Unit - 3

Linear Differential Eq^n with Cons^n Coefficients

A differential eq^n of form :

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = F(x) \quad \text{— (A)}$$

where $a_0, a_1, a_2 \cdots a_n$ are constants & $F(x)$ is a function of x, is called a Linear differential eq^n with cons^n coefficients. If $F(x) = 0$ then eq^n (A) is called Homogeneous otherwise non-homogeneous.

In particular, if $n = 2$ then :

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = F(x)$$

order = 2

If $f(x) = 0$ then

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

dividing by a_0

$$\frac{d^2 y}{dx^2} + \frac{a_1}{a_0} \frac{dy}{dx} + \frac{a_2}{a_0} y = 0$$

Date.
Page No.

$$\frac{a_1}{a_0} = P_1 \quad \& \quad \frac{a_2}{a_2} = P_2$$

OR

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad \text{— (B)}$$

Order = 2

) — (A)

ic -

us.

Solutionⁿ of LDP with consⁿ coeff.

Consider $\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$

clearly $y = 0$ is a solution of eqⁿ (B)
But $y = 0$ is a trivial solution. In order seek non trivial solⁿ of eqⁿ (B), we assume $y = e^{mx}$ is a solⁿ of eqⁿ (B)
{why}
(I) $e^{mx} \neq 0$
(II) derivative of e^{mx} is always consᵗ multiple of e^{mx}.

now $y = e^{mx}$ must satisfy eq (B)

$\Rightarrow m^2 e^{mx} + P_1 m e^{mx} + P_2 e^{mx} = 0$

$e^{mx}[m^2 + P_1 + P_2] = 0$

$\Rightarrow \boxed{m^2 + P_1 + P_2 = 0}$ {as we know that $e^{mx} \neq 0$

auxiliary eqⁿ

Date.
Page No.

Note that the above eqn thus obtained is algebraic in nature which is called as Auxiliary eqn. Three cases may arise :—

CASE I

If Auxiliary eqn real & distinct roots i.e $m_1 \neq m_2$ where both m_1 & m_2 are real. In this case there are 2 soln viz $e^{m_1 x}$ & $e^{m_2 x}$. Clearly both are linearly independent. The complete soln is given by their linear combination i.e

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

CASE II

If Auxiliary eqn has real & equal roots i.e $m_1 = m_2 = m$ (say m.)

In this case there will be only 1 soln.

In order to create 2nd linearly independent soln, multiply 1st soln by x {to be revealed later.}

i.e $x e^{mx}$ is the 2nd soln, the complete soln is given by their linear soln i.e $y = c_1 e^{mx} + c_2 e^{mx}$

$$y = e^{mx}(c_1 + c_2)$$

Case III If the roots of auxiliary eqⁿ are complex conjugates i.e $m_1 = \alpha + i\beta$

$$m_2 = \alpha - i\beta$$

Then solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$\Rightarrow e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right]$$

$$\Rightarrow e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\cos\beta x)\right]$$

$$= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x\right]$$

$$y = e^{\alpha x}\left[C_1\cos\beta x + C_2\sin\beta x\right]$$

where $C_1 = c_1 + c_2$ & $C_2 = ic_1 - ic_2$

Remarks

For a 3rd order LDE, suppose m_1, m_2 & m_3 are roots then following cases are possible -

CASE I $m_1 \neq m_2 \neq m_3$ then solⁿ is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$$

CASE II $m_1 = m_2 \neq m_3$ then the solⁿ is

$$y = [C_1 + C_2 x] e^{m_1 x} + C_3 e^{m_3 x}$$

CASE III $m_1 = m_2 = m_3$ then the solⁿ is

$$y = [C_1 + C_2 x + C_3 x^2] e^{m_1 x}$$

CASE IV $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ & m_3 is distinct & real. Then the solⁿ is

$$y = e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x)] + C_3 e^{m_3 x}$$

Ques (1) Solve the following -

(i) $$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 4y = 0$$

(ii) $$\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 4y = 0$$

(iii) $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

(iv) $$[D^3 - 3D^2 + 4]y = 0 \text{ where } D \equiv \frac{d}{dx}$$

(v) $[D^3 -$

(vi) $[D^4$

(vii) $[D^2$

(viii) $(D^2 +$

Ans (1)

(i)

(ii)

(iii)

Date.
Page No.

