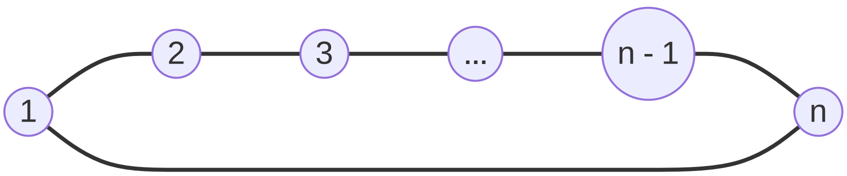
The Polya Enumeration Theorem

Problem Statement

$$\forall U: \text{ is a Set } \ni |U| < \infty$$
$$\forall G \subseteq \mathrm{Perm}[U]$$
$$\forall k \in \mathbb{P}$$
$$\forall c, c' : U \longrightarrow [k] \ni \exists \sigma \in G c'(\sigma(u \in U)) = c \iff c \sim_G c'$$
$$C_k(U) := \{f : U \longrightarrow [k]\}$$
$$G \backslash C_k(U) := \{\{\tau \in C_k(U) | \tau \sim_G \sigma\} | \sigma \in C_k(U)\}$$

Example

$$G = \langle \sigma \rangle = \{\sigma^i\}_{0 \leq i < \mathrm{ord}(\sigma)}$$



Given:

$$c \in C_k(U)$$

Let

$$\alpha_i := \#\{u \in U | c(u) = i\}$$
$$\alpha_c := (\alpha_i)_{1 \leq i \leq k} \in \mathbb{N}^k$$
$$\sum_{i=1}^{k} \alpha_i = |U|$$

α_c is called the shape of the coloring c.

$$c \sim_G c' \iff \alpha_c = \alpha_{c\prime}$$
$$\forall k < \infty \in \mathbb{N}, \forall \alpha_k \in \mathbb{N}^k$$
$$C_\alpha(U) := \{c \in C_k(U) | \alpha_c = \alpha\}$$

Problem Determine the number of G-equivalent classes of colourings of U with shape α.\

Solution

1. Power sum symmetric Polynomials Let $f : S^k \longrightarrow U \ni \forall \omega \in \mathrm{Perm}[S], \omega(S^k) \longmapsto u \in U$ For example $p_m(x_i)_{1 \leq i \leq k} = \sum_{i=1}^{k} x_i^m$ can be a possible value of f.