MATHEMATICAL INDUCTION

Let $P(n)$ be a statement for $n \in \mathbb{N}$ such that
(i) $P(1)$ is true.
(ii) $P(K+1)$ is true whenever $P(K)$ is true.
Then $P(n)$ is true for all $n \in \mathbb{N}$ (natural no.'s)

Ques 1 Prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$.

Sol Let $P(n): 1+2+3+\cdots+n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$.

Basis of Induction $P(1): 1 = \frac{1(1+1)}{2} \Rightarrow 1 = 1$ holds

Induction Hypothesis Let $P(K)$ be true i.e.

$$1+2+3+\cdots+K = \frac{K(K+1)}{2} \quad \text{——— (1)}$$

To prove: $P(K+1)$ is true i.e.

$$1+2+3+\cdots+(K+1) = \frac{(K+1)(K+2)}{2}$$

Induction Step $LHS = \underbrace{1+2+3+\cdots+K}+(K+1)$ ← NOTE THIS

$$= \frac{K(K+1)}{2} + (K+1) \quad \text{[Using (1)]}$$

$$= (K+1)\left[\frac{K}{2}+1\right]$$

$$= (K+1)\frac{(K+2)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= RHS$$

Ques 2 Prove: $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in \mathbb{N}$

Sol Let $P(n): 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Basis of Induction $P(1): 1^2 = \frac{1(1+1)(2+1)}{6} \Rightarrow 1 = \frac{1\times2\times3}{6}$ or $1=1$ holds

Induction Hypothesis Let $P(K)$ be true i.e.

$$1^2+2^2+3^2+\cdots+K^2 = \frac{K(K+1)(2K+1)}{6} \quad \text{——— (1)}$$

To prove $P(K+1)$ is true i.e.

$$1^2+2^2+3^2+\cdots+(K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}$$

Induction Step $LHS = \underbrace{1^2+2^2+3^2+\cdots+K^2}+(K+1)^2$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= (K+1)\left[\frac{K(2K+1)}{6}+(K+1)\right]$$

$$= (K+1)\left[\frac{2K^2+K+6K+6}{6}\right]$$

$$= (K+1)\left[\frac{2K^2+7K+6}{6}\right]$$

$$= (K+1)\left[\frac{2K^2+4K+3K+6}{6}\right]$$

$$= (K+1)\left[\frac{(2K+3)(K+2)}{6}\right]$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

$$= RHS$$

Ques 3) Prove $1+3+5+\cdots+(2n-1)=n^2$; $n \in \mathbb{N}$

Sol) Let $P(n): 1+3+5+\cdots+(2n-1)=n^2$

$\boxed{\text{Basis of Induction}}$ Put $n=1$, $P(1): 1=1^2$ or $1=1$ true

$\boxed{\text{Induction Hypothesis}}$ Let $P(K)$ be true i.e.

$$1+3+5\cdots+(2K-1)=K^2 \quad \text{——— (1)}$$

$\boxed{\text{To prove}}$ $P(K+1)$ is true i.e.

$$1+3+5+\cdots+(2K+1)=(K+1)^2$$

$\boxed{\text{Induction Step}}$ $LHS = \underbrace{1+3+5+\cdots-(2K-1)}+(2K+1) \leftarrow$ NOTE THIS

$= K^2+(2K+1)$ [Using (1)]

$=(K+1)^2$

$= RHS$

Ques 4) Prove $a+ar+ar^2+\cdots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1}$, $r>1$ and $n \in \mathbb{N}$

Sol) Let $P(n): a+ar+ar^2+\cdots+ar^{n-1}=\dfrac{a(r^n-1)}{r-1}$,

$\boxed{\text{Basis of Induction}}$ $P(1)$: $ar^0=\dfrac{a\cancel{(r-1)}}{\cancel{(r-1)}}$ or $a=a$ true

$\boxed{\text{Induction Hypothesis}}$ Let $P(K)$ be true i.e.

$$a+ar+ar^2+\cdots+ar^{K-1}=\frac{a(r^K-1)}{r-1} \quad \text{——— (1)}$$

$\boxed{\text{To prove}}$ $P(K+1)$ is true:

$$a+ar+ar^2+\cdots+ar^K=\frac{a(r^{K+1}-1)}{r-1}$$

$\boxed{\text{Induction Step}}$ $LHS = \underbrace{a+ar+ar^2+\cdots+ar^{K-1}}+ar^K$

$=\dfrac{a(r^K-1)}{r-1}+ar^K$ [Using (1)]

