Problem 3
Colour each point in the xy plane having integer coefficients Red or Blue. Then some rectangle has all its vertices the same colour.

Solution Suppose we looked at a small part of the infinite plane. A 1 by 3 tall column. This might seem like a offside tangent, but how many ways are there to colour this? There are 2^3 ways, leaving us with 8 colourings. Then by the Pigeon Hole Principle, each coloring always has two cells of the same color. This means, if we have a repeated column we effectively have our rectangle with each corner being the same colour.

For any infinite grid of of colourings on the xy plane, we can pick out a 9 by 3 tall area that will always contain a rectangle by The Pigeon Hole Principle. This is because there are only 8 possible 1 by 3 unique columns colourings each of which have 2 identical colours, but since we have 9 spaces, there has to be at least two columns of the same colouring. Giving us a rectangle.

Problem 4
Your are attending a "Happy New Schoo Year" party.

At some point a person next to you says:"Do you know that at this party there must be at least two people with the exactly same number of friends among the party attendees?" You are puzzled: "How would you prove something like that?

Solution A friendship is a two way interaction, person a is friends with b and vise versa. Then we can start to rephrase the question. Suppose we treat every party attendant as a vertex in a graph, and the friendships can be the edges. The question then is, for any simple graph G, do there exist two vertices with the same degree?

Lets look at some facts, suppose G has n vertices. Then for any vertex v, deg(v) (the number of edges connected to it) can be [0, 1, ..., $n-1$]. $n-1$ since if a vertex had n or more, then it would need to be connected to n or more vertices, however v only has access to $n-1$.

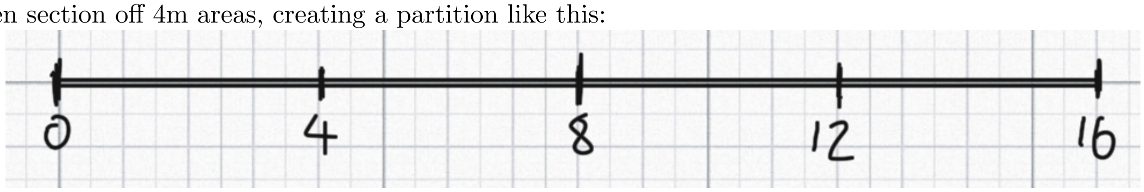
With this, we can start using the Pigeon Hole Principle, albeit a tad modified. Let G be a simple graph with n vertices, for any v in G the deg(v) = [0, 1, 2, ..., $n-1$] however with one big caveat. Let v_0 be a vertex in G, if v_0 has degree $n-1$, then there does not exist a v with degree 0, as v_0 is connected to every vertex. There are many other problems that bring the upper bound of the degrees lower, but with this we can already show that there will be two with the same degree. We now have $n-1$ degree options and n vertices, by the Pigeon Hole Principle there must be at least two vertices sharing the same degree.

$$deg(v) = \{1, 2, 3, ..., n-2, (n-1, 0)\}$$

Since you cant have both $n-1$ and 0.

Problem 5
Prove that it is impossible to seat 10 people around a circular table with the diameter of 5 meters and keep the COVID imposed rule of the minimal distance of 2 meters between any two people.

Solution Suppose we unravel the circular tables circumference to a line. Using simple circle formulas we find that the circumference is 15.7079...m, for simplicity lets say that line is no bigger than 16m. We can then section off 4m areas, creating a partition like this:



This gives us 4 spots that we can place people while keeping them 2m away from each other, however we have 10 people that need to be seated. By The Pigeon Hole Principle, we have 4 partitioned spots and 10 people, meaning there must be at least 3 people in one partitioned area.

It's impossible to impose COVID rules in that scenario. Although, you could probably just get another couple tables.

Problem 6
Let A be a set of an odd number of consecutive positive integers. Say

$$A = \{k, k+1, ..., k+2n\}$$

for some $k, n \, \epsilon \mathbb{N}$
Let B be any subset of the set A that contains at least $n+1$ elements. Prove that there must be $a, b \, \epsilon B$ such that $a + b = 2k + 2n$.

