The Pumping Lemma

Consider

$$L = 0^n 1^n : n \in \mathbb{N}$$

where 0^n means 0 repeated n times. Then L can be represented by the generated function

$$L(x) = \sum_{i=0}^{\infty} x^{2i} = \frac{1}{1-x^2}$$

L has a generating function however it isn't a regular language. This can be proven using the Pumping lemma.

Lemma:

$$\forall L \subseteq \Sigma^* \ni \text{regular}(L)$$
$$\implies \exists p \in \mathbb{P} \ni \forall w \in L, |w| \geq p$$
$$\implies \exists x, y, z \in \Sigma^*, (xyz = w) \wedge (|y| \in \mathbb{P}) \wedge (|xy| \leq p) \wedge (\forall n \in \mathbb{N}, xy^n z \in L)$$

Now take the original Language L, $\because$ this language can't be represented in the aforementioned way, we can conclude that it isn't a finite automaton. Proof: Let

$$w = xyz \ni w \in L \implies w = 0^n 1^n, n \in \mathbb{P}, \ni |xy| \leq n \wedge |y| > 0 \wedge x, y, z \in L$$

The only solution to this is $x, z = \emptyset \wedge y = w \because$ if $x \neq \emptyset$, This would imply that x is either all 0s, or y is all 1s, which both imply that $x \notin L$ and $y \notin L$ respectively. Which are both contradictions.

HOWEVER this then means that $\nexists p \neq 1 \ni y^p \in L \iff \not\forall p \in P, y^p \in L, \therefore$ the statement is proven by contradiction.

Proof for the Pumping Lemma

Let

- L be a regular Language over Σ character-set,
- recognized by the FSA $M = (Q, \Sigma, \delta : Q \times \Sigma \longrightarrow Q, q_0 \in Q, F \subseteq Q)$
- Pumping Length $p = |Q_M|$
- $w \in L \ni |w| \geq p$
- the ordered tuple of states visited to reach w, $(q_i)_{0 \leq i \leq p}$

We know that

$$\because \#((q_i)_{0 \leq i \leq p}) = p + 1$$
$$\implies \exists i, j \in [p] \ni i \neq j, q_i = q_j (\text{The pigeon hole principle})$$
$$\because i \neq j \wedge q_i = q_j$$

$\implies$ there exists a cycle from i^{th} to j^{th} state in the FSA. $\implies$ You can visit this cycle as many times as you want and the FSA will still recognize the word.