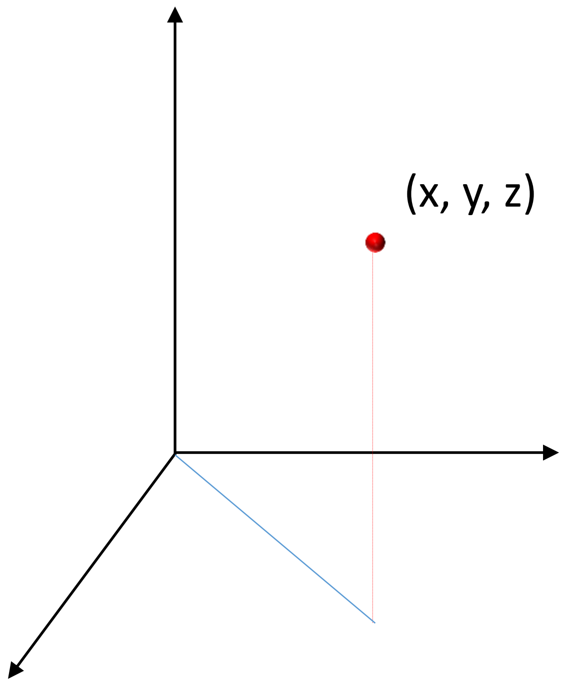
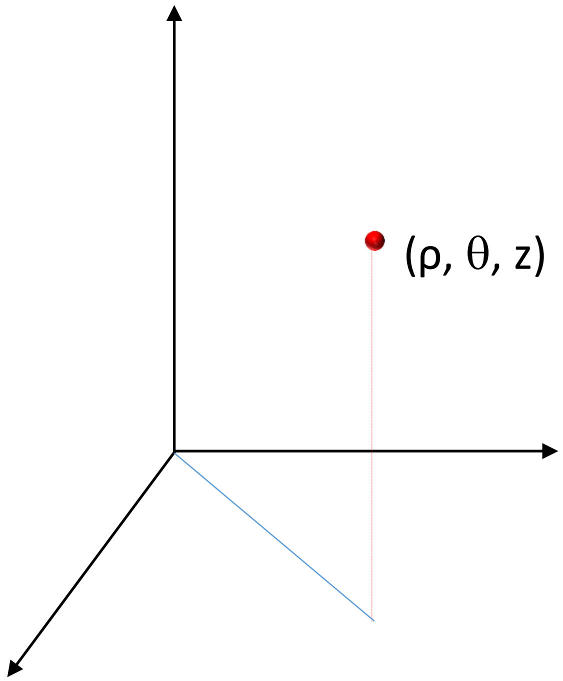
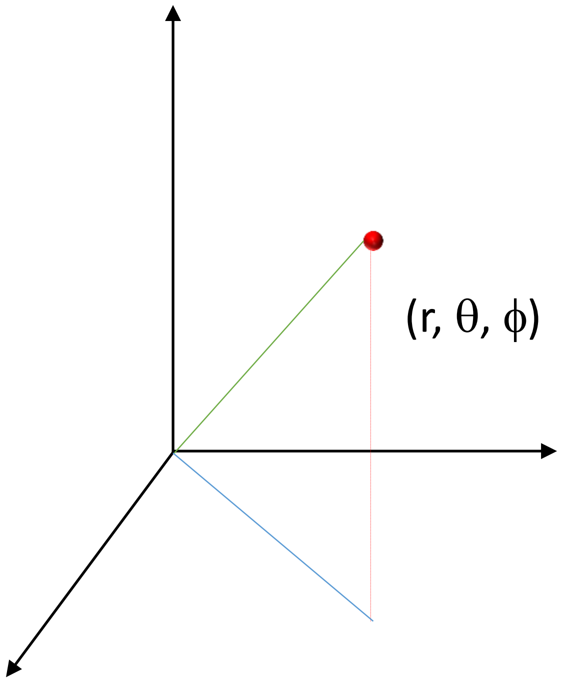
Coordinate system

