DOUBLE INTEGRALS

WHAT IS A DOUBLE INTEGRAL?

An integral of the form $\iint_R f(x,y)\,dx\,dy$

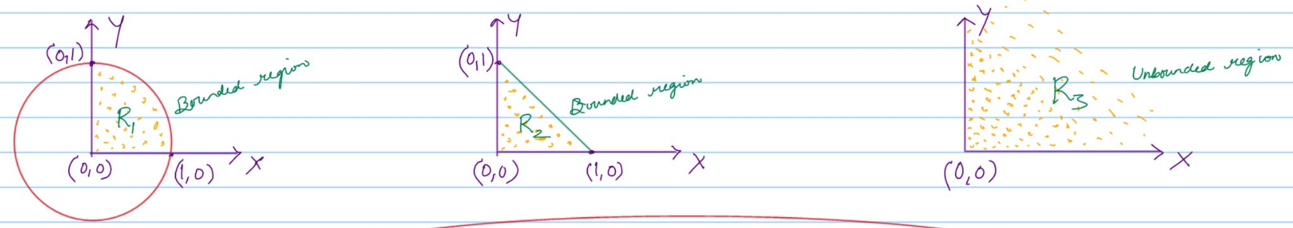
where $f(x,y)$ is a function of 2 variables x & y defined on a subset R of XY-plane.

Examples:

$$I_1 = \iint_{R_1} (x^2+y^2)\,dx\,dy \;;\; \text{where } R_1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$$

$$I_2 = \iint_{R_2} (x+y)\,dx\,dy \;;\; \text{where } R_2 = \{(x,y)\in\mathbb{R}^2 : x+y \le 1,\ x \ge 0,\ y \ge 0\}$$

$$I_3 = \iint_{R_3} (xy)\,dx\,dy \;;\; \text{where } R_3 = \{(x,y)\in\mathbb{R}^2 : x \ge 0,\ y \ge 0\}$$



★★★ DRAWING REGIONS IS IMPORTANT

★★★ FINDING LIMITS OF 'x' AND 'y' IS IMPORTANT

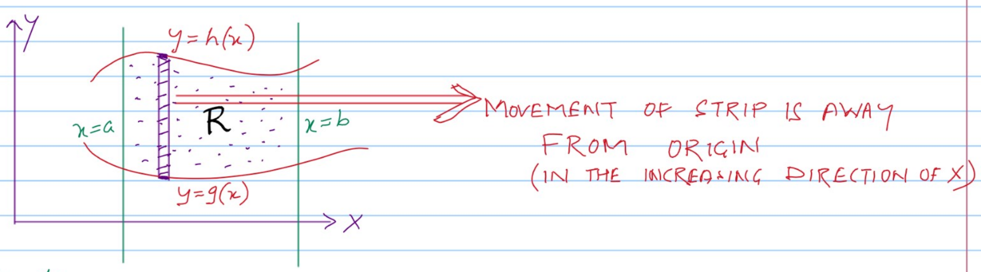
Double integral is broadly classified into 2 cases:

CASE 1: Draw a vertical strip (see figure)

Limits of y: $y=g(x)$ to $y=h(x)$

Limits of x: $x=a$ to $x=b$

Then $I = \int_{x=a}^{b}\int_{y=g(x)}^{h(x)} f(x,y)\,dy\,dx$

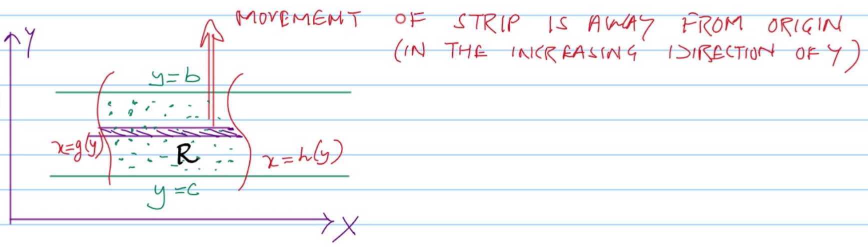


CASE 2: Draw a horizontal strip.

Limits of x: $x=g(y)$ to $x=h(y)$

Limits of y: $y=c$ to $y=d$

Then $I = \int_{y=c}^{d}\int_{x=g(y)}^{h(y)} f(x,y)\,dx\,dy$



NOTE:
1. The base & head of strip gives inner limits
2. The motion of strip gives the outer limits.
3. The base of strip is always near the origin
4. Motion is taken in the direction of increase in 'x' or 'y'.
5. Outer limits are always constant limits.

Ques (1) Evaluate $\int_0^3\int_0^1 (x^2+3y^2)\,dy\,dx$

SOL. Let $I = \int_{x=0}^{3}\int_{y=0}^{1}(x^2+3y^2)\,dy\,dx$

$$= \int_{x=0}^{3}\left[x^2y + y^3\right]_0^1 dx$$

$$= \int_{x=0}^{3}(x^2+1)\,dx$$

$$= \left[\frac{x^3}{3}+x\right]_0^3$$

$$= \frac{81}{3}+3$$

$$= 32$$

Ques (2) Evaluate: (a) $\int_2^a\int_2^b \frac{dx\,dy}{xy}$ (b) $\int_0^1\int_0^2 (xy)\,dx\,dy$ (c) $\int_0^1\int_0^x e^{y/x}\,dy\,dx$

SOL. (a) $I = \int_{y=2}^{a}\int_{x=2}^{b}\left(\frac{1}{xy}\right)dx\,dy$

$$= \int_{y=2}^{a}\frac{1}{y}\left[\log_e(x)\right]_2^b dy$$

$$= \left[\log_e(b)-\log_e(2)\right]\int_{y=2}^{a}\frac{dy}{y}$$

$$= \log_e\left(\frac{b}{2}\right)\left[\log_e(y)\right]_2^a$$

$$= \log_e\left(\frac{b}{2}\right)\log_e\left(\frac{a}{2}\right)$$

REMARK: $\int_{y=c}^{d}\int_{x=a}^{b} f(x).g(y)\,dx\,dy = \int_{y=c}^{d} g(y)\,dy \times \int_{x=a}^{b} f(x)\,dx$

(b) $I = \int_{y=0}^{1}\int_{x=0}^{2}(xy)\,dx\,dy = \int_{y=0}^{1} y\,dy \times \int_{x=0}^{2} x\,dx = \frac{1}{2}\times\frac{4}{2} = 1$

(c) $I = \int_{x=0}^{1}\int_{y=0}^{x} e^{y/x}\,dy\,dx$

$$= \int_{x=0}^{1}\left[\frac{e^{y/x}}{1/x}\right]_0^x dx$$

$$= \int_{x=0}^{1} x\,(e^1-e^0)\,dx = (e-1)\int_0^1 x\,dx = \frac{(e-1)}{2}$$

Ques (3) Evaluate (a) $\int_0^1\int_0^{\sqrt{1+y^2}}\frac{dx\,dy}{1+x^2+y^2}$ (b) $\int_0^1\int_0^{\sqrt{1-y^2}/2}\frac{dx\,dy}{1-x^2-y^2}$

SOL. (a) Let $I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1+y^2}}\frac{1}{x^2+(\sqrt{y^2+1})^2}\,dx\,dy$

$$= \int_{y=0}^{1}\left[\frac{1}{\sqrt{y^2+1}}\tan^{-1}\left(\frac{x}{\sqrt{y^2+1}}\right)\right]_0^{\sqrt{1+y^2}} dy \qquad \left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$= \int_{y=0}^{1}\frac{1}{\sqrt{y^2+1}}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dy$$

$$= \frac{\pi}{4}\int_{y=0}^{1}\frac{1}{\sqrt{y^2+1}}\,dy$$

$$= \frac{\pi}{4}\left[\log_e\left|y+\sqrt{y^2+1}\right|\right]_0^1 \qquad \left[\because \int\frac{dx}{\sqrt{x^2+a^2}} = \log_e\left[x+\sqrt{x^2+a^2}\right]+C\right]$$

$$= \frac{\pi}{4}\left[\log_e(1+\sqrt{2})\right]$$

(b) Let $I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}/2} \frac{1}{(\sqrt{1-y^2})^2 - x^2}\, dx\, dy$

$$= -\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}/2}\left[\frac{dx}{x^2-(\sqrt{1-y^2})^2}\right] dy \qquad \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log_e\left|\frac{x-a}{x+a}\right|\right]$$

$$= -\int_{y=0}^{1}\left[\frac{1}{2\sqrt{1-y^2}}\log_e\left|\frac{x-\sqrt{1-y^2}}{x+\sqrt{1-y^2}}\right|\right]_{x=0}^{\sqrt{1-y^2}/2} dy$$

$$= -\frac{1}{2}\int_{y=0}^{1}\frac{1}{\sqrt{1-y^2}}\left\{\log_e\left|\frac{-\sqrt{1-y^2}/2}{\frac{3}{2}\sqrt{1-y^2}}\right| - \log_e|-1|\right\} dy$$

$$= -\frac{1}{2}\int_{y=0}^{1}\frac{1}{\sqrt{1-y^2}}\left\{\log_e\left(\frac{1}{3}\right) - \underbrace{\cancel{\log_e(1)}}_{0}\right\} dy$$

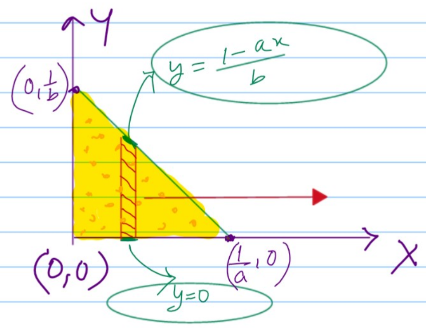
$$= -\frac{1}{2}\log_e\left(\frac{1}{3}\right)\int_{y=0}^{1}\frac{dy}{\sqrt{1-y^2}}$$

$$= +\frac{1}{2}\log_e(3)\left[\sin^{-1}(y)\right]_0^1$$

$$= \frac{1}{2}\log_e(3)\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}\log_e(3)$$

