Assignment 2

Due on Friday, October 8, 2021 at 11:59pm

1. Write a short essay (300-400 words) on the life and work of Bartel Leendert van der Waerden.

2. Prove the following statement: Suppose that nine people are gathered at a dinner party. Then there is a group of four people at the party who are all mutual acquaintances or there is a group of three people at the party who are all mutual strangers.

3. Recall the statement of Ramsey's theorem:

   > Given any r, n, and μ we can find an m_0 such that, if $m \geq m_0$ and the r- combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \ldots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.

   (a) By choosing appropriate values of r, n, and μ show that the Generalized Pigeonhole Principle:

   > Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ then at least one pigeonhole contains at least two pigeons.

   is a special case of Ramsey's theorem.

   (b) By choosing appropriate values of r, n, and μ show that the Pigeonhole Principle:

   > If n pigeons are sitting in k pigeonholes, where $n > k$, then there is at least one pigeonhole with at least $\lceil \frac{n}{k} \rceil$ pigeons.

   is a special case of Ramsey's theorem.

4. In this problem you will show that there is a 2-colouring of K_{17} with no monochromatic K_4.

   Label each vertex of K_{17} by one of the integers $0, 1, 2, \ldots, 16$.

   Let $[x, y]$ denote the edge between the vertices $x, y \in \{0, 1, 2, \ldots, 16\}$ with $x < y$.

   We define a 2-colouring of the edges of K_{17} in the following way:

$$C([x, y]) = \begin{cases} {\color{red}\bullet} & \text{if } y - x \in \{1, 2, 4, 8, 9, 13, 15, 16\} \\ {\color{blue}\blacksquare} & \text{if } y - x \in \{3, 5, 6, 7, 10, 11, 12, 14\}. \end{cases}$$

   (a) Prove that any K_4 that contains the edge $[0, 1]$ is not C-monochromatic.

   (b) Prove that for any $x, y \in \{0, 1, 2, \ldots, 16\}$, $x < y$,

$$C([x, y]) = C([0, y - x]).$$

   Explain why this fact implies that to check if K_4 with vertices x, y, z, w, $x < y < z < w$, is monochromatic, it is enough to check if K_4 with vertices $0, y - x, z - x, w - x$ is monochromatic.

   (c) Recall that you have established that if K_4 contains the edge $[0, 1]$ then it cannot be C-monochromatic.

   So, let us consider the complete graph K_4 with the vertices $0, x, y, z$, $2 \leq x < y < z$.

   i. Suppose that $C([0, x]) = C([0, y]) = C([0, z]) = {\color{red}\bullet}$. We will show that, under these conditions, K_4 is not C-monochromatic.

A. Show that $x \in \{2,4,8,9,13\}$, $y \in \{4,8,9,13,15\}$ and $z \in \{8,9,13,15,16\}$.

B. Show that if $x \in \{2,9,13\}$ then

$$C([x,y]) = {\color{blue}\blacksquare} \text{ or } C([x,z]) = {\color{blue}\blacksquare}$$

and conclude that K_4 is not monochromatic.

C. Show that if $x \in \{4,8\}$ and if

$$C([x,y]) = C([x,z]) = {\color{red}\bullet}$$

then

$$C(y,z) = {\color{blue}\blacksquare}$$

and conclude that K_4 is not monochromatic.

ii. Suppose that $C([0,x]) = C([0,y]) = C([0,z]) = {\color{blue}\blacksquare}$. We will show that, under these conditions, K_4 is not C-monochromatic.

A. Show that $x \in \{3,5,6,7,10,11\}$, $y \in \{5,6,7,10,11,12\}$, and $z \in \{5,6,7,10,11,12,14\}$.

B. Show that if $x \in \{10,11\}$ then

$$C([x,y]) = {\color{red}\bullet}.$$

and conclude that K_4 is not monochromatic.

C. Show that if $x \in \{3,5,6,7\}$ and if

$$C([x,y]) = C([x,z]) = {\color{blue}\blacksquare}$$

then

$$C([y,z]) = {\color{red}\bullet}$$

and conclude that K_4 is not monochromatic.

5. In the previous exercise we have established that there is a 2-colouring of K_{17} with no monochromatic K_4. Does this contradict the fact that $R(4,3) = R(3,4) = 9$? Why yes, or why not?

6. Prove that you need at least 18 people at a dinner party (including you) to make sure that there is a group of four people at the party who are all mutual acquaintances or there is a group of four people at the party who are all mutual strangers.

7. Recall that for $r \in \mathbb{N}$ and any set X we define $X^{(r)}$ to be the set of all subsets on X with exactly r elements:

$$X^{(r)} = \{A \subset X : |A| = r\}.$$

Also, recall that Ramsey's theorem guarantees that whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.

Let the colouring $C : \mathbb{N}^{(3)} \to \{{\color{red}\bullet}, {\color{blue}\blacksquare}\}$ be defined by

$$C(\{x<y<z\}) = \begin{cases} {\color{red}\bullet} & \text{if } x|z-y \\ {\color{blue}\blacksquare} & \text{otherwise.} \end{cases}$$

Find an infinite monochromatic set.

8. Consider an infinite word $\mathcal{W}$ on the alphabet $\{A, B, C\}$.

   This means that $\mathcal{W}$ is an infinite sequence of the letters from the set $\{A, B, C\}$.

   For example,

$$\mathcal{A} = ABBCCCCB \cdots BA \cdots AC \cdots CAA \cdots$$

   is an infinite word on the alphabet $\{A, B, C\}$.

   In this exercise we consider ANY infinite word $\mathcal{W}$ that uses letters A, B, and C.

   Prove that there is a 2-word XY on the alphabet $\{A, B, C\}$ that will appear in $\mathcal{W}$ at least 1000 times in a way that 999 words that separate consecutive appearances of XY are of the same length:

$$\mathcal{W} = \cdots XY \underbrace{\cdots}_{m} XY \underbrace{\cdots}_{m} XY \underbrace{\cdots}_{m} XY \ldots XY \underbrace{\cdots}_{m} XY \cdots$$