$^4C_2 = \frac{4\times3}{2}$ $\quad ^4C_3 = \frac{4\times3\times2}{3\times2}$

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Unit 4 - Probability

Random Variable

Let X is a random function where or whose domain is a sample space (S) & range is set of real numbers (R)

$$X : S \rightarrow R$$

Ques 1 Let X be random variable denoting total no. of head in 4 tosses of balanced coin. Find probability distribution.

Ans Sample space of coin -

HHHH, HHHT, HHTH, HTHH, THHH, HHTT, TTHH, TTTT, TTTH, TTHT, THTT, HTTT, HTTH, THHT, THTH, HTHT

Clearly X may take any value b/w 0 to 4

$$P(X=0) = \frac{1}{16}$$

$$P(X=1) = \frac{4}{16}$$

$$P(X=2) = \frac{6}{16}$$

$$P(X=3) = \frac{4}{16}$$

$$P(X=4) = \frac{1}{16}$$

Generalizing

Generally $P(X=x) = \frac{^4C_x}{16}$; $x = 0,1,2,3,4$

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Qu. Check weather func. given by
$$f(x) = \frac{x+2}{25} \quad \text{for} \quad x = 1,2,3,4,5$$
can serve as probability distribution of a discrete random variable

Ans $f(1) = \frac{3}{25}, \; f(2) = \frac{4}{25}, \; f(3) = \frac{5}{25}, \; f(4) = \frac{6}{25}, \; f(5) = \frac{7}{25}$

$f(x) \geq 0 \;\; \forall x$

Also $\sum_{x=1}^{5} f(x) = f(1) + f(2) + f(3) + f(4) + f(5)$

$= \frac{3+4+5+6+7}{25} = \frac{25}{25} = 1$

$\therefore$ Yes

Qu 3 Determine c so that function can serve as probability distribution of a random variable with given range.

(a) $f(x) = c\binom{5}{x}$ for $x = 0,1,2,3,4,5$

Ans $\because \sum_{x=0}^{5} f(x) = 1$

$\Rightarrow f(0) + f(1) + f(2) + f(3) + f(4) + f(5) = 1$

$\Rightarrow c\left[{}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5\right] = 1$

$= c\left[1 + 5 + 10 + 10 + 5 + 1\right] = 1$

$\Rightarrow 32c = 1 \Rightarrow \boxed{c = \frac{1}{32}}$

NOTE

(b)

NOTE

(c)

,4,5

a discrete

$(4) = \frac{6}{25}$, $f(5) = \frac{7}{25}$

$f(4) + f(5)$

as

variable

$5) = 1$

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NOTE $\boxed{{}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n}$

here $n = 5 \Rightarrow 2^5 = 32$

(b) $f(x) = cx$ for $x = 1, 2, 3, 4, 5$

$$\therefore \sum_{x=1}^{5} f(x) = 1$$

$$\Rightarrow c[1+2+3+4+5] = 1$$

$$\boxed{c = \frac{1}{15}}$$

NOTE $\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$

(c) $f(x) = cx^2$ for $x = 1, 2, 3, \cdots k$

$$\therefore \sum_{x=1}^{k} f(x) = 1$$

$$c[1^2 + 2^2 + \cdots + k^2] = 1$$

$$c \frac{k(k+1)(2k+1)}{6} = 1$$

$$\Rightarrow c = \frac{6}{k(k+1)(2k+1)}$$

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NOTE $\boxed{1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}}$

Probability Density Function

Qu If X has probability density function

$$f(x) = \begin{cases} Ke^{-3x} & \text{for } x>0 \\ 0 & \text{elsewhere} \end{cases}$$

Find K using the value of K find

(a) $P(0.5 \leq X \leq 1)$

(b) $P(0.5 < X \leq 1)$

(c) $P(X = 1)$

Ans $\because \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{\infty} Ke^{-3x}\,dx = 1$$

$$\Rightarrow K\int_{0}^{\infty} e^{-3x}\,dx = 1$$

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$\Rightarrow -\frac{K}{3}\left[e^{-3x}\right]_0^\infty = 1$

$\Rightarrow -\frac{K}{3}\left[\lim_{x\to\infty} e^{-3x} - e^0\right] = 1$

$\Rightarrow \frac{K}{3} = 1 \Rightarrow \boxed{K = 3}$

(a) $P(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)\,dx = \int_{0.5}^{1} 3e^{-3x}\,dx$

$3\int_{0.5}^{1} e^{-3x}\,dx = \frac{3}{-3}\left[e^{-3x}\right]_{0.5}^{1} =$

$-\left(e^{-3} - e^{-3/2}\right) = e^{-3/2} - e^{-3}$

(b) $P(0.5 < X \le 1) = e^{-3/2} - e^{-3}$

(c) $P(X = 1) = 0$

Qu: The p.d.f of random variable X is given by

$f(x) = \begin{cases} c/\sqrt{x} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$

Find (a) value of c

(b) $P(X < 1/2)$

(c) $P(X > 2)$

$$\frac{x^{-1/2+1}}{-\frac{1}{2}+1} = \frac{x^{1/2}}{\frac{1}{2}} = 2\sqrt{x}$$

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(a) $\because \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx = 1$$

