^{t\sigma} \mu_0$$

Finding C.E. allocations:

$$
\begin{aligned}
t^1(\alpha) &= \frac{2\alpha^\sigma}{(1+\alpha^\sigma)}\sum_{t=0}^{\infty}(\beta\gamma^\sigma)^t - 2\sum_{t=0}^{\infty}(\beta\gamma^\sigma)^{2t} \\
&= \frac{2\alpha^\sigma}{(1+\alpha^\sigma)}\frac{1}{(1-\beta\gamma^\sigma)} - \frac{2\beta\gamma^\sigma}{(1-\beta\gamma^\sigma)^2} \\
t^2\alpha &= \frac{2}{1+\alpha^\sigma}\frac{1}{(1-\beta\gamma^\sigma)} - \frac{2}{(1-\beta\gamma^\sigma)^2} \\
\frac{1}{1+\alpha^\sigma} &= \frac{1}{1+\beta\gamma^\sigma} \to \alpha = \beta^{\frac{1}{\sigma}}\gamma \\
&\boxed{c_t^1 = \frac{2\gamma^t\beta^{\frac{1}{\sigma}}\gamma}{1+\beta^{\frac{1}{\sigma}}\gamma}} \\
&\boxed{c_t^2 = \frac{2\gamma^t}{1+\beta^{\frac{1}{\sigma}}\gamma}}
\end{aligned}
$$

(e)

$$
\begin{aligned}
c_t^2 = \frac{2\gamma^t}{1+\alpha^\sigma} &= \gamma^t \\
\frac{2}{1+\alpha^\sigma} &= 1 \\
2 &= 1+\alpha^\sigma \\
\alpha &= 1 \\
t^2(1) &= \frac{1}{(1-\beta\gamma^\sigma)} - \frac{2}{(1-\beta\gamma^\sigma)^2} \\
&= \frac{1+\beta\gamma^\sigma-2}{(1-\beta\gamma^\sigma)(1+\beta\gamma^\sigma)} \\
&\boxed{t^2(1) = -\frac{2}{1-\beta\gamma^\sigma}} \\
&\boxed{t^1(1) = \frac{2}{1-\beta\gamma^\sigma}}
\end{aligned}
$$

This is the case since the transfers must add to 0.

(f)

$$
\begin{aligned}
g_t &= \frac{c_{t+1}^1 + c_{t+1}^2}{c_t^1 + c_t^2} \\
&= \frac{2\gamma^{t+1}(1+\beta^{\frac{1}{\sigma}}\gamma)}{2\gamma^t(1+\beta^{\frac{1}{\sigma}}\gamma)} \\
&= \gamma
\end{aligned}
$$

The first order condition of the growth formula w.r.t γ is 1, thus showing that growth is positively correlated with an increase in endowment. The first order condition of the growth formula with respect to σ is 0, showing that risk aversion plays no part.

(g)

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{i,1-\sigma} - 1}{1-\sigma})$$

s.t.

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leq e_t^i a_t^i$$
$$c_t^i \geq 0$$
$$a_{t+1}^i \geq -\bar{A}^i$$

Markets clear:

$$\sum_{i=1}^{2} \hat{c}_t^i = \sum_{i=1}^{2} \hat{e}_t^i \ \forall \ t$$
$$\sum_{i=1}^{2} \hat{a}_{t+1}^i = 0$$

Lagrangian and FOC's:

$$\mathcal{L} = \beta^t (\frac{c_t^{i,1-\sigma} - 1}{1-\sigma}) + \lambda_t (e_t^i + a_t^i - c_t^i - \frac{a_{t+1}^i}{1+r_{t+1}})$$
$$\mathcal{L}_{c_t} = \beta^t c_t^{-\sigma} = \lambda_t$$
$$\mathcal{L}_{c_{t+1}} = \beta^{t+1} c_{t+1}^{-\sigma} = \lambda_{t+1}$$
$$\mathcal{L}_{a_{t+1}} = -\lambda_t \frac{1}{1+r_{t+1}} + \lambda_{t+1}$$

Put equations together:

$$\beta^t \frac{1}{c_t^\sigma} = \frac{1}{1+r_{t+1}} + \beta^{t+1} \frac{1}{c_{t+1}^\sigma}$$
$$c_{t+1} = [\beta(1+r_{t+1})]^{-\sigma} c_t$$

