Normal-Form Games

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1 Introduction

Game theory is the study of strategic behavior. The key feature of game-theoretic settings is that they involve multiple agents who are strategically interdependent. In contrast, many of the problems we studied last quarter involved purely individual decisions about consumption or production. What is the difference? Imagine that two people walk into a restaurant and look at the menu. We can model this as two individual decision problems. But what if these people decide to each order an entree and share them? Now we have a game.

Since its introduction in the 1940s, game theory has become a central tool in many branches of economcs. To whet your appetite, consider just a few examples.

- Political Economy: How legislators vote depends on how others are voting, and how they expect their constituents and interest groups to react.
- Public Finance: The benefits from contributing to a public good depend on what everyone else contributes.
- Industrial Organization: Firm's profits depend on the prices they set and products they offer, but also on the prices and products offered by other firms.
- Labor Economics: Firms try to design incentive schemes and structure compensation to modify behavior.
- International Trade: Levels of imports and exports, and prices, depend on your tariffs, but also on the tariffs of others.

- Urban Economics: What time should a commuter leave home, knowing the traffic will depend on when others leave?

2 Normal Form Games

We will start by considering one representation of strategic environments — games in normal (or strategic) form.

Definition 1 *A normal form game G consists of:*

- *A finite set of agents $\mathcal{I} = \{1, 2, ..., I\}$*
- *Strategy sets $S_1, ..., S_I$.*
- *Payoff functions $u_i : S_1 \times ... \times S_I \to \mathbb{R}$ for each $i = 1, ..., I$.*

A *strategy profile* $s = (s_1, ..., s_I)$ specifies a strategy for each player. We'll write $S = S_1 \times S_2 \times ... \times S_I$ for the space of strategy profiles. It will also be useful to let s_{-i} denote the vector of strategies chosen by everyone except i, $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$.

3 Examples

Let's look at some examples. First, we consider two simple games.

Prisoners' Dilemma Two prisoners are taken into separate rooms to be interrogated. Each is told that they can either plead innocence or implicate the other prisoner. If just one prisoner implicates the other, the one who talks will be immediately released, while the other will hang. If both implicate the other, they will end up in jail. Finally, if both plead innocence, they will eventually be released, but not for a while.

We can represent this strategic situation as follows. Both players must choose independently from the strategy set $\{C, D\}$, where C represents the strategy of holding out and pleading innocent, and D represents the strategy of finking (or "defecting") on the other prisoner. The payoffs are as follows: if both hold out, then each get 1, so $u_i(C, C) = 1$. If i holds out, but j finks, then i gets -1, while j gets 2, so $u_i(C, D) = -1$ and $u_i(D, C) = 2$. Finally, if both fink, they each get 0, so $u_i(D, D) = 0$.

It is often convenient to summarize simple games like this in (bi-) matrix form.

	C	D
C	$1,1$	$-1,2$
D	$2,-1$	$0,0$

Note that the key feature of the prisoners' dilemma is that each player benefits by doing something bad to the other (playing D instead of C). This is true in many strategic interactions — for example arms' races (where building up a nuclear arsenal offers protection, but threatens other countries).

Battle of the Sexes On New Year's Eve, a couple finds themselves separated at a party as midnight approaches. There are two natural spots to meet — the bar, or the dance floor. All things equal, one prefers the bar, the other the dance floor. But most of all, they want to coordinate and end up in the same place at midnight.

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

We capture this as follows. Both must choose independently from the strategy set $\{B, F\}$. Both have a payoff 0 from miscoordinating, i.e. $u_i(B,F) = u_i(F,B) = 0$. If they meet at the bar, one gets a payoff of 2, the other a payoff of 1. Payoffs are reversed if they meet on the dance floor.

Now consider two models of duopolistic competition.

Cournot Competition In Cournot's (1838) model of imperfect competition, firms 1 and 2 choose quantities q_1, q_2. Firm i has a cost function $c_i(\cdot)$, and the market inverse demand curve is $P(Q)$, where $Q = q_1 + q_2$. The normal form of this game has $s_i = q_i$, $S_i = [0, \infty)$.

$$u_i(s_i, s_{-i}) = P(s_1 + s_2)\, s_i - c_i(s_i).$$

Bertrand Competition In Bertrand's model of competition, firms set prices rather than quantities. If there are two firms, they set prices p_1, p_2. Consumers all choose the firm with the lowest price, or, if the prices are the same split their purchases equally. Assume there is a mass Q of consumers, and firm i has per-unit costs $c_i(\cdot)$. To represent this as a normal form game, we take $s_i = p_i$, so that $S_i = [0, \infty)$ and define:

$$u_i(s_i, s_{-i}) = \begin{cases} 0 & \text{if } s_i > s_{-i} \\ s_i Q - c_i(Q) & \text{if } s_i < s_{-i} \\ s_i \frac{Q}{2} - c_i\left(\frac{Q}{2}\right) & \text{if } s_i = s_{-i} \end{cases} .$$

4 Dominant and Dominated Strategies

We now start to investigate the following question: how will people behave in strategic environments? We start by exploring the most basic implications of rationality for strategic play.

Definition 2 *A strategy s_i is **strictly dominated** if there exists some $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

If a given strategy is strictly dominated, there is some other strategy that does strictly better *regardless of opponents' play*. This suggests that a rational agent will not use a dominated strategy. To formalize this idea, we need to have a definition of what it means to behave rationally in a strategic environment.

Definition 3 *Player i is rational with beliefs μ_i if:*

$$s_i \in \arg\max_{s_i \in S_i} \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i, s_{-i}\right),$$

that is, if s_i maximizes i's expected payoff $\sum_{s_{-i}} u_i(s_i, s_{-i})\mu_i(s_{-i})$.

Proposition 1 *If player i is rational, then she wil not play a strictly dominated strategy.*

Proof. Suppose that s_i is strictly dominated by s_i'. Then for any beliefs μ_i,

$$\begin{aligned} \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i, s_{-i}\right) &= \sum_{s_{-i}} u_i(s_i, s_{-i})\mu_i(s_{-i}) \\ &< \sum_{s_{-i}} u_i(s_i', s_{-i})\mu_i(s_{-i}) = \mathbb{E}_{\mu_i(s_{-i})} u_i\left(s_i', s_{-i}\right). \end{aligned}$$

The key inequality holds because $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for *all* s_{-i}, and $\mu_i(s_{-i})$ must be strictly positive for some s_{-i}. It follows that s_i cannot be justified for any beliefs. *Q.E.D.*

Prisoners' Dilemma In the prisoners' dilemma, the strategy C is strictly dominated by the strategy D.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

If a player has one single strategy that strictly dominates all others, we say this strategy is *strictly dominant* (or sometimes that the player has a dominant strategy).

