PARTIAL DIFFERENTIATION

Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $y = f(x)$ OR simply $x \mapsto y$.
Here there is one input 'x' & one output 'y'

"A FUNCTION WITH MORE THAN ONE INPUT VARIABLES (OR OUTPUT VARIABLES) IS CALLED FUNCTION OF SEVERAL VARIABLES"

i.e. $f: \mathbb{R}^m \to \mathbb{R}^n$ (except $m=1$ & $n=1$ together)

(i) If $m=2, n=1$ then $f: \mathbb{R}^2 \to \mathbb{R}$.
example: $f(x,y) = x+y$

(ii) If $m=1, n=2$ then $f: \mathbb{R} \to \mathbb{R}^2$
example: $f(x) = (x, x^2)$

(iii) If $m=2, n=2$ then $f: \mathbb{R}^2 \to \mathbb{R}^2$
example: $f(x,y) = (x+y, xy)$

and so on.

PARTIAL DERIVATIVES OF $f: \mathbb{R}^m \to \mathbb{R}$ ($m \neq 1$)

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = e^{2x}\sin y$

The partial derivative of 'f' w.r.t. one input variable 'x' (say) by differentiating it (keeping other input variables as constant) w.r.t. x.
i.e.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2e^{2x}\sin y \\ \text{Similarly, } \frac{\partial f}{\partial y} &= e^{2x}\cos y \end{aligned}\right\} \text{ are partial derivatives of order one.}$$

Second order derivatives; $\frac{\partial^2 f}{\partial x^2} = 4e^{2x}\sin y$, $\frac{\partial^2 f}{\partial y^2} = -e^{2x}\sin y$

& $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(e^{2x}\cos y) = 2e^{2x}\cos y$$

& $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2e^{2x}\sin y) = 2e^{2x}\cos y$$

observe that $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$ if the domain of 'f' is connected

NOTE: $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$ & $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

dependent variable is 'f'

'x' & 'y' independent variables

Ques (1) If $z^3 - zx - y = 0$, find $\frac{\partial^2 z}{\partial x \partial y} \longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

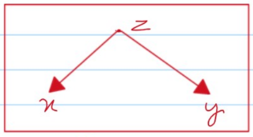
$\longrightarrow$ Implicit function of z

NOTE:
$x^2 + y + 1 = 0$ is an explicit function in x & y
$\because$ $\boxed{x = (-1-y)^{1/2}}$
OR
$\boxed{y = -1-x^3}$

Sol. NOTE: z is dependent variable; x & y are independent variables.

diff. partially w.r.t. 'y',

$$3z^2\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0$$



$$\Rightarrow \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x} \quad \text{--- (1)}$$

diff. partially (1) w.r.t. 'x', $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-1}{(3z^2-x)^2} \times \left(6z\cdot\frac{\partial z}{\partial x} - 1\right)$ --- (2)

Diff. $z^3 - zx - y = 0$ partially w.r.t. 'x';

$$3z^2\frac{\partial z}{\partial x} - \left(z\cdot 1 + x\frac{\partial z}{\partial x}\right) - 0 = 0$$

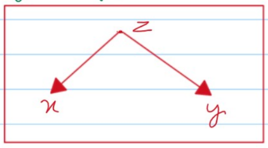
$$\Rightarrow \frac{\partial z}{\partial x} = \frac{z}{(3z^2 - x)} \quad \text{--- (3)}$$

(3) in (2) ; $$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{(3z^2-x)^2} \times \left(\frac{6z^2}{3z^2-x} - 1\right)$$
$$= \frac{-(3z^2+x)}{(3z^2-x)^2}$$

Ques 2) If $z = f(x^2y)$ then prove $x\frac{\partial z}{\partial x} = 2y\frac{\partial z}{\partial y}$, f is any differentiable funct^n.

Sol) Diff. partially w.r.t. x;

$$\frac{\partial z}{\partial x} = f'(x^2y) \times 2xy$$



Diff. partially w.r.t. y;

$$\frac{\partial z}{\partial y} = f'(x^2y) \times x^2$$

$$LHS = x\frac{\partial z}{\partial x} = 2x^2y\, f'(x^2y)$$

& $$RHS = 2y\frac{\partial z}{\partial y} = 2x^2y\, f'(x^2y)$$

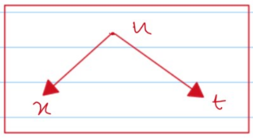
Clearly, $LHS = RHS$

Ques 3) If $u(x,t) = f(x+ct) + g(x-ct)$ where f and g are both differentiable functions then prove that $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$.

