Super Catalan or Schröder numbers

Problem statement

In how many ways can a string of n identical symbols be bracketed? Here x is a single character string and ++ refers to concatenation of two strings. Example: xxxx Can be bracketed as (x(xx)(xx))

Solution

Let Ix be an inductive type defined as following.

$$Ix := (x \in \Sigma^*)|(I \vee I)$$

Then we can represent the answer to the question as the following

$$S := \lambda n. |\{\sigma \in Ix : |\sigma| = n\}|$$

This set is isomorphic to a set of all possible trees with n leaf nodes, such that each node has a minimum of 2 children nodes if not zero.

n	$S\ n$	$\#(Sn)$
1	$\{x\}$	1
2	$\{(xx)\}$	1
3	$\{(xxx), (x(xx)), ((xx)x)\}$	3
4	$\{(xxxx), (x(xxx)), ((xxx)x), ((xx)(xx)), (x(x(xx))), ((x(xx))x),$ $(x((xx)x)), (((xx)x)x), (xx(xx)), (x(xx)x), ((xx)xx)\}$	11

$$S = \mathfrak{X} + Seq_{\geq}(S)$$

$$\implies S(x) = x + \frac{S(x)^2}{1 - S(x)}$$

$$\implies (S(x) - x)(1 - S(x)) - S(x)^2 = 0$$

$$\implies S(x) - S(x)^2 - x + x \cdot S(x) - S(x)^2 = 0$$

$$\implies 2 \cdot S(x)^2 - (x+1) \cdot S(x) + x = 0$$

$$\implies S(x) = \frac{(x+1) \pm ((x+1)^2 - 4 \cdot 2 \cdot x)^{0.5}}{4}$$

$$\implies S(x) = \frac{(x+1) \pm (x^2 + 2 \cdot x + 1 - 8 \cdot x)^{0.5}}{4}$$

$$\implies S(x) = \frac{(x+1) \pm (x^2 - 6 \cdot x + 1)^{0.5}}{4}$$

$\because$ the 0^{th} term of the generating function must be 0 (Definition)

$$\boxed{S(x) = \frac{(x+1) - (x^2 - 6 \cdot x + 1)^{0.5}}{4}}$$