TRIPLE INTEGRALS

WHAT IS A TRIPLE INTEGRAL?

An integral of the form $\iiint_D f(x,y,z)\,dx\,dy\,dz$

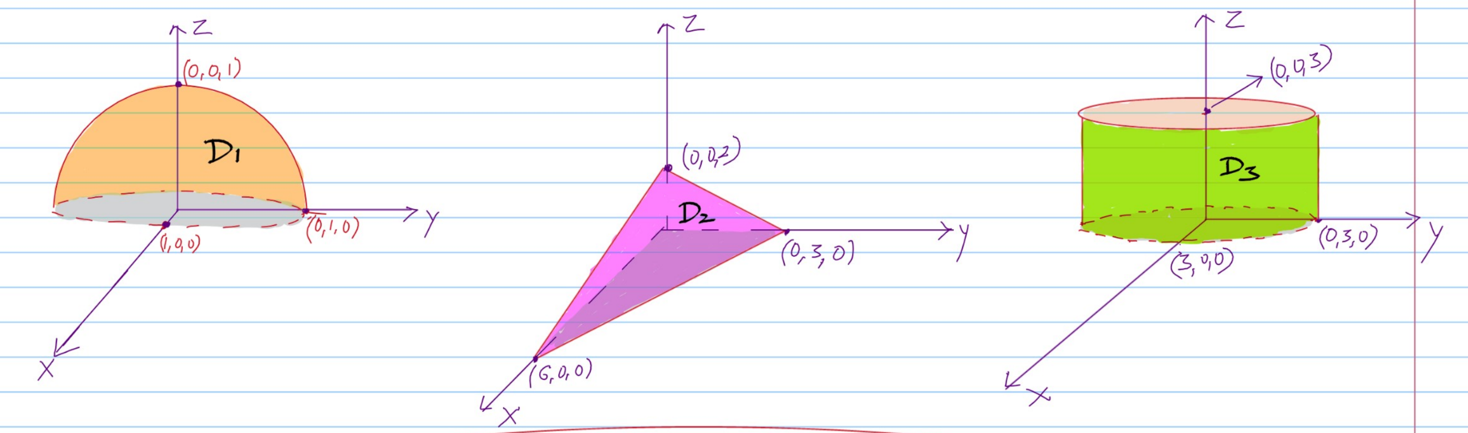
where $f(x,y,z)$ is a function of 3 variables x, y & z defined on a subset D of XYZ-space.

Examples:

$$I_1 = \iiint_{D_1} (x^2+y^2+z^2)\,dx\,dy\,dz; \text{ where } D_1 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 1,\ z \geq 0\}$$

$$I_2 = \iiint_{D_2} (x+y+z)\,dx\,dy\,dz; \text{ where } D_2 = \{(x,y,z)\in\mathbb{R}^3 : x+2y+3z \le 6,\ x\geq 0, y\geq 0, z\geq 0\}$$

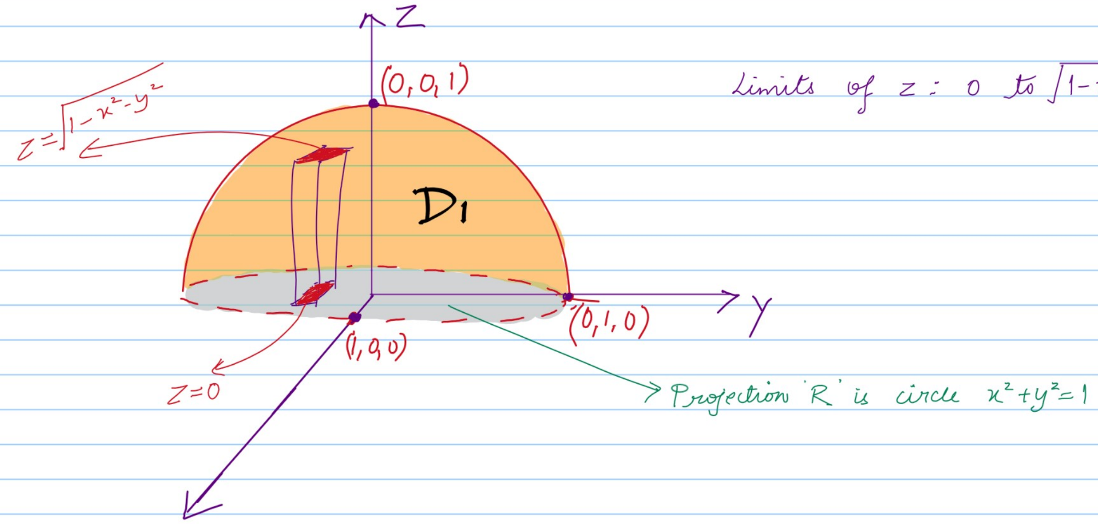
$$I_3 = \iiint_{D_3} (xy)\,dx\,dy\,dz; \text{ where } D_3 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 = 9,\ 0 \le z \le 3\}$$



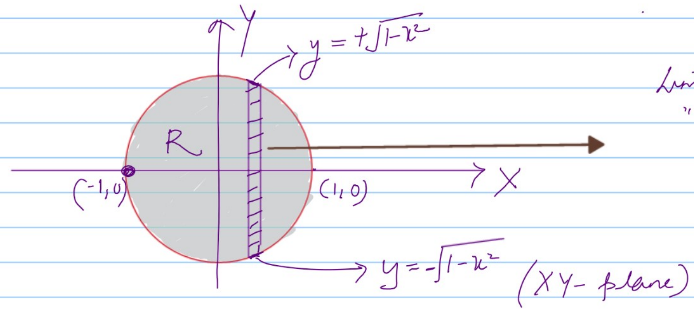
DRAWING REGIONS IN 3-SPACE IS IMPORTANT

FINDING LIMITS IS IMPORTANT

Draw an elementary volume (dV) (cuboidal shaped pillar) inside D. Suppose its base is on XY-plane (See figure). Then limits of z are determined by the surfaces on which the base and top of dV lie.
To determine the limits of 'x' and 'y', take orthogonal projection 'R' of top bounding surface on XY-plane and find limits of 'x' & 'y' by making strip inside region 'R' (as in double integral).



Limits of z : 0 to $\sqrt{1-x^2-y^2}$



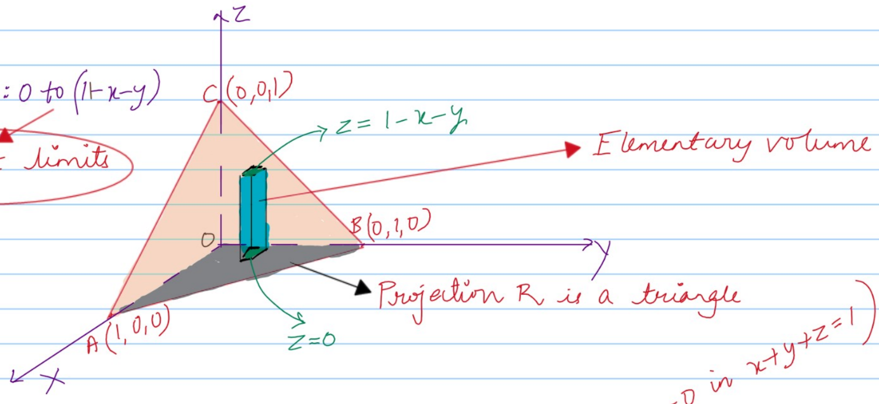
Limits of $y: -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$
" " $x: -1$ to 1.