(c) P

$$0 + \int_{0}^{4} \frac{c}{\sqrt{x}}\,dx + 0 = 1$$

$$c\int_{0}^{4} \frac{1}{\sqrt{x}} = 1 \Rightarrow 2c\left[\sqrt{x}\right]_{0}^{4} = 1$$

$$2c \times 2 = 1$$

$$c = \frac{1}{4}$$

(b) $P(X < 1/2) = P(-\infty < X < 1/2)$

$$= \cancel{\int_{-\infty}^{1/2} \frac{1}{4\sqrt{x}}\,dx = \frac{1}{4}\int_{-\infty}^{1/2} \frac{1}{\sqrt{x}}\,dx}$$

Ans

$$\cancel{= \frac{1}{4}\, 2\left[\sqrt{x}\right]_{-\infty}^{1/2}}$$

(i)

$$= \int_{-\infty}^{1/2} f(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1/2} f(x)\,dx$$

$$= 0 + \int_{0}^{1/2} \frac{1}{4\sqrt{x}}\,dx = \frac{1}{4_2}\, 2\left[\sqrt{x}\right]_{0}^{1/2}$$

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$= \frac{1}{2} \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$

(c) $P(X>2) = P(2<X<\infty)$

$= \int_2^\infty f(x) = \int_0^2 f(x)\,dx + \int_2^\infty f(x)\,dx$

$= \int_2^4 f(x)\,dx + \int_4^\infty f(x)\,dx = 1$

$= \frac{1}{\cancel{4}_2} \times \cancel{2} \left[\sqrt{x}\right]_2^4 + 0$

$= \frac{1}{2}(2-\sqrt{2}) = 1 - \frac{1}{\sqrt{2}}$

Que $f(x) = \begin{cases} \frac{1}{288}(36-x^2) & \text{for } -6<x<6 \\ 0 & \text{elsewhere} \end{cases}$

(i) atleast 2 minutes early.

$\Rightarrow P(X \leq -2) = P(-\infty < X < -2)$

$= \int_{-\infty}^{-2} f(x)\,dx = \int_{-\infty}^{-6} f(x)\,dx + \int_{-6}^{-2} f(x)\,dx$

$\int_1^\infty$ ~~$-1 < X < 3$~~

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$$= 0 + \int_{-6}^{-2} \frac{1}{288}(36 - x^2)\,dx$$

$$= \frac{1}{288}\left[36x - \frac{x^3}{3}\right]_{-6}^{-2}$$

$$= \frac{1}{288}\left[-72 + \frac{8}{3}\right] = \frac{97}{432}$$

(b) at least 1 minute late

$$P(X \geq 1) = P(1 \leq X \leq \infty)$$

$$= \int_1^6 f(x)\,dx + \int_6^\infty f(x)\,dx$$

$$= \frac{1}{288}\int_1^6 (36 - x^2)\,dx + 0$$

$$= \frac{325}{864}$$

(c) 1 to 3 min early

$$P(-3 \leq X \leq -1)$$

$$= \int_{-3}^{-1} f(x)\,dx = \cancel{\int_{-3}^{6} f(x)\,dx + \int f(x)\,dx}$$

$$= \frac{1}{288}\left[36x - \frac{x^3}{3}\right]_{-3}^{-1} = \frac{95}{432}$$

(d) ex

Step 2 at

K_2 =

K_3

K

K

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(d) exactly 5 minutes late

$$P(X = 5) = 0$$

Step 2 at $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$k_1 = hf(x_1, y_1) = 0.5(0.5 + 1.796875)$

$= \cancel{0.5} = 1.1484375$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.5\left(0.5 + \frac{0.5}{2}, 1.796875 + \frac{1.1484375}{2}\right)$

$= 1.156054688$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 1.663574219$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.5(0.5 + 0.5 + 1.796875 + 1.663574219)$

$= 2.230224609$

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.637824675$

$y_2 = y(x_2) = y(1) = y_1 + k$

$= 3.434699675$

Binomial Distribution

$P(X = r) = {}^nC_r\, p^r q^{n-r}$, n = num. of trials

r = no. of success

p = probability of success in each trial

$q = 1 - p$

Remarks → (i) mean, $\mu = np$

(ii) Standard deviation, $S.D = \sigma = \sqrt{npq}$

(iii) variance $= \sigma^2 = npq$

Que 1 If mean & S.D of binomial distribution are 12 & 3 respectively. Find n & p

Ans $\mu = np \Rightarrow 12 = np$ — (i)

$9 = npq$ — (ii)

(ii) ÷ (i)

$$\frac{9}{12} = q \Rightarrow q = \frac{\cancel{3}}{\cancel{4}} \Rightarrow \boxed{q = \frac{3}{4}}$$

$$p = 1 - q = \frac{1}{4} \Rightarrow \boxed{p = \frac{1}{4}}$$

$$\overset{3}{\cancel{9}} = n \times \frac{1}{4} \times \frac{\cancel{3}}{4} \Rightarrow \boxed{n = 48}$$

