EIGENVALUES AND EIGENVECTORS

Consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and a vector $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Observe $Av_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(1)+2(0) \\ 0(1)+3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{Av_1 = 1v_1}$

If $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then

$Av_2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(1)+2(1) \\ 0(1)+3(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$\Rightarrow Av_2 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \boxed{Av_2 = 3v_2}$

Observe that $\boxed{Av_i = \lambda_i v_i \quad , \quad i = 1, 2}$ —— (1)

But for $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have $Au = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \neq \lambda u$

i.e. vectors v_1 and v_2 are special vectors s.t. when they are multiplied (by right) to A, the result is just the scalar multiple of them. Such vectors are called eigenvectors of matrix A.

Definition: A non-zero vector 'v' is called as eigenvector of square matrix A if $Av = \lambda v$, where 'λ' is a real number (scalar) is called as eigenvalue of A.

From (1); $Av = \lambda v$

$\Rightarrow Av - \lambda v = 0$

$\Rightarrow \boxed{(A - \lambda I)v = 0}$ → Remember this also

$\Rightarrow \boxed{\det(A - \lambda I) = 0}$ (*Note this step) (WHY?)

Characteristic Equation

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

See $AB = 0$ (null matrix) but neither $A = 0$ nor $B = 0$

*If $AB = 0$ then $\det(A) = 0$ or $\det(B) = 0$

Thus $\det(A - \lambda I) = 0$ is the characteristic equation of matrix A in variable 'λ' which can be solved for λ (i.e. eigenvalue).

PROPERTIES OF EIGENVALUES

If λ is an eigenvalue of square matrix A with corresponding eigenvector v. Then

(1) The eigenvalue of A^T is also λ.
(2) Sum of eigenvalues of A = trace(A) i.e. sum of diagonal elements
(3) Product of eigenvalues of A = $\det(A)$
(4) The eigenvalue of A^n (n is an integer) is λ^n.
(5) If $f(A)$ is a matrix polynomial in A then $f(\lambda)$ is eigenvalue of $f(A)$.
(6) If A is triangular / diagonal matrix then eigenvalues of A are precisely the diagonal entries (principal) of A.
∴ Eigenvalues of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ are a & b.

Ques (1) Find the sum and product of eigenvalues of $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 6 & 7 \end{bmatrix}$.

Sol. Sum of eigenvalues $=$ trace (A)
$= 1+3+7$
$= 11$

Product of eigenvalues $= \det(A)$
$= 10$

Ques (2) Let $A = \begin{bmatrix} 3 & -3 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$. If sum of eigenvalues of A is 10 and product is 30. Find the value of a^2+b^2.

Sol. $\because$ Sum of eigenvalues $=$ trace (A)
$\Rightarrow 10 = 3+a+b$
$\Rightarrow a+b = 7$ —— (1)

$\because$ Product of eigenvalues $= \det(A)$
$\Rightarrow 30 =$ *product of diagonal entries of A
$\Rightarrow 30 = 3ab$
$\Rightarrow ab = 10$ —— (2)

* NOTE: If A is triangular then det (A) = product of diagonal entries of A

Solve (1) & (2) to $(a=2$ & $b=5)$ or $(a=5$ & $b=2)$

Then $a^2+b^2 = 29$

Ques (3) The product of 2 eigenvalues of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the 3rd eigenvalue.

Sol. Let α, β, γ be the eigenvalues of A.
$\because$ Product of eigenvalues $= \det(A)$
$\Rightarrow \alpha\beta\gamma = 32$
$\Rightarrow (\alpha\beta)\gamma = 32$

$\Rightarrow 16\gamma = 32$
$\Rightarrow \gamma = 2$ is 3rd eigenvalue.

Ques (4) Two eigenvalues of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are $\frac{1}{5}$ times the third. Find all eigenvalues.

Sol. Let α, β, γ be eigenvalues s.t. $\alpha = \frac{1}{5}\gamma$ & $\beta = \frac{1}{5}\gamma$

$\because$ Sum of eigenvalues $= tr(A)$

$\Rightarrow \frac{\gamma}{5} + \frac{\gamma}{5} + \gamma = 7$

$\Rightarrow \frac{7\gamma}{5} = 7$

$\Rightarrow \gamma = 5$

$\therefore$ Eigenvalues are $1, 1$ & 5.