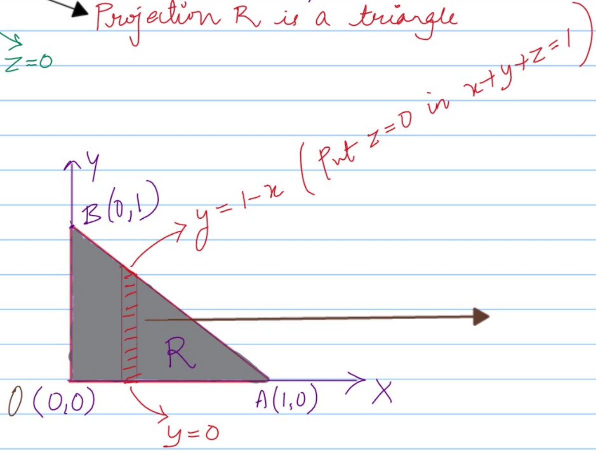
Ques ① Evaluate $\iiint\limits_D x^2\,dx\,dy\,dz$, where $D=\{(x,y,z)\in\mathbb{R}^3 : x+y+z\le 1,\ x\geq 0,\ y\geq 0,\ z\geq 0\}$

SOL.

Limits of $z: 0$ to $(1-x-y)$

Innermost limits





Limits of $y: 0$ to $(1-x)$
Limits of $x: 0$ to 1

Outer limits

$$\therefore\ I=\int_{x=0}^{1}\int_{y=0}^{(1-x)}\int_{z=0}^{(1-x-y)} x\ dz\,dy\,dx$$ ← NOTE THIS STEP

$$=\int_{x=0}^{1}\int_{y=0}^{1-x} x\,(1-x-y)\,dy\,dx$$

$$=\int_{x=0}^{1}\left[x(1-x)y-x\frac{y^2}{2}\right]_{y=0}^{1-x}dx$$

$$=\int_{x=0}^{1}\left[x(1-x)^2-\frac{x(1-x)^2}{2}\right]dx$$

$$=\frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

$$=\frac{1}{2}\int_0^1 (x+x^3-2x^2)\,dx$$

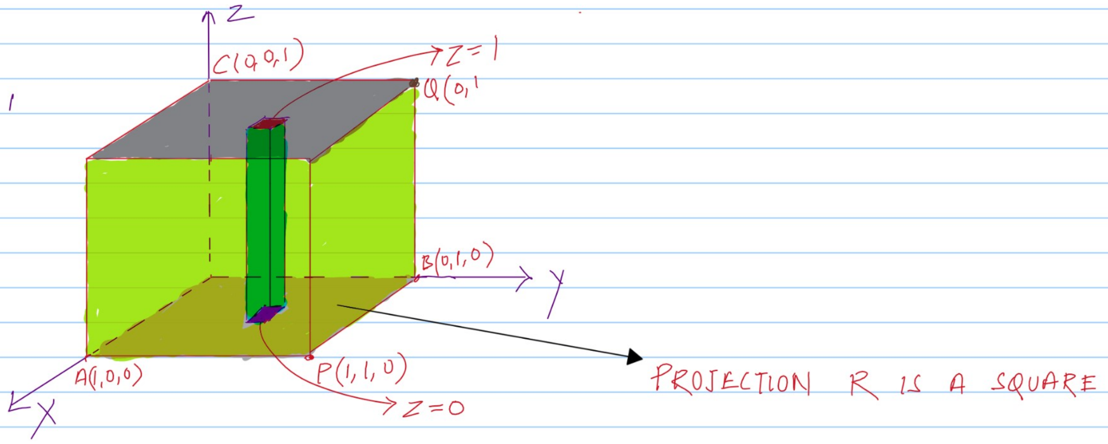
$$=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)$$

$$=\frac{1}{24}$$

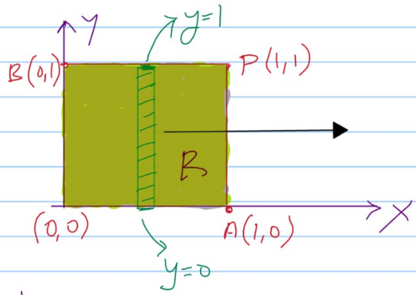
Ques ② Evaluate $\iiint\limits_D x^2y^3z\,dx\,dy\,dz$, where D is the region bounded by planes $x=0,\ y=0,\ z=0,\ x=1,\ y=1\ \&\ z=1$.

SOL.

Limits of z : 0 to 1



Limits of y : 0 to 1
Limits of x : 0 to 1



$$\therefore \quad I = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} x^2 y^3 z \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} x^2 \, dx \times \int_{y=0}^{1} y^3 \, dy \times \int_{z=0}^{1} z \, dz$$

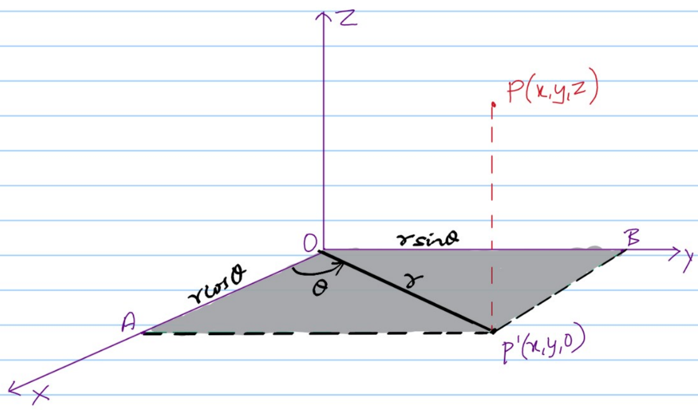
$$= \frac{1}{3} \times \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{24}$$

REMARK – Sometimes it is convenient to evaluate a triple integral using the conversion of cartesian coordinates to
(i) Cylindrical coordinates i.e. (r, θ, z) coordinate system
(ii) Spherical coordinates i.e. (r, θ, ϕ) coordinate system.

CYLINDRICAL COORDINATES

Let $P(x, y, z)$ be a point on the surface of region in 3-space (i.e. $\mathbb{R}^3$) bounded by a cylinder.