From intemporal budget constraint:

$$\frac{c_t^1 + c_t^2}{c_{t+1}^1 + c_{t+1}^2} = \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$\frac{c_t^1 + c_t^2}{(\beta(1+r_{t+1}))^{-\sigma}(c_1 + c_2)} = \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$\boxed{r_{t+1} = \frac{1}{\beta} (\frac{e_{t+1}^1 + e_{t+1}^2}{e_t^1 + e_t^2})^{\frac{1}{\sigma}} - 1}$$
$$\boxed{r_{t+1} = \frac{(2\gamma)^{\frac{1}{\sigma}}}{\beta} - 1}$$

So interest rate depends on endowment and risk aversion in this case. Endowment effects are weighted by level of risk aversion in the exponent. Thus, it can be said that regardless of level of endowment, risk aversion will determine interest rates.

Question 2

Problem

Consider the economy in question 1. Assume $\sigma = 1$ (so log utility) and $\gamma = 1$. Assume the following endowments

$$e_t^1 = (2, 0, 2, 0, 2, \ldots)$$
$$e_t^2 = (2, 2, 2, 2, 2, \ldots)$$

Parts

(a)

Find equilibrium allocations and prices. Are prices different than the one we derived during lecture? Why?

(b)

Define a sequential market competitive equilibrium. Find interest rate in this equilibrium.

Solutions

(a)

$$\max_{c_{t t=0}^{i \infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$
$$\hat{c}_t^i \geq 0$$
$$0 < \beta < 1$$

Markets Clear:

$$e_t^1 + e_t^2 = c_t^1 + c_t^2 = \begin{cases} 4, \text{if } t \text{ is odd} \\ 2, \text{if } t \text{ is even} \end{cases}$$

Lagrange and FOC's:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) + \lambda[p(e_t - c_t)]$$
$$\mathcal{L}_{c_t} = \frac{\beta^t}{c_t^i} = \lambda p_t$$
$$\mathcal{L}_{c_{t+1}} = \frac{\beta^{t+1}}{c_{t+1}^i} = \lambda p_{t+1}$$
$$\frac{p_t}{p_{t+1}} = \frac{\beta^t c_{t+1}}{\beta^{t+1} c_t}$$
$$p_{t+1} c_{t+1}^i = \beta p_t c_t^i$$
$$c_{t+1}^i = \beta \frac{p_t}{p_{t+1}} c_t^i$$
$$c_t^i = \beta^t \frac{p_0}{p_t} c_0^i$$

Plug back into constraint

$$\sum_{t=0}^{\infty} p_t \beta \frac{p_0}{p_t} c_0^i = \sum_{t=0}^{\infty} p_t e_t^i$$
$$p_0 c_0^i \sum_{t=0}^{\infty} \beta^t = p_t e_t^i$$
$$c_0^i = \frac{\sum_{t=0}^{\infty} p_t e_t^i}{p_0 \sum_{t=0}^{\infty} \beta^t}$$
$$c_0^i = \frac{(1-\beta)}{p_0} \sum_{t=0}^{\infty} p_t e_t^i$$

Plug back into solution for c_t^i

$$c_t^i = \beta^t \frac{p_0}{p_t} \frac{(1-\beta)}{p_0} \sum_{t=0}^{\infty} p_t e_t^i$$
$$c_t^i = \frac{\beta^t (1-\beta)}{p_t} \sum_{t=0}^{\infty} p_t e_t^i$$

Plug into market clearing

$$\frac{\beta^t(1-\beta)}{p_t}\sum_{t=0}^{\infty} p_t e_t^1 + \frac{\beta^t(1-\beta)}{p_t}\sum_{t=0}^{\infty} p_t e_t^2 = e_t^1 + e_t^2$$
$$\frac{\beta^{t+1}(1-\beta)}{p_{t+1}}\sum_{t=0}^{\infty} p_t e_t^1 + \frac{\beta^{t+1}(1-\beta)}{p_{t+1}}\sum_{t=0}^{\infty} p_t e_t^2 = e_{t+1}^1 + e_{t+1}^2$$

Divide the two

$$\frac{p_{t+1}}{\beta p_t} = \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$p_{t+1} = \beta p_t \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$\boxed{p_t = \beta p_0 \frac{e_0^1 + e_0^2}{e_t^1 + e_t^2}}$$