Battle of the Sexes In the battle of the sexes, neither strategy is strictly dominated. Playing B is best if you believe your opponent will play B; playing F is best if you believe your opponent will play F.

5 Iterated Strict Dominance

The line of argument we are following can be pushed further. Suppose that player j *knows* that player i is rational. If j himself is rational, and understands the above proposition, he then *knows* that i will never use a dominated strategy. Thus, when j forms his beliefs about i's play, he should put probability zero on i playing a dominated strategy. If follows that if j is rational, then not only will j not want to play a dominated strategy, j will not want to play a strategy that is dominated *conditional on i not playing a dominated strategy.*

Example Consider the following game:

	L	M	R
U	$2, 2$	$1, 1$	$4, 0$
D	$1, 2$	$4, 1$	$3, 5$

In this game, neither of Row's strategies are dominated. But M is dominated for Column. Once M is eliminated, then D is dominated for Row. But once both D is eliminated, then R is dominated for Column. We are left with the strategy profile (U, L).

To formalize this idea, we define the process of iterated strict deletion.

- Define $S_i^0 = S_i$.

- For each $k = 0, 1, 2....$, define:

$$S_i^{k+1} = \left\{ \begin{array}{c} s_i \in S_i^k \mid \nexists s_i' \in S_i^k \text{ with} \\ u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k \end{array} \right\}.$$

- Define $S_i^\infty = \cap_{k=1}^\infty S_i^k$.

To understand this process, observe that initially all strategies are allowed. At the first round, strictly dominated strategies are eliminated to obtain the restricted strategy sets S_i^1. At the second round, we eliminate those strategies that are strictly dominated conditional on opponents using strategies in S_{-i}^1. Eventually this process must converge since each set S_i^0, S_i^1, etc.. is getting progressively smaller.

Let $S^\infty = S_1^\infty \times ... \times S_I^\infty$ denote the set of strategy profiles that survive the entire process of iterated strict deletion. If S^∞ contains a single strategy profile, we say the game is *dominance solvable*. Some games (such as the prisoners' dilemma) are dominance solvable, but most games are not. For example, in battle of the sexes, no strategies can be eliminated, so $S^\infty = S$.

Note that each level of iterated dominance invokes one further implication of rationality. First, that each i is rational. Second, that each i is rational and knows everyone else is rational. And so on.... The full restriction to strategies in S^∞ depends on *common knowledge of rationality* (everyone is rational, knows everyone is rational, and so on).

Proposition 2 *If players $1, ..., I$ are rational, and this fact is common knowledge, then no player i will play a strategy that is eliminated by iterated strict deletion of dominated strategies.*

6 Rationalizability

In their respective dissertations, Bernheim (1984) and Pearce (1984) investigated the implications of rationality from an alternative perspective. Instead of asking what players won't do, they asked what sort of behavior could be *directly* justified in an environment where rationality was common knowledge. Their intuitive idea is that a strategy should be *rationalizable* if it is a best response to beliefs that are themselves consistent with the rationality of others.

Example Consider the following game.

	a	b	c	d
U	$4, 10$	$3, 0$	$1, 3$	$2, 6$
D	$0, 0$	$2, 10$	$10, 3$	$3, 6$

- U is rationalizable. It is a best response to a, which is itself a best response to U.
- D is also rationalizable. It is a best response to d. In turn, d is a best response if Column believes that U, D are equally likely (and we already know that U is rationalizable).

This logic also tells us that a, d are rationalizable. Similarly, b is rationalizable because it is a best-response to D (which we just showed was rationalizable). However, c is not rationalizable because it is not a best response to anything.

Now consider a formal definition.

Definition 4 *A subset $B_1 \times ... \times B_I \subset S$ is a* ***best reply set*** *if for all i and all $s_i \in B_i$, there exists a probability distribution over opponent strategies $\mu_i \in \Delta(B_{-i})$ such that s_i is a best reply to beliefs μ_i.*[1]

Definition 5 *The set of* ***rationalizable strategies*** *is the component by component union of all best reply sets:*

$$R = R_1 \times ... \times R_I = \bigcup_{\alpha} B_1^{\alpha} \times ... \times B_I^{\alpha}$$

where each $B^{\alpha} = B_1^{\alpha} \times ... \times B_I^{\alpha}$ is a best reply set.

It is not hard to show that R is itself a best reply set, and of course contains all the others. Thus R is the *maximal* best reply set.

In thinking about solution concepts, it is often useful to think about what these solution concepts imply about (i) how players form beliefs about opponents' behavior, and (ii) how players act given their beliefs. In these terms, rationalizability makes two essential assumptions about strategic behavior.

1. Each player is rational, and maximizes his payoff given his beliefs about his opponents' play.
2. Each player's beliefs do not conflict with others being rational, or being aware of others' rationality, and so on. However, beliefs need not be correct.

[1] For the cognoscenti, note that I have allowed beliefs μ_i to reflect correlation between opponent choices, i.e. $\mu_i \in \Delta(B_{-i})$ rather than $\mu_i \in \times_{j \neq i} \Delta(B_j)$. This leads to a definition of rationalizability that coincides exactly with ISD (as commented on below). The original papers by Bernheim and Pearce assume independence, which (in games of three or more players) leads to a slightly less permissive solution concept.

You might be wondering about the relationship between strategies that are rationalizable and those that survive iterated strict deletion. As we have defined ISD, these sets are not exactly the same. However, it turns out that once we allow for mixed strategies, rationalizability and iterated strict deletion yield identical solutions — that is, $R = S^{\infty}$!

7 Nash Equilibrium

So far, we have considered the implications of rationality (and common knowledge of rationality) for predicting how people will play games. However, while iterated strict dominance is an attractive solution concept that makes relatively weak assumptions about behavior, it does not have much bite in many games. Relatively few games are actually dominance solvable, and often virtually every strategy is rationalizable.

