EULER'S THEOREM

WHAT IS A HOMOGENEOUS FUNCTION ???

A function $f(x,y)$ is homogeneous in input variables x & y if each term is of degree equal to n. Mathematically,

$$\boxed{\text{If } f(tx,ty) = t^n f(x,y) \text{ then 'f' is homogeneous in 'x' \& 'y' of degree 'n'.}}$$

For example: $f(x,y) = x^2 \sin\left(\frac{x}{y}\right) + y^2 \log_e\left(\frac{x}{y}\right)$ is homogeneous in 'x' & 'y' of degree 2.

REASON:

$$f(tx,ty) = (tx)^2 \sin\left(\frac{\cancel{t}x}{\cancel{t}y}\right) + (ty)^2 \log_e\left(\frac{\cancel{t}x}{\cancel{t}y}\right)$$

$$= t^2\left[x^2 \sin\left(\frac{x}{y}\right) + y^2 \log_e\left(\frac{x}{y}\right)\right]$$

$$= t^2 f(x,y)$$

STATEMENT OF THEOREM:

If 'u' is a homogeneous funct^n in 'x' & 'y' of degree 'n' then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

PROOF: Let $u = x^n f\left(\frac{x}{y}\right)$ $\left[\text{or } x^n f\left(\frac{y}{x}\right) \text{ or } y^n f\left(\frac{y}{x}\right) \text{ or } y^n f\left(\frac{x}{y}\right)\right]$

Diff. partially w.r.t. 'x';

$$\frac{\partial u}{\partial x} = n x^{n-1} f\left(\frac{x}{y}\right) + x^n f'\left(\frac{x}{y}\right) \times \frac{1}{y}$$

Diff. partially w.r.t. 'y';

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{x}{y}\right) \times \frac{-x}{y^2}$$

$$LHS = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \left[n x^n f\left(\frac{x}{y}\right) + \cancel{\frac{x^{n+1}}{y} f'\left(\frac{x}{y}\right)}\right] - \cancel{\frac{x^{n+1}}{y} f'\left(\frac{x}{y}\right)}$$

$$= n x^n f\left(\frac{x}{y}\right)$$

$$= nu = RHS$$

REMARK: If u is homogeneous function in x, y & z of degree 'n' then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Ques 1 If $u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e x - \log_e y}{y^2}$. Prove $x u_x + y u_y + 2u = 0$.

Sol $\because u(tx,ty) = \frac{1}{(tx)^2} + \frac{1}{(tx)(ty)} + \frac{\log_e(tx/ty)}{(ty)^2}$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e(x/y)}{y^2}\right]$$

$$= t^{-2} u(x,y)$$

$\therefore$ u is homogeneous in 'x' & 'y' of degree '-2'.

Using Euler's theorem;

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$$

$$\Rightarrow x u_x + y u_y + 2u = 0$$

Ques 2 If $u(x,y) = e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{x}{y}\right)$. Find the value of $x u_x + y u_y$.

Sol $\because u(tx,ty) = e^{tx/ty}\sin\left(\frac{tx}{ty}\right) + e^{ty/tx}\cos\left(\frac{tx}{ty}\right)$

$$= e^{x/y}\sin\left(\frac{x}{y}\right) + e^{y/x}\cos\left(\frac{x}{y}\right)$$

$$= t^0 u(x,y)$$

$\therefore$ u is homogeneous in 'x' & 'y' of degree 0.

Use Euler's theorem

$$x u_x + y u_y = 0u$$
$$\Rightarrow x u_x + y u_y = 0$$

Ques 3 If $T(x,y) = \sin\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \frac{x^2y}{x+y}$. Find $xT_x + yT_y$.

Sol ⚠ CAUTION – Each individual term has different degree viz. 0, 1 and 2.

Let $T(x,y) = u(x,y) + v(x,y) + w(x,y)$ ——— (1)

where $u(x,y) = \sin\left(\frac{xy}{x^2+y^2}\right)$ is homogeneous of degree 0 $[\because u(tx,ty) = t^0 u(x,y)]$

By E.T.,

$$x u_x + y u_y = 0 \quad \text{——— (a)}$$

and $v(x,y) = \sqrt{x^2+y^2}$ is homogeneous of degree 1. $[\because v(tx,ty) = t\,v(x,y)]$

By E.T.,

$$x v_x + y v_y = v \quad \text{——— (b)}$$

and $w(x,y) = \frac{x^2y}{x+y}$ is homogeneous of degree 2. $[\because w(tx,ty) = t^2 w(x,y)]$