From figure,

$$\boxed{\begin{aligned} x &= OA = r\cos\theta \\ y &= OB = r\sin\theta \\ z &= z \end{aligned}}$$

so that $|J| = \left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$

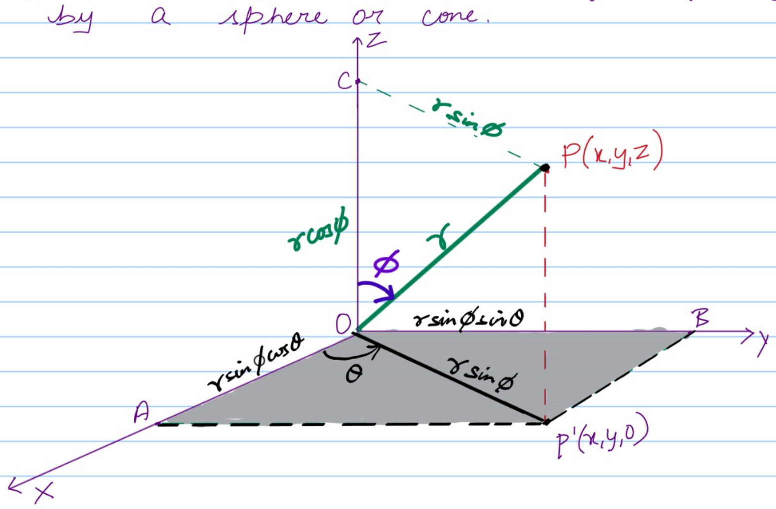
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

$$\boxed{\therefore\ dx\,dy\,dz = |J|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz}$$

SPHERICAL COORDINATES

Let $P(x,y,z)$ be a point on the surface of region in 3-space (i.e. $\mathbb{R}^3$) bounded by a sphere or cone.



From figure, $\boxed{\begin{aligned} x &= OA = r\sin\phi\cos\theta \\ y &= OB = r\sin\phi\sin\theta \\ z &= OC = r\cos\phi \end{aligned}}$

so that

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$= r^2\sin\phi \times \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{vmatrix}$$

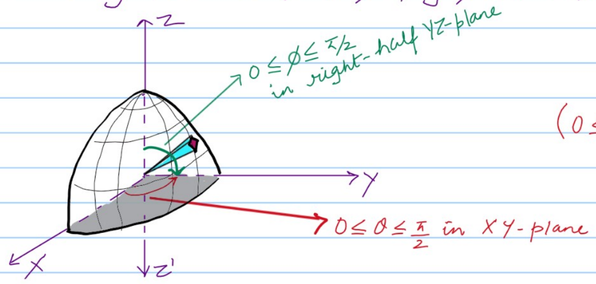
$$= r^2\sin\phi \times \left[-\sin^2\phi\cos^2\theta + \sin\theta\left(-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta\right) - \cos^2\phi\cos^2\theta\right]$$

$$= r^2\sin\phi \times \left[-\sin^2\phi(\cos^2\theta + \sin^2\theta) - \cos^2\phi(\sin^2\theta + \cos^2\theta)\right]$$
$$= r^2\sin\phi \times \left[-\sin^2\phi - \cos^2\phi\right]$$
$$= -r^2\sin\phi$$

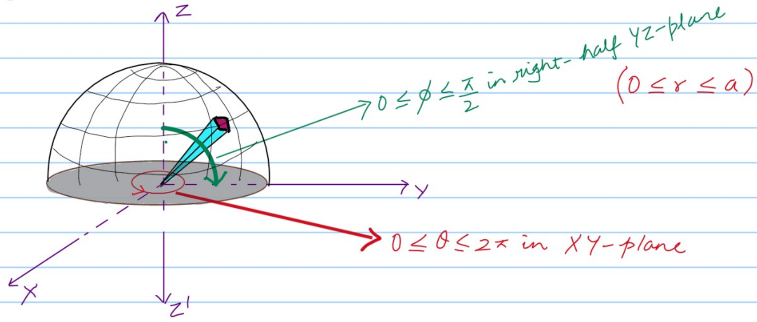
$$\boxed{\therefore\ dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2\sin\phi\,dr\,d\theta\,d\phi \quad (\text{Take + sign for } |J|)}$$

HOW TO FIND LIMITS OF r, θ, ϕ ???

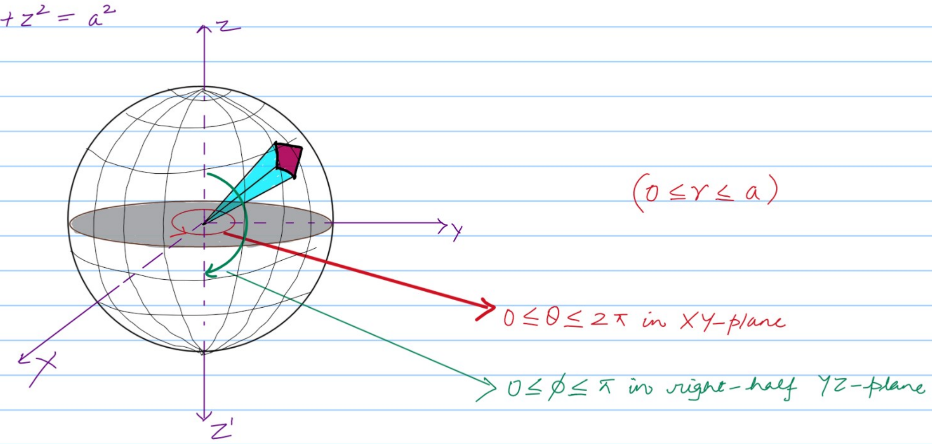
(1) Octant of sphere $x^2+y^2+z^2=a^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$