(v) $[D^3 - 7D - 6]y = 0$

(vi) $[D^4 - m^4]y = 0$

(vii) $[D^2 - 2D + 4]^2 y = 0$

(viii) $(D^2+1)y = 0$, $y(0) = 1$, $y(\pi/2) = 0$

Ans (i) $m^2 + 5m + 4 = 0$

$m^2 + 4m + m + 4 = 0$

$m(m+4) + 1(m+4) = 0$

$(m+1)(m+4) = 0$

$\Rightarrow m_1 = -4, m_2 = -1$

$\therefore y = C_1 e^{-4x} + C_2 e^{-x}$ is general sol^n

(ii) $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$\therefore y = (C_1 + C_2 x) e^{2x}$ is general sol^n

(iii) $m^2 + m + 1 = 0$

$m = \dfrac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2}$

$m = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$-1 \quad -8 \quad +14 \quad -6$ $\quad c_1 \cos\beta + i\sin\beta$ Date. $8-14-6$ Page No.

$$y = e^{-\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$ is general soln

(iv) $m^3 - 3m^2 + 4 = 0$

clearly $m = -1$ is one root

also $m = 2$ is 2nd root

$\Rightarrow$ sum of root $= \frac{-b}{a} = 3$

$-1 + 2 + m = 3 \Rightarrow m = 2 \; \{3^{rd} \text{ root}\}$

$\therefore -1, 2, 2$

$\therefore y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

(v) $m^3 - 7m - 6 = 0$

clearly $m = -1$ & $m = -2$

sum of roots $= \frac{-b}{a}$

$-1 - 2 + m = 0$

$m = 3$

$\therefore -1, -2, 3$ are roots

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$ is general solution.

$1-7-6$

$-1+7-6$

Date.

Page No.

(vi) $(m^2-2m+4)^2=0$

$$m=\frac{2\pm\sqrt{4-4\times1\times4}}{2}$$

$$m=\frac{2\pm\sqrt{-12}}{2}=\frac{2\pm i\sqrt{12}}{2}=\frac{2\pm2i\sqrt{3}}{2}$$

$$=1\pm i\sqrt{3}$$

$\therefore 1+i\sqrt{3},\ 1-i\sqrt{3},\ 1+i\sqrt{3},\ 1+i\sqrt{3}$

$$y=e^{x}\left[(C_1+C_2x)\cos\sqrt{3}x+(C_3+C_4x)\sin\sqrt{3}x\right]$$

is general soln

(vii) $p^4-m^4=0$

$(p^2)^2-(m^2)^2=0$

$(p^2-m^2)(p^2+m^2)=0$

$(p-m)(p+m)(p^2+m^2)=0$

$p=m,\ p=-m,\ p=\pm im$

$\therefore m,\ -m,\ im,\ -im$ are roots

$$\therefore y=C_1e^{mx}+C_2e^{-mx}+\left[C_3\cos(mx)+C_4\sin(mx)\right]$$

is general solution

Date.
Page No.

(viii) $(D^2+1)y = 0, \; y(0) = 1, \; y(\pi/2) = 0$

$m^2 + 1 = 0$

$m = \pm\sqrt{-1} \Rightarrow m = \pm i$ are roots

$y = C_1 \cos x + C_2 \sin x$ is general solution. $\rightarrow$ (i)

$y(0) = 1 \Rightarrow x = 0, \; y = 1$

$\boxed{1 = C_1}$

$\Rightarrow y = \cos x + C_2 \sin x$ — (ii)

$y(\pi/2) = 0 \Rightarrow x = \frac{\pi}{2}, \; y = 0$

$0 = 0 + C_2$

$\Rightarrow \boxed{C_2 = 0}$

$\boxed{y = \cos x}$ is sol^n

Solution of Non Homogeneous LDE i.e

Consider eq^n (A) $F(x) \neq 0$

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = F(x) \quad \text{(A)}$$

The sol^n of eq^n (A) is given by -

$$\boxed{y = CF + PI}$$

where CF is complementary function obtained by solving corresponding homogeneous LDE OR eq^n of (A) & PI is particular integral {OR particular sol^n} of eq^n (A) obtained as follows.

