Econ 8020

January 7th, 2025

Course Outline

1. Math Review and Monotone Comparative Statics
2. General Equilibrium
3. Auction Theory
4. Mechanism Design
5. Special Topics (if time permits)
    - Matching and Market Design
    - Bayesian Persuasion and Information Design

Syllabus: Materials

- Lecture slides
    - These are not to be distributed unless given permission
- Variety of supplemental lecture notes on eLC
- Recommended books:
    - *Microeconomic Theory* by Mas-colell, Whinston, Green
    - *Auction Theory* by Vijay Krishna
    - *Two-Sided Matching* by Alvin Roth and Marilda Sotomayor

Syllabus: Expectations

- Class Attendance is expected
- Please show up to class on time as coming in late can be distracting to other students.
- Please do not leave class early
    - Let me know in advance otherwise
    - Obvious exceptions include emergencies
- Please do not use phones & laptops to surf. It can be distracting to other students

Syllabus: Office Hours

- Bohdan and I will hold offices hours once a week
  - Time and location will be posted on eLC.

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- Questions and email policy
  - 48hr rule: give the individual at least 48hrs to respond
  - Exceptions: email on Friday, expectation should be Monday

Syllabus: Grading

- Assignments due approximately every two weeks (40%)
    - About one for each topic
- In-class "mid-term" (30%)
- Take-home exam (30%)

Syllabus: Cheating

- Assignments should be written up individually
- I do not mind you working with a friend but one should not simply copy the answers
- Abide by Honor Code
- University policy for handling cases of suspected dishonesty: `www.uga.edu/ovpi`

Syllabus: Mental Health

- UGA has several resources for students
  - https://www.uhs.uga.edu/bewelluga/bewelluga
  - Crisis Support: https://www.uhs.uga.edu/info/emergencies
- If need help managing stress, anxiety, relationships, etc., please visit BeWellUGA
  - List of free workshops, classes, mentoring, and health coaching
  - Additional resources can be accessed through the UGA App
- Take care of yourselves
- Look out for others

Mathematical Concepts

- Constrained Optimization
  - Karush-Kuhn-Tucker Conditions (KKT)
- Envelope Theorem
- Monotone Comparative Statics

Useful Terms

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- Convexity, Concavity, Quasi-Concavity
- Upper and lower hemicontinuity
    - Occasionally called upper and lower semicontinuity

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  - Examples: Wealth level in consumer optimization problem, Prices, wages

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    - Examples: Wealth level in consumer optimization problem, Prices, wages
- Choice set $\mathcal{X}(\theta)$
    - Can be dependent on parameter
- Objective function $f : \mathcal{X} \times \Theta \longrightarrow \mathbb{R}$

Environment

For a given $\theta \in \Theta$, the optimization problem we wish to solve is:

$$\max_{x \in \mathcal{X}(\theta)} f(x, \theta)$$

- This is what is known as constrained-optimization
- Myriad of economic problems are of this form

Environment

- $x^*(\theta) = arg\,\max_{x \in \mathcal{X}(\theta)} f(x, \theta)$ is called the solution set
- $V(\theta) = \max_{x \in \mathcal{X}(\theta)} f(x, \theta)$ is the value function

Environment

- Does a solution exist for each $\theta \in \Theta$?
- What are the properties of the solution set and value function?
  - Continuous, Differentiable etc.
- How can we compute the solution or value function?
- How does the solution and value change with the parameter?

Constrained Optimization: $\mathbb{R}^n$

- $\mathcal{X}(\theta)$ is a convex subset of $\mathbb{R}^n$
- There are functions $g_1, \ldots, g_K$ such that $\mathcal{X}(\theta) = \{x | g_1(x, \theta), \ldots, g_K(x, \theta) \leq 0\}$

$$\max_{x \in \mathbb{R}^n} f(x, \theta)$$

$$\text{subject to } g_j(x, \theta) \leq 0 \text{ for } j = 1, \ldots, K$$

KKT Theorem: Necessity

Given a parameter θ, suppose the following conditions hold:

1. f, $g_1, \ldots, g_K$ are continuously differentiable in x
2. $\mathcal{X}(\theta)$ is non-empty (i.e. the constraint inequalities can be satisfied)
3. For a solution x^*, the vectors in $\{\nabla g_j(x^*, \theta) | g_j(x^*, 0) = 0\}$ are linearly independent

Then there exist $\lambda_1, \ldots, \lambda_K \geq 0$ such that:

$$\nabla f(x^*, \theta) = \sum_{j=1}^{K} \lambda_j \nabla g_j(x^*, \theta)$$

$$\lambda_j g_j(x^*, \theta) = 0$$

KKT Theorem: Sufficiency

Suppose the following holds:

