$\boxed{\text{CASE - 3}}$ If $F(x) = p(x)$ (a polynomial of degree n)

Then

$$y_p = PI = \frac{1}{f(D)} p(x)$$

$$= [f(D)]^{-1} p(x)$$

Expand $[f(D)]^{-1}$ using binomial expansion by taking common the least power of D.

$\boxed{\text{REMEMBER}}$

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots$
2. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \cdots$
3. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
4. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
5. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
6. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$\boxed{\text{CASE - 4}}$ If $F(x) = e^{ax}\phi(x)$, where $\phi(x)$ is any function as discussed in previous cases

Then
$$y_p = PI = \frac{1}{f(D)} e^{ax}\phi(x)$$

$$= e^{ax} \frac{1}{f(D+a)} \phi(x)$$

Ques 1) $(D^2 - 2D + 1)y = x^2 + x + 1$

Sol) Aux. eqn $\rightarrow$ $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1 \,\&\, 1$

$CF = (c_1 + c_2 x)e^x$

$$PI = \frac{1}{D^2 - 2D + 1}(x^2 + x + 1)$$

$$= \frac{1}{(D-1)^2}(x^2 + x + 1)$$

$$= \frac{1}{(1-D)^2}(x^2 + x + 1)$$

$$= (1-D)^{-2}(x^2 + x + 1)$$

$$= [1 + 2D + 3D^2 + 4D^3 + \cdots](x^2 + x + 1)$$

$$= (x^2 + x + 1) + 2D(x^2 + x + 1) + 3D^2(x^2 + x + 1) + 4D^3(x^2 + x + 1) + \cdots$$

$$= (x^2 + x + 1) + 2(2x + 1) + 3(2) + 4(0) + 5(0) + \cdots$$

$$= x^2 + 5x + 9$$

$\therefore$ $y = CF + PI$ is soln.

Ques 2) $(D^2 + D + 2)y = x^2$

Sol) Aux. eqn $\rightarrow$ $m^2 + m + 2 = 0$

$\Rightarrow m = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

$$CF = e^{-x/2}\left[c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right]$$

$$PI = \frac{1}{D^2 + D + 2} x^2$$

$$= \frac{1}{2\left[1 + \left(\frac{D^2 + D}{2}\right)\right]} x^2$$

least power of D i.e. $2D^0$ is taken out common

$$= \frac{1}{2}\left[1+\left(\frac{D^2+D}{2}\right)\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+D}{2}\right)+\left(\frac{D^2+D}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{D}{2}+\frac{D^4}{4}+\frac{D^2}{4}+D^3\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D}{2}-\frac{D^2}{4}+D^3+\cdots\right] x^2$$

$$= \frac{1}{2}\left[x^2-\frac{D}{2}(x^2)-\frac{D^2}{4}(x^2)+D^3(x^2)+\cdots\right]$$