This leads us to the notion of Nash equilibrium, the most important concept for solving games. Nash equilibrium captures the idea that players ought to do as well as they can given the strategies actually chosen by the other players.

Definition 6 *A strategy profile $(s_1, ..., s_I)$ is a* ***(pure strategy) Nash equilibrium*** *of G if for every i, and every $s_i' \in S_i$,*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}).$$

A Nash equilibrium profile is an "equilibrium" in the following sense: given that everyone knows what everyone else is doing, no one wants to change their behavior. It is also useful to define the notion of a strict Nash equilibrium.

Definition 7 *A strategy profile $(s_1, ..., s_I)$ is a* ***strict Nash equilibrium*** *of G if for every i, and every $s_i' \in S_i$,*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

Consider two simple examples.

Prisoners' Dilemma

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

The unique (strict) Nash Equilibrium is (D, D).

Battle of the Sexes

	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

There are two (strict) pure Nash equilibria (B, B) and (F, F).

Nash equilibrium has an appealing internal consistency in that no player would unilaterally wish to deviate from a Nash profile. An important issue, however, is how the equilibrium arises in the first place. To see what is required, it is useful again to think of Nash equilibrium as imposing a requirement both on how players form beliefs about their opponents' play, and about how they act given their beliefs. Nash equilibrium supposes that:

1. Each player is rational and maximizes his payoff with respect to his beliefs about opponent play.

2. Each player's beliefs about opponent play are *correct*.

The second assumption is quite strong. How exactly could a player come to *know* what his opponents will play? Depending on the situation being studied, several possible interpretations of Nash equilibrium might be offered.

1. Introspection. In some games (for example, those that are dominance solvable, or those with a "focal" equilibrium), introspection may lead players to make correct conjectures about their opponents' behavior. But not all games (e.g. Battle of the Sexes) will have an obvious play.

2. Communication/Self-enforcing agreement. If players can communicate prior to play, and agree on a strategy profile, any Nash equilibrium profile will be *self-enforcing* in the sense that no one will unilaterally want to deviate if it is agreed on.

3. Result of Learning/Convention. If the strategic situation arises repeatedly in a given society, people over time may learn what typical behavior is. A Nash equilibrium would then constitute a stable (non-changing) social arrangement.

8 Applications & Examples

8.1 Cournot Competition

Let's consider a particular case of Cournot duopoly, with linear demand and linear costs. Assume that:

$$P(q_1 + q_2) = 1 - (q_1 + q_2),$$

and for some $1 > c > 0$,

$$C(q_i) = cq_i.$$

Firm i's payoff function is:

$$u_i(q_i, q_j) = q_i \left[1 - (q_i + q_j) - c\right]$$

Consider the problem facing player i given that he expects his opponent to produce q_j. He solves:

$$\max_{q_i} q_i \left[1 - (q_i + q_j) - c\right].$$

His marginal returns to higher quantity are:

$$\frac{\partial u_i(q_i, q_j)}{\partial q_i} = 1 - 2q_i - q_j - c$$

so using the first order condition, the "best response" to opponent quantity q_j is

$$BR(q_j) = \max\left\{\frac{1-c}{2} - \frac{q_j}{2}, 0\right\}.$$

To find a Nash equilibrium, we look for a pair (q_i, q_j) with the property that:

$$q_i \in BR_i(q_j) \qquad \text{and} \qquad q_j \in BR_j(q_i).$$

It is easy to see that there will be a unique Nash equilibrium.

Proposition 3 *The unique Nash equilibrium is for each firm to choose quantity $q = (1-c)/3$.*

Interestingly, it turns out that the Cournot duopoly model is also dominance solvable. To show this, we need one preliminary result.

Lemma 1 *If $q_{-i} \in [\underline{q}, \overline{q}]$, then any quantity $q_i > BR(\underline{q})$ or $q_i < BR(\overline{q})$ is strictly dominated.*

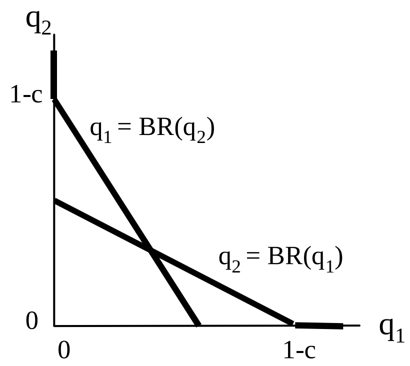

Figure 1: Cournot Duopoly

Proof. Note that the marginal returns to higher quantity are decreasing in both q_i and q_j. Thus, for all $q_j \leq \overline{q}$, and $q_i < BR(\overline{q})$, we have

$$\frac{\partial u_i(q_i, q_j)}{\partial q_i} > 0.$$

So $q_i < BR(\overline{q})$ will be strictly dominated by $BR(\overline{q})$, so long as $q_j \leq \overline{q}$. Similarly, we can show that $q_i > BR(\underline{q})$ will be strictly dominated by $BR(\underline{q})$ so long as $q_j \geq \underline{q}$. *Q.E.D.*

Now, let's use this claim to apply ISD.

- $S_i^0 = [0, \infty)$
- $S_i^1 = [0, BR(0)]$, where $BR(0)$ is the monopoly quantity $(1-c)/2$.
- $S_i^2 = [BR^2(0), BR(0)]$, where $BR^2(0) = (1-c)/2 - (1-c)/4$.

Continuing, we get a sequence of intervals $S_i^k = [\underline{q}^k, \overline{q}^k]$, where $\underline{q}^k = BR(\overline{q}^{k-1})$ and $\overline{q}^k = BR(\underline{q}^{k-1})$. Clearly, if either the sequence of upper or lower bounds converge, they both will converge to the same single point. Consider the lower bound:

$$\underline{q}^k = \frac{1-c}{2} - \frac{\overline{q}^{k-1}}{2} = \frac{1-c}{4} + \frac{\underline{q}^{k-2}}{4}.$$

It is easy to check that this sequence with initial value $q^0 = 0$, will converge to:

$$\underline{q} = \overline{q} = \frac{1-c}{3}.$$

Thus, $S^\infty = \left\{\frac{1-c}{3}, \frac{1-c}{3}\right\}$, and the game is dominance solvable.

