VENN DIAGRAM RELATIONS & FUNCTIONS

PRELIMINARIES

SET A set is a well-defined collection of objects.

Examples- ① a set of vowels in English alphabets $P = \{a, e, i, o, u\}$

Here $a \in P$ and $f \notin P$, etc.

② set of even positive integers $E = \{2, 4, 6, ----\}$

OR

$E = \{x \mid x \text{ is a positive even integer}\}$

③ $A = \{1, 4, 9, 16, 25\}$

OR

$A = \{x^2 \mid x \text{ is an integer from 1 to 5}\}$

SUBSETS If for each element $x \in A$; $x \in B$ holds then A is a subset of B. or B contains A or A is contained in B. We write

$A \subseteq B$

Examples – ① $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ then $A \subseteq B$,

More precisely $A \subset B$ i.e. A is proper subset of B.

② $E = \{2, 4, 6, 8, ---\}$ and $\mathbb{N} = \{1, 2, 3, 4, ---\}$ then $E \subseteq \mathbb{N}$ and $E \subset \mathbb{N}$

③ From every set A, $A \subseteq A$ but $A \subset A$ is false.

UNIVERSAL SET AND EMPTY SET

Any set A under investigation in set theory belongs to a large set called as universal set U.

Examples – ① If A = set of equilateral Δ's then U = set of all Δ's

② If $A = \mathbb{N}$ (natural numbers) then $U = \mathbb{R}$ (real numbers)

A set is empty if it has no elements. It is denoted by $\emptyset$ or $\{\ \}$.

Example – ⓐ Set of all real numbers whose square is negative

i.e. $\{x \in \mathbb{R} \mid x^2 < 0\}$ then the set is $\emptyset$.

ⓑ $A = \{x \in \mathbb{R} \mid x^2 + 1 = 0\}$ is an empty set $\emptyset$.

REMARKS -

① A^c is complement set of set A i.e. $A^c = U - A$

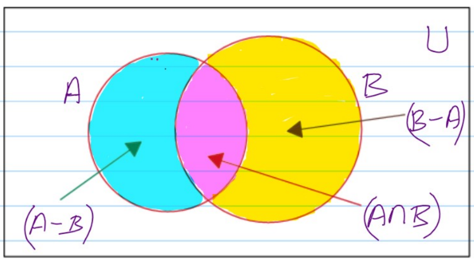
② $U^c = \emptyset$ and $\emptyset^c = U$

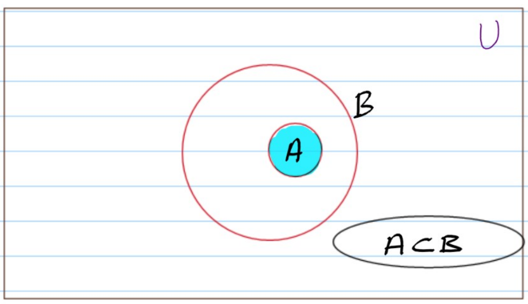
③ The sets $A = \{1, -1\}$ and $B = \{x \mid x^2 - 1 = 0\}$ are equal.

④ Sets $A = \{1, 2, 2\}$ and $B = \{2, 1\}$ are same.

⑤ $\mathbb{N} \subseteq \mathbb{N}_{\cup\{0\}} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

VENN DIAGRAM A pictorial representation of a set is Venn diagram.





VALIDITY OF AN ARGUMENT

An argument consists of a set of premises (assumptions / hypothesis) and a conclusion (deduction)

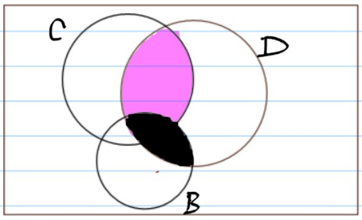
Example –

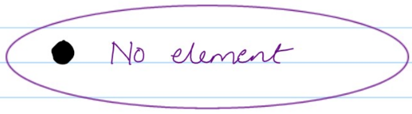
Some cats are dogs.
No dog is blue. } (Premises)

∴ No cat is blue. ⟶ Conclusion

Argument

(i) "No dog is blue"
Hence, Black out the region $D \cap B$ i.e. $D \cap B = \emptyset$.
(ii) "Some cats are dogs"
i.e. there is a non-empty intersection between sets C & D.



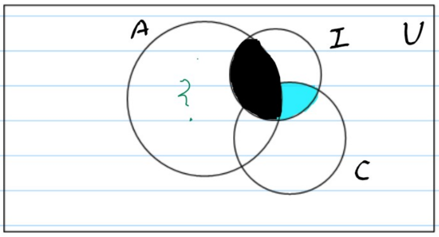


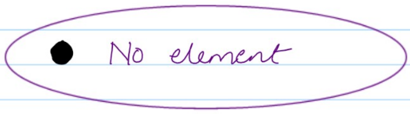
Now the conclusion "No cat is blue" i.e. $B \cap C = \emptyset$ which is false from Venn-diagram as the region $B \cap C$ is not blackened out.

Ques 1 Check the validity of argument:

A1: No addictive things are inexpensive
A2: Some cigarettes are inexpensive.
∴ Some addictive things are not cigarettes.

Sol

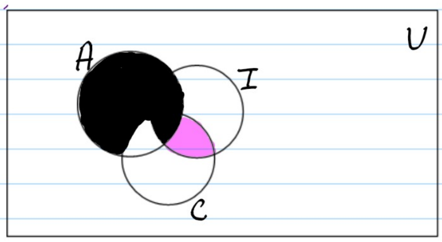




A1: No addictive things are inexpensive. $\Rightarrow A \cap I = \emptyset$
Color the region black

A2: Some cigarettes are inexpensive $\Longrightarrow C \cap I \neq \emptyset$
Color the region with pink

Thus, the conclusion, "Some addictive things are not cigarettes" or $A - C \neq \emptyset$ is invalid from the Venn-diagram due to the absence of any color in the region (?) $(A-C)$.
A possibility is this:

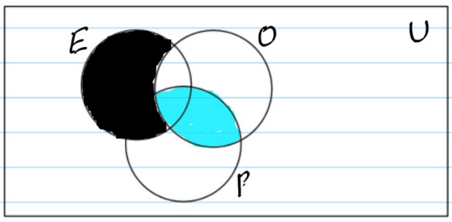


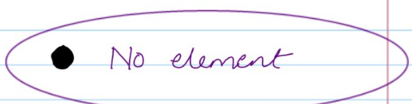
Ques 2 Check the validity of argument:

A1: Every even number is odd number
A2: Some odd numbers are prime
Thus, some even numbers are prime.