Let $PI = y_p$

Then y_p must satisfy eq^n (A) i.e

$$y_p\left[a_0 D^n + a_1 D^{n-1} + \dots a_n\right] = F(x)$$

$$\Rightarrow f(D)\, y_p = F(x)$$

$$\boxed{y_p = \frac{1}{f(D)} F(x)}$$

Depending on the nature of $F(x)$, we have the following cases :-

CASE I If $F(x) = e^{ax}$

Then $PI = y_p = \dfrac{1}{f(D)} e^{ax}$

$= \dfrac{e^{ax}}{f(a)}$ provided $f(a) \neq 0$

In case of failure i.e $f(a) = 0$

The $PI = y_p = \dfrac{1}{f(D)} e^{ax}$

$= \dfrac{x}{f'(D)} e^{ax}$

CASE II If $RHS = \sin ax$ or $\cos ax$

Then $PI = y_p = \dfrac{1}{f(D^2)} \sin ax$

$= \dfrac{\sin ax}{f(-a^2)}$ provided $f(-a^2) \neq 0$

In case of failure i.e $f(-a^2) = 0$

Then $PI = y_p = \dfrac{1}{f(D^2)} \sin ax$

$= \dfrac{x}{f'(D^2)} \sin ax$

$1-5+4=0 \quad 2-5=-3$

Date.
Page No.

Que 1 $(D^2+5D+4)y = e^{3x}$

Aux eq^n $\rightarrow$ $m^2+5m+4=0$

$\Rightarrow (m+1)(m+4)=0$

$\Rightarrow m_1 = -1$ & $m_2 = -4$

$\therefore CF = y_h = C_1 e^{-x} + C_2 e^{-4x}$

$$PE = y_p = \frac{1}{(D^2+5D+4)} e^{3x}$$

$$= \frac{1}{9+15+4} e^{3x} = \frac{e^{3x}}{28}$$

$$\therefore \quad y = CF + PE = C_1 e^{-x} + C_2 e^{-4x} + \frac{e^{3x}}{28}$$

Que 2 $(D^2-5D+4)y = e^x$

Aux. eq^n $\rightarrow$ $m^2-5m+4=0$

$\Rightarrow (m-1)(m-4)=0$

$\Rightarrow m=1$ & $m=4$

$\therefore CF = y_h = C_1 e^{x} + C_2 e^{4x}$

$$PI = \frac{1}{D^2-5D+4} e^x$$

$$= \frac{x}{2D-5} e^x = \frac{-x e^x}{3}$$

$$\therefore y = CF + PE = C_1 e^x + C_2 e^{4x} - \frac{xe^x}{3}$$

12 $25 - 50 + 25$ $1 - 10 + 25$ -16

82 -16

Date.
Page No.

Qus 3 $(D^2 - 10D + 25)y = e^{5x} + e^x$

Auxiliary eq^n $\rightarrow$ $m^2 - 10m + 25 = 0$

$\Rightarrow (m-5)^2 = 0$

$\Rightarrow m = 5, m = 5$

$CF = (C_1 + C_2 x)e^{5x}$

$$PI = \frac{1}{D^2 - 10D + 25} e^{5x} + e^x$$

$$PI = \frac{1}{D^2 - 10D + 25} e^{5x} + \frac{1}{D^2 - 10D + 25} e^x$$

$$= \frac{x}{2D - 10} e^{5x} + \frac{e^x}{1 - 10 + 25}$$

$$= \frac{x^2}{2} e^{5x} + \frac{e^x}{16}$$

$$\therefore y = (C_1 + C_2 x)e^{5x} + \frac{x^2}{2} e^{5x} + \frac{e^x}{16}$$

Qus 4 $(4D^2 + 4D - 3)y = \sin x$

Auxiliarly eq^n $4m^2 + 4m - 3 = 0$

$4m^2 + 6m - 2m - 3 = 0$

$2m(2m+3) - 1(2m+3) = 0$

$(2m-1)(2m+3) = 0$

$m = \frac{1}{2}, \quad m = \frac{-3}{2}$

$-4 - 4 - 3 \quad 4D^2 + 4D - 3$

$-4 + 4D - 3$

$4D - 7$

Date.

Page No.

$\therefore$ CF $= C_1 e^{\frac{1}{2}x} + C_2 e^{-\frac{3}{2}x}$

$$PI = \frac{1}{4D^2 + 4D - 3} \sin x$$

$$= \frac{1}{-4 + 4D - 3} \sin x = \frac{1}{4D - 7} \sin x$$

$$\Rightarrow \frac{4D + 7}{16D^2 - 7^2} \sin x$$

$$\Rightarrow \frac{1}{-16 - 49} \left[4D \sin x + 7 \sin x \right]$$

$$\Rightarrow \frac{-1}{65} \left[4 \cos x + 7 \sin x \right]$$

$\therefore y = CF + PI = C_1 e^{\frac{1}{2}x} + C_2 e^{-\frac{3}{2}x} - \frac{1}{65}\left[4 \cos x + 7 \sin x\right]$

Ques 5 $(D^2 + 4) y = \cos 2x$

Auxiliary eq^n $= m^2 + 4 = 0$

$m = \pm i2$

$y_h = CF = C_1 \cos 2x + C_2 \sin 2x$

$$PI = \frac{1}{D^2 + 4} \cos 2x$$

$$= \frac{x}{2D} \cos 2x = \frac{x}{2} \left(\frac{1}{D} \sin 2x \right)$$

$$= \frac{x \sin 2x}{2 \times 2} = \frac{x \sin 2x}{4}$$

Date.
Page No.

