RANK OF MATRIX

ECHELON FORM

A matrix $X=[x_{ij}]$ said to be in its $\boxed{\text{echelon form}}$ if :

- X is an upper triangular i.e. $x_{ij}=0$ for $i>j$
- The number of $\boxed{\text{leading zeros}}$ must increase as we move down along the first column of X.

The zeros which lie at the beginning of a row

Examples -
$$\begin{bmatrix}1&2\\0&3\end{bmatrix}, \begin{bmatrix}1&2\\0&0\end{bmatrix}, \begin{bmatrix}1&2&3\\0&2&1\\0&0&3\end{bmatrix}, \begin{bmatrix}1&2&3\\0&1&0\\0&0&3\end{bmatrix}, \begin{bmatrix}0&2&3\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&2&3\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$
$$\begin{bmatrix}2&3&1\\0&0&1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}2&0&0\end{bmatrix}, \text{etc.}$$

Non examples -
$$\begin{bmatrix}1&0\\2&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&2&3\\0&0&1\\0&0&5\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \text{etc.}$$

HOW TO ACHIEVE ECHELON FORM?

Suppose matrix A is given. In order to achieve A in its echelon form, we use elementary row operations of the form $R_i \to R_i + \alpha R_j$ where 'α' is real number.

Examples - (1) $A = \begin{bmatrix}1&2\\3&4\end{bmatrix}$

Target

Apply $R_2 \to R_2 - 3R_1$; $= \begin{bmatrix}1&2\\0&-2\end{bmatrix} = RREF(A)$ (Row reduced echelon form of A)

(2) $A = \begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}$

Targets

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - 3R_1$; $= \begin{bmatrix}1&2&3\\0&-1&-5\\0&-5&-7\end{bmatrix}$

Target

$R_3 \to R_3 - 5R_2$; $= \begin{bmatrix}1&2&3\\0&-1&-5\\0&0&18\end{bmatrix} = RREF(A)$

RANK OF MATRIX

Let X be a matrix. The rank of X i.e. $rank(X)$ is defined as the number of $\boxed{\text{linearly independent}}$ rows (or columns) in it.

To be discussed later!

OR

$\boxed{\text{The number of } \boxed{\text{non-zero rows}} \text{ in } RREF(X) \text{ gives } rank(X).}$

A row in which all elements are zero is a zero row otherwise, it is a non-zero row (with at least one non-zero element)

Ques 1 Find the rank of $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 5 & 1 & 0 \end{bmatrix}$

Targets

Sol $R_2 \to R_2 - 2R_1$;
$R_3 \to 2R_3 - 5R_1$;

$$A = \frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & -13 & 5 \end{bmatrix}$$

Target

$R_3 \leftrightarrow R_2$

$$= \frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 0 & -13 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3/2 & 1/2 \\ 0 & 13/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

rank (A) = no. of non-zero rows in RREF(A) = 2

Ques 2 Find the rank of $A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$, where $1, \omega$ & ω^2 are cube roots of unity.

HINTS - (a) $\omega^3 = 1$, $\omega^4 = \omega \cdot \omega^3 = \omega \cdot 1 = \omega$
(b) $1 + \omega + \omega^2 = 0$

Sol $R_3 \to R_1 + R_2 + R_3$;

$$= \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \omega R_1$;

$$= \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 1-\omega^3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

$\therefore$ rank(A) = 1

ALTERNATE

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix}$$

$R_2 \to R_2 - \omega R_1$;
$R_3 \to R_3 - \omega^2 R_1$;

$$= \begin{bmatrix} 1 & \omega & \omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

$\therefore$ rank(A) = 1

Ques 3 Find the rank of $A = \begin{bmatrix} 2 & 5 & 0 & 1 \\ 9 & -7 & 1 & 3 \\ 5 & -17 & 1 & 1 \\ 4 & 10 & 0 & 2 \end{bmatrix}$