Sol A1: Every even number is odd
i.e. $E - O = \emptyset$
Color the region black.

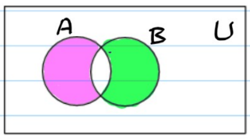
A2: Some odd numbers are prime.
i.e. $O \cap P \neq \emptyset$
Color the region pink.





Thus, the conclusion, "Some even numbers are prime" is valid due to the presence of pink color in a part of $E \cap P$ which shows $E \cap P \neq \emptyset$.

REMARKS (I) The Symmetric difference of sets A & B is $A \oplus B = (A - B) \cup (B - A)$
OR
$A \oplus B = (A \cup B) - (A \cap B)$



Venn Diagram for $A \oplus B$

② For any finite set A, $n(A)$ or $|A|$ denotes the number of elements in A.

③ For two sets A and B,

$$n(A\cup B) = n(A) + n(B) - n(A\cap B) \text{ (Inclusion - Exclusion principle)}$$

④ For three sets A, B and C,

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(C\cap A) + n(A\cap B\cap C)$$

Ques 1 In a class of 40 students; 32 have Mathematics, 18 have English. Find the number of students who have (i) both subjects (ii) Mathematics only (iii) English only.

Sol Here $n(M) = 32$, $n(E) = 18$, $n(M\cup E) = 40$

(i) $\because n(M\cup E) = n(M) + n(E) - n(M\cap E)$

$\Rightarrow 40 = 32 + 18 - n(M\cap E)$

$\Rightarrow n(M\cap E) = 10$

$\therefore$ 10 students have both subjects

(ii) $n(M-E) = n(M) - n(M\cap E)$

$= 32 - 10$

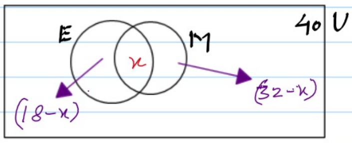
$= 22$

(iii) $n(E-M) = n(E) - n(M\cap E)$

$= 18 - 10$

$= 8$

OR



Let $n(E\cap M) = x$

Then $n(E-M) = 18 - x$ and $n(M-E) = 32 - x$

From Venn-diagram; $(18-x) + x + (32-x) = 40$

$\Rightarrow 50 - x = 40$

$\Rightarrow x = 10$

(i) $n(E\cap M) = 10$

(ii) From Venn-diagram; $n(M-E) = 32 - x = 32 - 10 = 22$

(iii) From Venn-diagram; $n(E-M) = 18 - x = 18 - 10 = 8$

RELATION

Consider two sets $A = \{b, c\}$ and $B = \{2, 3, 6\}$
The cartesian product of A and B is given by:
$$A \times B = \{(b,2), (b,3), (b,6), (c,2), (c,3), (c,6)\}$$
where in each ordered pair of the form $(x,y) \in A \times B$, we have $x \in A$ & $y \in B$.
Thus,
$$\boxed{A \times B = \{(x,y) \mid x \in A, y \in B\}}$$ (Cartesian product of sets A & B)

A **relation** R from set A to B is a subset of $A \times B$ i.e. $R \subseteq A \times B$
For $A = \{b, c\}$ and $B = \{2, 3, 6\}$
Suppose R is the relation from A to B defined by " is n^{th} alphabet", then
$R = \{(b,2), (c,3)\}$
Here $(b,2) \in R$ or $bR2$ but $(b,6) \notin R$ or $b \not{R} 6$.
Also, $dom(R) = \{b, c\}$ and $ran(R) = \{2, 3\}$
The **domain** of a relation R from set A to B is the set of all first elements of the ordered pairs $(x,y) \in R$, i.e.
$$\boxed{dom(R) = \{x \mid (x,y) \in R\}}$$
Similarly, **range** of R is:
$$\boxed{ran(R) = \{y \mid (x,y) \in R\}}$$

REMARKS –
1. The relation $R = A \times B$ is called universal relation from A to B.
2. Since empty set is always a subset of $A \times B$ hence ϕ is null relation from A to B.
3. $n(A) = p$ & $n(B) = q$ then $n(A \times B) = pq$. Then there are 2^{pq} relations possible from A to B (and B to A as well).

★ ∵ For a set A having 'n' elements, there are 2^n subsets possible for A.
e.g. $A = \{1, 2\}$ has 2 elements.
Then $2^2 = 4$ subsets of A are:
ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$

4. The inverse of a relation R from A to B is defined as:
$$\boxed{R^{-1} = \{(y,x) \mid (x,y) \in R\}}$$
is relation from B to A.
e.g. for $A = \{b, c\}$ and $B = \{2, 3, 6\}$
If $R = \{(b,2), (c,3)\}$
Then $R^{-1} = \{(2,b), (3,c)\}$ from B to A.
Also $dom(R) = \{b, c\} = ran(R^{-1})$
& $ran(R) = \{2, 3\} = dom(R^{-1})$

5. The complement relation R^c from A to B is defined as:
$R^c = (A \times B) - R$ [∵ $X^c = U - X$]
Equivalently, $$\boxed{R^c = \{(x,y) \mid (x,y) \notin R\}}$$

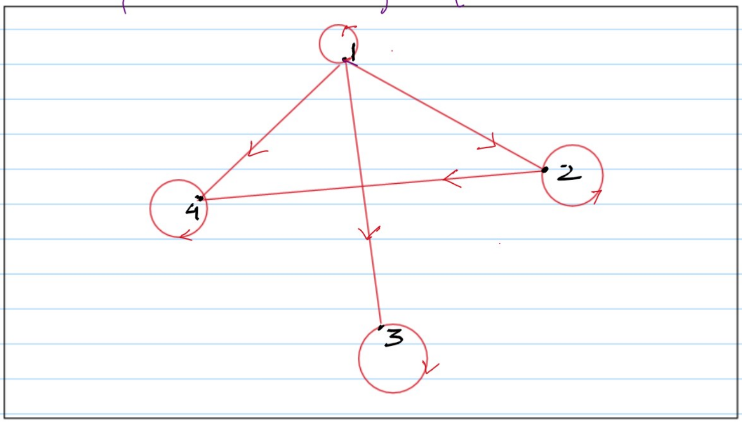
6. If $A = B$ then R is a relation from set A to itself. We say, R is a relation on set A.
7. For a given set A, the identity relation on set A is defined as $I_A = \{(x,x) \mid x \in A\}$
e.g. for $A = \{\alpha, \beta, \gamma\}$, we have $I_A = \{(\alpha,\alpha), (\beta,\beta), (\gamma,\gamma)\}$
Note that $(\alpha,\beta) \notin I_A$, $(\beta,\gamma) \notin I_A$, etc.