Ques (4) Evaluate $\iint_R e^{ax+by}\, dx\, dy$ where $R = \{(x,y)\in\mathbb{R}^2;\ x\geq 0,\ y\geq 0,\ ax+by\leq 1\}$

Sol. Draw region R as follows:



Limits of y: $y=0$ to $\left(\frac{1-ax}{b}\right)$

Limits of x: $x=0$ to $\frac{1}{a}$

$$I = \int_{x=0}^{1/a}\int_{y=0}^{\left(\frac{1-ax}{b}\right)} e^{ax+by}\, dy\, dx$$

$$= \int_{x=0}^{1/a}\int_{y=0}^{\left(\frac{1-ax}{b}\right)}\left[e^{ax}\cdot e^{by}\, dy\right] dx$$

$$= \int_{x=0}^{1/a} e^{ax}\left[\frac{e^{by}}{b}\right]_0^{\frac{1-ax}{b}} dx$$

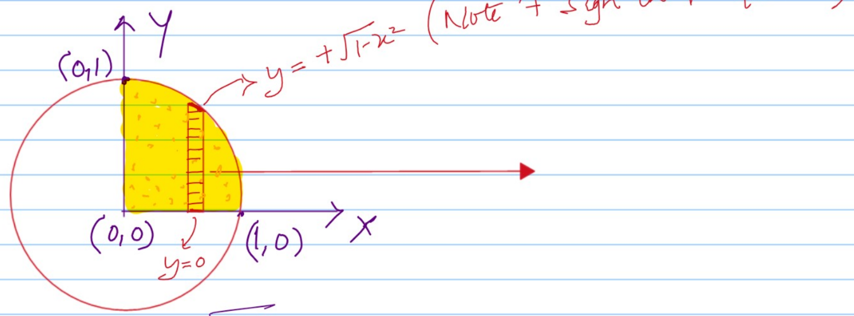
$$= \frac{1}{b}\int_{x=0}^{1/a} e^{ax}\left(e^{1-ax}-e^0\right) dx$$

$$= \frac{1}{b}\int_{x=0}^{1/a}\left(e-e^{ax}\right) dx = \frac{1}{b}\left[ex - \frac{e^{ax}}{a}\right]_0^{1/a} = \frac{1}{b}\left[\left(\frac{e}{a}-\frac{e}{a}\right)-\left(0-\frac{1}{a}\right)\right] = \frac{1}{ab}$$

Ques (5) Evaluate $\iint_R \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$, where $R = \{(x,y)\in\mathbb{R}^2 \mid x\geq 0,\ y\geq 0,\ x^2+y^2\leq 1\}$

Sol. Limit of y: $y=0$ to $+\sqrt{1-x^2}$

Limits of x: $x=0$ to 1



$$\therefore\ I=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x\left[\frac{y}{\sqrt{1-y^2}}\,dy\right]dx$$

$$=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[(1-y^2)^{-\frac{1}{2}}(-2y)\,dy\right]\left(\frac{-x}{2}\right)dx$$

$$=\int_{x=0}^{1}\left[\frac{(1-y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^{\sqrt{1-x^2}}\left(\frac{-x}{2}\right)dx \qquad \left[\because \int [f(x)]^n f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}+c\right]$$

$$=-\int_{x=0}^{1}(x-1)\,x\,dx$$

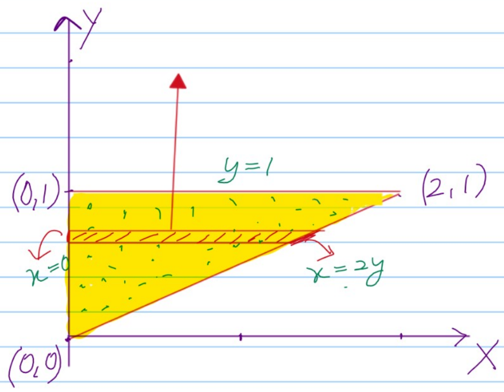
$$=-\int_{x=0}^{1}(x^2-x)\,dx$$

$$=-\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$=\frac{1}{6}$$

Ques 6) Evaluate $\iint_R e^{y^2}\,dx\,dy$, where R is region bounded by a triangle with vertices $(0,0)$, $(2,1)$ & $(0,1)$.

SOL.



CAUTION: Since $\int e^{y^2}dy$ is difficult but $\int e^{y^2}dx$ is easy, hence take inner limits of 'x' only. This indicates that the motion of strip must be parallel to Y-axis.

Limits of x : $x=0$ to $x=2y$

" " y : $y=0$ to $y=1$

Then $$I=\int_{y=0}^{1}\int_{x=0}^{2y}\left[e^{y^2}dx\right]dy$$

$$=\int_{y=0}^{1}e^{y^2}\left[x\right]_0^{2y}dy$$

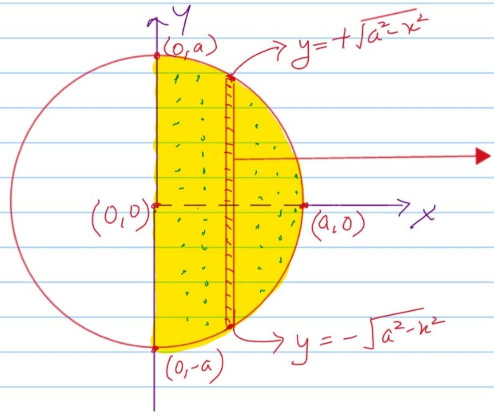
$$=\int_0^1 e^{y^2}(2y)\,dy$$

Let $y^2=t \Rightarrow 2y\,dy=dt$

$$\therefore\ I=\int_0^1 e^t\,dt = e^1-e^0=(e-1)$$

Ques (7) Evaluate $\iint_R (a-x)^2\,dx\,dy$, where $R = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ x^2+y^2 \leq a^2\}$

SOL.



Limits of y : $y = -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

" " x ; $x = 0$ to a

$$\therefore\ I = \int_{x=0}^{a} \left[\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a-x)^2\,dy\right] dx$$

$$I = \int_{x=0}^{a} (a-x)^2 \Big[y\Big]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx$$

$$I = 2\int_{x=0}^{a} (a-x)^2\sqrt{a^2-x^2}\,dx$$

Let $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

$$\therefore\ I = 2\int_0^{\pi/2} (a-a\sin\theta)^2\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$

$$= 2a^4\int_0^{\pi/2} (1-\sin\theta)^2\cos^2\theta\,d\theta$$

$$= 2a^4\int_0^{\pi/2} [\cos^2\theta + \sin^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta]\,d\theta$$

$$= 2a^4[I_1(\text{say}) + I_2(\text{say}) - 2I_3(\text{say})] \quad \text{—— (1)}$$

Here $I_1 = \int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta + \frac{1}{2}\int_0^{\pi/2}(\cos 2\theta)\,d\theta$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\Big[\sin 2\theta\Big]_0^{\pi/2} = \frac{\pi}{4}$$

& $I_2 = \int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$

$$= \frac{1}{4}\int_0^{\pi/2}(2\sin\theta\cos\theta)^2\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{8}\left[\frac{\pi}{2} - \underbrace{\int_0^{\pi/2}\cos(4\theta)\,d\theta}_{\to\ 0\ (\text{See }\star)}\right]$$

$$= \frac{\pi}{16}$$

★ REMEMBER

$$\int_0^{2\pi/n}(\cos n\theta)\,d\theta = 0 \quad (k \in \mathbb{N})$$

&

$$\int_0^{2\pi/n}(\sin n\theta)\,d\theta = 0 \quad (k \in \mathbb{N})$$

& $I_3 = \int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta$

Let $\cos\theta = t \Rightarrow \sin\theta\,d\theta = -dt$

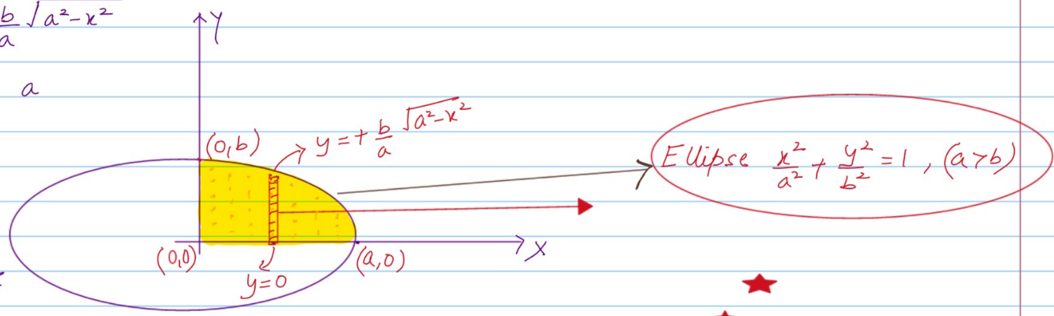
$$\therefore\ I_3 = \int_1^0 t^2(-dt) = \int_0^1 t^2\,dt = \frac{1}{3}$$

Substitute in (1) ; $I = 2a^4\left[\frac{\pi}{4} + \frac{\pi}{16} - \frac{2}{3}\right] = 2a^4\left(\frac{5\pi}{16} - \frac{2}{3}\right)$

Ques 8 Evaluate $\iint_R xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{n/2} dx\,dy$, where $R=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; x\geq 0,\ y\geq 0,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\right\}$

SOL. Limits of y : $y=0$ to $\frac{b}{a}\sqrt{a^2-x^2}$

Limits of x : $x=0$ to a



$$\therefore\ I=\int_{x=0}^{a}\left(\frac{b^2x}{2}\right)\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\left[\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{n/2}\left(\frac{2y}{b^2}\right)dy\right]dx \longleftarrow \boxed{\text{NOTE THIS STEP}}$$

$$=\int_{x=0}^{a}\left(\frac{b^2x}{2}\right)\left[\frac{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}}{\left(\frac{n}{2}+1\right)}\right]_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx \qquad \left[\because \int [f(x)]^n f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}+C\right]$$

$$=\int_{x=0}^{a}\left(\frac{b^2x}{2}\right)\left(\frac{2}{n+2}\right)\left[1-\left(\frac{x}{a}\right)^{n+2}\right]dx$$

$$=\left(\frac{b^2}{n+2}\right)\int_{x=0}^{a}\left(x-\frac{x^{n+3}}{a^{n+2}}\right)dx$$

$$=\left(\frac{b^2}{n+2}\right)\left[\frac{x^2}{2}-\frac{x^{n+4}}{a^{n+2}(n+4)}\right]_0^a$$

$$=\left(\frac{b^2}{n+2}\right)\left(\frac{a^2}{2}-\frac{a^2}{n+4}\right)$$

$$=\frac{a^2b^2}{2(n+4)} \quad (\text{provided } n\neq -4)$$

CHANGE OF ORDER

WHAT IS CHANGE OF ORDER?? WHY IS IT REQUIRED???

