Mathematical Cryptography

Group

- a set of elements or “numbers”
- with some operation whose result is also in the set (closure)
- obeys:
  - associative law: $(a.b).c = a.(b.c)$
  - has identity e: $e.a = a.e = a$
  - has inverses a^{-1}: $a.a^{-1} = e$
- if commutative $a.b = b.a$
  - then forms an **abelian group**

Cyclic Group

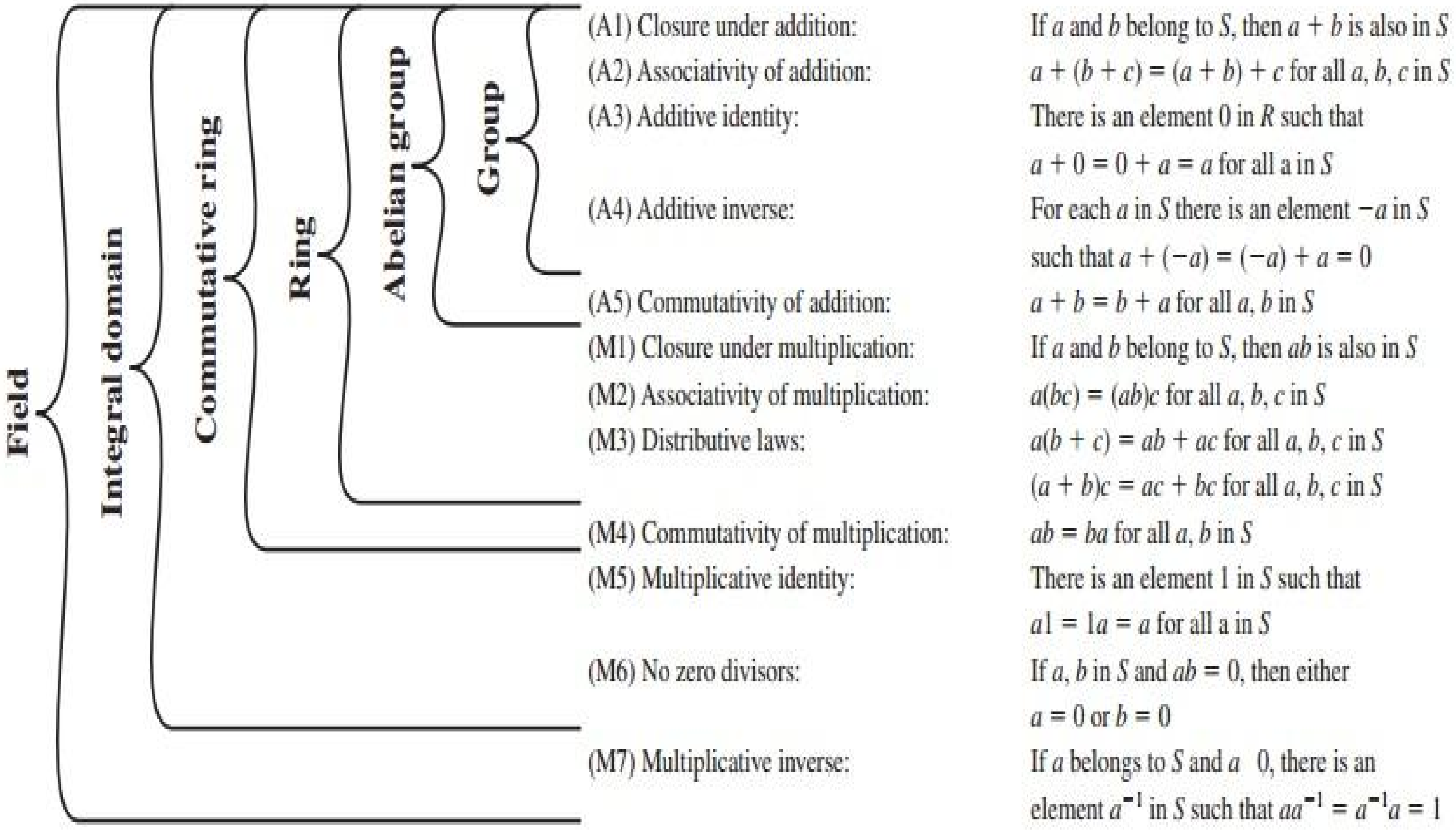
- define **exponentiation** as repeated application of operator
  - example: $a^3 = a.a.a$
- and let identity be: $e=a^0$
- a group is cyclic if every element is a power of some fixed element
  - ie $b = a^k$ for some a and every b in group
- a is said to be a generator of the group