$=a\left[\dfrac{r^K-1}{r-1}+r^K\right]$

$=a\left[\dfrac{r^K-1+(r-1)r^K}{r-1}\right]$

$=a\left[\dfrac{\cancel{r^K}-1+r\cdot r^K-\cancel{r^K}}{r-1}\right]$

$=a\left[\dfrac{r^{K+1}-1}{r-1}\right]$

$=\dfrac{a(r^{K+1}-1)}{r-1}=RHS$

Ques 5) Prove (i) $\sin(n\pi)=0$; $n \in \mathbb{N}$ (ii) $\cos(n\pi)=(-1)^n$, $n \in \mathbb{N}$

Sol) (i) $\boxed{\text{Basis of Induction}}$ Put $n=1$, $\sin\pi=0$ (true)

$\boxed{\text{Induction hypothesis}}$ $\sin K\pi=0$ ——— (1)

$\boxed{\text{To prove}}$ $\sin(K+1)\pi=0$

$\boxed{\text{Induction Step}}$ $LHS=\sin(K\pi+\pi)$

$=\sin(\pi+K\pi)$

$=-\sin K\pi$ $\boxed{\because \sin(\pi+\theta)=-\sin\theta}$

$=0=RHS$ [Using (1)]

(ii) $\boxed{\text{Basis of Induction}}$ Put $n=1$, $\cos\pi=(-1)^1=-1$ (true)

$\boxed{\text{Induction hypothesis}}$ $\cos K\pi=(-1)^K$ ——— (1)

$\boxed{\text{To prove}}$ $\cos(K+1)\pi=(-1)^{K+1}$

$\boxed{\text{Induction Step}}$ $LHS=\cos(K\pi+\pi)$

$=\cos(\pi+K\pi)$

$=-\cos K\pi$ $\boxed{\because \cos(\pi+\theta)=-\cos\theta}$

$=-(-1)^K=(-1)(-1)^K=(-1)^{K+1}=RHS$

Ques 6 Prove: $1+2+2^2+2^3+\cdots+2^{n-1}=2^n-1,\ n\in\mathbb{N}$

Sol $\boxed{\text{Basis of Induction}}$: Put $n=1$; $2^0=2^1-1$
$\Rightarrow 1=1$ (true)

Put $n=2$;
$1+2^1=2^2-1 \Rightarrow 3=3$ (true)
Put $n=3$,
$1+2+2^2=2^3-1 \Rightarrow 7=7$ (true)

Note this

$\boxed{\text{Induction Hypothesis}}$: $1+2+2^2+2^3+\cdots+2^{k-1}=2^k-1$ —— (1)

$\boxed{\text{To prove}}$: $1+2+2^2+\cdots+2^k=2^{k+1}-1$

$\boxed{\text{Induction step}}$: $LHS=\underbrace{1+2+2^2+\cdots+2^{k-1}}+2^k$

$=2^k-1+2^k$ [Using (1)]
$=2\cdot2^k-1$
$=2^{k+1}-1=RHS$

Ques 7 Prove that 5^n-1 is divisible by 4, $n\in\mathbb{N}$.

Sol Let $P(n)$: 5^n-1 is divisible by 4.

$\boxed{\text{Basis of Induction}}$: Put $n=1$;
$5^1-1=4$ (divisible by 4)
$\therefore P(1)$ is true.

$\boxed{\text{Induction hypothesis}}$: 5^k-1 is divisible by 4 i.e. $5^k-1=4m$ for some integer m
$\Rightarrow 5^k=4m+1$ —— (1)

$\boxed{\text{To prove:}}$ $5^{k+1}-1$ is divisible by 4.

$\boxed{\text{Induction Step}}$ We have $5^{k+1}-1=5^k\cdot5-1$
$=(4m+1)5-1$
$=20m+5-1$
$=20m+4$
$=4(5m+1)$ [divisible by 4]

Ques 8 Prove that $n(n+1)(n+2)$ is divisible by 3 for all natural numbers $n\in\mathbb{N}$.

Sol Let $P(n)$: $n(n+1)(n+2)$ is divisible by 3

$\boxed{\text{Basis of Induction}}$: Put $n=1$,
$1(1+1)(1+2)=6$ is divisible by 3.
$\therefore P(1)$ is true.

$\boxed{\text{Induction Hypothesis}}$: Let $P(k)$ be true i.e. $k(k+1)(k+2)$ is div. by 3
or $k(k+1)(k+2)=3m$ for some integer m. —— (1)

$\boxed{\text{To prove:}}$ $P(k+1)$ is true i.e. $(k+1)(k+2)(k+3)$ is div. by 3.