$$= \frac{1}{2}\left[x^2-x-\frac{1}{2}\right]$$

$$= \frac{x^2}{2}-\frac{x}{2}-\frac{1}{4}$$

$\therefore$ $y = CF + PI$ is sol^n

Ques 3 $(D^2-4D+4)y = e^{2x}\sin 3x$

Sol Aux. eq^n $\rightarrow$ $m^2-4m+4=0$

$\Rightarrow (m-2)^2=0$

$\Rightarrow m = 2 \;\&\; 2$

$CF = (c_1+c_2x)e^{2x}$

$$PI = \frac{1}{D^2-4D+4} e^{2x}\sin 3x$$

$$= \frac{1}{(D-2)^2} e^{2x}\sin 3x$$

$$= e^{2x}\frac{1}{(D+2-2)^2}\sin 3x$$

$$= e^{2x}\frac{1}{D^2}\sin 3x$$

$$= e^{2x}\frac{1}{D}\int \sin 3x\, dx$$

$$= e^{2x}\frac{1}{D}\left(-\frac{\cos 3x}{3}\right)$$

$$= -\frac{e^{2x}}{3}\frac{1}{D}\cos 3x$$

$$= -\frac{e^{2x}}{3}\int \cos 3x\, dx$$

$$= -\frac{e^{2x}}{3}\times\frac{\sin 3x}{3}$$

$$= -\frac{e^{2x}\sin 3x}{9}$$

$$\therefore\; y = CF + PI = (c_1+c_2x)e^{2x} - \frac{e^{2x}\sin 3x}{9}$$

Ques 4 $(D^2+2D+1)y = e^{-x}\,x\sin x$

Sol Aux eq^n $\rightarrow$ $m^2+2m+1=0$

$\Rightarrow (m+1)^2=0$

$\Rightarrow m = -1 \;\&\; -1$

$CF = (c_1+c_2x)e^{-x}$

$$PI = \frac{1}{(D+1)^2} e^{-x}(x\sin x)$$

$$= e^{-x}\frac{1}{(D-1+1)^2} x\sin x$$

$$= e^{-x}\frac{1}{D^2}(x\sin x)$$

$$= e^{-x}\frac{1}{D}\int x\sin x\, dx$$

$$= e^{-x}\frac{1}{D}\left[(x)(-\cos x)-(1)(-\sin x)\right]$$

$$= e^{-x}\frac{1}{D}\left[\sin x - x\cos x\right]$$

$$= e^{-x}\left[\frac{1}{D}(\sin x) - \frac{1}{D}(x\cos x)\right]$$

$$\boxed{\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx}$$

(Integration by parts)

TIP!

must be a polynomial

$$\int u v\,dx = (u)(v_1) - (u')(v_2) + (u'')(v_3) - \cdots$$

e.g. $\int x^3 e^x dx = (x^3)(e^x) - (3x^2)(e^x) + (6x)(e^x) - (6)(e^x)$

$= e^x\left[x^3-3x^2+6x-6\right]$

$$= e^{-x}\left[\int \sin x\,dx - \int x\cos x\,dx\right]$$
$$= e^{-x}\left[-\cos x - \{(x)(\sin x) - (1)(-\cos x)\}\right]$$
$$= e^{-x}\left[-2\cos x - x\sin x\right]$$

$$\therefore\ y = CF + PI = (c_1 + c_2x)e^{-x} - e^{-x}\left[2\cos x + x\sin x\right]$$

Ques 5) $(D^2+2D+1)y = \dfrac{e^{-x}}{x+1}$

Sol) $CF = (c_1 + c_2x)e^{-x}$

$$PI = \frac{1}{(D+1)^2}\, e^{-x}\left(\frac{1}{x+1}\right)$$
$$= e^{-x}\frac{1}{(D-1+1)^2}\left(\frac{1}{x+1}\right)$$
$$= e^{-x}\frac{1}{D^2}\left(\frac{1}{x+1}\right)$$
$$= e^{-x}\frac{1}{D}\int\frac{1}{x+1}\,dx$$
$$= e^{-x}\frac{1}{D}\log|x+1|$$
$$= e^{-x}\int\log(x+1)\,dx \qquad (\text{where } x+1>0)$$
$$= e^{-x}\int\log(x+1)\cdot 1\,dx$$
$$= e^{-x}\left[\log(x+1)\cdot x - \int\frac{1}{x+1}\cdot x\,dx\right]$$

How?

$\boxed{\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx \quad \text{(Integration by parts)}}$

$$= e^{-x}\left[x\log(x+1) - \int\frac{(x+1)-1}{x+1}\,dx\right]$$
$$= e^{-x}\left[x\log(x+1) - \int dx + \int\frac{dx}{x+1}\right]$$
$$= e^{-x}\left[x\log(x+1) - x + \log(x+1)\right]$$

$\therefore\ y = CF + PI = (c_1+c_2x)e^{-x} + e^{-x}\left[x\log(x+1) - x + \log(x+1)\right]$ is solⁿ