Consider $I=\int_0^2\int_{x/2}^{1} e^{y^2}\,dy\,dx$

Observe that the innermost limits are of 'y' which enforces the integration of function e^{y^2} w.r.t. 'y' first (a difficult task). On the other hand, integration of e^{y^2} w.r.t. 'x' first is easy.
Hence performing the integration w.r.t. 'x' first and then w.r.t. 'y' later is preferred and this is called as change of order of integration.

HOW TO CHANGE THE ORDER OF INTEGRATION???

1. Draw the region R of integration from the original limits.
2. Rotate the strip inside R through 90°.
3. Find new limits of 'x' & 'y' and integrate.

Applying this to $I=\int_{x=0}^{2}\int_{y=x/2}^{1} e^{y^2}\,dy\,dx$

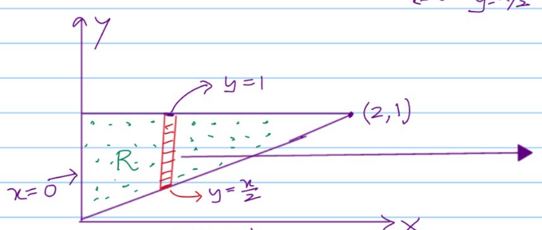


Fig (a)

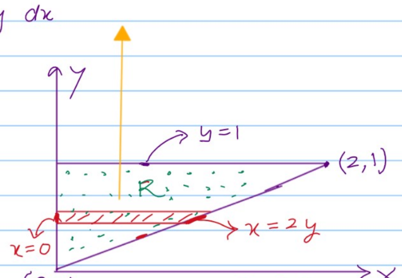


Fig (b)

Limits of x : 0 to $2y$
" " y : 0 to 1.

$$\therefore \quad I = \int_{y=0}^{1}\int_{x=0}^{2y} e^{y^2}\,dx\,dy$$

$$= \int_{y=0}^{1} e^{y^2}\left[x\right]_0^{2y} dy$$

$$= \int_{y=0}^{1} 2y\, e^{y^2}\,dy$$

Put $y^2 = t \Rightarrow 2y\,dy = dt$

$$\therefore \quad I = \int_0^1 e^t\,dt = \left[e^t\right]_0^1 = e^1 - e^0 = (e-1)$$

Ques (1) Evaluate: (a) $\displaystyle\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$ (b) $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$ (c) $\displaystyle\int_0^{\infty}\int_0^{x} x\, e^{-x^2/y}\,dy\,dx$

SOL. NOTE: Integration w.r.t. 'y' first is difficult but integration w.r.t. 'x' first is easy.

(a)

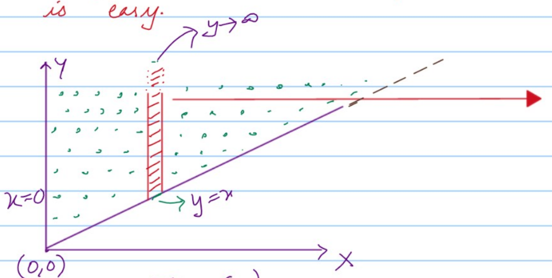


Fig. (a)

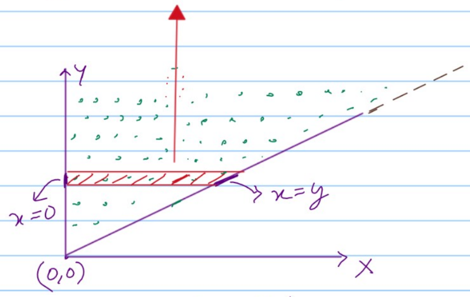


Fig. (b)

Limits of x : 0 to y
" " y : 0 to ∞

$$\therefore \quad I = \int_{y=0}^{\infty}\int_{x=0}^{y} \left(\frac{e^{-y}}{y}\right) dx\,dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y}\right)\left[x\right]_0^{y} dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}}\cdot\cancel{y}\;dy$$

$$= -\left[e^{-y}\right]_0^{\infty}$$

$$= -\left[\lim_{y\to\infty} e^{-y} - e^{-0}\right]$$

$$= 1 \qquad \left(\because \lim_{y\to\infty} e^{-y} = \lim_{y\to\infty}\frac{1}{e^y} = 0\right)$$

(b) $$I = \int_{x=0}^{\pi}\int_{y=x}^{\pi} \frac{\sin y}{y}\,dy\,dx$$

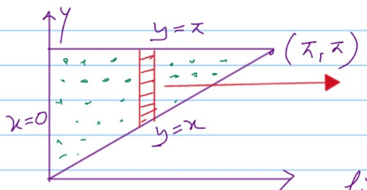


fig. (a)

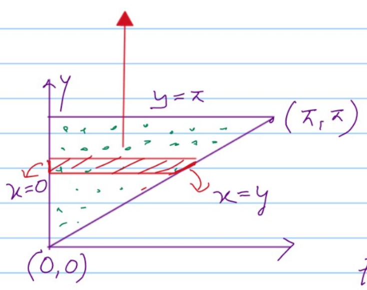
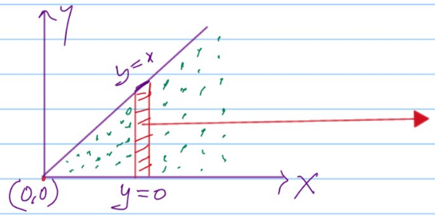
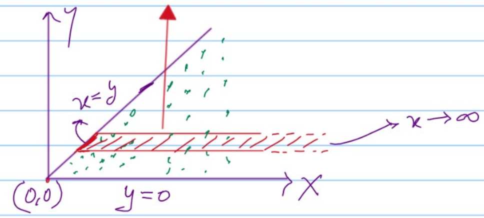


fig. (b)

$$\therefore \quad I = \int_{y=0}^{\pi}\int_{x=0}^{y} \left(\frac{\sin y}{y}\right) dx\,dy = \int_0^{\pi} \sin y\,dy = 2$$

(c) $$I = \int_{x=0}^{\infty}\int_{y=0}^{x} x\, e^{-x^2/y}\,dy\,dx$$





Limits of x : y to ∞
" " y : 0 to ∞

$$I = \int_{y=0}^{\infty}\int_{x=y}^{\infty} x\, e^{-x^2/y}\, dx\, dy$$

Let $\frac{x^2}{y} = t \Rightarrow \frac{2x}{y}\, dx = dt \Rightarrow x\, dx = \frac{y\, dt}{2}$

$$\therefore\; I = \int_{y=0}^{\infty}\left[\int_{t=y}^{\infty} e^{-t}\, \frac{dt}{2}\right] y\, dy$$

$$= -\frac{1}{2}\int_{y=0}^{\infty}\left[e^{-t}\right]_y^{\infty} y\, dy$$

$$= -\frac{1}{2}\int_0^{\infty}\left[0 - e^{-y}\right] y\, dy$$

$$= \frac{1}{2}\int_0^{\infty} y e^{-y}\, dy$$

$$= \frac{1}{2}\, {}^{\star\star}\!\left[(y)\left(\frac{e^{-y}}{-1}\right) - (1)\left(\frac{e^{-y}}{+1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\left\{-\lim_{y\to\infty}\left(\frac{y}{e^y}\right)^{\star} - \lim_{y\to\infty}\left(\frac{1}{e^y}\right)\right\} - \{0 - 1\}\right]$$

$$= \frac{1}{2}\left[\{0 - 0\} + 1\right]$$

$$= \frac{1}{2}$$

★ NOTE

$\because \lim_{y\to\infty}\left(\frac{y}{e^y}\right)$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{y\to\infty} \frac{1}{e^y}$ (LH-rule)

$= \frac{1}{\infty} \to 0$

★★ SHORT-CUT

$\int u v\, dx = (u)(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + (u'''')(v_5) - \ldots$

u : polynomial function

Ex- $\int x^2 \sin x\, dx$

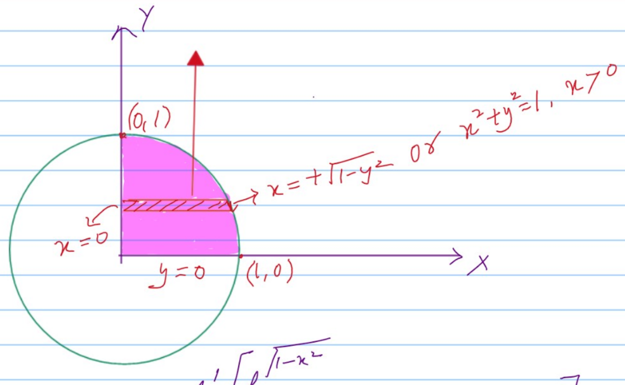
$= (x^2)(-\cos x) - (2x)(-\sin x) + (2)(+\cos x) - 0$