Ring

- a set of “numbers” with two operations (addition and multiplication) which are:
- an abelian group with addition operation
- multiplication:
  - has closure
  - is associative
  - distributive over addition: `a(b+c) = ab + ac`
- if multiplication operation is commutative, it forms a **commutative ring**
- if multiplication operation has inverses and no zero divisors, it forms an **integral domain**

Field

- a set of numbers with two operations:
  - abelian group for addition
  - abelian group for multiplication (ignoring 0)
  - ring



(A1) Closure under addition: If a and b belong to S, then $a + b$ is also in S

(A2) Associativity of addition: $a + (b + c) = (a + b) + c$ for all a, b, c in S

(A3) Additive identity: There is an element 0 in R such that $a + 0 = 0 + a = a$ for all a in S

(A4) Additive inverse: For each a in S there is an element $-a$ in S such that $a + (-a) = (-a) + a = 0$

(A5) Commutativity of addition: $a + b = b + a$ for all a, b in S

(M1) Closure under multiplication: If a and b belong to S, then ab is also in S

(M2) Associativity of multiplication: $a(bc) = (ab)c$ for all a, b, c in S

(M3) Distributive laws: $a(b + c) = ab + ac$ for all a, b, c in S
$(a + b)c = ac + bc$ for all a, b, c in S

(M4) Commutativity of multiplication: $ab = ba$ for all a, b in S

(M5) Multiplicative identity: There is an element 1 in S such that $a1 = 1a = a$ for all a in S

(M6) No zero divisors: If a, b in S and $ab = 0$, then either $a = 0$ or $b = 0$

(M7) Multiplicative inverse: If a belongs to S and a 0, there is an element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Figure 4.2 Groups, Ring, and Field

Modular Arithmetic

- define **modulo operator** `a mod n` to be remainder when a is divided by n
- use the term **congruence** for: $a \equiv b \bmod n$
  - when divided by *n,* a & b have same remainder
  - eg. 100 = 34 mod 11
- b is called the **residue** of a mod n
  - since with integers can always write: $a = qn + b$
- usually have $0 <= b <= n-1$

$-12 \bmod 7 \equiv -5 \bmod 7 \equiv 2 \bmod 7 \equiv 9 \bmod 7$

Modulo 7 Example

...

-21	-20	-19	-18	-17	-16	-15
-14	-13	-12	-11	-10	-9	-8
-7	-6	-5	-4	-3	-2	-1
0	**1**	**2**	**3**	**4**	**5**	**6**
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34

...

Divisors

- say a non-zero number b **divides** a if for some m have $a=mb$ (a,b,m all integers)
- that is b divides into a with no remainder
- denote this $b|a$
- and say that b is a **divisor** of a
- eg. all of 1,2,3,4,6,8,12,24 divide 24

