

Orbits, Fixed Points, and Stabilizers

Let $G \subseteq \text{Perm}[U]$ Then

$$\begin{aligned}\forall u \in U, G \cdot u &:= \{\sigma(u) \mid \sigma \in G\} \\ \forall \sigma \in G, U^\sigma &:= \{u \in U \mid \sigma(u) = u\} \\ \forall u \in U, G_u &:= \{\sigma \in G \mid \sigma(u) = u\}\end{aligned}$$

Orbit-Stabilizer Theorem

$\forall G \subset \text{Perm}[U], \ni G$ is a permutation group

$$|G| = |Gu| \cdot |G_u|$$

Proof

$$\begin{aligned}\forall u \in U \\ \forall v \in Gu, \sigma_v \in G \ni \sigma_v(u) = v \\ \phi : Gu \times G_u \longrightarrow G \\ \phi(v, \sigma_0) := \sigma_v \cdot \sigma_0\end{aligned}$$

Claim: -

$$\forall x, x' \in Gu, \tau, \tau' \in G_u, \phi(x, \tau) = \phi(x', \tau') \iff x = x', \tau = \tau'$$

-

$$\forall \sigma \in G, \exists x \in Gu, \tau \in G_u \ni \phi(x, \tau) = \sigma$$

Proof:

Let,

$$\begin{aligned}x, x' \in Gu, \\ \tau, \tau' \in G_u \\ \text{s.t. } \phi(x, \tau) = \phi(x', \tau')\end{aligned}$$

Then, We know that,

$$\begin{aligned}\phi(x, \tau) &= \sigma_x \cdot \tau \\ \phi(x', \tau') &= \sigma_{x'} \cdot \tau' \\ \therefore \phi(x, \tau) &= \phi(x', \tau') \\ \implies \phi(x, \tau)(u) &= \phi(x', \tau')(u) \\ \implies \sigma_x \cdot \tau(u) &= \sigma_{x'} \cdot \tau'(u) \\ \therefore \tau, \tau' &\in G_u \\ \implies \tau(u) &= \tau'(u) = u \\ \implies \sigma_x(u) &= \sigma_{x'}(u) \\ \iff \boxed{x = x'} &\end{aligned} \tag{1}$$

$$\begin{aligned}\therefore \sigma_x &= \sigma_{x'} \\ \implies \cancel{\sigma_x} \cdot \tau &= \cancel{\sigma_x} \cdot \tau' \\ \iff \boxed{\tau = \tau'} &\end{aligned} \tag{2}$$

$$\boxed{(1), (2) \implies \phi \text{ is an injective function}}$$

Let,

$$\begin{aligned}\sigma \in G, \\ u, v \in U \ni \sigma(u) = v\end{aligned}$$

Then, we know that,

$$\exists \tilde{\sigma} \in G \ni \sigma \cdot \tilde{\sigma} = \tilde{\sigma} \cdot \sigma = \lambda x.x$$

$$\begin{aligned}
& \lambda x.x \in G_u \\
& \because \sigma \cdot (\lambda x.x) = \sigma \in G \\
\Rightarrow & \boxed{\forall \sigma \in G \exists u \in U, \tau \in G_u, \exists \phi(u, \tau) = \sigma}
\end{aligned} \tag{3}$$

$$\boxed{(1), (2), (3) \Rightarrow \phi \text{ is a injective and surjective(bijective) function}}$$