PICTORIAL REPRESENTATION OF A RELATION

Consider a relation R on a finite set A defined as "xRy iff $x|y$ (x divides y)" where $x, y \in A$.

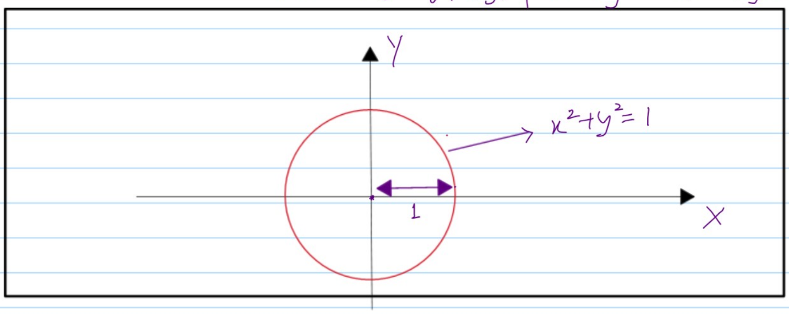
$$A = \{1, 2, 3, 4\}$$
$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

The pictorial representation of R is:



ⓐ For a pair (x,x), there is a self-loop at vertex 'x'.
ⓑ For a pair (x,y), there is a directed edge from vertex 'x' to 'y'.
This is called as **directed graph** of a relation R.

For an infinite set like $\mathbb{R}$, a relation R on $\mathbb{R}$ is defined as: "xRy iff $x^2+y^2=1$"
i.e. $R = \{(x,y) \mid x^2+y^2=1,\ x, y \in \mathbb{R}\}$

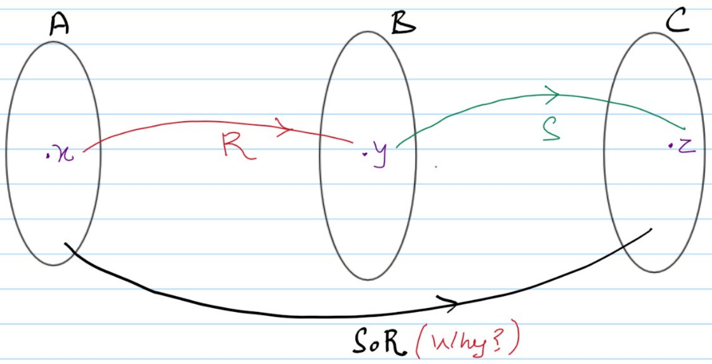


This is called as **graph** of a relation on an infinite set

COMPOSITION OF RELATIONS

Suppose R is a relation from set A to B and S is a relation from set B to C. Then the composition of relations R and S is defined as:

$$S \circ R = \{(x,z) \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in B\}$$



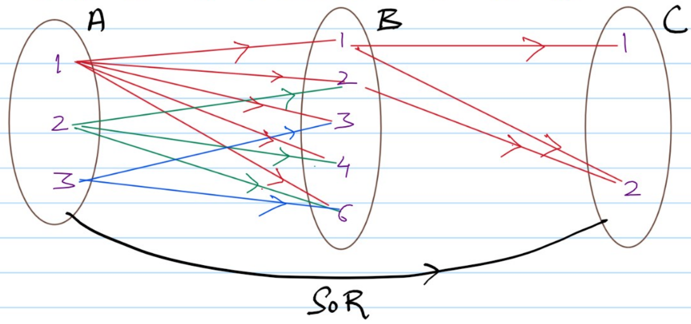
Note that $S \circ R$ is a relation from set A to C.

Example - Let R be the relation of 'divides' from set A to B and S be the relation of '$\leq$' from set B to C
where $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 6\}$ & $C = \{1, 2\}$

Then $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6)\}$

$S = \{(1,1),(1,2),(2,2)\}$

Then $S \circ R$ is a relation A to C.



$\therefore S \circ R = \{(1,1),\ (1,2),\ (2,2)\}$. (from above picture).

TYPES OF RELATION

Let R be a relation on a non-empty set A.
Suppose $x, y, z \in A$.
The relation R on set A is:

① REFLEXIVE – If for all $x \in A$, we have $(x,x) \in R$.
e.g. for $A = \{1,2,3\}$
$R = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$, is reflexive.

② IRREFLEXIVE – If for all $x \in A$, we have $(x,x) \notin R$
e.g. for $A = \{1,2,3\}$
$R = \{(1,2),(2,3)\}$ is irreflexive.

③ NON-REFLEXIVE – If it is neither reflexive nor irreflexive.
i.e. $(x,x) \in R$ for some $x \in A$ but not for all $x \in A$.
e.g. for $A = \{1,2,3\}$
$R = \{(1,1),(2,2),(1,2),(2,3)\}$ is non-reflexive.

④ SYMMETRIC – If $(x,y) \in R$ then $(y,x) \in R$
e.g. for $A = \{1,2,3\}$
$R = \{(1,2),(2,1),(1,3),(3,1)\}$ is symmetric

⑤ ASYMMETRIC – [If $(x,y) \in R$ then $(y,x) \notin R$] and [$(x,x) \notin R$ for any $x \in A$]
e.g. for $A = \{1,2,3\}$
$R = \{(1,2),(1,3),(2,3)\}$ is asymmetric.

⑥ ANTISYMMETRIC – $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$
e.g. for $A = \{1,2,3\}$
$R = \{(1,2),(1,1),(2,3)\}$ is antisymmetric.

★ Alternatively, if $(x,y) \in R$ $(x \neq y)$ then $(y,x) \notin R$

⑦ TRANSITIVE – If $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$.
e.g. for $A = \{1,2,3\}$
$R = \{(1,2),(2,3),(1,3)\}$ is transitive

CLASSIFICATION WITH THE HELP OF DIRECTED GRAPH

Let G be the directed graph of relation R on set A. Then R is:

① REFLEXIVE – If there is a self-loop at each vertex of graph then R is reflexive.