(2) Hemisphere $x^2+y^2+z^2=a^2, \quad z \geq 0$

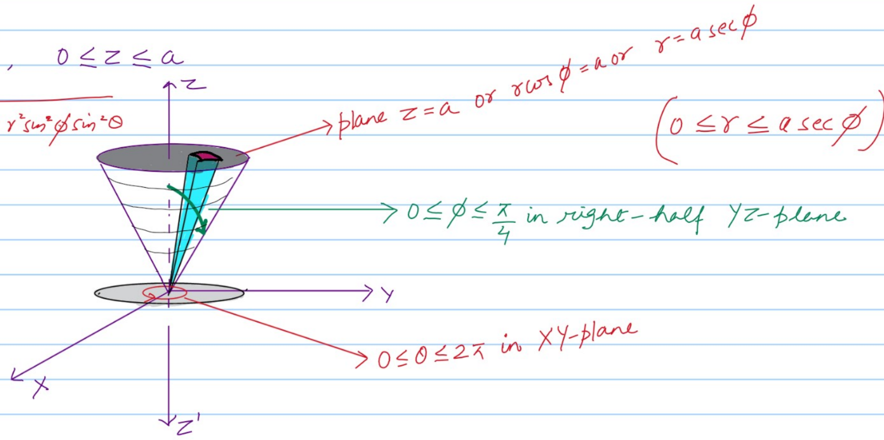


(3) Sphere $x^2+y^2+z^2=a^2$

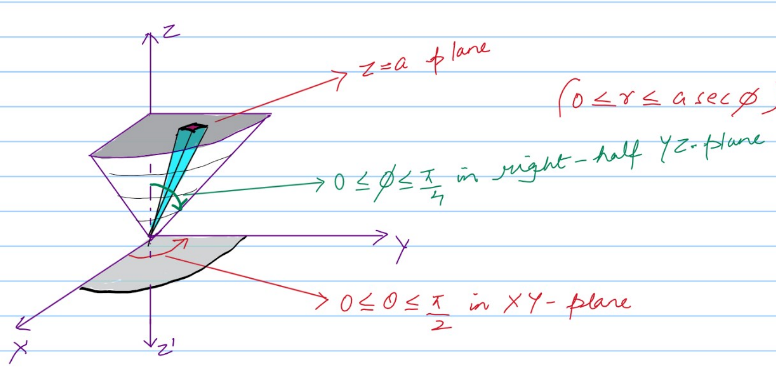


(4) Cone $z=\sqrt{x^2+y^2}, \quad 0 \le z \le a$

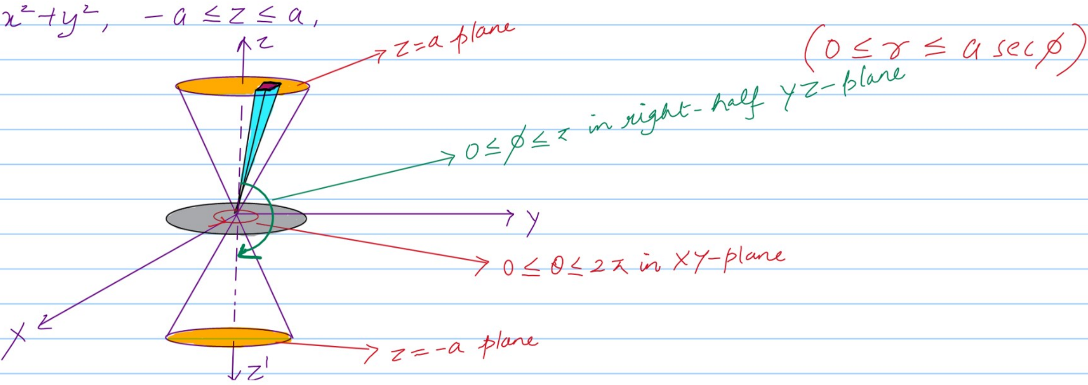
NOTE: $r\cos\phi = \sqrt{r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta}$
$\Rightarrow r\cos\phi = r\sin\phi$
$\Rightarrow \tan\phi = 1$
$\Rightarrow \phi = \pi/4$



(5) Cone $z=\sqrt{x^2+y^2}, \quad 0 \le z \le a, \quad x \geq 0, \quad y \geq 0$



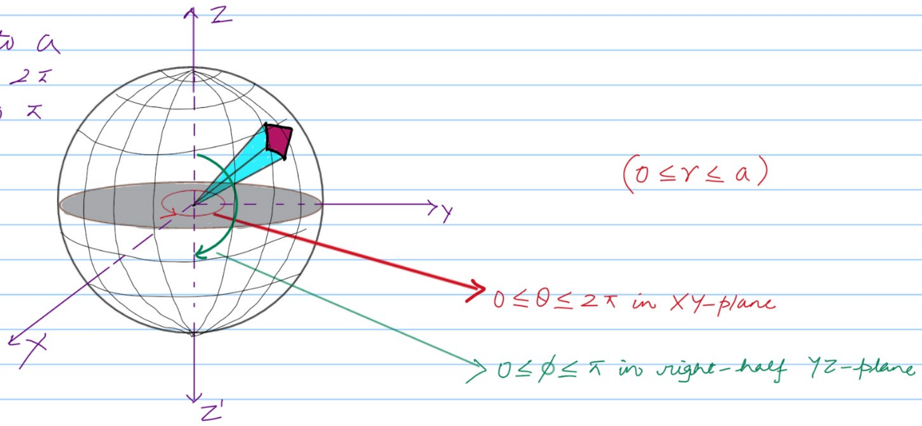
(6) Cone $z^2=x^2+y^2, \quad -a \le z \le a,$



Ques (1) Evaluate $\iiint_D \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$, where D is the region bounded by sphere $x^2+y^2+z^2=a^2$.

SOL. Using spherical coordinates ; $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$
& $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\phi\,dr\,d\theta\,d\phi$

Limits of r : 0 to a
" " θ : 0 to 2π
" " ϕ : 0 to π



$$I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{\sqrt{a^2-r^2}}$$

$$= \int_{\phi=0}^{\pi}\sin\phi\,d\phi \times \int_{\theta=0}^{2\pi} d\theta \times \int_{r=0}^{a}\frac{r^2\,dr}{\sqrt{a^2-r^2}}$$

$$= 2 \times 2\pi \times \int_{r=0}^{a}\frac{r^2\,dr}{\sqrt{a^2-r^2}}$$

Let $r = a\sin t \Rightarrow dr = a\cos t\,dt$

$$I = 4\pi \times \int_{0}^{\pi/2}\frac{a^2\sin^2 t\,(a\cos t\,dt)}{a\cos t}$$

$$= 4a^2\pi \times \int_0^{\pi/2}\sin^2 t\,dt$$

$$= 4a^2\pi \times \frac{\pi}{4}$$

$$= a^2\pi^2$$

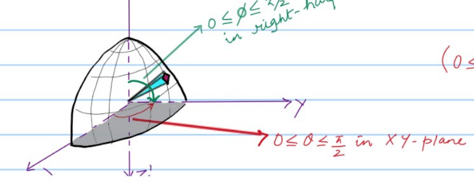
Ques (2) Evaluate $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$

SOL. See $z = \sqrt{a^2-x^2-y^2}$
$\Rightarrow z^2 = a^2-x^2-y^2$

$\Rightarrow x^2+y^2+z^2 = a^2$ (sphere)

Note that $z>0 \Longrightarrow$ sphere in upper half

Also $x>0$ & $y>0$
Thus sphere in +ve octant only.