Ques 6 $(D^2+a^2)^2 y = \sin ax$

Auxiliary eqⁿ

$(m^2+a^2)(m^2+a^2)=0$

$m = \pm ia$

$m = \pm ia$

CF $= y_h = (C_1+C_2x)\cos ax + (C_3+C_4x)\sin ax$

$y_h = (C_1+C_2x)\cos ax + (C_3+C_4x)\sin ax$

PI $= \frac{1}{(D^2+a^2)^2}\sin ax$

$= \cancel{\frac{x}{2D}\sin ax = \frac{xD}{2D^2}\sin ax}$

$= \frac{1}{2(D^2+a^2)2D}\sin ax = \frac{x}{4D^3+4a^2D}\sin(ax)$

$= \frac{x^2}{12D^2+4a^2}\sin ax$

$= \frac{-x^2}{8a^2}\sin(ax)$

$\therefore y = (C_1+C_2x)\cos ax + (C_3+C_4x)\sin ax - \frac{x^2}{8a^2}\sin(ax)$ Ans

CASE III

If

Expand by

Remark

(i) (1+x)

(ii) (1-

(iii) (1+

(iv) (1-

(v) (1+

(vi) (1+

(D

Auxil

CF

Date. ______
Page No. ______

CASE III If $F(x) = p(x)$ {a polinomial of degree n}

$$y_p = PI = \frac{1}{f(D)} p(x)$$

$$= [f(D)]^{-1} p(x)$$

Expand $[f(D)]^{-1}$ using binomial expansion by taking the least power of D.

Remark

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2.1} x^2 + \dots$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2.1} x^2 - \dots$

(iii) $(1+x)^{-1} = 1 - x + x^2 - x^3 +$

(iv) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

(v) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

(vi) $(1+x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Ques $(D^2 - 2D + 1) y = x^2 + x + 1$

Auxiliary eq^n $\rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, m = 1$

$CF = (C_1 + C_2 x) e^x$

$1-4\times 2 \quad 2x$

Date. ____
Page No. ____

$$PI = \frac{1}{D^2-2D+1}(x^2+x+1)$$

$$= \frac{1}{(D-1)^2}(x^2+x+1) = \frac{1}{(1-D)^2}(x^2+x+1)$$

$$= (D-1)^{-2}(x^2+x+1)$$

$$= [1+2D+3D^2+4D^3+\cdots](x^2+x+1)$$

$$= (x^2+x+1)+2D(x^2+x+1)+3D^2(x^2+x+1)$$
$$+4D^3(x^2+x+1)+\cdots$$

$$= (x^2+x+1)+2(2x+1)+3(2)+4(0)+5(0)+\cdots$$
$$= x^2+5x+9$$

$$y = CF+PI = (C_1+C_2x)e^x + x^2+5x+9$$

Que $(D^2+D+2)y = x^2$

Auxiliary eq^n $\quad m^2+m+2=0$

$$m = \frac{-1\pm\sqrt{-7}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{7}}{2}$$

$$CF = e^{-x/2}\left[C_1 \cos\frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x\right]$$

$$PI = \frac{1}{(D^2+D+2)}x^2$$

$\frac{1}{(1-D)^2}(x^2+x+1)$

$^2+x+1)$

$D^2(x^2+x+1)$

$)+4(0)+5(0)+\dots$

$+x^2+5x+9$

$=0$

$=\frac{-1}{2}\pm\frac{i\sqrt{7}}{2}$

$\sin\frac{\sqrt{7}x}{2}]$

$\frac{2D^3}{4\cdot 2}$ $\frac{D^4+D^2+2D^3}{4}$

Date. $\frac{D^4}{4}+\frac{D^2}{4}+\frac{D^3}{2}$

Page No.