② IRREFLEXIVE – If there is no self-loop at any vertex of graph then R is irreflexive.

③ NON-REFLEXIVE – If some (but not all) loops are present then R is non-reflexive.

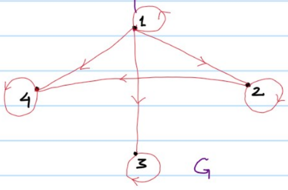
④ SYMMETRIC – If for every forward directed arrow from 'x' to 'y' (say), there should be a backward directed arrow from 'y' to 'x', then R is symmetric.

⑤ ASYMMETRIC – If for every forward directed arrow from 'x' to 'y' (say), the backward directed arrow from 'y' to 'x' is absent plus no self loops then R is asymmetric.

★ ⑥ ANTISYMMETRIC – If there is atmost one directed arrow between every pair of vertices then R is antisymmetric.

⑦ $\boxed{\text{TRANSITIVE}}$ — If for every edge from 'x' to 'y' and 'y' to 'z', then there is a direct edge from 'x' to 'z', then R is transitive.

For example — $A = \{1, 2, 3, 4\}$
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
The pictorial representation of R is :



R is reflexive, antisymmetric and transitive (Why?)
$\because$ $(1,2) \in R$ and $(2,4) \in R \Rightarrow (1,4) \in R$
& $(1,1) \in R$ and $(1,2) \in R \Rightarrow (1,2) \in R$
& $(1,3) \in R$ and $(3,3) \in R \Rightarrow (1,3) \in R$

ADJACENCY MATRIX FOR A RELATION

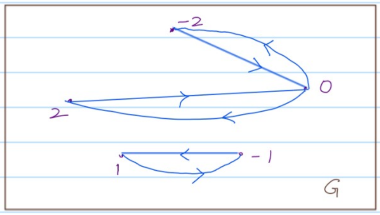
Consider a set $A = \{1, 2, 3\}$ with a relation $R = \{(1,1), (1,3), (2,3)\}$. The adjacency matrix is given by

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

- If M has entry 1 at all principal diagonal positions then R is reflexive.
- If M is symmetric matrix i.e. $M^T = M$ then R is symmetric
- If in $M = [m_{ij}]$, R is transitive if $m_{ij} = m_{jk} = 1 \Rightarrow m_{ik} = 1$.

Ques 1) Consider the relation R defined by:
"xRy iff $|x-y| = 2$"
on set $A = \{-2, -1, 0, 1, 2\}$. Classify R into reflexive, irreflexive, non-reflexive, symmetric, asymmetric, antisymmetric and transitive.

Sol) Here $R = \{(-2,0), (-1,1), (0,2), (1,-1), (0,-2), (2,0)\}$
ⓐ R is irreflexive.
ⓑ R is symmetric
ⓒ R is not transitive
b'coz $(-2,0) \in R$ & $(0,2) \in R$ but $(-2,2) \notin R$.
ⓓ R is not antisymmetric by directed graph:



Ques 2) Consider a relation $R = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1), (3,3)\}$ on set $A = \{1,2,3\}$. Find the adjacency matrix M and comment on the properties of R.

Sol)
$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

① $\because m_{ii} = 1$ everywhere thus R is reflexive
② $\because M^T = M$ thus R is symmetric.
③ $\because m_{31} = 1$ & $m_{12} = 1$ but $m_{32} = 0 \neq 1$, $\therefore$ R is not transitive.

EQUIVALENCE RELATION

A relation R on set A is an $\boxed{\text{equivalence relation}}$ if it is reflexive, symmetric and transitive.

★ The relation "xRy iff $x|y$" (where $x|y$ stands for x divides y) on set $\mathbb{N}$ is not an equivalence relation because
- $x|x \;\forall x \in \mathbb{N} \Rightarrow xRx \;\forall x \in \mathbb{N}$ (reflexive)
- If $x|y$ then $y \nmid x$ (not symmetric)
- If $x|y$ & $y|z$ then $x|z$ (transitive)

- $\forall$: for all
- $\exists$: there exists

★ The relation "xRy iff $x = y$" on set $\mathbb{R}$ is an equivalence relation b'coz
- $x = x \;\forall x \in \mathbb{R}$ i.e. $xRx \;\forall x \in \mathbb{R}$ (reflexive)
- If xRy i.e. $x = y \Rightarrow y = x$ i.e. yRx (symmetric).
- If xRy & yRz i.e. $x = y$ & $y = z \Rightarrow x = z$ i.e. xRz (Transitive)

★ The relation "xRy iff $m|(x-y)$ (where $m \in \mathbb{N}$)" is an equivalence relation on set $\mathbb{Z}$ i.e. $x, y \in \mathbb{Z}$, because
- Since $m|(x-x)$ i.e. m divides $0 \Rightarrow xRx \;\forall x \in \mathbb{Z}$ (reflexive)
- Let xRy i.e. $m|(x-y)$
$\Rightarrow m|-(x-y)$ (Why?)
$\Rightarrow m|(y-x)$

$\Rightarrow yRx$ (symmetric)

• Let xRy & $yRz \Longrightarrow m|(x-y)$ & $m|(y-z)$
$\Rightarrow m|(x-y)+(y-z)$ (Why?)
$\Rightarrow m|x-z)$
$\Rightarrow xRz$ (transitive)

Ques 1) Consider a relation R defined as
"xRy iff $x^y = y^x$"
on the set $\mathbb{R}^+$ (positive real numbers). Prove that R is an equivalence relation.

Sol
◆ Reflexive: $\because x^x = x^x$ holds for all $x \in \mathbb{R}^+$
$\therefore xRx \ \forall x \in \mathbb{R}^+$
so, R is reflexive.

◆ Symmetric: Let $xRy \Longrightarrow x^y = y^x$
$\Rightarrow y^x = x^y$
$\Rightarrow yRx$
so, R is symmetric.

◆ Transitive: Let xRy & $yRz \Longrightarrow x^y = y^x$ & $y^z = z^y$
Taking log on both sides: $y \log x = x \log y$ — (1)
& $z \log y = y \log z$ — (2)

(1) X (2) ; $\cancel{y} z \log x \cdot \cancel{\log y} = x \cancel{y} \cancel{\log y} \log z$

$\Rightarrow z \log x = x \log z$
$\Rightarrow \log x^z = \log z^x$
$\Rightarrow x^z = z^x$
$\Rightarrow xRz$, so R is transitive

Thus, R is an equivalence relation.