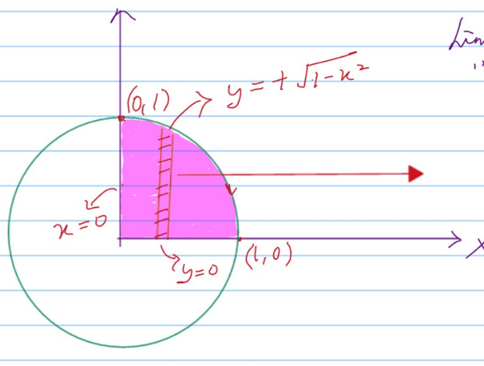
$= -x^2\cos x + 2x\sin x + 2\cos x$

Ques② Prove that ⓐ $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dx\, dy = \frac{\pi^3}{16}$

ⓑ $\displaystyle\int_0^{1/2}\int_0^{\sqrt{1-4y^2}} \frac{(1+x^2)}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dx\, dy = \frac{\pi^2}{8}$ (DO YOURSELF)

Sol. ⓐ $\displaystyle I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dx\, dy$





Limits of y : 0 to $\sqrt{1-x^2}$
" " x : 0 to 1

$$\therefore\; I = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}} \frac{1}{\sqrt{\left(\sqrt{1-x^2}\right)^2 - y^2}}\, dy\right] \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^{1}\left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_{y=0}^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2}\int_{\pi/2}^{0} u(-du) \quad \text{where } u = \cos^{-1}x \Rightarrow du = -dx/\sqrt{1-x^2}$$

$$= \frac{\pi}{2}\left[\frac{u^2}{2}\right]_0^{\pi/2} = \frac{\pi^3}{16}$$

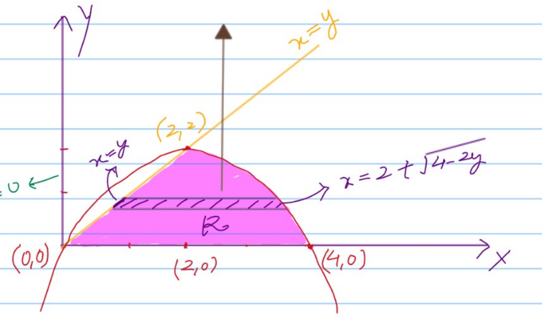
Ques 3 Apply change of order of integration

(a) To evaluate $\int_0^2 \int_y^{2+\sqrt{4-2y}} f(x,y)\, dx\, dy$.

(b) To prove $\int_{-a}^{a} \int_0^{y^2/a} f(x,y)\, dx\, dy = \int_0^a \int_{-a}^{-\sqrt{ax}} f(x,y)\, dy\, dx + \int_0^a \int_{\sqrt{ax}}^{a} f(x,y)\, dy\, dx$

(DO YOURSELF)

SOL. (a) Let $I = \int_{y=0}^{2} \int_{x=y}^{2+\sqrt{4-2y}} f(x,y)\, dx\, dy$



From upper inner limit:

$x = 2 + \sqrt{4-2y}$

$\Rightarrow x-2 = \sqrt{4-2y}$

$\Rightarrow (x-2)^2 = 4-2y$

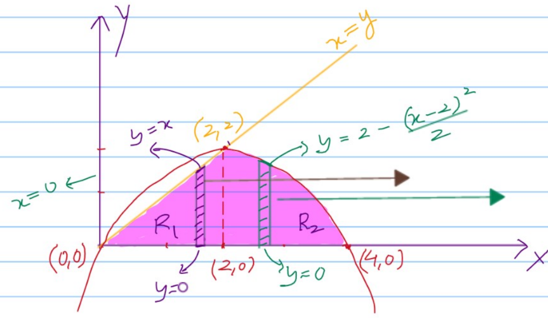
$\Rightarrow (x-2)^2 = -2(y-2)$

$\Rightarrow X^2 = -4aY$ parabola with vertex $X=0,\ Y=0$

$\Rightarrow x-2=0,\ y-2=0$

$\Rightarrow x=2,\ y=2$

$\therefore$ Vertex at $(2,2)$



$\boxed{\text{NOTE:}}$ There are 2 subregions R_1 (where the top of the strip lies below line) and R_2 (where top of the strip lies below parabola).

For Region R_1; limits of y: 0 to x

" " x: 0 to 2

For Region R_2; limits of y: 0 to $2-\frac{(x-2)^2}{2}$

" " ": 2 to 4

$$\therefore I = \iint_{R_1} f(x,y)\, dy\, dx + \iint_{R_2} f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{x} f(x,y)\, dy\, dx + \int_{x=2}^{4} \int_{y=0}^{2-\frac{(x-2)^2}{4}} f(x,y)\, dy\, dx.$$

Ques 4 Evaluate $\iint_R xy\, dx\, dy$; where $R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2-2x \geq 0,\ y^2 \leq 2x,\ y \geq x\}$

SOL. $\because x^2+y^2-2x = 0$

$\Rightarrow x^2-2x+1+y^2 = 1$

$\Rightarrow (x-1)^2 + y^2 = 1^2$ i.e. $(x-h)^2+(y-k)^2 = r^2$ a circle centred at $(1,0)$ & radius 1

Also, $y^2 = 2x$ (a parabola with X-axis as its axis)

& $y = x$ (a line through origin)

NOTE: Find the points of intersection of circle and parabola:

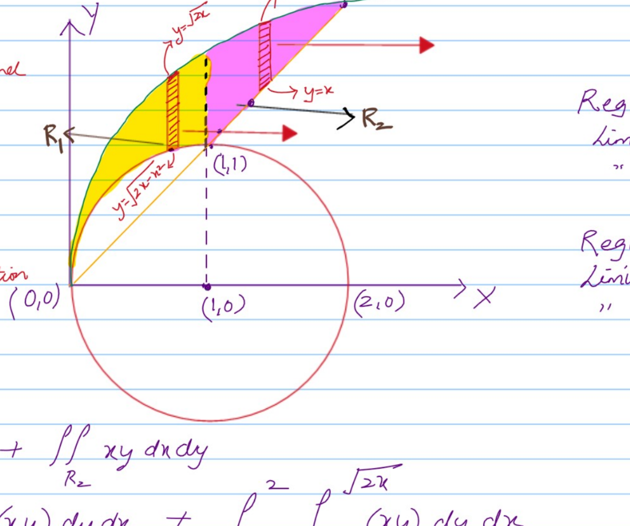
$x^2+y^2-2x=0$ —— ①

$y^2 = 2x$ —— ②

② in ①; $x^2 = 0 \Rightarrow x = 0$

$x=0$ in ②; $y=0$

$\therefore (0,0)$ is pt. of intersection



Region R_1:

Limits of y: $\sqrt{2x-x^2}$ to $\sqrt{2x}$

" " x: 0 to 1

Region R_2:

Limits of y: x to $\sqrt{2x}$

" " x: 1 to 2

$$\therefore I = \iint_{R_1} xy\, dx\, dy + \iint_{R_2} xy\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=\sqrt{2x-x^2}}^{\sqrt{2x}} (xy)\, dy\, dx + \int_{x=1}^{2} \int_{y=x}^{\sqrt{2x}} (xy)\, dy\, dx$$

$$= \int_0^1 \left(\frac{2x}{2} - \frac{2x-x^2}{2}\right) x\,dx + \int_{x=1}^{2} x\left(\frac{2x}{2} - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \frac{x^3}{2}\,dx + \int_1^2 \left(x^2 - \frac{x^3}{2}\right) dx$$

$$= \frac{1}{8} + \left(\frac{7}{3} - \frac{15}{8}\right)$$

$$= \frac{7}{3} - \frac{7}{4}$$

$$= \frac{7}{12}$$

Que ⑤ Evaluate $\iint_R \frac{dx\,dy}{x^4+y^2}$, where R is bounded by $y \geq x^2$ and $x \geq 1$.

SOL. Limits of y: x^2 to ∞

" " x: 1 to ∞

$$\therefore I = \int_{x=1}^{\infty}\int_{y=x^2}^{\infty} \frac{dx\,dy}{y^2+(x^2)^2}\,dy\,dx$$

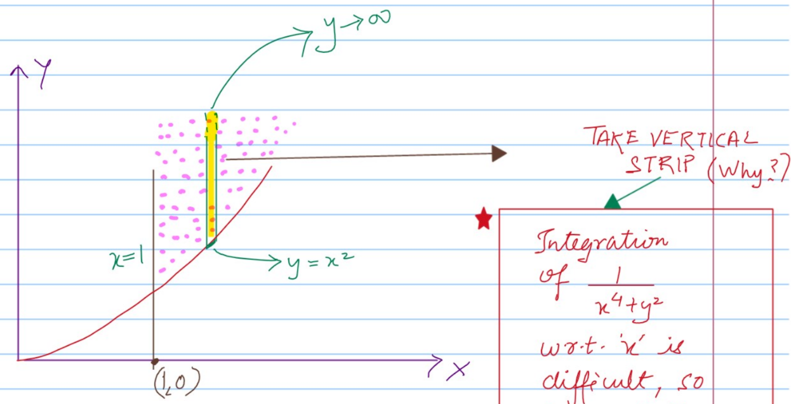
$$= \int_{x=1}^{\infty} \left[\frac{1}{x^2}\tan^{-1}\left(\frac{y}{x^2}\right)\right]_{y=x^2}^{\infty} dx$$

$$= \int_1^{\infty} \frac{1}{x^2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4}\left(\frac{-1}{x}\right)_1^{\infty}$$

$$= \frac{\pi}{4}\left[\lim_{x\to\infty}\left(\frac{-1}{x}\right) - \left(\frac{-1}{1}\right)\right]$$

$$= \frac{\pi}{4}$$



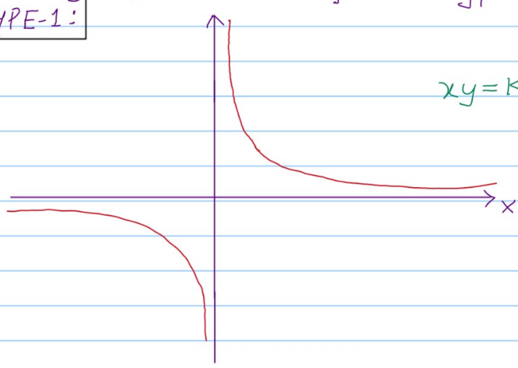
Integration of $\frac{1}{x^4+y^2}$ w.r.t. 'x' is difficult, so inner limits should be of y (for which integration is easy).