Some points

Single point representation in different system

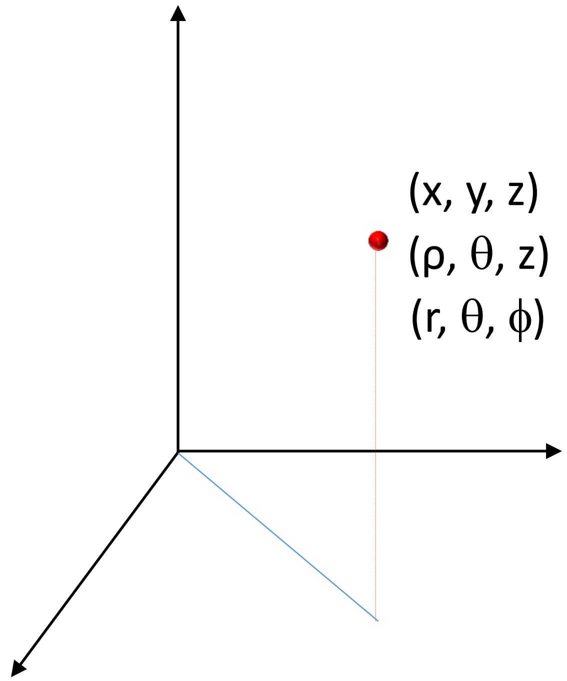






COORDINATE SYSTEMS

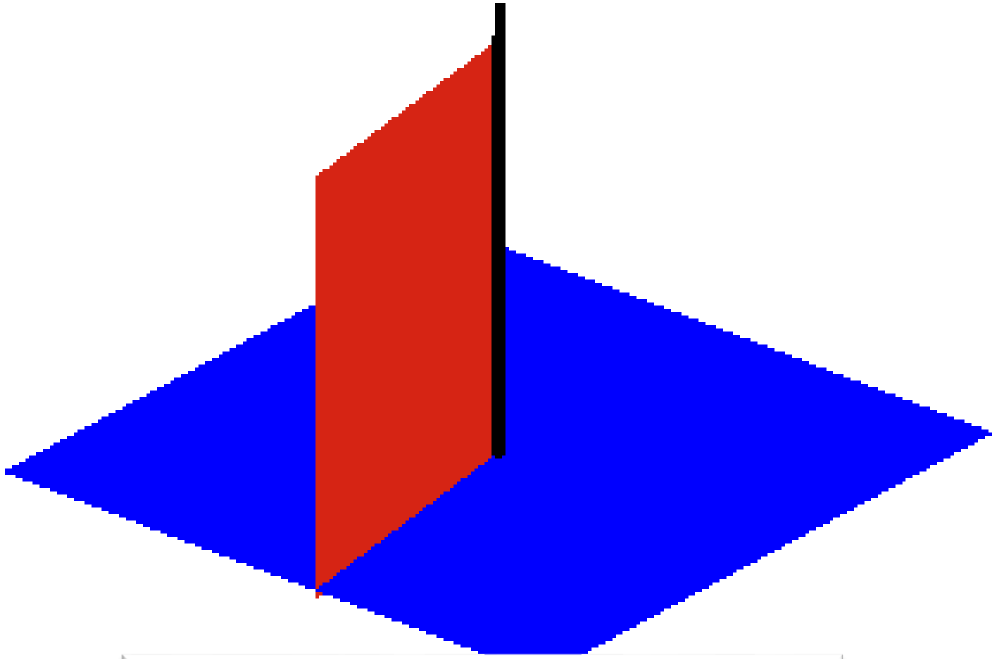
- Cartesian coordinate systems (x, y, z)
- Cylindrical coordinate systems (ρ, θ, z)
- Spherical coordinate systems (r, θ, ϕ)



RECTANGULAR or Cartesian
CYLINDRICAL
SPHERICAL

Choice is based on symmetry of problem

Examples:
Sheets – RECTANGULAR
Wires/Cables – CYLINDRICAL
Spheres – SPHERICAL



Spherical Symmetry

Cylindrical Symmetry

CARTESIAN COORDINATE SYSTEMS

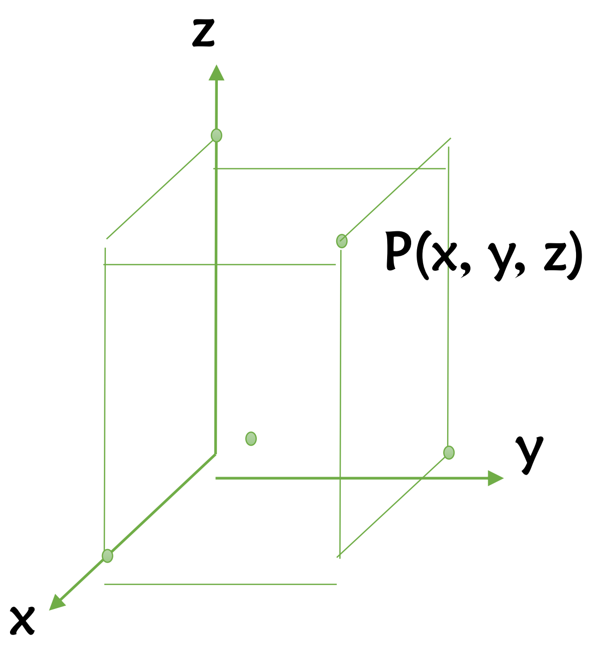
Four basic elements;

1. Choice of origin…..
2. Choice of axis…….
3. Choice of positive direction for each axis…..
4. Choice of unit vectors for each axis…..

1. Origin; Spherical point, may be the mid point of the given body.

2. Axis; From the origin, a set of axis can be chosen. Simplest set of axis is Cartesian axis; x-axis, y-axis and z-axis.

Each point p in space may be assigned triplet values (x_P, y_P, z_P) as Cartesian coordinates of P.

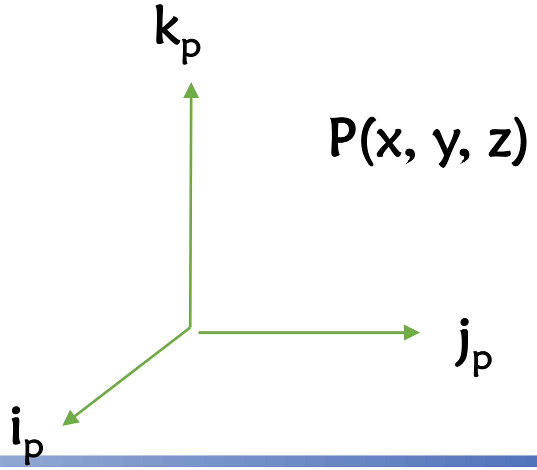


The range of these variables may be given as; **P (x, y, z)**

$$-\infty < x < \infty, \qquad -\infty < y < \infty, \qquad -\infty < z < \infty$$

3. Positive direction; In the plane of paper, the horizontal direction from left to right is positive x-axis, vertical direction from bottom to top is taken as positive y-axis and bottom to upward as positive z-axis. All axis are mutually perpendicular to each other.
For the best fit of the given problem, axis and positive direction my be chosen in any manner.

3. Unit vectors; Point p is associated with three unit directions called unit vectors (i_p, j_p, k_p). Each unit vector has magnitude 1.
The direction of i_p is in the direction of increasing x-coordinates to point p and so on…



Any vector **A** in Cartesian coordinates can be written as;

$$(A_x, A_y, A_z) \quad \text{or} \quad A_x a_x + A_y a_y + A_z a_z$$

where a_x, a_y and a_z are unit vectors along x, y and z-directions.