Soln $R_2 \to 2R_2 - 9R_1$;
$R_3 \to 2R_3 - 5R_1$;
$R_4 \to R_4 - 2R_1$;

$$= \frac{1}{4}\begin{bmatrix} 2 & 5 & 0 & 1 \\ 0 & -59 & 2 & -3 \\ 0 & -59 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$;

$$= \frac{1}{4}\begin{bmatrix} 2 & 5 & 0 & 1 \\ 0 & -59 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 5/4 & 0 & 1/4 \\ 0 & -59/4 & 1/2 & -3/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

$\therefore$ rank(A) = 2

REMARKS
1. $\text{Rank}(A) = \text{Rank}(A^T)$, where A^T is transpose of A.
2. If A is invertible then $\text{Rank}(A^{-1}) = \text{Rank}(A)$.
3. For every square matrix A, $\text{Rank}(\text{adj}(A)) = \text{Rank}(A)$
4. For a square matrix of order n, if $\det(A) = 0$ then $\text{rank}(A) < n$.

OR

More precisely, $\det(A) = 0 \iff A$ doesn't have full rank

5. If A is of order $m \times n$ then $\text{rank}(A) \leq \min\{m, n\}$

Example – Let A be a matrix of order 50×99

Then $\text{rank}(A) \leq \min\{50, 99\}$ i.e. $\text{rank}(A) \leq 50$.

Ques 4 Find the rank of $A = \begin{bmatrix} 5 & 9 \\ 3 & 7 \\ 11 & -17 \\ 22 & 1 \end{bmatrix}$

Sol Here $A^T = \begin{bmatrix} 5 & 3 & 11 & 22 \\ 9 & 7 & -17 & 1 \end{bmatrix}$

$R_2 \to 5R_2 - 9R_1$;

$= \frac{1}{5}\begin{bmatrix} 5 & 3 & 11 & 22 \\ 0 & 8 & -184 & -193 \end{bmatrix} = \text{RREF}(A^T)$

$\therefore \text{rank}(A) = \text{rank}(A^T) = 2$

Ques 5 Find the rank of matrix $A = [a_{ij}]_{5\times5}$ where $a_{ij} = \begin{cases} ij, & \text{if } i > j \\ 0, & \text{otherwise} \end{cases}$

Sol $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 \\ 4 & 8 & 12 & 0 & 0 \\ 5 & 10 & 15 & 20 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 8 & 10 \\ 0 & 0 & 0 & 12 & 15 \\ 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A^T)$

$\therefore \text{rank}(A^T) =$ no. of non-zero rows $= 4$

$\Rightarrow \text{rank}(A) = 4$.

Ques 6 Find the value of 'K' if the rank of matrix $A = \begin{bmatrix} 1 & 3 & K \\ 1 & 0 & -1 \\ 2 & 5 & 4 \end{bmatrix}$ is not 3.

Sol Using Remark 4;

Since A doesn't have rank = 3 (full rank)

$\Rightarrow \det(A) = 0$

$\Rightarrow 1(0+5) - 3(4+2) + K(5-0) = 0$

$\Rightarrow 5 - 18 + 5K = 0$

$\Rightarrow K = \frac{13}{5}$

Ques 7 Find the value of rational number 'α' so that the matrix $\begin{bmatrix} \alpha & \pi/7 \\ \pi/7 & \alpha \end{bmatrix}$ has rank less than 2.

Sol Using Remark 4;

Since A doesn't have rank = 2 (full rank)

$\Rightarrow \det(A) = 0$

$\Rightarrow \alpha^2 - \frac{\pi^2}{7^2} = 0$

$$= \left(\alpha - \frac{\pi}{7}\right)\left(\alpha + \frac{\pi}{7}\right) = 0$$

$\Rightarrow \alpha = \pm \frac{\pi}{7}$ (which is irrational)

$\therefore$ No such rational 'α' exists so that rank$(A) < 2$.

Ques 8) Find the rank of matrix A where $A = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$ and $b \neq 0$, $a \in \mathbb{R}$.

Sol)
$$A = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$;
$$= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = RREF(A)$$

$\because b \neq 0$ so the first row is non-zero

$\therefore$ Rank$(A) = 1$

NORMAL FORM

A matrix X is said to be in its normal form if it is in any of the forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. each non-diagonal element is zero and a diagonal element may be either 1 or 0.

