Wave Mechanics

[1.] $\lambda = \dfrac{12400}{2.24} = 5535.7\,\text{Å} \simeq 5536\,\text{Å}$

[2] The equations for Photoelectric effect are

$$\frac{hc}{\lambda_1} = E_1 + \phi \quad -① \qquad ; \ \phi = \text{work function}$$

$$\frac{hc}{\lambda_2} = E_2 + \phi \quad -②$$

Subtracting ① from ②

$$E_2 - E_1 = hc\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right]$$

$$\boxed{h = \frac{(E_2 - E_1)\lambda_1 \lambda_2}{c(\lambda_1 - \lambda_2)}} \quad -③$$

Put ③ in ①

$$\frac{\cancel{c}}{\cancel{\lambda_1}} \frac{(E_2 - E_1)\cancel{\lambda_1}\lambda_2}{\cancel{c}(\lambda_1 - \lambda_2)} = E_1 + \phi$$

$$\phi = \frac{E_2\lambda_2 - E_1\lambda_2}{\lambda_1 - \lambda_2} - E_1$$

$$\phi = \frac{E_2\lambda_2 - E_1\lambda_2 - E_1\lambda_1 + E_1\lambda_2}{\lambda_1 - \lambda_2}$$

$$\boxed{\phi = \frac{E_2\lambda_2 - E_1\lambda_1}{\lambda_1 - \lambda_2}}$$

3.

$$eV_0 = \frac{hc}{\lambda} - \phi$$

$$= \frac{6.623\times10^{-34}\times 3\times10^{8}}{800\times10^{-9}\times1.6\times10^{-19}} - 1.25$$

$$= 1.55 - 1.25$$

$$eV_0 = 0.3\ eV$$

$$\boxed{V_0 = 0.3V}$$

$$K.E = 0.3eV$$

$$\frac{1}{2}mv^2 = 0.3\times1.6\times10^{-19}$$

$$v = \sqrt{\frac{1.6\times3\times10^{-19}\times 2}{9.1\times10^{-31}}} = \sqrt{0.10549\times10^{12}}$$

$$= 0.324\times10^{6}$$

$$v = 3.2\times10^{5} m/s$$

4.

Compton's scattering :-

$$\lambda' - \lambda = \frac{h}{m_0 c}[1-\cos\theta] \qquad [\cos\theta = -1]$$

$$\Delta\lambda_{max} = \frac{2h}{m_0 c}$$

$$= 2\times0.0243\ \mathring{A} = 0.0486\ \mathring{A}$$

For $1\mathring{A}$ :-

$$\frac{\Delta\lambda}{\lambda}\times100 = \frac{0.0486\ \mathring{A}}{1\mathring{A}}\times100 = 4.8\%$$ → More change

For $10\mathring{A}$ :-

$$\frac{\Delta\lambda}{\lambda}\times100 = \frac{0.0486\mathring{A}}{10\mathring{A}}\times100 = 0.48\%$$ → less change is found with ↑ in wavelength.

$E = h\nu - h\nu'$

$= h\nu\left[1 - \frac{\nu'}{\nu}\right]$ —— (1)

We know $\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)$

$$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_0 c} \cdot 2\sin^2\theta/2$$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{2h}{m_0 c^2} \sin^2\theta/2$$

$$\frac{\nu}{\nu'} = 1 + \frac{2h\nu}{m_0 c^2} \sin^2\theta/2 \quad \text{—— (2)}$$

Put (2) in (1)

$$KE = h\nu\left[1 - \frac{1}{1 + \frac{2h\nu}{m_0 c^2}\sin^2\theta/2}\right]$$

$$= h\nu\left[\frac{\frac{h\nu}{m_0 c^2} \cdot 2\sin^2\theta/2}{1 + \frac{2h\nu}{m_0 c^2}\sin^2\theta/2}\right]$$