REMARK Consider the set $\mathbb{N}$ (two copies) i.e. $\mathbb{N} = \{1,2,3,---\}$, $\mathbb{N} = \{1,2,3,---\}$
Then the Cartesian product $\mathbb{N} \times \mathbb{N} = \begin{Bmatrix} (1,1), (1,2), (1,3), --- \\ (2,1), (2,2), (2,3), --- \\ (3,1), (3,2), (3,3), --- \\ ------- \end{Bmatrix}$

An element of $\mathbb{N} \times \mathbb{N}$ is a 2-tuple (or ordered pair) (a,b).

Ques 2) Consider a relation R on $\mathbb{N} \times \mathbb{N}$ defined as:
$(a,b) R (c,d)$ iff $a+d = b+c$;
Prove that R is an equivalence relation.

Sol
• Reflexive: $(a,b) R (a,b) \Longrightarrow a+b = b+a$ true $\forall (a,b) \in \mathbb{N} \times \mathbb{N}$
• Symmetric: Let $(a,b) R (c,d) \Longrightarrow a+d = b+c$
$\Rightarrow b+c = a+d$
$\Rightarrow c+b = d+a$
$\Rightarrow (c,d) R (a,b)$
• Transitive: Let $(a,b) R (c,d)$ & $(c,d) R (e,f)$
$\Rightarrow a+d = b+c$ — (1) & $c+f = d+e$ — (2)
(1) + (2) ; $(a+\cancel{d}) + (\cancel{c}+f) = (b+\cancel{c}) + (\cancel{d}+e)$
$\Rightarrow a+f = b+e$
$\Rightarrow (a,b) R (e,f)$
$\therefore$ R is an equivalence relation of $\mathbb{N} \times \mathbb{N}$

Ques 3) Consider a relation R on $\mathbb{N} \times \mathbb{N}$ defined as:
$(a,b) R (c,d)$ iff $ad = bc$;
Prove that R is an equivalence relation.

Sol
• Reflexive: $(a,b) R (a,b) \Longrightarrow ab = ba$ true $\forall (a,b) \in \mathbb{N} \times \mathbb{N}$
• Symmetric: Let $(a,b) R (c,d) \Longrightarrow ad = bc$
$\Rightarrow bc = ad$
$\Rightarrow cb = da$
$\Rightarrow (c,d) R (a,b)$

- Transitive : Let $(a,b)R(c,d)$ & $(c,d)R(e,f)$

$$\Rightarrow ad = bc \quad \text{—— (1)} \quad \& \quad cf = de \quad \text{—— (2)}$$

(1) × (2) ;

$$a\cancel{d} \times \cancel{c}f = b\cancel{c} \times \cancel{d}e$$
$$\Rightarrow af = be$$
$$\Rightarrow (a,b)R(e,f)$$

$\therefore$ R is an equivalence relation of $\mathbb{N} \times \mathbb{N}$

Ques 4) Consider a relation R on $\mathbb{N} \times \mathbb{N}$ defined as:

$$(a,b)R(c,d) \text{ iff } ad(b+c) = bc(a+d)$$

Prove that R is an equivalence relation.

Sol)
- Reflexive : $(a,b)R(a,b) \Rightarrow ab(b+a) = ba(a+b)$ true $\forall (a,b) \in \mathbb{N} \times \mathbb{N}$
- Symmetric : Let $(a,b)R(c,d) \Rightarrow ad(b+c) = bc(a+d)$

$$\Rightarrow bc(a+d) = ad(b+c)$$
$$\Rightarrow cb(d+a) = da(c+b)$$
$$\Rightarrow (c,d)R(a,b)$$

- Transitive : Let $(a,b)R(c,d)$ & $(c,d)R(e,f)$

$$\Rightarrow ad(b+c) = bc(a+d) \quad \& \quad cf(d+e) = de(c+f)$$
$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \quad \& \quad \frac{d+e}{de} = \frac{c+f}{cf}$$
$$\Rightarrow \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a} \quad \text{—— (1)} \quad \& \quad \frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c} \quad \text{—— (2)}$$

(1) + (2) ;

$$\left(\cancel{\frac{1}{c}} + \frac{1}{b}\right) + \left(\frac{1}{e} + \cancel{\frac{1}{d}}\right) = \left(\cancel{\frac{1}{d}} + \frac{1}{a}\right) + \left(\frac{1}{f} + \cancel{\frac{1}{c}}\right)$$
$$\Rightarrow \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$$
$$\Rightarrow \frac{e+b}{be} = \frac{f+a}{af}$$
$$\Rightarrow af(e+b) = be(f+a)$$
$$\Rightarrow af(b+e) = be(a+f)$$
$$\Rightarrow (a,b)R(e,f)$$

$\boxed{\text{REMARK}}$ The relation of 'divides' on set $\mathbb{N}$ is antisymmetric while on set $\mathbb{Z}$ it is not antisymmetric.

$\boxed{\text{REASON}}$ — On set $\mathbb{N}$; $a|b$ & $b|a \Rightarrow a = b$ only

$\therefore$ '|' is antisymmetric on $\mathbb{N}$.

While on set $\mathbb{Z}$; $2|(-2)$ & $-2|2$ but $2 \neq -2$

$\therefore$ '|' is not antisymmetric on $\mathbb{Z}$

PARTIAL ORDER RELATION

A relation R on a set A is called $\boxed{\text{partial order relation}}$ (or partial ordering) if R is reflexive, antisymmetric and transitive.

★ The relation of '$\leq$' (less than or equal to) is a partial order relation on $\mathbb{Z}$, because
- $x \leq x \quad \forall x \in \mathbb{Z}$ (reflexive)
- $x \leq y$ & $y \leq x \Rightarrow x = y$ (antisymmetric)
- If $x \leq y$ & $y \leq z$ then $x \leq z$ (transitive)

★ The relation of '|' on set $\mathbb{N}$ is partial order relation, because
- $x|x \quad \forall x \in \mathbb{Z}$ (reflexive)
- $x|y$ & $y|x \Rightarrow x = y$ (antisymmetric)
- If $x|y$ & $y|z \Rightarrow x|z$ (transitive)

$\boxed{\text{REMARK}}$ For a given set $X = \{a, b, c\}$ (say), we can construct a set of all subsets of X, which is called as power set $P(X)$,

$$P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\} = X\}$$

Note that the number of elements in $P(X)$ is 2^n, where set X has 'n' distinct elements.