OR

A matrix X is said to be in its normal form if it contains a submatrix I_r, $(r \geq 0)$ and rest of the elements are precisely zero.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}, \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}_{I_2}, \begin{bmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix}_{I_2} \text{ and } \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{I_1}$$

> In order to reduce a matrix X into its normal form, both row and column elementary operations are allowed.

RANK BY NORMAL FORM

Suppose X is a matrix. The rank(X) is equal to 'r' where I_r is the largest submatrix contained inside X in its normal form.

$$\text{Let } X = \begin{bmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}_{I_2} \end{bmatrix}$$

Then Rank$(X) = 2$

Ques 1) Find the rank of $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 5 & 1 & 0 \end{bmatrix}$ by reducing it to normal form.

Sol)
$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 5 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$;
$R_3 \to 2R_3 - 5R_1$;
$$= \frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & -13 & 5 \end{bmatrix}$$

$R_3 \longleftrightarrow R_2$;
$$= -\frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 0 & -13 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 2C_3 + C_1$;
$C_2 \to 2C_2 - 3C_1$
$$= -\frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -26 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 26C_3 + 10C_2$
$$= \frac{-1}{208}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -26 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$ & $R_2 \to -\frac{1}{26}R_2$
$$= \frac{26 \times 2}{108}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to I_2$$

$\therefore \text{Rank}(X) = 2$

Ques 2) Find the rank of $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & -1 \\ 4 & 13 & -1 \end{bmatrix}$ by reducing it into normal form.

Sol) $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & -1 \\ 4 & 13 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - 4R_1$;
$$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$;
$$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$;
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + C_2$;
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to I_2$$

$\therefore \text{Rank}(A) = 2.$

Ques 3) Find the rank of $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \end{bmatrix}$ by reducing it to normal form.

Sol) $R_2 \to R_2 - 2R_1$;
$$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$;
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$;
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$R_2 \to (-)R_2$
$$= -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to I_2$$

$\therefore \text{Rank}(A) = 2$

SOLUTION OF SYSTEM OF EQUATIONS

Consider a system of 'n' equations in 'm' variables:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}\right\} \text{—— (A)}$$

If atleast one $b_i \neq 0$ $(i = 1, 2, \ldots n)$ then system (A) is non-homogeneous

If $b_i = 0$ for each i then system (A) is homogeneous.

SOLUTION OF NON-HOMOGENEOUS SYSTEM $AX = B$

For simplicity, take $m = n = 3$ in (A);

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Rewriting the above system as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{AX = B}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$X = A^{-1}B$ is the sol^n by old method

Does it always work?

"No"

NEW METHOD Construct an augmented matrix $C = [A \,\vdots\, B]$

$$C = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Convert C into its echelon form. Then there are 3 possibilities:

Case 1: If $\text{rank}(A) = \text{rank}(C) = r$ (say) $= m$ (no. of variables) then the system is consistent and has a unique solution

Case 2: If $\text{rank}(A) = \text{rank}(C) = r$ (say) $< m$ (no. of variables) then the system is consistent and has infinitely many solutions.

Exactly $(m-r+1)$ solutions are linearly independent (obtained by assuming $(m-r)$ variables arbitrary)

Case 3: If $\text{rank}(A) \neq \text{rank}(C)$ then the system is inconsistent and has no solution.

Ques 1 Test the consistency and find the solution, if it is consistent.

$3x - y + z = 3$, $x + 2y + z = 4$ and $5x - y - 2z = 2$

Sol The augmented matrix;

$C = [A \,\vdots\, B]$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 5 & -1 & -2 & 2 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$; $R_3 \to 3R_3 - 5R_1$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 0 & 7 & 2 & 9 \\ 0 & 2 & -11 & -9 \end{array}\right]$$

$R_3 \to 7R_3 - 2R_2$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & -81 & -81 \end{array}\right] = RREF\,(C)$$

$\because$ Rank(A) = Rank(C) = 3 (= r) = m (= 3) (no. of variables)

$\therefore$ the system is consistent and has a unique solution

HOW TO FIND SOLUTION?