Ques (5) Let A be a singular matrix of order 3X3 with 2 & 3 as its two eigenvalues. Find the 3rd eigenvalue.
[HINT: If A is singular then $\det(A) = 0$ & vice-versa]

Sol. $\because$ A is singular
$\therefore$ $\det(A)=0$
$\Rightarrow$ Product of eigenvalues $=0$
$\Rightarrow$ $2\times 3\times \gamma = 0$ (where 'γ' is 3rd eigenvalue)
$\Rightarrow$ $6\gamma=0$
$\Rightarrow$ $\gamma=0$

Ques ⑥ Find 'a' & 'b' such that $A=\begin{bmatrix} a & 4 \\ 1 & b\end{bmatrix}$ has eigenvalues 3 & −2.

Sol. $\because$ Sum of eigenvalues $= tr(A)$

$$\Rightarrow 3+(-2) = a+b$$
$$\Rightarrow a+b = 1 \quad \text{——— (1)}$$

$\because$ Product of eigenvalues $= \det(A)$
$$\Rightarrow (3)(-2) = ab-4$$
$$\Rightarrow ab=-2 \quad \text{——— (2)}$$

On solving (1) & (2), $a=2$, $b=-1$.

Ques ⑦ If $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2\end{bmatrix}$ has eigenvalues 1, 1 & 5. Find the eigenvalues of (i) A^T (ii) $5A^{-1}$ (iii) A^2+5A+I

Sol. (i) Eigenvalues of $A^T = 1, 1$ & 5.

(ii) Eigenvalues of $A^{-1} = 1^{-1}, 1^{-1}$ & 5^{-1} [Property ④]
$= 1, 1$ & $\frac{1}{5}$

$\Rightarrow$ Eigenvalues of $5A^{-1} = 5(1), 5(1)$ & $5\left(\frac{1}{5}\right) = 5, 5$ & 1

(iii) Here $f(A) = A^2+5A+I$

$\because$ Eigenvalues of $f(A)$ are $f(\lambda)$ [Property ⑤]

$\therefore$ $f(\lambda)=\lambda^2+5\lambda+1$ [$\because$ 1 is the eigenvalue of identity matrix I]

$\therefore$ $f(1)$, $f(1)$ & $f(5)$ are eigenvalues of $f(A)$

$\Rightarrow$ $1^2+5(1)+1$, $1^2+5(1)+1$ & $5^2+5(5)+1$ are eigenvalues of $f(A)$

$\Rightarrow$ 7, 7 & 51 are eigenvalues of $f(A)$.

HOW TO FIND EIGENVALUES AND EIGENVECTORS?

Suppose A is a square matrix of order n. The steps involved are:

① Write the characteristic equation of A i.e. $\boxed{\det(A-\lambda I)=0}$ (Derived earlier!)

② Solve characteristic eqn for λ.

③ For each λ_i where $i=1,2,\dots n$ determine v_i using $\boxed{(A-\lambda_i I)v_i=0}$ (Derived earlier!)

RECALL – λ_i is eigenvalue and v_i is eigenvector of A

TIP! – ★ If A is of order 2×2 then characteristic eqn is:
$$\lambda^2 - tr(A)\lambda + \det(A)=0$$

★ If A is of order 3×3 then characteristic eqn is:
$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - \det(A) = 0$$
where A_{ii} = cofactor of diagonal element a_{ii}.

If $A=\begin{bmatrix} 1 & 3 & 2 \\ 6 & 5 & 1 \\ 3 & 2 & 0\end{bmatrix}$ then cofactor of 5 ($=a_{22}$) is given by:
$$A_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 0\end{vmatrix} = -6$$

Ques 1) Find the eigenvalues and eigenvectors of:

(i) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

Sol (i) Here $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

The characteristic eq^n : $\det(A-\lambda I)=0$

ALTERNATE

$$\Rightarrow \det\left\{\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0$$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$

$$\Rightarrow \det\begin{bmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) + 2 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 3\lambda + 6 = 0$$
$$\Rightarrow \lambda(\lambda-2) - 3(\lambda-2) = 0$$
$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$
$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3 \text{ are eigenvalues}$$

