# LINEAR DEPENDENCE AND INDEPENDENCE

What is a vector?

Row vector $\longrightarrow [2 \quad 3 \quad 1]$

Column vector $\rightarrow \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

## BASIC INTUITION

Consider $X_1 = [3 \quad 2 \quad 5]$ & $X_2 = [6 \quad 4 \quad 10]$

Note $X_2 = 2X_1$

$\Rightarrow 2X_1 - 1X_2 = 0$

Here $X_1$ and $X_2$ are linearly dependent.

Consider $X_1 = [3 \quad 2 \quad 5]$, $X_2 = [0 \quad 1 \quad 4]$ & $X_3 = [3 \quad 1 \quad 1]$

Note $X_1 - X_2 = X_3$

$\Rightarrow 1X_1 - 1X_2 - 1X_3 = 0$

Here $X_1, X_2$ & $X_3$ are linearly dependent.

On the other hand, if $X_1 = [1 \quad 0 \quad 0]$ & $X_2 = [0 \quad 0 \quad 1]$

Note that $X_1 \neq K X_2$ for any real $K$.

Here $X_1$ and $X_2$ are linearly independent.

## LINEARLY INDEPENDENT VECTORS

A system of vectors $X_1, X_2, ---, X_n$ are linearly independent if
$c_1X_1 + c_2X_2 + --- + c_nX_n = 0$ gives
$c_1 = c_2 = \cdots = c_n = 0$

If atleast one $c_i \neq 0$ then system is linearly dependent.

$\boxed{\text{Remark:}}$ The system $X_1 = [1 \quad 2 \quad 3]$ & $X_2 = [0 \quad 0 \quad 0]$ is linearly dependent
b'coz $c_1X_1 + c_2X_2 = 0$ gives
$c_1 = 0$ & $c_2 =$ any non-zero real number.

Ques (1) Examine whether the system of vectors $X_1 = [3 \quad 1 \quad 1]$, $X_2 = [2 \quad 0 \quad -1]$, $X_3 = [4 \quad 2 \quad 1]$ is LI or LD.

Sol. Let $c_1X_1 + c_2X_2 + c_3X_3 = 0$

$\Rightarrow c_1[3,1,1] + c_2[2,0,-1] + c_3[4,2,1] = [0,0,0]$

$\Rightarrow [3c_1 + 2c_2 + 4c_3, \; c_1 + 2c_3, \; c_1 - c_2 + c_3] = [0,0,0]$

This gives:

$$\left.\begin{aligned} 3c_1 + 2c_2 + 4c_3 &= 0 \\ c_1 + 2c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0 \end{aligned}\right\} \text{homogeneous system in variables } c_1, c_2 \text{ \& } c_3.$$

Augmented matrix

$C = [A : 0]$

$$\Rightarrow C = \begin{bmatrix} 3 & 2 & 4 & : 0 \\ 1 & 0 & 2 & : 0 \\ 1 & -1 & 1 & : 0 \end{bmatrix}$$

1

2

$R_2 \to 3R_2 - R_1$;
$R_3 \to 3R_3 - R_1$;

$$C = \begin{bmatrix} 3 & 2 & 4 & : 0 \\ 0 & -2 & 2 & : 0 \\ 0 & -5 & -1 & : 0 \end{bmatrix}$$

3

$R_3 \to 2R_3 - 5R_2$;

$$C = \begin{bmatrix} 3 & 2 & 4 & : 0 \\ 0 & -2 & 2 & : 0 \\ 0 & 0 & -12 & : 0 \end{bmatrix} = RREF(C)$$

$\because$ Rank (A) = Rank (C) = 3 (= r) = m (no. of variables)

$\therefore$ Unique solution exists (or Trivial solution exists)

i.e. $c_1 = c_2 = c_3 = 0$

$\therefore$ LI

**Ques ②** Examine whether the given system of vectors are linearly independent or not. If they are linearly dependent then determine the relationship among them.

ⓐ $X_1 = [3, 1, -4]$, $X_2 = [2, 2, -3]$, $X_3 = [0, -4, 1]$

ⓑ $X_1 = [1, 2, -1, 0]^T$, $X_2 = [1, 3, 1, 2]^T$, $X_3 = [4, 2, 1, 0]^T$, where the symbol 'T' denotes the transpose of given vector.