From last matrix ; $-81z = -81$

$\Rightarrow z = 1$

& $7y + 2z = 9$

$\Rightarrow 7y = 7$

$\Rightarrow y = 1$

& $3x - y + z = 3$

$\Rightarrow x = 1$

$\therefore$ $x = 1, y = 1, z = 1$ is the unique solution.

Ques 2 Test the consistency and find the solution, if it is consistent.

$3x - y + z = 3$, $x + 2y + z = 4$ and $4x + y + 2z = 7$

Sol The augmented matrix;

$C = [A \vdots B]$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 1 & 2 & 7 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$;

$R_3 \to 3R_3 - 4R_1$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 0 & 7 & 2 & 9 \\ 0 & 7 & 2 & 9 \end{array}\right]$$

$R_3 \to R_3 - R_2$;

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] = RREF(C)$$

$\because$ Rank (A) = Rank (C) = 2 (= r) < 3 (= m)

$\therefore$ the system is consistent and has infinite solutions.

From the last matrix; $3x - y + z = 3$

& $7y + 2z = 9$

Assume $m - r = 3 - 2 = 1$ variable as arbitrary real number

Let $z = K$ (say)

Then $7y + 2K = 9$ gives $y = \frac{9 - 2K}{7}$

and $3x - y + z = 3$ gives $3x = 3 + \left(\frac{9 - 2K}{7}\right) - K \Rightarrow x = \frac{30 - 9K}{21} = \frac{10 - 3K}{7}$

$\therefore$ $x = \frac{10 - 3K}{7}$, $y = \frac{9 - 2K}{7}$ and $z = K$, where K is arbitrary real number.

Ques 3 Test the consistency and find the solution, if it is consistent.

$x - y + z = 2$, $14x - 14y + 14z = 28$ and $3x - 3y + 3z = 6$

Sol The augmented matrix;

$C = [A \vdots B]$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 14 & -14 & 14 & 28 \\ 3 & -3 & 3 & 6 \end{array}\right]$$

$R_2 \to R_2 - 14R_1$;

$R_3 \to R_3 - 3R_1$;

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = RREF(C)$$

$\because$ Rank(A) = Rank$(C) = 1\ (=r) < 3\ (=m)$ (no. of variables)
$\therefore$ System is consistent and has infinite solutions.

HOW TO FIND INFINITE SOLUTIONS?

From last matrix; $x - y + z = 2$
Assume $m - r = 3 - 1 = 2$ variables as arbitrary real numbers.
Let $z = K_1$ (say) and $y = K_2$ (say); $K_1 \in \mathbb{R}$ and $K_2 \in \mathbb{R}$
Then $x - K_2 + K_1 = 2$
$\Rightarrow x = 2 - K_1 + K_2$
$\therefore x = 2 - K_1 + K_2$, $y = K_2$ and $z = K_1$; $K_1 \in \mathbb{R}$ and $K_2 \in \mathbb{R}$

TIP! — If $K_1 = 1$, $K_2 = 1$ then $x = 2, y = 1, z = 1$ is solution
If $K_1 = 0$, $K_2 = -1$ then $x = 1, y = -1, z = 0$ is solution
If $K_1 = \sqrt{2}$, $K_2 = \sqrt{3}$ then $x = 2 - \sqrt{2} + \sqrt{3}$, $y = \sqrt{3}$, $z = \sqrt{2}$ is solution, etc.

