Homework 3

ECON 8050: Macroeconomics II
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Problem 1: Dynamic Programming

Consider the following model with a disability shock. There are three sources of uncertainty:

- Out-of-pocket medical shock evolving according to transition matrix $\Psi(x_t|x_{t-1})$.
- Productivity evolving according to $T(z_t|z_{t-1})$.
- Disability shock.

The timing of events is as follows: At the beginning of the period, an individual with savings k_t learns their productivity z_t and medical shock x_t. Then they decide whether to work ($l_t = 0$ or $l_t > 0$). If working, labor income is wz_tl_t. Then, they decide about consumption c_t and savings k_{t+1}.

At the end of the period, the disability shock is realized with probability d. Disabled individuals stay permanently disabled, do not work, receive constant benefits DI, and make only consumption/savings decisions. Medical spending for disabled individuals is fully covered by public insurance.

(1) Write down the dynamic programming problem of a non-disabled individual, denoting the value function as V_t.

(2) Write down the dynamic programming problem of a disabled individual, denoting the value function as V_t^d.

(3) Modify the problem assuming disabled individuals can recover with probability f. Recovered individuals draw new productivity realizations from the invariant distribution.

(4) Extend the model to allow non-disabled individuals to falsely claim disability benefits, introducing the value function for falsely disabled V_t^{fd}.

Problem 2: Consumption-Savings Model

A consumer with infinite life maximizes quadratic utility:

$$u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$$

where future utility is discounted at rate β and borrowing/savings occur at interest rate r with $\beta(1+r) = 1$.

The consumer's endowment y_t is i.i.d. with values y_H and y_L occurring with probabilities p_H and p_L respectively. The budget constraint is:

$$c_t = a_t(1+r) + y_t - a_{t+1}.$$

(1) Solve for the consumption and saving functions. Provide intuition on when savings are positive or negative.

(2) Introduce a borrowing constraint $a_{t+1} \geq 0$. Solve the consumer's problem in recursive form numerically using given parameters.

(3) Plot policy functions a_{t+1} and c_t as functions of current assets a_t for cases with and without borrowing constraints.

(4) Simulate the income process and optimal decision rules over $T = 100$ periods. Compare results with and without borrowing constraints.

Solution 1

Part (i)

The Bellman for an able bodied person with probability of becoming disabled d is as follows:

$$V_t(k_t, x_t, z_t) = \max_{c_t, l_t, k_{t+1}} \{u(c_t, l_t) + \beta(1-d)\sum_{x_t}\sum_{z_t} \Psi(x_t|x_{t-1})\mathrm{T}(z_t|z_{t-1})V_{t+1}(k_{t+1}, x_{t+1}, z_{t+1})$$
$$+\beta d \sum_{x_t} \Psi(x_t|x_{t-1})V_{t+1}(k_{t+1}, x_{t+1}, z_{t+1})\}$$

s.t.

$$c_t + k_{t+1} + x_t = wz_t l_t + k_t(1+r)$$

Part (ii)

The Bellman for an individual who is disabled and has no probability of recovery can be represented as follows:

$$V_t^d(k_t, x_t) = \max_{c_t, k_{t+1}} \{u(c_t, 0) + \beta \sum_{x_t} \Psi(x_t|x_{t-1})V_{t+1}^d(k_{t+1}, x_{t+1})\}$$

s.t.

$$c_t + k_{t+1} = DI + k_t(1+r)$$

Part (iii)

The Bellman equation for an individual who is disabled but has a probability of recovery is as follows:

$$V_t^{df}(k_t, x_t) = \max_{c_t, k_{t+1}} \{u(c_t, l_t) + \beta f \sum_{x_t|x_{t-1}} \sum_{z_t|z_{t-1}} \Psi(x_t|x_{t-1})\mathrm{T}(z_t|z_{t-1})V_{t+1}^d(k_{t+1}, x_{t+1}, z_{t+1})$$
$$+\beta(1-f)\sum_{x_t} \Psi(x_t|x_{t-1})V_{t+1}^d(k_{t+1}, x_{t+1})\}$$

s.t.

$$c_t + k_{t+1} + x_t = DI + k_t(1+r)$$

Part (iv)

Finally, the Bellman for someone who has the option to fake disability is as follows:

$$V_t^{df}(k_t, x_t, z_t) = \max_{c_t, k_{t+1}, l_t} \{u(c_t, l_t) + \beta f \sum_{x_t|x_{t-1}} \sum_{z_t|z_{t-1}} \Psi(x_t|x_{t-1})\mathrm{T}(z_t|z_{t-1})V_{t+1}^d(k_{t+1}, x_{t+1}, z_{t+1})$$
$$+\beta(1-f)\sum_{x_t} \Psi(x_t|x_{t-1})V_{t+1}^d(k_{t+1}, x_{t+1})$$
$$+\beta f \mathbb{1}_{fake=1} \sum_{x_t}(x_t|x_{t-1})V_{t+1}^{fd}(k_{t+1}, x_{t+1})\}$$

s.t.

$$c_t + k_{t+1} + x_t = wz_t l_t + DI\mathbb{1}_{D=1 or fake=1} + k_t(1+r)$$

Solution 2:

0.1 Part (i)

$$\sum_{t=0}^{\infty} \beta^t [-\frac{1}{2}(c_t - \bar{c})^2]$$

s.t.

$$c_t + a_{t+1} = (1+r)a_t + y_t$$

Dynamic Programming is as follows:

$$V(a_t, y_t) = \max_{c_t, a_{t+1}} \{-\frac{1}{2}(c_t - \bar{c})^2 + \beta \mathbb{E}_t[V(a_{t+1}, y_{t+1})]\}$$

Because the utility is quadratic and $\beta(1+r) = 1$, we know the Euler is as follows:

$$c_t = \mathbb{E}_t[c_{t+1}]$$

Plugging in from the constraint,

$$c_t = \mathbb{E}[y_t] + (1+r)\mathbb{E}[a_t] - \mathbb{E}[a_{t+1}]$$

Expectations can be represented as follows:

$$\mathbb{E}[y_t] = p_H y_H + p_L y_L$$
$$\mathbb{E}[a_{t+1}] = a_t \text{ smoothes consumption due to instability of income}$$

From the FOC w.r.t to a_{t+1}

$$c_t - \bar{c} = \mathbb{E}_t[c_{t+1} - \bar{c}]$$
$$c_t = \bar{c} + (1+r)a_t + y_t - a_t$$
$$c_t = \bar{c} + ra_t + y_t$$

This is the consumption function when there is no borrowing constraint and infinite lifetime. Let's find the saving function below.

$$a_{t+1} = (1+r)a_t + y_t - c_t$$
$$a_{t+1} = (1+r)a_t + y_t - (\bar{c} + ra_t + y_t)$$
$$a_{t+1} = a_t - \bar{c}$$

This is the optimal savings function in the case of no borrowing constraint and infinite lifetime.

The savings function states that depending on the consumer's state of assets, they will either save or borrow. This depends on the relationship between $\bar{c}$ and a_t. When $a_t > \bar{c}$, the agent will consume more as they do not have to worry about their savings. In the opposite case, they will save more, trying to replenish their savings.