Ques 1) Prove that the relation '$\subseteq$' (set inclusion) is a partial order relation on power set $P(X)$ for any set X.

Sol) Let $X_1, X_2, X_3 \in P(X)$
- Since $X_1 \subseteq X_1$ is true for all $X_1 \in P(X)$. So reflexive

- $X_1 \subseteq X_2$ & $X_2 \subseteq X_1 \Longrightarrow X_1 = X_2$. So antisymmetric
- Let $X_1 \subseteq X_2$ & $X_2 \subseteq X_3 \Longrightarrow X_1 \subseteq X_3$. So transitive

  Thus '$\subseteq$' is a partial order relation on $P(X)$.

FUNCTION

A function 'f' from set A to set B is a relation satisfying the following properties:

- Each element of A must be associated to some element of B.
- No element of A must be associated to two or more elements of B.

Example – Let $A = \{a, b, c\}$ & $B = \{1, 2, 3\}$. Then consider the following relations from A to B.

$R_1 = \{(b,2), (c,3)\}$ (not a function)

$R_2 = \{(a,1), (b,3), (b,2)\}$ (not a function)

$R_3 = \{(a,3), (b,1), (c,3)\}$ (a function)

$R_4 = \{(a,2), (b,2), (c,2)\}$ (a function)

$R_5 = \{(a,3), (b,1), (c,2)\}$ (a function)

Here $\mathrm{dom}(R_3) = \{a,b,c\} = A$ & $\mathrm{ran}(R_3) = \{1,3\} \subseteq B$ $(\mathrm{codom}(R_3))$

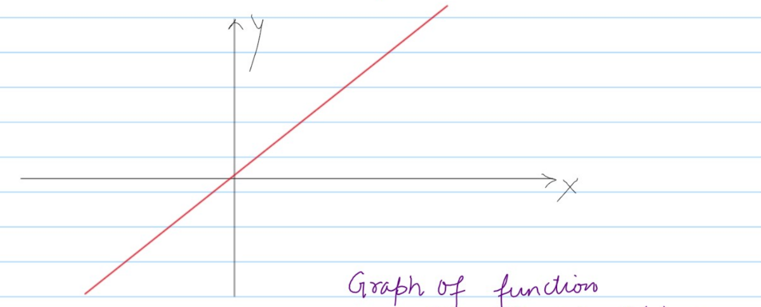
& $\mathrm{dom}(R_4) = \{a,b,c\} = A$ & $\mathrm{ran}(R_4) = \{2\} \subseteq B$ $(\mathrm{codom}(R_4))$

& $\mathrm{dom}(R_5) = A$ & $\mathrm{ran}(R_5) = B$ $(\mathrm{codom}(R_5))$

REMARKS

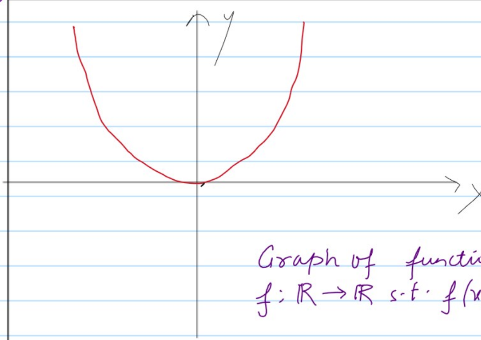
(a) For function $f: A \to B$, $\mathrm{dom}(f) = A$, $\mathrm{codom}(f) = B$ and $\boxed{\mathrm{ran}(f) \subseteq \mathrm{codom}(f)}$

(b) If each element of domain has a distinct image in codomain B then $f: A \to B$ is one-one (injective); otherwise many-one.



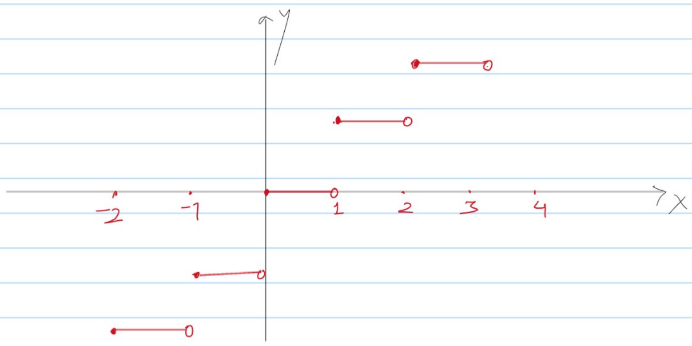
Graph of function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x$

★Each horizontal line cuts the graph (red) at atmost 1 point. Hence one-one (HORIZONTAL LINE TEST)



Graph of function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^2$

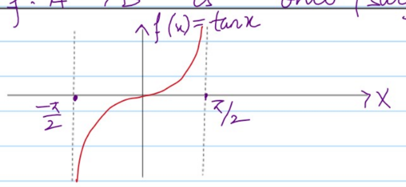
The graph is not one-one by the horizontal line test (Hence many-one)



$f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = [x]$ (greatest integer function)

(Many-one) (Why?)

(c) If each element of codomain B has a pre-image in domain A then $f: A \to B$ is onto (surjective); otherwise into.



★Each horizontal line cuts the graph (red) in at least 1 point.

(d) If $f: A \to B$ is both one-one and onto then f is called as bijection.
Example: $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x, x^3, x^5$, etc.; $g: (-\pi/2, \pi/2) \to \mathbb{R}$ s.t. $f(x) = \tan x$.

(e) If $f: A \to B$ is a bijection then f is invertible i.e. $f^{-1}: B \to A$ exists which is also a bijection.

(f) If $f: A \to B$ and $g: B \to C$ then the composition of f and g is given by $gof: A \to C$ defined by $gof(x) = g(f(x)) \ \forall x \in A$.

(g) $(fog)^{-1} = g^{-1} o f^{-1}$

★ Let 'g' puts socks & 'f' puts shoes, fog is applied in morning. Then $g^{-1} o f^{-1}$ is applied in evening.