Ques 4) Test the consistency of the system and find the solution if it is consistent:
$3x - y + z = 4$, $x + 2y - z = 2$, $x + y + z = 6$ and $2x - 2y + z = 2$

Sol) Augmented matrix, $C = [A \vdots B]$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 6 \\ 2 & -2 & 1 & 2 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$;
$R_3 \to 3R_3 - R_1$;
$R_4 \to 3R_4 - 2R_1$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 0 & 7 & -4 & 2 \\ 0 & 4 & 2 & 14 \\ 0 & -4 & 1 & -2 \end{array}\right]$$

$R_3 \to 7R_3 - 4R_2$;
$R_4 \to 7R_3 + 4R_2$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 0 & 7 & -4 & 2 \\ 0 & 0 & 30 & 90 \\ 0 & 0 & -9 & -6 \end{array}\right]$$

$R_4 \to 10R_4 + 3R_3$;

$$= \left[\begin{array}{ccc|c} 3 & -1 & 1 & 4 \\ 0 & 7 & -4 & 2 \\ 0 & 0 & 30 & 90 \\ 0 & 0 & 0 & 210 \end{array}\right] = \text{RREF}(C)$$

$\because$ Rank$(A) = 3 \neq 4$ = Rank(C)
$\therefore$ the system is inconsistent and has NO SOLUTION.

Ques 5) Investigate for what values of λ & μ, the system of equations
$x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$
has (i) a unique solution
(ii) infinitely many solutions
(iii) no solution.

Sol) Augmented matrix $C = [A \vdots B]$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$;
$R_3 \to R_3 - R_1$;

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$R_3 \to R_3 - R_2$;

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] = RREF(C)$$

(i) For unique sol^n, we need Rank(A) = Rank(C) = 3 (= no. of variables)

$\Rightarrow$ Rank(A) = 3 & Rank(C) = 3

$\Rightarrow$ $\lambda - 3 \neq 0$ & 'μ' is any real no.

$\Rightarrow$ $\lambda \neq 3$ & 'μ' is any real no.

(ii) For infinite many sol^n's, we need Rank(A) = Rank(C) < 3 (= no. of variables)

$\Rightarrow$ Rank(A) = Rank(C) = 2 (Why not 1?)

$\Rightarrow$ Rank(A) = 2 & Rank(C) = 2

$\Rightarrow$ $\lambda - 3 = 0$ & $\mu - 10 = 0$

$\Rightarrow$ $\lambda = 3$ & $\mu = 10$

(iii) For no sol^n, we need Rank(A) $\neq$ Rank(C)

$\Rightarrow$ Rank(A) = 2 and Rank(C) = 3 (Why not other choices?)

$\Rightarrow$ $\lambda - 3 = 0$ and $\mu - 10 \neq 0$

$\Rightarrow$ $\lambda = 3$ and $\mu \neq 10$

Ques 6 Investigate for what values of K, the system
$x+y+z=1$, $2x+y+4z=K$ and $4x+y+10z=K^2$
has infinite number of solutions.

Sol Augmented matrix ;

$$C = [A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - 4R_1$;

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & -3 & 6 & K^2-4 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$;

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right] = RREF(C)$$

For infinite no. of sol^n's, we need Rank(A) = Rank(C) < 3 (= n)

$\Rightarrow$ Rank(A) = 2 and Rank(C) = 2 (Why not 1?)

$\Rightarrow$ $K^2 - 3K + 2 = 0$

$\Rightarrow$ $(K-2)(K-1) = 0$

$\Rightarrow$ $K = 2$ or $K = 1$

The required values of K is 1 or 2.

Ques 7 Determine the values of λ for which the system
$3x - y + \lambda z = 0$, $2x + y + z = 2$, $x - 2y - \lambda z = -1$
will fail to have unique solution. For this value of λ, is the system consistent?

Sol Augmented matrix ; $C = [A \mid B]$

$$= \left[\begin{array}{ccc|c} 3 & -1 & \lambda & 0 \\ 2 & 1 & 1 & 2 \\ 1 & -2 & -\lambda & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$;

$$= \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 0 \\ 1 & -2 & -\lambda & -1 \end{array}\right]$$

$R_2 \to 2R_2 - 3R_1$;
$R_3 \to 2R_3 - R_1$;

$$= \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & -5 & 2\lambda-3 & -6 \\ 0 & -5 & -2\lambda-1 & -4 \end{array}\right]$$

$R_3 \to R_3 - R_2$;

$$= \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & -5 & 2\lambda-3 & -6 \\ 0 & 0 & -4\lambda+2 & 2 \end{array}\right] = RREF(C)$$

For non-existence of unique sol^n, either there is no sol^n or infinite sol^n's.
But Rank(C) = 3 always implies Rank(C) < 3 is not possible which further implies that only no sol^n exists.

