JACOBIANS

A FUNCTION $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Consider
(a) $f_1(x,y) = (x+y,\ x-y)$
(b) $f_2(x,y) = (x+2y,\ x^2y)$
(c) $f_3(x,y) = \left(xy,\ \frac{x+y}{2}\right)$

Each of the above function can be represented as:

(a) $(x,y) \longmapsto (u,v)$ s.t. $u = x+y$, $v = x-y$
(b) $(x,y) \longmapsto (u,v)$ s.t. $u = x+2y$, $v = x^2y$
(c) $(x,y) \longmapsto (u,v)$ s.t. $u = xy$, $v = \frac{x+y}{2}$

WHAT IS JACOBIAN???

For some $(x,y) \longmapsto (u,v)$, the matrix containing first order partial derivatives of 'u' & 'v' w.r.t. 'x' & 'y' s.t.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$ is called Jacobian matrix.

Further, $|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ is called Jacobian determinant.

PROPERTIES OF $|J|$

1. If $|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$. Then $|J^*| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|J|} \Rightarrow |J|\,|J^*| = 1$

2. If $u = f(x,y)$ & $v = g(x,y)$ and $x = \phi(r,\theta)$ & $y = \psi(r,\theta)$ then
$$\left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \times \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$$

3. $|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 0 \iff$ u & v are functionally dependent i.e. u & v are related to each other by some functional equation of the form $h(u,v) = 0$

4. If u & v are functions of x & y connected by implicit relations
$$f_1(u,v,x,y) = 0$$
& $$f_2(u,v,x,y) = 0$$
then
$$|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{(-1)^2\left|\partial(f_1,f_2)/\partial(x,y)\right|}{\left|\partial(f_1,f_2)/\partial(u,v)\right|}$$

Similarly, if u, v & w are functions of x, y & z connected by implicit relations
$$f_1(u,v,w,x,y,z) = 0$$
$$f_2(u,v,w,x,y,z) = 0$$
& $$f_3(u,v,w,x,y,z) = 0$$

then

$$|J| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = (-1)^3 \frac{\left|\partial(f_1,f_2,f_3)/\partial(x,y,z)\right|}{\left|\partial(f_1,f_2,f_3)/\partial(u,v,w)\right|}$$

Ques ① If $u = xyz$, $v = x^2y^2z^2$ & $w = x+y+z$. Find $|J| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|$

SOL.

$$|J| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ 2xy^2z^2 & 2yx^2z^2 & 2zx^2y^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (2xyz) \begin{vmatrix} yz & xz & xy \\ yz & xz & xy \\ 1 & 1 & 1 \end{vmatrix}$$

$= 0$ (WHY?)

Ques ② If $x = a(u+v)$, $y = b(u-v)$ & $u = r^2\cos 2\theta$, $v = r^2 \sin 2\theta$. Find $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$.

SOL.

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \times \left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| \quad \text{—— ①}$$

Now
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} a & a \\ b & -b \end{vmatrix}$$

$$= -2ab$$

&
$$\left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2r\cos 2\theta & -r^2 2\sin 2\theta \\ 2r \sin 2\theta & r^2 2\cos 2\theta \end{vmatrix}$$

$$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$$
$$= 4r^3$$

Put in ①; $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = -2ab \times 4r^3 = -8abr^3$

Ques ③ Examine whether u, v and w are functionally dependent or not. If YES then find the relationship among them.

ⓐ $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$

ⓑ $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}(x) + \tan^{-1}(y)$

ⓒ $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z$

SOL ⓐ $u = \frac{x-y}{x+y}$ & $v = 1 + \frac{y}{x}$

$$|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = 0$$

$\therefore$ u & v are functionally dependent.

RELATION : $\because$ $u = \dfrac{x-y}{x+y}$

$$\Rightarrow u = \frac{1-y/x}{1+y/x} = \frac{1-(v-1)}{1+(v-1)} \Rightarrow u = \frac{2-v}{v} \Rightarrow \boxed{uv+v-2=0}$$

(b) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}(x) + \tan^{-1}(y)$

$$|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= 0$$

$\therefore$ u & v are functionally dependent.

RELATION : $\because$ $v = \tan^{-1}(x) + \tan^{-1}(y)$

$$\Rightarrow v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow v = \tan^{-1}(u) \Longrightarrow \boxed{\tan(v) - u = 0}$$

(c) $u = xy+yz+zx$, $v = x^2+y^2+z^2$ & $w = x+y+z$

$$|J| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2$; $$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore$ u, v & w are functionally dependent.

RELATION : $w^2 = (x+y+z)^2$

$$\Rightarrow w^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

$$\Rightarrow \boxed{w^2 = v+2u}$$

Ques (4) If $u+v+w = x+y+z$, $uv+vw+wu = x^2+y^2+z^2$ & $uvw = \dfrac{x^3+y^3+z^3}{3}$. Find $\left|\dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|$.

