## EMT Tute sheet Solution

1. From Gauss's law

$$\phi_E = \oint \vec{E} \cdot \hat{n}\, ds = \frac{q_{enc}}{\varepsilon_0}$$

$$q_{enc} = \text{Total enclosed charge}$$
$$= \sum_i q_i$$
$$= (3-2+2+4-1) \times 10^{-9}\ C$$
$$= 6 \times 10^{-9}\ C$$

and $\phi_E = \dfrac{q_{enc}}{\varepsilon_0} = \dfrac{6 \times 10^{-9} C}{8.85 \times 10^{-12} C/Vm} = 678\ Vm$

2. Charge on a line charge of length $L$ is given by

$$q = \lambda L$$

Thus $\quad \phi_E = \dfrac{q_{enc}}{\varepsilon_0} = \dfrac{\lambda L}{\varepsilon_0}$

$$L = \frac{\phi_E \varepsilon_0}{\lambda}$$

Since $L$ is twice the radius of the sphere

$$2R = L = \frac{\phi_E \varepsilon_0}{\lambda}$$

$$R = \frac{\phi_E \varepsilon_0}{2\lambda} = \frac{1.13 \times 10^{-3}\ Vm \times 8.85 \times 10^{-12}\ C/Vm}{2 \times 10^{-12}\ C/m}$$

$$= 5 \times 10^{-3}\ m.$$

3. Using $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $x^2+y^2=\rho^2$

$\vec{D} =$ $\quad$ " $= \frac{1}{\rho}(\cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y)$

$$D_\rho = \vec{D} \cdot \hat{a}_\rho = \frac{1}{\rho}\left[\cos\phi\,(\hat{a}_x \cdot \hat{a}_\rho) + \sin\phi\,(\hat{a}_y \cdot \hat{a}_\rho)\right]$$

$$= \frac{1}{\rho}(\cos^2\phi + \sin^2\phi)$$

$$= \frac{1}{\rho}$$

$$D_\phi = \vec{D}\cdot\hat{a}_\phi = \frac{1}{\rho}\left[\cos\phi\,(\hat{a}_x\cdot\hat{a}_\phi) + \sin\phi\,(\hat{a}_y\cdot\hat{a}_\phi)\right]$$
$$= \frac{1}{\rho}\left[\cos\phi(-\sin\phi) + \sin\phi\cos\phi\right]$$
$$= 0$$

$$\therefore\ \vec{D} = D_\rho\,\hat{a}_\rho = \frac{1}{\rho}\,\hat{a}_\rho$$

4. $$\text{Volume} = \int_3^{4.5}\int_{100^\circ}^{130^\circ}\int_3^{5} \rho\,d\rho\,d\phi\,dz = 2\pi = 6.28$$

$$\text{Total area} = 2\int_{100^\circ}^{130^\circ}\int_3^{5} \rho\,d\rho\,d\phi + \int_3^{4.5}\int_{100^\circ}^{130^\circ} 3\,d\phi\,dz + \int_3^{4.5}\int_{100^\circ}^{130^\circ} 5\,d\phi\,dz$$
$$+ 2\int_3^{4.5}\int_3^{5} d\rho\,dz = 20.7$$

5. $$\text{Volume} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}\int_2^{4} r^2\sin\theta\,dr\,d\theta\,d\phi = 2.91$$

$$\text{Area} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ} (4^2+2^2)\sin\theta\,d\theta\,d\phi + \int_2^{4}\int_{20^\circ}^{60^\circ} r(\sin 30^\circ + \sin 50^\circ)\,dr\,d\phi$$
$$+ 2\int_{30^\circ}^{50^\circ}\int_2^{4} r\,dr\,d\theta = 12.61$$

6. At point P: $x = -2,\ y = 6,\ z = 3$

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.32$$
$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43^\circ$$
$$z = 3$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = 7$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62^\circ$$

Thus $P(-2,6,3) = P(6.32, 108.43^\circ, 3) = P(7, 64.62^\circ, 108.43^\circ)$

$\vec{A} = 6\hat{a}_x + \hat{a}_y$ at point P in Cartesian System

For vector $\vec{A}$, $A_x = y$, $A_y = x+2$, $A_z = 0$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+2 \\ 0 \end{bmatrix}$$

$$A_\rho = y\cos\phi + (x+2)\sin\phi$$

$$A_\phi = -y\sin\phi + (x+2)\cos\phi$$

$$A_z = 0$$

Substituting $x = \rho\cos\phi$, $y = \rho\sin\phi$

$$\vec{A} = (A_\rho, A_\phi, A_z) = [\rho\cos\phi\sin\phi + (\rho\cos\phi+2)\sin\phi]\,\hat{a}_\rho$$
$$+ [-\rho\sin^2\phi + (\rho\cos\phi+2)\cos\phi]\,\hat{a}_\phi$$

