# Homework 4

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An ECON - 8040 Homework Assignment

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# 1 Question 1

## 1.1 Problem

Consider the following planning problem

$$w(k_0) = \max_{k_{t+1}, c_t \geq 0} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{-\sigma}}{1-\sigma}$$

s.t.

$$c_t + k_{t+1} \leq z k_t + (1-\delta) k_t$$
$$k_0 \text{ given}$$

(a) Write this problem as a dynamic programming problem.

(b) Solve the Bellman equation you wrote using guess-and-verify method (hint: try $v(k) = A\frac{k_t^{1-\sigma}}{1-\sigma}$).

(c) What is the growth rate of $k_t$ and $c_t$ in this economy? (use you answer in part (b) to answer this). Under what condition does the economy grow?

# 2 Question 2

## 2.1 Problem

Let $n_t$ denote hours worked. Consider the following planning problem ($n_t$ is hours worked)

$$w(k_0) = \max \sum_{t=0}^{\infty} \beta^t [\theta \log c_t + (1-\theta) \log(1-n_t)]$$

s.t.

$$c_t + k_{t+1} \leq z k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t$$
$$k_{t+1}, c_t \geq 0$$
$$1 \geq n_t \geq 0$$
$$k_0 \text{ given}$$

We want to transform this problem to a formulation that we can solve using dynamic programming (Note that if $\alpha = 1$ this problem is a special case of question 1. Also, if $\theta = 1$ and $\delta = 1$ this transforms to the example we solved in class). To do this, we break it down to few steps. Consider the problem of choosing optimal hours worked, given current capital $k$ and future capital $k'$. Write down the optimality condition for $n$ in the following

$$F(k, k') = \max_n \theta \log(z k^{\alpha} n^{1-\alpha} + (1-\delta) k - k') + (1-\theta) \log(1-n)$$

you won't be able to find a closed form solution for $n$ (yet). Call the optimal solution to the above problem $n(k, k')$.

(a) Write the planning problem as a dynamic programming problem (hint: utilize the function $F(k, k')$ above which is equivalent to period utility function for optimally chosen $c$ and $n$, taking $k$ and $k'$ as given).

(b) From now on, assume $\delta = 1$. Now guess the value function $V(k) = A + B \log k$. Derive optimal condition for $k'$ (just like we did for example with $\theta = 1$ in class). Solve for $k'$ as function of parameters of $B$ and $n$.

(c) Rewrite the optimality condition in (a), replace for $k'$ from (c) and solve for optimal $n$ (this won't depend on $k$, only on parameters and $B$). Remember to impose $\delta = 1$.

(d) Now, replace optimal $k'$ and $n$ in the Bellman equation and solve for coefficient $B$.

(e) Write the optimal policy functions for $n, k'$, and $c$ as a function of $k$.

# 3 Question 3

## 3.1 Problem

Consider an economy with two types of capital. Let $k_t$ denote physical capital and $h_t$ denote human capital. Output is produced using physical and human capital

$$y_t = z k_t^{\alpha} h_t^{1-\alpha}.$$

The final good can be consumed, or can be invested in making physical or human capital. Assume both types of capital depreciate at rate $\delta$. Assume households have log preferences over consumption (and labor is inelastically supplied).

(a) Write the planning problem that maximizes the welfare of the representative household for an initial stock of physical and human capital $(k_0, h_0)$. Denote the value to the planner as $w(k_0, h_0)$.

(b) Write the planning problem recursively. This means write a Bellman equation that corresponds to the planning problem in part (a).

(c) Assume full depreciation $\delta = 1$. Solve the Bellman equation using guess and verify. Find the optimal policy functions for future physical and human capital as function of current physical and human capital.

# 4 Question 4

## 4.1 Problem

Consider the following planning problem

$$w(\bar{k}_0) = \max_{\{(c_t, k_{t+1}, x_t)\}_{t=0}^{\infty}} \sum^{\infty} t = 0 \beta^t \theta_t \log c_t$$

s.t.

$$c_t + k_{t+1} = k_t$$
$$k_0 = \bar{k}_0$$

and

$$\theta_t = \begin{cases} \theta_H & t\ iseven \\ \theta_L & t\ isodd \end{cases}$$

(a) Write this problem only in terms of a sequence of capital $\{k_{t+1}\}_{t=0}^{\infty}$.

(b) Write the problem in part (a) as a Bellman equation(s). (hint: Note that we have an additional (exogenous) state variable $\theta_t$. So you need to write two equations, one for $v(k, \theta_L)$ and one for $v(k, \theta_H)$. Also, we know how $\theta_t$ evolves.)

(c) Solve the Bellman equation in part (b) using guess and verify method.

(d) Write the formula for optimal policy functions $g(k, \theta_L)$ and $g(k, \theta_H)$.

(e) Start from $k_0 = 1$. Describe how you will simulate the optimal path of capital stock $\{k_{t+1}\}_0^\infty$. Write down $k_t$ for $k = 1, 2, 3$.