**Sol.** Let $c_1 X_1 + c_2 X_2 + c_3 X_3 = \mathbf{0}$

$$\Rightarrow c_1[3, 1, -4] + c_2[2, 2, -3] + c_3[0, -4, 1] = [0, 0, 0]$$

$$\Rightarrow [3c_1 + 2c_2, \ c_1 + 2c_2 - 4c_3, \ -4c_1 - 3c_2 + c_3] = [0, 0, 0]$$

This gives
$$\left.\begin{aligned} 3c_1 + 2c_2 &= 0 \\ c_1 + 2c_2 - 4c_3 &= 0 \\ -4c_1 - 3c_2 + c_3 &= 0 \end{aligned}\right\} \text{Homogeneous system in } c_1, c_2 \text{ \& } c_3$$

Augmented matrix $C = [A : 0]$

$$\Rightarrow C = \left[\begin{array}{ccc:c} 3 & 2 & 0 & 0 \\ 1 & 2 & -4 & 0 \\ -4 & -3 & 1 & 0 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$;
$R_3 \to 3R_3 + 4R_1$;
$$C = \left[\begin{array}{ccc:c} 3 & 2 & 0 & 0 \\ 0 & 4 & -12 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$R_3 \to 4R_3 + R_2$;
$$C = \left[\begin{array}{ccc:c} 3 & 2 & 0 & 0 \\ 0 & 4 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = RREF(C)$$

Observe Rank (A) = Rank (C) = 2 (= r) < 3 (= m)

Thus non-trivial solution exists

$\therefore$ vectors are linearly dependent.

FINDING RELATIONSHIP

This gives: $3c_1 + 2c_2 = 0$ —— ①

& $4c_2 - 12c_3 = 0$ —— ②

Assume $m - r = 3 - 2 = 1$ variable as arbitrary.

Let $c_3 = K$ (say)

From ②; $c_2 = 3K$

From ①; $c_1 = -2K$

Plug these values of $c_1, c_2$ & $c_3$ in $c_1 X_1 + c_2 X_2 + c_3 X_3 = 0$

$$\Rightarrow (-2K)X_1 + (3K)X_2 + (K)X_3 = 0$$
$$\Rightarrow \boxed{-2X_1 + 3X_2 + X_3 = 0}$$

ⓑ $X_1 = [1, 2, -1, 0]^T$, $X_2 = [1, 3, 1, 2]^T$, $X_3 = [4, 2, 1, 0]^T$ are vectors of 4-dimensional space i.e. $\mathbb{R}^4$.

Let $c_1 X_1 + c_2 X_2 + c_3 X_3 = \mathbf{0}$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + 4c_3 \\ 2c_1 + 3c_2 + 2c_3 \\ -c_1 + c_2 + c_3 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives a homogeneous system:

$$c_1 + c_2 + 4c_3 = 0$$
$$2c_1 + 3c_2 + 2c_3 = 0$$
$$-c_1 + c_2 + c_3 = 0$$
$$2c_2 = 0$$

Augmented matrix : $C = \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 2 & 3 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$

① ③

$R_2 \to R_2 - 2R_1$ ;
$R_3 \to R_3 + R_1$ ;

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$$

③ ④

$R_3 \to R_3 - 2R_2$ ;
$R_4 \to R_4 - 2R_2$ ;

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

⑤

$R_4 \to 13R_4 - 8R_3$ ;

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = RREF(C)$$

Observe $Rank(A) = Rank(C) = 3 (=r) = m$ (no. of variables)

Thus trivial solution exists

$$c_1 = c_2 = c_3 = 0$$

$\therefore$ vectors are linearly independent.

Ques ③ If $X_1 = [1,2,1]$, $X_2 = [3,0,-1]$ & $X_3 = [a,b,c]$ are linearly dependent. Find the relationship among $a, b$ & $c$.

Sol. $\because$ $X_1, X_2$ & $X_3$ are L.D.

$\therefore$ The homogeneous system given by $K_1X_1 + K_2X_2 + K_3X_3 = 0$ must have a $\boxed{\text{non-trivial solution}}$

$$\Rightarrow K_1[1,2,1] + K_2[3,0,-1] + K_3[a,b,c] = [0,0,0]$$

$$\Rightarrow [K_1 + 3K_2 + aK_3,\ 2K_1 + bK_3,\ K_1 - K_2 + K_3 c] = [0,0,0]$$

This gives a homogeneous system:

$$K_1 + 3K_2 + aK_3 = 0$$
$$2K_1 + bK_3 = 0$$
$$K_1 - K_2 + cK_3 = 0$$

must have a non-trivial solution.

Then $\det \begin{bmatrix} 1 & 3 & a \\ 2 & 0 & b \\ 1 & -1 & c \end{bmatrix} = 0$

FACTS ★★
① NTS $\Leftrightarrow$ $\det(A) = 0$
② TS $\Leftrightarrow$ $\det(A) \neq 0$

$$\Rightarrow 1(0+b) - 3(2c-b) + a(-2-0) = 0$$
$$\Rightarrow b - 6c + 3b - 2a = 0$$
$$\Rightarrow 4b - 2a - 6c = 0$$
$$\Rightarrow \boxed{2b - a - 3c = 0}$$ is the required relation.

Ques ④ If $X_1 = [0,1,1]$, $X_2 = [1,0,1]$ & $X_3 = [1,1,0]$ are LI. Express the vector $[\alpha, \beta, \gamma]$ as the linear combination of $X_1, X_2$ & $X_3$.

