

Combinatorial Species

Definition

Let

•

$$\Omega := \{S \text{ is a Set} : |S| < \infty\}$$

•

$$\Gamma := \{\sigma : U \longrightarrow V \mid U, V \in \Omega, |U| = |V|, \forall x, y \in U, \sigma(x) = \sigma(y) \iff x = y\}$$

It's a rule F which:

•

$$\forall U \in \Omega \exists F[U] \in \Omega (\text{Set of } F \text{ structures on } U)$$

•

$$\forall \sigma : U \longrightarrow V \in \Gamma \exists F[\sigma] \in \Gamma, F[\sigma] : F[U] \longrightarrow F[V] (\text{Transport of } F \text{ structure along } \sigma)$$

•

$$\forall \sigma : U \longrightarrow V, \tau : V \longrightarrow W \in \Gamma, F[\sigma \circ \tau] = F[\sigma] \circ F[\tau], F[id_U] = id_{F[U]}$$

Examples

- \mathcal{G} : The Species of simple Graphs
 - $\forall U \in \Omega, \mathcal{G}[U]$: A graph with vertex set U .
 - $\forall \sigma \in \Gamma, \mathcal{G}[\sigma]$: A graph transformation, where σ is applied to all values.
- \mathcal{S} : The Species of all Permutations
 - $\mathcal{S}[U] = \{\sigma : U \longrightarrow U \mid \forall x, y \in U, x = y \iff \sigma(x) = \sigma(y)\}$
 - $\mathcal{S}[\sigma](\alpha)(\sigma(u)) = \sigma(\alpha(u))$

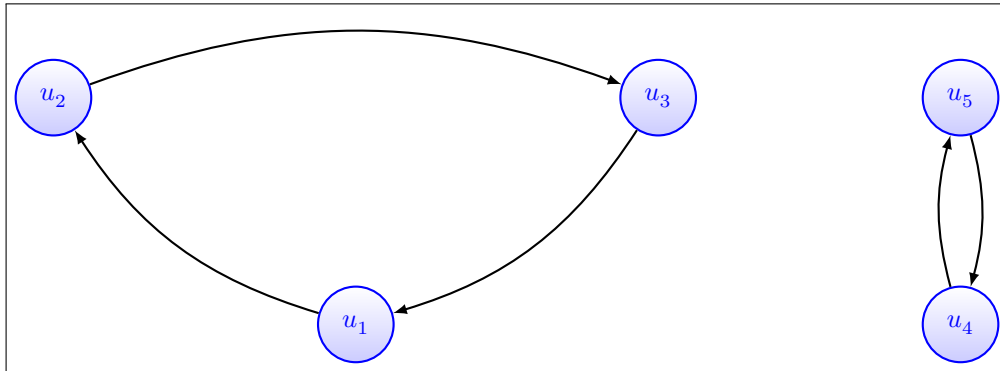


Figure: α

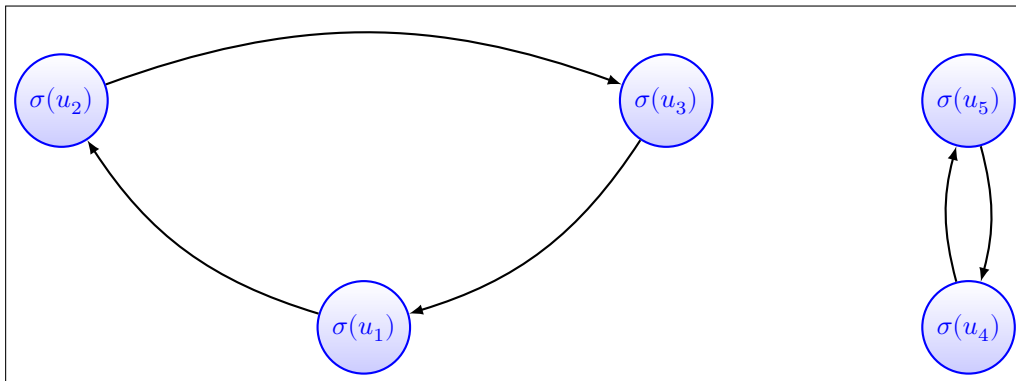


Figure: $\sigma(\alpha)$

Some Standard Species

- \mathcal{G} : Simple Graphs
- \mathcal{G}^0 : Connected Simple Graphs
- a : Trees
- \mathcal{A} : Rooted Trees
- \mathcal{D} : Directed Graphs
- $\mathcal{P}ar$: Set Partitions
- \mathcal{P} : Power-set construction
- $\mathcal{E}nd$: Endo-functions
- $\mathcal{I}nv$: Involutions
- \mathcal{S} : Permutations
- \mathcal{C} : Species of Cycles
- \mathcal{L} : Species of Linear Orders
- $\forall U \in \Omega, E[U] := \{U\}$: Species of Sets
- $\forall U \in \Omega, \varepsilon[U] := U$: Species of set-elements
- $\forall U \in \Omega, X[U] := \begin{cases} \{U\} & \text{iff } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$: Species of singleton sets
- $\forall U \in \Omega, 1[U] := \begin{cases} \{U\} & \text{iff } U = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$
- $\forall U \in \Omega, 0[U] := \emptyset$
- $\forall U \in \Omega, E_k(U) := \begin{cases} \{U\} & \text{iff } |U| = k \\ \emptyset & \text{otherwise} \end{cases}$