# Homework 6

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An ECON - 8040 Homework Assignment

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# Question 1

## Problem

Consider the example of our endowment economy with two types of households. Assume utility function is

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \text{ for } \sigma > 0$$

Also assume endowments growth at constant rate, i.e.,

$$e_t^1 = \begin{cases} 2\gamma^t & \text{if t is even} \\ 0 & \text{if t is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 & \text{if t is even} \\ 2\gamma^t & \text{if t is odd} \end{cases}$$

Let $0 < \beta < 1$ be discount rate and assume $\beta\gamma^{1-\sigma} < 1$. (Note: for $\sigma = 1$, $u(c) = \log(c)$. Therefore the example we studied in class was a special case of this problem with $\gamma = 1$ and $\sigma = 1$)

## Parts

### (a)

Define (Arrow-Debreu) competitive equilibrium.

### (b)

Drive household's Euler equation.

### (c)

Write down a social planner problem for some Pareto weights and solve for Pareto efficient allocations (treat Pareto weights as parameters). hint: it is easier to solve everything in terms of the ratio of weights $\alpha \equiv \frac{\alpha_2}{\alpha_1}$.

### (d)

Use Negishi Method to solve for competitive equilibrium allocations and prices.

### (e)

Find Pareto weights that generate the following allocation $(c_t^1, c_t^2) = (\gamma^t, \gamma^t)$. Find transfers needed to implement this allocation in a competitive equilibrium.

### (f)

How does growth rate of consumption depend on $\gamma$ and $\sigma$? How do equilibrium prices depend on $\gamma$ and $\sigma$?

### (g)

Define a sequential market competitive equilibrium. Find interest rate in this equilibrium. How do equilibrium interest rate depend on $\gamma$ and $\sigma$?

## Solutions

### (a)

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left( \frac{{c_t^i}^{1-\sigma} - 1}{1-\sigma} \right)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$
$$\hat{c}_t^i \geq 0$$
$$0 < \beta < 1$$

Markets Clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = \hat{e}_t^1 + \hat{e}_t^2 \ \forall \ t$$

### (b)

$$c_t^i : \frac{\beta^t}{{c_t^i}^{\sigma}} - \lambda p_t$$
$$c_{t+1}^i : \frac{\beta^{t+1}}{{c_{t+1}^i}^{\sigma}} - \lambda p_{t+1}$$
$$\beta \frac{{c_{t+1}^i}^{\sigma}}{{c_t^i}^{\sigma}} = \frac{p_t}{p_{t+1}}$$
$$<<<<<<< HEAD {c_{t+1}^i}^{\sigma} p_{t+1} = \beta {c_t^i}^{\sigma} p_t$$
$${c_{t+1}^i}^{\sigma} = \beta \frac{p_t}{p_{t+1}} {c_t^i}^{\sigma} ======= {c_{t+1}^i}^{\sigma} = \beta \frac{p_t}{p_{t+1}} {c_t^i}^{\sigma} >>>>>>> 9176d0f066c99492f90177ff1154f1153280797b$$

### (c)

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[ \alpha_1 \frac{{c_t^1}^{1-\sigma} - 1}{1-\sigma} + \alpha_2 \frac{{c_t^2}^{1-\sigma}}{1-\sigma} \right]$$

s.t.

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 = 2\gamma^t ; \frac{\mu}{2}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \alpha \sum_{t=0}^{\infty} \beta^t \frac{{c_t^i}^{1-\sigma} - 1}{1-\sigma} + \frac{\mu}{2}[\sum_{t=0}^{\infty} e_t^i - \sum_{t=0}^{\infty} c_t^i]$$
$$\mathcal{L}_c = \alpha\beta c^{-\sigma} - \frac{\mu}{2} = 0$$
$$c_t^1 : \alpha_1 \beta c_t^{1-\sigma} - \frac{\mu}{2}$$
$$c_t^2 : \alpha_2 \beta c_t^{2-\sigma} - \frac{\mu}{2}$$