Que 2 ~~The~~ 10 percent of screw produced in certain factory are defective. If 5 screws are drawn at random, find P(A) that

(a) exactly 2 are defective

(b) atleast 2 are defective

(c) at ~~least~~ most 2 are defective

(d) less than 2 are defective

Ans $p = 10\% = \frac{10}{100} = \frac{1}{10}$

$$q = 1 - \frac{1}{10} = \frac{9}{10} \text{ and } n = 5$$

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(a) $P(X=2) = {}^5C_2\, p^2 q^3 = 10 \times \left(\frac{1}{10}\right)^2 \times \left(\frac{9}{10}\right)^3$

$= 0.0729$

(b) $P(X \geq 2) = P(2) + P(3) + P(4) + P(5)$

$= {}^5C_2\, p^2 q^3 + {}^5C_3\, p^3 q^2 + {}^5C_4\, p^4 q + {}^5C_5\, p^5 q^0$

$= 1 - [P(0) + P(1)]$

$= 1 - \left[{}^5C_0\, p^0 q^5 + {}^5C_1\, p^1 q^4\right]$

$= 0.08145$

(c) $P(X < 2) = P(0) + P(1)$

$= {}^5C_0\, p^0 q^5 + {}^5C_1\, p^1 q^4$

$= \left(\frac{9}{10}\right)^5 + 5 \times \frac{1}{10} \left(\frac{9}{10}\right)^4$

$= \left(\frac{9}{10}\right)^4 \left[\frac{9}{10} + \frac{5}{10}\right] = \left(\frac{9}{10}\right)^4 \left(\frac{14}{10}\right)$

$= \frac{91854}{10^5} = 0.91854$

(c) $P(X \leq 2) = P(0) + P(1) + P(2)$

$= {}^5C_0\, p^0 q^5 + {}^5C_1\, p^1 q^4 + {}^5C_2\, p^2 q^3$

$= 0.99144$

$\frac{\overset{15}{\cancel{9}}\times 8}{120}$, $\frac{10\times 9}{2}$ $P(X\le 3)$ $\overset{5}{}\ \frac{10\times 9^{3}\times 8}{3\times 2}$

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Que 3 $p = 10\% = \frac{1}{10}$, $q = \frac{9}{10}$, $n = 10$

(a) $P(X\ge 1) = P(1) + P(2) + \cdots + P(10)$

$= 1 - P(0) = 1 - {}^{10}C_0 p^0 q^{10}$

$= 1 - \left(\frac{9}{10}\right)^{10} = 0.6514$

(b) $n = 10$

$P(\text{missing}) = \frac{9}{10}$, $q = 1 - \frac{9}{10} = \frac{1}{10}$

$P(x\ge 1) = P(1) + P(2) + \cdots + P(10) = 1 - P(0)$

$= 1 - {}^{10}C_0 p^0 q^{10} = 1 - \left(\frac{1}{10}\right)^{10}$

$= 0.9999999999$

Que $N = 100$, $n = 10$, $P = 20\% = \frac{20}{100} = \frac{1}{5}$, $q = \frac{4}{5}$

$P(X\le 3) = P(0) + P(1) + P(2) + P(3)$

$= {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9 + {}^{10}C_2 p^2 q^8 + {}^{10}C_3 p^3 q^7$

$= \left(\frac{4}{5}\right)^{10} + 10\ \frac{1}{5}\left(\frac{4}{5}\right)^9 + 45\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8 + 120\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7$

$= 0.8791$

no. of investigate $= N \times P(X\le 3)$

$= 100 \times 0.8791$

$= 87.91 \approx 88$

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Qu $\mu = 5$, variance $= \frac{10}{3}$, $n = ?$

$\mu = np$

$\sigma^2 = npq$

$\sigma^2 = 5q \Rightarrow \frac{\cancel{10}\,2}{3} = \cancel{5}q \Rightarrow \boxed{q = \frac{2}{3}}$

$p = 1 - \frac{2}{3} = \frac{1}{3}$

$5 = n \times \frac{1}{3} \Rightarrow n = 15$

$\therefore P(X = r) = {}^{n}C_r\, p^r q^{n-r} = {}^{15}C_r\, p^r q^{15-r}$

$$\boxed{= {}^{15}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{15-r};\ r = 0, 1, 2, \cdots 15.}$$

Qu $p = 0.5\% = \frac{\cancel{5}^{\,1}}{\cancel{100}_{\,20}} = 0.005$, $q = 0.995$

$n = 100$

$P(X \leq 3) = P(0) + P(1) + P(2) + P(3)$

${}^{100}C_0\, p^0 q^{100} + {}^{100}C_1\, p q^{99} + {}^{100}C_2\, p^2 q^{98} + {}^{100}C_3\, p^3 q^{97}$

$=$

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Root Group Finding Methods

A real no $x = \alpha$ is called a root of eq^n $f(x) = 0$, if $x = \alpha$ satisfies the equation i.e $f(\alpha) = 0$

For example $x = 1$ is a root of eq^n

$$x^3 - x^2 + x - 1 = 0$$

Apart from usual direct methods, there are some indirect method { approximation method } which are used to find the root of a given eq^n $f(x) = 0$.

