Möbius Function on the set of Ideals

$$\forall A, B \in J(P) \ni A \subset B$$

We know that,

$$\mu_{J(P)}\ A\ B = \mu_{[A,B]}\ \hat{0}\ \hat{1}$$

Lemma:

$$[A, B] \cong J(B/A)$$

Proof:

$$f_{[A,B]} = X \longmapsto X/A$$

$$f^{-1}_{[A,B]} = X \longmapsto X \cup A$$

$$\implies \boxed{\mu_{[A,B]}\ \hat{0}\ \hat{1} = \mu_{J(B/A)}\ \hat{0}\ \hat{1}}$$

Then by Weisner's Theorem:

$$\mu_{J(P)}\ \hat{0}\ \hat{1} = -\sum_{\substack{I \neq \hat{0} \\ I \cap A = \emptyset}} \mu_{J(P)}(I, \hat{1})$$

Theorem:

$$\mu_{J(P)}\ X\ Y = \begin{cases} -1^{|Y/X|} & \text{iff } \forall Z \in Y/X, |Z| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case 1 : $\exists x \in X, y \in max(X) \ni x < y$ Take $A = P/\{y\}$, then

$$X \cap A = \emptyset \implies X \subseteq y$$

case I: $X = \emptyset = \hat{0}$ case II: $X = \{y\}$ which it can't be because then X wouldn't be an Ideal.

$$\implies X = \emptyset$$

$\boxed{\implies \mu_{P(E)}\ \hat{0}\ \hat{1} = 0}$ if X isn't an antichain

Case 2 : $\nexists x \in X, y \in max(X) \ni x < y$

$$\implies J(P) = 2^P$$

$$\implies \mu_{J(P)}\ \hat{0}\ \hat{1} = -1^{|P|}$$

And in case of X and Y

$$\implies \boxed{\mu_{J(P)}\ X\ Y = -1^{|Y/X|}}$$

$$\blacksquare$$