At P

$$\rho = \sqrt{40}, \quad \tan\phi = \frac{6}{-2}$$

Hence $\cos\phi = \dfrac{-2}{\sqrt{40}}$, $\sin\phi = \dfrac{6}{\sqrt{40}}$

$$\vec{A} = -\frac{6}{\sqrt{40}}\hat{a}_\rho - \frac{38}{\sqrt{40}}\hat{a}_\phi = -0.9487\,\hat{a}_\rho - 6.008\,\hat{a}_\phi$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+2 \\ 0 \end{bmatrix}$$

$$A_r = y\sin\theta\cos\phi + (x+2)\sin\theta\sin\phi$$

$$A_\theta = y\cos\theta\cos\phi + (x+2)\cos\theta\sin\phi$$

$$A_\phi = -y\sin\phi + (x+2)\cos\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$

$$\vec{A} = (A_r, A_\theta, A_\phi)$$

$$= r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\hat{a}_r$$

$$+ r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\hat{a}_\theta$$

$$+ r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\hat{a}_\phi$$

At P

$$r = 7, \quad \tan\phi = \frac{6}{-2}, \quad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence $\cos\phi = \frac{-2}{\sqrt{40}}$, $\sin\phi = \frac{6}{\sqrt{40}}$, $\cos\theta = \frac{3}{7}$, $\sin\theta = \frac{\sqrt{40}}{7}$

$$\therefore \vec{A} = -\frac{6}{7}\hat{a}_r - \frac{18}{7\sqrt{40}}\hat{a}_\theta - \frac{38}{\sqrt{40}}\hat{a}_\phi$$

$$\vec{A} = -0.8571\hat{a}_r - 0.4066\hat{a}_\theta - 6.008\hat{a}_\phi$$

9. The magnitude of electric field due to either of the two line charges is $|\vec{E}| = \frac{\rho_l}{2\pi\varepsilon_0 r}$. The resultant electric field $E_r$ is then

$$E_r = |\vec{E}|\cos 45^\circ\hat{a}_y = 2\frac{0.4\times10^{-6}}{2\pi\left(\frac{10^{-9}}{36\pi}\right)(4\sqrt{2})}\frac{1}{\sqrt{2}}\hat{a}_y$$

$$= 1800\hat{a}_y \text{ V/m}$$

10.

$$\vec{R} = -x\hat{a}_x - y\hat{a}_y + 3\hat{a}_z, \quad dQ = \rho_s\, dS = 2\times10^{-9}|\vec{R}|^3 dS$$

$$\therefore d\vec{E} = \frac{dQ}{4\pi\varepsilon_0|\vec{R}|^3}\vec{R} = \frac{10^{-9}\, dS}{2\pi\varepsilon_0}(-x\hat{a}_x - y\hat{a}_y + 3\hat{a}_z)$$

As a result of symmetry, only the z component of $\vec{E}$ exists:

$$\vec{E} = \left(\frac{3\times10^{-9}}{2\pi\varepsilon_0}\hat{a}_z\frac{\text{V}}{\text{m}^3}\right)(4\text{ m})^2 = 864\hat{a}_z \text{ V/m}$$

$\vec{E} = -\vec{\nabla} V = -\frac{dV}{dz}\hat{a}_z = 600z\,\hat{a}_z$

$\vec{D} = \varepsilon_0\varepsilon_r\vec{E} = \frac{10^{-9}}{36\pi}(2.4)\,600z\,\hat{a}_z = 12.73z\,\hat{a}_z\ nC/m^2$

$\rho_v = \vec{\nabla}\cdot\vec{D} = \frac{\partial D_z}{\partial z} = 12.73\ nC/m^3$

(b) $\chi_e = \varepsilon_r - 1 = 1.4$

$\vec{P} = \chi_e\varepsilon_0\vec{E} = \frac{\chi_e\vec{D}}{\varepsilon_r} = \frac{1.4}{2.4}(12.73z\,\hat{a}_z)$

$= 7.427z\,\hat{a}_z\ nC/m^2$

$\rho_{pv} = -\vec{\nabla}\cdot\vec{P} = -7.427\,nC/m^3$

12. $I = \int\vec{J}\cdot d\vec{S} = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{500}{\rho}\,\rho\,d\phi\,d\rho = 500(2\pi a)$