Differential Length, Area and Volume; CARTESIAN COORDINATE SYSTEMS

1. Differential displacement;

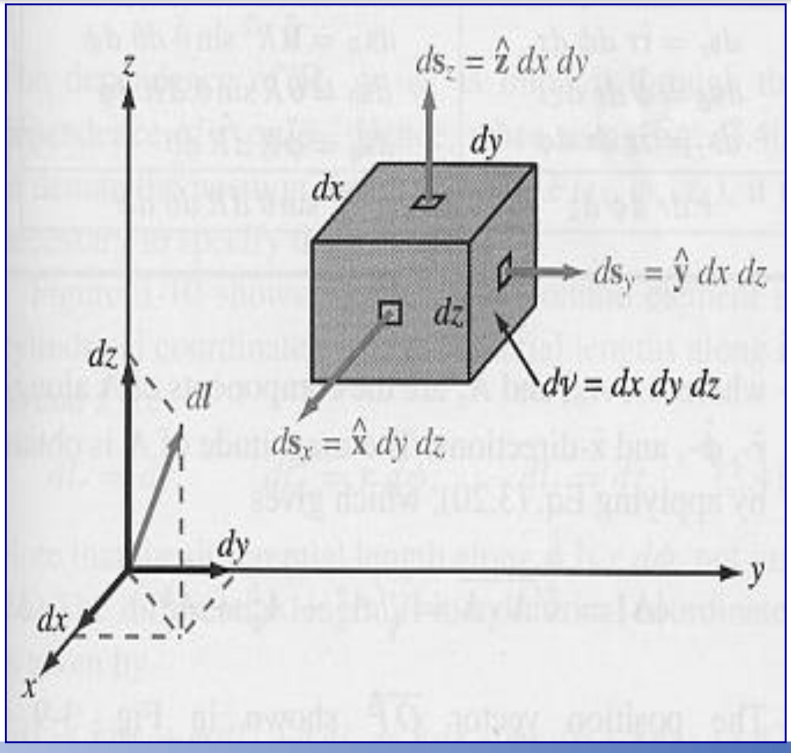
$$dl = dxa_x + dya_y + dza_z$$

2. Differential area;

$$dS = dydza_x = dxdza_y = dxdya_z$$

3. Differential volume;

$$dV = dxdydz$$



Cylindrical Coordinates; (ρ, Φ, z)

Any point P in cylindrical coordinate system is represented as (ρ, Φ, z). Out of these variables;

ρ is radius of cylinder passing through point P or radial distance from z-axis.

Φ is the azimuthal angle measured from the x-axis in x-y plane and z is similar to the Cartesian coordinates.

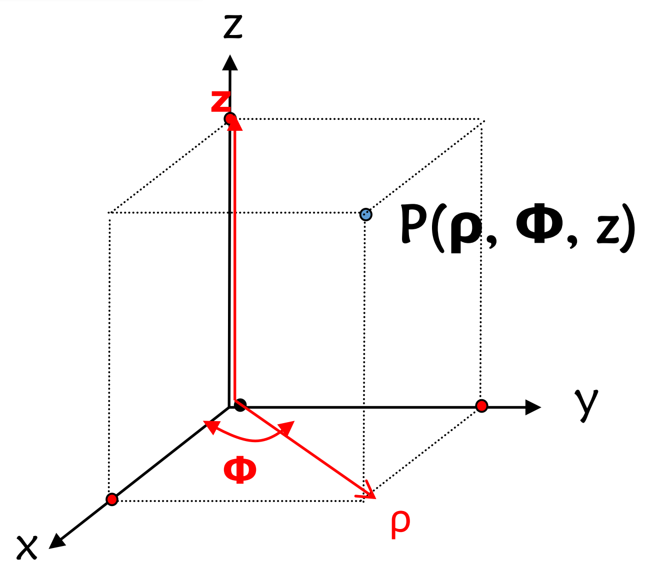
The range of these variables may be given as;

P (ρ, Φ, z);

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$



Cylindrical Coordinates; (ρ, Φ, z)

Any vector **A** in Cylindrical coordinates can be written as;

$$(A_\rho + A_\phi + A_z) \qquad \text{or} \qquad A_\rho a_\rho + A_\phi a_\phi + A_z a_z$$

where a_ρ, a_Φ and a_z are unit vectors along ρ, Φ and z-directions.

It may be noted that the unit vectors; a_ρ, a_Φ and a_z are mutually perpendicular simply because of our coordinate system which is orthogonal i.e., a_ρ pointed in the direction of increasing ρ while a_Φ pointed in the direction of increasing Φ and a_z pointed in the direction of increasing positive z-directions.

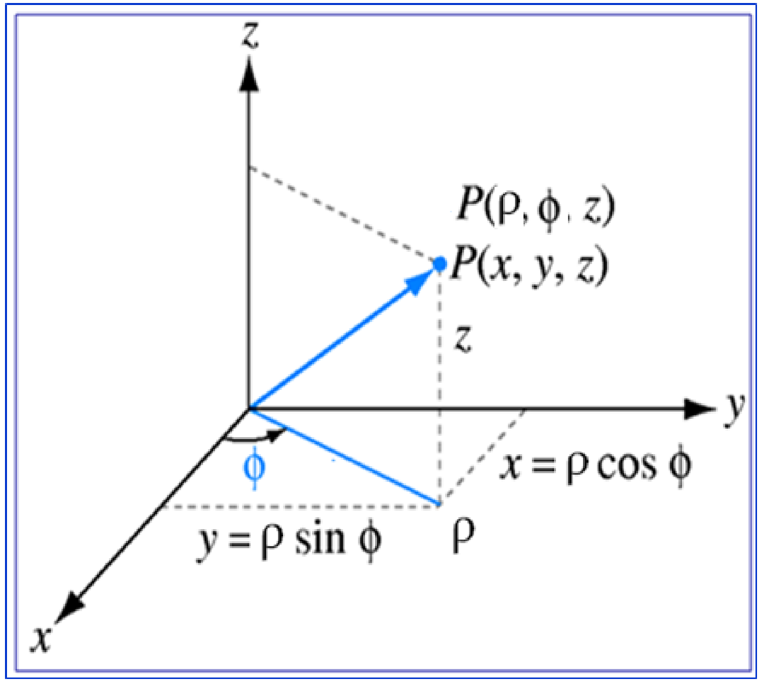
Further; $a_\rho.a_\rho = a_\Phi.a_\Phi = a_z.a_z = 1$