Divide the two FOC's

$$\frac{\alpha_1 \beta c_t^{1-\sigma} - \frac{\mu}{2}}{\alpha_2 \beta c_t^{2-\sigma} - \frac{\mu}{2}}$$
$$1 = \frac{\alpha_1}{\alpha_2} (\frac{c_t^1}{c_t^2})^{-\sigma}$$
$$(\frac{\alpha_2}{\alpha_1})^{\sigma} = \frac{c_t^1}{c_t^2}$$
$$\alpha \equiv \frac{\alpha_2}{\alpha_1} \Rightarrow c_t^1 = c_t^2 \alpha^{\sigma}$$
$$\alpha^{\sigma} c_t^2 + c_t^2 = 2\gamma$$
$$c_t^2 = \frac{2\gamma}{1+\alpha^{\sigma}}$$
$$c_t^1 = \frac{2\gamma\alpha^{\sigma}}{1+\alpha^{\sigma}}$$

**(d)**

Solving for $\mu$

$$c_t^i : \frac{\mu_t}{2\alpha} = \frac{\beta^t}{c_t^{\sigma}}$$
$$c_{t+1}^i : \frac{\mu_{t+1}}{2\alpha} = \frac{\beta^{t+1}}{c_{t+1}^{\sigma}}$$

$$\frac{\mu_{t+1}}{\mu_t} = \beta \frac{c_t}{c_{t+1}} =$$

# Question 2

## Problem

Consider the economy in question 1. Assume $\sigma = 1$ (so log utility) and $\gamma = 1$. Assume the following endowments

$$e_t^1 = (2, 0, 2, 0, 2, \ldots)$$
$$e_t^2 = (2, 2, 2, 2, 2, \ldots)$$

## Parts

### (a)

Find equilibrium allocations and prices. Are prices different than the one we derived during lecture? Why?

### (b)

Define a sequential market competitive equilibrium. Find interest rate in this equilibrium.

## Solutions

### (a)

$$\max_{c^{i\,\infty}_{t\,t=0}} \sum_{t=0}^{\infty} \beta^t \ln(c^i_t)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c^i_t \leq \sum_{t=0}^{\infty} \hat{p}_t e^i_t$$
$$\hat{c}^i_t \geq 0$$
$$0 < \beta < 1$$

Markets Clear:

$$e^1_t + e^2_t = c^1_t + c^2_t = \begin{cases} 4, \text{if } t \text{ is odd} \\ 2, \text{if } t \text{ is even} \end{cases}$$

# Question 3

## Problem

Consider the economy in question 2. Assume the following endowments

$$e^1_t = (2, 1, 2, 1, 2, \ldots)$$
$$e^2_t = (2, 1, 2, 1, 2, \ldots)$$

Find equilibrium allocations and prices. Does any trade happen in this equilibrium? Why?

## Solution

Trade is not beneficial since prices are equal for both agents. Thus, they are able to be self-sufficient, not requiring the other to consumption smooth.

# Question 4

## Problem

Consider an economy with two types of household with the following preferences

$$\sum_{t=0}^{\infty} \beta_i^t \log(c_t^i) \text{ for } i = 1, 2$$

The households are different in their discount factor $0 < \beta_1 < \beta_2 < 1$. Both households have endowment of $e_t = 1$ every period.

## Parts

### (a)

Define (Arrow-Debreu) competitive equilibrium. And derive household's Euler equation.

### (b)

Write down a social planner problem for some Pareto weights and solve for Pareto efficient allocations (treat Pareto weights as parameters).

### (c)

Use Negishi Method to solve for competitive equilibrium allocations and prices. (hint: you can find weights that correspond to equilibrium by examining feasibility at $t = 0$ (or $t = \infty$))

### (d)

How does $\frac{c_t^1}{c_t^2}$ move over time? What it its limit?

### (e)

Explain what is happening here.

## Solutions

### (a)

# Question 5 (Bonus)

## Problem

Consider a two period endowment economy with a single consumption good, $c$. The economy is comprised of 2 types of households with identical preferences over consumption in periods $t = 0, 1$

$$u(c_0^k) + u(c_1^k), k = 1, 2$$

Fraction $\eta$ of households are of type 1 and have endowments $(e_0^1, e_1^2) = (1, 0)$. The rest of the households are of type 2 and have endowments $(e_1^2, e_1^2) = (0, 1)$. There is no production technology and consumption good is non-storable.

## Parts

### (a)

Let $a_k$ be one period arrow security in this economy $(k = 1, 2)$ with associate interest rate $i$. Households have no initial assets. There are no other assets in the economy. Define sequential market competitive equilibrium.

### (b)

Assume $u(c) = \log c$. Find interest rate $i$ as function of $\eta$. What happens to interest rate as $\eta$ increases?

### (c)

Discuss your answer to part (b). What is the intuition?