SUCCESSIVE DIFFERENTIATION

Consider a function $y=f(x)$. The derivatives of y w.r.t. x are:

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \cdots\cdots,\ \frac{d^ny}{dx^n},\ \text{etc.}$$

These are called as successive derivatives.

An alternate notation is:

$$y',\ y'',\ y''',\ \cdots\cdots,\ y^{(n)},\ \text{etc.}$$

OR

$$y_1,\ y_2,\ y_3,\ \cdots\cdots,\ y_n,\ \text{etc.}$$ (Remember)

n^{th} DERIVATIVE OF SOME STANDARD FUNCTIONS

① $y = e^{ax}$

Differentiate w.r.t. x;

$$y_1 = a e^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$\vdots$$

Generalize, $$\boxed{y_n = a^n e^{ax}}$$

② $y=(ax+b)^m$

Differentiate w.r.t. x;

$$y_1 = am(ax+b)^{m-1}$$

$$y_2 = a^2m(m-1)(ax+b)^{m-2}$$

$$y_3 = a^3m(m-1)(m-2)(ax+b)^{m-3}$$

$$\vdots$$

Generalize, $$\boxed{y_n = a^n m(m-1)(m-2)\cdots[m-(n-1)](ax+b)^{m-n}}$$

③ $y = \log_e(ax+b)$

Differentiate w.r.t. x;

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = a \times \frac{-1}{(ax+b)^2} \times a = -\frac{a^2}{(ax+b)^2} = \frac{(-1)^1\, 1!\, a^2}{(ax+b)^2}$$

$$y_3 = -a^2 \times \frac{-2}{(ax+b)^3} \times a = +\frac{2a^3}{(ax+b)^3} = \frac{(-1)^2\, 2!\, a^3}{(ax+b)^3}$$

$$y_4 = 2a^3 \times \frac{-3}{(ax+b)^4} \times a = -\frac{3\cdot 2\, a^4}{(ax+b)^4} = \frac{(-1)^3\, 3!\, a^4}{(ax+b)^4}$$

$$\vdots$$

Generalize, $$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}}$$

④ $y = \sin(ax+b)$

Differentiate w.r.t. x; $\quad y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+ax+b\right)$

$$y_2 = a^2 \cos\left(\frac{\pi}{2} + ax+b\right) = a^2 \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax+b\right)$$

$$= a^2 \sin\left(\frac{2\pi}{2} + ax+b\right)$$

★ $\cos\theta = \sin\left(\frac{\pi}{2}+\theta\right)$

Generalize, $\boxed{y_n = a^n \sin\left(\frac{n\pi}{2} + ax+b\right)}$

(5) $y = \cos(ax+b)$

Differentiate w.r.t. x; $y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2} + ax+b\right)$

$$y_2 = -a^2 \sin\left(\frac{\pi}{2} + ax+b\right) = a^2\cos\left(\frac{2\pi}{2} + ax+b\right)$$

★ $-\sin\theta = \cos\left(\frac{\pi}{2}+\theta\right)$

Generalize, $\boxed{y_n = a^n \cos\left(\frac{n\pi}{2} + ax+b\right)}$

(6) $y = e^{ax}\sin(bx+c)$

Differentiate w.r.t. x; $y_1 = e^{ax} b\cos(bx+c) + a e^{ax}\sin(bx+c)$

$$= e^{ax}\left[b\cos(bx+c) + a\sin(bx+c)\right]$$

Let $b = r\sin\theta$ & $a = r\cos\theta$

★ $\sin A\cos B + \cos A \sin B = \sin(A+B)$

Squaring and adding; $a^2+b^2 = r^2$

$$\Rightarrow \boxed{r = \sqrt{a^2+b^2}}$$

Dividing:

$$\frac{b}{a} = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\therefore y_1 = e^{ax}\left[r\sin\theta\cos(bx+c) + r\cos\theta\sin(bx+c)\right]$$

$$= r e^{ax}\sin\left[\theta + (bx+c)\right]$$

Generalize, $\boxed{y_n = r^n e^{ax}\sin\left[n\theta + (bx+c)\right] \text{ where } r = \sqrt{a^2+b^2} \ \& \ \theta = \tan^{-1}\left(\frac{b}{a}\right)}$