KE → KEmax when $\theta = 180°$, $\sin\theta/2 \to 1$

$$KE_{max} = \frac{2h^2\nu^2}{m_0 c^2}\left[\frac{1}{1 + \frac{2h\nu}{m_0 c^2}}\right]$$

$E = h\nu$ = Energy of incident photon

$E_0 = m_0 c^2$ = Energy of $e^-$ when $e^-$ is at rest

$$\boxed{KE_{max} = \frac{2E^2}{E_0}\left[\frac{1}{1 + 2E/E_0}\right]}$$

[6] given $[KE]_{max} = \frac{E}{2}$ — ①

We know $[KE]_{max} = \frac{2E^2}{E_0}\left[\frac{1}{1+2E/E_0}\right]$ — ②

Compare ① & ②

$$\frac{\cancel{E}}{2} = \frac{2E^{\cancel{2}}}{E_0}\left[\frac{1}{1+2E/E_0}\right]$$

$$\frac{1}{2} = \frac{E}{E_0+2E}$$

$$E_0 + 2E = 2E$$

$$E = \frac{E_0}{2} \simeq \frac{511\ keV}{2}$$

$$\boxed{E \simeq 256\ keV}$$

[7] $$\lambda = \frac{h}{\sqrt{2meV}}$$

$$= \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 6.644 \times 10^{-27} \times 1.6 \times 2 \times 10^{-19} \times 2000}}$$

$$= \frac{6.62 \times 10^{-34}}{291.6 \times 10^{-23}} = 2.27 \times 10^{-13}$$

$$\simeq 2.3 \times 10^{-3}\ \mathring{A}$$

100 keV

$$\lambda_{NR} = \frac{h}{\sqrt{2m_0E}} = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times10^{5}\times1.6\times10^{-19}}}$$

$= 3.87\,pm$

$E = KE = mc^2 - m_0c^2$

$m - m_0 = \frac{E}{c^2}$

$m = m_0 + \frac{E}{c^2}$

$= 9.1\times10^{-31} + \frac{100\times10^{3}\times1.6\times10^{-19}}{9\times10^{16}}$

$= (9.1 + 1.7)\times10^{-31}$

$= 10.8\times10^{-31}\,kg$

With Relativistic mass:—

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.62\times10^{-34}}{\sqrt{2\times10.8\times10^{-31}\times100\times10^{3}\times1.6\times10^{-19}}}$$

$$= \frac{6.62\times10^{-34}}{\sqrt{3.456\times10^{-47}}} = 3.56\times10^{-12} = 3.56\,pm$$

Now, $\frac{\Delta\lambda}{\lambda}\times100 = \frac{0.31}{3.87}\times100 = 8\%$

$\boxed{9}$ $\lambda_e = \frac{h}{m_ev_e}$ ; $\lambda_p = \frac{h}{m_pv_p}$

$v_p = \frac{v_e}{2}$ (given)

$$\frac{\lambda_e}{\lambda_p} = \frac{h}{m_ev_e}\times\frac{m_pv_p}{h} = \frac{m_p}{2m_e} \simeq \frac{\cancel{1836m_e}}{2m_e} \simeq \cancel{918}$$

$$= \frac{1.67\times10^{-27}}{2\times9.1\times10^{-31}} \simeq 918$$

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$$\lambda_c = \frac{h}{m_0 c}$$

$$\lambda_d = \frac{h}{mv}$$

$$\frac{\lambda_c}{\lambda_d} = \frac{h}{m_0 c} \times \frac{mv}{h} = \frac{m_0/\sqrt{1-v^2/c^2}\,.\,v}{m_0\, c}$$

$$= \frac{v}{c\sqrt{1-v^2/c^2}}$$

Put $\phi_0 = v/c$

$$\boxed{\frac{\lambda_c}{\lambda_d} = \frac{\phi_0}{\sqrt{1-\phi_0^2}}}$$

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$$\Delta t = 10\,ns = 10^{-8}\,sec.$$

$$E = \frac{hc}{\lambda}$$

$$\frac{\Delta E}{\Delta \lambda} = -\frac{hc}{\lambda^2}$$

[neglecting -ve sign]

