Formal Power series

$$\forall(F,\oplus,\odot,0,1)\ni\forall x\in F\exists\tilde{x}\ni x\oplus\tilde{x}=0\wedge\forall y\in F/\{0\}\exists\hat{y}\ni y\odot\hat{y}=1$$

Let

$$F[\![x]\!]:=\left\{\sum_{i=0}^{\infty}a_i\odot x^i\middle|\forall i,a_i\in F\right\}$$

Definitions

$$\forall f,g\in F[\![x]\!]$$
$$f\oplus g:=\sum_{i=0}^{\infty}(a_{g\cdot i}\oplus a_{f\cdot i})\odot x^i$$
$$f-g:=f\oplus\tilde{g}$$
$$f\odot g:=\sum_{i=0}^{\infty}\sum_{j+k=i}(a_{g\cdot j}\odot a_{f\cdot k})\odot x^i$$
$$\text{val}:=\lambda f.\min\{n|a_n\neq 0\}$$
$$\forall f\in F[\![x]\!],|f|:=\begin{cases}2^{-\text{val }f} & \text{iff } f\neq 0\\ 0 & \text{o/w}\end{cases}$$
$$d:=\lambda f.\lambda g.|f-g|$$

Theorem

$\forall(F,\oplus,\odot,0,1)\ni\forall x\in F\exists\tilde{x}\ni,x\oplus\tilde{x}=0\wedge\forall y\in F\exists\hat{y}\ni y\odot\hat{y}=1,F[\![x]\!]$ is 1. a commutative unital RING and, 2. a Metric Space

Lemmas

- $\forall f\in F[\![x]\!],f\oplus\sum_{i=0}^{\infty}0\odot x^i=f$
- $\forall f\in F[\![x]\!],f\odot\sum_{i=0}^{\infty}a_i\odot x^i(\ni a_0=1\wedge a_{i>0}=0)=f$
- $\forall f,g\in F[\![x]\!],f\oplus g=g\oplus f$
- $\forall f,g\in F[\![x]\!]f\odot g=g\odot f$
- $\forall f,\in F[\![x]\!]\exists\tilde{f}\in F[\![x]\!]\ni f\oplus\tilde{f}=\sum_{i=0}^{\infty}0\odot x^i$
- $\forall f,g\in F[\![x]\!],d\ f\ g=d\ g\ f$
- $\forall f,g\in F[\![x]\!]\ni f=g\iff d\ f\ g=0$
- $\forall f,g\in F[\![x]\!]d\ f\ g\in\{x\in\mathbb{Q}^+|0\leq x\leq 1\}$
- $\forall f,g,h\in F[\![x]\!]\ni f\neq g\neq h,d\ f\ g\leq\max(d\ f\ h)(d\ g\ h)\leq d\ f\ h+d\ g\ h$

Proofs

1. $$\forall f\in F[\![x]\!],f\oplus\sum_{i=0}^{\infty}0\odot x^i=f$$

   By definition of $f\oplus g$

   $$f\oplus\sum_{i=0}^{\infty}0\odot x^i=\sum_{i=0}^{\infty}(a_{f\cdot i}+0)\odot x^i$$

   Since, 0 is the addition identity of F, We can simplify the expression to

   $$\boxed{\sum_{i=0}^{\infty}a_{f\cdot i}\odot x^i}$$

Which is just the definition of f

$\blacksquare$

2.

$$\forall f \in F[\![x]\!], f \odot \sum_{i=0}^{\infty} a_i \odot x^i (\ni a_0 = 1 \wedge a_{i>0} = 0) = f$$

By the definition of $f \odot g$,

$$f \odot \sum_{i=0}^{\infty} a_i \odot x^i (\ni a_0 = 1 \wedge a_{i>0} = 0) = \sum_{i=0}^{\infty} \sum_{j+k=i} (a_{f \cdot j} \odot a_k) \odot x^i (\ni a_0 = 1, a_{k>1} = 0)$$

$\because$ 0 is the product irreducible element of F, $\implies \forall k > 0, a_{f \cdot j} \odot a_k = 0 \because$ 0 is the additive identity element of F, $\implies \sum_{j+k=i} (a_{f \cdot j} \odot a_k) = a_{f \cdot i}$

$$\implies \boxed{\sum_{i=0}^{\infty} \sum_{j+k=i} (a_{f \cdot j} \odot a_k) \odot x^i (\ni a_0 = 1, a_{k>1} = 0) = \sum_{i=0}^{\infty} a_{f \cdot i} \odot x^i}$$

Which is just the definition of f

$\blacksquare$

3.

$$\forall f, g \in F[\![x]\!], f \oplus g = g \oplus f$$

Simplifying using the definition of $f \oplus g$ LHS:

$$f \oplus g = \sum_{i=0}^{\infty} (a_{f \cdot i} \oplus a_{g \cdot i}) \odot x^i$$

RHS:

$$g \oplus f = \sum_{i=0}^{\infty} (a_{g \cdot i} \oplus a_{f \cdot i}) \odot x^i$$

We know that $\forall a, b \in F a \oplus b = b \oplus a$

$$\implies \forall i, a_{g \cdot i} \oplus a_{f \cdot i} = a_{f \cdot i} \oplus a_{g \cdot i}$$

Hence we can just change the terms in the summation to be equal. Since two summations are equal, if all of their co-efficients are equal, this proves the lemma.