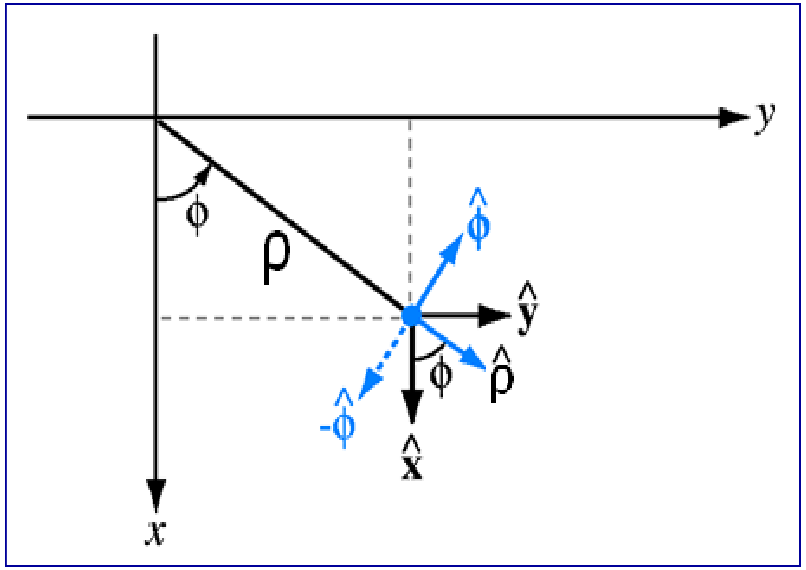
while; $a_\rho.a_\Phi = a_\Phi.a_z = a_z.a_\rho = 0$

& $a_\rho \times a_\Phi = a_z$

$a_\Phi \times a_z = a_\rho$

$a_z \times a_\rho = a_\Phi$ Obtained in cyclic order.





From these relations;

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x}, z = z$$

Relation b/w Cartesian (x, y, z) and cylindrical coordinate system (ρ, Φ, z) can be obtained from fig;

x= ρ cos Φ, y=ρ sin Φ, z=z

First eq. is used for (ρ, Φ, z) to (x, y, z) transformations, while other is used for (x, y, z) to (ρ, Φ, z) transformations.

Point conversion (Cartesian to Cylindrical)

- 1. The cylindrical coordinate system is also referred to as
  - Cartesian system
  - Circular system
  - Spherical system
  - Space system

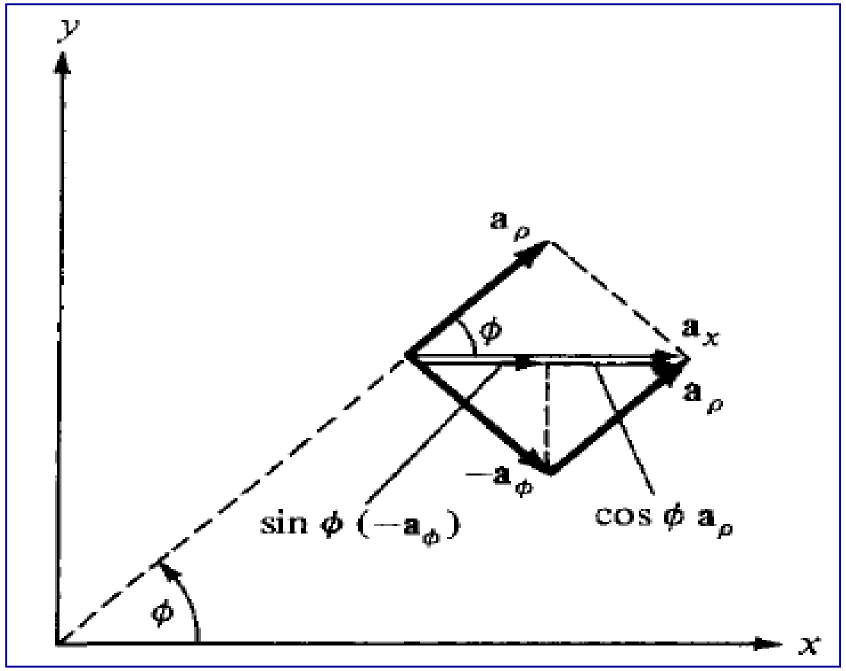
- A charge located at point A ***(5,30^0,2)*** is said to be in which coordinate system?
  - a) Cartesian system
  - b) Cylindrical system
  - c) Spherical system
  - d) Space system

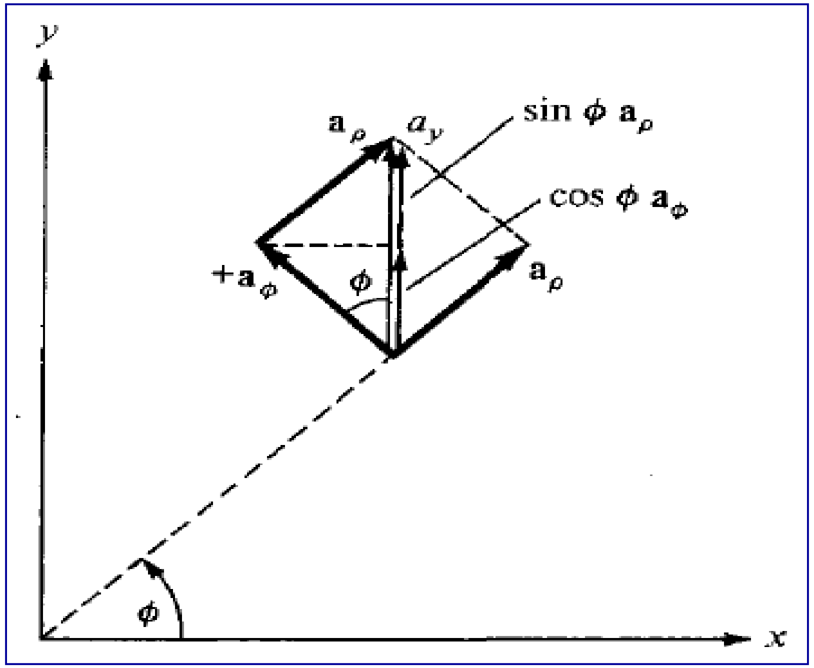
Point conversion (Cartesian to Cylindrical)

- Convert the given rectangular coordinates ***A (2,3,1)*** into corresponding cylindrical coordinates.

Point conversion (Cartesian to Cylindrical)

- Convert the given rectangular coordinates ***A (4,-3,-5)*** into corresponding cylindrical coordinates. Give answers ***for*** **ρ and ϕ** as positive values. Round to two decimal places if needed.