The fundamental principle used in root finding method is as follows as –

" For a given eq^n $f(x) = 0$ if $f(a) \cdot f(b) < 0$ then a root of eq^n must lie the interval (a, b) "

Bisection Method

Consider the eq^n $f(x) = 0$ whose root to be determined. Suppose $f(a) < 0$ & $f(b) > 0$

Then root lie in the interval (a, b).

Bisection Method suggests that root is at the midpoint at (a, b).

$$x_1 = \frac{a+b}{2} \quad \{1^{st} \text{ approximation}\}$$

suppose $f(x_1) > 0$

Therefore root lies in the interval (a, x_1)

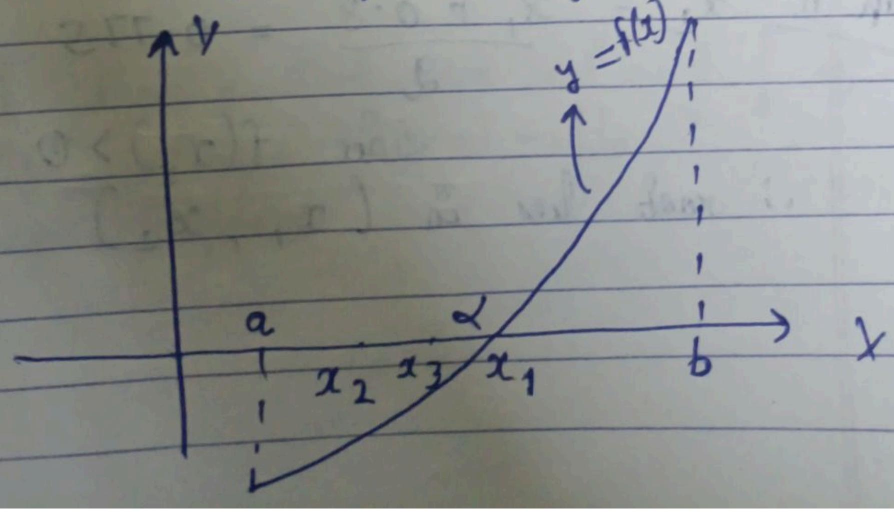
$$x_2 = \frac{a+x_1}{2} \quad \{\text{II approx}\}$$

suppose $f(x_2) < 0$

$\therefore$ root lie in the interval (x_2, x_1)

Then $x_3 = \dfrac{x_2 + x_1}{2}$ $\{\text{III approx}\}$

Continue this process untill the end pts of braketing interval are same upto desired degree of accuracy.



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$1+1-1$

$-1+1-1$

Ques Find a +ve root of a equation $x^3+x^2-1=0$ correct upto 2 decimal places

Ans Here $f(x)=x^3+x^2-1$

clearly $f(0)<0$

$f(1)=1>0$

$\therefore$ root lies in $(0,1)$

$$\left\{\begin{array}{ccccc} (-) & (-) & (-) & (+) & (+) \\ 0 & 0.5 & 0.7 & 0.8 & 1 \end{array}\right\} \text{secret calculation}$$

since $f(0.7)<0$ & $f(0.8)>0$

$\therefore$ root lie in interval $(0.7, 0.8)$

Step I $x_1 = \frac{0.7+0.8}{2} = 0.75$

since $f(x_1)<0$

$\therefore$ root lies in $(x_1, 0.8)$

Step II $x_2 = \frac{x_1+0.8}{2} = 0.775$

since $f(x_2)>0$

$\therefore$ root lies in (x_1, x_2)

Step III

Step IV

Ques

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Step III $x_3 = \frac{x_1 + x_2}{2} = \frac{0.75 + 0.775}{2}$
$= 0.7625$

since $f(x_3) > 0$

$\therefore$ root lies in (x_1, x_3)

Step IV $x_4 = \frac{x_1 + x_3}{2} = 0.75625$

since $f(x_4) > 0$

$\therefore$ root lies in (x_1, x_4)

since the end pt for final bracketing interval (x_1, x_4) are same upto 2 decimal place so their midpoint will definately contain 0.75 as its first 2 digits.

Hence the root is 0.75 { correct a 2 decimal places }

Ques Find the smallest +ve real root of eq^n $xe^x - 1 = 0$ correct upto 2 decimal place

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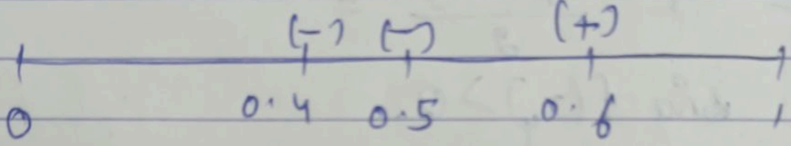
Here $f(x) = xe^{x} - 1$

clearly $f(0) = -1 < 0$

and

$f(1) = e - 1 = +ve > 0$

So root lies in interval $(0, 1)$



since $f(0.5) < 0$ & $f(0.6) > 0$

$\therefore$ root lies in between $(0.5, 0.6)$

Step I $\quad x_1 = \dfrac{0.5 + 0.6}{2} = 0.55$

$f(x_1) < 0$

Step II $\quad x_2 = \dfrac{0.55 + 0.6}{2} = 0.575$

$f(x_2) > 0$

Step III $\quad x_3 = \dfrac{0.55 + 0.575}{2} = 0.5625$

$f(x_3) < 0$

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Step IV $x_4 = \frac{0.575 + 0.5625}{2} = 0.56875$

