Assignment 1

Due Date: January 23rd, 11:59pm
Please Show Your Work and Circle Your Final Answer
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Question 1: Envelope Theorem

Consider a constrained optimization problem:

$$x^*(\theta) = \arg\max_{x \in X(\theta)} f(x, \theta), \quad V(\theta) = \max_{x \in X(\theta)} f(x, \theta)$$

where $X(\theta) = \{x \in \mathbb{R}^n | g_j(x) \leq \theta_j\}$.

(a) Derive the envelope theorem, computing $\frac{\partial V(\theta)}{\partial \theta_j}$ for each $j \in \{1, \ldots, m\}$.

(b) Interpret your answer in (a), especially regarding KKT multipliers.

(c) Apply the result to consumer maximization. Define $f(x, \theta)$, constraints g_j, and interpret θ and KKT multipliers.

Question 2: Topkis' Theorem (Single-Dimension)

(a) Provide a full proof of Topkis' theorem using the approach from class.

(b) Analyze when $q^*(\theta)$ is nondecreasing for a monopolist with inverse demand $p(q)$ and cost $c(q, \theta)$.

(c) For a firm minimizing costs with production $f(k, \phi l) = q$, find conditions where optimal k^* is weakly increasing/decreasing in ϕ.

Question 3: Topkis' Theorem (Multi-Dimensional)

(a) Interpret the assumption $\frac{\partial^2 f}{\partial l \partial k} < 0$.

(b) Analyze how optimal labor choice changes as wage w increases.

Question 4: Putting it All Together

(a) For utility $u(x, y, z) = x^{1/2}y^{1/2} + z$, show optimum T equals 0 or W.

(b) For $u(x, y, z) = x^\alpha y^\alpha + z$, derive T in terms of prices and W.

(c) For $u(x, y, z) = x^\alpha y^\beta + h(z)$, show T is weakly increasing in W.

Solution 1:

(a)

Using the envelope theorem on the optimization problem:

$$V(\theta) = \begin{matrix} \max \\ x \in \mathcal{X} \end{matrix} f(x, \theta)$$
$$\text{where} X(\theta) = \{x \in \mathbb{R}^n | g_j(x) \leq \theta_j\}$$

Proof. $\frac{\partial v}{\partial \theta_j} = \frac{\partial f(x^*(\theta),\theta_j)}{\partial \theta_j} + \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}\frac{\partial x^*(\theta_j)}{\partial \theta_j} = \frac{\partial f}{\partial \theta_j} + \sum_{j=1}^{m} \lambda \frac{\partial X(\theta_j)}{\partial x_j} \cdot \frac{\partial x^*(\theta)}{\partial \theta_j}$

$\sum_{j=1}^{m} \lambda \frac{\partial X(\theta_j)}{\partial x_j} \cdot \frac{\partial x^*(\theta)}{\partial \theta_j} = \sum_{j=1}^{m} \frac{\partial X(\theta)}{\partial (x_j)} \cdot \Delta x_j + \frac{\partial X(\theta)}{\partial \theta_j} \cdot \Delta \theta_j$

Now, we can apply the following:

$$\frac{1}{\Delta \theta_j} \cdot [\sum_{j=1}^{m} \frac{\partial X(\theta)}{\partial x_j} \Delta x_j + \frac{\partial X(\theta_j)}{\partial \theta_j}]$$
$$\sum_{j=1}^{m} \frac{\partial X(\theta)}{\partial x_j} \frac{\Delta x_j}{\Delta \theta_j} + \frac{\partial X(\theta)}{\partial \theta_j}$$

Finally, we can yield an end result:

$$\frac{V(\theta)}{\partial \theta_j} = \frac{\partial f}{\partial \theta_j} + \lambda[-\frac{\partial X(\theta)}{\partial \theta_j}]$$

This is the result since, at the max, $x^* = 0$ thus eliminating the first term in the summation and leaving the derivative of the constraint with respect to θ_j. $\square$

(b)