By E.T.,

$$x w_x + y w_y = 2w \quad \text{——— (c)}$$

(a) + (b) + (c); $\quad x(u_x+v_x+w_x) + y(u_y+v_y+w_y) = 0 + v + 2w$

$$\Rightarrow x T_x + y T_y = \sqrt{x^2+y^2} + \frac{2x^2y}{x+y}$$

Ques 4 If $v(r,\theta) = \frac{1}{r}f(\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$. Show that $xv_x + yv_y + v = 0$

Sol Given $x = r\cos\theta$ & $y = r\sin\theta$

Squaring & adding;

$$x^2+y^2 = r^2 \Longrightarrow \boxed{r = \sqrt{x^2+y^2}}$$

Dividing, $\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

Then $v(x,y) = \frac{1}{\sqrt{x^2+y^2}} f\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

Now $v(tx,ty) = \frac{1}{\sqrt{t^2x^2+t^2y^2}} f\left(\tan^{-1}\left(\frac{ty}{tx}\right)\right)$

$$= t^{-1} v(x,y)$$

$\therefore$ v is homogeneous in x & y of degree -1.

By E.T.;

$$x v_x + y v_y = -1v$$
$$\Rightarrow x v_x + y v_y + v = 0$$

REMARK If $f\circ g$ is non-homogeneous but g is homogeneous of degree n

Let $v(x,y) = f\circ g(x,y)$

Then $f^{-1}(v) = g$ is homogeneous in x & y of degree n.

By E.T; $x\dfrac{\partial g}{\partial x} + y\dfrac{\partial g}{\partial y} = ng$

$$\Rightarrow x\frac{\partial}{\partial x} f^{-1}(v) + y\frac{\partial}{\partial y} f^{-1}(v) = n f^{-1}(v)$$

Ques 5 Let $u(x,y) = \sin\left(\dfrac{x^2+y^2}{x-y}\right)$. Find $xu_x + yu_y$.

Sol NOTE → $u(tx,ty) = \sin\left(\dfrac{(tx)^2+(ty)^2}{tx-ty}\right) = \sin t\left(\dfrac{x^2+y^2}{x-y}\right) \neq t^n u(x,y)$

Here

$v(\text{say}) = \sin^{-1}u = \dfrac{x^2+y^2}{x-y}$ is homogeneous in x & y of degree 1.

Then by E.T.,

$$x v_x + y v_y = 1v$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin^{-1}u) + y\frac{\partial}{\partial y}(\sin^{-1}u) = \sin^{-1}u$$

$$\Rightarrow x\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x} + y\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} = \sin^{-1}u$$

$$\Rightarrow xu_x + yu_y = \sqrt{1-u^2}\,\sin^{-1}u$$

Ques 6 Let $u(x,y) = \log_e\left(\dfrac{x^3+y^3+x^2y}{\sqrt{x+y}}\right)$. Find $xu_x + yu_y$.

Sol

Here

$v(\text{say}) = e^u = \dfrac{x^3+y^3+x^2y}{\sqrt{x+y}}$ is homogeneous in x & y of degree $\frac{5}{2}$.

$\because v(tx,ty) = \dfrac{(tx)^3+(ty)^3+(tx)^2(ty)}{\sqrt{tx+ty}}$

$= t^{5/2}v$

By E.T.,

$$xv_x + yv_y = \frac{5}{2}v$$

$$\Rightarrow x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = \frac{5}{2}e^u$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = \frac{5}{2}e^u$$

$$\Rightarrow xu_x + yu_y = \frac{5}{2}$$

AN IMPORTANT DEDUCTION

For a function $u(x,y)$, suppose $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = G(u)$

Then

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)\,[G'(u)-1]}$$

PROOF Suppose $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = G(u)$ ——— (1)

Diff. (1) partially w.r.t. 'x',

$$\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + y\frac{\partial^2 u}{\partial x\partial y} = G'(u)\frac{\partial u}{\partial x} \quad \text{——— (a)}$$

Diff. (1) partially w.r.t. 'y',

$$x\frac{\partial^2 u}{\partial y\partial x} + \left(y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\right) = G'(u)\frac{\partial u}{\partial y} \quad \text{——— (b)}$$