$\boxed{\text{Induction Step}}$ We have $(k+1)(k+2)(k+3)=(k+3)\left[(k+1)(k+2)\right]$
$=k\left[(k+1)(k+2)\right]+3\left[(k+1)(k+2)\right]$
$=3m+3(k+1)(k+2)$ Using (1)
$=3\left[\underbrace{m+(k+1)(k+2)}_{\text{an integer } q \text{ (say)}}\right]$
$=3q$ (divisible by 3)

Ques 9 Prove: $2n\le2^n$ for all $n\in\mathbb{N}$,

Sol $\boxed{\text{Basis of Induction}}$: Put $n=1$; $2\le2$ (true)

$\boxed{\text{Induction Hypothesis}}$: Let $2k\le2^k$ —— (1)

$\boxed{\text{To prove:}}$ $2(k+1)\le2^{k+1}$

$\boxed{\text{Induction Step}}$: From (1); $2k\le2^k$
$\Rightarrow 2k+2\le2^k+2$
$\Rightarrow 2(k+1)\le2^k+2$ —— (2)

Also, $2^K = 2^K$

$\Rightarrow 2^K + 2 \le 2^K \cdot 2$ ← NOTE THIS STEP

$\Rightarrow 2^K + 2 \le 2^{K+1}$ —— (3)

From (2) & (3),

$2(K+1) \le 2^K + 2 \le 2^{K+1}$

$\Rightarrow 2(K+1) \le 2^{K+1}$

Ques 10 Prove $1 + 2^n < 3^n$ for $n \geq 2$.

Sol $\boxed{\text{Basis of Induction}}$: Put $n = 2$; $1 + 2^2 < 3^2 \Rightarrow 5 < 9$ (true)

$\boxed{\text{Induction hypothesis}}$: Let $1 + 2^K < 3^K$ —— (1)

$\boxed{\text{To prove :}}$ $1 + 2^{K+1} < 3^{K+1}$

$\boxed{\text{Induction Step}}$: From (1), $1 + 2^K < 3^K$

$\Rightarrow 2(1 + 2^K) < 2 \cdot 3^K$

$\Rightarrow 2 + 2^{K+1} < 2 \cdot 3^K$

$\Rightarrow 1 + 2^{K+1} < 2 \cdot 3^K - 1$ —— (2)

Also, $3^K = 3^K$

$\Rightarrow 2 \cdot 3^K < 3 \cdot 3^K$ ← NOTE THIS STEP

$\Rightarrow 2 \cdot 3^K < 3^{K+1}$

$\Rightarrow 2 \cdot 3^K - 1 < 3^{K+1}$ —— (3)

From (2) & (3); $1 + 2^{K+1} < 2 \cdot 3^K - 1 < 3^{K+1}$

$\Rightarrow 1 + 2^{K+1} < 3^{K+1}$

Ques 11 Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$; $n \in \mathbb{N}$

Sol Let $P(n) : 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$\boxed{\text{Basis of Induction}}$: Put $n = 1$; $1^3 = \left[\frac{1(1+1)}{2}\right]^2 \Rightarrow 1 = 1$ (true)

$\therefore P(1)$ is true.

$\boxed{\text{Induction hypothesis}}$: Let $P(K)$ be true i.e.

$1^3 + 2^3 + 3^3 + \cdots + K^3 = \left[\frac{K(K+1)}{2}\right]^2$ —— (1)

$\boxed{\text{To prove :}}$ $P(K+1)$ is true i.e.

$1^3 + 2^3 + 3^3 + \cdots + (K+1)^3 = \left[\frac{(K+1)(K+2)}{2}\right]^2$

$\boxed{\text{Induction step}}$: $LHS = \underbrace{1^3 + 2^3 + 3^3 + \cdots + K^3} + (K+1)^3$

$= \left[\frac{K(K+1)}{2}\right]^2 + (K+1)^3$ [using (1)]

$= (K+1)^2 \left[\frac{K^2}{4} + (K+1)\right]$

$= (K+1)^2 \left[\frac{K^2 + 4K + 4}{4}\right]$

$= (K+1)^2 \frac{(K+2)^2}{4}$

$= \left[\frac{(K+1)(K+2)}{2}\right]^2 = RHS.$

Ques 12 Prove : $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$, $n \in \mathbb{N}$

Sol Let $P(n) : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

$\boxed{\text{Basis of Induction}}$: Put $n = 1$, $\frac{1}{2} = 1 - \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{2}$ (true)

Put $n = 2$, $\frac{1}{2} + \frac{1}{2^2} = 1 - \frac{1}{2^2}$

$\Rightarrow \frac{3}{4} = \frac{3}{4}$ (true)

