Magneto-static

Magneto-statics

Up to this we have limited our study to electro-static field characterized by E and D. Now we will study on static magnetic field characterized by H and B.

There are similarities and dissimilarities between electric and magnetic fields. As E and D are related as $D = \varepsilon E$ for linear isotropic material space, H and B are also related as $B = \mu H$.

A link b/w electric and magnetic fields was established by Orested in 1820, as an electrostatic field is produced by stationary charges. If the charges are moving with constant velocity, a static magnetic field is produced. A magneto-static field is produced by a constant current flow.

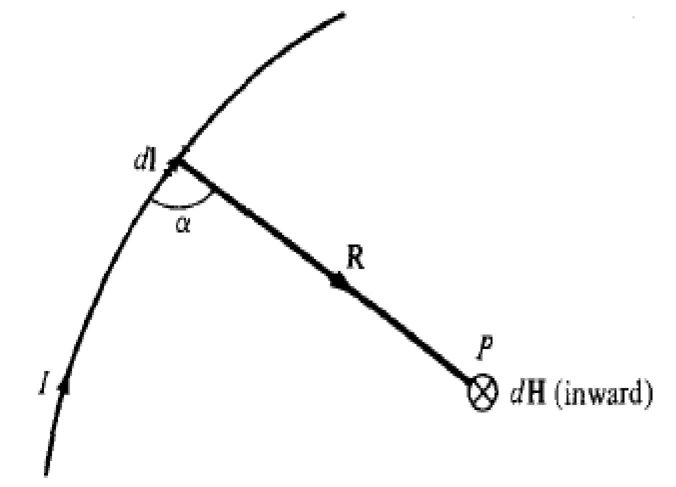
There are two major laws governing magneto-statics; (i) Biot-Savart's law and (ii) Ampere's circuital law.
Like Coulomb's law, Biot-Savart's law is the general law of magneto-statics. As the Gauss's law is the special case of Coulomb's law, similarly Ampere's circuital law is the special case of Biot-Savart's law and is easily applicable in problems involving symmetrical current distributions.

It states that the magnetic field intensity *dH* produce at a point P by the differential current element *Idl* is proportional to the product of current element *Idl* and the sine of angle α between the element and line joining Point P to the current element and is inversely proportional to the square of distance *R* between P and the element.

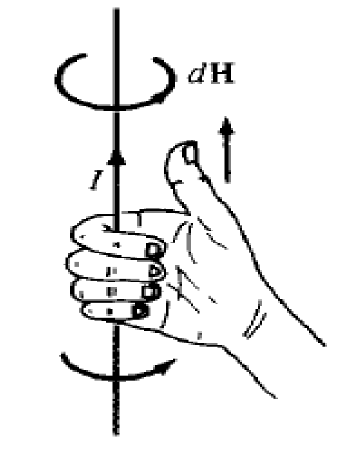
$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$

or

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$



By the definition of cross-product, we have

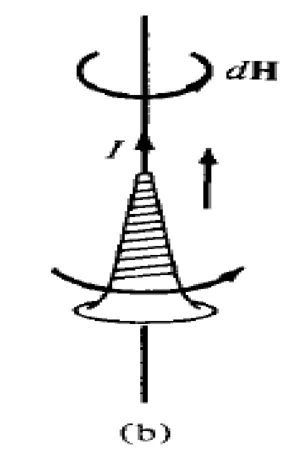


(a)

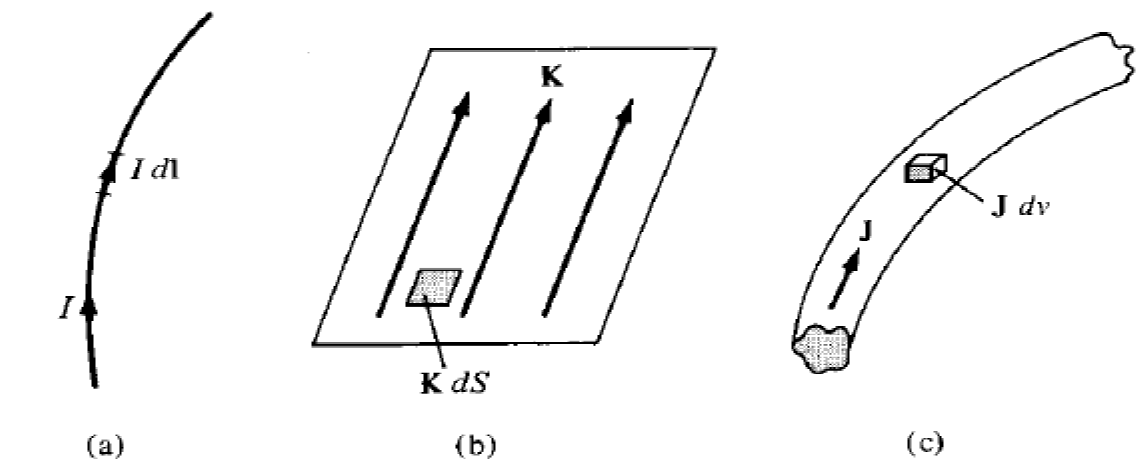
$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

The direction of *dH* can be determined by the right hand thumb rule with the right hand thumb pointing in the direction of the current, the right hand fingers encircling the wire in the direction of *dH.* (fig a)

Alternatively, The right hand screw rule may be used to determine the direction of ***dH,*** with the screw placed along the wire and pointed in the direction of current flow, then the direction of advance of the screw is the direction of ***dH.*** (fig b)



As we have different charge distributions, likely we can have different current distributions as line current, surface current and volume current (fig a, b & c).