$\therefore$ Rank(A) $\neq$ Rank(C)(= 3)

$\Rightarrow$ Rank(A) = 2 (Why not 1?)

$\Rightarrow -4\lambda + 2 = 0$

$\Rightarrow \lambda = \frac{1}{2}$

For $\lambda = 1/2$, the RREF(C) =
$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & -5 & -2 & -6 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$\because$ Rank(A) = 2 $\neq$ 3 = Rank(C)

$\therefore$ the system is inconsistent.

SOLUTION OF HOMOGENEOUS SYSTEM AX=0

Consider the homogeneous system for $m = n = 3$.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

Rewriting the above system as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{AX = 0}$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Augmented matrix, $C = [A \mid 0]$

$$= \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right]$$

Convert C into echelon form. Two cases arise:

Case 1: If rank(A) = rank(C) = r(say) = m (no. of variables) then the system is consistent and has a unique solution given $x_1 = x_2 = x_3 = 0$ (Trivial solution)

Case 2: If rank(A) = rank(C) = r(say) < m (no. of variables) then the system is consistent and has infinitely many solutions. (Non-trivial solution)
Exactly $(m - r)$ solutions are linearly independent.

Ques 1 Solve the system : $x + 2y + 3z = 0,\ 2x - y + 3z = 0,\ 3x + y + 5z = 0$

Sol Augmented matrix $C = [A \mid 0]$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - 3R_1$;

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & -5 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$;
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] = RREF(C)$$

$\because$ Rank(A) = Rank(C) = 3 = m (no. of variables)
$\therefore$ the system is consistent and has a unique solution given by $x = y = z = 0$

Ques 2 Solve the system : $x + 2y + 3z = 0$, $2x - y + 3z = 0$, $3x + y + 6z = 0$

Sol Augmented matrix $C = [A \mid 0]$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & 6 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$;
$R_3 \to R_3 - 3R_1$;
$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & -5 & -3 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$;
$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = RREF(C)$$

$\because$ Rank(A) = Rank(C) = 2 $(= r) < 3$ $(= m)$
$\therefore$ The system is consistent and has infinitely many solutions.

From last matrix;
$$x + 2y + 3z = 0 \quad \text{——— (1)}$$
$$-5y - 3z = 0 \quad \text{——— (2)}$$

Assume $m - r = 3 - 2 = 1$ variable as arbitrary real number.
Let $z = K$ (say) ; $K \in \mathbb{R}$

From (2); $y = -\frac{3K}{5}$

From (1); $x - \frac{6K}{5} + 3K = 0$

$\Rightarrow x = -\frac{9K}{5}$

Thus $x = -9K/5$, $y = -3K/5$, $z = K$; (where $K \in \mathbb{R}$ is arbitrary) is solution.

Ques 3 Solve the system $x + 2y + 5z = 0$ and $5x - y - z = 0$.

Sol Here $C = [A \mid 0]$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 5 & -1 & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 - 5R_1$;
$$= \left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -11 & -26 & 0 \end{array}\right] = RREF(C)$$

$\because$ Rank(A) = Rank(C) = 2 $(= r) < 3$ $(= m)$
$\therefore$ System is consistent and has infinitely many solutions.

From last matrix;
$$x + 2y + 5z = 0 \quad \text{——— (1)}$$
$$-11y - 26z = 0 \quad \text{——— (2)}$$

Assume $m - r = 3 - 2 = 1$ variable as arbitrary.
Let $z = K$ (say)

From (2), $y = \frac{-26K}{11}$

From (1), $x - \frac{52K}{11} + 5K = 0$

$\Rightarrow x = -\frac{3K}{11}$

$\therefore x = -3K/11$, $y = -26K/11$ & $z = K$, (where $K \in \mathbb{R}$ is arbitrary) is solution.