The relationships between (a_x, a_y, a_z) and (a_ρ, a_Φ, a_z) are;

$$a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi$$

$$a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi$$

$$a_z = a_z$$

The relationships between (a_ρ, a_Φ, a_z) and (a_x, a_y, a_z) are;

$$a_\rho = \cos\phi\, a_x + \sin\phi\, a_y$$

$$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$$

$$a_z = a_z$$

A vector in Cartesian coordinate can be written as;

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z$$

In order to get the relationships between (A_x,A_y, A_z) and (A_ρ, A_Φ, A_z), putting the value of (a_x, a_y, a_z) in the above vector and collecting the term in terms of a_ρ, a_Φ and a_z; we have

$$\vec{A} = (A_x \cos\phi + A_y \sin\phi)a_\rho + (-A_x \sin\phi + A_y \cos\phi)a_\phi + A_z a_z$$

Comparing the magnitude components;

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$A_z = A_z$$

or

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\hat{z} = \hat{z}$$

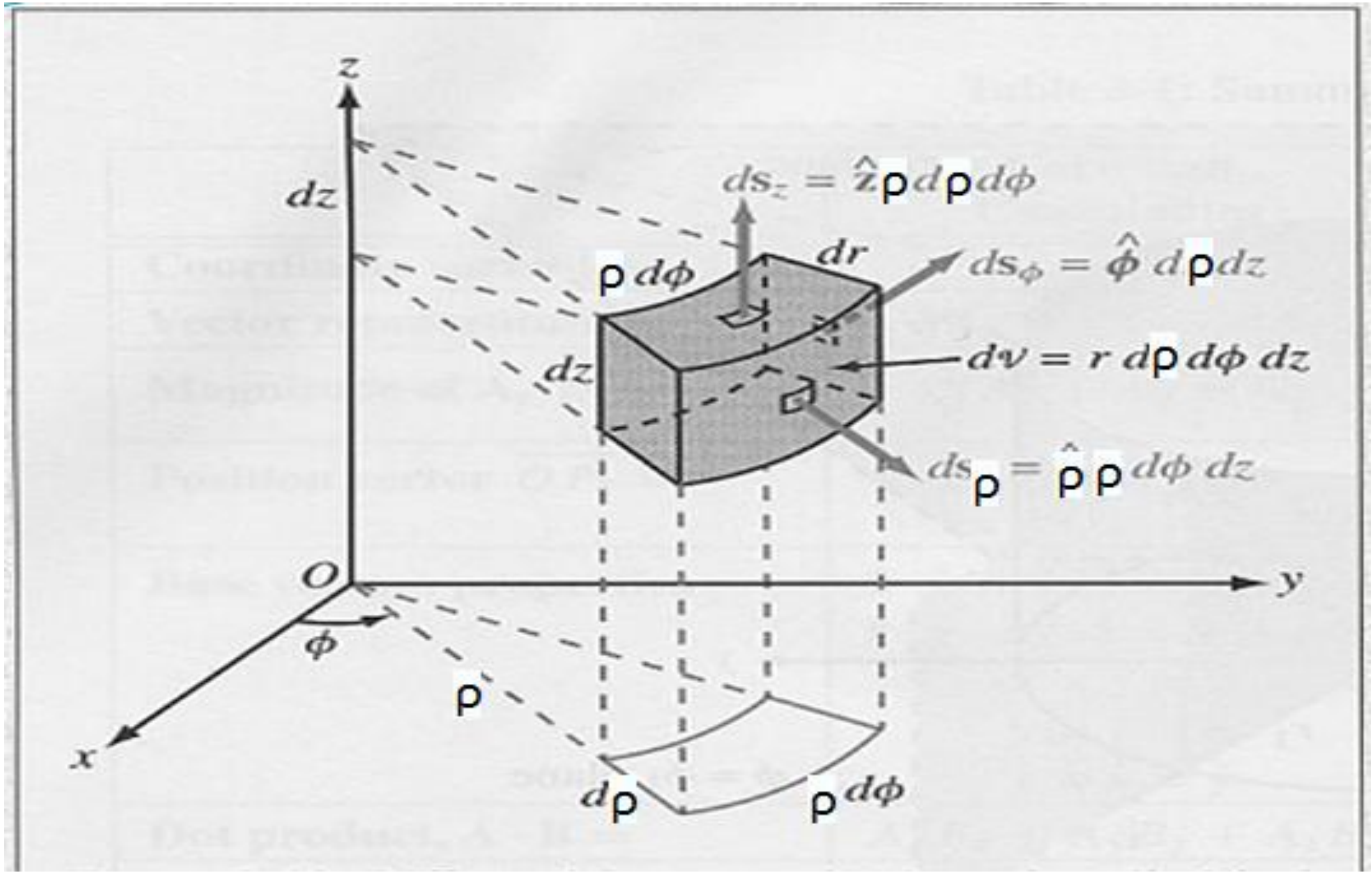
The transformations of vector A from (A_x, A_y, A_z) to (A_ρ, A_Φ, A_z), can be written in matrix form as;

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Inverse of transformations of vector A from (A_ρ, A_Φ, A_z) to (A_x, A_y, A_z);

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Differential Length, Area and Volume in Cylindrical Coordinates