$D^3(x^2)$

Least power of D is taken out common.

$$= \frac{1}{2\left(\frac{D^2}{2}+\frac{D}{2}+1\right)}x^2$$

$$= \frac{1}{2\left(1+\left(\frac{D^2+D}{2}\right)\right)}x^2 = \frac{1}{2}\left[1+\left(\frac{D^2+D}{2}\right)\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+D}{2}\right)+\left(\frac{D^2+D}{2}\right)^2\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{D}{2}+\frac{D^4}{4}+\frac{D^2}{4}+\frac{D^3}{2}\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{D}{2}+\frac{D^4}{4}+\frac{D^2}{4}+\frac{D^3}{2}\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{D}{2}+\frac{D^2}{4}\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{2}{2}-\frac{2x}{2}+\frac{2}{4}\right]$$

$$= \frac{1}{2}\left[x^2+\cancel{1}-\cancel{1}-x+\frac{1}{2}\right] = \frac{1}{2}\left[\frac{x^2+1}{2}-x\right]$$

$$= \frac{x^2}{2}-\frac{1}{4}-\frac{x}{2}$$

$\therefore \cancel{y} = \frac{1}{(D^2+D+2)}$

$y = CF + PI$

$D^2 - y = 0 \quad \frac{d^2x}{dy^2} - y = \sin x \, e^2$

$D +$

Date
Page No.

CASE IV If $f(x) = e^{ax}\phi(x)$ where $\phi(x)$ is any function as discussed then

$$y_p = PI = \frac{1}{f(D)} e^{ax}\phi(x)$$

$$= e^{ax}\frac{1}{f(D+a)}\phi(x)$$

Que $(D^2 - 4D + 4)\, y = e^{2x}\sin 3x$

Auxiliary eqn $\rightarrow m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2,\ m = 2$

$CF = (C_1 + C_2 x)\, e^{2x}$

$$PI = \frac{1}{D^2 - 4D + 4} e^{2x}\sin 3x$$

$$= \frac{1}{(D-2)^2} e^{2x}\sin 3x$$

$$= \cancel{e^{2x}\frac{1}{D-2+2}} = e^{2x}\frac{1}{(D+2-2)^2}\sin 3x$$

$$= e^{2x}\frac{1}{D^2}\sin 3x$$

$$= e^{2x}\frac{1}{D}\int \sin 3x$$

$$= -e^{2x}\frac{1}{D}\frac{\cos 3x}{3} = \frac{-e^{2x}\sin 3x}{9}$$

$f(uv\,dx) = (u)(v_1) - (u')(v_2) + (u'')(v_3) - \cdots$

eg $\int x^3 e^x dx = (x^3)(e^x) - (3x^2)(e^x) + (6x)(e^x) + 6e^x$

Date

Page No.

$\therefore y = (C_1 + C_2 x)e^{2x} + \left(\frac{-e^{2x}\sin 3x}{9}\right)$

Que $(D^2 + 2D + 1)y = e^{-x} x \sin x$

Auxiliary Eqⁿ → $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, m = 1$

$C.F = (C_1 + C_2 x) e^{-x}$

$PI = \frac{1}{(D^2 + 2D + 1)} e^{-x} x \sin x$

$= \frac{1}{(D+1)^2} e^{-x} x \sin x$

$= e^{-x} \frac{1}{(D-1+1)^2} x \sin x$

$= e^{-x} \frac{1}{D^2} x \sin x = e^{-x} \frac{1}{D} \int x \sin x$

$= e^{-x} \frac{1}{D} [-x \cos x + \sin x]$

$= e^{-x} \frac{1}{D} [\sin x - x \cos x]$

$= e^{-x} [-\cos x - [x \sin x + \cos x]]$

$= e^{-x} [-2\cos x - x \sin x]$

$y = CF + PI$

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Date.
Page No.