$f(x_4) > 0$

Ans. $\boxed{0.56}$ It will lie approx 0.52 at 2 decimal.

$$\frac{0.4121803177}{2.4730819061}$$

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Newton Raphson Method

Consider $f(x)=0$ whose root is to be determined, suppose $f(a)f(b)<0$
Then $x_0 \in (a,b)$ is chosen as initial approximation.
The next approximation are computed using formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \quad n = 0,1,2,\cdots$$

Continue using above formula until you get $x_i = x_{i+1}$ upto desired degree of accuracy.

Remarks: (i) If $f'(x_n)=0$ for any $n \in N$ then NR-method fail.
(ii) If $f'(x_n)$ is large than method converges quickly.

Qy 1 Find the smallest posn root of $xe^x - 1 = 0$ correct to 4 decimal places.

Ans In this $f(x) = xe^x - 1$
Clearly $f(0) = -1 < 0$ & $f(1) = e-1 \ (>0)$
Let $x_0 = 0.5 \in (0,1)$

$\cancel{0.75}$
$-(++$

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$$f'(x) = e^x + xe^x$$

Using NR formula

$$x_{n+1} = x_n - \frac{(x_n e^{x_n} - 1)}{e^{x_n} + x_n e^{x_n}}$$

$$\Rightarrow x_{n+1} = \frac{e^{x_n}\cancel{x_n} + x_n^2 e^{x_n} - \cancel{x_n}e^{x_n} + 1}{e^{x_n} + x_n e^{x_n}}$$

$$\Rightarrow x_{n+1} = \frac{x_n^2 e^{x_n} + 1}{e^{x_n} + x_n e^{x_n}} = \frac{x_n^2 e^{x_n} + 1}{e^{x_n}(1+x_n)}$$

Put $n=0$; $x_1 = \frac{x_0^2 e^{x_0} + 1}{e^{x_0}(1+x_0)} = \frac{0.25 \times e^{0.5} + 1}{e^{0.5}(1+0.5)}$

$= 0.57102$

Put $n=1$

$$x_2 = \frac{x_1^2 e^{x_1} + 1}{e^{x_1}(1+x_1)} = \frac{(0.57102)^2 e^{0.57102} + 1}{e^{0.57102}(1+0.57102)}$$

$= 0.56715$

Put $n=2$, $x_3 = \frac{x_2^2 e^{x_2} + 1}{e^{x_2}(1+x_2)} = 0.56714$

$\because x_2 = x_3$ (upto 4 d.p)

$\therefore$ Root is 0.5671 (correct upto 4 d.p)

3   12.25 $\frac{24.25}{7}$

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Qny Find the value of $\sqrt{12}$ correct upto 3 d.p

Ans Let $x = \sqrt{12}$

$\Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$

here $f(x) = x^2 - 12$

$f'(x) = 2x$

$f(3) = 9 - 12 = -3 < 0$

$f(4) = 16 - 12 = 4 > 0$

Let $x_0 = 3.5 \in (3, 4)$

Using NR - formula

$$x_{n+1} = x_n - \frac{(x_n^2 - 12)}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 12}{2x_n}} \quad (1)$$

Put $n = 0$, $x_1 = \frac{x_0^2 + 12}{2x_0} = \frac{(3.5)^2 + 12}{7}$

$= 3.4642$

Put $n = 1$, $x_2 = \frac{x_1^2 + 12}{2x_1} = 3.4641$

$\therefore x_1 = x_2$ upto 3 d.places

$\therefore \sqrt{12} = 3.4641$

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Newton Forward Formula {Interpolation formula}

Consider an unknown function $y = f(x)$ whose values are tabulated at discrete points $x = x_0, x_1, x_2 \cdots x_n$ such that $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots$
$x_n - x_{n-1} = h$.

We are required to determine

(1) The value of the fun. at some point x_i (not tabulated).

OR

The polynomial approximation of $y = f(x)$

To achieve these goals we need Newton Forward Interpolation formula.

$$f(x) = f(a) + \frac{u}{1!} \Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \cdots$$

where $u = \frac{x-a}{h}$, a = first value of x
h = difference b/w successive values of x.

$x^2 - 3x - x + 3$

$x^2 - 4x + 3$

$x^3 - 4x^2 + 3x$

Que Consider 3^x at $x = 0, 1, 2, \& 3$. Approximate it by same polynomial.