Current distributions:
(a) line current, **(b)** surface current, **(c)** volume current.

Biot-Savart's law

If we define *K* as the surface current density in Ampere/meter, *J* as volume current density in Ampere/square meter then source elements are related as;

$$I\, d\mathbf{l} \equiv \mathbf{K}\, dS \equiv \mathbf{J}\, dv$$

In terms of the distributed current sources, the Biot-Savart's law becomes

$$\mathbf{H} = \int_L \frac{I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \int_L \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

(line current)

$$\mathbf{H} = \int_S \frac{\mathbf{K}\, dS \times \mathbf{a}_R}{4\pi R^2} = \int_S \frac{\mathbf{K}\, dS \times \mathbf{R}}{4\pi R^3}$$

(Surface current)

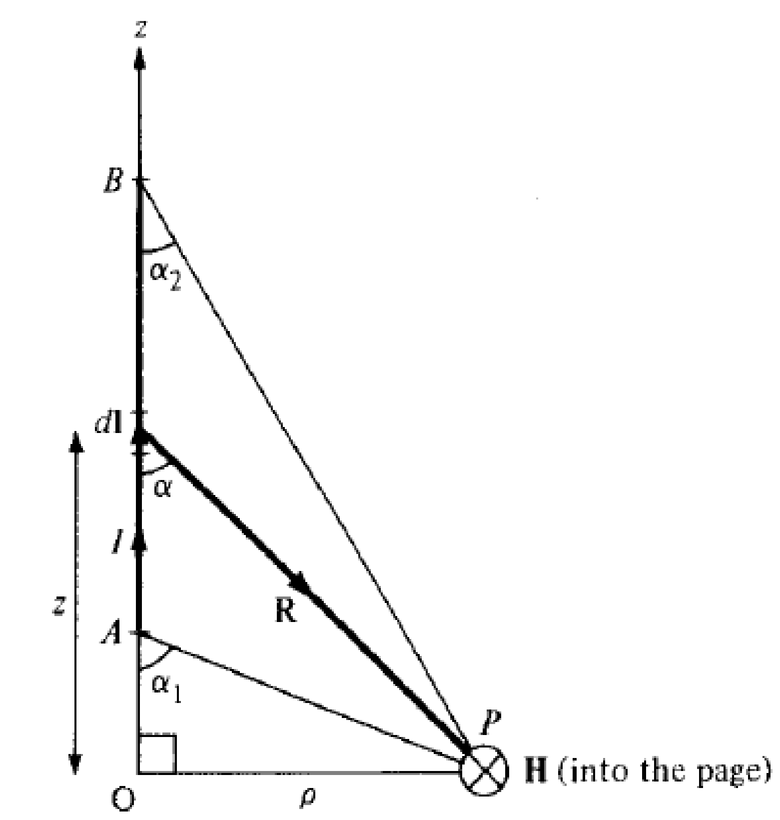
$$\mathbf{H} = \int_v \frac{\mathbf{J}\, dv \times \mathbf{a}_R}{4\pi R^2} = \int_v \frac{\mathbf{J}\, dv \times \mathbf{R}}{4\pi R^3}$$

(Volume current)

Now let us apply Biot-Savart's law to determine the field due to a straight current carrying filament conductor of finite length AB;

The conductor is along z-axis. The lower and upper end of AB make angles α_1 and α_2 with point P at which *H* is to be determine. Current flows from point A, where $\alpha = \alpha_1$ to the point B where $\alpha = \alpha_2$.
So, from B-S law;

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$



If we consider the contribution d*H* at point P due to the line element *dl* at (0, 0, z), then $\mathbf{dl} = \mathbf{dz.a_z}$ and $\boldsymbol{R} = \rho.a_\rho$ - z.$\mathbf{a_z}$ (from fig.)

So, $\mathbf{dl} \times \mathbf{R} = (\mathbf{dz.a_z}) \times (\rho.a_\rho - z.\mathbf{a_z}) = \rho\mathbf{dz}\,(\mathbf{a_z}\times a_\rho) = \rho\mathbf{dz.\ a_\Phi}$

So, putting all above terms, the magnetic field is;

$$\mathbf{H} = \int \frac{I\rho\, dz}{4\pi[\rho^2 + z^2]^{3/2}}\,\mathbf{a}_\phi$$

Letting $z = \rho \cot \alpha$, $dz = -\rho\, \text{cosec}^2\, \alpha\, d\alpha$, and eq. (7.11) becomes

$$\mathbf{H} = -\frac{1}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\, \text{cosec}^2\, \alpha\, \text{d}\alpha}{\rho^3\, \text{cosec}^3\, \alpha}\,\mathbf{a}_\phi$$

$$= -\frac{I}{4\pi\rho}\,\mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha$$

or

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi}$$

Special Case 1: when the conductor is *semiinfinite* (with respect to P) so that point A is now at $O(0, 0, 0)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 90°$, $\alpha_2 = 0°$

$$\mathbf{H} = \frac{I}{4\pi\rho}\,\mathbf{a}_\phi$$

Special Case 2: when the conductor is *infinite* in length. For this case, point A is at $(0, 0, -\infty)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 180°$, $\alpha_2 = 0°$.

