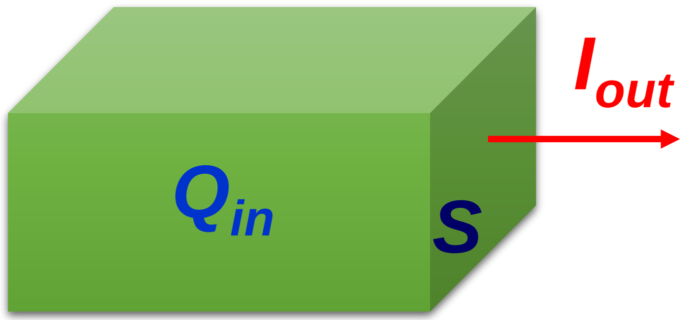
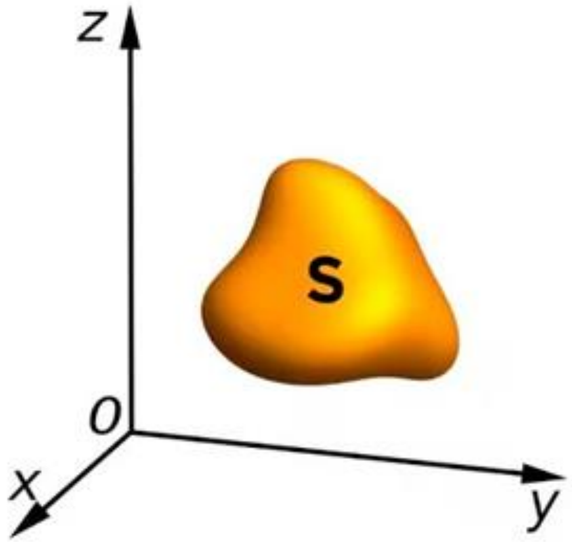
Continuity equation and relaxation time


I_{out}
Q_{in}
S

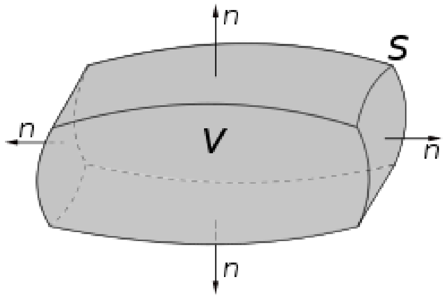
The Divergence Theorem

The divergence theorem states that the surface integral of the normal component of a vector point function **"F"** over a closed surface **"S"** is equal to the volume integral of the divergence of **"F"** taken over the volume "V" enclosed by the surface S.



$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V \nabla.\vec{F}\, dV$$

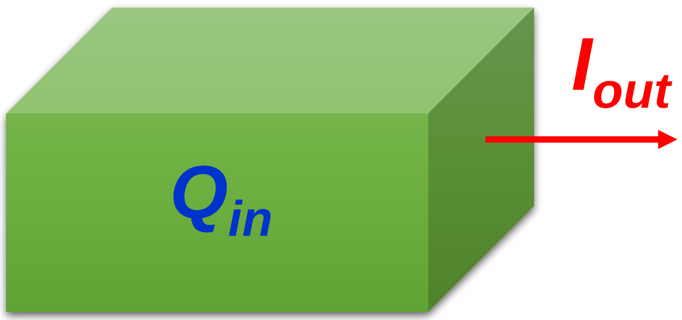
$\hat{n}$ is the outward pointing unit normal at each point on the boundary dV.



According to principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

The current I_{out} coming out of the closed surface;

$$I_{out} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{in}}{dt} \qquad \text{(i)}$$



where Q_{in} is the total charge enclosed by the closed surface.

Using divergence theorem

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{J}\, dv$$

But

$$\frac{-dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

Equation (i) now becomes

$$\int_v \nabla \cdot \mathbf{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

or $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad \text{(ii)}$$

This is called the **continuity of current equation**. And states that there can be no accumulation of charge at any point.

For steady current, $\frac{\partial \rho_v}{\partial t} = 0$ and hence $\nabla \cdot \mathbf{J} = 0$ showing that the total charge leaving a volume is the same as total charge entering it, showing the validity of Kirchoff's law.