Ans Forward differe Table

x	$f(x) = 3^x$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	2	4	8
1	3	6	12	
2	9	18		
3	27			

here $a = 0,\ h = 1,\ x = x$

$u = \frac{x-a}{h} = x$

Using Newton Formula

$$= 1 + \frac{x}{1} \times 2 + \frac{x(x-1)}{2} \times \cancel{4}^{2} + \frac{x(x-1)(x-2)}{\cancel{6}_{3}} \times \cancel{8}^{4}$$

$$= 2x + 2x(x-1) + \frac{4x(x-1)(x-2)}{3}$$

$$= \cancel{2x} + 2x^2 - \cancel{2x} + \frac{4x(x-1)(x-3)}{3}$$

$$= \frac{6x^2 + 4(x^3 - 4x^2 + 3x)}{3}$$

$$= \frac{4}{3}x^3 - 2x^2 + \frac{8}{3}x + 1$$

$5\overset{0.4}{\overline{)20}}$ $\frac{2}{5}$

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Que Find value of sin (2°) from following table

x	0	5	10	15
$\sin(x^\circ)$	0	0.0871	0.1736	0.2588

Ans

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0	0.0871	−0.0006	−0.0007
5	0.0871	0.0865	−0.0013	
10	0.1736	0.0852		
15	0.2588			

Here $a=0$ $h=5$ $x=2$

$$u = \frac{x-0}{h} = \frac{2-0}{5} = 0.4$$

$$f(x) = f(0) + \frac{u}{1}\Delta f(0) + \frac{u(u-1)}{2}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{6}\Delta^3 f(x)$$

$$= 0 + 0.4\times 0.0871 + \frac{0.4\times 0.3}{2} \times -0.0006$$

$$+ \frac{0.4\times 0.3\times 0.2}{6} \times -0.007$$

$$= 0.03484 - 0.000036 - 0.00028$$

$$= 0.034524$$

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~~Newton's Backward Interpolation Formula~~

~~Consider an unknown~~

Numerical Integration

Consider definite integration.

$$I = \int_a^b f(x)\, dx$$

The above integral can be evaluated using indirect method in which the range of integration i.e $\{a, b\}$ is divided into n equal part each of width

$$h = \frac{b-a}{n}$$

x	x_1	x_2	x_3	---	x_n
y	y_1	y_2	y_3	---	y_n

Based on value of n we have following values

(1) Trapezoidal Rule –

$$I = \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots y_{n-1})\right\}$$

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This rule is applicable for all n but it is least accurate.

(ii) Simpson's (1/3) method Rule -

$$I = \frac{h}{3}\left\{(y_0 + y_n) + 4\{y_1 + y_3 + y_5 + \cdots + y_{n-1}\} + 2\{y_2 + y_4 + y_6 - \cdots y_{n-2}\}\right\}$$

This rule is applicable when n is multiple of 2

(iii) Simpson's (3/8) rule -

$$I = \frac{3h}{8}\left[\{y_0 + y_n\} + 3\{y_1 + y_2 + y_4 + y_5 + y_7 + \cdots y_{n-2} + y_{n-1}\} + 2\{y_3 + y_6 + y_9 + y_{n-3}\}\right]$$

This rule is applicable when n is a multiple of 3

Ques 1 Evaluate $\int_0^1 x^2\,dx$ by dividing the range of integration in 5 equal parts, also compute exact value & absolute error in your answer.

x^2 $\frac{1}{1+(\frac{5}{6})^2}$ $\frac{1}{36+25}$

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$$\boxed{h = \frac{b-a}{n}}$$

$$\left\{ I = \int_a^b f(x) \cdot dx \right.$$

, n is no parts divided

$a = 0, b = 1, n = 5$

$y = x^2$

$\therefore \quad h = \frac{1-0}{5} = \frac{1}{5}$

x	0	1/5	2/5	3/5	4/5	1
$y = x^2$	0	1/25	4/25	9/25	16/25	1
	y_0	y_1	y_2	y_3	y_4	y_5

Using Trapizoidal rule

$$I = \frac{h}{2}\left[\{y_0 + y_n\} + 2\{y_1 + y_2 + y_3 + \cdots y_{n-1}\}\right]$$

$$= \frac{1}{2 \times 5}\left[(0+1) + 2\left(\frac{1}{25} + \frac{4}{25} + \frac{9}{25} + \frac{16}{25}\right)\right]$$

$$= \frac{1}{10}\left[1 + 2\left(\frac{30}{25}\right)\right] = \frac{1}{10}\left[\frac{25+60}{25}\right]$$

$$= \frac{85}{250} = \frac{17}{50}$$

Absolute value $= |$exact value $-$ computed value$|$

$$= \left|\frac{1}{3} - \frac{17}{50}\right| = \frac{1}{150}$$

$\frac{1}{\frac{36+1}{36}}$ $\frac{36}{3}$

Ans 2

Ans

$\frac{1}{\frac{36+1}{36}}$ $\frac{36}{37}$ $\frac{1}{1+\frac{1}{36}}$ $\frac{1}{1+\frac{1}{4}}$ $\frac{1}{\frac{5}{4}}$

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Que 2 Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Simpson's 1/3 rule
(ii) Simpson's 3/8 rule

Hence compute approx value of π in each case

Ans

Take $n = 6$ (multiple of both 2 & 3)

