$$\forall (R, \oplus, \odot, 0, 1) \ni \forall a \in R \exists \tilde{a} \ni a \oplus \tilde{a} = 0,$$
$$\forall (P, \leq),$$

Definitions:

$$\forall x \in P, I\ x = \{y \in P | y \leq x\}$$
$$M_P := \{f \colon P \to R\}$$
$$A_P := \{f \colon P \to P \to R | \forall x, y \in P, f\ x\ y \neq 0 \iff x \leq y\}$$

Assuming, $|I\ x| < \infty$

$$\forall f, g \in A_P, f * g = \lambda x. \lambda y. \sum_{z=x}^{y} (f\ x\ z) \odot (g\ z\ y)$$
$$\forall f \in A_P, \forall g \in M_P, g * f = \lambda x. \sum_{y \in P}^{x} (f\ y\ x) \odot (g\ y)$$

Lemma:

$$\forall f, g \in A_P, \forall \xi \in M_P, \xi * (f * g) = (\xi * f) * g$$

Proof:

$$\xi * (f * g) = \lambda x. \sum_{y \in P}^{x} (f * g\ y\ x) \odot (\xi\ y)$$
$$= \lambda x. \sum_{y \in P}^{x} \left(\sum_{z=y}^{x} (f\ y\ z) \odot (g\ z\ x) \right) \odot (\xi\ y)$$

By the distributive property of multiplication on R

$$= \lambda x. \sum_{y \in P}^{x} \sum_{z=y}^{x} (f\ y\ z) \odot (g\ z\ x) \odot (\xi\ y) \tag{i}$$

Similarly,

$$(\xi * f) * g = \lambda x. \sum_{y \in P}^{x} (g\ y\ x) \odot (\xi * f\ y)$$
$$= \lambda x. \sum_{y \in P}^{x} (g\ y\ x) \odot \left(\sum_{z \in P}^{y} (f\ z\ y) \odot (\xi\ z) \right)$$

By the distributive property of multiplication on R

$$= \lambda x. \sum_{y \in P}^{x} \sum_{z \in P}^{y} (g\ y\ x) \odot (f\ z\ y) \odot (\xi\ z) \tag{ii}$$

To show that (i) = (ii) Let:

$$\forall x \in P,$$
$$\urcorner_x := \{\langle \alpha, \beta \rangle \in I\ x \times I\ x | \alpha \leq \beta\}$$
$$\sqsupset_x := \{\langle \alpha, \beta \rangle \in I\ x \times I\ x | \beta \leq \alpha\}$$
$$\implies \sqsupset_x = \{\langle \beta, \alpha \rangle \mid \langle \alpha, \beta \rangle \in \urcorner_x\}$$

Then, (i) can be re-written as:

$$\lambda x. \sum_{\langle \alpha, \beta \rangle \in \urcorner_x} (f\ \alpha\ x) \odot (g\ \beta\ x) \odot (\xi\ \alpha)$$

Similarly (ii) can be re-written as:

$$\lambda x. \sum_{\langle \alpha, \beta \rangle \in \sqsupset_x} (f\ \beta\ x) \odot (g\ \alpha\ x) \odot (\xi\ \beta)$$

However, $\because, \beth_x = \{\langle \beta, \alpha \rangle \mid \langle \alpha, \beta \rangle \in \daleth_x\}$

$$\implies \sum_{\langle \alpha, \beta \rangle \in \beth_x} (f\ \beta\ x) \odot (g\ \alpha\ x) \odot (\xi\ \beta) = \sum_{\langle \beta, \alpha \rangle \in \daleth_x} (f\ \beta\ x) \odot (g\ \alpha\ x) \odot (\xi\ \beta)$$

$$\boxed{\sum_{\langle \beta, \alpha \rangle \in \daleth_x} (f\ \beta\ x) \odot (g\ \alpha\ x) \odot (\xi\ \beta) = \sum_{\langle \alpha, \beta \rangle \in \daleth_x} (f\ \alpha\ x) \odot (g\ \beta\ x) \odot (\xi\ \alpha)}$$

$\blacksquare$