If t is even:

$$p_t = \beta^t p_0 \frac{2}{2}$$
$$\boxed{p_t = \beta^t p_0}$$
$$c_t^i = \frac{\beta^t(1-\beta)}{\beta^t p_0}\sum_{t=0}^{\infty} \beta^t p_0 e_t^i$$
$$c_t^i = (1-\beta)\sum_{t=0}^{\infty} \beta^t e_t^i$$

If t is odd:

$$p_t = \beta^t p_0 \frac{4}{2}$$
$$\boxed{p_t = 2\beta^t p_0}$$
$$c_t^i = \frac{\beta^t(1-\beta)}{2\beta^t p_0}\sum_{t=0}^{\infty} 2\beta^t p_0 e_t^i$$
$$c_t^i = (1-\beta)\sum_{t=0}^{\infty} \beta^t e_i^t$$

Prices are different due to different endowments being listed. Thus, prices will vary depending on if the period is an odd or even one due to the different endowments.

(b)

Equilibrium:

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

s.t.

$$c_t^i + \frac{a_{t+1}^i}{(1+\hat{r}_{t+1})} \leq e_t^i + a_t^i$$
$$c_t^i \geq 0$$
$$a_{t+1}^i \geq -\bar{A}^i$$

Markets clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = \hat{e}_t^1 + \hat{e}_t^2 \forall t$$
$$\hat{a}_{t+1}^1 + \hat{a}_{t+1}^2 = 0$$

Lagrangian and FOC's:

$$\mathcal{L} = \beta^t \ln(c_t^i) + \lambda_t(e_t^i + a_t^i - c_t^i - \frac{a_{t+1}^i}{1+r_{t+1}})$$
$$\mathcal{L}_{c_t} = \beta^t \frac{1}{c_t^i} = \lambda_t$$
$$\mathcal{L}_{c_{t+1}} = \beta^{t+1} \frac{1}{c_{t+1}} = \lambda_{t+1}$$
$$\mathcal{L}_{a_{t+1}} = -\lambda_t \frac{1}{1+r_{t+1}} + \lambda_{t+1}$$

Put equations together:

$$\beta^t \frac{1}{c_t^i} = \frac{1}{1+r_{t+1}} + \beta^{t+1} \frac{1}{c_{t+1}}$$
$$\boxed{c_{t+1}^i = \beta(1+r_{t+1})c_t^i}$$

From intertemporal budget equation:

$$\frac{c_t^1 + c_t^2}{c_{t+1}^1 + c_{t+1}^2} = \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$\frac{c_t^1 + c_t^2}{\beta(1+r_{t+1}(c_t^1 + c_t^2)}) = \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2}$$
$$\boxed{r_{t+1} = \frac{1}{\beta} \frac{e_t^1 + e_t^2}{e_{t+1}^1 + e_{t+1}^2} - 1}$$

If t is odd:

$$r_{t+1} = \frac{1}{\beta}\frac{2}{4} - 1$$
$$\boxed{= \frac{1}{2\beta} - 1}$$

If t is even:

$$r_{t+1} = \frac{1}{\beta}\frac{4}{2} - 1$$
$$\boxed{\frac{2}{\beta} - 1}$$

Question 3

Problem

Consider the economy in question 2. Assume the following endowments

$$e_t^1 = (2, 1, 2, 1, 2, \ldots)$$
$$e_t^2 = (2, 1, 2, 1, 2, \ldots)$$

Find equilibrium allocations and prices. Does any trade happen in this equilibrium? Why?

Solution

Given this problem is identical to the last, just with different endowments, I will skip ahead for the sake of brevity.

$$p_t = \frac{e_0^1 + e_0^2}{e_t^1 + e_t \beta^t p_0}$$
$$c_t^i = \frac{\beta^t(1-\beta)}{p_t} \sum_{t=0}^{\infty} p_t e_t^i$$

When t is odd:

$$\boxed{p_t = 2\beta^t p_0}$$
$$\boxed{c_t^i = (1-\beta)\sum_{t=0}^{\infty}}$$

When t is even:

$$\boxed{p_t = \beta^t p_0}$$
$$\boxed{c_t^i = (1-\beta)\sum_{t=0}^{\infty} \beta^t e_t^i}$$

Prices are the same as in 2a. Trade is not beneficial since endowments equal for both agents in each period. Thus, they are able to be self-sufficient, not requiring the other to consumption smooth.