8.2 Bertrand Competition

Consider the Bertrand model of competition with linear costs, $C_i(q_i) = cq_i$, with $1 > c > 0$, and a total market demand of $Q \geq 1$.

Proposition 4 *The unique Nash equilibrium is for both firms to set a price $p = c$.*

Proof. We first show that (c, c) is a Nash equilibrium, i.e. that $p_i = c$ is a best response to $p_j = c$. Note that if $p_j = c$, then i has three options. He can set $p_i < p_j$ and make Q sales, each at a loss. He can set $p_i < p_j$ and make no sales, or he can set $p_i = c$ and make $Q/2$ sales, each a zero profit. So $p_i = c$ is among his best responses, and it follows that (c, c) is a NE.

To see that (c, c) is the unique equilibrium, we have to consider different cases.

- Suppose $p_i \leq p_j < c$. Firm i makes positive sales, each at a loss, and would prefer to raise his price above p_j. So this cannot be a NE.
- Suppose $p_i < c \leq p_j$. Firm i again makes positive sales, each at a loss, and would prefer to raise his price above c. So this cannot be a NE.
- Suppose $c \leq p_i < p_j$. Firm i makes positive sales, but can always find a price $p \in (p_i, p_j)$ at which he would make the same number of sales, but at a higher price. Also, Firm j makes no sales, so if $c < p_i$, he would like to undercut Firm i. For both reasons, this cannot be a NE. *Q.E.D.*

Unlike Cournot duopoly, Bertrand duopoly is not dominance solvable. It is fairly easy to see setting $p_i = 0$ is strictly dominated by $p_i = c_i$ (since the former results in a demand of at least $Q/2$ and negative profits, while the latter always gives zero profits). However, no other strategies (not even prices below marginal cost) can be elimated by ISD! To see why, consider two prices $p_i, p_i' > 0$. Neither is dominated by the other because there will always be some opponent price $p_j < p_i, p_i'$ against which p_i, p_i' yield identical (zero) profits.

8.3 Bertrand Competition with Differentiated Products

The Bertrand model of competition has the appealing feature that firms choose prices, but the unappealing feature that firm's end up pricing at marginal cost even if there are only two of them. A more realistic model emerges if we assume that consumers have preferences for one firm or the other, so that an ε-price difference does not lead to wild swings in demand. In particular, suppose that firms have constant marginal costs c, and choose prices $p_1, p_2 \in [0, \infty)$, and that the demand for Firm i's product is:

$$D_i(p_i, p_j) = 1 - p_i + \lambda p_j.$$

Then Firm i's chooses his price to maximize:

$$\max_{p_i} (p_i - c) [1 - p_i + \lambda p_j].$$

The first order condition for this problem is:

$$1 - 2p_i + \lambda p_j + c = 0$$

leading to a best response function:

$$BR_i(p_j) = \frac{1+c}{2} + \frac{\lambda p_j}{2}.$$

Proposition 5 *The unique Nash equilibrium is for both firms to set a price* $p = \frac{1+c}{2-\lambda}$.

Again, a graphical derivation is illuminating.

8.4 Games with Multiple Equilibria

Even if one believes strongly in the Nash equilibrium concept, in some games, there may be a great many such equilibria. This raises the difficult problem of *equilibrium selection*: which Nash equilibrium is most likely to be played?

Bargaining: Nash Demand Game Imagine two parties who must bargain over a pie of size 1. Each player can make a demand x_i. If $x_1 + x_2 > 1$, then both parties get zero. If $x_1 + x_2 \leq 1$, the demands are compatible, and player i can take home a share x_i of the pie.

Claim. Any split of the pie $(x, 1 - x)$ with $x \in [0, 1]$ is a Nash equilibrium outcome.

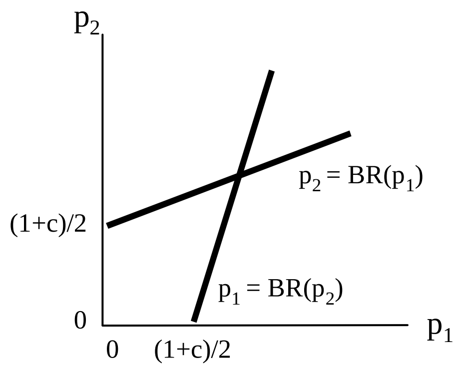

Figure 2: Bertrand-Nash Duopoly

Proof. Suppose that j demands x_j. Then i's best-response is to demand $x_i = 1 - x_j$. Thus, it must be that in any Nash equilibrium, $x_i + x_j = 1$, but there are no further constraints on the values of x_i, x_j. *Q.E.D.*

Stag Hunt This game has two pure strategy Nash equilibria, (A, A) and (B, B). Both are strict.

	A	B
A	$9, 9$	$-5, 8$
B	$8, -5$	$7, 7$

It is natural to ask which of these two equilibria is more likely. One view is that (A, A) should be played, since it is the Pareto-preferred equilibrium. For instance, shouldn't the players meet before the game and agree to play (A, A)? There is a question, however, of whether this sort of communication would be credible. Even if Row was planning to play B, he would want to convince Column to play A by claiming that he himself would play A. In fact, Harsanyi and Selten (1988) argue that (B, B) is in fact more likely, because it is *risk-dominant* (i.e. B is a best reply if you are completely uncertain and assign probability $1/2$ to each opponent strategy).

9 Mixed Strategies

While pure strategy Nash equilibrium is a natural solution concept, there are some situations where a pure strategy Nash equilibrium profile does not exist.

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

This game has *no* pure strategy Nash equilibrium. At a matched profile (H, H) or (T, T), Column wants to switch strategies, while at a mismatched profile (H, T) or (T, H), Row wants to switch strategies.

An important feature of matching pennies is that it is a game of *pure conflict*, or a zero-sum game. Any gain for i is a loss for j. Zero-sum games were the original class of games studied by game theorists. Other examples include parlor games such as poker, sporting events such as soccer or football, and military conflicts.

In games of pure conflict, if player i knows his opponent's strategy, he can exploit this to his advantage. Fearing this, his opponent may prefer to be unpredictable and to choose a *randomized* or *mixed strategy*. For example, in matching pennies, Column could decide to play H with probability 1/2 and T with probability 1/2.