Ques 6) $(D^2+4D-12)y = (x-1)e^{2x}$

Sol) Aux eqⁿ $\rightarrow m^2+4m-12=0$
$\Rightarrow m = -6\ \&\ 2$

$CF = c_1e^{-6x} + c_2e^{2x}$

$$PI = \frac{1}{D^2+4D-12}\, e^{2x}(x-1)$$
$$= e^{2x}\frac{1}{(D+2)^2+4(D+2)-12}(x-1)$$
$$= e^{2x}\frac{1}{D^2+8D}(x-1)$$

Take least power of D common

$$= e^{2x}\frac{1}{8D\left[1+\frac{D}{8}\right]}(x-1)$$
$$= \frac{e^{2x}}{8}\frac{1}{D}\left[1+\frac{D}{8}\right]^{-1}(x-1)$$
$$= \frac{e^{2x}}{8}\cdot\frac{1}{D}\left[1-\frac{D}{8}+\frac{D^2}{8^2}-\cdots\right](x-1)$$
$$= \frac{e^{2x}}{8}\frac{1}{D}\left[(x-1)-\frac{D}{8}(x-1)\right]$$
$$= \frac{e^{2x}}{8}\frac{1}{D}\left[x-1-\frac{1}{8}\right]$$
$$= \frac{e^{2x}}{8}\frac{1}{D}\left(x-\frac{9}{8}\right)$$

$$= \frac{e^{2x}}{8} \int \left(x - \frac{9}{8}\right) dx$$

$$= \frac{e^{2x}}{8} \left(\frac{x^2}{2} - \frac{9}{8}x\right)$$

$\therefore$ $y = c_1 e^{-6x} + c_2 e^{2x} + \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9}{8}x\right)$ is solⁿ

Ques 7) $(D^2 - 4) y = x \sinh(x)$

Sol) Aux. eqn $\rightarrow$ $m^2 - 4 = 0$

$\Rightarrow m = 2$ & -2

$CF = c_1 e^{2x} + c_2 e^{-2x}$

NOTE:

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

HYBERBOLIC FUNCTIONS

$$PI = \frac{1}{D^2 - 4} x \sinh(x)$$

$$= \frac{1}{D^2-4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}(e^x . x) - \frac{1}{D^2-4}(e^{-x} . x)\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}(x) - e^{-x}\frac{1}{(D-1)^2 - 4}(x)\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D-3}(x) - e^{-x}\frac{1}{D^2-2D-3}(x)\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3\left\{1 - \left(\frac{D^2+2D}{3}\right)\right\}}(x) - \frac{e^{-x}}{-3\left\{1 - \left(\frac{D^2-2D}{3}\right)\right\}}(x)\right]$$

$$= -\frac{1}{6}\left[e^x\left\{1 - \left(\frac{D^2+2D}{3}\right)\right\}^{-1} x - e^{-x}\left\{1 - \left(\frac{D^2-2D}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^x\left\{1 + \left(\frac{D^2+2D}{3}\right) + \cdots\right\} x - e^{-x}\left\{1 + \left(\frac{D^2-2D}{3}\right) + \cdots\right\} x\right]$$

$$= -\frac{1}{6}\left[e^x\left\{x + \frac{D^2}{3}(x) + \frac{2}{3}D(x)\right\} - e^{-x}\left\{x + \frac{D^2}{3}(x) - \frac{2}{3}D(x)\right\}\right]$$

$$= -\frac{1}{6}\left[e^x\left(x + \frac{2}{3}\right) - e^{-x}\left(x - \frac{2}{3}\right)\right]$$

$$= -\frac{1}{6}\left[x\left(e^x - e^{-x}\right) + \frac{2}{3}\left(e^x + e^{-x}\right)\right]$$

$$= -\frac{1}{3}\left[x\left(\frac{e^x - e^{-x}}{2}\right) + \frac{2}{3}\left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$$= -\frac{1}{3}\left[x\sinh(x) + \frac{2}{3}\cosh(x)\right]$$

$\therefore$ $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}\left[x \sinh(x) + \frac{2}{3}\cosh(x)\right]$ is solⁿ.