$= 1000\pi\times 1.6\times 10^{-3} = 1.6\pi = 5.026\ A$

13. ~~Ele~~ Relaxation time $T_r = \frac{\varepsilon}{\sigma}$

14. (a) $\vec{H} = \vec{H_x} + \vec{H_y} = 2\vec{H_x}$

$\vec{H_x} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\hat{a}_\phi$

where $\hat{a}_\phi = -\hat{a}_x\times\hat{a}_y = -\hat{a}_z$

$\alpha_1 = 180^\circ,\quad \alpha_2 = 45^\circ$

$\vec{H_x} = \frac{5}{4\pi(\rho)}(\cos45^\circ - \cos180^\circ)(-\hat{a}_z) = -0.6792\,\hat{a}_z\ A/m$

(b) $\vec{H} = \vec{H_x} + \vec{H_y} = \frac{5}{4\pi(2)}(1-0)\,\hat{a}_\phi,\quad \hat{a}_\phi = -\hat{a}_x\times -\hat{a}_y = \hat{a}_z$

$= 198.9\,\hat{a}_z\ mA/m$

$\vec{H_y} = 0$ since $\alpha_1 = \alpha_2 = 0$

$\vec{H} = 0.1989\,\hat{a}_z\ A/m$

(c) $\vec{H_x} = \frac{5}{4\pi(2)}(1-0)(-\hat{a}_x\times\hat{a}_z) = 198.9\,\hat{a}_y\ mA/m$

$\vec{H_y} = \frac{5}{4\pi(2)}(1-0)(\hat{a}_y\times\hat{a}_z) = 198.9\,\hat{a}_x\ mA/m$

$\vec{H} = (0.1989\,\hat{a}_x + 0.1989\,\hat{a}_y)\ A/m$

15. (a) $\vec{J} = \vec{\nabla} \times \vec{H}$ $\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z(x^2+y^2) & -y^2xz & -4x^2y^2 \end{vmatrix}$

$$= (8x^2y + xy^2)\hat{a}_x + [y(x^2+y^2) - 4xy^2]\hat{a}_y + [-y^2z - z(x^2+y^2)]\hat{a}_z$$

At $(5,2,-3)$, $x = 5$, $y = 2$, $z = -3$

$$\vec{J} = 420\,\hat{a}_x - 22\,\hat{a}_y + 99\,\hat{a}_z \quad A/m^2$$

(b) $I = \int \vec{J} \cdot d\vec{s} = \iint (8x^2y + xy^2)\, dy\, dz \,\big|_{x=-1}$

$$= \int_0^2 dz \int_0^2 (8y - y^2)\, dz = 2\left(4y^2 - \frac{y^3}{3}\right)\Big|_0^2$$

$$= 4\left(16 - \frac{8}{3}\right) = 53.33\ A$$

(c) $\vec{B} = \mu \vec{H}$, $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{\nabla} \cdot \vec{H} = 0$

16. $V = -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{s} = -\frac{\partial \vec{B}}{\partial t} \cdot \vec{S}$

$$= 3770 \sin 377t \times \pi (0.2)^2 \times 10^{-3}$$

$$= 0.4738 \sin 377t \quad V$$

17. $V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$, $d\vec{l} = d\rho\, \hat{a}_\rho$, $\vec{u} = \rho \frac{d\phi}{dt} \hat{a}_\phi = \rho w \hat{a}_\phi$

$$\vec{u} \times \vec{B} = \rho w\, \hat{a}_\phi \times B_0 \hat{a}_z = B_0 \rho w\, \hat{a}_\rho$$

$$V_{emf} = \int_{\rho=0}^{l} B_0 \rho w \hat{a}_\rho \cdot d\rho\, \hat{a}_\rho = B_0 w \frac{\rho^2}{2}\Big|_0^l = \frac{1}{2} B_0 w l^2$$

18. $V_{emf} = -\frac{d\psi}{dt} = -\frac{dB}{dt} S = 40 \times 10^4 \sin(10^4 t) \cdot 10^{-3} \times 20 \times 10^{-4}$

$$= 0.8 \sin 10^4 t$$

$$I = \frac{V_{emf}}{R} = 0.2 \sin 10^4 t \quad A$$

19. (a) $V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$, $d\vec{l} = dy\, \hat{a}_y$, $\vec{u} \times \vec{B} = 2\hat{a}_x \times 0.1\hat{a}_z$
$= -0.2\, \hat{a}_y$

$V_{emf} = -\int 0.2\, dy = -0.2y$, $y = ut = 2t$, $V_{emf} = -0.4t$ V

(b) $\vec{u} \times \vec{B} = 2\hat{a}_x \times 0.5x\, \hat{a}_z = -x\hat{a}_y \Rightarrow V_{emf} = -\int x\, dy = -\int y\, dy$
$= -\frac{y^2}{2}$

But $x = y = ut = 2t$ $\therefore V_{emf} = -2t^2$ V