Differential Length, Area and Volume

Cylindrical Coordinates

Differential displacement

$$dl = d\rho a_{\rho} + \rho d\phi a_{\phi} + dz a_{z}$$

Differential area

$$dS = \rho d\phi dz a_{\rho} = d\rho dz a_{\phi} = \rho d\rho d\phi a_{z}$$

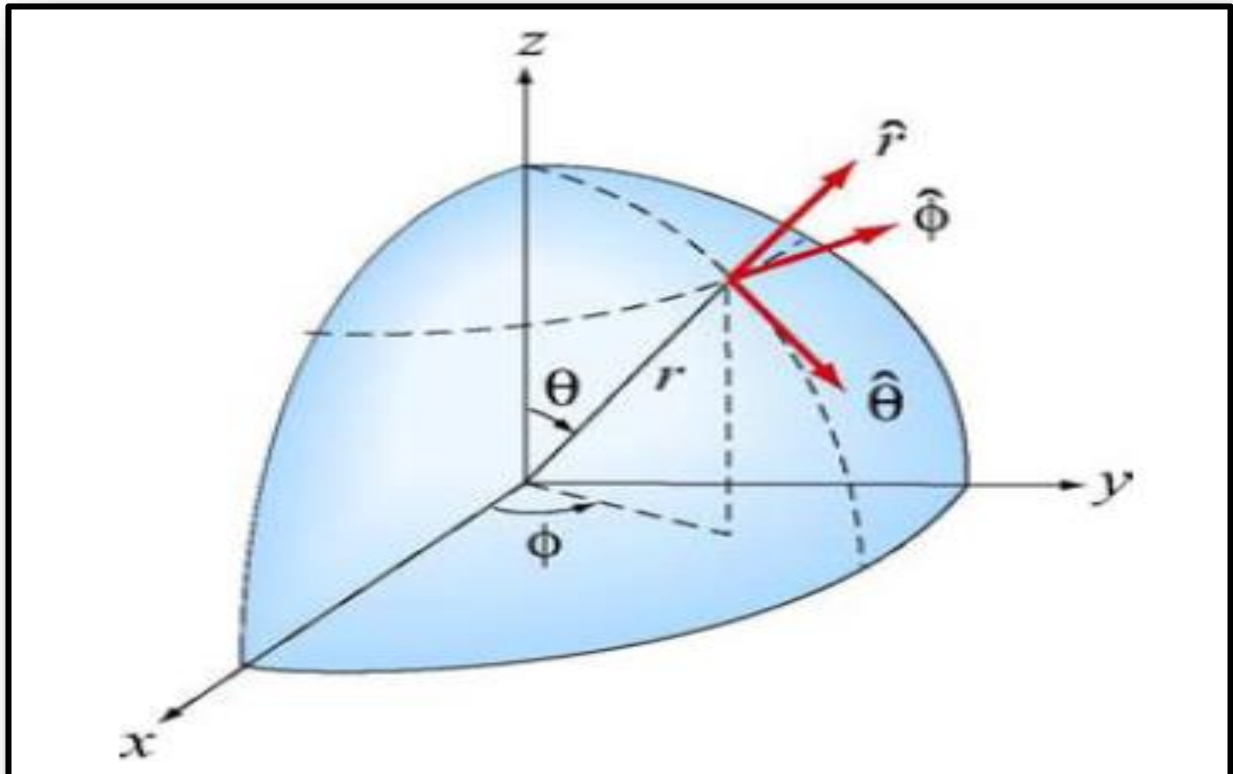
Differential Volume

$$dV = \rho \, d\rho d\phi dz$$

Dealing with the spherical symmetry

Any point P in the spherical coordinates is represented by (r, θ, Φ).

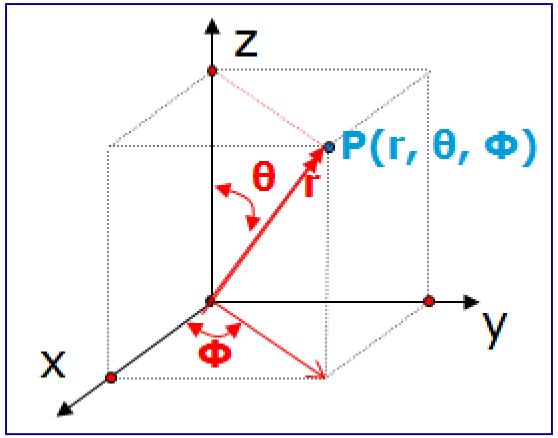
From fig. r is the distance from the origin to the point P or, measures the radial distance from the origin to the point P.
θ is the angle between the z-axis and the position vector of P, while Φ is measured from x-axis (similar to the azimuthal angle as in cylindrical coordinates.)



Spherical Coordinates (r, θ, Φ)

The range of these variables in spherical coordinates

$$0 \le r < \infty, \quad 0 \le \theta \le \pi \quad and \quad 0 \le \phi < 2\pi$$



A **vector A** in Spherical coordinates can be written as

$$(A_r, A_\theta, A_\phi) \quad \text{or} \quad A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

where a_r, a_θ, and a_Φ are unit vectors along r, θ, and Φ-directions.

Further; $a_r . a_r = a_\theta . a_\theta = a_\Phi . a_\Phi = 1$

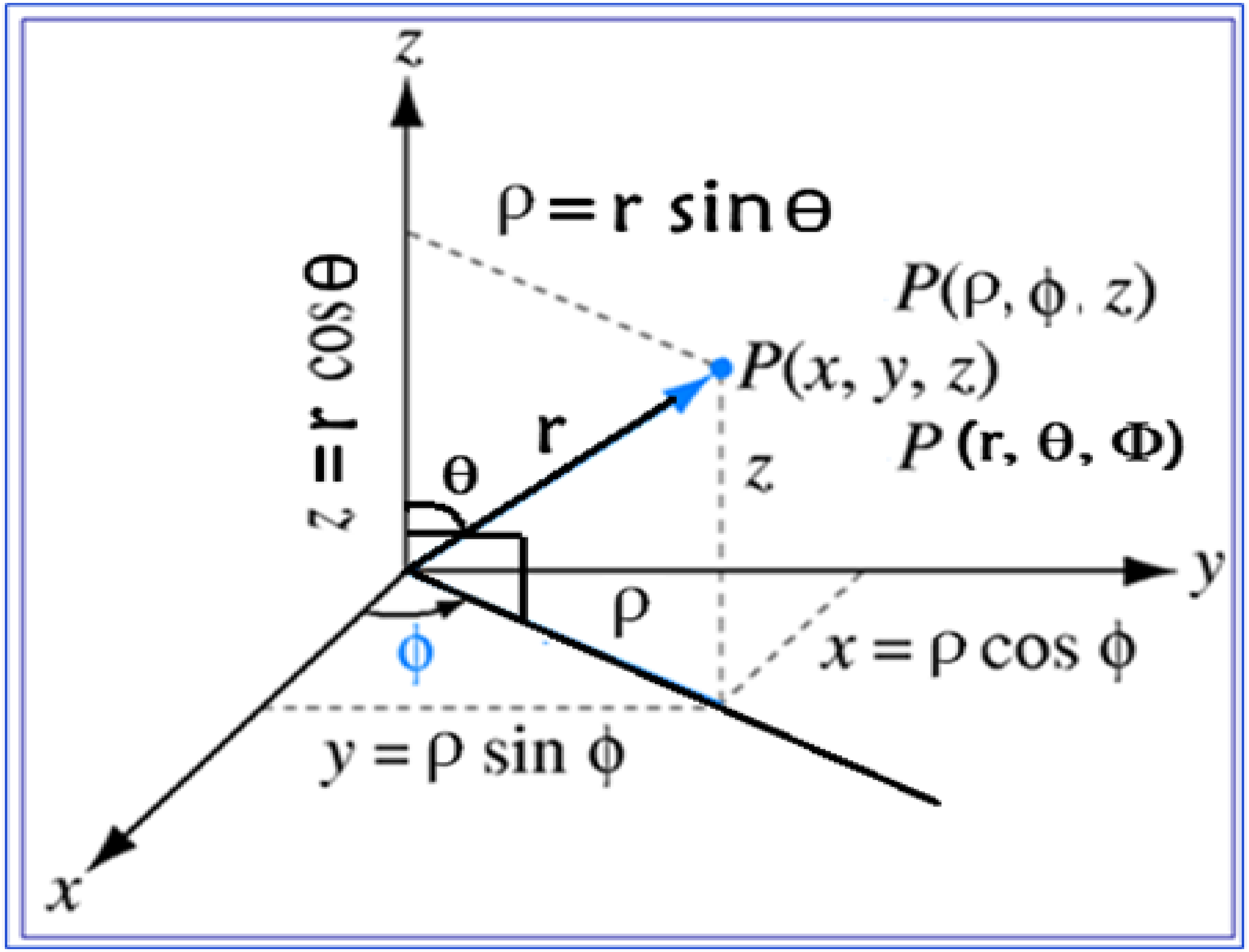
while; $a_r . a_\theta = a_\theta . a_\Phi = a_\Phi . a_r = 0$