Question 4

Problem

Consider an economy with two types of household with the following preferences

$$\sum_{t=0}^{\infty} \beta_i^t \log(c_t^i) \text{ for } i = 1, 2$$

The households are different in their discount factor $0 < \beta_1 < \beta_2 < 1$. Both households have endowment of $e_t = 1$ every period.

Parts

(a)

Define (Arrow-Debreu) competitive equilibrium. And derive household's Euler equation.

(b)

Write down a social planner problem for some Pareto weights and solve for Pareto efficient allocations (treat Pareto weights as parameters).

(c)

Use Negishi Method to solve for competitive equilibrium allocations and prices. (hint: you can find weights that correspond to equilibrium by examining feasibility at $t = 0$ (or $t = \infty$))

(d)

How does $\frac{c_t^1}{c_t^2}$ move over time? What it its limit?

(e)

Explain what is happening here.

Solutions

(a)

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t \log c_t^i$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

$$c_t^i \geq 0$$

$$0 < \beta_1 < \beta_2 < 1$$

Markets clear:

$$\sum_{i=1}^{2} \hat{c}_t^i = \sum_{i=1}^{2} e_t^i$$

Lagrangian and FOC's:

$$\mathcal{L} = \beta_i^t \log(c_t^i) + \lambda(p_t(e_t^i - c_t^i))$$
$$\mathcal{L}_{c_t} = \beta_i^t \frac{1}{c_t^i} = \lambda p_t$$
$$\mathcal{L}_{c_{t+1}} = \beta_i^{t+1} \frac{1}{c_{t+1}^i} = \lambda p_{t+1}$$

Divide the two:

$$\frac{\beta_t}{t+1} \frac{c_{t+1}}{c_t} = \frac{p_t}{p_{t+1}}$$
$$c_{t+1}^i = \beta_i \frac{p_t}{p_{t+1}} c_t^i$$
$$\boxed{c_t^i = \beta_i^t \frac{p_0}{p_t} c_0^i}$$

(b)

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{i=1}^{2} \alpha \sum_{t=0}^{\infty} \beta_i^t \log(c_t^i)$$

s.t.

$$\sum_{t=0}^{\infty} c_t^i = \sum_{t=0}^{\infty} e_t^i \ \forall t$$
$$c_t^i \geq 0 \ \forall t \text{ and } i = 1, 2$$

Lagrangian and FOC's:

$$\mathcal{L} = \sum_{i=1}^{2} \alpha_i \sum_{t=0}^{\infty} \beta_i^t \log(c_i^t) + \frac{\mu}{2}(\sum_{t=0}^{\infty} e_t^i - \sum_{t=0}^{\infty} c_t^i)$$
$$\mathcal{L}_{c_t} = \alpha_i \beta_i^t \frac{1}{c_t^i} = \frac{\mu}{2}$$
$$\frac{\alpha_1}{\alpha_2} \frac{\beta_1}{\beta_2} \frac{c_t^2}{c_t^1} = 1$$
$$\alpha \equiv \frac{\alpha_1}{\alpha_2}; \ \beta \equiv \frac{\beta_1}{\beta_2}$$
$$c_t^1 = \alpha \beta c_t^2$$

Plugging into constraint:

$$\alpha\beta c_t^2 + c_t^2 = 2$$

$$\boxed{c_t^2 = \frac{2}{1+\alpha\beta}}$$

$$\boxed{c_t^1 = \frac{2\alpha\beta}{1+\alpha\beta}}$$

(c)

Solving for μ:

$$c_t^i : \frac{\mu_t}{2\alpha_i} = \frac{\beta_i^t}{c_t^i}$$
$$c_{t+1} : \frac{\mu_{t+1}}{2\alpha_i} = \frac{\beta_i^{t+1}}{c_{t+1}^i}$$
$$\frac{\mu_{t+1}}{\mu_t} = \frac{\beta_i^{t+1}}{\beta_i^t}\frac{c_t^i}{c_{t+1}^i}$$
$$\mu_t = \beta_i^t(\frac{1+\alpha(\frac{\beta_1}{\beta_2})^t}{1+\alpha})\mu_0$$