Que ⑥ Evaluate $\iint_R x^2\,dx\,dy$; where $R = \{(x,y) \in \mathbb{R}^2 \mid xy \leq 16,\ y \leq x,\ 0 \leq x \leq 8,\ y \geq 0\}$

SOL. PREREQUISITE

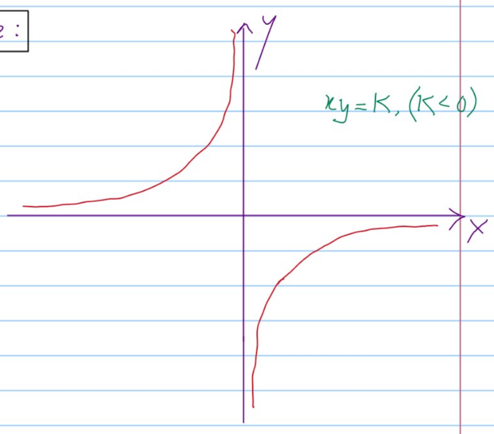
$xy = K$ is rectangular hyperbola, K is constant.

TYPE-1:



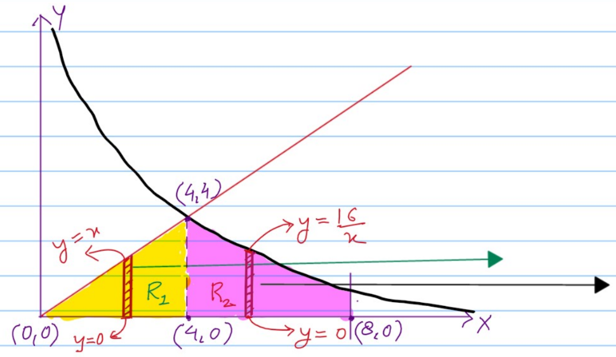
$xy = K,\ (K>0)$

TYPE-2:



$xy = K,\ (K<0)$

Drawing region R:



For region R_1:

Limits of y: 0 to x

" " x: 0 to 4

For region R_2:

Limits of y: 0 to $16/x$

" " x: 4 to 8

$$\therefore\ I = \iint_{R_1} x^2\,dx\,dy + \iint_{R_2} x^2\,dx\,dy$$

$$= \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$$

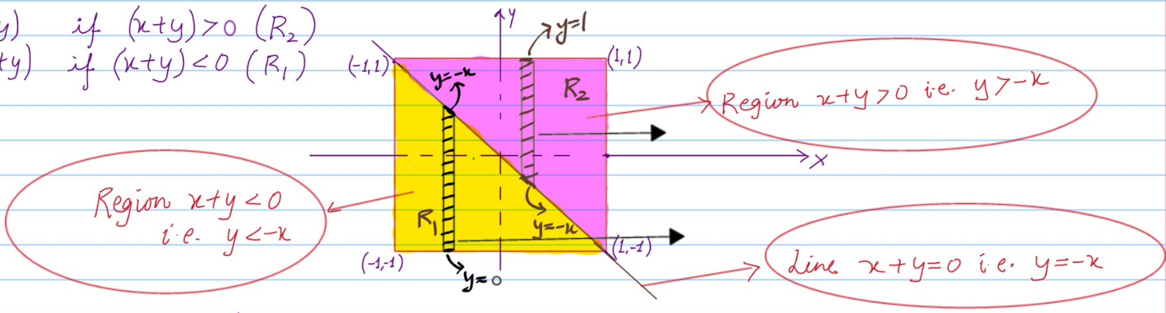
$$= \int_{x=0}^{4} x^3\,dx + \int_{x=4}^{8} (16x)\,dx$$

$$= 64 + 8(8^2 - 4^2)$$

$$= 448$$

Ques ⑦ Evaluate $\int_{-1}^{1}\int_{-1}^{1} |x+y|\,dx\,dy$

SOL. $|x+y| = \begin{cases} (x+y) & \text{if } (x+y) > 0\ (R_2) \\ -(x+y) & \text{if } (x+y) < 0\ (R_1) \end{cases}$



$$I = \iint_{R_1} -(x+y)\,dx\,dy + \iint_{R_2} (x+y)\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{-x} (-x-y)\,dy\,dx + \int_{x=-1}^{1}\int_{y=-x}^{1} (x+y)\,dy\,dx$$

$$= -\int_{x=-1}^{1} \left[xy + \frac{y^2}{2}\right]_{-1}^{-x} dx + \int_{x=-1}^{1} \left(xy + \frac{y^2}{2}\right)_{-x}^{1} dx$$

$$= -\int_{-1}^{1} \left[\left(-x^2 + \frac{x^2}{2}\right) - \left(-x + \frac{1}{2}\right)\right] dx + \int_{-1}^{1} \left[\left(x + \frac{1}{2}\right) - \left(-x^2 + \frac{x^2}{2}\right)\right] dx$$

$$= \int_{1}^{-1} \left(-\frac{x^2}{2} + x - \frac{1}{2}\right) dx + \int_{-1}^{1} \left(x + \frac{1}{2} + \frac{x^2}{2}\right) dx$$

$$= \left(\frac{1}{3} + 1\right) + \left(1 + \frac{1}{3}\right)$$

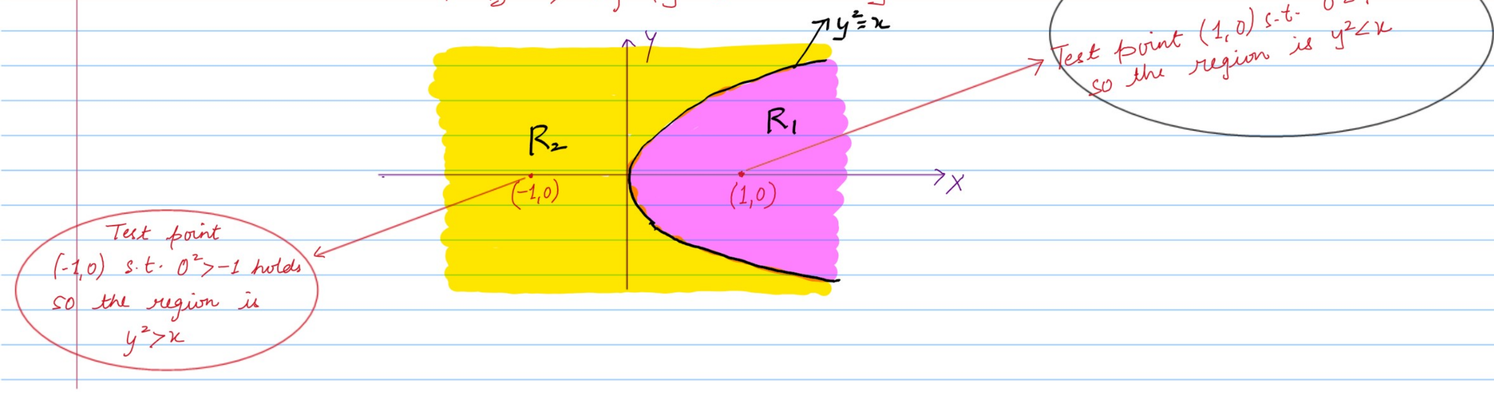
$$= \frac{8}{3}$$

TIP – HOW TO LOCATE REGIONS FOR $|f(x,y)|$?

① Draw the curve on which $f(x,y) = 0$
② Locate the regions for which $f(x,y) > 0$ and $f(x,y) < 0$.

Example : $|y^2 - x|$

$$|y^2 - x| = \begin{cases} (y^2 - x) & \text{if } (y^2 - x) > 0 \text{ or } y^2 > x \\ -(y^2 - x) & \text{if } (y^2 - x) < 0 \text{ or } y^2 < x \end{cases}$$



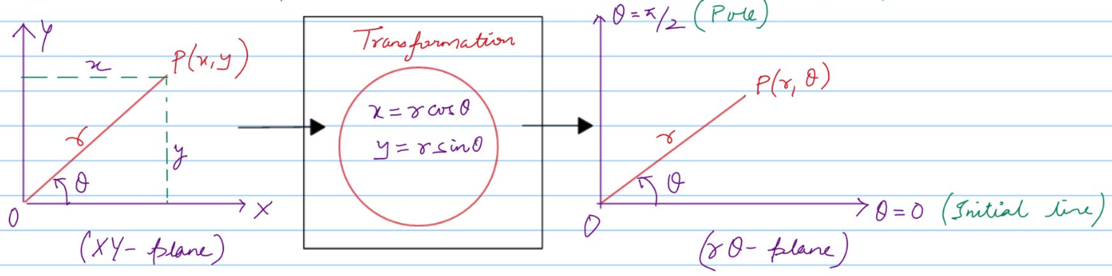
DOUBLE INTEGRATION IN POLAR COORDINATES

What are the polar coordinates?

(r, θ) coordinate of a point P is called its polar coordinate. It is denoted by $P(r, \theta)$.