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi}$$

Ampere's circuit *Law*

The line integral of the tangential component of **H** around a close path is the same as the net current I_{inc} enclosed by the path.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$$

Using Stoke's law

$$I_{enc} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

But

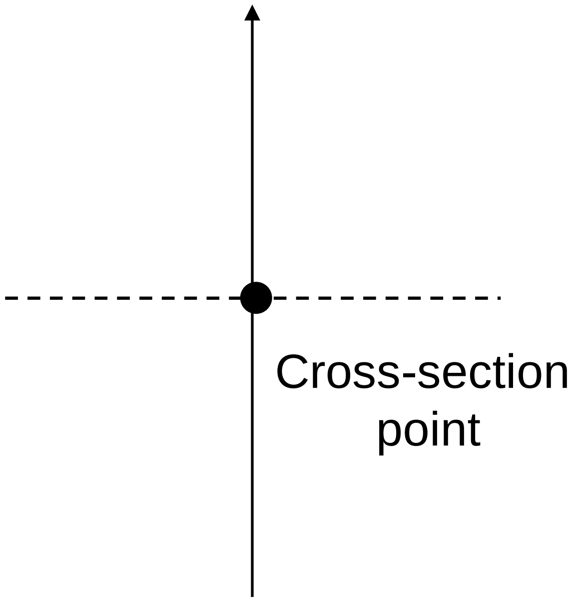
$$I_{enc} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Comparing we get