(7) $y = e^{ax}\cos(bx+c)$

Differentiate w.r.t. x; $y_1 = -e^{ax} b\sin(bx+c) + a e^{ax}\cos(bx+c)$

$$= e^{ax}\left[a\cos(bx+c) - b\sin(bx+c)\right]$$

Let $b = r\sin\theta$ & $a = r\cos\theta$

★ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

Squaring and adding; $a^2+b^2 = r^2$

$$\Rightarrow \boxed{r = \sqrt{a^2+b^2}}$$

Dividing:

$$\frac{b}{a} = \tan\theta \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\therefore y_1 = e^{ax}\left[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)\right]$$

$$= r e^{ax}\cos\left[\theta + (bx+c)\right]$$

Generalize, $\boxed{y_n = r^n e^{ax} \cos\left[n\theta + (bx+c)\right] \text{ where } r = \sqrt{a^2+b^2} \text{ \& } \theta = \tan^{-1}\left(\frac{b}{a}\right)}$

Ques 1 Find the n^{th} derivative of:

(a) $\dfrac{5+2x}{x^2-5x+6}$ (b) $\dfrac{x^3}{x^2-1}$ (c) $\sin 5x \cos 3x$ (d) $\dfrac{x^n-1}{x-1},\ n \in \mathbb{Z}^+$

Sol (a) Let $y = \dfrac{5+2x}{x^2-5x+6}$

$$= \frac{5+2x}{(x-2)(x-3)}$$

Let $\dfrac{5+2x}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$ (by partial fractions)

$$\Rightarrow \frac{5+2x}{(x-2)(x-3)} = \frac{A(x-3) + B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow 5+2x = A(x-3) + B(x-2)$$

Put $x=3$, $11 = B$

Put $x=2$ $-9 = A$

$$\therefore \quad y = \frac{-9}{x-2} + \frac{11}{x-3}$$

Can be hidden

Differentiate w.r.t. x;

$$y_1 = 9\left[\frac{1}{(x-2)^2}\right] + 11\left[\frac{-1}{(x-3)^2}\right]$$

$$y_2 = 9\left[\frac{-(2\cdot 1)}{(x-2)^3}\right] + 11\left[\frac{-(2\cdot 1)}{(x-3)^3}\right]$$

$$y_3 = 9\left[\frac{(-1)^2(3\cdot 2\cdot 1)}{(x-2)^4}\right] + 11\left[\frac{(-1)^2(3\cdot 2\cdot 1)}{(x-3)^4}\right]$$

$$\vdots$$

Generalize, $$y_n = \frac{9\,(-1)^{n-1}\, n!}{(x-2)^{n+1}} + \frac{11\,(-1)^{n-1}\, n!}{(x-3)^{n+1}}$$

(b) Let $y = \dfrac{x^3}{x^2-1}$

$$\Rightarrow y = x + \frac{x}{x^2-1}$$

$$\Rightarrow y = x + \frac{x}{(x-1)(x+1)}$$

$$= x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

★ $x^2-1 \overline{)\ \cancel{x^3}\ }\ x$ (Quotient)

$\underline{\cancel{(-)x^3} \ \ \overset{-}{(+)}x}$

x (Remainder)

Then $\dfrac{x^3}{x^2-1}$ = Quotient + $\dfrac{\text{Remainder}}{\text{Divisor}}$

Differentiate w.r.t. x;

$$y_1 = 1 + \frac{1}{2}\left[\frac{-1}{(x-1)^2} + \frac{(-1)}{(x+1)^2}\right]$$

$$y_2 = \frac{1}{2}\left[\frac{(-1)^2\, 2!}{(x-1)^3} + \frac{(-1)^2\, 2!}{(x+1)^3}\right]$$

$$\vdots$$

Generalize, $$y_n = \frac{1}{2}(-1)^n n! \left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

(c) Let $y = \sin 5x \cos 3x$

$$\Rightarrow y = \frac{1}{2}(2\sin 5x \cos 3x)$$

$$= \frac{1}{2}\left[\sin 8x + \sin 2x\right]$$

$\boxed{★\ 2\sin A \cos B = \sin(A+B) + \sin(A-B)}$

Differentiate w.r.t. x 'n' times;

$$y_n = \frac{1}{2}\left[8^n \sin\left(\frac{n\pi}{2} + 8x\right) + 2^n \sin\left(\frac{n\pi}{2} + 2x\right)\right]$$

USE RESULT FOR $D^n \sin(ax+b)$

(a) Let $y = \dfrac{x^n - 1}{x-1}$, n is a positive integer

$$x-1 \;\big)\; x^n - 1 \;\big(\; x^{n-1} + x^{n-2} + \cdots\cdots + 1$$

$$\begin{array}{l}
\cancel{x^n} - 1 \\
\underline{\cancel{\overset{(-)}{x^n}} \quad \overset{-}{(+)}x^{n-1}} \\
\cancel{x^{n-1}} - 1 \\
\underline{\cancel{\overset{(-)}{x^{n-1}}} \quad \overset{-}{(+)}x^{n-2}} \\
x^{n-2} - 1 \\
\vdots \\
\cancel{x} - \cancel{1} \\
\underline{\cancel{\overset{(-)}{x}} \quad \cancel{\overset{-}{(+)}1}} \\
0
\end{array}$$