Using spherical coordinates ; $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$
& $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\phi\,dr\,d\theta\,d\phi$

$$\therefore I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\sin^2\phi\cos\phi\sin\theta\cos\theta\,(r^2\sin\phi\,dr\,d\theta\,d\phi)$$

$$= \int_{r=0}^{a} r^5\,dr \times \int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,d\theta \times \int_{\phi=0}^{\pi/2}\sin^3\phi\cos\phi\,d\phi$$

Let $\sin\phi = t$
$\cos\phi\,d\phi = dt$
$\therefore \int_0^1 t^3\,dt = \frac{1}{4}$

$$= \frac{a^6}{6}\times\frac{1}{2}\times\frac{1}{4}$$

$$= \frac{a^6}{48}$$

Ques (3) Evaluate $\displaystyle\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$

SOL. $\because$ $x>0,\ y>0,\ z>0 \Rightarrow 1^{st}$ octant of XYZ-space
or
1^{st} octant of a sphere of infinite radius

Using spherical coordinates ; $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$
& $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\phi\,dr\,d\theta\,d\phi$

Limits of r : 0 to ∞
" " θ : 0 to $\pi/2$
" " ϕ : 0 to $\pi/2$.

$$\therefore I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty}\frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$= \int_{\phi=0}^{\pi/2}\sin\phi\,d\phi \times \int_{\theta=0}^{\pi/2}d\theta \times \int_{r=0}^{\infty}\frac{r^2\,dr}{(1+r^2)^2}$$

Let $r = \tan t \Rightarrow dr = \sec^2 t\,dt$

$$= 1\times\frac{\pi}{2}\times\int_0^{\pi/2}\frac{\tan^2 t\sec^2 t\,dt}{\sec^4 t}$$

$$= \frac{\pi}{2}\times\int_0^{\pi/2}\sin^2 t\,dt$$

$$= \frac{\pi}{2}\times\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{\pi^2}{8}$$

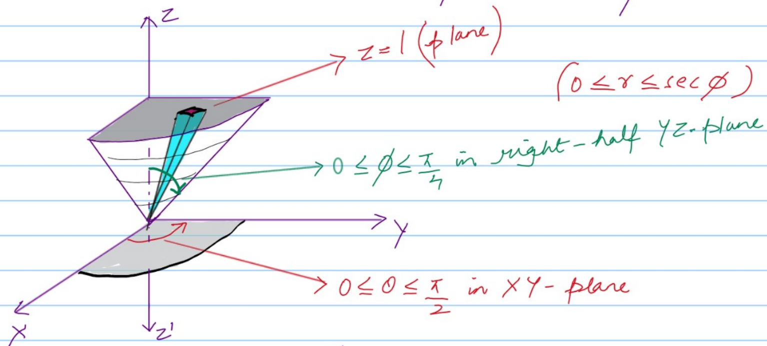
Ques (4) Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=\sqrt{x^2+y^2}}^{1}\frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$

SOL. See $z=\sqrt{x^2+y^2}$ is cone
Also $\sqrt{x^2+y^2} < z < 1$, $x>0$ & $y>0$

$\therefore$ Cone in 1^{st} octant only bounded by plane $z=1$

Using spherical coordinates ; $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$\&$ $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\phi\,dr\,d\theta\,d\phi$



$$\therefore I = \int_{\phi=0}^{\pi/4}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\sec\phi} \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{\sqrt{r^2}}$$

$$= \int_{\phi=0}^{\pi/4}\int_{\theta=0}^{\pi/2} \sin\phi\,\frac{\sec^2\phi}{2}\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/4} \sin\phi\sec^2\phi \times \int_{\theta=0}^{\pi/2} \frac{1}{2}\,d\theta$$

$$= \int_{\phi=0}^{\pi/4} \tan\phi\sec\phi\,d\phi \times \frac{\pi}{4}$$

$$= \Big[\sec\phi\Big]_0^{\pi/4} \times \frac{\pi}{4}$$

$$= \frac{\pi}{4}\left(\sqrt{2}-1\right)$$

Ques ⑤ Evaluate $\iiint_D \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^2}$, where D is the region bounded between spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ $(a>b)$.

SOL. Using spherical coordinates ; $x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$

$\&$ $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$
$= r^2\sin\phi\,dr\,d\theta\,d\phi$

Limits of r : b to a
" " θ : 0 to 2π
" " ϕ : 0 to π

$$\therefore I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=b}^{a} \frac{r^2\sin\phi\,dr\,d\theta\,d\phi}{r^4}$$

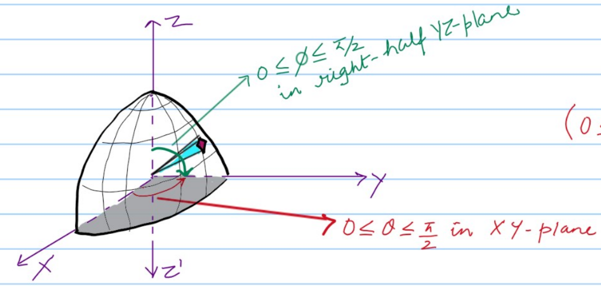
$$= \int_{r=b}^{a}\frac{1}{r^2}\,dr \times \int_{\theta=0}^{2\pi} d\theta \times \int_{\phi=0}^{\pi}\sin\phi\,d\phi$$

$$= \left(\frac{1}{b}-\frac{1}{a}\right) \times 2\pi \times 2$$

$$= 4\pi\left(\frac{a-b}{ab}\right)$$

Ques ⑥ Evaluate $\iiint_D xyz \, dx\,dy\,dz$, where $D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$

SOL. Using spherical coordinates ;
$$x = r \sin\phi \cos\theta$$
$$y = r \sin\phi \sin\theta$$
$$z = r\cos\phi$$
$$\& \quad dx\,dy\,dz = |J|\, dr\, d\theta\, d\phi$$
$$= r^2 \sin\phi \, dr\, d\theta\, d\phi$$



$(0 \leq r \leq 2)$

Limits of r : 0 to 2
" " θ : 0 to $\frac{\pi}{2}$
" " ϕ : 0 to $\frac{\pi}{2}$

$$\therefore I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \cos\phi (r^3 \sin^2\phi \sin\theta\cos\theta)\, r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \int_{r=0}^{2} r^5 \, dr \times \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta \, d\theta \times \int_{\phi=0}^{\pi/2} \cos\phi \sin^3\phi \, d\phi$$

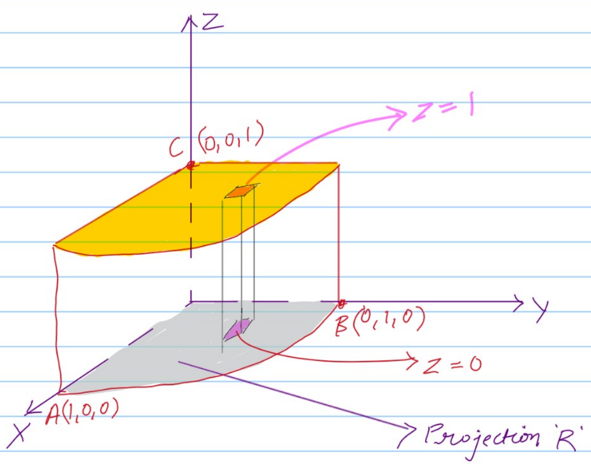
$$= \frac{64}{6} \times \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{4}{3}$$

$I_1 = \int_0^{\pi/2} \cos\phi \sin^3\phi \, d\phi$
Let $\sin\phi = t$
$\cos\phi \, d\phi = -dt$
$\therefore I_1 = \int_0^1 t^3 \, dt$
$= \frac{1}{4}$

Ques ⑦ Evaluate $\iiint_D xyz \, dx\,dy\,dz$ over the region D bounded by the planes $x=0, y=0, z=0, z=1$ and the cylinder $x^2+y^2=1$.