$$\nabla \times \mathbf{H} = \mathbf{J}$$

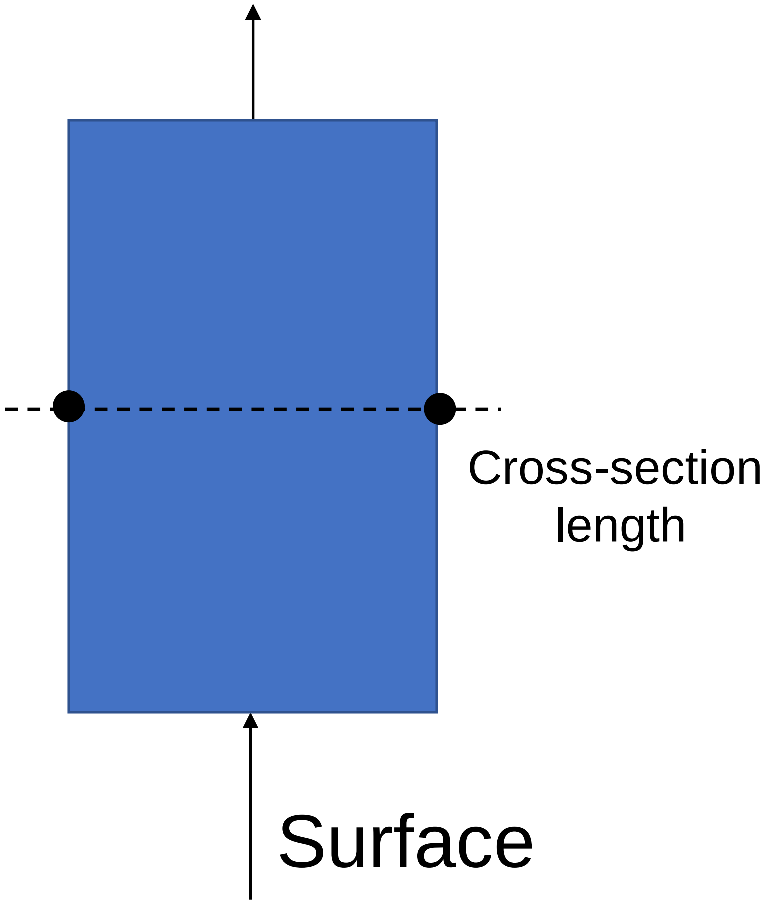
This is third maxwell equation

1-D



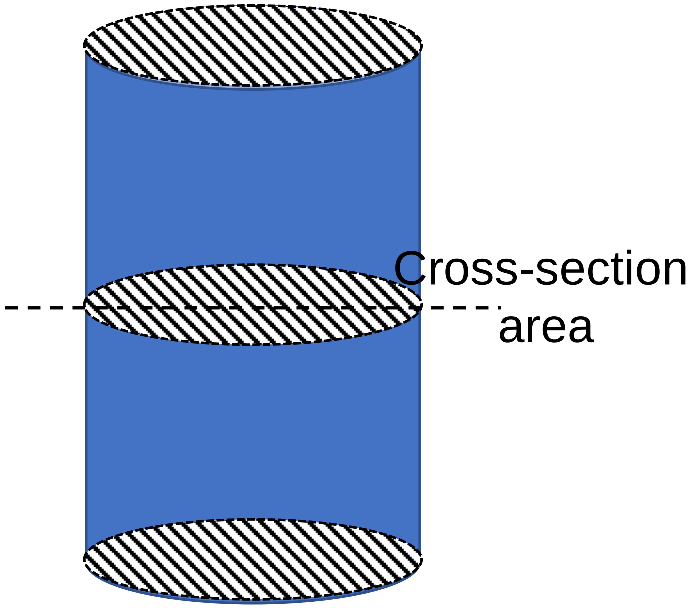
Filamentary current

2-D



Surface Current

3-D



Volume current

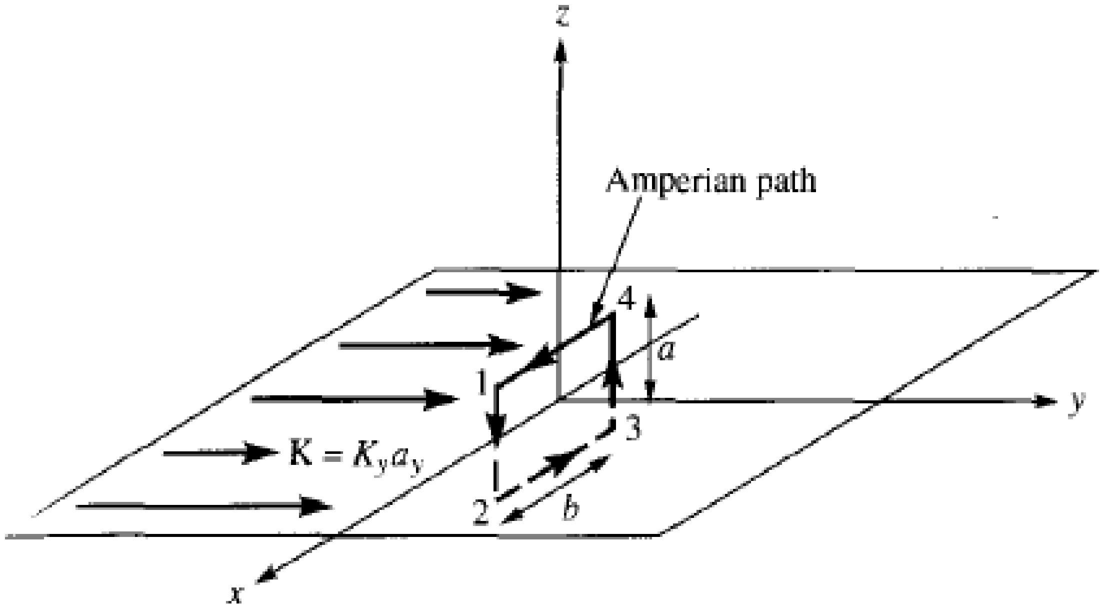
Application of Ampere's law : Infinite Sheet *Current*

Consider an infinite current sheet in z = 0 plane.
If the sheet has a uniform current density then

$$\vec{K} = K_y \hat{a}_y$$

Applying Ampere's Law on closed rectangular path 1-2-3-4-1 (Amperian path) we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b \qquad \text{(i)}$$



To solve integral we need to know how **H** is like

We assume the sheet comprising of filaments d**H** above and below the sheet due to pair of filamentary current.

The resultant $d\mathbf{H}$ has only an x-component.

Also **H** on one side of sheet is the negative of the other.

Due to infinite extent of the sheet, it can be regarded as consisting of such filamentary pairs so that the characteristic of **H** for a pair are the same for the infinite current sheets

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases} \qquad \text{(ii)}$$

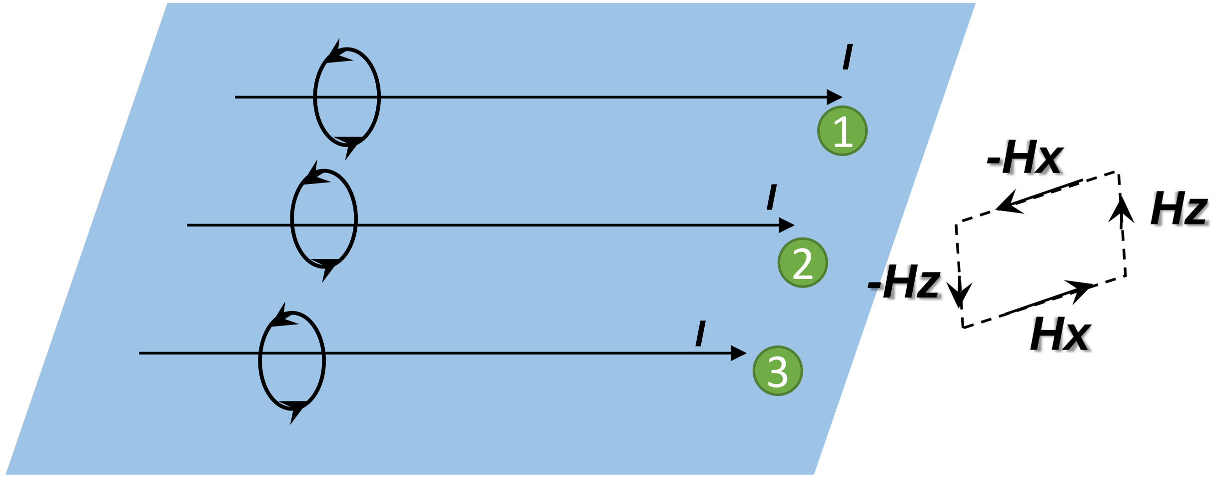
where H_o is to be determined.