Fractional line broadening

$$\frac{\Delta \lambda}{\lambda} = \frac{\Delta E\,.\,\lambda}{hc} \quad \text{—①}$$

$$\Delta E = \frac{hc}{\lambda}$$

$$\frac{\Delta E}{\Delta \lambda} = \frac{-hc}{\lambda^2}$$

Now, $\Delta E\, \Delta t = \frac{\hbar}{2}$

$$\Delta E = \frac{\hbar}{2\times 10^{-8}} \quad \text{—②}$$

Put ② in ①

$$\frac{\Delta \lambda}{\lambda} = \frac{\cancel{h}}{2\pi \times 2\times 10^{-8}} \cdot \frac{\lambda}{\cancel{h}\times 3\times 10^{8}}$$

$$\boxed{\frac{\Delta \lambda}{\lambda} = \frac{\lambda}{12\pi}}$$

$v_p = \frac{\omega}{k} = \frac{E}{p}$

$= \frac{mc^2}{mv} = \frac{c^2}{v}$

$v_p$ - phase velocity
$v_g$ - group velocity
$v$ - velocity of Particle

Now $v_g = v$

$v_p = \frac{c^2}{v_g}$

$$\boxed{v_p \times v_g = c^2}$$

If $v = 0.9c$

$v_g = 0.9c$ ✓

$v_p = \frac{c^2}{v_g} = \frac{c^2}{0.9c} = \frac{c}{0.9}$ ✓

13.

$\Delta v = 1.05 \times 10^4 \times \frac{0.01}{100} = 1.05$

$\Delta x \, m \Delta v = \frac{\hbar}{2}$

$\Delta x = \frac{6.62 \times 10^{-34}}{2 \times 2 \times 3.14 \times 9.1 \times 10^{-31} \times 1.05}$

$= 0.0551 \times 10^{-34+31}$

$$\boxed{\Delta x = 5.5 \times 10^{-5} m}$$

$\psi = A\sec x$

$\sec x \to \infty$ at $x = 90^\circ$

It is not admissible wave function

(b) $Y = A\tan x$

$\tan x \to \infty$ at $x = 90^\circ$

Not admissible wave function

(c) $Y = Ae^{x^2}$

$e^{x^2} \to \infty$ at $x \to \pm\infty$

Not admissible wave function

(d) $\psi = Ae^{-x^2}$

$x \to \pm\infty$ ; $e^{-x^2} \to 0$

Acceptable wave function

15. $Y = A\cos w\left(t - \frac{x}{v}\right)$ —— (1)

OR

Just prove LHS = RHS for any 'y' to be solution of given wave equation

Take partial derivative of y w.r.t. x

$$\frac{\partial y}{\partial x} = A\left(-\frac{w}{v}\right)\sin\left[w\left(t - \frac{x}{v}\right)\right]$$

Taking second partial derivative

$$\frac{\partial^2 y}{\partial x^2} = \frac{A\omega^2}{v^2}\cos\left[\omega\left(t - \frac{x}{v}\right)\right]$$

$$= y\frac{\omega^2}{v^2} \quad \text{—— (2)}$$

Take partial diff. of (1) w.r.t. 't'

$$\frac{\partial y}{\partial t} = A\omega\sin\left[w\left(t - \frac{x}{v}\right)\right]$$

Double derivative w.r.t 't'

$$\frac{\partial^2 y}{\partial t^2} = A\omega^2 \cos\left[\omega\left(t - \frac{x}{u}\right)\right]$$

$$= \omega^2 y \quad \text{—} (3)$$

Put (3) in (2)

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 y}{\partial t^2}$$

which is a wave equation of a stretched string

$y$ = displacement of the string along x-axis

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(a) $\Psi = Nx \exp(-x^2/2)$

