Properties of Asymptotic Notations - Step-by-Step Notes

These notes break down the lecture into core properties with examples, proofs via examples, and key takeaways for easy revision.

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1. Constant Multiple Property (General Property)

If f(n) = O(g(n)), **then** c \cdot f(n) = O(g(n)) for any **constant** c > 0.

- Applies to **Big O**, **Omega**, and **Theta** notations.
- **Example**: f(n) = 2n^2 + 5 = O(n^2).
  Then 7 \cdot f(n) = 14n^2 + 35 = O(n^2).
- **Why?** Multiplying by a constant doesn't change the **growth rate** (dominant term remains n^2).

> A function's Big O remains the same when multiplied by a constant.

Key Takeaway: 🔑 **Constants are ignored** in asymptotic analysis.

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2. Reflexive Property

Every function is an asymptotic bound of itself:

- f(n) = O(f(n))
- f(n) = Omega(f(n))
- f(n) = Theta(f(n)).
- **Example**: f(n) = n^2, so n^2 = O(n^2).
- **Intuition**: A function is always **greater than or equal** to itself (upper/lower bound).

Key Takeaway: 📌 **Functions bound themselves** - trivial but foundational.

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3. Transitive Property

If f(n) = O(g(n)) **and** g(n) = O(h(n)), **then** f(n) = O(h(n)).

- Applies to **Big O**, **Omega**, and **Theta**.
- **Example**:
  - f(n) = n = O(n^2) (g(n))
  - n^2 = O(n^3) (h(n))
  - Thus, n = O(n^3).
- **Intuition**: If g "sandwiches" f and h "sandwiches" g, then h sandwiches f.

Key Takeaway: 💡 **Asymptotic relations chain together**.

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4. Symmetric Property (Theta Only)

If f(n) = Theta(g(n)), **then** g(n) = Theta(f(n)).

- **True only for Theta** (tight bound).
- **Example**: f(n) = n^2, g(n) = n^2 → symmetric.
- **Not for Big O/Omega**: One-way relationships.

Key Takeaway: 🔑 **Theta is bidirectional** - perfect match both ways.

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5. Transpose Symmetry (Big O ↔ Omega)

If f(n) = O(g(n)), **then** g(n) = Omega(f(n)) (and vice versa).

- **Example**: f(n) = n, g(n) = n^2.
  - n = O(n^2) (upper bound)
  - n^2 = Omega(n) (lower bound).
- **Intuition**: If f ≤ g (Big O), then g ≥ f (Omega).

Key Takeaway: 📌 **Upper bound flips to lower bound** when roles reverse.

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6. Theta from Big O + Omega

If f(n) = O(g(n)) **and** f(n) = Omega(g(n)), **then** f(n) = Theta(g(n)).

- **Same g(n) acts as both upper and lower bound** → tight bound.
- **Intuition**: g "sandwiches" f perfectly.

Key Takeaway: 💡 **Theta = Big O ∧ Omega** (with same function).

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7. Sum Property

If f(n) = O(g(n)) **and** h(n) = O(i(n)), **then** f(n) + h(n) = O(max(g(n), i(n))).

- **Result dominated by the faster-growing function**.
- **Example**: f(n) = n = O(n), h(n) = n^2 = O(n^2) → n + n^2 = O(n^2).
- **Reverse case**: n^2 + n = O(n^2) (still max term).

Key Takeaway: 🔑 **Sum takes the maximum growth rate**.

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8. Product Property

If f(n) = O(g(n)) **and** h(n) = O(i(n)), **then** f(n) \cdot h(n) = O(g(n) \cdot i(n)).

- **Growth rates multiply**.
- **Example**: f(n) = n = O(n), h(n) = n^2 = O(n^2) → n \cdot n^2 = n^3 = O(n \cdot n^2).

Key Takeaway: 📌 **Product multiplies the bounds**.

Quick Revision Summary

Property	Applies To	Key Idea	Example
Constant Multiple	O, Ω, Θ	c \cdot f(n) same bound	7(2n^2+5) = O(n^2)
Reflexive	All	f(n) = O(f(n))	n^2 = O(n^2)
Transitive	All	Chains: f→g→h	n = O(n^3)
Symmetric	Θ only	Bidirectional	n^2 = Theta(n^2)
Transpose	O↔Ω	Flip roles	n^2 = Omega(n)
Theta Combo	Θ	O + Ω → Θ	Sandwich
Sum	O	Max term wins	n + n^2 = O(n^2)
Product	O	Multiply bounds	n \cdot n^2 = O(n^3)