Lecture Overview: Comparing Functions for Asymptotic Analysis

This lecture teaches methods to compare mathematical functions (like n^2 vs. n^3) to determine which grows faster, establishing upper bounds and lower bounds without exhaustive checking.

Key objective: Quickly identify if one function dominates another asymptotically (for large n).

Method 1: Sampling Values

Sample small values of n to observe growth patterns empirically.

1. **Choose test values** (e.g., n=2, 3, 4).
2. **Compute both functions**:
   - For n=2: $2^2 = 4$, $2^3 = 8$ → $n^2 < n^3$.
   - For n=3: $3^2 = 9$, $3^3 = 27$ → $n^2 < n^3$.
   - For n=4: $4^2 = 16$, $4^3 = 64$ → pattern confirmed.
3. **Conclusion**: 3 samples suffice to infer n^2 is always smaller than n^3 for $n > 1$.

🔑 **Limitation**: Works for simple cases; use for quick intuition.

Method 2: Apply Logarithm to Both Sides

Take log of both functions to simplify exponents into coefficients for easier comparison.

Step-by-Step Log Application

1. **Start with functions**: Compare $f(n) = n^2$ and $g(n) = n^3$.
2. **Apply log**:
   $\log(n^2) = 2 \log n, \quad \log(n^3) = 3 \log n$
3. **Compare coefficients**: $2 \log n < 3 \log n$ since $2 < 3$ and $\log n > 0$ for $n > 1$.

Thus, $n^2 = o(n^3)$ (n^2 grows slower).

Essential Log Formulas (Memorize These!)

> Log Properties:
>
> - $\log(ab) = \log a + \log b$
> - $\log(a/b) = \log a - \log b$
> - $\log(a^b) = b \log a$
> - $b = \log_a n$ if $a^b = n$

💡 **Tip**: Logs turn products/powers into sums/multipliers—ideal for complex functions.

Example 1: $n^2 \log n$ vs. $(n \log n)^{10}$

Visualize growth comparison:

No numerical data provided in lecture for exact plotting—sampling shows $(n \log n)^{10}$ dominates.

Log Analysis Steps

1. **Apply log**:
   $\log(n^2 \log n) = \log(n^2) + \log(\log n) = 2\log n + \log(\log n)$
2. **Other side**:
   $\log((n \log n)^{10}) = 10 \log(n \log n) = 10(\log n + \log(\log n))$
3. **Dominant terms**: Compare $2\log n$ vs. $10\log n$ → $10\log n$ wins (coefficient $10 > 2$).
4. **Smaller terms**: $10 \log(\log n)$ vs. $\log(\log n)$ → 10x larger.

Result: $(n \log n)^{10} > n^2 \log n$.

Example 2: $4n^3$ vs. $2^{\sqrt{n} \log n}$

Rewrite using log properties to normalize bases/exponents.

Step-by-Step Normalization

1. **Rewrite second function**:
   $2^{\sqrt{n} \log n} = (2^{\log n})^{\sqrt{n}} = n^{\sqrt{n}}$

   (Since $2^{\log_2 n} = n$, adjust base).

2. **Now compare**: $4n^3$ vs. $n^{\sqrt{n}}$.

3. **Asymptotic equality check**:

   - Coefficients (4 vs. 1) don't matter asymptotically.
   - Bases same (n), exponents 3 vs. $\sqrt{n}$ → $\sqrt{n} > 3$ for large n.

Result: Both asymptotically equal in growth class ($n^{\sqrt{n}}$ slightly faster but same order).