Homework 5

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An ECON - 8070 Homework Assignment

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Question 15.3

Problem

Let $\mathbf{z}_i$ be a vector of variables. Let z_2 be a continuous variable, and let d_1 be a dummy variable.

(a)

In the model

$$\mathbb{P}(y = 1|\mathbf{z}_1, z_2) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2),$$

find the partial effects of z_2 on the response probability. How would you estimate this partial effect?

(b)

In the model

$$\mathbb{P}(y = 1|\mathbf{z}_1, z_2, d_1) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 d_1 + \gamma_3 z_2 d_1),$$

find the partial effects of z_2. How would you estimate the effect of d_1 on the response probability? How would you estimate these effects?

(c)

Describe how you would obtain the standard errors of the estimated partial effects from parts a and b.

Solution

(a)

To get the partial effect of z_2 on the response probability, we will take the partial derivative with respect to z_2

$$G(z) = \Phi(z) \equiv \int_{-\infty}^{(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)} \phi(z)dz$$

$$\phi(z) = (2\pi)^{-\frac{1}{2}} \exp(-\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)^2/2)$$

$$\frac{\partial p(z)}{\partial z_2} = \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)(\gamma_1 + 2\gamma_2 z_2)$$

This partial effect can be estimated by Probit regression.

(b)

To get the partial effect of z_2, we follow the same protocol as in (a)

$$G(z) = \Phi(z) \equiv \int_{-\infty}^{(\mathbf{z}_1\boldsymbol{\delta}_1+\gamma_1 z_2+\gamma_2 d_1+\gamma_3 z_2 d_1)} \phi(z)dz$$

$$\phi(z) = (2\pi)^{-\frac{1}{2}} \exp(-\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 d_1 + \gamma_3 z_2 d_1)^2/2)$$

$$\frac{\partial p(z)}{\partial z_2} = \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)(\gamma_1 + 2\gamma_3 d_1)$$

To estimate the effects of d_1, we would then evaluate the partial effects at $d_1 = 1, 0$.

$$d_1 = 1 \to \phi(z)(\gamma_1 + \gamma_3)$$
$$d_1 = 0 \to \phi(z)(\gamma_1)$$

This would be estimated, again, by a Probit model, being sure to include the interaction term $z_2 \times d_1$

(c)

Using the delta method, we would be able to ascertain the standard errors for both parts.

Question 15.5

Problem

Consider the probit model

$$\mathbb{P}(y = 1|\mathbf{z}, q) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma z_2 q),$$

where q is independent of $\mathbf{z}$ and distributed as $N(0, 1)$; the vector $\mathbf{z}$ is observed but the scalar $\mathbf{q}$ is not.

(a)

Find the partial effect of z_2 on the response probability, namely,

$$\frac{\partial \mathbb{P}(y = 1|\mathbf{z}, q)}{\partial z_2}$$

(b)

Show that $\mathbb{P}(y = 1|\mathbf{z}) = \Phi[\mathbf{z}_1\boldsymbol{\delta}_1/(1 + \gamma_1\gamma_2)^{\frac{1}{2}}]$

(c)

Define $\rho_1 \equiv \gamma_1^2$. How would you test $\mathrm{H}_0 : \rho_1 = 0$?

(d)

If you have a reason to believe $\rho_1 > 0$, how would you estimate $\boldsymbol{\delta}_1$ along with ρ_1?

Solution

(a)

Following a similar protocol to the questions in 15.3,

$$G(z) = \Phi(z) \equiv \int\limits_{-\infty}^{\mathbf{z}_1\boldsymbol{\delta}+\gamma_1 z_2 q} \phi(z)dz$$

$$\phi(z) = (2\pi)^{-\frac{1}{2}} \exp(\mathbf{z}_1\boldsymbol{\delta} + \gamma_1 z_2 q)$$

$$\frac{\partial p(z)}{\partial z_2} = \phi(\mathbf{z}_1\boldsymbol{\delta} + \gamma_1 z_2 q)(\gamma_1 q)$$

(b)

$$y = \mathbb{1}\{z_1\delta_1 + \gamma_1 z_2 q + \epsilon\}$$

$$e = \gamma_1 z_2 q + \epsilon$$

$$e \sim N(0, 1 + \gamma_1^2 z_2^2)$$

$$\mathbb{P}(e \geq \mathbf{z}_1\boldsymbol{\delta}_1)$$

$$\mathbb{P}(e \leq \mathbf{z}_1\boldsymbol{\delta}_1)$$

$$= \Phi(\frac{(\mathbf{z}_1\boldsymbol{\delta}_1)}{(\gamma_1 z_2)^{\frac{1}{2}}})$$

(c)

To test $H_0: \ \rho_1 = 0$, such that $\rho_1 \equiv \gamma_1^2$, we would use the Langrange multiplier test.

