Permutation Groups

$$\forall U : \text{Set} \ni |U| < \infty, \text{Perm}[U] := \{\sigma : U \longrightarrow U | \forall x, y \in U, \sigma(x) = \sigma(y) \iff x = y\}$$

Lemmas

$$\exists \text{id}_U \in \text{Perm}[U] \ni \forall x \in U, \text{id}_U(x) = x$$
$$\forall \sigma, \tau \in \text{Perm}[U], \sigma \cdot \tau \in \text{Perm}[U]$$
$$\forall \sigma \in \text{Perm}[U] \exists \tilde{\sigma} \in \text{Perm}[U] \ni \sigma \cdot \tilde{\sigma} = \tilde{\sigma} \cdot \sigma = \text{id}_U$$
$$\forall \sigma, \tau, \mu \in \text{Perm}[U] \sigma \cdot (\tau \cdot \mu) = (\sigma \cdot \tau) \cdot \mu = \sigma \cdot \tau \cdot \mu$$
$$\forall \sigma \in \text{Perm}[U], \sigma \cdot \text{id}_U = \text{id}_U \cdot \sigma = \sigma$$
$$\forall \sigma, \tau, \tau' \in \text{Perm}[U], \sigma \cdot \tau = \sigma \cdot \tau' \iff \tau = \tau'$$
$$\forall \sigma, \sigma', \tau \in \text{Perm}[U], \sigma \cdot \tau = \sigma' \cdot \tau \iff \sigma = \sigma'$$

Permutation Group

Definition

$$G \subseteq \text{Perm}[U] \ni$$

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$$G \neq \emptyset$$

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$$\forall \sigma, \tau \in G, \sigma \cdot \tau \in G$$

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$$\forall \sigma \in G, \exists \tilde{\sigma} \in G \ni \sigma \cdot \tilde{\sigma} = \tilde{\sigma} \cdot \sigma = \text{id}_U$$

$$\forall \sigma \in \text{Perm}[U], k \in \mathbb{N}, \sigma^k := \begin{cases} id_U & \text{iff } k = 0, \\ \sigma \cdot (\sigma^{k-1}) & \text{iff } k > 0 \end{cases}$$
$$\forall \sigma \in \text{Perm}[U] \text{ord}\sigma := \min(\{t \ni \sigma^t = \text{id}_U, t \neq 0\})$$

Lemma -

$$G \ni \text{id}_U$$

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$$\forall \sigma \in Perm[U], \exists k < l \ni \sigma^k = \sigma^l$$
$$\implies \text{id}_U = \sigma^{l-k}$$

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$$\forall \sigma \in Perm[U] \{\sigma^k | k \in \mathbb{N}, k < \text{ord}(\sigma)\} \in \{G\}$$

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$$\forall \sigma \in Perm[U], \text{ord}\sigma = \text{lcm}(\lambda_i)_{i \leq n}$$

Proof ($\forall G, G \ni \text{id}_U$)

$$\because G \neq \emptyset, |G| \geq 1$$
$$\implies \exists \sigma \in \text{Perm}[U] \ni \sigma \in G$$
$$\because \sigma \in G \iff \tilde{\sigma} \in G$$
$$\implies \{\sigma, \tilde{\sigma}\} \subseteq G$$
$$\because \forall \sigma, \tau \in G, \sigma \cdot \tau \in G$$
$$\implies \sigma \cdot \tilde{\sigma} \in G$$
$$\implies \text{id}_U \in G$$
$$\blacksquare$$

Graph Automorphism Group

$$\forall V, E \Gamma := (V, E) \ni E \subseteq V \times V$$

$$\mathrm{Aut}(\Gamma) := \{\sigma \in \mathrm{Perm}[V] | \forall \{x, y\} \in E \iff \{\sigma(x), \sigma(y)\} \in E\}$$

Benzene Group

$$\mathrm{Aut}(\Gamma) \subset S_6$$

This group has - 12 elements - 6 rotations - 6 reflections

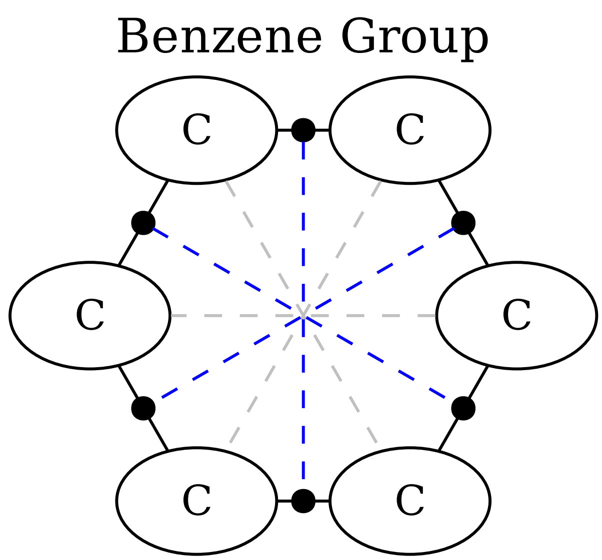

Figure 1: