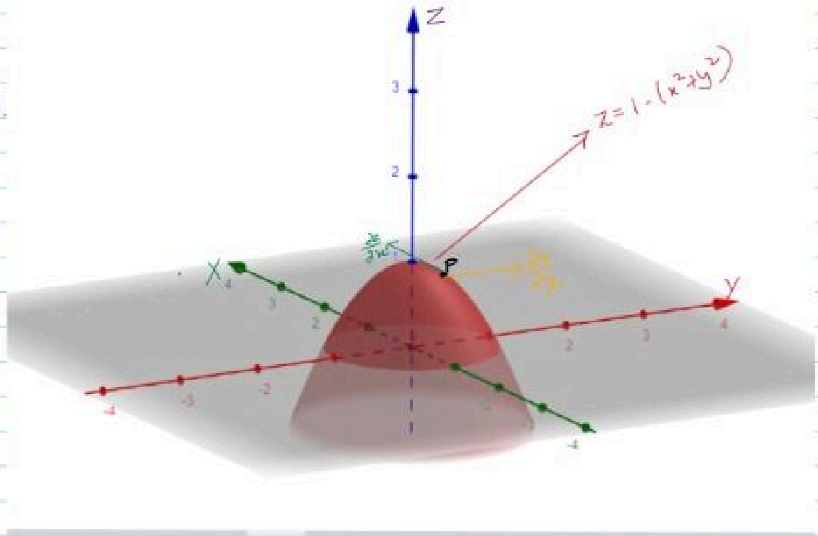
Maxima And Minima

Visualization of $f: \mathbb{R}^2 \to \mathbb{R}$

Consider $f(x,y) = 1-(x^2+y^2)$
Represent the output f by variable 'z' so that in 3D-space ($\mathbb{R}^3$) the vertical axis gives the output (height) [See figure]

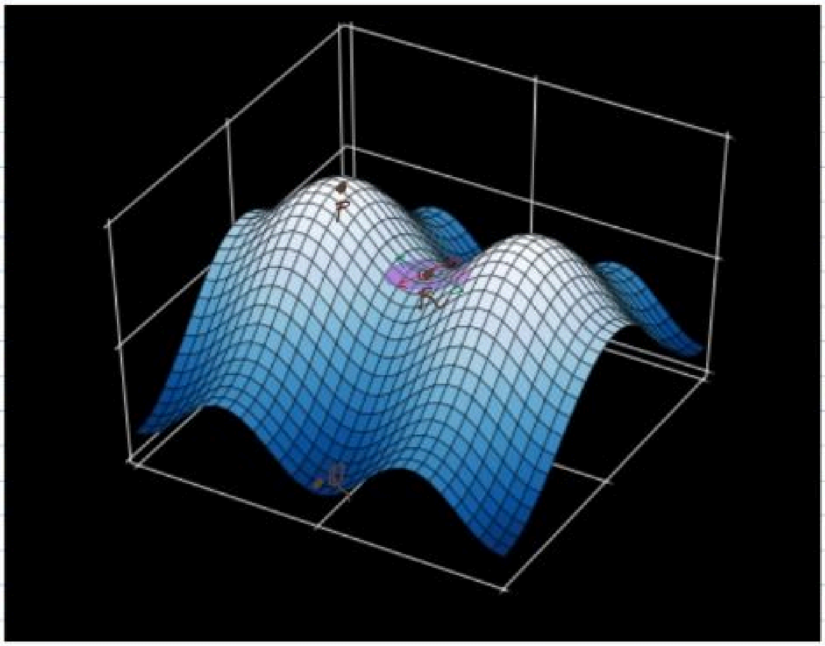
The partial derivative $\left(\frac{\partial z}{\partial x}\right)_P$ represents the tangent at point $P(x,y)$ in the direction of x-axis.
Similarly $\left(\frac{\partial z}{\partial y}\right)_P$ represents the tangent at P in direction of y-axis.



$\frac{\partial^2 z}{\partial x^2}$ is change of $\frac{\partial z}{\partial x}$ in direction of x-axis.
$\frac{\partial^2 z}{\partial y^2}$ is change of $\frac{\partial z}{\partial y}$ in direction of y-axis.
$\frac{\partial^2 z}{\partial x \partial y}$ is change of $\frac{\partial z}{\partial y}$ in direction of x-axis.

Maxima/Minima of $z = f(x,y)$

See the graph of some $z = f(x,y)$ below:



At P,
$\boxed{z_P \geqslant z_D}$ where D is domain of f i.e. xy-plane or $\mathbb{R}^2$
OR $\boxed{f(P) \geqslant f(x,y)}$
$\therefore$ P is point of maxima

At Q, $z_Q \leq z_D$
OR
$\boxed{f(Q) \leq f(x,y)}$
$\therefore$ Q is point of minima.

At R, $z_R \geqslant z_{D_1}$ where D_1 is part of domain parallel to the direction (in red).
Also, $z_R \leq z_{D_2}$ where D_2 is part of domain parallel to direction (in green).
$\therefore$ R is point of neither maxima nor minima (i.e. saddle point)

Test for Maxima/Minima for $z = f(x,y)$

1. Find $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ & $t = \frac{\partial^2 z}{\partial y^2}$.
2. Put $p=0$ & $q=0$ to obtain the stationary points $P(\alpha, \beta)$ (say)
3. If at point $P(\alpha,\beta)$, we have $rt - s^2 > 0$ and:
   (a) $r < 0$ then P is point of maxima
   (b) $r > 0$ then P is point of minima.
4. If at point $P(\alpha, \beta)$, we have $rt - s^2 < 0$ then P is saddle point.

⑤ If at point $P(\alpha,\beta)$, we have $rt-s^2=0$ then the test fails to conclude the nature of P. Further investigation based on theory is required.

Ques① Find the extremum values of $f(x,y)=x^3+3xy^2-3x^2-3y^2+7$.

SOL Here $p=\frac{\partial f}{\partial x}=3x^2+3y^2-6x$

$$q=\frac{\partial f}{\partial y}=6xy-6y$$

$$r=\frac{\partial^2 f}{\partial x^2}=6x-6\ ,\ s=\frac{\partial^2 f}{\partial x\partial y}=6y\ ,\ t=\frac{\partial^2 f}{\partial y^2}=6x-6$$

Put $p=0$ and $q=0$

$3x^2+3y^2-6x=0$ and $6xy-6y=0$

$\Rightarrow x^2+y^2-2x=0$ — ① and $xy-y=0$

$\Rightarrow y(x-1)=0$

$\Rightarrow y=0$ or $x=1$ — ②

If $y=0$, ① gives: $x^2-2x=0$

$\Rightarrow x(x-2)=0$

$\Rightarrow x=0$ or $x=2$

$$\boxed{\therefore (0,0)\ \&\ (2,0) \text{ are stationary points}}$$

If $x=1$, ① gives: $y^2-1=0 \Rightarrow y=\pm 1$

$$\boxed{\therefore (1,1)\ \&\ (1,-1) \text{ are stationary points.}}$$

⚠ CAUTION: Verify that the points obtained satisfy $p=0$ & $q=0$.

Stationary point	$r=6x-6$	$s=6y$	$t=6x-6$	$rt-s^2$	CONCLUSION
$(0,0)$	$-6\ (<0)$	0	-6	$36\ (>0)$	Pt. of maxima
$(2,0)$	$6\ (>0)$	0	6	$36\ (>0)$	Pt. of minima
$(1,1)$	0	6	0	$-36\ (<0)$	Saddle point
$(1,-1)$	0	-6	0	$-36\ (<0)$	Saddle point

Extremum values of f:

$$f_{min}=f(2,0)=3$$

$$f_{max}=f(0,0)=7$$

Ques② Find the extremum values of x^3+y^3-3axy, $(a>0)$.

SOL Let $f(x,y)=x^3+y^3-3axy$, $(a>0)$

$$p=3x^2-3ay\ ,\ q=3y^2-3ax\ ,\ r=12x\ ,\ s=-3a\ ,\ t=6y$$

Put $p=0$ and $q=0$

$\Rightarrow x^2-ay=0$ — ① and $y^2-ax=0$ — ②

From ②, $x=\frac{y^2}{a}$ — ③

③ in ①; $\frac{y^4}{a^2}-ay=0$

$\Rightarrow y^4-a^3y=0$

$\Rightarrow y(y^3-a^3)=0$

$\Rightarrow y=0,\quad y^3-a^3=0$

$\Rightarrow \boxed{y=0}$ $\quad\Big|\quad$ $(y-a)(y^2+a^2+ay)=0$

$\boxed{y=a,}$ $\quad y = \dfrac{-a \pm \sqrt{a^2-4a^2}}{2} = \dfrac{-a \pm a\sqrt{-3}}{2}$

$= \dfrac{-1 \pm i\sqrt{3}}{2a} \notin \mathbb{R}$

Neglect

From ③;

If $y=0,\ x=0 \Longrightarrow (0,0)$ is stationary point.

If $y=a,\ x=a \Rightarrow (a,a)$ is stationary point.

Stationary point	$r=12x$	$s=-3a$	$t=6y$	$rt-s^2$	CONCLUSION	f
$(0,0)$	0	$-3a$	0	$-9a^2(<0)$	SADDLE POINT	—
(a,a)	$12a$	$-3a$	$6a$	$63a^2(>0)$	MINIMA	$f_{min}=-a^3$

Ques③ Find the maximum and minimum value of $f(x,y)=x^3y^2(12-3x-4y)$

SOL. Here $f(x,y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$

so that $p = \dfrac{\partial f}{\partial x} = 36x^2y^2 - 12x^3y^2 - 12x^2y^3$

& $q = \dfrac{\partial f}{\partial y} = 24x^3y - 6x^4y - 12x^3y^2$

$r = \dfrac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 72x^2y - 24x^3y - 36x^2y^2$

$t = \dfrac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y$

Put $p=0$ & $q=0$

$12x^2y^2(3-x-y)=0$ & $6x^3y(4-x-2y)=0$

$\Rightarrow \boxed{x=0,\ y=0,\ x+y=3}$ & $\boxed{x=0,\ y=0,\ x+2y=4}$

TWO DIFFERENT SETS OF 6 EQUATIONS, EACH CONTAINS 3 EQUATIONS

Possibilities are:

(i) $x=0$ & $y=0 \Rightarrow (0,0)$ is st. pt.
(ii) $x=0$ & $x+2y=4 \Rightarrow (0,2)$ is st. pt.
(iii) $y=0$ & $x+2y=4 \Rightarrow (4,0)$ is st. pt.
(iv) $x+y=3$ & $x=0 \Longrightarrow (0,3)$ is st. pt.
(v) $x+y=3$ & $y=0 \Rightarrow (3,0)$ is st. pt.
(vi) $x+y=3$ & $x+2y=4 \Rightarrow (2,1)$ is st. pt

NOTE

$x=0$ gives $p=q=0$ & $y=0$ gives $p=q=0$

$x=0$ & $x=0$ gives $x=0$ i.e. Y-axis

$\Rightarrow$ All points of Y-axis are stationary points.

Similarly $y=0$ & $y=0$ gives $y=0$

$\Rightarrow$ All pts on X-axis are stationary points

STATIONARY POINTS	r	s	t	$rt-s^2$	CONCLUSION	EXTREME VALUES OF f
$(0,0)$	0	0	0	0	Test fails	See investigation below
$(0,2)$	0	0	0	0	Test fails	"
$(4,0)$	0	0	0	0	Test fails	"
$(0,3)$	0	0	0	0	Test fails	"
$(3,0)$	0	0	16270	0	Test fails	"
$(2,1)$	$-48(<0)$	-96	-48	$2400(>0)$	Maxima	$f_{max}=f(2,1)=16$