Sol. $\boxed{\text{Key point}}$: If $X_1, X_2, --- X_n$ are LI vectors in $\mathbb{R}^n$ (n-dimensional space) then any arbitrary vector $X$ can be expressed as a linear combination of $X_1, X_2, --- X_n$ i.e.

$$X = c_1X_1 + c_2X_2 + ---- + c_nX_n$$

let $[\alpha, \beta, \gamma] = c_1[0,1,1] + c_2[1,0,1] + c_3[1,1,0]$ ——— (1)

$\Rightarrow [\alpha, \beta, \gamma] = [c_2+c_3,\ c_1+c_3,\ c_1+c_2]$

This gives a non-homogeneous system

$$c_2+c_3=\alpha$$
$$c_1+\ \ c_3=\beta$$
$$c_1+c_2=\gamma$$

To solve this; augmented matrix $C=[A:B]$

$$\Rightarrow C = \left[\begin{array}{ccc:c} 0 & 1 & 1 & \alpha \\ 1 & 0 & 1 & \beta \\ 1 & 1 & 0 & \gamma \end{array}\right]$$

$R_1 \longleftrightarrow R_2$, $\quad C = \left[\begin{array}{ccc:c} 1 & 0 & 1 & \beta \\ 0 & 1 & 1 & \alpha \\ 1 & 1 & 0 & \gamma \end{array}\right]$

$R_3 \to R_3 - R_1$ ; $\quad C = \left[\begin{array}{ccc:c} 1 & 0 & 1 & \beta \\ 0 & 1 & 1 & \alpha \\ 0 & 1 & -1 & \gamma-\beta \end{array}\right]$

$R_3 \to R_3 - R_2$ ; $\quad C = \left[\begin{array}{ccc:c} 1 & 0 & 1 & \beta \\ 0 & 1 & 1 & \alpha \\ 0 & 0 & -2 & \gamma-\beta-\alpha \end{array}\right] = RREF(C)$

$\because$ Rank$(A)$ = Rank$(C) = 3\ (=m)$

$\therefore$ Unique solution exists.

This gives

$$-2c_3 = \gamma-\beta-\alpha$$
$$\Rightarrow \boxed{c_3 = \frac{\alpha+\beta-\gamma}{2}}$$

& $\quad c_2+c_3=\alpha$

$$\Rightarrow c_2 = \alpha - \left(\frac{\alpha+\beta-\gamma}{2}\right)$$
$$\Rightarrow \boxed{c_2 = \frac{\alpha-\beta+\gamma}{2}}$$

& $\quad c_1+c_3=\beta$

$$\Rightarrow c_1 = \beta - \left(\frac{\alpha+\beta-\gamma}{2}\right)$$
$$\Rightarrow \boxed{c_1 = \frac{\beta-\alpha+\gamma}{2}}$$

From (1) ; $\therefore [\alpha,\beta,\gamma] = \left(\frac{\beta-\alpha+\gamma}{2}\right)X_1 + \left(\frac{\alpha-\beta+\gamma}{2}\right)X_2 + \left(\frac{\alpha+\beta-\gamma}{2}\right)X_3$.

Ques (5) Let $u, v, w$ are LI vectors. Determine whether or not the following vectors are LI.

(a) $u-2v+w$, $u+v$ & $u-v$
(b) $u$, $u+v$ & $u+v+w$
(c) $u-v$, $v-w$ & $w-u$
(d) $u+v$, $2v+w$ & $w+u$

Sol. Since $u, v$ & $w$ are LI hence the equation:

$\boxed{c_1u + c_2v + c_3w = \mathbf{0} \text{ gives } c_1=c_2=c_3=0}$ (i.e. trivial solution) —— 1

(a) Let $\quad K_1(u-2v+w) + K_2(u+v) + K_3(u-v) = 0$

$\Rightarrow (K_1+K_2+K_3)u + (-2K_1+K_2-K_3)v + K_1w = 0$

From (1), it is clear that

$$\left.\begin{array}{r} K_1+K_2+K_3=0 \\ -2K_1+K_2-K_3=0 \\ K_1=0 \end{array}\right\} \text{ a homogeneous system}$$

Since $\det \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} = -2 \neq 0$

$\therefore$ Trivial sol$^n$ i.e. $K_1 = K_2 = K_3 = 0$

$\therefore$ LI.

ⓒ Let $K_1(u-v) + K_2(v-w) + K_3(w-u) = 0$

$\Rightarrow (K_1 - K_3)u + (-K_1 + K_2)v + (-K_2 + K_3)w = 0$

From ①, it is clear that
$$\left.\begin{aligned} K_1 - K_3 &= 0 \\ -K_1 + K_2 &= 0 \\ -K_2 + K_3 &= 0 \end{aligned}\right\} \text{homogeneous system}$$

Since, $\det \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = 1(1-0) - 1(1-0) = 0$

$\therefore$ Non-trivial solution exists

$\therefore$ LD.

ⓑ DO YOURSELF!

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★ $\boxed{\text{REMARKS}}$: ① Any system containing zero vector is LD.

② A system of $(n+1)$ vectors in n-dimensional space i.e. $\mathbb{R}^n$ is dependent.

For eg. the system $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,2,3]$ is LD as these are 4 vectors in 3-dimensional space i.e. $\mathbb{R}^3$.