(a) For $\lambda_1 = 2$,

$$(A-\lambda_1 I)v_1 = 0$$
$$\Rightarrow (A-2I)v_1 = 0$$
$$\Rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$;

$$\begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here $\text{rank}(A-2I) = 1 (= r)$

and $n = 2$ (no. of variables)

$\therefore n-r = 2-1 = 1$ LI eigenvectors exists

From above

$$-x - 2y = 0$$

Assume $n-r=1$ variable as arbitrary i.e. $y = K$ (say) $\in \mathbb{R}$

$$\Rightarrow x = -2K$$

Thus $v_1 = \begin{bmatrix} -2K \\ K \end{bmatrix} = K\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (if $K=1$) is an eigenvector.

(b) For $\lambda_2 = 3$,

$$(A-\lambda_2 I)v_2 = 0$$
$$\Rightarrow (A-3I)v_2 = 0$$
$$\Rightarrow \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$;

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here $\text{rank}(A-3I) = 1 (= r)$

and $n = 2$ (no. of variables)

$\therefore n-r = 2-1 = 1$ LI eigenvectors exists

From above

$$-2x - 2y = 0$$

Assume $n-r=1$ variable as arbitrary i.e. $y = K$ (say) $\in \mathbb{R}$

$$\Rightarrow x = -K$$

Thus $v_2 = \begin{bmatrix} -K \\ K \end{bmatrix} = K\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\therefore v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (if $K=1$) is an eigenvector.

(ii) Here $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

The characteristic eq^n : $\det(A-\lambda I) = 0$

ALTERNATE

$$\Rightarrow \det\left\{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0$$

$$\Rightarrow \det\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$
$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$$\Rightarrow (3-\lambda)(1-\lambda) + 1 = 0$$
$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$
$$\Rightarrow \lambda^2 - 2\lambda - 2\lambda + 6 = 0$$
$$\Rightarrow \lambda(\lambda-2) - 2(\lambda-2) = 0$$
$$\Rightarrow (\lambda-2)(\lambda-2) = 0$$
$\Rightarrow \lambda_1 = 2$, $\lambda_2 = 2$ are eigenvalues

(a) For $\lambda_1 = 2$,

$$(A - \lambda_1 I) v_1 = 0$$
$$\Rightarrow (A - 2I) v_1 = 0$$
$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$;

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here rank $(A - 2I) = 1 (= r)$
and $n = 2$ (no of variables)
$\therefore n - r = 2 - 1 = 1$ LI eigenvector exists
From above

$$-x - y = 0$$

Assume $y = K$ (say) then $x = -K$

$$v_1 = \begin{bmatrix} -K \\ K \end{bmatrix} = K \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\therefore v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the only eigenvector

Ques 2 Find the eigenvalues and eigenvectors of:

(i) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$

Sol (i) Characteristic eqn: $\det(A - \lambda I) = 0$

$$\Rightarrow \lambda^3 - tr(A)\lambda^2 + [A_{11} + A_{22} + A_{33}]\lambda - \det(A) = 0$$
$$\Rightarrow \lambda^3 - 4\lambda^2 + [(-1) + (-6) + 6]\lambda - (-4) = 0$$
$$\Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

By hit and trial, $\lambda_1 = 1$ is a root.
Also, $\lambda_2 = -1$ is a root.
$\because$ Sum of eigenvalues $= tr(A)$
$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 4$$
$$\Rightarrow \lambda_3 = 4$$
$\therefore$ 1, -1 and 4 are eigenvalues.

(a) for $\lambda_1 = 1$,

$$(A - 1I) v_1 = 0$$
$$\Rightarrow \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$;
$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$;
$$\begin{bmatrix} 3 & 6 & 6 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$;
$$\begin{bmatrix} 3 & 6 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ Rank $(A - I) = 2 (= r)$ & $m = 3$