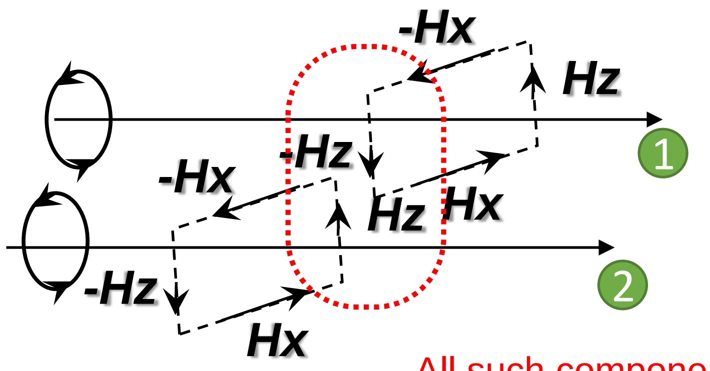
Evaluating the line integral of **H** along the closed path

$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l}$$

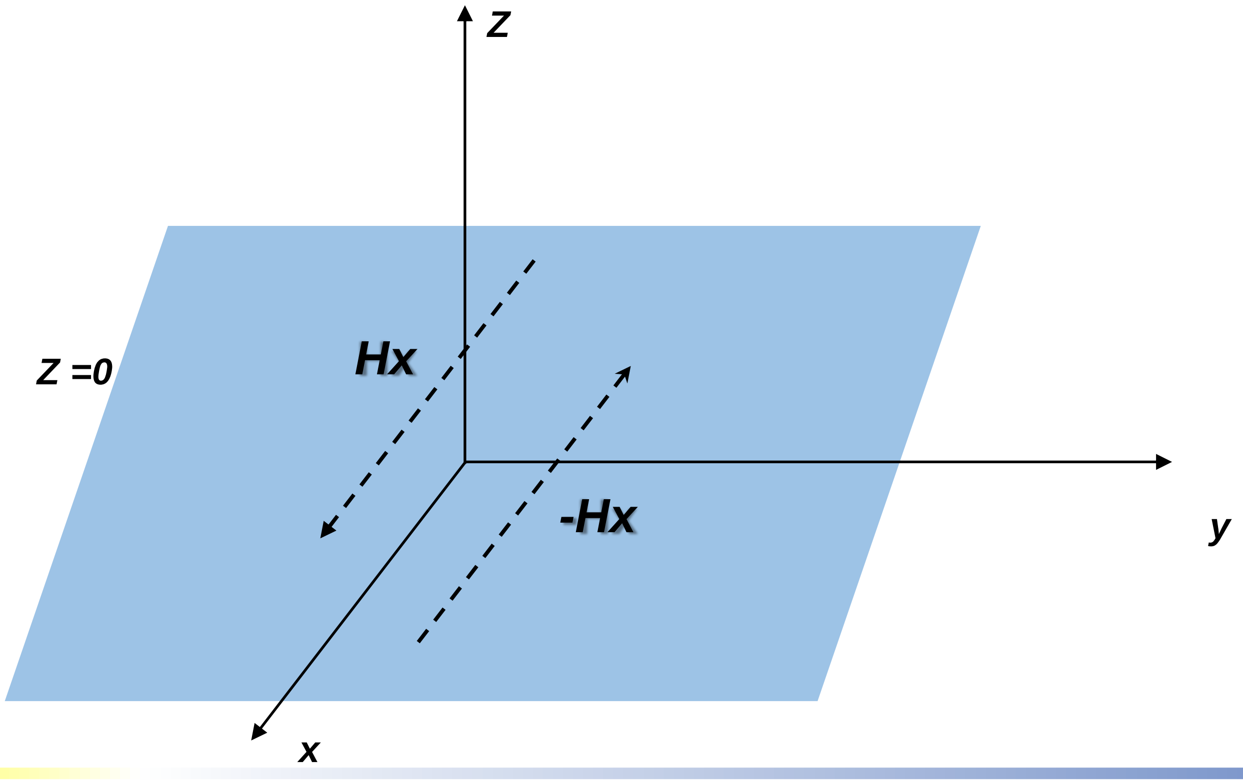
$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b)$$

$$= 2H_o b \qquad \text{(iii)}$$





All such components are cancel with each other



$$\boldsymbol{H}= \begin{cases} Hx\ \boldsymbol{a_x}, & z>0 \\ Hx\ (-\boldsymbol{a_x}), & z<0 \end{cases}$$

Comparing (i) and (iii), we get

$$H_o = \frac{1}{2} K_y \qquad \text{(iv)}$$

Using (iv) in (ii), we get

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

Generally, for an infinite sheet of current density **K**

$$\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n$$

where a_n is a unit normal vector directed from the current sheet to the point of interest.