Effect of introducing charge at some interior point of a conductor/dielectric

According to Ohm's law $$\mathbf{J} = \sigma \mathbf{E}$$

According to Gauss's law $$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}$$

Equation (ii) now becomes

$$\nabla \cdot \sigma \mathbf{E} = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

or $$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

This is homogeneous liner ordinary differential equation. By separating variables we get

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t$$

Integrating both sides

$$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{vo}$$

where $\ln \rho_{vo}$ is a constant of integration

$$\rho_v = \rho_{vo} e^{-t/T_r} \qquad \text{(iii)}$$

where

$$T_r = \frac{\varepsilon}{\sigma}$$

ρ_{vo} is the initial charge density (i.e., ρ_v at $t = 0$)

Equation (iii) shows that as a result of introducing charge at some interior point of the material there is a decay of the volume charge density ρ_v.

The time constant T_r is known as the **relaxation time**.

Relaxation time is the time in which a charge placed in the interior of a material to drop to e^{-1} = 36.8 % of its initial value.

For Copper $T_r = 1.53 \times 10^{-19}$ sec (short for good conductors)
For fused Quartz T_r = 51.2 days (large for good dielectrics)

Boundary Conditions

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions.

These conditions are helpful in determining the field on one side of the boundary when the field on other side is known.

We will consider the boundary conditions at an interface separating;

1. Dielectric (ϵ_{r1}) and Dielectric (ϵ_{r2})
2. Conductor and Dielectric
3. Conductor and free space

For determining boundary conditions we will use Maxwell's equations

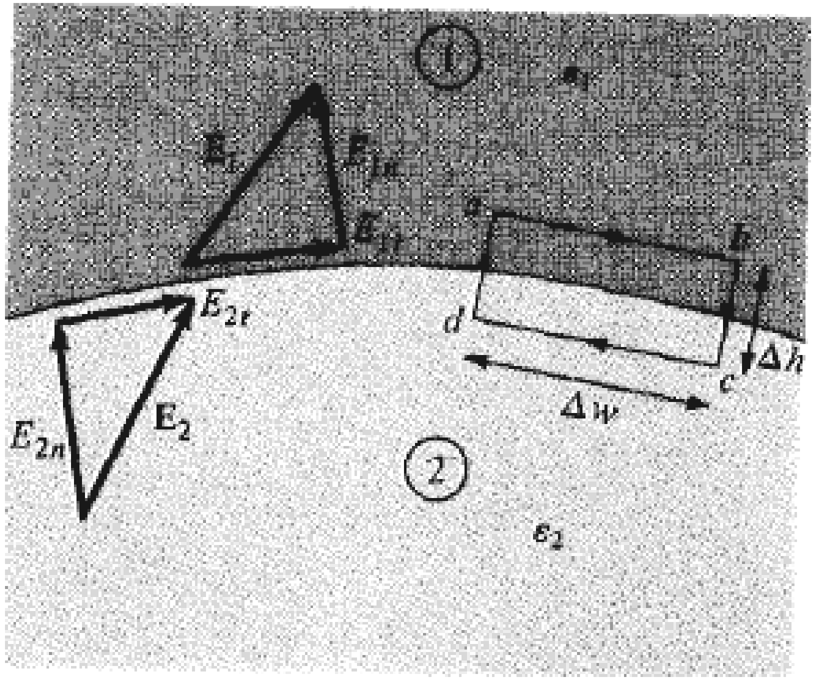
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{and} \quad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

Where Q_{enc} is free charge enclosed in surface. Further we have to split electric field intensity in to two orthogonal components;

$E = E_t + E_n$ (tangential and normal components at interface)

Boundary Conditions (Between two different dielectrics)

Consider the **E** field existing in a region consisting of two different dielectrics characterized by $\epsilon_1 = \epsilon_0\,\epsilon_{r1}$ and $\epsilon_2 = \epsilon_0\,\epsilon_{r2}$



$\mathbf{E_1}$ and $\mathbf{E}_2$ in the media 1 and 2 can be written as

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n} \quad \text{and} \quad \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

But $$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Assuming that the path abcda is very small with respect to the variation in **E**

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

As $\Delta h \longrightarrow 0$

$$E_{1t} = E_{2t}$$

Thus the tangential components of **E** are the same on the two sides of the boundary. **E** is continuous across the boundary.