$\therefore$ $m - r = 3 - 2 = 1$ LI eigenvector exists

From above;

$$3x + 6y + 6z = 0 \quad \text{—— (1)}$$
$$-2y - 2z = 0 \quad \text{—— (2)}$$

Let $z = K$ (say) be arbitrary

From (2); $y = -K$

From (1) $x = 0$

$$\therefore v_1 = \begin{bmatrix} 0 \\ -K \\ K \end{bmatrix}$$

If $K = 1$, $v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector

(b) for $\lambda_2 = -1$,

$$(A + 1I) v_2 = 0$$

$$\Rightarrow \begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$; $\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to 5R_2 - R_1$; $\begin{bmatrix} 5 & 6 & 6 \\ 0 & 14 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\because$ Rank $(A + I) = 2 (= r)$ & $m = 3$

$\therefore$ $m - r = 3 - 2 = 1$ LI eigenvector exists

From above; $5x + 6y + 6z = 0$ —— (3)

$14y + 4z = 0$ —— (4)

Let $z = K$ be arbitrary

From (4); $y = -\frac{2K}{7}$

From (3); $5x - \frac{12}{7}K + 6K = 0$

$$\Rightarrow x = -\frac{6K}{7}$$

$$\therefore V_2 = \begin{bmatrix} -6K/7 \\ -2K/7 \\ K \end{bmatrix}$$

If $K = 7$, $V_2 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$ is an eigenvector

(c) for $\lambda_3 = 4$,

$$(A - 4I) v_3 = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_1$; $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_1$; $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to 6R_3 + 5R_2$;

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ Rank $(A-4I) = 2 (= r)$ & $m = 3$

$\therefore$ $m - r = 3 - 2 = 1$ LI eigenvector exists

From above ; $x - y + 2z = 0$ ——— (5)

$6y + 6z = 0$ ——— (6)

Let $z = K$ be arbitrary

From (6) ; $y = -K$

From (5) ; $x = -3K$

$$\therefore \quad v_3 = \begin{bmatrix} -3K \\ -K \\ K \end{bmatrix}$$

If $K = 1$, $v_3 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector

(ii) Characteristic eqn: $\det(A - \lambda I) = 0$

$\Rightarrow \lambda^3 - tr(A)\lambda^2 + [A_{11} + A_{22} + A_{33}]\lambda - \det(A) = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 + [(1) + (0) + 4]\lambda - (2) = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

By hit and trial, $\lambda_2 = 1$ is a root

Also $\lambda_1 = 2$ is a root

Now $\lambda_1 + \lambda_2 + \lambda_3 = tr(A)$

$\Rightarrow 1 + 2 + \lambda_3 = 4$

$\Rightarrow \lambda_3 = 1$

$\therefore$ 1, 1 and 2 are eigenvalues

(a) for $\lambda_1 = 2$, $(A - 2I)v_1 = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$;

$$\Rightarrow \begin{bmatrix} 2 & 1 & -4 \\ 0 & 1 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$;

$$\Rightarrow \begin{bmatrix} 2 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$;

$$\Rightarrow \begin{bmatrix} 2 & 1 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ Rank $(A-2I) = 2 (= r)$ & $m = 3$

$\therefore$ $m - r = 3 - 2 = 1$ LI eigenvector exists

From above ; $2x + y - 4z = 0$ ——— (1)

$y - 2z = 0$ ——— (2)

Let $z = K$ be arbitrary.

From (2) ; $y = 2K$

From (1) ; $x = K$

$$\therefore \quad v_1 = \begin{bmatrix} K \\ 2K \\ K \end{bmatrix}$$

If $K = 1$, $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector

(b) For $\lambda_2 = 1$, $(A - 1I)v = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - R_1$; $\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\because$ Rank $(A - I) = 1 (= r)$ & $n = 3$

$\therefore$ $n - r = 3 - 1 = 2$ LI eigenvectors exist.

From above, $x + y - 2z = 0$ —— (3)

Let $z = K_1$ & $y = K_2$ be arbitrary

From (3); $x = 2K_1 - K_2$

$$\therefore \quad v = \begin{bmatrix} 2K_1 - K_2 \\ K_2 \\ K_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2K_1 \\ 0 \\ K_1 \end{bmatrix} + \begin{bmatrix} -K_2 \\ K_2 \\ 0 \end{bmatrix}$$

$$= K_1 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{v_2} + K_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_3}$$

$\therefore$ $v_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ (if $K_1 = 1, K_2 = 0$) & $v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (if $K_2 = 1, K_1 = 0$) are eigenvectors.