Magnetic Flux Density

The magnetic flux density **B** is similar to the electric flux density **D**

Therefore, the magnetic flux density **B** is related to the magnetic field intensity **H**

$$\mathbf{B} = \mu_o \mathbf{H}$$

where μ_o is a constant and is known as the permeability of free space.

Its unit is Henry/meter (H/m) and has the value

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

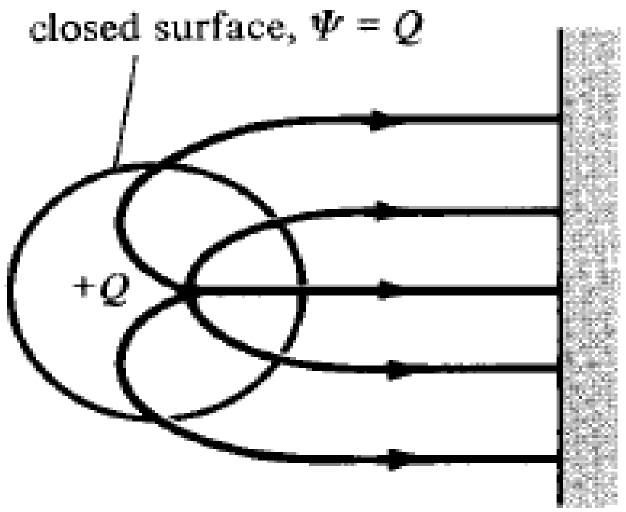
The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

where the magnetic flux ψ is in webers (Wb) and the magnetic flux density is in weber/ square meter or Teslas.

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed.

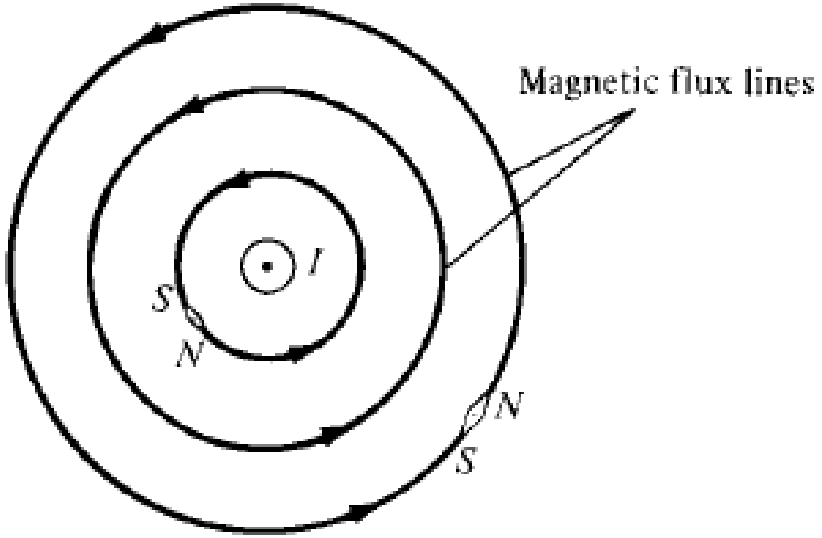
$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$$



Thus it is possible to have an isolated electric charge.

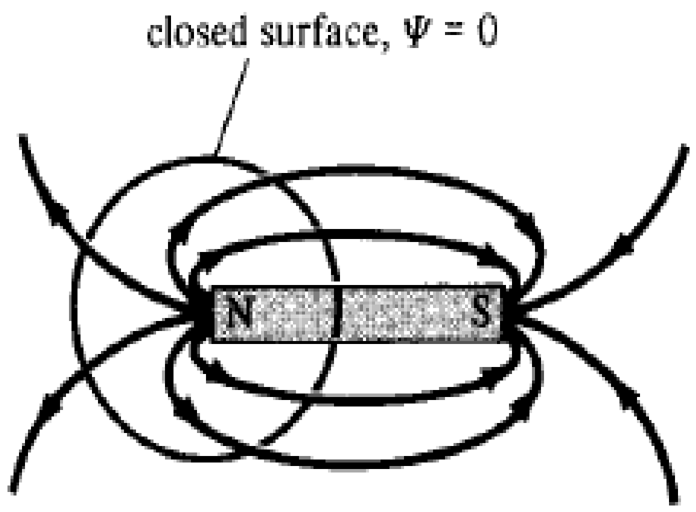
Also the electric flux lines are not necessarily closed.

Magnetic flux lines due to a straight wire with current coming out of the page.



Each magnetic flux line is closed with no beginning and no end and are also not crossing each other.

Magnetic flux lines are always close upon themselves.



So it is not possible to have an isolated magnetic pole (or magnetic charges)

An isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero.

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

This equation is known as the law of conservation of magnetic flux or Gauss's Law for Magneto-static fields.