Que $(D^2+2D+1)y = \dfrac{e^{-x}}{x+1}$

$CF = m^2+2m+1 = (m+1)^2$

$\Rightarrow m = -1, m = -1$

$CF = (C_1 + C_2x)e^{-x}$

$PI = \dfrac{1}{D^2+2D+1}\dfrac{e^{-x}}{x+1}$

$= \dfrac{1}{(D+1)^2}\dfrac{e^{-x}}{x+1}$

$= e^{-x}\dfrac{1}{(D-1+1)^2}\dfrac{1}{x+1}$

$= e^{-x}\dfrac{1}{D^2}\dfrac{1}{x+1} = e^{-x}\dfrac{1}{D}\int\dfrac{1}{x+1}dx$

$= e^{-x}\dfrac{1}{D}\log|x+1| = e^{-x}\int\log|x+1|\,dx$

$= e^{-x}\log(x+1)x - \int\dfrac{x}{x+1}dx$

$= e^{-x}\left[\log(x+1)x - \int dx + \int\dfrac{dx}{x+1}\right]$

$= e^{-x}\left[\log(x+1)x - x + \log(x+1)\right]$

$y \Rightarrow CF + PI$ is sol^n

$m^2 + 6m - 2m - 12$

$m(m+6) - 2(m+6)$

$(m-2)(m+6)$

Date. ________

Page No. ________

Que $(D^2 + 4D - 12)y = (x-1)e^{2x}$

Auxilary eqⁿ $\rightarrow$ $m^2 + 4m - 12 = 0$

$(m-2)(m+6) = 0$

$m = 2, \; m = -6$

$CF = C_1 e^{2x} + C_2 e^{-6x}$

$$PI = \frac{1}{D^2 + 4D - 12}(x-1)e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 + 4(D+2) - 12}(x-1)$$

$$= e^{2x} \frac{1}{D^2 + 4 + 4D + 4D + 8 - 12} x - 1$$

$$= e^{2x} \frac{1}{D^2 + 8D}(x-1)$$

$$= e^{2x} \frac{1}{8D\left(\frac{D}{8} + 1\right)}(x-1)$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left(1 + \frac{D}{8}\right)^{-1}(x-1)$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left[1 - \frac{D}{8} + \frac{D^2}{8^2} - \dots\right](x-1)$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left[(x-1) - \frac{D}{8}(x-1)\right]$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left[x - 1 - \frac{1}{8}\left(\frac{x^2}{2} - x\right)\right]$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left[x - 1 - \frac{x^2}{16} - \frac{x}{8}\right]$$

$$= \frac{e^{2x}}{8} \frac{1}{D}\left[x - 1 - \frac{1}{8}\right]$$

$$= \frac{e^{2x}}{8}\left(\frac{1}{D}(x) - \frac{1}{D}\left(\frac{9}{8}\right)\right)$$

$$= \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9}{8}x\right)$$

$$y = C_1 e^{-5x} + C_2 e^{2x} + \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9}{8}x\right)$$ is solⁿ

Ques $(D^2 - 4)y = x \sinh(x)$

Auxiliary eqⁿ $m^2 - 4 = 0$

$m = 2, m = -2$

$CF = C_1 e^{2x} + C_2 e^{-2x}$

$$PI = \frac{1}{(D^2 - 4)} x \sinh(x)$$

NOTE

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

HYBERBOLIC FUNCTIONS

$D^2+1+2D-4 \qquad D^2+1-2D-4$

$D^2+2D-3 \qquad D^2-2D-3$

Date.
Page No.

$$\Rightarrow PI = \frac{1}{D^2-4}\, x\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{D^2-4}(e^{x}\cdot x) - \frac{1}{D^2-4}(e^{-x}\cdot x)\right]$$

$$\Rightarrow \frac{1}{2}\left[e^{x}\frac{1}{(D+1)^2-4}\,x - e^{-x}\frac{1}{(D-1)^2-4}\,x\right]$$

$$\Rightarrow \frac{1}{2}\left[e^{x}\frac{1}{D^2+2D-3}(x) - e^{-x}\frac{1}{D^2-2D-3}(x)\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{e^{x}}{-3\left(1-\left(\frac{D^2+2D}{3}\right)\right)}(x) - \frac{e^{-x}}{-3\left(1-\left(\frac{D^2-2D}{3}\right)\right)}(x)\right]$$

$$= \frac{1}{-6}\left[e^{x}\left(1-\left(\frac{D^2+2D}{3}\right)\right)^{-1}(x) - e^{-x}\left(1-\left(\frac{D^2-2D}{3}\right)\right)(x)\right]$$

$$= \frac{1}{-6}\left[e^{x}\left\{1+\frac{D^2+2D}{3}+\dots\right\}x - e^{-x}\left\{1+\frac{D^2-2D}{3}+\dots\right\}x\right]$$

$$= \frac{1}{-6}\left[e^{x}\left\{x+\frac{D^2}{3}(x)+\frac{2D(x)}{3}\right\} - e^{-x}\left\{x+\frac{D^2}{3}(x)+\frac{2D(x)}{3}\right\}\right]$$

$$= -\frac{1}{6}\left[e^{x}\left\{x+0+\frac{2}{3}\right\} - e^{-x}\left\{x+0-\frac{2}{3}\right\}\right]$$

$$= -\frac{1}{6}\left[e^{x}\left\{x+\frac{2}{3}\right\} - e^{-x}\left\{x-\frac{2}{3}\right\}\right]$$