$\therefore\; y = x^{n-1} + x^{n-2} + \cdots\cdots + 1$

Differentiate 'n' times, $\quad y_n = 0$

LEIBNITZ THEOREM

Let 'u' and 'v' be functions of 'x' whose n^{th} derivatives are known. Then

$$\boxed{D^n(uv) = {}^nC_0 (D^n u)(D^0 v) + {}^nC_1 (D^{n-1}u)(D^1 v) + {}^nC_2 (D^{n-2}u)(D^2 v) + \cdots\cdots + {}^nC_n (D^0 u)(D^n v)}$$

PROOF We know that $D(uv) = u_1 v + u v_1 = {}^1C_0 D(u) D^0(v) + {}^1C_1 D^0(u) D(v)$

$$\begin{aligned}
D^2(uv) = D[D(uv)] &= D(u_1 v) + D(u v_1) \\
&= u_2 v + u_1 v_1 + u_1 v_1 + u v_2 \\
&= u_2 v + 2u_1 v_1 + u v_2 \\
&= {}^2C_0 D^2(u) D^0(v) + {}^2C_1 D(u) D(v) + {}^2C_2 D^0(u) D^2(v)
\end{aligned}$$

$\vdots$

Generalize, $D^n(uv) = {}^nC_0 (D^n u)(D^0 v) + {}^nC_1 (D^{n-1}u)(D^1 v) + {}^nC_2 (D^{n-2}u) + \cdots\cdots + {}^nC_n (D^0 u)(D^n v)$

REMARKS

- $D^0(u) = u$, $D^0(v) = v$ & $D^n(y_k) = y_{k+n}$.
- ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ i.e. ${}^nC_0 = 1$, ${}^nC_1 = n$, ${}^nC_2 = \dfrac{n(n-1)}{2!}$, ${}^nC_3 = \dfrac{n(n-1)(n-2)}{3!}$, etc.
- ${}^nC_r = {}^nC_{n-r}$ i.e. ${}^nC_n = {}^nC_0 = 1$, ${}^nC_{n-1} = {}^nC_1 = n$, etc.
- The polynomials are generally taken as 'v'.
- $(y_n)_0$ denote the value of y_n evaluated at $x = 0$

Ques 1 If $y = x^2 e^{2x}$, prove that $(y_n)_0 = 2^{n-2} n(n-1)$

Sol Diff. n times using LT; $\quad D^n(y) = D^n(e^{2x} \cdot x^2)$

$$\begin{aligned}
\Rightarrow y_n &= {}^nC_0 (D^n e^{2x})(D^0 x^2) + {}^nC_1 (D^{n-1} e^{2x})(D^1 x^2) + {}^nC_2 (D^{n-2} e^{2x})(D^2 x^2) \\
&= (2^n e^{2x})(x^2) + n\,(2^{n-1} e^{2x})(2x) + \frac{n(n-1)}{2!}(2^{n-2} e^{2x})(2)
\end{aligned}$$

$$\Rightarrow y_n = 2^n x^2 e^{2x} + 2^n n x e^{2x} + n(n-1)\, 2^{n-2} e^{2x}$$

Put $x = 0$,

$$(y_n)_0 = n(n-1)\, 2^{n-2}$$

Ques 2) Find the n^{th} derivative of:

(a) $x^2 \sin x$ (b) $x^2 e^x \sin x$

Sol (a) $D^n[\sin x . x^2] = {}^nC_0 (D^n \sin x)(D^0 x^2) + {}^nC_1 (D^{n-1} \sin x)(D^1 x^2) + {}^nC_2 (D^{n-2} \sin x)(D^2 x^2)$

$= \sin\left(\frac{n\pi}{2} + x\right)(x^2) + n\left[\sin \frac{(n-1)\pi}{2} + x\right](2x) + n(n-1) \sin\left[\frac{(n-2)\pi}{2} + x\right]$

HOW?