Magneto-static field is not conservative but magnetic flux is conserved.

Applying Divergence theorem, we get

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \, dv = 0$$

or

$$\nabla \cdot \mathbf{B} = 0$$

This is Maxwell's fourth equation.

This equation suggests that magnetostatic fields have no source or sinks.
Also magnetic flux lines are always continuous.

Faraday's law

According to Faraday a time varying magnetic field produces an induced voltage (called electromotive force or emf) in a closed circuit, which causes a flow of current.

The induced emf (V_{emf}) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is Faraday's Law and can be expressed as

$$V_{\mathrm{emf}} = -N\frac{d\Psi}{dt}$$

where N is the number of turns in the circuit and ψ is the flux through each turn.

The negative sign shows that the induced voltage acts in such a way to oppose the flux producing in it. This is known as Lenz's Law.

Transformer and Motional EMF

For a circuit with a single turn (N = 1)

$$V_{emf} = -\frac{d\Psi}{dt}$$

In terms of **E** and **B** this can be written as

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(i)}$$

where ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by a closed path L.

The equation says that in time-varying situation, both electric and magnetic fields are present and are interrelated.

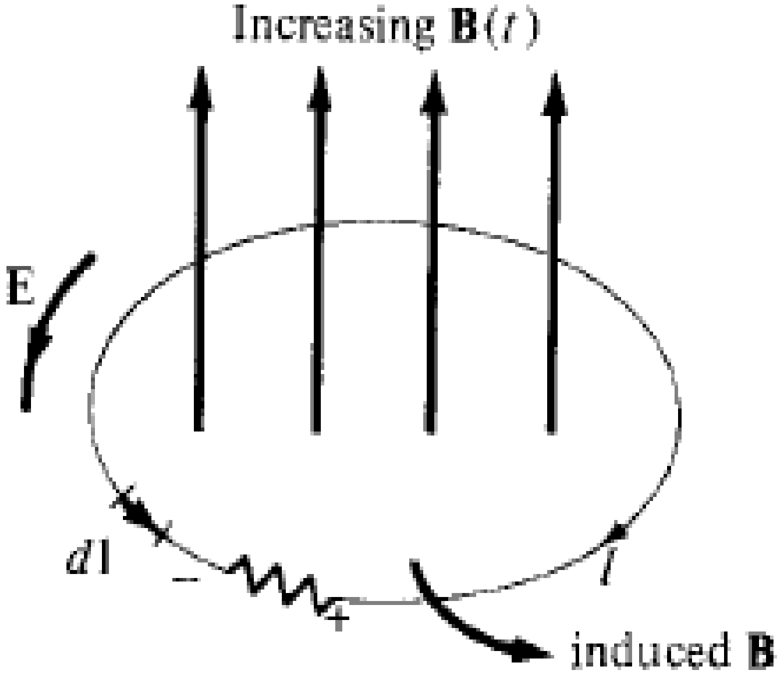
The variation of flux with time may be caused in three ways.

1. By having a stationary loop in a time-varying **B** field.
2. By having a time-varying loop area in a static **B** field.
3. By having a time-varying loop area in a time-varying **B** field.

Stationary loop in a time-varying B field (Transformer emf)

Consider a stationary conducting loop in a time-varying magnetic **B** field. The equation (i) becomes

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$



This emf induced by the time-varying current in a stationary loop is often referred to as transformer emf in power analysis since it is due to the transformer action.

By applying Stokes's theorem to the middle term, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Thus $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is one of the Maxwell's equations for time-varying fields.

It shows that the time-varying field is not conservative.

$$\nabla \times \mathbf{E} \neq 0$$

2. Moving loop in static B field (Motional emf)

When a conducting loop is moving in a static **B** field, an emf is introduced in the loop.

The force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

The motional electric field $\mathbf{E}_m$ is defined as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

Consider a conducting loop moving with uniform velocity **u,** the emf induced in the loop is

$$V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(i)}$$

This kind of emf is called the motional emf or flux-cutting emf. Because it is due to the motional action. eg,. Motors, generators

By applying Stokes's theorem to equation (i), we get

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})$$

3. Moving loop in time-varying field

Consider a moving conducting loop in a time-varying magnetic field. Then both transformer emf and motional emf are present.

Thus the total emf will be the sum of transformer emf and motional emf

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

also $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

Displacement Current

For static EM fields $\nabla \times \mathbf{H} = \mathbf{J}$ (i)

But the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \qquad \text{(ii)}$$

But the continuity of current requires

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad \text{(iii)}$$

Equation (ii) and (iii) are incompatible for time-varying conditions

So we need to modify equation (i) to agree with (iii)

Add a term to equation (i) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \qquad \text{(iv)}$$

where J_d is to defined and determined.