$\blacksquare$

4.

$$\forall f, g \in F[\![x]\!], f \odot g = g \odot f$$

Simplifying by defintion LHS:

$$f \odot g = \sum_{i=0}^{\infty} \sum_{j+k=i} (a_{f \cdot j} \odot a_{g \cdot k}) x^i$$

RHS:

$$g \odot f = \sum_{i=0}^{\infty} \sum_{j+k=i} (a_{g \cdot j} \odot a_{f \cdot k}) x^i$$

$\because j, k \in \mathbb{N} \implies \{j | j + k = i\} = \{k | j + k = i\}, \wedge \forall x, y, z \in F, x \oplus (y \oplus z) = (x \oplus y) \oplus z$

$$\implies LHS = RHS$$

$\blacksquare$

5.

$$\forall f \in F[\![x]\!] \exists \tilde{f} \in F[\![x]\!] \ni f \oplus \tilde{f} = \sum_{i=0}^{\infty} 0 \odot x^i$$

Simplifying by deffinition

$$f \oplus \tilde{f} = \sum_{i=0}^{\infty} (a_{f \cdot i} \oplus a_{\tilde{f} \cdot i}) \odot x^i$$

To prove that,

$$\forall i, a_{f \cdot i} \exists a_{\tilde{f} \cdot i} \ni a_{f \cdot i} \oplus a_{\tilde{f} \cdot i} = 0$$

Remember,

$$\forall i, a_{f \cdot i}, a_{\tilde{f} \cdot i} \in F \implies a_{\tilde{f} \cdot i} = \widetilde{a_{f \cdot i}}$$

$$\blacksquare$$

6.

$$\forall f, g \in F[\![x]\!], d\ f\ g = d\ g\ f$$

Case 1 ($f = g$):

$$d\ f\ g = d\ g\ f = 0$$

so the lemma holds Case 2 ($f \neq g$): LHS:

$$d\ f\ g = |f - g| = 2^{-\text{val}(f-g)} = 2^{-\min\{n | a_{f \cdot n} - a_{g \cdot n} \neq 0\}}$$

RHS:

$$d\ g\ f = |g - f| = 2^{-\text{val}(g-f)} = 2^{-\min\{n | a_{g \cdot n} - a_{f \cdot n} \neq 0\}}$$

$\because \forall a, b \in \mathbb{Z}, a - b \neq 0 \iff b - a \neq 0$ This proves the lemma.

$$\blacksquare$$

7.

$$d\ f\ g = 0 \iff f = g$$

$$d\ f\ g = |f - g| = |f \oplus \tilde{g}| = 0 \iff f \oplus \tilde{g} = 0$$

$$\implies \tilde{g} = \tilde{f}$$

$$\implies g = f$$

$$\blacksquare$$

8.

$$\forall f, g \in F[\![x]\!], d\ f\ g \in \{x \in \mathbb{Q}^+ | 0 \leq x \leq 1\}$$

This is trivial, since it can just be proven by simplifying the definition of d.

9.

$$\forall f, g, h, \in F[\![x]\!] d\ f\ g \leq \max(d\ f\ h)(d\ g\ h) \leq d\ f\ h + d\ g\ h$$

Proving first inequality
LHS:

$$d\ f\ g = |f - g| = 2^{-\text{val}(f-g)} = 2^{-\min\{n | a_{f \cdot n} - a_{g \cdot n} \neq 0\}}$$

RHS:

$$\max(d\ f\ h)(d\ g\ h) = \max(2^{-(\text{val}\ f-h)})(2^{-(\text{val}\ g-h)}) = \max 2^{-\min\{n | a_{f \cdot n} - a_{h \cdot n} \neq 0\}}\ 2^{-\min\{n | a_{g \cdot n} - a_{h \cdot n} \neq 0\}}$$

$$2^{\max\ (-\min\{n | a_{f \cdot n} - a_{h \cdot n} \neq 0\})\ (-\min\{n | a_{g \cdot n} - a_{h \cdot n} \neq 0\})}$$

Taking $\log_2$ on both sides, we can rewrite the inequality to be

$$-\min\{n | a_{f \cdot n} - a_{g \cdot n} \neq 0\} \leq \max\{-\min\{n | a_{f \cdot n} - a_{h \cdot n} \neq 0\}, -\min\{n | a_{g \cdot n} - a_{h \cdot n} \neq 0\}\}$$