SOL.



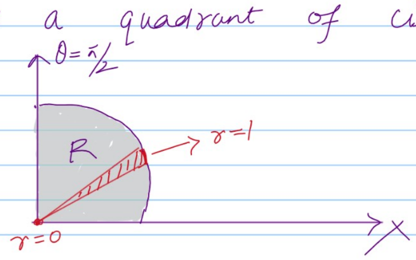
Limits of z : 0 to 1

$$\therefore I = \underbrace{\iint}_{R} \left[\int_{z=0}^{1} xyz \, dz \right] dx\,dy$$

$$\text{or } I = \frac{1}{2} \iint_R xy \, dx\,dy$$

Clearly the projection 'R' is a quadrant of circle $x^2+y^2=1$.

Using (r,θ) coordinates,
Put $x = r\cos\theta$
$y = r\sin\theta$
$\& \ dx\,dy = |J|\, dr\, d\theta$
$= r\, dr\, d\theta$



Limits of r : 0 to 1
" " θ : 0 to $\pi/2$

$$\therefore \quad I = \frac{1}{2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2 \sin\theta\cos\theta\,(r\,dr\,d\theta)$$

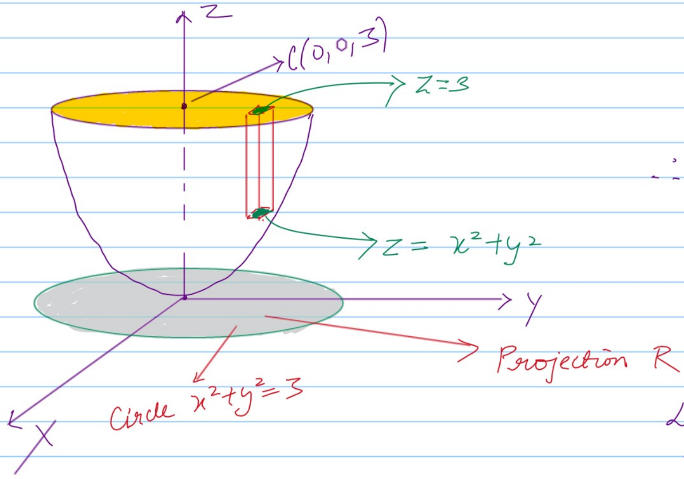
$$= \frac{1}{2}\int_{r=0}^{1} r^3\,dr \times \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{16}$$

Ques (8) Evaluate $\iiint_D (x^2+y^2)\,dx\,dy\,dz$, D is the region bounded by paraboloid $z = x^2+y^2$ and the plane $z=3$.

SOL.



Limits of z: x^2+y^2 to 3

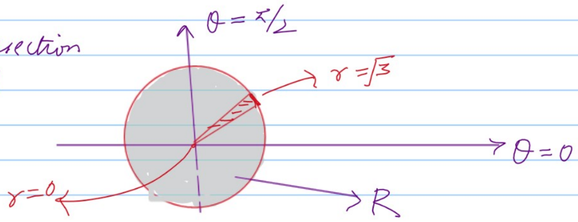
$$\therefore \quad I = \underbrace{\iint}_{R}\left[\int_{z=x^2+y^2}^{3} (x^2+y^2)\,dz\right]dx\,dy$$

$$= \iint_R (x^2+y^2)\left[3-(x^2+y^2)\right]dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$
& $dx\,dy = |J|\,dr\,d\theta$
$= r\,dr\,d\theta$

Circle is obtained by the intersection of plane $z=3$ with paraboloid $x^2+y^2=z$.

$\therefore$ $x^2+y^2 = 3$ is circle
or $(r\cos\theta)^2 + (r\sin\theta)^2 = 3$
or $r^2 = 3$
or $r = \sqrt{3}$ (take +ve sign)



Limit of r: 0 to $\sqrt{3}$
" " θ: 0 to 2π

$$\therefore \quad I = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{3}} r^2\left[3-r^2\right] r\,dr\,d\theta$$

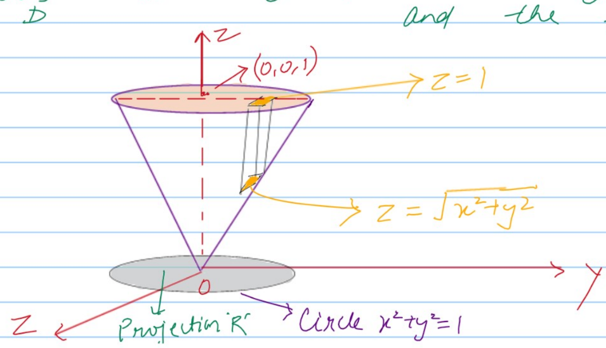
$$= \int_{\theta=0}^{2\pi}\left(\frac{27}{4} - \frac{27}{6}\right)d\theta$$

$$= \frac{9}{4} \times 2\pi$$

$$= \frac{9\pi}{2}$$

Ques (9) Evaluate $\iiint_D \sqrt{x^2+y^2}\,dx\,dy\,dz$, D is region bounded by cone $x^2+y^2=z^2$ and the planes $z=0$, $z=1$.

SOL.



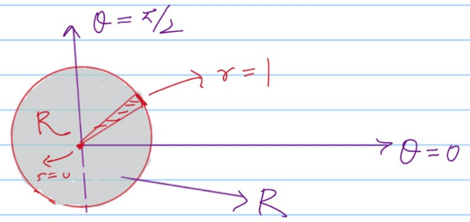
Limits of z: $\sqrt{x^2+y^2}$ to $z=1$

$$\therefore \quad I = \iint_R\left[\int_{z=\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\,dz\right]dx\,dy$$

$$I = \iint_R \sqrt{x^2+y^2}\left[1-\sqrt{x^2+y^2}\right]dx\,dy$$

Let $x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



Limits of r : 0 to 1

" " θ : 0 to 2π

$$\therefore I = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r[1-r]\, r\, dr\, d\theta$$

$$= \int_{r=0}^{1}(r^2 - r^3)\,dr \times \int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{1}{12} \times 2\pi$$