Again the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \qquad \text{(v)}$$

In order for equation (v) to agree with (iii)

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or $$\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt} \qquad \text{(vi)}$$

Putting (vi) in (iv), we get

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

This is Maxwell's equation (based on Ampere Circuital Law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as displacement current density and **J** is the conduction current density.

$$\mathbf{J} = \sigma \mathbf{E}$$

Maxwell's Equations in Final Form

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

Forces due to Magnetic Fields

There are at least three ways in which force due to magnetic field s can be experienced.

a) Due to a moving charge particle in a magnetic field
b) On a current element in an external magnetic field
c) Between two current elements

Force on a Charged particle

The force on a stationary or moving electric charge Q in an electric field is

$$\mathbf{F}_e = Q\mathbf{E}$$

The force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

Force on a Charged particle

The force on a moving charge Q in the presence of both electric field and magnetic field is given by

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

This is known as Lorentz force equation.

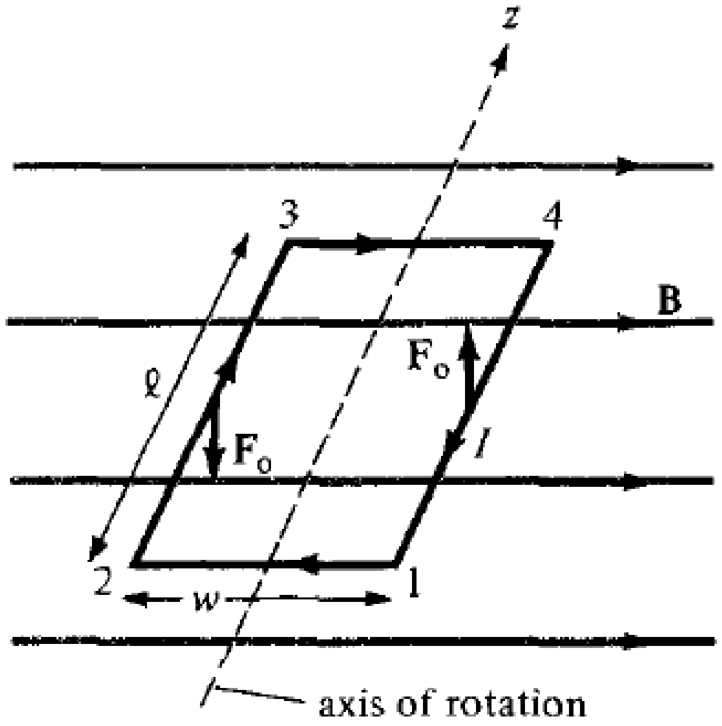
If the mass of the charged particle moving in E and B fields is m, by Newton's law of motion

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetic Moment and Torque

Consider a rectangular loop of length l and width w placed in a uniform magnetic field **B**.

Here d**l** is parallel to **B** along sides 12 and 34 of the loop and hence no force is exerted on these sides



$$\mathbf{F} = I\int_2^3 d\mathbf{l} \times \mathbf{B} + I\int_4^1 d\mathbf{l} \times \mathbf{B}$$

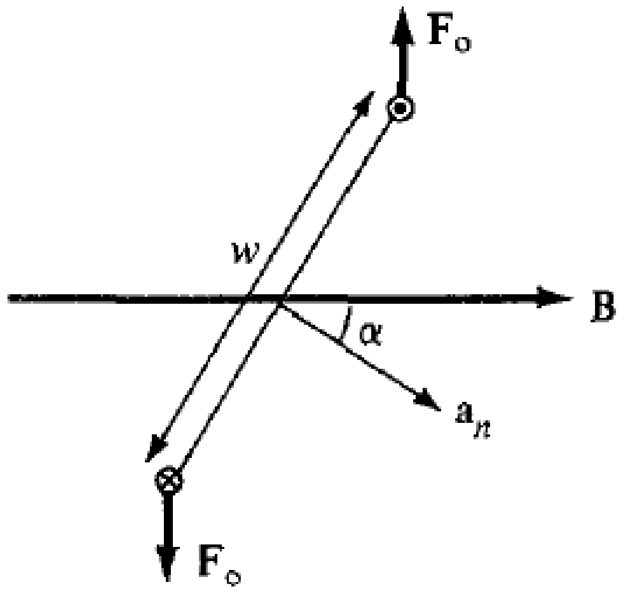
$$= I\int_0^\ell dz\,\mathbf{a}_z \times \mathbf{B} + I\int_\ell^0 dz\,\mathbf{a}_z \times \mathbf{B}$$

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0$$

Magnetic Moment and Torque

Where $|\mathbf{F}_o| = IB\ell$ because B is uniform.

Thus no force is exerted on the loop as a whole. However $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points, creating a couple.



If normal to the plane of the loop makes and angle α with **B**, the torque on the loop is

$$|\mathbf{T}| = |\mathbf{F}_o|\, w \sin \alpha$$

$$T = BI\ell w \sin \alpha$$

But lw = S, area of the loop

$$T = BIS \sin \alpha$$

Magnetic Moment and Torque

Now we define

$$\mathbf{m} = IS\mathbf{a}_n$$

as the dipole moment (in A.m^2) of the loop.

The magnetic dipole moment is the product of current and area of the loop and its direction is normal to the loop.

$$\mathbf{T} = \mathbf{m} \times \mathbf{B}$$

We can apply this equation for loop of any arbitrary shape. The only limitation is that the magnetic field should be uniform.