Definition 8 *A **mixed strategy** σ_i for player i is a probability distribution on S_i, i.e. for S_i finite, a mixed strategy is a function $\sigma_i : S_i \to \mathbb{R}_+$ such that*

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

We will use several different notations for mixed strategies. For example, the strategy of mixing equally between Heads and Tails could be represented alternatively as: $\sigma_1(H) = \frac{1}{2}$, $\sigma_1(T) = \frac{1}{2}$; or as $(\sigma_1(H), \sigma_1(T)) = (\frac{1}{2}, \frac{1}{2})$, or finally as $\sigma_1 = \frac{1}{2}H + \frac{1}{2}T$.

We use $\Sigma_i = \Delta(S_i)$ to represent the set of mixed strategies available to player i, and Σ to represent the set of mixed strategy profiles. And we write $u_i(\sigma_i, \sigma_{-i})$ to represent i's payoff given a profile σ:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i, s_{-i}} u_i(s_i, s_{-i})\sigma_i(s_i)\sigma_{-i}(s_{-i}).$$

Definition 9 *A mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ is a Nash equilibrium of G if for all $\sigma_i' \in \Sigma_i$,*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

A somewhat thorny question is what is means exactly for a player to play a mixed strategy. The most obvious interpretation is that players *explicitly* randomize, perhaps by flipping a coin. However, this is somewhat unsatisfactory, given that we don't often see people going around flipping precisely weighted coins to make decisions. Another interpretation, due to Harsanyi, is that mixed strategies reflect small uncertainties in payoffs, and the probability of playing a given strategy is the probability that this small uncertainty will resolve in favor of that strategy. We will return to this idea in a few weeks. A third interpretation, which we will return to sooner is that mixed strategy equilibria describe a stable state for a *population* of players, each of whom uses a pure strategy. Finally, the interpretation favored by some current theorists (e.g. Aumann and Brandenburger, 1995) is that people don't actually randomize, but that mixed strategy equilibria can be thought of as equilibria in *beliefs*. That is, given a mixed strategy equilibrium (σ_1, σ_2) we think of σ_i simply as j's belief about what i will do, but *not necessarily* as the analyst's prediction about what i will do, or even as i's belief about what i will do.[2]

9.1 Characterizing Mixed Strategy Equilibria

An important feature of mixed strategy equilibria is that if (σ_i, σ_{-i}) is a mixed Nash equilibrium profile, then i must be indifferent between every strategy s_i in the support of σ_i.

Definition 10 *In a finite game, the support of a mixed strategy σ_i is the set of pure strategies to which σ_i assigns positive probability.*

$$supp(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}.$$

Proposition 6 *If σ is a mixed strategy Nash equilibrium, and $s_i, s_i' \in supp(\sigma_i)$, then:*

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}).$$

[2] If this seems confusing, don't worry.

This suggest a method for identifying mixed strategy equilibria in a given game. First identify the support of each player's mixed strategy profile. Second, identify the mixture that j must be using to keep i indifferent between all pure strategies in the support of his mixed strategy.

Example Consider the following game:

	L	R
U	$1,2$	$3,1$
D	$2,1$	$0,2$

This game has no pure strategy Nash equilibrium. We look for a mixed equilibrium where Row randomizes between U, D. For this to happen, it must be that Row is indifferent between U, D. But this can only happen if Column is playing the mixed strategy $\frac{3}{4}L + \frac{1}{4}R$. But for Column to be mixing, it must be that Column is indifferent between L, R. This can only happen if Row is playing the mixed strategy $\frac{1}{2}U + \frac{1}{2}D$. It follows that the unique NE is $(\frac{1}{2}U + \frac{1}{2}D, \frac{3}{4}L + \frac{1}{4}R)$.

There is also a nice graphical way to identify this same equilibrium. Let $pU+(1-p)D$ denote an arbitrary strategy for Row (with $p \in [0,1]$) and $qL + (1-q)R$ denote an arbitrary strategy for Column. Then

$$BR_R(qL + (1-q)R) = \begin{cases} U & \text{if } q < 3/4 \\ pU + (1-p)D & \text{if } q = 3/4 \\ D & \text{if } q > 3/4 \end{cases}$$

and

$$BR_C(pU + (1-p)D) = \begin{cases} L & \text{if } p > 1/2 \\ qL + (1-q)R & \text{if } p = 1/2 \\ R & \text{if } p < 1/2 \end{cases}$$

We find a Nash equilibrium, we need a profile (σ_R, σ_C) with the property that:

$$\sigma_R \in BR_R(\sigma_C) \qquad \text{and} \qquad \sigma_C \in BR_C(\sigma_R)$$

This is shown in Figure 3.

Note that if σ is a mixed equilibrium, then σ cannnot be a strict NE. As we just noted, if σ_i is mixed, then i must be indifferent between every strategy in the support of σ_i. But why should a player play precisely his mixed

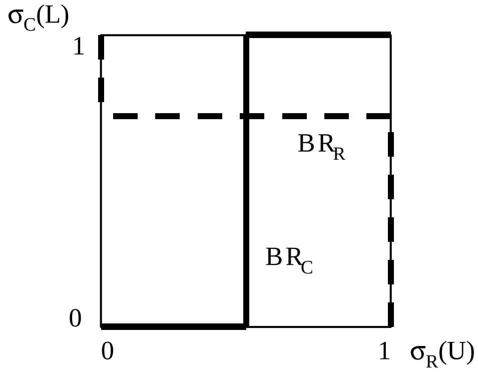

Figure 3: Graphical Derivation of Mixed NE

strategy, given that he is indifferent between that an many other strategies. For instance, in our example, Row is indifferent between playing U, playing D and playing any randomization. Why should Row play precisely $\frac{1}{2}U + \frac{1}{2}D$ rather than some other strategy? How problematic this is depends on how one wishes to interpret or justify Nash equilibrium. It does not seem terribly problematic if we view Nash equilibrium as a self-enforcing agreement, or as a stable social configuration. It seems more problematic if we want to think about Nash equilibrium as arising just from introspection.

9.2 Iterated Dominance Revisited

Now that we have defined mixed strategies, it is possible to re-define iterated strict dominance in a way that more completely exploits rationality.

Example Consider the following game:

	L	M	R
U	$4, 10$	$3, 0$	$1, 3$
D	$0, 0$	$2, 10$	$10, 3$

In this game, no strategy is strictly dominated by another pure strategy. However, for Column, R is strictly dominated by the mixed strategy $(1/2, 1/2, 0)$ that assigns equal probability to L, M. Once R is deleted, then U strictly dominates D. It follows that the game is dominance solvable giving the profile (U, L).