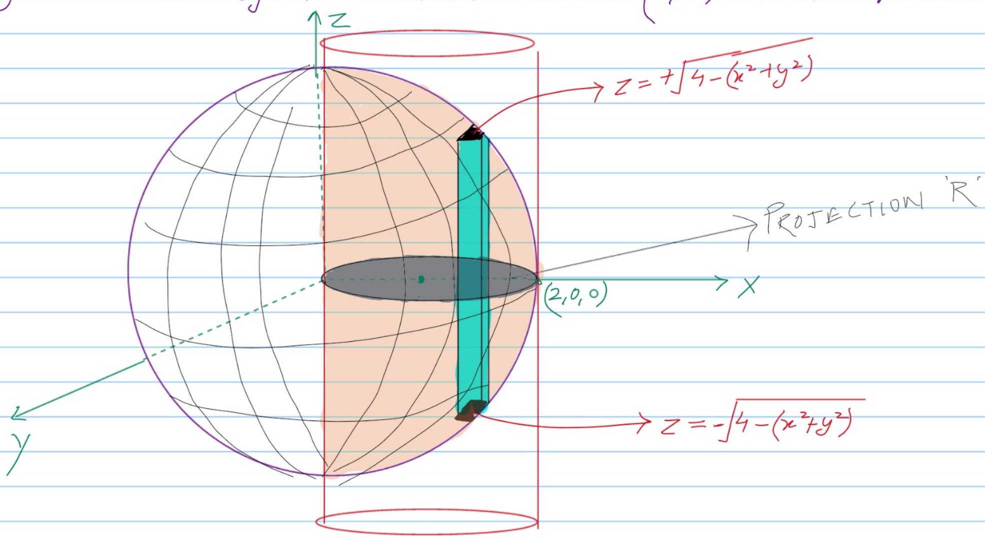
$$= \frac{\pi}{6}$$

Ques (10) Evaluate $\iiint_D z^2\,dx\,dy\,dz$, D is the region common to sphere $x^2+y^2+z^2=4$ and the cylinder $x^2+y^2-2x=0$

SOL. $x^2+y^2-2x=0$

or $\underbrace{x^2-2x+1^2}-1^2+y^2=0$

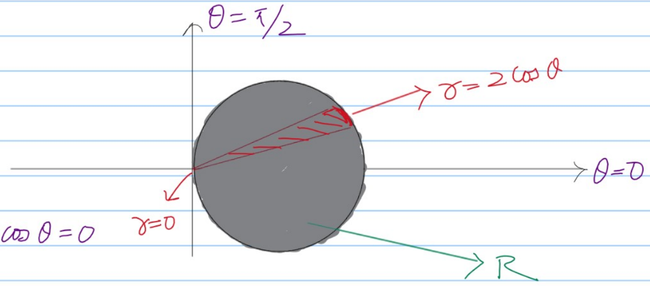
or $(x-1)^2+y^2=1^2$ is cylinder with center $(1,0)$ and radius $= 1$



Limits of z : $-\sqrt{4-x^2-y^2}$ to $\sqrt{4-x^2-y^2}$

$$\therefore I = \iint_R \left[\int_{z=-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} z^2\,dz\right] dx\,dy$$

$$= \frac{2}{3}\iint_R (4-x^2-y^2)^{3/2}\,dx\,dy$$



Let $x = r\cos\theta,\ y = r\sin\theta$

& $dx\,dy = r\,dr\,d\theta$

Then $x^2+y^2-2x=0$

$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r(r-2\cos\theta) = 0$

$\Rightarrow r - 2\cos\theta = 0$ (as $r \neq 0$)

$\Rightarrow \boxed{r = 2\cos\theta}$

Limits of r : 0 to $2\cos\theta$
 " " $\theta : -\pi/2$ to $\pi/2$

$$\therefore \quad I = \frac{2}{3}\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2\cos\theta}(4-r^2)^{3/2}\, r\, dr\, d\theta$$

$$= -\frac{1}{3}\int_{\theta=-\pi/2}^{\pi/2}\left[\int_{r=0}^{2\cos\theta}(4-r^2)^{3/2}(-2r)\, dr\right] d\theta$$

$$\boxed{\int [f(r)]^n f'(r)\, dr = \frac{[f(r)]^{n+1}}{n+1} + c}$$

$$= -\frac{1}{3}\int_{\theta=-\pi/2}^{\pi/2}\left[\frac{(4-r^2)^{5/2}}{5/2}\right]_{r=0}^{2\cos\theta} d\theta$$

$$= -\frac{2}{15}\times 32\int_{\theta=-\pi/2}^{\pi/2}(\sin^5\theta - 1)\, d\theta$$

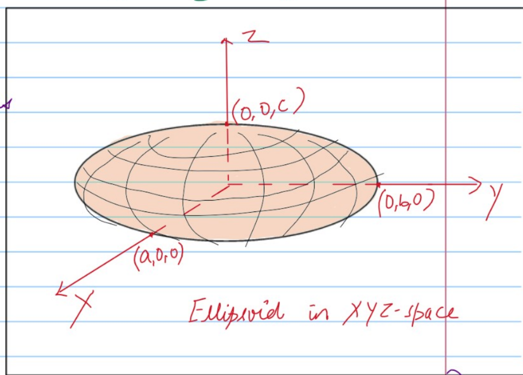
$$= -\frac{64}{15}(0-\pi) \quad --- \left[\because \int_{-a}^{a} f(x)\, dx = 0, \text{ if } f(x) \text{ is odd funct}^n\right]$$

$$= \frac{64}{15}\pi$$

Ques (11) Evaluate $\iiint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\, dx\, dy\, dz$, D is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

SOL. Let $\frac{x}{a}=u$, $\frac{y}{b}=v$, $\frac{z}{c}=w$

$\Rightarrow x = au$, $y = bv$ & $z = cw$ are transformations



$$|J| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \neq 0$$

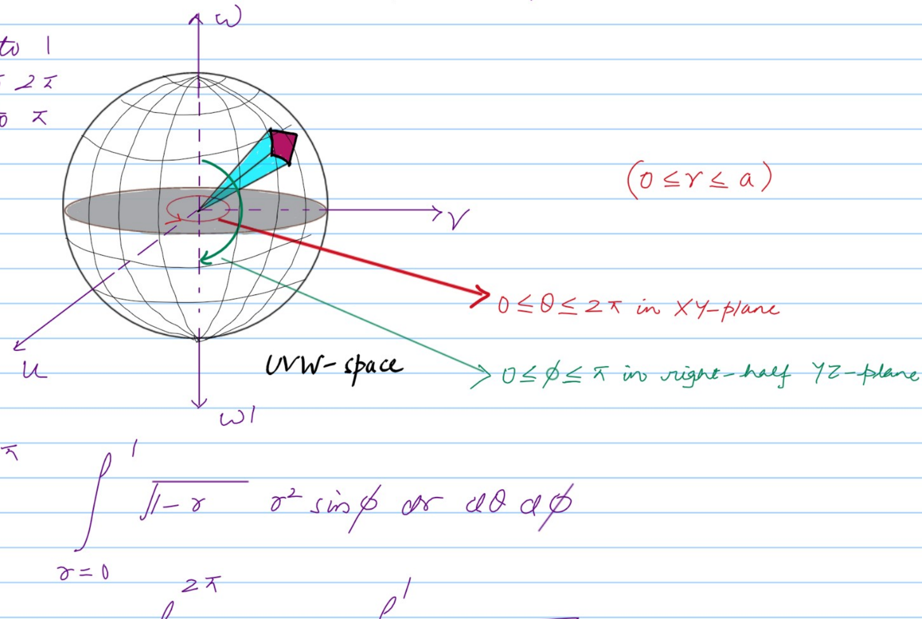
Then $I = \iiint_{D'} \sqrt{1-u^2-v^2-w^2}\, |J|\, du\, dv\, dw$