★ $D^n \sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

Put $a = 1, b = 0$

$D^n(\sin x) = \sin\left(\frac{n\pi}{2} + x\right)$

(b) $D^n[e^x \sin x . x^2]$

$= {}^nC_0 D^n(e^x \sin x)(x^2) + {}^nC_1 D^{n-1}(e^x \sin x) D(x^2) + {}^nC_2 D^{n-2}(e^x \sin x). D^2(x^2)$

$= 2^{n/2} e^x \sin\left(n\frac{\pi}{4} + x\right)x^2 + n 2^{\frac{n-1}{2}} e^x \sin\left[\frac{(n-1)\pi}{4} + x\right](2x) + \frac{n(n-1)}{2!} 2^{\frac{n-2}{2}} e^x \sin\left[\frac{(n-2)\pi}{4} + x\right](2)$

$= e^x\left[x^2 2^{n/2} \sin\left(\frac{n\pi}{4} + x\right) + nx 2^{\frac{n+1}{2}} \sin\left[\frac{(n-1)\pi}{4} + x\right] + n(n-1) 2^{\frac{n-2}{2}} \sin \frac{(n-2)\pi}{4} + x\right]$

HOW?

★ $D^n e^{ax} \sin(bx+c) = r^n e^{ax} \sin[n\theta + (bx+c)]$

where $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Put $a = 1, b = 1, c = 0$

$D^n[e^x \sin x] = 2^{1/2} e^x \sin\left[n\frac{\pi}{4} + x\right]$

Ques 3) If $y = x^n \log x$, prove that $y_{n+1} = \frac{n!}{x}$

★ y_{n+r} — Diff. 'r' times as earlier; Use LT to diff. further n times

Sol $y = x^n \log x$

diff. w.r.t. x, $y_1 = x^n \times \frac{1}{x} + n x^{n-1} . \log x$

$\Rightarrow x y_1 = x^n + n x^n \log x$

$\Rightarrow x y_1 = x^n + n y$ ← NOTE THIS STEP

Diff. n times using LT; $D^n[y_1 x] = D^n[x^n] + n D^n[y]$

$\Rightarrow [{}^nC_0 (D^n y_1)(D^0 x) + {}^nC_1 (D^{n-1} y_1)(D^1 x)] = n! + n y_n$

$\Rightarrow x y_{n+1} + \cancel{n y_n} = n! + \cancel{n y_n} \Longrightarrow y_{n+1} = \frac{n!}{x}$

Ques 4) If $y = (x^2-1)^n$, prove that $(x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$.

Sol $y = (x^2-1)^n$

diff. w.r.t. x; $y_1 = n(x^2-1)^{n-1} \times 2x$

$\Rightarrow y_1 = 2nx \frac{(x^2-1)^n}{(x^2-1)}$

$\Rightarrow (x^2-1) y_1 = 2nxy$ ← NOTE THIS STEP

diff. w.r.t. x;

$(x^2-1) y_2 + 2x y_1 = 2n(x y_1 + y)$

$\Longrightarrow (x^2-1) y_2 + 2x y_1 - 2nx y_1 - 2ny = 0$

diff. n times using L.T.,

$D^n[y_2(x^2-1)] + 2 D^n[y_1 x] - 2n D^n[y_1 x] - 2n D^n[y] = D^n[0]$

$\Rightarrow [{}^nC_0 (D^n y_2)(x^2-1) + {}^nC_1 (D^{n-1} y_2) D^1(x^2-1) + {}^nC_2 (D^{n-2} y_2) D^2(x^2-1)]$

$+ 2[{}^nC_0 (D^n y_1)(x) + {}^nC_1 (D^{n-1} y_1)(Dx)] - 2n[{}^nC_0 (D^n y_1)(x) + {}^nC_1 (D^{n-1} y_1)(Dx)] - 2n y_n = 0$

$\Rightarrow \left[y_{n+2}(x^2-1) + n y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n (2)\right] + 2[y_{n+1} x + n y_n] - 2n[y_{n+1} x + n y_n] - 2n y_n = 0$

$\Rightarrow (x^2-1) y_{n+2} + 2x y_{n+1} + (n^2 - n - 2n^2) y_n = 0$

$\Rightarrow (x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$

Ques 5) If $y = \sin[\log(x^2+2x+1)]$, prove that $(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

Sol) $y = \sin[\log(x+1)^2] = \sin[2\log(x+1)]$

diff. w.r.t x; $y_1 = \cos[2\log(x+1)] \times \left(\frac{2}{x+1}\right)$

$\Rightarrow (x+1)y_1 = 2\cos[2\log(x+1)]$

diff. again;

$(x+1)y_2 + y_1 = -2\sin[2\log(x+1)] \times \frac{2}{(x+1)}$

$\Rightarrow (x+1)^2 y_2 + (x+1)y_1 = -4y$

diff n times using L.T.; $D^n[y_2(x+1)^2] + D^n[y_1(x+1)] = -4D^n[y]$

$\Rightarrow [{}^nC_0(D^n y_2)(x+1)^2 + {}^nC_1(D^{n-1}y_2)D(x+1)^2 + {}^nC_2(D^{n-2}y_2)D^2(x+1)^2]$