Formally, we can expand our definition of ISD as follows:

- $S_i^0 = S_i$
- $S_i^{k+1} = \left\{ \begin{array}{c} s_i \in S_i^k \mid \nexists \sigma_i \in \Delta(S_i^k) \text{ such that} \\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k \end{array} \right\}.$
- $S_i^\infty = \cap_{k=0}^\infty S_i^k$.

Proposition 7 *Suppose σ is a Nash Equilibrium of G, and that $\sigma_i^*(s_i) > 0$. Then s_i is not removed by ISD, $s_i \in S_i^\infty$.*

Proof. Let σ be a Nash equilibrium of G, and suppose (by way of contradiction) that not every pure strategy in its support survives ISD. Let s_i be he first pure strategy with $\sigma_i(s_i) > 0$ that is removed during ISD, and suppose this occurs at round k. This means that there exists some $\sigma_i' \in \Delta(S_i)$ with the property that:

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}^k.$$

However, since s_i is the *first* strategy in the support of the NE σ to be removed, it must be that for any $s_{-i} \in$supp(σ_{-i}), it is also the case that $s_{-i} \in S_{-i}^k$. Thus,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$$

which contradicts the fact that σ is a Nash Equilibrium. *Q.E.D.*

9.3 Application: Equilibrium Price Dispersion

Some compelling applications of mixed strategy Nash equilibrium come in games with a large population of players. Consider the following model of price competition. There are a continuum of firms, $i \in [0, 1]$, each of whom posts a price p_i. Once firms have posted prices, consumers make purchase decisions. There is a mass $m > 1$ of consumers, each of whom has willingness to pay v. A fraction μ of these consumers sample a single firm's price and buy if the price is below v, while the remainder $1 - \mu$ sample two prices and purchase from the firm that sets the lower price, conditional on this price being below v. All firms have constant marginal cost equal to $c < v$.

Consider the problem facing firm i. Suppose that i sets a price p_i. A fraction μ of consumers sample only one price, meaning that Firm i can expect a mass μm of these consumers to arrive and purchase if and only if $p_i < v$. Firm i can also expect a mass $(1 - \mu)m$ of consumers to arrive, who also will see a second price. Let $F_i(p_i)$ denote the probability that such a consumer will see a price below p_i as her second price. Then Firm i's demand is:

$$D_i(p_i) = \begin{cases} \mu m + (1-\mu)m\,[1 - F_i(p_i)] & \text{if } p_i \leq v \\ 0 & \text{if } p_i > v \end{cases}$$

Let's look for a Nash equilibrium of this model. First, note that so long as $\mu \in (0,1)$, there cannot be an equilibrium where all firms set the same price p. If all firms set the same price $p > c$, then by dropping its price by ε, firm i can gain $(1-\mu)m/2$ new sales for new profit of $(p-c-\varepsilon)(1-\mu)m/2$, and incur a loss of only $\varepsilon\mu m$. On the other hand, if all firms set $p \leq c$, then by charging $p_i = v$ a deviating firm can go from at best zero profits to $\mu m(v-c)$ in profits.

Instead, we look for a mixed equilibrium, where each firm uses all prices between $\underline{p}$ and v, for some $\underline{p}$, and where the probability of setting a price below p is $F(p)$. For this to be an equilibrium, we need firms to realize the same profits for any price in use. Since

$$u_i(p_i = v, F) = (v-c)\,\mu m$$

we have that for all $p \in [\underline{p}, v]$,[3]

$$u_i(p, F) = (p-c)\,[\mu m + (1-\mu)m\,(1-F(p))] = (v-c)\mu m.$$

This allows us to solve for $F(\cdot)$:

$$F(p) = 1 - \frac{\mu}{1-\mu}\frac{v-p}{p-c},$$

in turn, we solve $F(\underline{p}) = 0$ to find that:

$$\underline{p} = \mu v + (1-\mu)c.$$

Note that if $\mu = 1$, then all consumers look at only a single price, and hence there is a pure strategy equilibrium where all firms set the monopoly price $p = v$. If $\mu = 0$, then all consumers look at two prices, so there is effectively Bertrand competition and there is a pure strategy equilibrium in which all

[3] You might be wondering why v is the highest price used. Clearly it makes no sense to set a price above v, since no consumer will purchase. On the other hand, if the highest price used was $\overline{p} < v$, then a firm that priced at $\overline{p}$ would sell only to consumers who looked at only one price. Given that it sells only to these consumers, it does better to price at v than $\overline{p} < v$.

firms price a marginal cost $p = c$. Finally, for $\mu \in (0,1)$, we have equilibrium price dispersion.

Observe that this price dispersion can be interpreted in two ways. One is that each firm literally plays a mixed strategy, randomizing in its choice of price. Alternatively, we can think of each firm as playing a different pure strategy, but as the population distribution of prices being $F(\cdot)$. Either way, we have an equilibrium!

10 Properties of Nash Equilibrium

We now take up a few of the important properties of the Nash equilibrium solution concept.

10.1 Existence of Nash Equilibrium

The first is the celebrated existence theorem from Nash's Ph.D. thesis (1951).

Proposition 8 *(Nash) At least one Nash equilibrium exists in any finite game.*

To prove this, we will rely on a characterization of Nash equilibrium that we have already been using informally. Define the *best response correspondence* of player i as follows:

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}).$$

Lemma 2 *σ is a Nash equilibrium if and only if for all i, $\sigma_i \in BR_i(\sigma_{-i})$.*

This observation permits us to identify Nash equilibria as *fixed points* of the correspondence $BR : \Sigma \rightrightarrows \Sigma$ given by

$$BR(\sigma) = (BR_1(\sigma_{-1}), ..., BR_I(\sigma_{-I})).$$

To proceed further, we need to take a short mathematical detour to discuss fixed points of functions and correspondences.[4]

Let $X, Y \subset \mathbb{R}^n$, and let $r : X \rightrightarrows Y$ be a correspondence. Recall that X is *convex* if and only if $x, x' \in X$ implies that $\lambda x + (1-\lambda)x' \in X$ for all $\lambda \in [0,1]$. Recall also that X is *compact* if and only if it is closed and bounded.