$a = 0, b = 1, y = \frac{1}{1+x^2}, n = 6$

$h = \frac{1-0}{6} \Rightarrow h = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = \frac{1}{1+x^2}$	1	36/37	9/10	4/5	9/13	36/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using Simpson's 1/3 rule

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots y_{n-1}) + 2(y_2+y_4+y_6+\cdots y_{n-2})\right]$$

$$I = \frac{1}{18}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right) + 2\left(\frac{9}{10}+\frac{9}{13}\right)\right]$$

$$I = \frac{1}{18}\left[\frac{3}{2} + 4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right) + 2\left(\frac{9}{10}+\frac{9}{13}\right)\right]$$

$I = 0.785397945 2$

$\frac{3}{6}+1$ $\quad 1+\frac{2}{6}$ $\quad \frac{8}{6}$ $\quad \frac{6}{8}$ $\quad \frac{9}{6}$

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Using Simsons 3/8 rule

$$I = \frac{3h}{8}\Big[\{(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots y_{n-2}, y_{n-1})$$
$$+ 2(y_3+y_6+y_6+\cdots y_{n-3})\Big]$$ Ans

$$= \frac{3}{2\cdot 6\times 8}\left[\left(1+\frac{1}{2}\right)+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)+2\left(\frac{4}{5}\right)\right]$$

$$= \frac{1}{16}\left[\frac{3}{2}+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)+\frac{8}{5}\right]$$

$$= 0.7853958624$$

Exact value —

$$I = \int_0^1 \frac{1}{1+x^2} = \left[\tan^{-1}(x)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

We wish

(I) from part (I) $\pi/4 = 0.785397945 2 \Rightarrow \pi = 3.145917808$

(II) from part (II) $\pi/4 = 0.7853958624 \Rightarrow \pi = 3.14158 34496$

Qu 3 Evaluate $\int_0^1 \frac{dx}{1+x}$ by dividing range of integration into 7 equidistant parts (hint $n=6$)

$1 + \frac{1}{6}$ $\quad \frac{1}{\frac{7}{6}} = \frac{6}{7}$

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Also compute approximate value $\log_e 2$

$$h = \frac{b-a}{n} \Rightarrow h = \frac{1-0}{6} \Rightarrow h = \frac{1}{6}$$

$$y = \frac{1}{1+x}$$

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	1
$y = \frac{1}{1+x}$	1	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{1}{2}$

By using Trapezoidal Rule

$$I = \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots y_{n-1})\right\}$$

$$= \frac{1}{12}\left\{\left(1+\frac{1}{2}\right) + 2\left(\frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11}\right)\right\}$$

$$= \frac{1}{12}\left\{\frac{3}{2} + 2\times 6\left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right)\right\}$$

$$= \frac{1}{12}\left\{\frac{3}{2} + 12\left(\frac{8+7}{7\times 8} + \frac{9+10}{9\times 10} + \frac{1}{11}\right)\right.$$

$$= \frac{1}{12}\left\{\frac{3}{2} + 12\left(\frac{15}{56} + \frac{19}{90} + \frac{1}{11}\right)\right.$$

$$= \frac{3}{12}\left\{\frac{1}{2} + \cancel{4}\times\frac{15}{\cancel{56}_{14}} + \cancel{4}^{2}\times\frac{19}{\cancel{90}_{45}} + \frac{4}{11}\right\}$$

$$= \frac{1}{4}\left[\frac{1}{2} + \frac{15}{14} + \frac{38}{45} + \frac{4}{11}\right]$$

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$$= \frac{1}{4}\left[\frac{7 + 15}{14} + \frac{38 \times 11 + 45 \times 4}{45 \times 11}\right]$$

$$= \frac{1}{4}\left[\frac{22}{14} + \frac{418 + 180}{495}\right]$$

$$= \frac{1}{4}\left[\frac{22}{14} + \frac{598}{495}\right]$$

$$= \frac{1}{4}\left[\frac{496 \times 22 + 598 \times 14}{14 \times 495}\right]$$

$$= \frac{1}{4}\left[\frac{10912 + 8372}{6930}\right] = \frac{19284}{4} \times \frac{1}{6930}$$

$$= \frac{4821}{6930} = 0.6956709956709$$

~~Exact value of~~ $\cancel{\int_0^1 \frac{1}{1+x} = \int \frac{1}{\sqrt{x^2} + \sqrt{1}^2}}$

$\cancel{= \tan^{-1}\sqrt{1} - \tan^{-1}\sqrt{0}}$

$\cancel{= \frac{\pi}{4} - 0 = \frac{\pi}{4}}$

~~∴ From four~~ :

exact value $\int_0^1 \frac{1}{1+x}\, dx$

Let $u = 1 + x \Rightarrow du = dx$

$$\Rightarrow \int_0^1 \frac{1}{u}\, du$$

$$[\ln|u|]_0^1 \Rightarrow \ln(2) - \ln(1)$$
$$\Rightarrow \ln(2) - \ln(1)$$
$$\Rightarrow 0.693147806$$

$\therefore$ The value of $\log_e(2) = 0.693147806$

Numerical Solution

A differential eq^n
↳ (first order) $\frac{dy}{dx} = f(x,y)$

Together with the initial condition
↳ $y(x_0) = y_0$

The sol^n of IVP can be obtained using indirect methods as follows —

(1) Picard's Method —

since $\frac{dy}{dx} = f(x,y)$

$dy = f(x,y)\,dx$

integrating both sides using suitable limits

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

(1) PC

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$$y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

There approx are given as —

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_2 = y_0 + \int_{x_1}^{x} f(x, y_1)\,dx$$

And so on.