(d)

Using maximum likelihood estimation, we could estimate $\boldsymbol{\delta}_1$ and ρ_1.

Question 19.7

Problem

Suppose in section 19.6.1, we replace assumption 19.1d with

$$\mathbb{E}(u_1 \mid v_2) = \gamma_1 v_2 + \gamma_2(v_2^2 - 1).$$

(We subtract unity from v_2^2 to ensure that the second term has zero expectation.)

(a)

Using the fact that $\mathrm{Var}(v_2 \mid v_2 > -a) = 1 - \lambda(a)[\lambda(a) + a]$, show that

$$\mathrm{E}(y_1 \mid \mathbf{x}, y_2 = 1) = \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1\lambda(\mathbf{x}\boldsymbol{\delta}_2) - \gamma_2\lambda(\mathbf{x}\boldsymbol{\delta}_2)\mathbf{x}\boldsymbol{\delta}_2.$$

[Hint: Take $a = \mathbf{x}\boldsymbol{\delta}_2$ and use the fact that $\mathrm{E}(v_2^2 \mid v_2 > -a) = \mathrm{Var}(v_2 \mid v_2 > -a) + [\mathrm{E}(v_2 \mid v_2 > -a)]^2$.]

(b)

Explain how to correct for sample selection in this case.

Solutions

(a)

Proof. We are given a few definitions which will be valuable to list. Firstly, $a = \mathbf{x}\boldsymbol{\delta}_2$. This is useful given $\mathbb{E}(v_2^2|v_2 > -a) = \text{var}(v_2|v_2 > -a) + \mathbb{E}(v_2|v_2 > -a)^2$ and $\text{var}(v_2|v_2 > -a) = 1 - \lambda(a)[\lambda(a) + a]$. When substituting for a, $\text{var}(v_2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) = 1 - \lambda(\mathbf{x}\boldsymbol{\delta}_2)[\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \mathbf{x}\boldsymbol{\delta}_2]$. So, $\mathbb{E}(v_2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) = \lambda(\mathbf{x}\boldsymbol{\delta}_2)$ and $\mathbb{E}(v_2^2|v_2 > -a) = 1 - \lambda(\mathbf{x}\boldsymbol{\delta}_2)[\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \mathbf{x}\boldsymbol{\delta}_2] + [\lambda(\mathbf{x}\boldsymbol{\delta}_2)^2]$. Thus, we can move on, saying that $\mathbb{E}(u|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) = \gamma_1 \mathbb{E}(v_2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) + \gamma_2 \mathbb{E}(v_2^2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2)$. When plugging in results, we get $\mathbb{E}(u|v_2) = \gamma_1 \lambda(\mathbf{x}\boldsymbol{\delta}_2) + \gamma_2([1 - \lambda(\mathbf{x}\boldsymbol{\delta}_2)(\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \mathbf{x}\boldsymbol{\delta}_2) + \lambda(\mathbf{x}\boldsymbol{\delta}_2)^2] - 1)$. After collecting terms, we are left with $\mathbb{E}(u|v_2) = \gamma_1 \lambda(\mathbf{x}\boldsymbol{\delta}_2) - \gamma_2 \lambda(\mathbf{x}\boldsymbol{\delta}_2)\mathbf{x}\boldsymbol{\delta}_2$. Then, we can conclude by saying that $\mathbb{E}(y|x, y_2 = 1) = (x)_1(\beta)_1 + \gamma_1 \lambda(\mathbf{x}\boldsymbol{\delta}_2) - \gamma_2 \lambda(\mathbf{x}\boldsymbol{\delta}_2)\mathbf{x}\boldsymbol{\delta}_2$ $\square$

(b)

First, obtain the probit estimate for y_2. Then obtain the inverse mills ratio for each observation. Then, run OLS on the equation we derived in part a.