Sol) Diff. partially w.r.t. x;

$$\frac{\partial u}{\partial x} = f'(x+ct) + g'(x-ct) \quad \text{--- (1)}$$

Diff. partially (1) w.r.t. x; $\frac{\partial^2 u}{\partial x^2} = f''(x+ct) + g''(x-ct)$



Diff. partially w.r.t. t;

$$\frac{\partial u}{\partial t} = f'(x+ct)\times c + g'(x-ct)\times -c$$
$$= c\left[f'(x+ct) - g'(x-ct)\right] \quad \text{--- (2)}$$

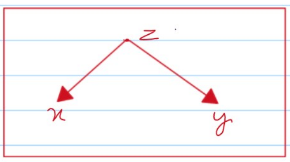
Diff. partially (2) w.r.t. t; $$\frac{\partial^2 u}{\partial t^2} = c\left[f''(x+ct)\times c + g''(x-ct)\times c\right]$$
$$= c^2\left[f''(x+ct) + g''(x-ct)\right]$$
$$= c^2\frac{\partial^2 u}{\partial x^2} \quad [\text{from (1)}]$$

or $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Ques 4) If $x^x y^y z^z = c$ Prove $\frac{\partial^2 z}{\partial x \partial y} = -[x \log_e(ex)]^{-1}$ at $x = y = z$

Sol) Taking log both sides;

$$x\log x + y\log y + z\log z = \log c \quad \text{--- (1)}$$



Diff. partially w.r.t. y;

$$0 + \left(y\times\frac{1}{y} + \log y\right) + \left(z\times\frac{1}{z}\frac{\partial z}{\partial y} + \log z\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)} \quad \text{--- (2)}$$

Diff. partially (2) w.r.t. x;

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y) \times \frac{-1}{(1+\log z)^2} \times \frac{1}{z}\frac{\partial z}{\partial x} \quad \text{--- (3)}$$

Due to symmetry in (1), $\frac{\partial z}{\partial x}$ can be written directly from $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad (4)$$

(4) in (3);

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y) \times \frac{-1}{(1+\log z)^2} \times \frac{1}{z} \times -\frac{(1+\log x)}{(1+\log z)}$$

$$= -\frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

At $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)^2}{x(1+\log x)^3}$

$$= \frac{-1}{x(1+\log x)}$$

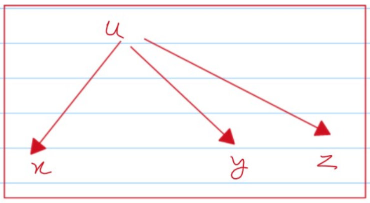
$$= -\frac{1}{x[\log_e e + \log_e x]}$$

$$= -\frac{1}{x \log_e(ex)}$$

$$= -[x \log_e(ex)]^{-1}$$

Ques 5 If $u = (x^2+y^2+z^2)^{n/2}$ is $\boxed{\text{harmonic function}}$ in x, y and z then find n.

i.e. the sum of its second order pure partial derivatives is zero or $u_{xx} + u_{yy} + u_{zz} = 0$



Sol Diff. partially w.r.t. x;

$$u_x = \frac{n}{2}\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x$$

$$= nx(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

Diff. partially w.r.t. 'x' again;

$$u_{xx} = n\left[x\left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{\frac{n}{2}-2} \times 2x + (x^2+y^2+z^2)^{\frac{n}{2}-1} \times 1\right]$$

$$= n(n-2)x^2(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

Similarly,

$$u_{yy} = n(n-2)y^2(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

and

$$u_{zz} = n(n-2)z^2(x^2+y^2+z^2)^{\frac{n}{2}-2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$\because$ u is harmonic function so

$$u_{xx} + u_{yy} + u_{zz} = 0$$

$$\Rightarrow n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-2}(x^2+y^2+z^2) + 3n(x^2+y^2+z^2)^{\frac{n}{2}-1} = 0$$

$$\Rightarrow n(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-1} + 3n(x^2+y^2+z^2)^{\frac{n}{2}-1} = 0$$

$$\Rightarrow (x^2+y^2+z^2)^{\frac{n}{2}-1}[n(n-2)+3n] = 0$$

$$\Rightarrow n(n-2)+3n = 0$$

$$\Rightarrow n^2+n = 0$$

$$\Rightarrow n(n+1) = 0$$

$$\Rightarrow n = 0 \quad \text{or} \quad n = -1$$