But $$\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{D}_t + \mathbf{D}_n$$

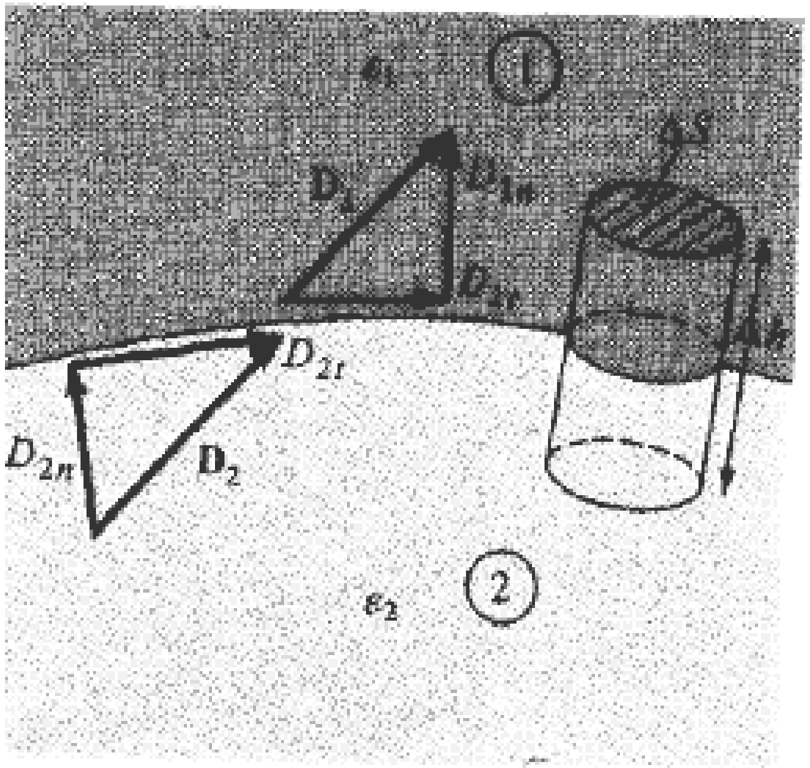
Thus

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

or $$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Here $\mathbf{D_t}$ undergoes some change across the surface and is said to be discontinuous across the surface.

Applying $\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$



Putting $\Delta h \rightarrow 0$ gives

$$\Delta Q = \rho_S \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

$$D_{1n} - D_{2n} = \rho_S$$

Where ρ_s is the free charge density placed deliberately at the boundary

If there is no charge on the boundary i.e. $\rho_s = 0$ then

$$D_{1n} = D_{2n}$$

Thus the normal components of **D** is continuous across the surface.

Thus the normal component of **D** is continuous across the interface; that is, D_n undergoes no change at the boundary. Since **D** = ε**E**, eq. (5.60) can be written as

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \tag{5.61}$$

showing that the normal component of **E** is discontinuous at the boundary. Equations (5.57) and (5.59), or (5.60) are collectively referred to as *boundary conditions*; they must be satisfied by an electric field at the boundary separating two different dielectrics.

As mentioned earlier, the boundary conditions are usually applied in finding the electric field on one side of the boundary given the field on the other side. Besides this, we can use the boundary conditions to determine the "refraction" of the electric field across the interface. Consider $\mathbf{D}_1$ or $\mathbf{E}_1$ and $\mathbf{D}_2$ or $\mathbf{E}_2$ making angles θ_1 and θ_2 with the *normal* to the interface as illustrated in Figure 5.11. Using eq. (5.57), we have