$$= -\frac{1}{6}\left[e^{x}x + e^{x}\frac{2}{3} - e^{-x}x + e^{-x}\frac{2}{3} \right]$$

$$= -\frac{1}{6}\left[\frac{2}{3}\left(e^{x} + e^{-x}\right) + x\left(e^{x} - e^{-x}\right) \right]$$

$$= -\frac{1}{3}\left[x\left(\frac{e^{x} - e^{-x}}{2}\right) + \frac{2}{3}\left(\frac{e^{x} + e^{-x}}{2}\right) \right]$$

$$\Rightarrow -\frac{1}{3}\left[x \sinh(x) + \frac{2}{3}\cosh(x) \right]$$

$$\therefore y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{3}\left[x \sinh(x) + \frac{2}{3}\cosh(x) \right]$$

Methods of Variation of Parameters

Que 1 $\cancel{D^2 + 9^2}$ $(D^2 + a^2)y = \sec(ax)$

Auxiliary equation - $m^2 + a^2 = 0$

$m = \pm ia$

$\therefore$ C.F $= C_1 \cos(ax) + C_2 \sin(ax)$

here $y_1 = \cos(ax)$ & $y_2 = \sin(ax)$

$$\therefore W(x) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a(\sin^2 ax + \cos^2 ax) = a \neq 0$$

Date:
Page No.

$\therefore W(x) = a$

Now $u(x) = \int \frac{-y_2 R(x)}{W(x)} dx + A$

$= -\int \frac{\sin(ax) \cdot \sec(ax)}{a} dx + A$

$= -\frac{1}{a} \int \frac{\sin ax}{\cos ax} \times \cancel{1} \, dx + A$

$= -\frac{1}{a} \int \tan(ax) \, dx + A$

$= -\frac{1}{a} \times \frac{\log|\sec ax|}{a} + A$

$= \frac{1}{a^2} \log|\cos ax| + A$

Now $v(x) = \int \frac{y_1 R(x)}{W(x)} dx + B$

$= \int \frac{\cos ax \ \sec ax}{a} dx + B$

$= \frac{x}{a} + B$

$\therefore y = u(x) y_1 + v(x) y_2$

$= A \cos(ax) + B \sin(ax) + \frac{\cos ax}{a^2} \log \cos(ax)$

$+ \frac{x \sin ax}{a}$

$(x)]$

$\frac{e^{-x}}{1+e^{x}} \times \frac{e^{-x}}{e^{-x}} \quad \frac{e^{-2x}}{e^{-x}+1}$

Date.
Page No.

Ques 2 $(D^2-1)y = \frac{1}{1+e^x}$

Auxiliarly eq^n $\longrightarrow$ $m^2 - 1 = 0$

$m^2 = 1 \Rightarrow m = 1, m = -1$

$\therefore CF = C_1 e^{x} + C_2 e^{-x}$

$y_1 = e^{x}$ & $y_2 = e^{-x}$

$W(x) = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix}$

$= -e^{-x+x} - e^{-x+x}$

$= -1-1 = -2 \neq 0$

$u(x) = \frac{-y_2 R(x)}{W(x)} dx + A$

$= -\int \frac{e^{-x} \times \frac{1}{1+e^{x}}}{-2} dx + A$

$= \frac{1}{2}\int \frac{e^{-2x}}{1+e^{-x}} dx + A$ {multiply by e^{-x}}

Let $e^{-x} = t$

$e^{-x} dx = -dt$

$= \frac{1}{2}\int \frac{e^{2} e^{-x}}{e^{-x}+1} dx + A$

Date.
Page No.

$1, m = -1$

$\log e^{-x} \}$

$$= -\frac{1}{2}\int \frac{t}{t+1}\,dt + A$$

$$= -\frac{1}{2}\int \frac{(t+1)-1}{t+1}\,dt + A$$

$$= -\frac{1}{2}\left[\int dt\right] + \frac{1}{2}\left[\int\left[\frac{1}{t+1}\right]dt + A\right.$$

$$= -\frac{t}{2} + \frac{1}{2}\log|t+1| + A$$

$$= \frac{-e^{-x}}{2} + \frac{1}{2}\log|e^{-x}+1| + A$$

$$v(x) = \frac{y_1 R(x)}{W(x)}\,dx + A$$

$$= \int \frac{e^x}{\frac{1+e^x}{-2}}\,dx + B = \frac{-1}{2}\int \frac{e^x}{e^x+1}\,dx + B$$

$$\int \frac{f'(x)}{f(x)} = \log f(x)$$

$$\Rightarrow -\frac{1}{2}\log|e^x+1| + B$$

$$\therefore y = u(x)y_1 + v(x)y_2$$
$$= Ae^x + Be^{-x} - \frac{1}{2} + \frac{e^x}{2}\log(e^{-x}+1) - \frac{e^{-x}}{2}\log|e^x+1|$$

Date.
Page No.