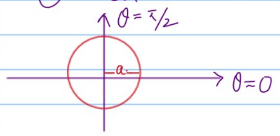
How to convert cartesian coordinates into polar coordinates??

Let $P(x, y)$ be the point in cartesian plane (XY-plane).

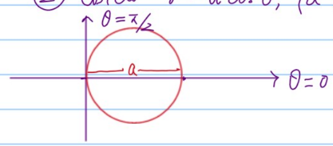


Few standard polar curves :

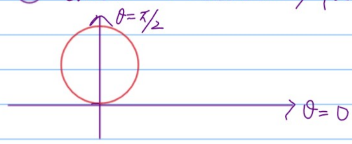
① Circle $r = a$, $(a > 0)$



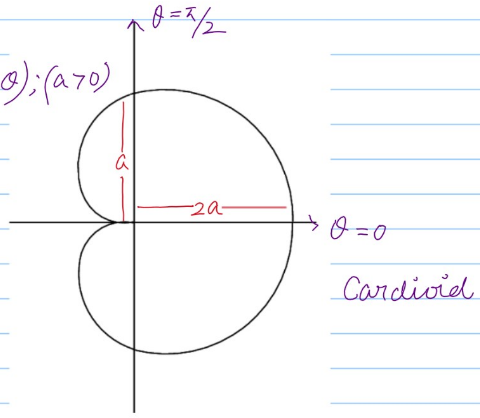
② Circle $r = a\cos\theta$; $(a > 0)$



③ Circle $r = a\sin\theta$; $(a > 0)$

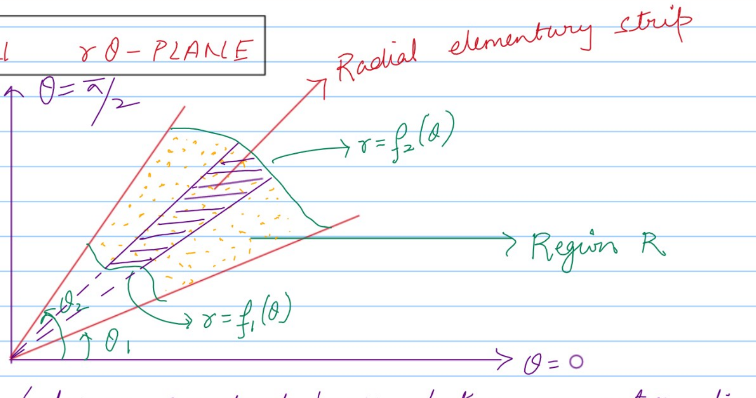


④ $r = a(1+\cos\theta)$; $(a > 0)$



HOW TO FIND LIMITS IN $r\theta$-PLANE

Draw elementary radial strip inside region R (see figure). The limits of 'r' are given by the base and top of the strip while the limits of 'θ' are given by the rotation of strip.



Limits of r: $f_1(\theta)$ to $f_2(\theta)$

Limits of θ: θ_1 to θ_2 (always constant hence taken as outer limits)

Ques ① Evaluate $\iint_R r^2\, dr\, d\theta$, where R is region bounded between $r = a\sin\theta$ & $r = 2a\sin\theta$.

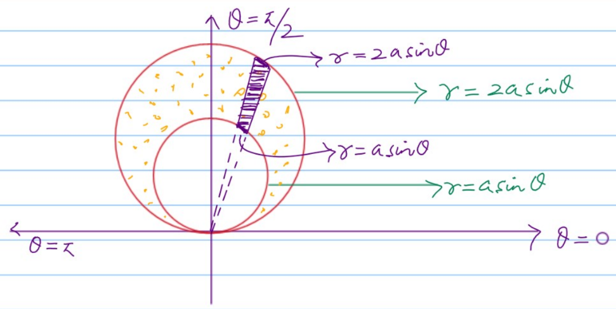
SOL. Limits of r : $a\sin\theta$ to $2a\sin\theta$

Limits of θ : 0 to π

$$\therefore I = \int_{\theta=0}^{\pi}\int_{r=a\sin\theta}^{2a\sin\theta} r^2\, dr\, d\theta$$

$$= \frac{7a^3}{3}\int_{\theta=0}^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi}\left(\frac{3\sin\theta - \sin 3\theta}{4}\right) d\theta$$



$$= \frac{7a^3}{12}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi}$$

$$= \frac{7a^3}{12}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right]$$

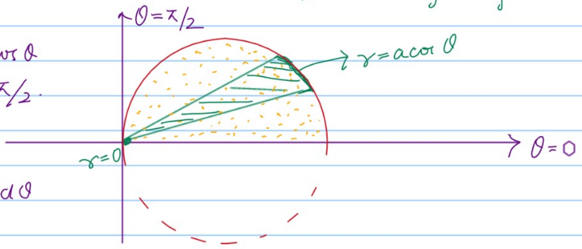
$$= \frac{28a^3}{9}$$

Ques ② Evaluate $\iint r\sqrt{a^2-r^2}\, dr\, d\theta$ over the upper half of circle $r = a\cos\theta$.

SOL.

Limits of r : 0 to $a\cos\theta$

Limits of θ : 0 to $\pi/2$.



$$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

$$= \left(-\frac{1}{2}\right)\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} (a^2-r^2)^{\frac{1}{2}}(-2r)\, dr\, d\theta \longleftarrow \text{NOTE THIS STEP}$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{a\cos\theta} d\theta \qquad \left[\because \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= -\frac{1}{3}\int_{\theta=0}^{\pi/2} a^3(\sin^3\theta - 1)\, d\theta$$

$$= -\frac{a^3}{3}\int_0^{\pi/2}\left(\frac{3\sin\theta - \sin 3\theta}{4} - 1\right) d\theta$$

$$= -\frac{a^3}{3}\left[\frac{-3\cos\theta}{4} + \frac{\cos 3\theta}{12} - \theta\right]_0^{\pi/2}$$

$$= -\frac{a^3}{3}\left[\left(-\frac{\pi}{2}\right) - \left(-\frac{3}{4} + \frac{1}{12}\right)\right]$$

$$= \frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)$$

EVALUATION OF DOUBLE INTEGRAL BY CHANGING INTO POLAR COORDINATES

Consider $I = \iint_R F(x,y)\, dx\, dy$.

Sometimes, it is relatively easier to evaluate the above integral in $r\theta$-plane. To do this,

Put $\boxed{x = r\cos\theta, \quad y = r\sin\theta}$

Then the Jacobian ; $|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r (\neq 0)$

$\because$ $|J| = \frac{dx\, dy}{dr\, d\theta}$ i.e. $|J|$ represents the ratio of change in area.

or, $\boxed{dx\, dy = |J|\, dr\, d\theta}$

Thus, $I = \iint_R F(r\cos\theta, r\sin\theta)\, |J|\, dr\, d\theta$

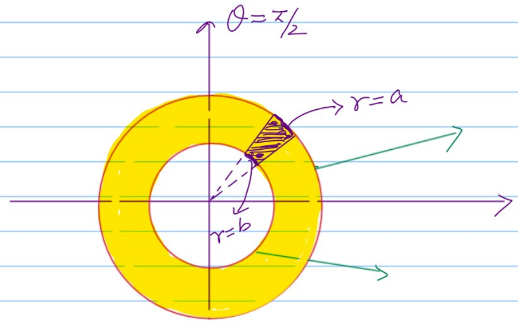
$$\text{or, } \boxed{I = \int_{\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

Ques ① Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\, dx\, dy$; where $R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le a^2,\ x^2+y^2 \ge b^2,\ a > b\}$

SOL. NOTE: Integration of $\frac{x^2y^2}{x^2+y^2}$ is difficult w.r.t. either variable x or y first.

Let $x = r\cos\theta, \quad y = r\sin\theta$

Then $dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$



Now, $x^2 + y^2 = b^2$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = b^2$

$\Rightarrow r^2 = b^2$

$\Rightarrow r = \pm b$

$\Rightarrow \boxed{r = b}$ (negative sign is ignored)

Similarly, $\boxed{r = a}$

Limits of r : b to a
" " θ : 0 to 2π

$$\therefore I = \int_{\theta=0}^{2\pi}\int_{r=b}^{a} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}(r\,dr\,d\theta)$$

$$= \int_b^a r^3\,dr \times \int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta$$

$$= \left(\frac{a^4-b^4}{4}\right) \times \frac{1}{4}\int_0^{2\pi}\sin^2 2\theta\,d\theta$$

$$= \left(\frac{a^4-b^4}{16}\right) \times \int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \left(\frac{a^4-b^4}{32}\right) \times 2\pi$$

$$= \pi\left(\frac{a^4-b^4}{16}\right)$$

Ques ② Evaluate $\iint_R \frac{4xy}{x^2+y^2}\cdot e^{-x^2-y^2}\,dx\,dy$, where $R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2-x \le \;, x \geq 0, y \geq 0\}$

SOL. Let $x = r\cos\theta, \quad y = r\sin\theta$

Then $dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

Circle is $x^2 + y^2 - x = 0$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) - r\cos\theta = 0$

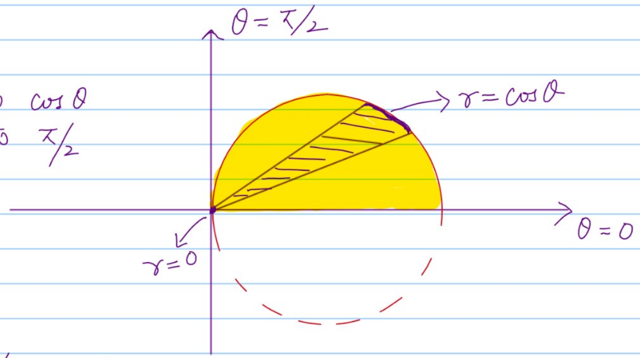
$\Rightarrow r^2 - r\cos\theta = 0$

$\Rightarrow r(r - \cos\theta) = 0$

$\Rightarrow r - \cos\theta = 0 \quad (\because r \neq 0)$

$\Rightarrow \boxed{r = \cos\theta}$

Limits of r : 0 to $\cos\theta$
" " θ : 0 to $\pi/2$



$$\therefore I = \int_{\theta=0}^{\pi/2}\int_0^{\cos\theta} \frac{4r^2\cos\theta\sin\theta}{r^2}e^{-r^2}(r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\cos\theta} 2r e^{-r^2} \sin 2\theta \; dr \; d\theta$$

