Orbits, Fixed Points, and Stabilizers

Let $G \subseteq \text{Perm}[U]$ Then

$$\forall u \in U, G\ u := \{\sigma(u) | \sigma \in G\}$$
$$\forall \sigma \in G, U^{\sigma} := \{u \in U | \sigma(u) = u\}$$
$$\forall u \in U, G_u := \{\sigma \in G | \sigma(u) = u\}$$

Orbit-Stabilizer Theorem

$\forall G \subset \text{Perm}[U], \ni G$ is a permutation group

$$|G| = |Gu| \cdot |G_u|$$

Proof

$$\forall u \in U$$
$$\forall v \in Gu, \sigma_v \in G \ni \sigma_v(u) = v$$
$$\phi : Gu \times G_u \longrightarrow G$$
$$\phi(v, \sigma_0) := \sigma_v \cdot \sigma_0$$

Claim: -

$$\forall x, x' \in Gu, \tau, \tau' \in G_u, \phi(x, \tau) = \phi(x', \tau') \iff x = x', \tau = \tau'$$

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$$\forall \sigma \in G, \exists x \in Gu, \tau \in G_u \ni \phi(x, \tau) = \sigma$$

Proof:
Let,

$$x, x' \in Gu,$$
$$\tau, \tau' \in G_u$$
$$\text{s.t. } \phi(x, \tau) = \phi(x', \tau')$$

Then, We know that,

$$\phi(x, \tau) = \sigma_x \cdot \tau$$
$$\phi(x', \tau') = \sigma_{x'} \cdot \tau'$$
$$\because \phi(x, \tau) = \phi(x', \tau')$$
$$\implies \phi(x, \tau)(u) = \phi(x', \tau')(u)$$
$$\implies \sigma_x \cdot \tau(u) = \sigma_{x'} \cdot \tau'(u)$$
$$\because \tau, \tau' \in G_u$$
$$\implies \tau(u) = \tau'(u) = u$$
$$\implies \sigma_x(u) = \sigma_{x'}(u)$$
$$\iff \boxed{x = x'} \tag{1}$$
$$\because \sigma_x = \sigma_{x'}$$
$$\implies \cancel{\sigma_x} \cdot \tau = \cancel{\sigma_x} \cdot \tau'$$
$$\iff \boxed{\tau = \tau'} \tag{2}$$
$$\boxed{(1), (2) \implies \phi \text{ is an injective function}}$$

Let,

$$\sigma \in G,$$
$$u, v \in U \ni \sigma(u) = v$$

Then, we know that,

$$\exists \tilde{\sigma} \in G \ni \sigma \cdot \tilde{\sigma} = \tilde{\sigma} \cdot \sigma = \lambda x.x$$

$$\lambda x.x \in G_u$$

$$\because \sigma \cdot (\lambda x.x) = \sigma \in G$$

$$\implies \boxed{\forall \sigma \in G \exists u \in U, \tau \in G_u, \ni \phi(u,\tau) = \sigma} \tag{3}$$

$$\boxed{(1),(2),(3) \implies \phi \text{ is a injective and surjective(bijective) function}}$$