Que 3 $(D^2 - 1)y = \frac{2}{\sqrt{1-e^{-2x}}}$

$m = \pm 1$

~~CF ⇒ $y_1 = e^x$~~

$CF = c_1 e^x + c_2 e^{-x}$

here $y_1 = e^x$ & $y_2 = e^{-x}$

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$= -2$

$$u(x) = \int \frac{-y_2 R(x)}{W(x)}\,dx + A = \int \frac{\frac{-e^{-x} \times 2}{\sqrt{1-e^{-2x}}}}{-2}\,dx + A$$

$$= \frac{1}{\cancel{2}} \int \frac{\cancel{2}\, e^{-x}}{\sqrt{1-e^{-2x}}}\,dx + A = \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx + A$$

$e^{-x} = t$

$-e^{-x}\,dx = dt$

$$\Rightarrow -\int \frac{dt}{\sqrt{1-t^2}} + A = -\sin^{-1}(t) + A$$

$$= -\sin^{-1}(e^{-x}) + A$$

$$v(x) = \int \frac{y_1 R(x)}{w(x)}\,dx + B$$

$$= \int \frac{\cancel{2}e^{x}}{\sqrt{1-e^{-2x}}} \times \frac{1}{-\cancel{2}}\,dx + B$$

$$= -\int \frac{e^{2x}}{\sqrt{1-\frac{1}{e^{2x}}}}\,dx + B$$

$$= -\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}\,dx + B$$

Let $e^{2x} - 1 = t^2$

$$e^{2x}\,\cancel{2}\,dx = \cancel{2}\,t\,dt$$

$$e^{2x}\,dx = t\,dt$$

$$v(x) = \int \frac{-\cancel{t}\,dt}{\cancel{t}} + B$$

$$= -t + B = -\sqrt{e^{2x}-1} + B$$

$$\therefore y = u(x)\,y_1 + v(x)\,y_2$$

$$= Ae^{x} + Be^{-x} - e^{x}\sin^{-1}(e^{-x}) - e^{-x}\sqrt{e^{2x}-1}$$

$-e^{-x}(x+1)$

$-e^{-x}(x-1)$

Date.
Page No.

Que 4 $(D^2+2D+1)y = e^{-x}\log x$

Auxiliary eq^n → $m^2+2m+1=0$

$\Rightarrow (m+1)^2 = 0$

$CF = (C_1+C_2x)e^{-x} = C_1e^{-x}+C_2xe^{-x}$

$y_1 = e^{-x} \quad y_2 = xe^{-x}$

$$W(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} \end{vmatrix}$$

$= -e^{-2x}(x-1) + xe^{-2x}$

$\Rightarrow xe^{-2x} - e^{-2x}(x-1)$

$\Rightarrow e^{-2x}(1) = e^{-2x}$

$$u(x) = \int \frac{-y_2R(x)}{W(x)}\,dx + A$$

$$= \int \frac{-x\cancel{e^{-x}}}{\cancel{e^{-2x}}} \cdot \cancel{e^{-x}}\log x\,dx + A$$

$$= -\int x\log x\,dx + A$$

$$= -\left[\log x\,\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{3}\,dx\right] + A$$

$$= \frac{-x^2}{2}\log x + \frac{x^2}{4} + A$$

v(x)

∴ y

$e^{-2x} \quad e^{-x}(1-x) + \}$

Date.
Page No.

$$v(x) = \int \frac{e^{-\cancel{x}} \cdot \cancel{e^{-x}} \log x}{\cancel{e^{-2x}}} dx + B$$

$$= \int \log x \cdot 1 \cdot dx + B$$

$$= \left[\log x \cdot x - \int \frac{1}{x} \cdot x \, dx\right] + B$$

$$= \log x - x + B$$

$$\therefore y = (Ae^{-x} + Bxe^{-x}) + e^{-x}\left(\frac{-x^2}{2}\log x + \frac{x^2}{4}\right) + xe^{-x}(\log x - x)$$

$x + C_2 x e^{-x}$

$y_2 = xe^{-x}$

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