Que Consider IVP $\frac{dy}{dx} = x + y$ with $y(0) = 1$. Compute $y(1)$ using picards method of succesive approximation.

Ans Here $f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$

Formula : —

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \qquad y_{n+1} = y_0 \int_{x_0}^{x} f(x, y_n)$$

If $n = 0$

$$y_1 = y_0 + \int_{x_0}^{x} (x + y_0)\,dx$$

$$y_1 = 1 + \frac{x^2}{2} + x$$

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$\Rightarrow (y(1)) = 1 + \frac{1}{2} + 1 = 2.5$ is 1st approximation

If $n = 1$

$$y_2 = y_0 + \int_{x_0}^{x} (x + y_1)\, dx$$

$$1 + \frac{x^2}{2} + 2.5x$$

$$y_2 = 1 + \int_{x_0}^{x} \left(x + 1 + \frac{x^2}{2} + x\right) dx$$

$$y_2 = 1 + \int_{x_0}^{x} \frac{x^2}{2} + 2x + 1 = 1 + \frac{x^3}{6} + x^2 + x$$

$$(y_2)(1) = 3.166$$

If $n = 2$

$$y_3 = y_0 + \int_{x_0}^{x} \left(2x + 1 + \frac{x^3}{6} + x^2\right) dx$$

$$y_3 = 1 + x^2 + x + \frac{x^4}{24} + \frac{x^3}{3}$$

$$y_3(1) = \frac{27}{8} = 3.375$$

mpute

uation.

$y_0 \int f(x, y_n)$

$1+1+1+\frac{1}{3}+\frac{1}{24}+\frac{1}{120}$

Date. $\frac{360+46}{120}$

$3+\frac{39}{248}+\frac{1}{120}$ $\quad \frac{46}{120}+3$ Page No. $\frac{406}{120}$

If $n=3$

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\, dx$$

$$= 1 + \int_{x_0}^{x} \left(2x + 1 + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \frac{x^5}{120}$$

$y_4(1) = 3.5083$

If $n = 4$

$$y_5 = y_0 + \int_{x_0}^{x} f(x, y_4)\, dx$$

$$= 1 + \int_{x_0}^{x} \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \frac{x^5}{120}\right) dx$$

$$= 1 + x^2 + x + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^6}{720}$$

$y(5)(1) = 3.4347$

If $n = 5$

$$y_6 = y_0 + \int_{x_0}^{x} f(x, y_5)\, dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \frac{x^7}{5040}$$

$y_6(1) = 3.4363$

$y_5(1) = y_6(1)$ upto 3 decimal places

$\therefore y(1) = 3.43$ is correct upto 2 decimals

Runge Kutta Method

Consider IVP $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ to compute value of y at $x_1 = x_0 + h$ following formative are used.

Step 1 : at $x_1 = x_0 + h$,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

Then $k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

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$y_1 = y(x) - y_0 + K$

Similarly to compute value of y at $x_2 = x_1 + h$ we use following setup formula at $x_2 = x_1 + h$,

$k_1 = hf(x_1, y_1)$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$, $k_4 = hf(x_1 + h, y_1 + k_3)$

Then $K = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y_2 = y(x_2) = y_1 + K$

Que Consider IVP $\frac{dy}{dx} = x + y$ with $y(0) = 1$ find $y(1)$ using R.K method with $h = 0.5$

Ans $f(x, y) = x + y$

$x_0 = 0,\ y_0 = 1,\ h = 0.5$

Step I at $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$k_1 = hf(x_0, y_0) = 0.5(0+1) = 0.5$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.5(0.25 + 1.25)$
$= 0.75$

$k_3 = hf\left(x^0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.5(0.25 + 1.375)$
$= 0.8125$

$k_4 = hf(x^0 + h, y^0 + k_3) = 0.5(0.5 + 1.8125)$
$= 1.15625$

Then $K = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.796875$

$y_1 = y(x_1) = y(0.5) = y_0 + K = 1 + 0.796875 = 1.796875$

Step 2 at $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$$K_1 = hf(x_1, y_1) = 0.5(0.5 + 1.796875)$$

$$\cancel{= 0.5} = 1.1484375$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right) = 0.5\left(0.5 + \frac{0.5}{2}, 1.796875 + \frac{1.1484375}{2}\right)$$

$$= 1.156054688$$

$$K_3 = hf\left(x_1 + \frac{h}{2} + y_1 + \frac{K_2}{2}\right) = 1.663574219$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.5(0.5 + 0.5 + 1.796875 + 1.663574219)$$

$$= 2.230224609$$

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 1.637824675$$

$$y_2 = y(x_2) = y(1) = y_1 + K$$

$$= 3.434699675$$