← NOTE THIS

$\therefore P(1)$ is true

Induction Hypothesis : Let $P(K)$ be true :

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^K}=1-\frac{1}{2^K} \quad \text{——— (1)}$$

To prove : $P(K+1)$ is true i.e. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{K+1}}=1-\frac{1}{2^{K+1}}$

Induction Step : $LHS = \underbrace{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^K}}+\frac{1}{2^{K+1}}$

$$= \left(1-\frac{1}{2^K}\right)+\frac{1}{2^{K+1}} \qquad [\text{using } (1)]$$

$$= 1-\frac{1}{2^K}+\frac{1}{2^{K+1}}$$

$$= 1-\frac{1}{2^K}\left(1-\frac{1}{2}\right)$$

$$= 1-\frac{1}{2^K}\cdot\frac{1}{2}$$

$$= 1-\frac{1}{2^{K+1}} = RHS.$$

Ques 13 Prove $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1).(3n+2)}=\frac{n}{6n+4}$, $n\in\mathbb{N}$.

Sol **Basis of Induction** : Put $n=1$,

$$\frac{1}{2.5}=\frac{1}{6+4}$$

$$\Rightarrow \frac{1}{10}=\frac{1}{10} \quad (\text{true})$$

Induction Hypothesis : $\frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3K-1)(3K+2)}=\frac{K}{6K+4}$ ——— (1)

To prove: $\frac{1}{2.5}+\frac{1}{5.8}+\cdots\cdots+\frac{1}{(3K+2)(3K+5)}=\frac{K+1}{6K+10}$

Induction Step : $LHS = \underbrace{\frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3K-1)(3K+2)}}+\frac{1}{(3K+2)(3K+5)}$

$$= \frac{K}{6K+4}+\frac{1}{(3K+2)(3K+5)} \qquad [\text{using } (1)]$$

$$= \frac{1}{(3K+2)}\left[\frac{K}{2}+\frac{1}{(3K+5)}\right]$$

$$= \frac{1}{(3K+2)}\left[\frac{3K^2+5K+2}{2(3K+5)}\right]$$

$$= \frac{1}{(3K+2)}\left[\frac{3K^2+3K+2K+2}{2(3K+5)}\right]$$

$$= \frac{1}{\cancel{(3K+2)}}\left[\frac{\cancel{(3K+2)}(K+1)}{2(3K+5)}\right]$$

$$= \frac{K+1}{6K+10} = RHS$$

Ques 14 Prove that $3^{2n+2}-8n-9$ is divisible by 8, $n\in\mathbb{N}$.

Sol Let $P(n)$: $3^{2n+2}-8n-9$ is divisible by 8

Basis of Induction : Put $n=1$; $3^4-8-9=64$ (div. by 8)
$\therefore P(1)$ is true.

Induction Hypothesis : Let $P(K)$ be true i.e. $3^{2K+2}-8K-9$ is div. by 8
or, $3^{2K+2}-8K-9=8m$ for some integer 'm'
$\Rightarrow 3^{2K+2}=8m+8K+9$ ——— (1)

To prove : $P(K+1)$ is true i.e. $3^{2K+4}-8(K+1)-9$ is div. by 8.

$\boxed{\text{Induction Step}}$: $3^{2k+4} - 8(k+1) - 9 = 3^{2k+2} \cdot 3^2 - 8(k+1) - 9$

$= (8m + 8k + 9)3^2 - 8(k+1) - 9$

$= 72m + 72k + 81 - 8(k+1) - 9$

$= 72m + 72k - 8(k+1) + 72$

$= 8[\underbrace{9m + 9k - (k+1) + 9}_{\text{an integer } q \text{ (say)}}]$

$= 8q$ (divisible by 8)

Ques 15 Prove : $n < 2^n$ for all $n \in \mathbb{N}$.

Sol $\boxed{\text{Basis of Induction}}$; Put $n = 1$, $1 < 2^1$ (true)

$\boxed{\text{Induction Hypothesis}}$; Let $k < 2^k$ —— (1)

$\boxed{\text{To prove :}}$ $(k+1) < 2^{k+1}$

$\boxed{\text{Induction Step}}$: From (1); $k < 2^k$

Add 1 to both sides ; $k+1 < 2^k + 1$ —— (2)

Since, $2^k = 2^k$

$\Rightarrow 2^k + 1 < 2^k \cdot 2$ NOTE THIS STEP

$\Rightarrow 2^k + 1 < 2^{k+1}$ —— (3)

From (2) & (3);

$(k+1) < 2^k + 1 < 2^{k+1}$

$\Rightarrow (k+1) < 2^{k+1}$