$$E_1 \sin \theta_1 = E_{1t} = E_{2t} = E_2 \sin \theta_2$$

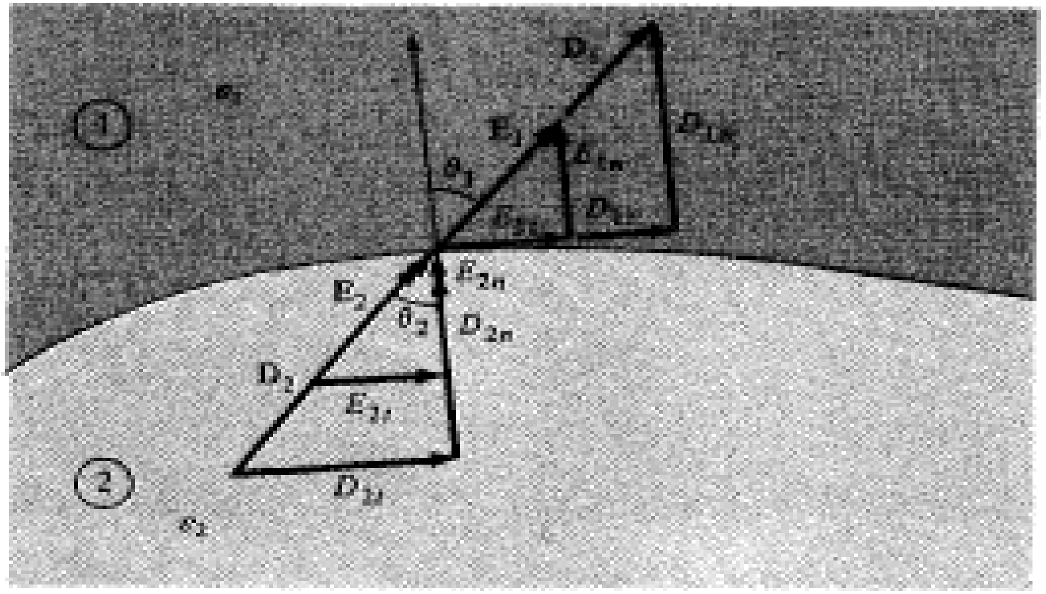


Figure 5.11 Refraction of **D** or **E** at a dielectric–dielectric boundary.

or

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \tag{5.62}$$

Similarly, by applying eq. (5.60) or (5.61), we get

$$\varepsilon_1 E_1 \cos \theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos \theta_2$$

or

$$\varepsilon_1 E_1 \cos \theta_1 = \varepsilon_2 E_2 \cos \theta_2 \tag{5.63}$$

Dividing eq. (5.62) by eq. (5.63) gives

$$\frac{\tan \theta_1}{\varepsilon_1} = \frac{\tan \theta_2}{\varepsilon_2} \tag{5.64}$$

Since $\varepsilon_1 = \varepsilon_o \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_o \varepsilon_{r2}$, eq. (5.64) becomes

$$\boxed{\frac{\tan \theta_1}{\tan \theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \tag{5.65}$$

This is the *law of refraction* of the electric field at a boundary free of charge (since $\rho_S = 0$ is assumed at the interface). Thus, in general, an interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the sides of the interface.

Term	Electric	Magnetic
Basic laws	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^2}\,\mathbf{a}_r$	$d\mathbf{B} = \dfrac{\mu_o I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$
	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$
Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
Source element	dQ	$Q\mathbf{u} = I\, d\mathbf{l}$
Field intensity	$E = \dfrac{V}{\ell}$ (V/m)	$H = \dfrac{I}{\ell}$ (A/m)
Flux density	$\mathbf{D} = \dfrac{\Psi}{S}$ (C/m^2)	$\mathbf{B} = \dfrac{\Psi}{S}$ (Wb/m^2)
Relationship between fields	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
Potentials	$\mathbf{E} = -\nabla V$	$\mathbf{H} = -\nabla V_m\ (\mathbf{J} = 0)$
	$V = \displaystyle\int \dfrac{\rho_L dl}{4\pi\varepsilon r}$	$\mathbf{A} = \displaystyle\int \dfrac{\mu\, I\, d\mathbf{l}}{4\pi R}$
Flux	$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$	$\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\dfrac{dV}{dt}$	$V = L\dfrac{dI}{dt}$
Energy density	$w_E = \dfrac{1}{2}\mathbf{D} \cdot \mathbf{E}$	$w_m = \dfrac{1}{2}\mathbf{B} \cdot \mathbf{H}$