Let $r^2 = t \Rightarrow 2r\,dr = dt$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{t=0}^{\cos^2\theta} e^{-t} \, dt \right] \sin 2\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(1 - e^{-\cos^2\theta}\right) \sin 2\theta \; d\theta$$

Let $p = -\cos^2\theta$

$\Rightarrow dp = 2\cos\theta \sin\theta \, d\theta$

$$I = \int_{-1}^{0} \left(1 - e^{p}\right) dp$$

$$= 1 - \left[e^{p}\right]_{-1}^{0}$$

$$= e^{-1}$$

Ques (3) Evaluate $I = \int_{x=0}^{1} \int_{y=\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xy\, e^{-(x^2+y^2)}}{x^2+y^2} \, dy \, dx$

SOL. Let $x = r\cos\theta$, $y = r\sin\theta$ & $dx\,dy = r\,dr\,d\theta$

$\because$ $y = \sqrt{x-x^2}$

$\Rightarrow y^2 = x - x^2$

$\Rightarrow x^2 + y^2 - x = 0$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) - r\cos\theta = 0$

$\Rightarrow r^2 - r\cos\theta = 0$

$\Rightarrow r(r - \cos\theta) = 0$

$\Rightarrow r - \cos\theta = 0 \quad (\because r \neq 0)$

$\Rightarrow \boxed{r = \cos\theta}$

and $y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1$

$\Rightarrow r^2 = 1$

$\Rightarrow \boxed{r = 1}$

$\because$ $x = 0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \cos\theta = 0 \quad (\text{as } r \neq 0)$

$\Rightarrow \boxed{\theta = \frac{\pi}{2}}$

and $x = 1$

$\Rightarrow r\cos\theta = 1$

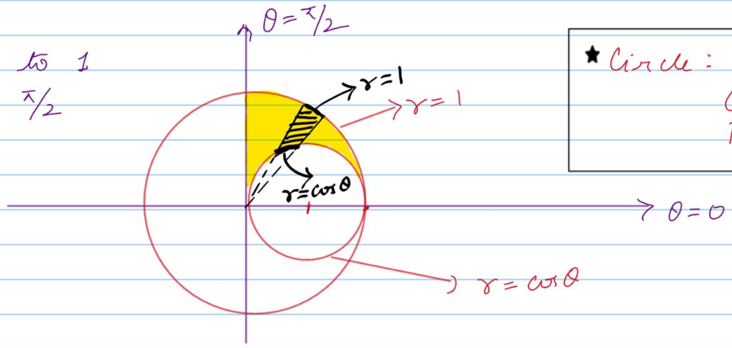
$\Rightarrow \cos\theta = 1$ or $\cos^2\theta = 1$

$\Rightarrow \boxed{\theta = 0}$ or $\cos\theta = 1$

$\theta = 0$

Limits of r : $\cos\theta$ to 1

" " θ : 0 to $\pi/2$



★ Circle : $x^2 + y^2 + 2gx + 2fy + c = 0$

Center $C\ (-g, -f)$

Radius $r = \sqrt{g^2 + f^2 - c}$

$$\therefore I = \frac{1}{4} \int_{\theta=0}^{\pi/2} \int_{r=\cos\theta}^{1} 2r e^{-r^2} \sin 2\theta \; dr \; d\theta$$

$$= \frac{1}{4}\left(1 - \frac{2}{e}\right)$$

EVALUATION OF DOUBLE INTEGRAL BY CHANGE OF VARIABLES

Consider $I = \iint_{R_1} F(x,y)\, dx\, dy$.

Sometimes, it is relatively easier to evaluate the above integral by using the transformation;

Put $\boxed{x = \phi_1(u,v) \ \& \ y = \phi_2(u,v)}$

Then the Jacobian; $|J| = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0$

$\because$ $|J| = \dfrac{dx\,dy}{du\,dv}$ i.e. $|J|$ represents the ratio of change in area. (a positive quantity)

or, $\boxed{dx\,dy = |J|\, du\, dv}$

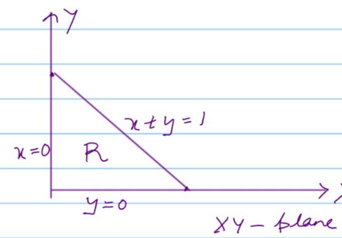
Thus, $\boxed{I = \iint_{R_2} F(\phi_1(u,v), \phi_2(u,v))\, |J|\, du\, dv}$

where R_2 is the region in uv-plane

Ques (1) Evaluate $\iint_R \cos\left(\dfrac{x-y}{x+y}\right) dx\, dy$; where R is region bounded by $x=0,\ y=0,\ x+y=1$.

Use substitution $x-y=u$ & $x+y=v$.

SOL. (a) Draw region R.



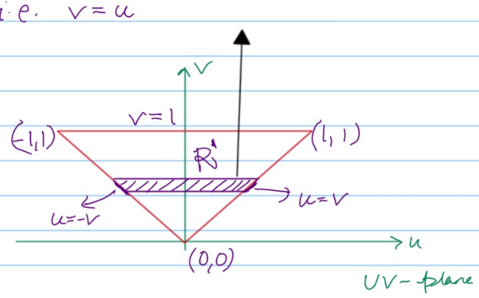
(b) Calculation of $|J|$

$\because$ $|J| = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|} = \dfrac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \dfrac{1}{2}$

(c) Drawing transformed region R'.

(i) $x=0$ maps to $(u=-y \ \& \ v=y)$ i.e. $v=-u$
(ii) $y=0$ maps to $(u=x \ \& \ v=x)$ i.e. $v=u$
(iii) $x+y=1$ maps to $v=1$

Limits of u: $-v$ to v
Limits of v: 0 to 1.



(d) Evaluation of integral

$$I = \int_{v=0}^{1} \int_{u=-v}^{v} \cos\left(\frac{u}{v}\right) |J|\, du\, dv$$

$$= \int_{v=0}^{1} \left[\frac{\sin(u/v)}{1/v}\right]_{u=-v}^{v} \frac{1}{2}\, dv$$

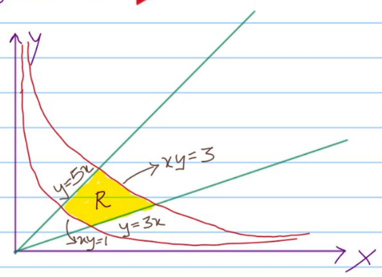
$$= \frac{1}{2} \int_{v=0}^{1} v\left[\sin(1) - \sin(-1)\right] dv$$

$$= \frac{2\sin(1)}{2} \int_{v=0}^{1} v\, dv$$

$$= \frac{\sin(1)}{2}$$

Ques ② Use change of variables to evaluate $\iint_R (xy)\,dx\,dy$ where R is region bounded by $xy=1$, $xy=3$, $y=3x$ and $y=5x$ in first quadrant.

SOL. ⓐ Draw region R.



XY-plane

ⓑ Calculation of $|J|$

Let the transformations be $\boxed{xy = u \quad \& \quad \frac{y}{x} = v}$

★ NOTE THIS STEP

Take hint from the boundaries of region R

i.e. $\underbrace{xy=1,\ xy=3}_{u}$, $\underbrace{\frac{y}{x}=3,\ \frac{y}{x}=5}_{v}$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$= \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$$

$$= \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix}}$$

$$= \frac{x}{2y}$$

ⓒ Draw transformed region R'

ⓐ $xy=1$ maps to $\boxed{u=1}$

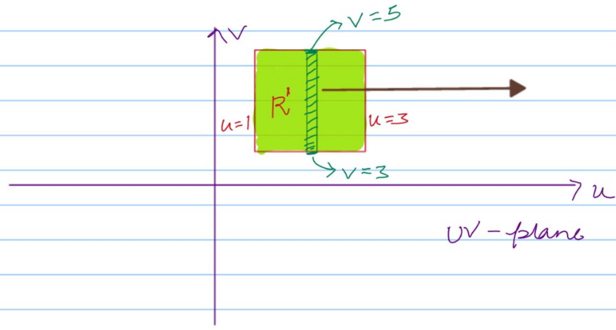
ⓑ $xy=3$ maps to $\boxed{u=3}$

ⓒ $\frac{y}{x}=3$ maps to $\boxed{v=3}$

ⓓ $\frac{y}{x}=5$ maps to $\boxed{v=5}$

Limits of v : 3 to 5

Limits of u : 1 to 3



UV-plane

ⓓ Evaluation of integral

$$I = \int_{u=1}^{3}\int_{v=3}^{5} (u)\ |J|\ du\,dv$$

$$= \int_{u=1}^{3}\int_{v=3}^{5} (u)\left(\frac{x}{2y}\right) du\,dv$$

$$= \int_{u=1}^{3}\int_{v=3}^{5} (u)\ \frac{1}{2(x/y)}\ du\,dv$$

$$= \frac{1}{2}\int_{u=1}^{3}\int_{v=3}^{5} \left(\frac{u}{v}\right) du\,dv \qquad \left(\because \frac{x}{y} = v\right)$$

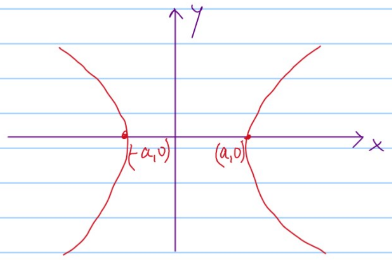
$$= \frac{1}{2}\int_{u=1}^{3} [\log(5) - \log(3)]\, u\, du$$

$$= \frac{\log(5/3)}{2}\int_{u=1}^{3} u\, du$$

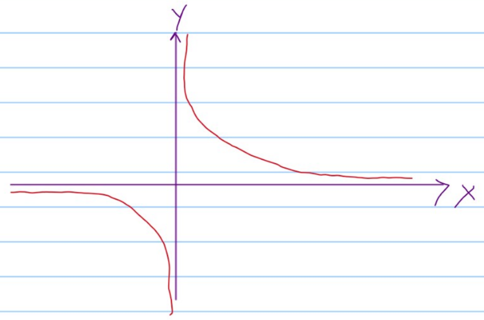
$$= 2\log\left(\frac{5}{3}\right)$$

Que ③ Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, where R is the region bounded by $x^2-y^2=1$, $x^2-y^2=2$, $xy=2$ and $xy=4$ in 1^{st} quadrant.

