

## Assignment 1

### Tate Mason

Due Date: January 23rd, 11:59pm

### Question 1: Envelope Theorem

Consider a constrained optimization problem:

$$x^*(\theta) = \arg \max_{x \in X(\theta)} f(x, \theta), \quad V(\theta) = \max_{x \in X(\theta)} f(x, \theta)$$

where  $X(\theta) = \{x \in \mathbb{R}^n | g_j(x) \leq \theta_j\}$ .

- (a) Derive the envelope theorem, computing  $\frac{\partial V(\theta)}{\partial \theta_j}$  for each  $j \in \{1, \dots, m\}$ .
- (b) Interpret your answer in (a), especially regarding KKT multipliers.
- (c) Apply the result to consumer maximization. Define  $f(x, \theta)$ , constraints  $g_j$ , and interpret  $\theta$  and KKT multipliers.

### Question 2: Topkis' Theorem (Single-Dimension)

- (a) Provide a full proof of Topkis' theorem using the approach from class.
- (b) Analyze when  $q^*(\theta)$  is nondecreasing for a monopolist with inverse demand  $p(q)$  and cost  $c(q, \theta)$ .
- (c) For a firm minimizing costs with production  $f(k, \phi l) = q$ , find conditions where optimal  $k^*$  is weakly increasing/decreasing in  $\phi$ .

### Question 3: Topkis' Theorem (Multi-Dimensional)

- (a) Interpret the assumption  $\frac{\partial^2 f}{\partial l \partial k} < 0$ .
- (b) Analyze how optimal labor choice changes as wage  $w$  increases.

### Question 4: Putting it All Together

- (a) For utility  $u(x, y, z) = x^{1/2}y^{1/2} + z$ , show optimum  $T$  equals 0 or  $W$ .
- (b) For  $u(x, y, z) = x^\alpha y^\alpha + z$ , derive  $T$  in terms of prices and  $W$ .
- (c) For  $u(x, y, z) = x^\alpha y^\beta + h(z)$ , show  $T$  is weakly increasing in  $W$ .

### Solution 1:

(a)

Using the envelope theorem on the optimization problem:

$$V(\theta) = \max_{x \in \mathcal{X}} f(x, \theta)$$

$$\text{where } X(\theta) = \{x \in \mathbb{R}^n | g_j(x) \leq \theta_j\}$$

$$\begin{aligned}
 \text{Proof. } \frac{\partial v}{\partial \theta_j} &= \frac{\partial f(x^*(\theta), \theta_j)}{\partial \theta_j} + \sum_{j=1}^m \frac{\partial f}{\partial x_j} \frac{\partial x^*(\theta_j)}{\partial \theta_j} = \frac{\partial f}{\partial \theta_j} + \sum_{j=1}^m \lambda \frac{\partial X(\theta_j)}{\partial x_j} \cdot \frac{\partial x^*(\theta)}{\partial \theta_j} \\
 \sum_{j=1}^m \lambda \frac{\partial X(\theta_j)}{\partial x_j} \cdot \frac{\partial x^*(\theta)}{\partial \theta_j} &= \sum_{j=1}^m \frac{\partial X(\theta)}{\partial x_j} \cdot \Delta x_j + \frac{\partial X(\theta)}{\partial \theta_j} \cdot \Delta \theta_j
 \end{aligned}$$

Now, we can apply the following:

$$\begin{aligned}
 \frac{1}{\Delta \theta_j} \cdot \left[ \sum_{j=1}^m \frac{\partial X(\theta)}{\partial x_j} \Delta x_j + \frac{\partial X(\theta_j)}{\partial \theta_j} \right] \\
 \sum_{j=1}^m \frac{\partial X(\theta)}{\partial x_j} \frac{\Delta x_j}{\Delta \theta_j} + \frac{\partial X(\theta)}{\partial \theta_j}
 \end{aligned}$$

Finally, we can yield an end result:

$$\boxed{\frac{V(\theta)}{\partial \theta_j} = \frac{\partial f}{\partial \theta_j} + \lambda \left[ -\frac{\partial X(\theta)}{\partial \theta_j} \right]}$$

This is the result since, at the max,  $x^* = 0$  thus eliminating the first term in the summation and leaving the derivative of the constraint with respect to  $\theta_j$ .  $\square$

**(b)**

The result in part (a) is interpreted as the derivative of the value function with respect to the constraint  $\theta_j$  is equal to the derivative of the objective function with respect to  $\theta_j$  plus the derivative of the constraint with respect to  $\theta_j$  times the Lagrange multiplier. This is interpreted as the marginal change in the value function with respect to a change in the constraint  $\theta_j$ . The KKT multipliers are interpreted as the marginal change in the value function with respect to a change in the constraint  $\theta_j$ .