& $a_r \times a_\theta = a_\Phi$

$a_\theta \times a_\Phi = a_r$

$a_\Phi \times a_r = a_\theta$ Obtained in cyclic order.

Relation between space variables (x, y, z), (ρ, Φ, z) and (r, θ, Φ).



We have

$x = \rho \cos \Phi$,
$y = \rho \sin \Phi$
$Z = r \cos \theta$ &
$\rho = r \sin \Phi$

So, We have;

$x = r \sin \theta \cos \Phi$,
$y = r \sin \theta \sin \Phi$
$Z = r \cos \theta$

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \phi = \tan^{-1} \frac{y}{x}$$

Point conversion (Cartesian to Spherical)

- Convert the given rectangular coordinates ***A (-2,-1,5)*** into corresponding spherical coordinates.

Point conversion (Cartesian to Spherical)

- Convert the given rectangular coordinates ***A (-2,-1,5)*** into corresponding spherical coordinates.

Above equations (in last slide) are used to transform Cartesian coordinate system to the spherical coordinates system.

The relationships between the cartesian coordinates (a_x,a_y, a_z) and spherical coordinates (a_r,a_θ,a_Φ) are

$$a_x = \sin\theta\cos\phi a_r + \cos\theta\cos\phi a_\theta - \sin\phi a_\phi$$

$$a_y = \sin\theta\sin\phi a_r + \cos\theta\sin\phi a_\theta + \cos\phi a_\phi$$

$$a_z = \cos\theta a_r - \sin\theta a_\theta$$

or

$$a_r = \sin\theta\cos\phi a_x + \sin\theta\sin\phi a_y + \cos\theta a_z$$

$$a_\theta = \cos\theta\cos\phi a_x + \cos\theta\sin\phi a_y - \sin\theta a_z$$

$$a_\phi = -\sin\phi a_x + \cos\phi a_y$$

Then the relationships between (A_x,A_y, A_z) and (A_r, A_θ, A_Φ)are

$$A = (A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta)a_r$$
$$+(A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta)a_\theta$$
$$+(-A_x \sin\phi + A_y \cos\phi)a_\phi$$

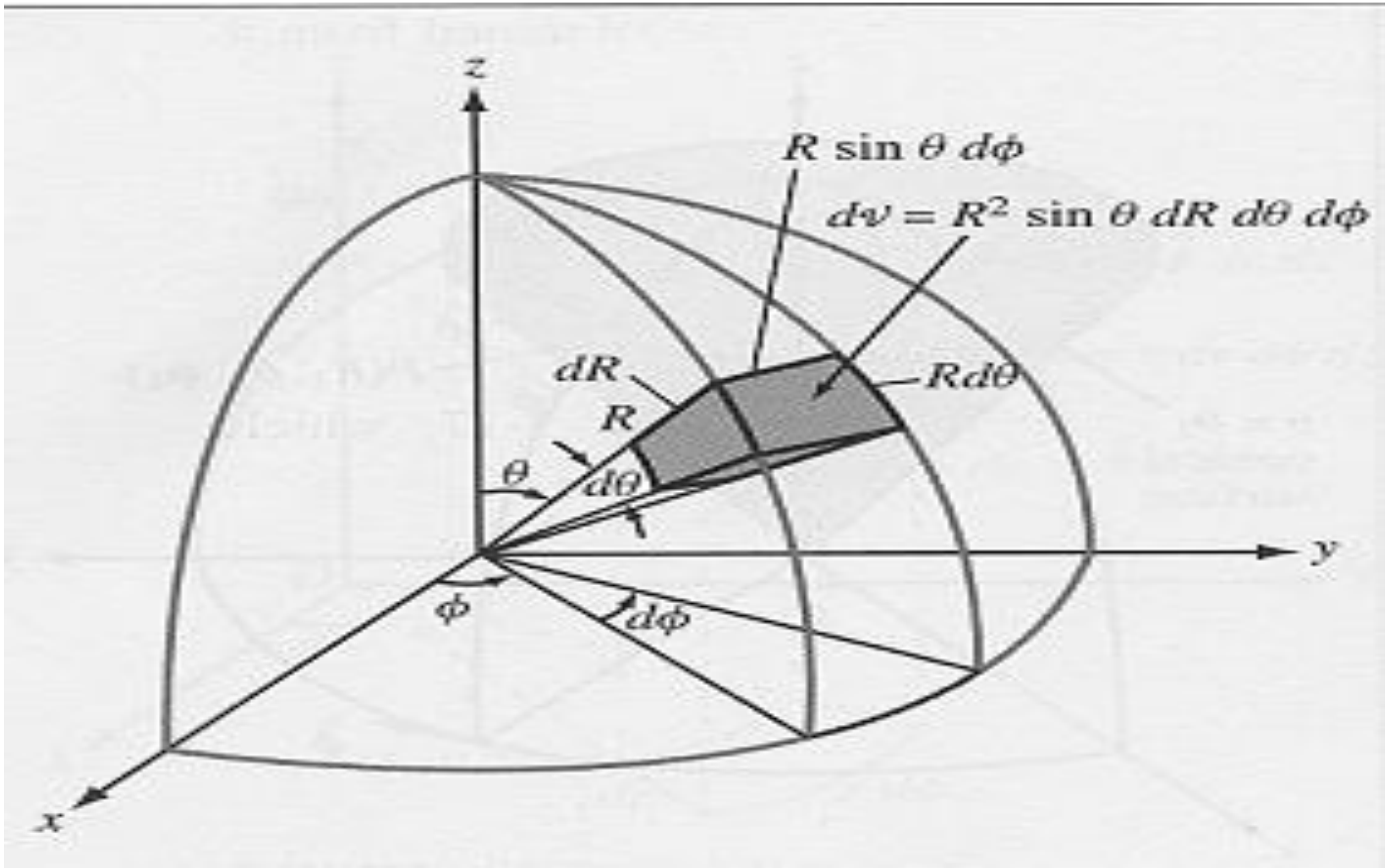
$$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$$
$$A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