Solution We can first simplify this and remove the k element of A to make $A' = \{0, 1, 2, ..., 2n\}$, then in a informal way means $A = A' + k$. We can also change out our question to show for $B \subset A'$ where $|B| = n+1$, that there exists $a, b \epsilon B$ such that $a + b = 2n$

Now let's build up some pairs that would add up to $2n$. Let P be the set of all 2 pairs that add up to $2n$ in A'

$$P = \{(0, 2n), (1, 2n-1), (2, 2n-2), ..., (n, n)\}$$

We are left with $|P| = n+1$ however we have a special pair at the end (n, n). If $n \epsilon B$ then we already have a pairing that adds up to $2n$, a and $b = n$, $a + b = 2n$, all while fitting the condition that $a, b \epsilon B$. In terms of the Pigeon Hole principle we have n pigeon holes, plus one special pigeon hole, and $n+1$ pigeons.

At least one pigeon hole will have two pigeons, or one pigeon will sit in the special pigeon hole. Meaning we always have number that add up to $2n$. One big glaring question is what about the k value we took out? We can simply add it back in. Each element in A' gets and additional k amount. Meaning for all a and b in our equation $a + b = 2n$ it gets an additional $2k$. Thus proving the original statement. $a, b \epsilon B$ where $B \subset A$ then $a + b = 2n + 2k$

Problem 7
Consider the set
$$A = \{1, 11, 111, 1111, ...\}$$
the set that contains all natural numbers whose decimal expression uses only the digit, 1.
Prove that the set A contains and element that is divisible by 2021.

Solution For convenience let $a_k \epsilon A$ where $a_0 = 1, a_1 = 11, a_2 = 111, ...$ Then for another convenience let $|A| >= 2021$. We can then use the Pigeon Hole Principle to show that either some a_k is divisible by 2021, or there are numbers in A such that there difference is divisible 2021.

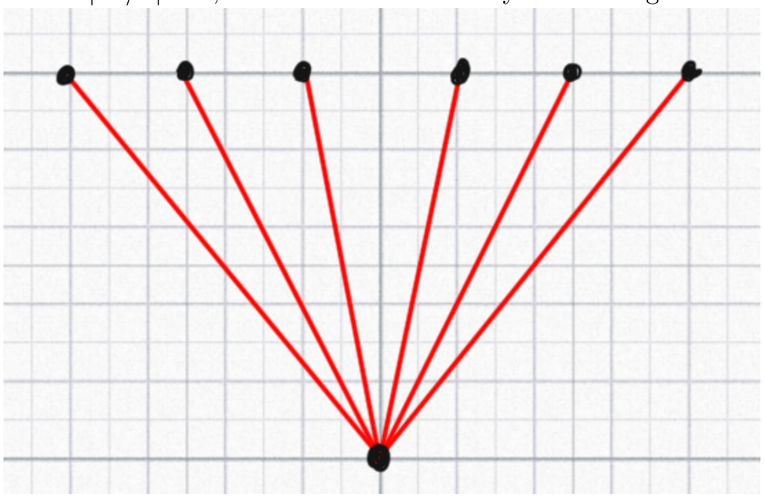
If a_k divides 2021, our work is done. Else, we can use congruence classes mod 2021 to make a set of our remainders $\{0, 1, 2, 3, ..., 2020\}$, if a number has no remainder, then its divisible, so lets remove 0 from our remainder possibilities. We are then left with $\{1, 2, 3, ..., 2020\}$ 2020 different remainder options and a list of the 2021 or more A values. By the Pigeon Hole Principle two numbers must have the same remainder, if we treat the remainders as the pigeon holes and pigeons as the A elements.

What does this mean for our divisibility though? Since A has a regular pattern we can probably simplify it. Suppose $k > i$ and a_k and a_i have the same remainder, meaning their difference would be divisible by 2021.
$$a_k - a_i = 111....11100...00$$
We have $k - i$ ones from the left, and i zeros from the right. This can then be simplified to
$$a_k - a_i = a_{k-i-1}10^i$$
Then from our fact above, $a_k - a_i$ is divisible by 2021, so then $a_{k-i-1}10^i$ is divisible by 2021. 10^i shares no common factors with 2021, as 2021 has factors 43 and 47. Therefore 2021 must share factors with a_{k-i}. Then there does exist some element in A which is divisible by 2021.