SOL. Let $f_1 \equiv u+v+w-x-y-z = 0$

$f_2 \equiv uv+vw+wu-x^2-y^2-z^2 = 0$

$f_3 \equiv 3uvw - x^3-y^3-z^3 = 0$

$$\because \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = (-1)^3 \frac{\left|\partial(f_1,f_2,f_3)/\partial(x,y,z)\right|}{\left|\partial(f_1,f_2,f_3)/\partial(u,v,w)\right|} \quad \text{—— (1)}$$

Now, $$\left|\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\right| = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -3x^2 & -3y^2 & -3z^2 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \longrightarrow \text{(VANDERMONDE'S DETERMINANT)}$$

$$= -6\,(x-y)(y-z)(z-x)$$

& $$\left|\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}\right| = \begin{vmatrix} \partial f_1/\partial u & \partial f_1/\partial v & \partial f_1/\partial w \\ \partial f_2/\partial u & \partial f_2/\partial v & \partial f_2/\partial w \\ \partial f_3/\partial u & \partial f_3/\partial v & \partial f_3/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ 3vw & 3uw & 3uv \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$C_1 \to C_1 - C_2$;
$C_2 \to C_2 - C_3$; $$= 3 \begin{vmatrix} 0 & 0 & 1 \\ v-u & w-v & u+v \\ w(v-u) & u(w-v) & uv \end{vmatrix}$$

$$= 3(v-u)(w-v) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & u+v \\ w & u & uv \end{vmatrix}$$

$$= 3(v-u)(w-v)(u-w)$$
$$= -3(u-v)(v-w)(w-u)$$

Put in ①; $$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Ques ⑤ If u, v & w are the roots of equation in 'l' given by:
$(l-x)^3 + (l-y)^3 + (l-z)^3 = 0$
Then find $\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|$.

SOL. $\because (l-x)^3 + (l-y)^3 + (l-z)^3 = 0$
$\Rightarrow l^3 - x^3 - 3lx(l-x) + l^3 - y^3 - 3ly(l-y) + l^3 - z^3 - 3lz(l-z) = 0$

$\Rightarrow 3l^3 - 3(x+y+z)l^2 + 3(x^2+y^2+z^2)l - (x^3+y^3+z^3) = 0$ ——— ①
i.e. $al^3 + bl^2 + cl + d = 0$

Given that u, v & w are roots of ①;

Sum of roots $= \frac{-b}{a} = \frac{3(x+y+z)}{3}$

$\Rightarrow u+v+w = x+y+z$

Let $f_1 \equiv u+v+w-x-y-z = 0$ ——— (i)

Product of roots taken in pair $= \frac{c}{a} = \frac{3(x^2+y^2+z^2)}{3}$

$\Rightarrow uv+vw+wu = x^2+y^2+z^2$

Let $f_2 \equiv uv+vw+wu-x^2-y^2-z^2 = 0$ ——— (ii)

Product of roots $= -\frac{d}{a} = \frac{x^3+y^3+z^3}{3}$

$\Rightarrow uvw = \frac{x^3+y^3+z^3}{3}$

Let $f_3 \equiv 3uvw - x^3 - y^3 - z^3 = 0$ ——— (iii)

(Similar to Ques ④); $$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Ques 6 If $x = e^v \cdot \sec u$, $y = e^v \cdot \tan u$. Then verify $|J||J^*| = 1$ where $|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ & $|J^*| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

SOL. $\because$ $x = e^v \cdot \sec u$ & $y = e^v \cdot \tan u$

$$\therefore \ |J^*| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$$

$$= e^{2v} \sec u \left(\tan^2 u - \sec^2 u\right)$$

$$= -e^{2v} \sec u \qquad (\because \sec^2\theta - \tan^2\theta = 1)$$

Express u & v in terms of x & y as follows:

On dividing ; $\frac{y}{x} = \frac{\tan u}{\sec u} = \sin u$

$$\Rightarrow \boxed{u = \sin^{-1}\left(\frac{y}{x}\right)}$$

Again ; $\frac{x}{e^v} = \sec u$ & $\frac{y}{e^v} = \tan u$

Square & subtract ; $\left(\frac{x}{e^v}\right)^2 - \left(\frac{y}{e^v}\right)^2 = \sec^2 u - \tan^2 u$

$$\Rightarrow \boxed{x^2 - y^2 = e^{2v}}$$

$$\Rightarrow \log_e (x^2 - y^2) = 2v$$

$$\Rightarrow \boxed{v = \frac{1}{2} \log_e (x^2 - y^2)}$$

Then $|J| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \cdot \left(\frac{-y}{x^2}\right) & \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \cdot \left(\frac{1}{x}\right) \\ \frac{x}{x^2 - y^2} & \frac{-y}{x^2 - y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \\ \frac{x}{x^2 - y^2} & \frac{-y}{x^2 - y^2} \end{vmatrix}$$

$$= \frac{1}{(x^2 - y^2)^{3/2}} \begin{vmatrix} -y/x & 1 \\ x & -y \end{vmatrix}$$

$$= \frac{y^2 - x^2}{x(x^2 - y^2)^{3/2}}$$

$$LHS = |J|\,|J^*| = \frac{y^2 - x^2}{x(x^2 - y^2)^{3/2}} \times -e^{2v} \sec u$$

$$= \frac{x^2 - y^2}{\cancel{x}(x^2 - y^2)^{3/2}} \times (x^2 - y^2) \times \frac{\cancel{x}}{e^v}$$

$$= \frac{(x^2 - y^2)^2}{(x^2 - y^2)^{3/2}} \times \frac{1}{\sqrt{x^2 - y^2}}$$

$$= \frac{(x^2 - y^2)^2}{(x^2 - y^2)^2}$$

$$= 1 = RHS.$$