In matrix form we can write

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Spherical Coordinates

Differential Length, Area and Volume



Differential Length, Area and Volume

Spherical Coordinates

Differential displacement

$$dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi$$

Differential area

$$dS = r^2 \sin\theta\, d\theta\, d\phi\, a_r = r \sin\theta\, dr\, d\phi\, a_\theta = r\, dr\, d\theta\, a_\phi$$

Differential Volume

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Line, Surface and Volume Integrals

Line Integral

$$\oint_L A.dl$$

Surface Integral

$$\psi = \int_S A.dS$$

Volume Integral

$$\int_V \rho_v dv$$

Scalar and Vector Fields

- Every physical quantity can be expressed as a continuous function of position of a point in the region of space. Such a function is called point function and the region in which it specify the physical quantity is called field.

- A scalar field is a function that gives us a single value of some variable for every point in space.

  voltage, current, energy, temperature

- A vector is a quantity which has both a magnitude and a direction in space.

  velocity, momentum, acceleration and force

The Del Operator

Del operator is basically a vector differential operator denoted by;

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \quad \text{or} \quad \nabla = \frac{\partial}{\partial x} \boldsymbol{a}_x + \frac{\partial}{\partial y} \boldsymbol{a}_y + \frac{\partial}{\partial z} \boldsymbol{a}_z$$

This is also known as gradient operator and useful for the following functions by;
Gradient of a scalar function f is a vector quantity; ∇f

Divergence of a vector function **A** is a scalar quantity and given by; $\nabla \, dot \, A \;\; or \;\; \nabla . A$

Curl of a vector function **A** is a vector quantity given by; $\nabla \times A$

The Laplacian of a scalar function A is given by; $\nabla^2 A$

Cartesian Coordinates

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

Cylindrical Coordinates

$$\nabla = \frac{\partial}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} a_\phi + \frac{\partial}{\partial z} a_z$$

Spherical Coordinates

$$\nabla = \frac{\partial}{\partial r} a_r + \frac{1}{r} \frac{\partial}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} a_\phi$$

The gradient of a scalar field V is a vector that represents whose the magnitude at any point is equal to the maximum rate of change of scalar function (increase of) V with respect to the space variables and has the direction of that change.

Cartesian Coordinates

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

Cylindrical Coordinates

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$$

Spherical Coordinates

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r} \frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} a_\phi$$

Gradient of a Scalar

The following computation formulas on gradient, which are easily proved, should be noted:

(a) $\nabla(V + U) = \nabla V + \nabla U$

(b) $\nabla(VU) = V\nabla U + U\nabla V$

(c) $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$

(d) $\nabla V^n = nV^{n-1}\nabla V$

where U and V are scalars and n is an integer.

Divergence of a Vector

The divergence of A at a given point P is the outward flux per unit small volume surrounding the point P.

$$divA = \nabla . A = \lim_{\Delta v \to 0} \frac{\oint_S A.dS}{\Delta v}$$

<u>Cartesian Coordinates</u>

$$\nabla . A = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}$$

<u>Cylindrical Coordinates</u>

$$\nabla . A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

<u>Spherical Coordinates</u>

$$\nabla . A = \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Divergence of a Vector

Note the following properties of the divergence of a vector field:

1. It produces a scalar field (because scalar product is involved).
2. The divergence of a scalar V, div V, makes no sense.
3. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
4. $\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$

Curl of a Vector

The curl of a vector field A at any point is defined as a vector quantity having magnitude equal to the maximum line integral per unit area along the boundary of an infinitesimal test area at that point and direction perpendicular to the test area.

The curl of A is an axial vector whose magnitude is the maximum circulation of A per unit area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

$$curl A = \nabla \times A = \left(\lim_{\Delta s \to 0} \frac{\oint_L A.dl}{\Delta S} \right)_{\max} a_n$$

Where ΔS is the area bounded by the curve L and a_n is the unit vector normal to the surface ΔS

$$\nabla \times A = \begin{bmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

Cylindrical Coordinates

$$\nabla \times A = \frac{1}{\rho}\begin{bmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{bmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\mathbf{a}_z$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{bmatrix} a_r & r a_\theta & r\sin\theta a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{bmatrix}$$

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{a}_r \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta \\ & + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\mathbf{a}_\phi \end{aligned}$$

Note the following properties of the curl:

1. The curl of a vector field is another vector field.
2. The curl of a scalar field V, $\nabla \times V$, makes no sense.
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$
6. The divergence of the curl of a vector field vanishes, that is, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
7. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$.