$= abc \iiint_{D'} \sqrt{1-u^2-v^2-w^2}\, du\, dv\, dw$, ($D'$ is a sphere $u^2+v^2+w^2=1$ in uvw-space)

Using spherical coordinates: $u = r\sin\phi\cos\theta$, $v = r\sin\phi\sin\theta$, $w = r\cos\phi$

& $du\,dv\,dw = r^2\sin\phi\, dr\, d\theta\, d\phi$

Limits of r : 0 to 1
 " " θ : 0 to 2π
 " " ϕ : 0 to π



$$\therefore \quad I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{1-r}\; r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi}\sin\phi\, d\phi \times \int_{\theta=0}^{2\pi} d\theta \times \int_{r=0}^{1} r^2\sqrt{1-r}\, dr$$

$$= 2 \times 2\pi \times I_1 \text{ (say)} \quad \text{——— (1)}$$

Here $I_1 = \int_0^1 r^2 \sqrt{1-r}\, dr$

$$\Rightarrow I_1 = \int_0^1 (1-r)^2 \sqrt{1-(1-r)}\, dr \quad --- \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \sqrt{r}\,(1-r)^2\, dr$$

$$= \int_0^1 \left(r^{1/2} + r^{5/2} - 2r^{3/2}\right) dr$$

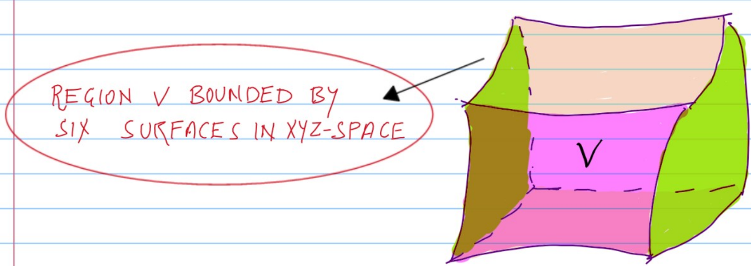
$$= \frac{2}{3} + \frac{2}{7} - \frac{4}{5}$$

$$= \frac{2}{35}$$

Thus, (1) gives; $\quad I = \dfrac{8\pi}{35}$

Ques (12) Evaluate $\iiint_V x^2y^2z^2\, dV$, where V is the region bounded by surfaces $xy=4$, $xy=8$, $yz=1$, $yz=4$, $zx=25$ and $zx=49$.

SOL.

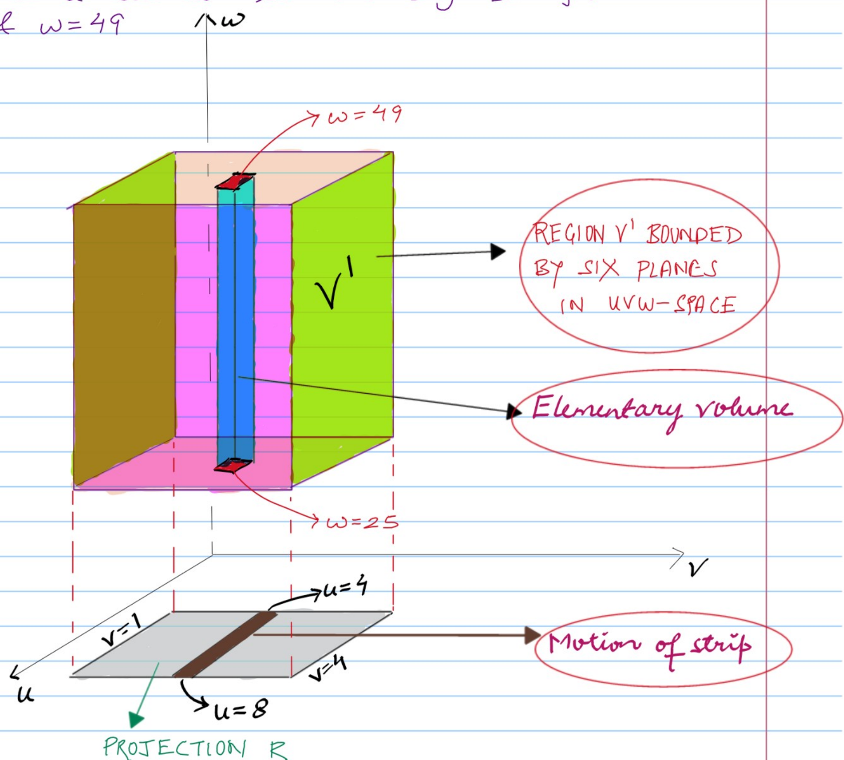


Let $xy = u$, $yz = v$ & $zx = w$ be the transformations

Then $$|J| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|} = \frac{1}{\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}} = \frac{1}{\begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix}} = \frac{1}{2xyz}$$

Then, the region V' in uvw-space is a cuboid bounded by six planes:
$u=4$, $u=8$, $v=1$, $v=4$, $w=25$ & $w=49$

& $dx\,dy\,dz = |J|\, du\,dv\,dw = \dfrac{1}{2xyz}\, du\,dv\,dw$



$$\therefore I = \int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} (x^2y^2z^2)\,|J|\, du\,dv\,dw$$

$$= \int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} (x^2y^2z^2)\left(\frac{1}{2xyz}\right) du\,dv\,dw$$

$$= \frac{1}{2}\int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} (xyz)\, du\,dv\,dw$$

$$= \frac{1}{2}\int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} \sqrt{(xyz)^2}\, du\,dv\,dw$$

$$= \frac{1}{2}\int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} \sqrt{(xy)(yz)(zx)}\, du\,dv\,dw$$

$$= \frac{1}{2}\int_{u=4}^{8}\int_{v=1}^{4}\int_{w=25}^{49} \sqrt{uvw}\, du\,dv\,dw$$

$$= \frac{1}{2}\int_4^8 u^{1/2}\,du \times \int_1^4 v^{1/2}\,dv \times \int_{25}^{49} w^{1/2}\,dw$$

$$= \frac{1}{2} \times \frac{2}{3}\left(8^{3/2} - 4^{3/2}\right) \times \frac{2}{3}\left(4^{3/2} - 1\right) \times \frac{2}{3}\left(49^{3/2} - 25^{3/2}\right)$$

$$= \frac{48832}{27}\left(\sqrt{2} - 1\right)$$

REMARKS

1. Area bounded by region R in 2-space is: (a) $\iint_R dx\,dy$ in xy-plane

   (b) $\iint_R r\,dr\,d\theta$ in $r\theta$-plane

2. Volume bounded by region D in 3-space is: (a) $\iiint_D dx\,dy\,dz$ in XYZ-space

   (b) $\iiint_D r^2 \sin\phi\,dr\,d\theta\,d\phi$ in $r\theta\phi$-space