Normalizing $\mu_0 = 1$ and making the conjecture $\mu_t = p_t$, we can use the transfer function to find our allocations:

$$\begin{aligned}
t^2(\alpha) &= \sum_{t=0}^{\infty}\beta_2^t(\frac{1+\alpha(\frac{\beta_1}{\beta_2})^t}{1+\alpha})\frac{2}{1+\alpha\beta} - \sum_{t=0}^{\infty}\beta_2^t(\frac{1+\alpha(\frac{\beta_1}{\beta_2})^t}{1+\alpha})\mu_0 \\
&= \frac{2}{1+\alpha}\sum_{t=0}^{\infty}\beta_2^t - \frac{1}{1+\alpha}\sum_{t=0}^{\infty}\beta_2^t - \frac{\alpha}{1+\alpha}\sum_{t=0}^{\infty}\beta_1^t \\
&= \frac{1}{1+\alpha}\sum_{t=0}^{\infty} - \frac{\alpha}{1+\alpha}\sum_{t=0}^{\infty}\beta_1^t \\
&= \frac{1}{1+\alpha}\frac{1}{1-\beta_2^t} = \frac{\alpha}{1+\alpha}\frac{1}{1-\beta_1^t} \\
&= \frac{1}{\alpha} = \frac{1-\beta_2^t}{1-\beta_1^t}
\end{aligned}$$

$$\boxed{\alpha = \frac{1-\beta_1^t}{1-\beta_2^t}}\ \therefore$$

$$\boxed{c_t^2 = \frac{2}{1+\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}}$$

$$\boxed{c_t^1 = \frac{2\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}{1+\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}}$$

(d)

Taking the ratio:

$$\frac{c_t^1}{c_t^2} = \frac{\frac{2\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}{1+\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}}{\frac{2}{1+\frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}}}$$

$$\frac{c_t^1}{c_t^2} = \frac{\beta_1(1-\beta_1)}{\beta_2(1-\beta_2)}$$

$$\boxed{= (\frac{\beta_1}{\beta_2})^t \frac{1-\beta_1}{1-\beta_2}}$$

Since $\beta_1 < \beta_2$:

$$\begin{array}{c}\lim \\ t \to \infty\end{array} \frac{c_t^1}{c_t^2} = \begin{array}{c}\lim \\ t \to \infty\end{array} (\frac{\beta_1}{\beta_2})^t \frac{1-\beta_1}{1-\beta_2} = 0$$

(e)

HH1 will have lower consumption through their lifetime since β_1 is perpetually lower and since their consumption is weighted by the ratio of β_i.

Question 5 (Bonus)

Problem

Consider a two period endowment economy with a single consumption good, c. The economy is comprised of 2 types of households with identical preferences over consumption in periods $t = 0, 1$

$$u(c_0^k) + u(c_1^k), k = 1, 2$$

Fraction η of households are of type 1 and have endowments $(e_0^1, e_1^2) = (1, 0)$. The rest of the households are of type 2 and have endowments $(e_1^2, e_1^2) = (0, 1)$. There is no production technology and consumption good is non-storable.

Parts

(a)

Let a_k be one period arrow security in this economy ($k = 1, 2$) with associate interest rate i. Households have no initial assets. There are no other assets in the economy. Define sequential market competitive equilibrium.

(b)

Assume $u(c) = \log c$. Find interest rate i as function of η. What happens to interest rate as η increases?

(c)

Discuss your answer to part (b). What is the intuition?

Solutions

(a)

$$\max_{c_0, c_1, u} u(c_0) + \beta u(c_1)$$

s.t.

$$c_0 + \frac{a}{1+i} = e_0^k$$
$$c_1 = e_1^k + a$$
$$c_0, c_1 \geq 0$$

Markets clear:

$$\eta c_0^1 + (1-\eta)c_0^2 = \eta e_0^1 + (1-\eta)e_0^2$$
$$\eta c_1^1 + (1-\eta)c_1^2 = \eta e_1^1 + (1-\eta)e_1^2$$
$$\eta a^1 + (1-\eta)a^2 = 0$$

Got kind of lost on how to proceed. I know this was an exam question, so will look at how it solved in the key.