[4] For those of you who have seen the proof of the existence of Walrasian equilibria in exchange economies, much of this will look familiar.

The correspondence r is *non-empty valued* if for all $x \in X$, $r(x) \neq \emptyset$. The correspondence r is *convex-valued* if for all $x \in X$, $r(x)$ is a convex set. Finally, we can define the *graph* of r as follows:

$$Graph(r) = \{(x, y) \in X \times Y \mid y \in r(x)\}.$$

The correspondence r has a *closed graph* if $Graph(r)$ is a closed subset of $X \times Y$.

Lemma 3 *(Kakutani's Fixed Point Theorem) Suppose that X is a non-empty, compact, convex subset of $\mathbb{R}^n$ and that $r : X \rightrightarrows X$ is a non-empty and convex-valued correspondence with a closed graph. Then r has a fixed point, i.e. there exists some $x \in X$ with the property that $x \in r(x)$.*

To prove Nash's Theorem, we apply Kakutani's Fixed Point Theorem to the best response correspondence $BR : \Sigma \rightrightarrows \Sigma$.

Proof of Nash's Theorem. Consider the best response correspondence $BR : \Sigma \rightrightarrows \Sigma$. We proceed to verify the conditions for Kakutani's Theorem.

1. Σ is a compact, convex subset of $\mathbb{R}^I$.

2. BR is non-empty valued. To see this, recall that

$$BR_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$$

   Since $u_i(\sigma_i, \sigma_{-i})$ is continuous in σ_i, and Σ_i is compact, a maximizer exists, so $BR_i(\sigma_{-i})$ exists for all σ_{-i} and hence $BR(\sigma)$ exists for all σ.

3. BR is convex-valued. To see this, suppose $\sigma_i, \sigma_i' \in BR_i(\sigma_{-i})$. Then, any s_i in the support of σ_i gives the same payoff, and this payoff is the same as that given by any s_i in the support of σ_i'. It follows that i is indifferent to any randomization amongst these pure strategies. So $BR_i(\sigma_{-i})$ must be convex, and hence so is $BR(\sigma)$.

4. BR has a closed graph. Suppose that $(\sigma^k, \hat{\sigma}^k) \in Graph(BR)$, and that $\sigma^k \to \sigma$ and $\hat{\sigma}^k \to \hat{\sigma}$. We want to show that $\hat{\sigma} \in BR(\sigma)$, that is, for all i that $\hat{\sigma}_i \in BR_i(\sigma_{-i})$. To see this, note that for all s_i, and all k,

$$u_i(\hat{\sigma}_i^k, \sigma_i^k) \geq u_i(s_i, \sigma_i^k).$$

   Furthermore, u_i is continuous in all arguments. So by taking limits:

$$u_i(\hat{\sigma}_i, \sigma_i) \geq u_i(s_i, \sigma_i).$$

Applying Kakutani's Theorem, there exists some profile σ such that $\sigma \in BR(\sigma)$, and hence σ is a Nash Equilibrium. *Q.E.D.*

Nash's Theorem establishes the existence of a *mixed strategy equilibrium* in *finite games*. It is also interesting to inquire whether there are conditions under which *pure strategy equilibrium* are known to exist. This is clearly not the case in arbitrary finite games, as we have already seen examples (such as matching pennies) where there is no pure strategy Nash equilibrium. However, pure strategy equilibrium will certainly exist in certain well–behaved games with continuous strategy spaces.

Proposition 9 *(Debreu-Fan-Glicksberg) Let G be a normal form game. Suppose that each strategy set S_i is a non-empty, compact, convex subset of $\mathbb{R}^n$ and that each payoff function u_i is continuous in s_i, s_{-i} and quasi-concave in s_i. Then G has a pure strategy Nash equilibrium.*

There are also mixed strategy existence theorems for games with continuous (and infinite) strategy spaces. Problems with existence typically arise because the payoff functions are discontinuous. In this case, best responses may not necessarily exist. A recent and beautiful paper on the existence of equilibrium in games with discontinuous payoff functions is by Reny (2000).

10.2 Other Properties of Nash Equilibrium

A useful property of Nash equilibrium is that as the payoff functions change, the set of Nash equilibria changes in an upper hemi-continuous way.

Proposition 10 *Consider a sequence of games G^k with finite strategy space S, but differing in the payoff functions u^k. Suppose that for all i, $u_i^k \to u_i$ as $k \to \infty$, and let G denote the game (S, u). If $\sigma^k \in NE(G^k)$ for all k, and $\sigma^k \to \sigma$, then $\sigma \in NE(G)$.*

Proof. This is just like the proof that BR has a closed graph, and it left as an exercise. *Q.E.D.*

Another useful result is that finite games typically have only a finite (and odd) number of equilibria. To state this result, we need a notion of genericity. First, letting G and G' be two games with finite strategy space S and payoff functions u, u', we define the *distance* between G, G' as:

$$\|G' - G\| = \left(\sum_{s \in S} \sum_{i \in I} \left(u_i'(s) - u_i(s) \right)^2 \right)^{1/2}.$$

Definition 11 *A property holds generically for finite normal form games if it holds for a set X such that: (i) if $G \in X$, then there is some $\varepsilon > 0$ such that if $\|G' - G\| < \varepsilon$, then $G' \in X$, and (ii) if $G \notin X$, then for any $\varepsilon > 0$, there exists some $G' \in X$ with $\|G' - G\| < \varepsilon$.*

Proposition 11 *(Wilson) Finite normal form games generically have a finite and odd number of Nash equilibria.*

11 Evolution and Nash Equilibrium

We now briefly take up the idea that Nash equilibrium might arise as the end result of "evolutionary pressure" or adaptive behavior. To do this, we imagine a population of N players who are randomly matched against each other over time to play a given game. It is useful to think of N as very large (or even infinite). When N is large, players are unlikely to ever meet their opponent again, so each interaction is truly a one-time event. We focus on symmetric $n \times n$ games, meaning that each player $i = 1, 2$ has the same strategy set and that $u_i(s_i, s_j) = u_j(s_j, s_i)$ for all $s_i, s_j \in S = \{s_1, ..., s_n\}$.