1. $f(\cdot, \theta)$ is quasi-concave
2. $g_1(\cdot, \theta), \ldots, g_K(\cdot, \theta)$ are quasi-convex

Then any x^* satisfying the KKT conditions is a solution

Incorporating equality constraints

- Multiplier can be positive or negative
- No complementary slackness condition needed
- Gradient of equality constraints and active inequality constraints are linearly independent

Envelope Theorem

$$V(\theta) = \max_{x \in \mathcal{X}(\theta)} f(x, \theta)$$

- How does $V(\theta)$ change as θ change?
- This type of question comes up a lot in economics
- In particular, focus on case where $\mathcal{X}(\theta)$ is independent of θ

$$V(\theta) = \max_{x \in \mathcal{X}} f(x, \theta)$$

Envelope Theorem: Perfect Conditions

- To build intuition, let's assume everything is "perfect"
- By "perfect" I mean:
  - Θ is a convex subset of $\mathbb{R}$
  - $\mathcal{X}$ is a non-singleton, convex subset of $\mathbb{R}$
  - Everything is differentiable

Envelope Theorem: Perfect Conditions

$$V(\theta) = \max_{x \in \mathcal{X}} f(x, \theta)$$

Let $x^*(\theta)$ be the solution to the optimization problem. Therefore:

$$V(\theta) = f(x^*(\theta), \theta)$$

Let's take the derivative of V with respect to θ:

$$V'(\theta) = \frac{df(x^*(\theta), \theta)}{d\theta} = \frac{\partial f(x^*(\theta), \theta)}{\partial \theta} + \frac{\partial f(x^*(\theta), \theta)}{\partial x} \cdot \frac{dx^*(\theta)}{d\theta}$$

Envelope Theorem: Perfect Conditions

$$V'(\theta) = \frac{df(x^*(\theta),\theta)}{d\theta} = \frac{\partial f(x^*(\theta),\theta)}{\partial \theta} + \frac{\partial f(x^*(\theta),\theta)}{\partial x} \cdot \frac{dx^*(\theta)}{d\theta}$$

- Now, what do we know about $\frac{\partial f(x^*(\theta),\theta)}{\partial x}$?
- Has to be 0 by first-order condition (maximize a function means derivative is 0)

$$\implies V'(\theta) = \frac{\partial f(x^*(\theta),\theta)}{\partial \theta}$$

Envelope Theorem

When choice set is independent of parameter & conditions are "perfect" (e.g. differentiable):

$$\frac{\partial V(\theta)}{\partial \theta_j} = \frac{\partial f(x^*(\theta), \theta)}{\partial \theta_j}$$

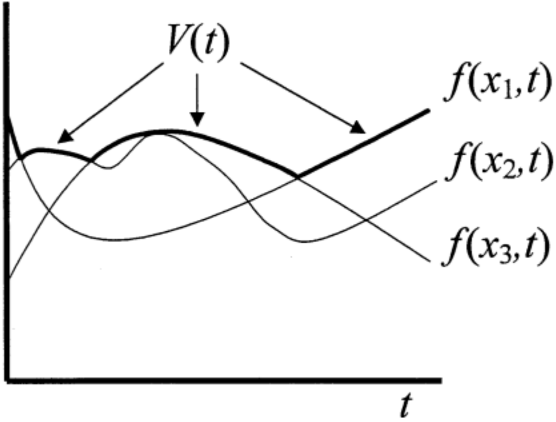

FIGURE 1

Envelope Theorem: Milgrom and Segal (2002)

- Θ is a convex subset of $\mathbb{R}$ (i.e. $\Theta = [\underline{\theta}, \overline{\theta}]$)
- $f(x, \theta)$ is absolutely continuous in θ
- $f(x, \theta)$ is differentiable in θ and $\frac{\partial f(x,\theta)}{\partial \theta} < m(\theta)$ for some integrable function m

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Theorem

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial f(x^*(t), t)}{\partial \theta} dt$$

Monotone Comparative Statics

- KKT $\longrightarrow$ technique to characterize optimal solution
- Envelope theorem $\longrightarrow$ how value function changes with parameter
- Monotone Comparative Statics $\longrightarrow$ how optimal choice changes with parameter

1. Single-variable/One-dimensional case (e.g. the choice and parameter are real numbers)
   - $\mathcal{X}$ is a closed subset of $\mathbb{R}$
2. Multi-variable case (e.g. the choice x is a *vector*; the parameter θ is a *vector*)

MCS: One-Dimensional

$$x^*(\theta) = \arg\max_{x \in \mathcal{X}} f(x, \theta)$$

- How does $x^*(\theta)$ change with θ?
- Why is this a tricky problem?
- x^* need not be differentiable or continuous
- May not even be single-valued!