Normalization Condition:—

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$$

$$N^2 \int_{-\infty}^{+\infty} x^2 e^{-x^2} dx = 1$$

$$2N^2 \int_0^{\infty} x^2 e^{-x^2} dx = 1$$

$$\boxed{\text{Using} \int_0^{\infty} x^2 e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}}$$

$$2N^2 \cdot \frac{1}{4}\sqrt{\frac{\pi}{1^3}} = 1$$

$$N^2 = \frac{2}{\sqrt{\pi}}$$

$$\boxed{N = \frac{\sqrt{2}}{(\pi)^{1/4}}}$$

$$\psi = N\exp\left(\frac{-x^2}{2a^2}\right)\exp(-ikx)$$

$$\int_{-\infty}^{+\infty} \psi\psi^* dx = 1$$

$$N^2 \int_{-\infty}^{+\infty} e^{-x^2/a^2} e^{(i-i)kx} dx = 1$$

$$\boxed{\text{using } \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}}$$

$$N^2\sqrt{\frac{\pi}{1/a^2}} = 1$$

$$N^2 . a . \sqrt{\pi} = 1$$

$$\boxed{N = \frac{1}{\sqrt{a}\,\pi^{1/4}}}$$

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$$P = \int_x^{x+\Delta x} |\psi|^2 dx$$

$$= \frac{2}{L}\int_x^{x+\Delta x} \sin^2\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{2}{2L}\int_x^{x+\Delta x}\left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right]dx$$

$$= \frac{1}{L}\left\{(x+\Delta x - x) - \left[\frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_x^{x+\Delta x}\right\}$$

$$= \frac{\Delta x}{L} - \frac{1}{\cancel{L}}\cdot\frac{\cancel{L}}{2n\pi}\left[\sin\frac{2n\pi(x+\Delta x)}{L} - \sin\left(\frac{2n\pi x}{L}\right)\right]$$

As $n \to \infty$, second term $\to 0$

$$\boxed{P = \Delta x / L}$$

[18] $P = N^2 \left(\sqrt{\frac{2}{L}}\right)^2 \sin^2\left(\frac{n\pi x}{L}\right)$

For max & mini in probability

$$\frac{dP}{dx} = 0$$

$$2\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\cdot\left(\frac{n\pi}{L}\right)\cdot\frac{2}{L} = 0$$

$$\sin\left(\frac{2n\pi x}{L}\right) = 0 = \sin m\pi$$

$$\Rightarrow \frac{2n\cancel{\pi}x}{L} = m\cancel{\pi}$$

$n = 2$ (given)

$$\frac{4x}{L} = m \Rightarrow x = \frac{mL}{4}$$

For $m = 0$, $x = 0$ first boundary of box (1-D)

$m = 1$, $x = \frac{L}{4}$ — Max

$m = 2$, $x = \frac{L}{2}$ — Min

$m = 3$, $x = \frac{3L}{4}$ — Max

$m = 4$, $x = L$ second boundary of 1-D Box

$\therefore$ Max are 2
Mini is 1

[19] (a) $E = \frac{(n_x^2 + n_y^2 + n_z^2) h^2}{8mL^2}$

$$= \frac{(1+1+1)\times(6.62\times10^{-34})^2}{8\times9.1\times10^{-31}\times(10^{-10})^2} = 1.80\times10^{-17}\ J$$

$k_B T = \frac{n^2 h^2}{8 m L^2}$

$$T = \frac{1.8 \times 10^{-17} \times 2}{3 \times 1.38 \times 10^{-23}}$$

$$T = 8.7 \times 10^{5}\ K$$

20.

$n =$ no. of nodes $+ 1$

$n = 6 + 1 = 7$

$$E = \frac{7^2 \times (6.6 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (2 \times 10^{-9})^2}$$

$$= 7.329 \times 10^{-19}\ J$$

$$= 4.58\ eV$$