Cayley- Hamilton Theorem

Consider a matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ ∵

characteristic equation of A : $\det(A - \lambda I) = 0$

$\Rightarrow \lambda^2 - 5\lambda + 7 = 0$

Now consider the matrix equation $\boxed{A^2 - 5A + 7I = 0}$

Find $A^2 = A \cdot A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$

If $f(A) = A^2 - 5A + 7I$

Then $f(A) = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3-10+7 & 5-5 \\ -5+5 & 8-15+7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$= 0$ (null matrix)

"EVERY SQUARE MATRIX SATISFIES ITS CHARACTERISTIC EQUATION."

OR

If $f(\lambda)$ represents the characteristic polynomial of square matrix A then $f(A) = 0$ (null matrix)

⚠ CAUTION :

∵ characteristic polynomial of A is $f(\lambda) = \det(A - \lambda I)$ — ①

Replace 'λ' by 'A' in ①

$f(A) = \det(A - AI)$

FALSE PROOF!

$= \det(A - A)$

$= \det(0)$

$= 0$ (zero number)

Ques ① If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Verify Cayley - Hamilton theorem. Also find A^{-1} & A^{-2}.

Sol. For verification, refer to the example in theory. From CH theorem,

∵ $A^2 - 5A + 7I = 0$

$\Rightarrow 7I = 5A - A^2$

$\Rightarrow I = \frac{1}{7}(5A - A^2)$

Pre-multiply by A^{-1}; $A^{-1} = \frac{1}{7}(5I - A)$ —— ①

$= \frac{1}{7}\left[\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\right]$

$= \frac{1}{7}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

Pre-multiplying ① by A^{-1};

$A^{-2} = \frac{1}{7}(5A^{-1} - I)$

$$= \frac{1}{7}\left[\frac{5}{7}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{49}\begin{bmatrix} 14 & -5 \\ 5 & 9 \end{bmatrix}$$

Ques 2 If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Use Cayley-Hamilton theorem to find the value of $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol. Characteristic eq^n: $\det(A - \lambda I) = 0$

$$\Rightarrow \lambda^3 - 5\lambda^2 + [2 + 3 + 2]\lambda - 3 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem;

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{——— (1)}$$

$$\because A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^5(0) + A(0) + A^2 + A + I \quad (\text{using } (1))$$

$$= A^2 + A + I \quad (\text{quadratic polynomial in } A)$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Ques 3 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Also find A^{50}.

Sol. Ch. eq^n: $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

By C.H. Theorem, $A^3 - A^2 - A + I = 0$ ——— (1)

TO PROVE: $\boxed{A^n = A^{n-2} + A^2 - I, \ (n \geq 3)}$

Basis of Induction: Put $n = 3$, $A^3 = A + A^2 - I$

or $A^3 - A^2 - A + I = 0$ (which is true by (1))

Induction Hypothesis: Let $A^k = A^{k-2} + A^2 - I$ ——— (2)

To prove: $A^{k+1} = A^{k-1} + A^2 - I$ i.e. the result is also true for $n = k+1$.

Induction Step: $LHS = A^{k+1}$

$$= A^k \cdot A$$

$$= (A^{k-2} + A^2 - I) \cdot A \quad [\text{using } (2)]$$

$$= A^{k-1} + A^3 - A$$

$$= A^{k-1} + (A^2 - I) \quad [\text{using } (1);\ A^3 - A = A^2 - I]$$

$$= A^{k-1} + A^2 - I$$

$$= RHS$$

COMPUTATION OF A^{50}:

$\because \boxed{A^n = A^{n-2} + A^2 - I}$ (Proved) ——— (3)

Put $n = 50$,

$$A^{50} = A^{48} + A^2 - I$$

$$= (A^{46} + A^2 - I) + A^2 - I \quad [\text{Put } n = 48 \text{ in } (3)]$$

$$= A^{46} + 2(A^2 - I)$$

$$= (A^{44} + A^2 - I) + 2(A^2 - I) \quad [\text{Put } n = 46 \text{ in } (3)]$$

$$= A^{44} + 3(A^2 - I)$$

Generalize,

$$A^{50} = A^2 + 24(A^2 - I)$$

$$= 25A^2 - 24I$$

Now $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\therefore A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$