SOL. PRE-REQUISITE

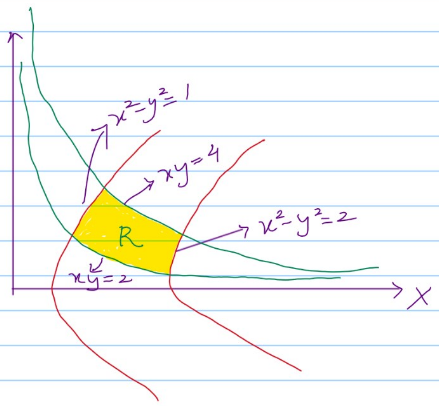


Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



Rectangular hyperbola $xy=K$, $(K>0)$

ⓐ Draw region R



ⓑ Calculation of $|J|$

Let $x^2-y^2=u$ & $xy=v$. Then

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$$

$$= \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}}$$

$$= \frac{1}{2(x^2+y^2)}$$

ⓒ Transformed region R'

(i) $x^2-y^2=1$ maps to $u=1$

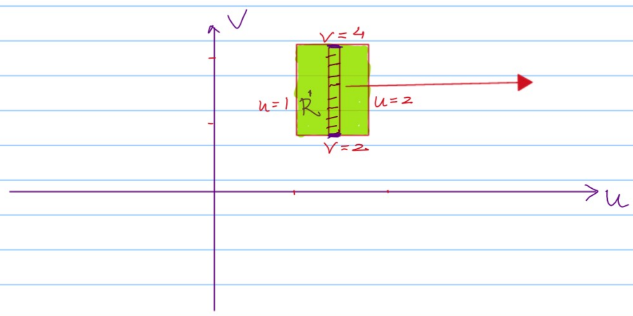
(ii) $x^2-y^2=2$ maps to $u=2$

(iii) $xy=2$ maps to $v=2$

(iv) $xy=4$ maps to $v=4$

Limits of v : 2 to 4

" " u : 1 to 2



ⓓ Evaluation of integral

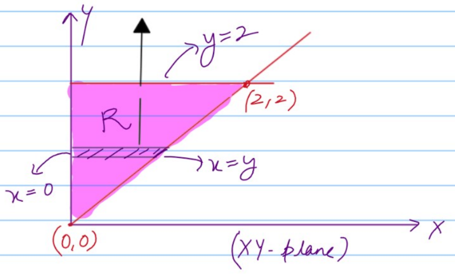
$$I = \int_{u=1}^{2}\int_{v=2}^{4} (x^2+y^2)\,|J|\,du\,dv$$

$$= \int_{u=1}^{2}\int_{v=2}^{4} \cancel{(x^2+y^2)}\cdot\frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv$$

$$= \frac{1}{2}\int_{u=1}^{2} du \times \int_{v=2}^{4} dv = \frac{1}{2}\times(2-1)\times(4-2) = 1$$

Ques ④ Using $x = u(1+v)$ & $y = v(1+u)$, $u \geq 0$ & $v \geq 0$, evaluate $\displaystyle\int_{y=0}^{2}\int_{x=0}^{y}\left[(x-y)^2 + 2(x+y)+1\right]^{-1/2} dx\, dy$

SOL. ⓐ Drawing region R



ⓑ Computation of $|J|$

Given $x = u(1+v)$ & $y = v(1+u)$

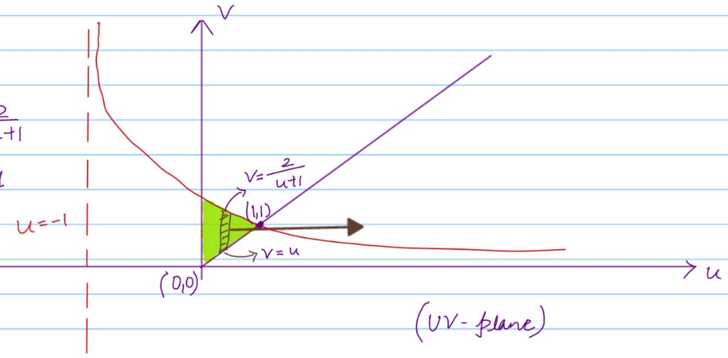
$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$$

ⓒ Drawing region R'

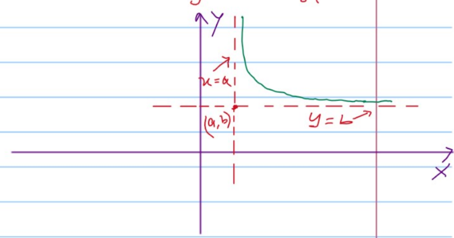
(i) $x = 0$ maps to $u(1+v) = 0$ i.e. $u = 0$ or $v = -1$ (neglect as $v \geq 0$)
(ii) $x = y$ maps to $u(1+v) = v(1+u)$ i.e. $v = u$
(iii) $y = 2$ maps to $v(u+1) = 2$

Limits of v: u to $\frac{2}{u+1}$

" " u : 0 to 1



NOTE → $(x-a)(y-b) = K$, $(a>0, b>0, K>0)$ is rectangular hyperbola



ⓓ Evaluation of integral

$$I = \int_{u=0}^{1}\int_{v=u}^{2/u+1}\left[(u-v)^2 + 2(u+v+2uv)+1\right]^{-1/2}(1+u+v)\, du\, dv$$

$$= \int_{u=0}^{1}\int_{v=u}^{2/u+1}\left[u^2+v^2+1^2+2uv+2v+2u\right]^{-1/2}(1+u+v)\, du\, dv$$

$$= \int_{u=0}^{1}\int_{v=u}^{2/u+1}\left[(u+v+1)^2\right]^{-\frac{1}{2}}(1+u+v)\, du\, dv$$

$$= \int_{u=0}^{1}\int_{v=u}^{\frac{2}{u+1}} dv\, du$$

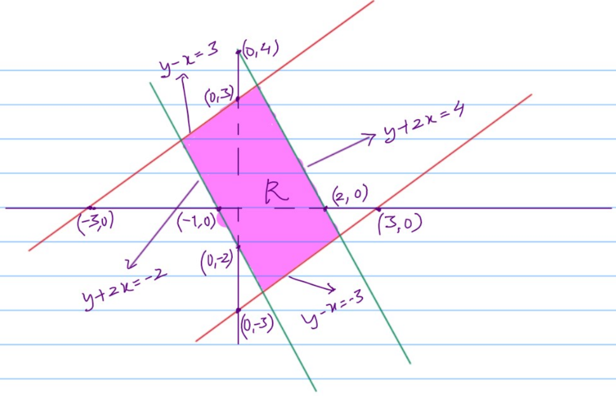
$$= \int_{u=0}^{1}\left(\frac{2}{u+1} - u\right) du$$

$$= \left[2\log_e|u+1| - \frac{u^2}{2}\right]_0^1$$

$$= 2\log_e 2 - \frac{1}{2}$$

Ques ⑤ Evaluate $\iint_R (x+2y)\, dx\, dy$, where R is bounded by $y-x=-3$, $y-x=3$, $y+2x=-2$ and $y+2x=4$

SOL ⓐ Drawing region R



(b) Computation of $|J|$

Let $y-x=u$ & $y+2x=v$

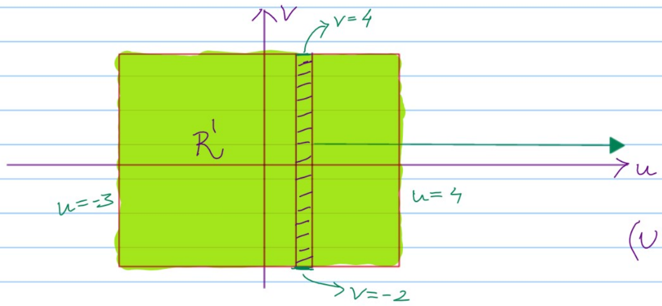
Then

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{-3} = \frac{1}{3}$$ (Take positive sign)

(c) Drawing region R'

(i) $y-x=-3$ maps to $u=-3$
(ii) $y-x=3$ maps to $u=3$
(iii) $y+2x=-2$ maps to $v=-2$
(iv) $y+2x=4$ maps to $v=4$

Limits of v: -2 to 4
" " u: -3 to 3



(uv-plane)

(d) Evaluation of integral

$$I = \int_{u=-3}^{3}\int_{v=-2}^{4} (u+v)\left(\frac{1}{3}\right) dv\, du$$

$$= \frac{1}{3}\int_{u=-3}^{3}\left[uv + \frac{v^2}{2}\right]_{v=-2}^{4} du$$

$$= \frac{1}{3}\int_{-3}^{3}(6u+6)\, du$$

$$= 12$$