Fun fact, let $a_k \epsilon A$ who can be divided by 2021. Then $a_k/2021$ is a number with 962 digits. Something way to large to imagine.

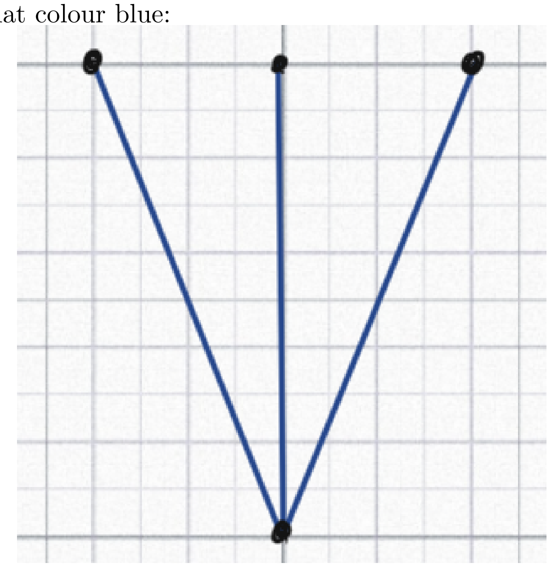
Problem 8
Show that $R(3,3,3) \leq 17$.

Solution Suppose for our k_{17} we start at one vertex then colour the edges connecting it to the others using three different colours, say red,green and blue. Using the Pigeon Hole Principle, we have 16 edges and 3 colours. $\lceil 16/3 \rceil = 6$, this means we will always have 6 edges of the same colour, lets make that color red:

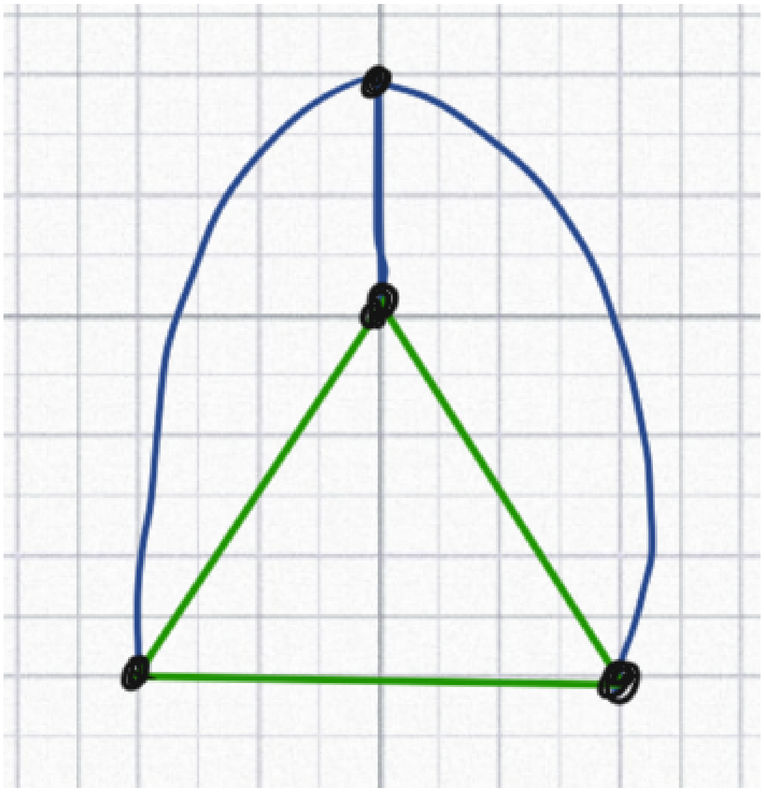


Lets group those top vertices together, into a graph called k_6, since those vertices will have connections between each and everyone of them. If either of those connections are red, then we've formed a red triangle. However what about the other colours, blue and green?

We can apply the Pigeon Hole Principle again, if we start at one vertex and start colouring all its connections, we find that with 5 edges and 2 colours we have $\lceil 5/2 \rceil = 3$ edges of the same colour. Lets make that colour blue:



If either of those top vertices have a blue edge then we have our blue triangle, but what if they are green instead?



Then we have green triangle! Meaning no matter what we do, we will always have an k_3 of either red blue or green in k_17