INVESTIGATION AT (0,0): At pt. $P(0,0)$

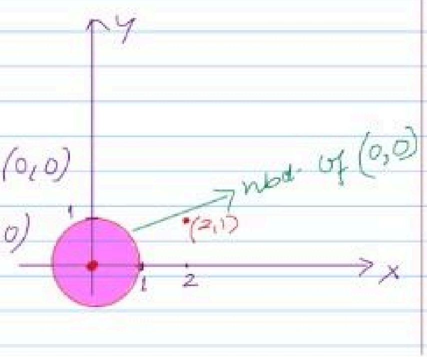
$f(P) = f(0,0) = 0$

Also, $f(x,y) = x^3y^2(12-3x-4y)$

$\because f\left(\frac{1}{3},\frac{1}{2}\right) = \left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right)^2\left[12-1-2\right] > f(0,0)$

& $f\left(-\frac{1}{3},\frac{1}{2}\right) = \left(\frac{-1}{3}\right)^3\left(\frac{1}{2}\right)^2\left[12+3-2\right] < f(0,0)$

$\therefore (0,0)$ is saddle point.



Ques(4) Investigate the nature of critical points of $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

Sol. Put $p=0$ & $q=0$

$4x^3-4x+4y=0$ & $4y^3+4x-4y=0$

$\Rightarrow x^3-x+y=0$ — (1) & $y^3+x-y=0$ — (2)

(1) + (2); $x^3+y^3=0$

$\Rightarrow (x+y)(x^2+y^2-xy)=0$

$\Rightarrow x+y=0$, $x^2-xy+y^2=0$ (quadratic in x for fixed y)

$\Rightarrow \boxed{y=-x}$ — (3), $x = \dfrac{y \pm \sqrt{y^2-4y^2}}{2} = \dfrac{y \pm yi\sqrt{3}}{2} \notin \mathbb{R}$

(Rejected)

(3) in (1); $x^3-2x=0 \Rightarrow x(x^2-2)=0$

$\Rightarrow x=0, x=\sqrt{2}, x=-\sqrt{2}$

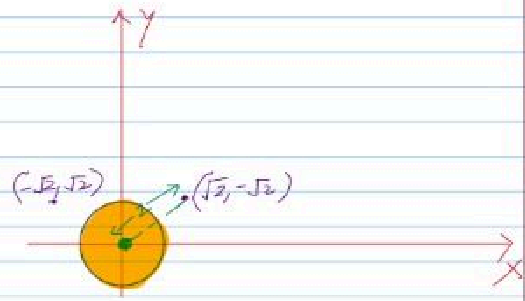
If $x=0$, then (3) gives $y=0 \Rightarrow (0,0)$ is st. pt.

If $x=\sqrt{2}$ then (3) gives $y=-\sqrt{2} \Rightarrow (\sqrt{2},-\sqrt{2})$ is st. pt.

If $x=-\sqrt{2}$ then (3) gives $y=\sqrt{2} \Rightarrow (-\sqrt{2},\sqrt{2})$ is st. pt

STATIONARY POINTS	$r=4(3x^2-1)$	$s=4$	$t=4(3y^2-1)$	$rt-s^2$	CONCLUSION	EXTREME VALUES OF f
$(0,0)$	$-1(<0)$	4	-1	0	Test fails	See investigation below
$(\sqrt{2},-\sqrt{2})$	$20(>0)$	4	20	$384(>0)$	Minima	$f_{min} = f(\sqrt{2},-\sqrt{2}) = 8$
$(-\sqrt{2},\sqrt{2})$	$20(>0)$	4	20	$384(>0)$	Test fails	$f_{min} = 8$

INVESTIGATION AT P(0,0)



$f(P) = f(0,0) = 0$

$\because f(x,y) = x^4+y^4-2(x^2-2xy+y^2)$

or $f(x,y) = (x^4+y^4) - 2(x-y)^2$

$\because f(1,1) = 2 > f(P)$

& $f(-1,1) = -6 < f(P)$

$\therefore (0,0)$ is saddle point.