Combinatorial Species

Definition

Let

- $$\Omega := \{S \text{ is a Set} : |S| < \infty\}$$
- $$\Gamma := \{\sigma : U \longrightarrow V | U, V \in \Omega, |U| = |V|, \forall x, y \in U, \sigma(x) = \sigma(y) \iff x = y\}$$

It's a rule F which:

- $$\forall U \in \Omega \exists F[U] \in \Omega (\text{Set of F structures on U})$$
- $$\forall \sigma : U \longrightarrow V \in \Gamma \exists F[\sigma] \in \Gamma, F[\sigma] : F[U] \longrightarrow F[V] (\text{Transport of F structure along } \sigma)$$
- $$\forall \sigma : U \longrightarrow V, \tau : V \longrightarrow W \in \Gamma, F[\sigma \circ \tau] = F[\sigma] \circ F[\tau], F[id_U] = id_{F[U]}$$

Examples

- $\mathcal{G}$: The Species of simple Graphs
  - $\forall U \in \Omega, \mathcal{G}[U]$: A graph with vertex set U.
  - $\forall \sigma \in \Gamma, \mathcal{G}[\sigma]$: A graph transformation, where σ is applied to all values.
- $\mathcal{S}$: The Species of all Permutations
  - $\mathcal{S}[U] = \{\sigma : U \longrightarrow U | \forall x, y \in U, x = y \iff \sigma(x) = \sigma(y)\}$
  - $\mathcal{S}[\sigma](\alpha)(\sigma(u)) = \sigma(\alpha(u))$

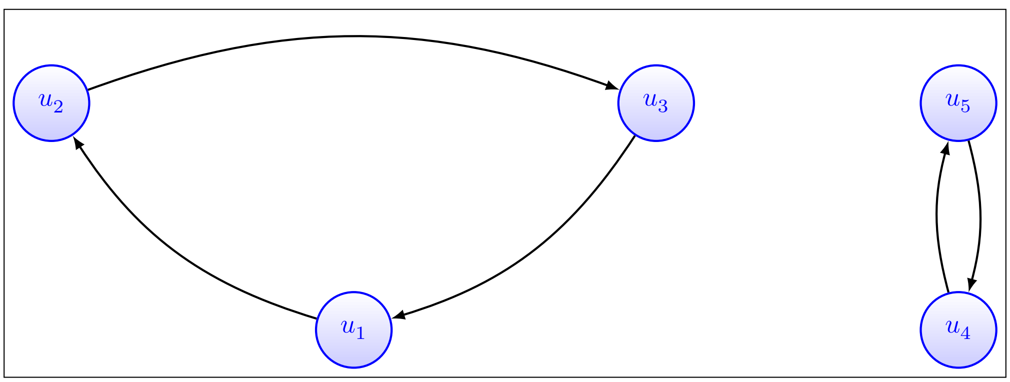

Figure: α

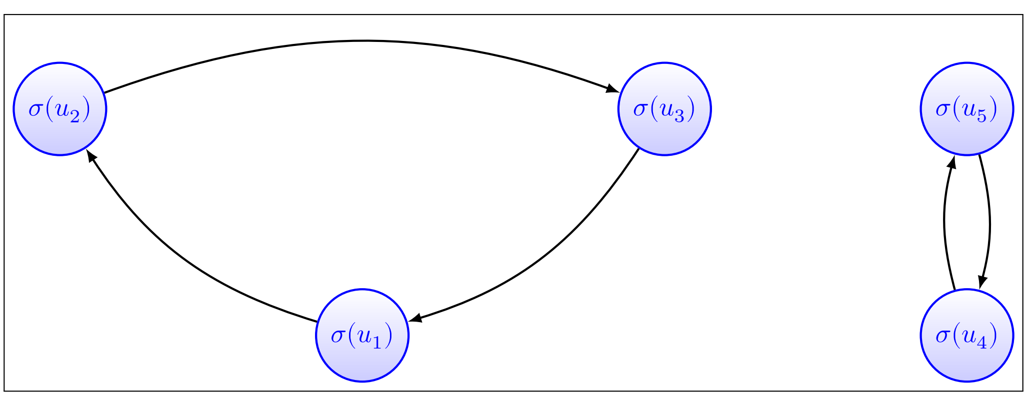

Figure: $\sigma(\alpha)$

Some Standard Species

- $\mathcal{G}$: Simple Graphs
- $\mathcal{G}^0$: Connected Simple Graphs
- a: Trees
- $\mathcal{A}$: Rooted Trees
- $\mathcal{D}$: Directed Graphs
- $\mathcal{P}ar$: Set Partitions
- $\mathcal{P}$: Power-set construction
- $\mathcal{E}nd$: Endo-functions
- $\mathcal{I}nv$: Involutions
- $\mathcal{S}$: Permutations
- $\mathcal{C}$: Species of Cycles
- $\mathcal{L}$: Species of Linear Orders
- $\forall U \in \Omega, E[U] := \{U\}$: Species of Sets
- $\forall U \in \Omega, \varepsilon[U] := U$: Species of set-elements
- $\forall U \in \Omega, X[U] := \begin{cases} \{U\} & \text{iff } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$: Species of singleton sets
- $\forall U \in \Omega, 1[U] := \begin{cases} \{U\} & \text{iff } U = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$
- $\forall U \in \Omega, 0[U] := \emptyset$
- $\forall U \in \Omega, E_k(U) := \begin{cases} \{U\} & \text{iff } |U| = k \\ \emptyset & \text{otherwise} \end{cases}$