To specify an evolutionary model, we specify a rule about how players change their behavior over time, and then track the fraction of players who are using each strategy at each point in time.[5] Let $x_1^t, ..., x_n^t$ denote the fraction of players who are using strategy $s_1, ..., s_n$ at time t. It will always be the case that $x_i^t \geq 0$ for all i and that $\sum_i x_i^t = 1$. Let $x^t = (x_1^t, ..., x_n^t)$ denote the vector of population shares. Note that we can easily interpret x as a *mixed strategy* of the underlying game (denoted G).

Evolutionary models typically have two forces: *selection* and *mutation.* Selection means that strategies that perform well will tend to spread. Mutation captures that idea that there may be some noise in this process — players may randomly try new strategies for no particular reason. We start by looking at one of the simplest versions of evolution, the *replicator dynamics.*

Under replicator dynamics, the fraction of players playing a given strategy s_i evolves according to:

$$x_i^{t+1} - x_i^t = x_i^t \left[u_i \left(s_i, x^t \right) - u_i(x^t, x^t) \right],$$

where $u_i(s_i, x)$ denotes the expected payoff of a player using strategy s_i given that the population is playing x, and $u(x, x) = \sum u_i(s_i, x) x_i$ denotes

[5] In evolutionary biology, it is sometimes useful to think about a strategy as being expressed by a gene. Then a given player never changes her strategy, but strategies are passed down from generation to generation.

the average population payoff given population play x. A starting point for this dynamic process is a vector $x^0 = (x_1^0, ..., x_n^0)$.

Lemma 4 *The population shares always sum to one, i.e. for all t,* $\sum_i x_i^t = 1$.

Proof. To see this, note that:

$$\begin{aligned}\sum_i x_i^{t+1} &= \sum_i x_i^t + \sum_i x_i^t \left[u_i\left(s_i, x^t\right) - u_i(x^t, x^t)\right] \\ &= \sum_i x_i^t + \left[u_i\left(x^t, x^t\right) - u_i(x^t, x^t)\right] = \sum_i x_i^t.\end{aligned}$$

So long as $\sum_i x_i^0 = 1$, we are ok. *Q.E.D.*

There are several important features to note about the replicator dynamics. First, they are deterministic – there is selection, but no mutations. Second, they are monotonic — strategies that do better (have a higher expected payoff) realize a larger percentage gain in their population share. Finally, growth is proportionate to the present share — so if $x_i^0 = 0$ for some strategy s_i, then $x_i^t = 0$ for all t.

We are interested in looking at social configurations, or population shares, that might be *stable* under these dynamics. We first give a definition of what this means.

Definition 12 *A population distribution x is a* ***steady-state*** *under the replicator dynamics if $x^t = x$ for some t implies that $x^T = x$ for all $T > t$.*

Proposition 12 *If a strategy profile σ is a symmetric Nash equilibrium of G, then the population distribution $x = \sigma$ is a steady-state under the replicator dynamics.*

Proof. If σ is a symmetric Nash equilibrium, then if $\sigma(s_i) > 0$, $u_i(s_i, \sigma) = u_i(\sigma, \sigma)$, so σ will also be a steady-state under the replicator dynamics. *Q.E.D.*

Note that the converse is not true. For example, it is a steady-state for all players to cooperate in the prisoner's dilemma.

Example In matching pennies, the *only* steady-state is the mixed Nash equilibrium, $x = (1/2, 1/2)$.

A stronger concept is the idea of stability under replicator dynamics.

Definition 13 *A steady-state population distribution x is **stable** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x^t - x| < \delta$, then $|x^T - x| < \varepsilon$ for all $T > t$.*

We say that x is *asymptotically stable* if there is some $\delta > 0$ such that whenever $|x^t - x| < \delta$ for some t, then $x^T \to x$ as $T \to \infty$.

Proposition 13 *If a steady-state population distribution x is stable, then the profile $\sigma = x$ is a Nash equilibrium.*

Proof. Assume σ is stable, but not a Nash equilibrium. Then there exists some s_i such that $u_i(s_i, \sigma) - u_i(\sigma, \sigma) = a > 0$. By continuity, it follows that there is some $\varepsilon > 0$ such that if $|x^t - \sigma| < \varepsilon$, then $u_i(s_i, x^t) - u_i(x^t, x^t) > a/3$. Now, because σ is stable, we can find some δ such that if $|x^t - \sigma| < \delta$, then $|x^T - \sigma| < \varepsilon$ for all $T > t$. Fix $x^t = (1 - \frac{\delta}{2})\sigma + \frac{\delta}{2} s_i$. Then for all $T > t$, $x_i^{T+1} - x_i^T > x_i^T \frac{a}{3}$, or $x_i^{T+1} > x_i^T (1 + \frac{a}{3})$. But then

$$x_i^T > \frac{\delta}{2}\left(1 + \frac{a}{3}\right)^{T-t} \qquad \Rightarrow \qquad x_i^T \to \infty \text{ as } T \to \infty,$$

which yields a contradiction. *Q.E.D.*

Stability focuses attention on the selection aspect of evolution. The idea of an *evolutionary steady state* focuses more on the role of mutations.

Definition 14 *A population distribution x is an **evolutionary steady state (ESS)** if, for all $y \neq x$, there exists some $\overline{\varepsilon}$ such that $u(x, (1-\varepsilon)x + \varepsilon y) > u(y, (1-\varepsilon)x + \varepsilon y)$ for all $0 < \varepsilon < \overline{\varepsilon}$.*

Suppose the population starts at state x, and a fraction ε mutate to play y. The original state x is an ESS if, whatever the nature of this mutation, the average payoff of the non-mutators is higher than the payoff of the mutators. Note that this definition is independent of the specific dynamics through which the population distribution changes. It is defined directly on the underlying game G.

Proposition 14 *A population distribution x is an ESS if and only if for all $y \neq x$ either (a) $u(x, x) > u(y, x)$ or (b) $u(x, x) = u(y, x)$ and $u(x, y) > u(y, y)$.*

There is a close link between Nash equilibria and evolutionary steady-states.

Proposition 15 *If x is an ESS, then it is also a Nash equilibrium. Conversely, if x is a strict Nash equillibrium, then it is an ESS.*

Our final result establishes a link between ESS and stability under the replicator dynamics.

Proposition 16 *If x is an ESS, then it is asymptotically stable under the replicator dynamics.*

You will be asked to work out one or two examples on the homework.