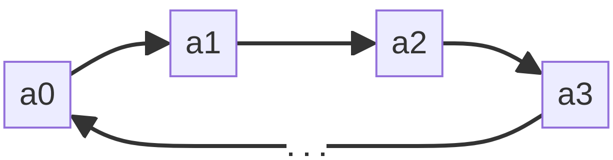
Permutations and their cycles

$$\mathbb{U} := \{X : X \text{ is a set}, |X| < \infty\}$$
$$\forall U \in \mathbb{U}$$
$$\mathrm{Perm}[U] := \{\sigma : U \to U | \forall x, z \in U, x \neq z \iff \sigma(x) \neq \sigma(z)\}$$

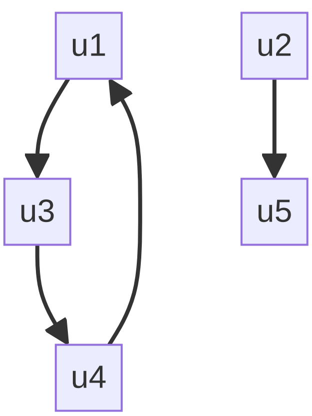
Permutations can be represented by a directed graph $\Gamma_\sigma := (U, \sigma)$.

A consequence of this is that every connected component of Γ_σ is a k-cycle $k \in \mathbb{P}$



Cycle structure of σ

The lengths of the cycles of σ written in weakly decreasing order. Integer partition of $|U|$ $(3, 2)$ integer partition of 5 cycle decomposition.



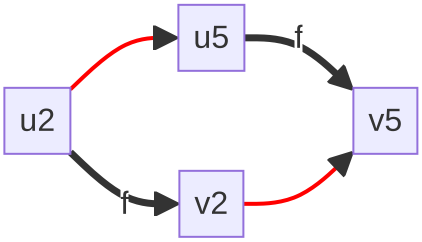
Transfer of structure in Permutations.

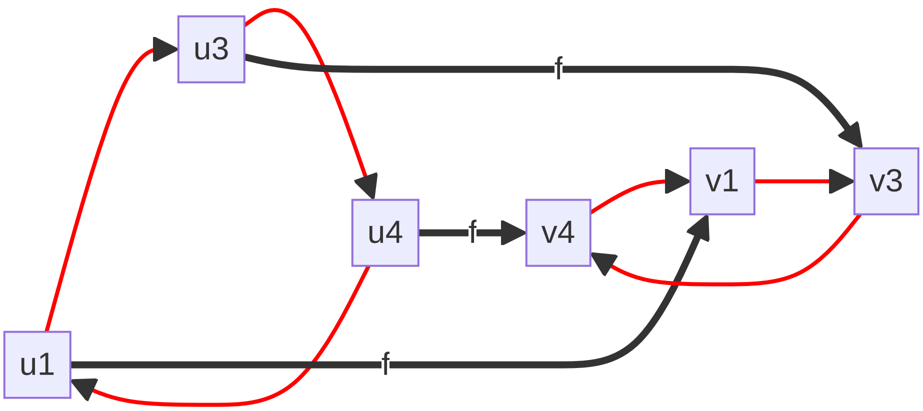
$$\forall f : U \longrightarrow V \ni \forall x, y \in D_f, x = y \iff f(x) = f(y)$$
$$\sigma \in \mathrm{Perm}[U]$$

Define

$$f[\sigma](f(u)) := f(v) \iff \sigma(u) = v$$

This is called transfer of Permutations from U to V.





$\sigma \longmapsto f[\sigma]$ is a bijection.

Unsigned Stirling number of the first kind

$$\forall n, k \in \mathbb{N}$$

$$s(n,k) := \#(\{\sigma \in S_n | \sigma \text{ has } k \text{ cycles}\})$$

Lemma

$$s(n,k) = \begin{cases} s(n-1,k-1) + (n-1) \cdot s(n-1,k) & \text{iff } k \in [1, n-1] \\ 1 & \text{iff } k = n \\ 0 & \text{otherwise} \end{cases}$$

Proof

- $s(n,n) = 1$, the proof is trivial, If there are n vertices and every cycle has n vertices of-course you only have one cycle.
- $s(n,k) \ni k \leq 0 \vee k > n = 0$, the proof is again trivial, since no cycle has 0 vertices, and n vertices can't make cycles of length greater than n, we just get the set as $\emptyset$ which has a definitional cardinality of 0.
- $s(n,k) \ni k \in [1, n-1] = s(n-1,k-1) + (n-1) \cdot s(n-1,k)$

Going from n nodes to $n+1$ nodes, there are two ways to add a node such that the number of edges remains number of vertices plus one. We must also add an edge.

However to do so, we can 1. Add the node in a cycle, so the cycle goes from having y nodes to $y+1$ nodes. 2. Add the node in a cycle with itself, so that the number of cycles goes from k to $k+1$ 3. Point the edge from the new vertex to a vertex in cycle or vice versa (not adding it in a cycle)

However, the 3rd way of doing this, leads to the new value no longer being a part of the set S_{n+1} because, by definition $S_n[X] := \{\sigma : X \longrightarrow X | \forall x, yx = y \iff \sigma(x) = \sigma(y)\}$ and having 2 incoming edges or two outgoing edges contradicts that.

Lemma

$$\forall n, \sum_{k=0}^{n} s(n,k) = n!$$

$$\forall n, \sum_{k=0}^{n} s_2(n,k) = B_n$$