$+ [{}^nC_0(D^n y_1)(x+1) + {}^nC_1(D^{n-1}y_1)D(x+1)] = -4y_n$

$\Rightarrow \left[(x+1)^2 y_{n+2} + n y_{n+1} 2(x+1) + \frac{n(n-1)}{2!} y_n (2)\right] + \left[(x+1)y_{n+1} + n y_n\right] + 4y_n = 0$

$\Rightarrow (x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + [n(n-1) + n + 4]y_n = 0$

$\Rightarrow (x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

Ques 6) If $y = a\cos(\log x) + b\sin(\log x)$, prove $x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$

Sol) Diff. w.r.t. x;

$y_1 = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$

$\Rightarrow x y_1 = -a\sin(\log x) + b\cos(\log x)$

Diff. again,

$x y_2 + y_1 = -a\cos(\log x) \times \frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$

$\Rightarrow x^2 y_2 + x y_1 = -y \longleftarrow$ NOTE THIS STEP

Diff. n times using L.T.; $D^n[y_2 x^2] + D^n[y_1 x] = -D^n[y]$

$\Rightarrow [{}^nC_0(D^n y_2)(x^2) + {}^nC_1(D^{n-1}y_2)(Dx^2) + {}^nC_2(D^{n-2}y_2)(D^2x^2)] + [{}^nC_0(D^n y_1)(x) + {}^nC_1(D^{n-1}y_1)(Dx)] = -y_n$

$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$

Ques 7) If $y = \log(x + \sqrt{x^2+1})$, prove that

(a) $(y_{2n})_0 = 0$ (b) $(y_{2n+1})_0 = (-1)^n [1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2]$

Sol) Here $y = \log(x + \sqrt{x^2+1})$

Diff. w.r.t. x; $y_1 = \left(\frac{1}{x+\sqrt{x^2+1}}\right) \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right) = \frac{1}{\sqrt{x^2+1}}$

or $y_1 \cdot \sqrt{x^2+1} = 1$ ——— (1)

Squaring; $(x^2+1)y_1^2 = 1 \longleftarrow$ GET RID OF FRACTIONAL POWERS!

Diff. again,

$(x^2+1) 2y_1 y_2 + 2x y_1^2 = 0$

$\Rightarrow y_2(x^2+1) + x y_1 = 0$ ——— (2)

Diff. using L.T;

$D^n[y_2 \cdot (x^2+1)] + D^n[y_1 \cdot x] = D^n[0]$

$\Rightarrow \left[D^n(y_2)\cdot(x^2+1) + n D^{n-1}(y_2)\cdot D(x^2+1) + \frac{n(n-1)}{2} D^{n-2}(y_2)\cdot D^2(x^2+1)\right] + \left[D^n(y_1)\cdot(x) + n D^{n-1}(y_1)\cdot D(x)\right] = 0$

$\Rightarrow \left[y_{n+2}\cdot(x^2+1) + n y_{n+1}\cdot(2x) + \frac{n(n-1)}{2} y_n (2)\right] + \left[y_{n+1}\cdot(x) + n y_n\right] = 0$

$\Rightarrow (x^2+1)y_{n+2} + (2n+1)x y_{n+1} + n^2 y_n = 0$ ——— (3)

Put $x = 0$ in ①, ② and ③;

$$(y_1)_0 = 1 \quad \text{(a)}$$

$$\& \quad (y_2)_0 = 0 \quad \text{(b)}$$

$$\& \quad (y_{n+2})_0 = -n^2 (y_n)_0$$

Do this until you see a pattern

Put $n=1$, $(y_3)_0 = -1^2 (y_1)_0 = -1^2$ — (c)

Put $n=2$, $(y_4)_0 = -2^2 (y_2)_0 = 0$ — (d)

Put $n=3$, $(y_5)_0 = -3^2 (y_3)_0 = -3^2 \times -1^2 = 3^2 \cdot 1^2$ — (e)

Put $n=4$, $(y_6)_0 = -4^2 (y_4)_0 = 0$ — (f)

Put $n=5$, $(y_7)_0 = -5^2 (y_5)_0 = -5^2 \cdot 3^2 \cdot 1^2$ — (g)

$\vdots$

Generalizing, $(y_{2n})_0 = 0$ from (b), (d) and (f)

$\&$ $(y_{2n+1})_0 = (-1)^n \left[1^2 \cdot 3^2 \cdot 5^2 \cdots\cdots (2n-1)^2 \right]$ from (a), (c), (e) & (g)