$x \times$ (a) $+ y \times$ (b) ; $\left(x^2\dfrac{\partial^2 u}{\partial x^2} + x\dfrac{\partial u}{\partial x} + xy\dfrac{\partial^2 u}{\partial x\partial y}\right) + \left(xy\dfrac{\partial^2 u}{\partial y\partial x} + y^2\dfrac{\partial^2 u}{\partial y^2} + y\dfrac{\partial u}{\partial y}\right) = G'(u)\left[x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right.$

$$\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G'(u)\,(x u_x + y u_y) - (x u_x + y u_y)$$

$$\Rightarrow x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G'(u)\, G(u) - G(u) \qquad [\text{using } 1]$$

$$\Rightarrow \boxed{x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1]}$$

REMARK If u is homogeneous then $G(u) = nu$ & $G'(u) = n$

Then $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = nu(n-1) = n(n-1)u$

Ques 7 If $z = \left(\frac{x}{y}\right)^{y/x}$, prove that $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = 0$

Sol Here $z(tx, ty) = \left(\frac{tx}{ty}\right)^{ty/tx} = t^0 z(x,y)$

$\therefore$ z is hom. in x & y of degree 0.

By E.T.,

$$x z_x + y z_y = 0 \cdot z = 0 = G(u) \text{ (say)}$$

Then

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] = 0$$

Ques 8 If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$ prove that $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} + x z_x + y z_y = n^2 z$

Sol Let $z = u + v$ (say) ——— 1

where $u = x^n f\left(\frac{y}{x}\right)$ is hom. in x & y of degree n

By E.T., $x u_x + y u_y = nu = G(u)$ (say) ——— a

& $v = y^{-n} f\left(\frac{x}{y}\right)$ is hom. in x & y of degree $-n$

By E.T., $x v_x + y v_y = -nv = G(v)$ (say) ——— b

From a; $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = G(u)\,[G'(u) - 1] = nu(n-1)$

$\Rightarrow x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = n(n-1)u$ ——— c

From b; $x^2 v_{xx} + 2xy\, v_{xy} + y^2 v_{yy} = G(v)\,[G'(v) - 1] = -nv(-n-1)$

$\Rightarrow x^2 v_{xx} + 2xy\, v_{xy} + y^2 v_{yy} = n(n+1)v$ ——— d

a + b + c + d;

$$x^2(u_{xx} + v_{xx}) + 2xy\,(u_{xy} + v_{xy}) + y^2(u_{yy} + v_{yy}) + x(u_x + v_x) + y(u_y + v_y) = nu - nv + n(n-1)u + n(n+1)v$$

$$\Rightarrow x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} + x z_x + y z_y = n^2(u+v) = n^2 z$$

Ques 9 If $z = \frac{(x^2+y^2)^n}{2n(2n-1)} + x f\left(\frac{y}{x}\right) + \phi\left(\frac{x}{y}\right)$. Prove $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = (x^2+y^2)^n$

Sol Let $z = u$ (say) $+ v$ (say) $+ w$ (say) ——— 1

where

$u = \frac{(x^2+y^2)^n}{2n(2n-1)}$ is hom. in x & y of degree $2n$.

By E.T, $x u_x + y u_y = 2nu = G(u)$ (say)

Then

$$\begin{aligned} x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} &= G(u)\,[G'(u) - 1] \\ &= 2nu\,[2n-1] \\ &= 2n(2n-1)u \end{aligned}$$ ——— a

Also ; $v = x f\left(\frac{y}{x}\right)$ is hom. in x & y of degree 1

By E.T, $x v_x + y v_y = 1 \cdot v = v = G(v)$ (say)

Then $x^2 v_{xx} + 2xy v_{xy} + y^2 v_{yy} = G(v)\left[G'(v) - 1\right]$
$= v\left[1-1\right]$
$= 0$ ——— (b)

Also, $w = \phi\left(\frac{x}{y}\right)$ is hom. in x & y of degree 0.

By E.T.,
$x w_x + y w_y = 0 \cdot w = 0 = G(w)$ (say)

$x^2 w_{xx} + 2xy w_{xy} + y^2 w_{yy} = G(w)\left[G'(w) - 1\right]$
$= 0$ ——— (c)

(a) + (b) + (c) ;

$$x^2(u_{xx} + v_{xx} + w_{xx}) + 2xy(u_{xy} + v_{xy} + w_{xy}) + y^2(u_{yy} + v_{yy} + w_{yy}) = 2n(2n-1)u$$

$$\Rightarrow x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = \frac{\cancel{2n(2n-1)}(x^2+y^2)^n}{\cancel{2n(2n-1)}}$$

$$\Rightarrow x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = (x^2+y^2)^n$$