Ques 1 Consider $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x^2 + 1$ and $g: \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = 2x + 3$. Find fog and gof.

Sol. Clearly $fog: \mathbb{R} \to \mathbb{R}$. $\because gof: \mathbb{R} \to \mathbb{R}$

$$fog(x) = f(g(x)) = f(\underbrace{2x+3}_{y \text{ (say)}}) = (2x+3)^2 + 1 = 4x^2 + 12x + 10$$

&

$$gof(x) = g(f(x)) = g(\underbrace{x^2+1}_{y \text{ (say)}}) = 2(x^2+1) + 3 = 2x^2 + 5$$

Ques 2 Prove that the function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 3x + 5$ is invertible. Also find the inverse function.

Sol $\boxed{f \text{ is one-one:}}$ Suppose $f(x_1) = f(x_2)$

$$\Rightarrow 3x_1 + 5 = 3x_2 + 5$$
$$\Rightarrow 3x_1 = 3x_2$$
$$\Rightarrow x_1 = x_2$$

$\boxed{f \text{ is onto:}}$ Let $f(x) = y$, where $y \in \mathbb{R}$ (codomain)

$$\Rightarrow 3x + 5 = y$$
$$\Rightarrow \boxed{x = \frac{y-5}{3}}$$

Clearly $\forall y \in \mathbb{R}$ (codomain), $\frac{y-5}{3} \in \mathbb{R}$ (domain) $\Rightarrow x \in \mathbb{R}$ (domain)

Thus f is a bijection $\Rightarrow f^{-1}$ exists

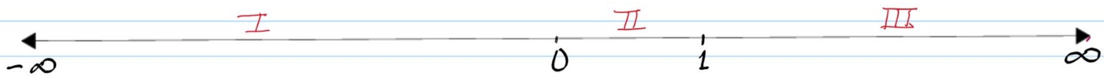
The inverse is given by replacing x by $f^{-1}(y)$ in $x = \frac{y-5}{3}$

i.e. $\boxed{f^{-1}(y) = \frac{y-5}{3}}$

Ques 3 Let $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = |x|$ and $g: \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1 \end{cases}$

Find (i) fog. (ii) $gof(-2)$ (iii) $fog(1)$ (iv) f^2 or fof

Sol $\because f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ & $g(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1 \end{cases}$



(i) for I i.e. $(-\infty, 0)$;

$$fog(x) = f(g(x)) = f(\underbrace{x^2}_{(x^2 \geq 0)}) = x^2$$

for II i.e. $[0,1]$;

$$fog(x) = f(g(x)) = f(\underbrace{x^2}_{(x^2 \geq 0)}) = x^2$$

for III i.e. $(1,\infty)$;

$$fog(x) = f(g(x)) = f(\underbrace{x+1}_{\substack{x+1 \geq 0 \\ \text{i.e. } x \geq -1}}) = x+1$$

Thus

$$fog(x) = \begin{cases} x^2 , & -\infty < x < 0 \\ x^2 , & 0 \leq x \leq 1 \\ x+1 , & 1 < x < \infty \end{cases}$$

$$= \begin{cases} x^2 , & -\infty < x \leq 1 \\ x+1 , & 1 < x < \infty \end{cases}$$

$$= \begin{cases} x^2 ; & x \leq 1 \\ x+1 ; & x > 1 \end{cases}$$

(ii) $gof(-2) = g(f(-2)) = g(|-2|) = g(2) = 2+1 = 3$

(iii) $fog(1) = f(g(1)) = f(2) = |2| = 2$

(iv) $f^2(x) = fof(x) = f(f(x)) = f(|x|) = ||x|| = |x|$

COUNTABLE AND UNCOUNTABLE SETS

A set X is countable if it can be put in one-one correspondence (bijection) with a subset of natural numbers $\mathbb{N}$.

A finite set is always countable as it can be put in one-one correspondence to some finite subset of $\mathbb{N}$.

★ Intuitively, if there exists a counting scheme such that none of the elements of set X is missed out; then we say that X is a countable set (or infinitely countable)

The set of natural numbers $\mathbb{N} = \{1, 2, 3, ----\}$ is countable trivially by the definition.

Clearly $f: \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n$ (Identity function) is a bijection.

EXAMPLES

1. The set $X = \{2, 4, 6, 8, ---\}$ is countable because there exists $f: \mathbb{N} \longrightarrow X$ defined by $f(n) = 2n$ is a bijection.

2. The set of integers $\mathbb{Z} = \{----, -3, -2, -1, 0, 1, 2, 3, ---\}$ is countable because there exists a scheme:

$$\begin{array}{ccccccccccccc} \mathbb{Z}: & \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{N}: & \cdots & 9 & 7 & 5 & 3 & 1 & 2 & 4 & 6 & 8 & \cdots \end{array}$$

Mathematically, $f: \mathbb{Z} \longrightarrow \mathbb{N}$ defined by $f(n) = \begin{cases} 2n , & n > 0 \\ -2n+1 , & n \leq 0 \end{cases}$ is bijection.

3. $\boxed{\text{The set of positive rational numbers } \mathbb{Q}^+ \text{ is countable.}}$

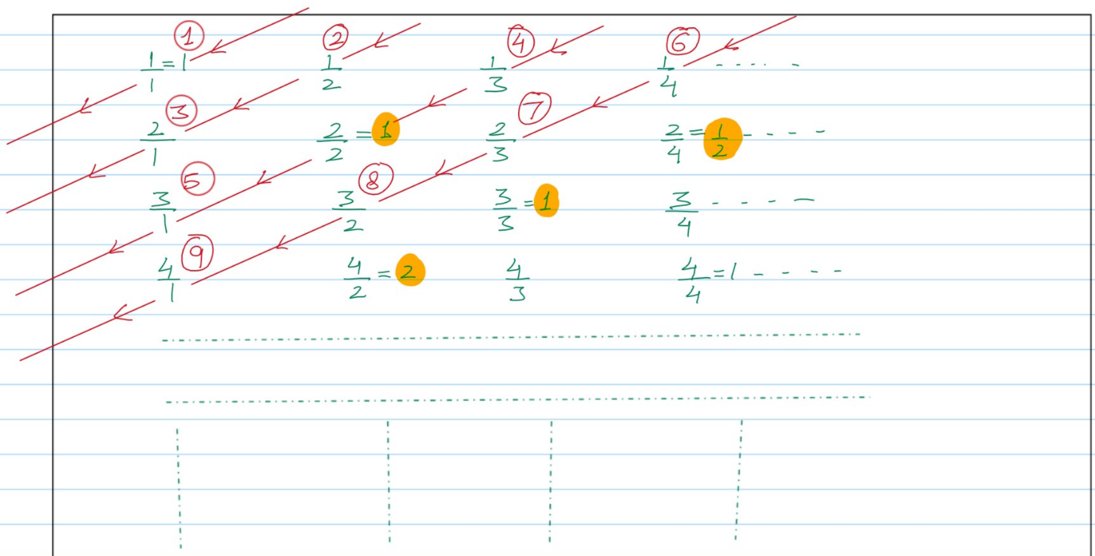
Each positive rational number in $\mathbb{Q}^+$ is of the form:

$$\frac{m}{n} ; \; m \in \mathbb{N}, n \in \mathbb{N}$$

The scheme is as follows:

- Count the numbers for which $m+n = 2$. (1st diagonal)
- Count the numbers for which $m+n = 3$. (2nd diagonal)
- Count the numbers for which $m+n = 4$. (3rd diagonal)

-- and so on.



Skip $\frac{2}{2}=1$, $\frac{2}{4}=\frac{1}{2}$, $\frac{3}{3}=1$, $\frac{4}{2}=2$, etc. to avoid overcounting

④ The finite union of countable sets is countable i.e. if $X_1, X_2, \ldots X_n$ each countable then $X_1 \cup X_2 \cup \cdots \cup X_n$ is countable.

⑤ The set of all rationals $\mathbb{Q}$ is countable because $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$ and $\mathbb{Q}^+$ is countable, $\{0\}$ is finite (hence countable). Further $\mathbb{Q}^-$ is also countable (proof is similar to that one for $\mathbb{Q}^+$)

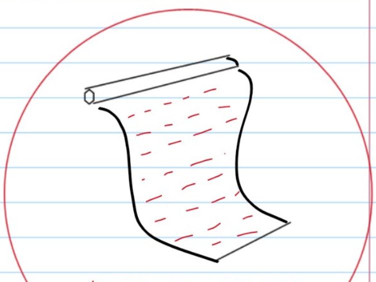
⑥ A set which is not countable is uncountable.

SET OF REAL NUMBERS IN INTERVAL (0,1) IS UNCOUNTABLE

Suppose contradiction i.e. $(0,1)$ is countable. This implies that all the real numbers in $(0,1)$ can be written completely in a list

The list consists of numbers of the form:

$$
\begin{array}{l}
0.a_{11}\, a_{12}\, a_{13} \cdots \\
0.a_{21}\, a_{22}\, a_{23} \cdots \\
0.a_{31}\, a_{32}\, a_{33} \cdots \\
\vdots \\
0.a_{n1}\, a_{n2}\, a_{n3} \cdots a_{nn} \cdots \\
\vdots
\end{array}
$$



where $a_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$ for all i & j

Construct a number

$$x = 0.b_{11}\, b_{22}\, b_{33} \cdots b_{nn} \cdots$$

s.t. $b_{11} \neq a_{11}$, $b_{22} \neq a_{22}$, $b_{33} \neq a_{33}$, $\ldots$ $b_{nn} \neq a_{nn}$, and so on.

Clearly, the number 'x' doesn't match to any number in the above list. Thus the real numbers in $(0,1)$ is uncountable.

REMARKS

① The symbol $\aleph_0$ (aleph not) is used to denote the cardinality of set $\mathbb{N}$ i.e. $|\mathbb{N}| = \aleph_0$

② Since $(0,1)$ is a subset of $\mathbb{R}$ and $(0,1)$ is uncountable thus $\mathbb{R}$ is uncountable.

③ The symbol c (continuum) is used to denote the cardinality of set $\mathbb{R}$ i.e. $|\mathbb{R}| = c$

④ Any real interval (a,b) is equipotent (i.e. cardinally equivalent) to set $\mathbb{R}$ i.e. $|(a,b)| = |\mathbb{R}|$

⑤ The set of irrational numbers $\mathbb{Q}^c$ is uncountable because $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable s.t. $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$.

RECURSIVELY DEFINED FUNCTION

A function is said to be recursively defined if the function definition refers to itself.
Mathematically,

- There is an argument for which the function doesn't refer to itself called base value. Generally it is for $n=0$ or 1.
- Each time the function refers to itself, the argument must be closer to the base value.

EXAMPLES

① Factorial function $n!$

- $n! = 1$ for $n=0$ i.e. $0! = 1$ (base value)
- $n! = n(n-1)!$ for $n \geq 1$

For $n=5$, $5! = 5\cdot 4! = 5\cdot 4\cdot 3! = 5\cdot 4\cdot 3\cdot 2! = 5\cdot 4\cdot 3\cdot 2\cdot 1! = 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 0!$
$= 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 1$
$= 120$

② Fibonacci sequence F_n

- $F_n = n$ for $n=0$ and $n=1$ i.e. $F_0 = 0$ & $F_1 = 1$ (base values)
- $F_n = F_{n-2} + F_{n-1}$ for $n \geq 2$

i.e. $\underbrace{0, 1}_{\text{base values}}, 1, 2, 3, 5, 8, 13, \ldots$

③ Ackermann function $A(m,n)$

- If $m=0$ then $A(m,n) = n+1$. (Base value)
- If $m \neq 0$ but $n=0$ then $A(m,n) = A(m-1, 1)$
- If $m \neq 0$ and $n \neq 0$ then $A(m,n) = A(m-1, A(m, n-1))$

For $m=2$, $n=1$.

Then $A(2,1) = A(1, A(2,0))$
$= A(1, A(1,1))$
$= A(1, A(0, A(1,0)))$
$= A(1, A(0, A(0,1)))$
$= A(1, A(0,2))$
$= A(1,3)$
$= A(0, A(1,2))$
$= A(0, A(0, A(1,1)))$
$= A(0, A(0, A(0, A(1,0))))$
$= A(0, A(0, A(0, A(0,1))))$
$= A(0, A(0, A(0,2)))$
$= A(0, A(0,3))$
$= A(0,4)$
$= 5$

For $m=1$, $n=1$

Then $A(1,1) = A(0, A(1,0))$
$= A(0, A(0,1))$
$= A(0,2)$
$= 3$