MCS: One-Dimensional

- Again, for intuition assume perfect conditions:
  1. x^* is single-valued (i.e. unique solution for each θ)
  2. x^* is differentiable
- Sufficient condition for #1 and #2: f is strictly concave, twice differentiable

MCS: One-Dimensional

First-order condition means $\frac{\partial f(x^*(\theta),\theta)}{\partial x} = 0$

$$\implies \frac{\partial^2 f(x^*(\theta),\theta)}{\partial x \partial \theta} + \frac{\partial^2 f(x^*(\theta),\theta)}{\partial x^2} \cdot \frac{\partial x^*(\theta)}{\partial \theta} = 0$$

$$\implies \frac{\partial x^*(\theta)}{\partial \theta} = -\frac{f_{x\theta}(x^*,\theta)}{f_{xx}(x^*,\theta)}$$

$$\implies sign\left(\frac{\partial x^*(\theta)}{\partial \theta}\right) = sign(f_{x\theta}(x^*,\theta))$$

MCS: One-Dimensional

$$x^*(\theta) = \arg\max_{x \in \mathcal{X}} f(x, \theta)$$

- Relax assumptions from before
- Assume $x^*(\theta)$ exists but is not necessarily unique
- No assumptions on differentiability

MCS: One-Dimensional

- What does one mean by $x^*(\theta)$ increases/decreases if it is not single-valued?
- Since $x^*(\theta)$ is a correspondence (i.e. set-valued), need a slightly different notion
  - Strong-set order
  - Fancy way of saying a set "lies above" another set

Strong-Set Order

Definition

Given two sets A and B, A is less than B in the **strong set order** (denoted by $A \leq_S B$) if for any $a \in A$ and $b \in B$, $min\{a, b\} \in A$ and $\max\{a, b\} \in B$.

MCS: One-Dimensional

$$x^*(\theta) = \arg\max_{x \in \mathcal{X}} f(x, \theta)$$

- Relax assumptions from before
  - Assume $x^*(\theta)$ exists but is not necessarily unique
  - No assumptions on differentiability
- Want to know if $x^*(\theta)$ is increasing or decreasing in the strong-set order

MCS: One-Dimensional

Definition

A function $f(x,\theta)$ satisfies increasing differences if:

$$\text{For every } x' > x,\ f(x',\theta) - f(x,\theta) \text{ is non-decreasing in } \theta$$

- When f is twice-differentiable, this is just $\frac{\partial^2 f}{\partial x \partial \theta} \geq 0$

MCS: One-Dimensional

- Sometimes you will see Increasing Differences called Supermodularity
    - E.g. $f(x, \theta)$ is supermodular in (x, θ)
- This is not technically correct
- Supermodularity $\implies$ Increasing Differences, but the opposite isn't always true
- Equivalent when choice set is cartesian product of convex subsets of $\mathbb{R}$

MCS: One-Dimensional

Topkis Theorem

Suppose $f(x, \theta)$ satisfies increasing differences in (x, θ). Then $x^*(\theta)$ is non-decreasing in θ in the strong-set order.

- Think of increasing differences as saying that the two objects x and θ are complements

How to Prove?

- Consider $\theta' > \theta$ and $x' \in x^*(\theta')$ and $x \in x^*(\theta)$
- Want to show that $x^*(\cdot)$ is increasing in the strong-set order
- Need to show that $\min\{x, x'\} \in x^*(\theta)$ and $\max\{x, x'\} \in x^*(\theta')$

Example

Suppose a monopolist faces an inverse demand curve of $p(q)$. The cost to produce q units is given by the cost function $c(q, \theta)$. One can interpret θ as a parameter affecting the monopolists cost structure (e.g. a price of a key input). The monopolists problem is:

$$\max_q p(q)q - c(q, \theta)$$

When is $q^*(\theta)$ nondecreasing?

Topkis Theorem: General

- What if x is a vector (e.g., many choice variables)?

THEOREM

Let $x = (x_1, \ldots, x_n)$. If the choice set is a lattice, $f(x, \theta)$ satisfies increasing differences in (x, θ) and is supermodular in $(x_1, \ldots, x_n)$, then $x^*(\theta)$ is non-decreasing in θ.

- Lattice is technical condition
  - Any cartesian product of closed subsets of $\mathbb{R}$ is a lattice
- Increasing differences and supermodular are both used (that is because they are different)
  - Equivalent when choiec set is cartesian product of closed subsets of $\mathbb{R}$
  - In that case, need to check increasing difference between every pair in $(x_1, \ldots, x_n, \theta)$