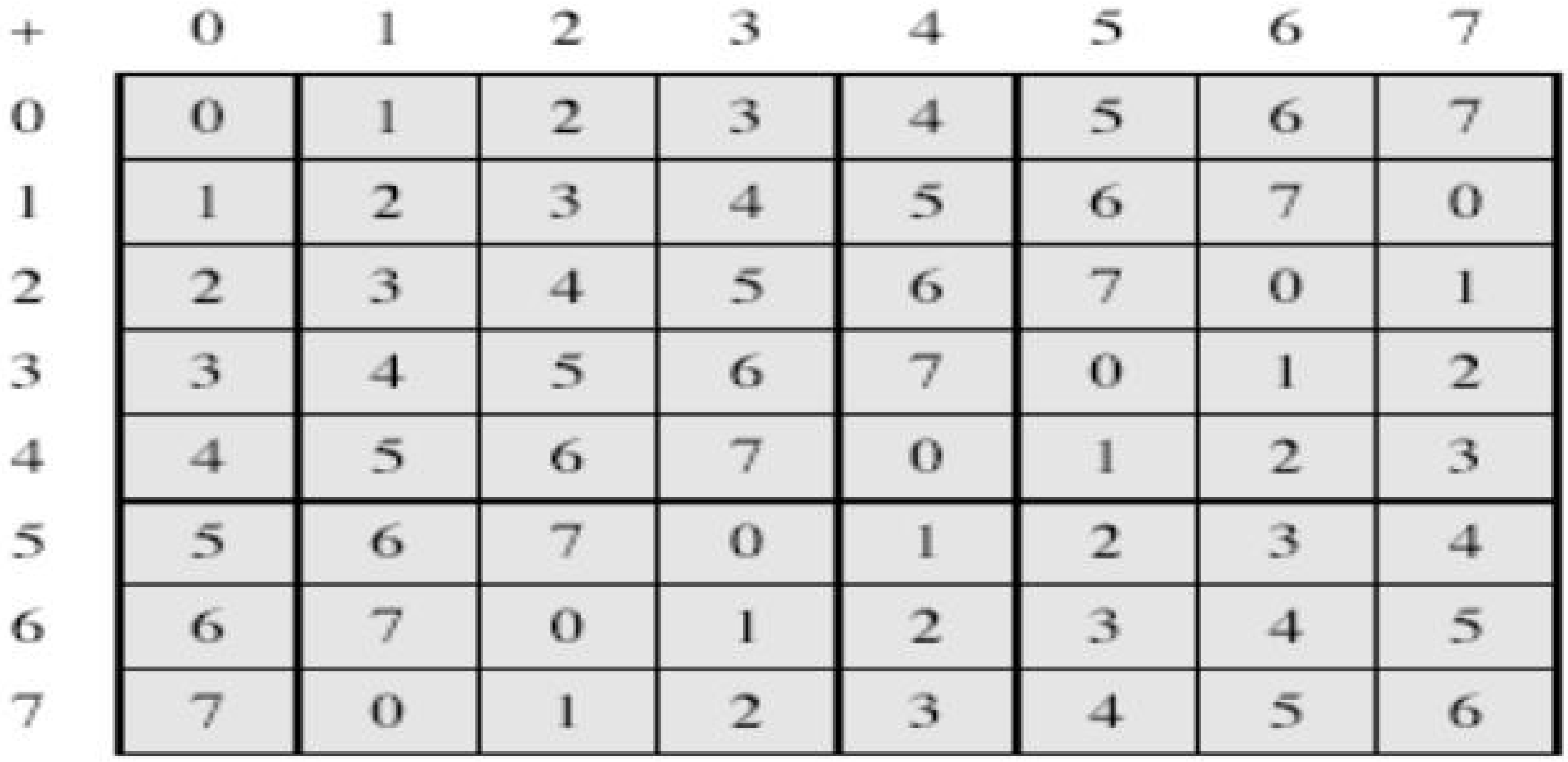
Modular Arithmetic Operations

- uses a finite number of values, and loops back from either end
- modular arithmetic is when do addition & multiplication and modulo reduce answer
- can do reduction at any point, i.e.
  - `a+b mod n = [a mod n + b mod n] mod n`

Modular Arithmetic

- can do modular arithmetic with any group of integers: $Z_n = \{0, 1, \ldots, n-1\}$
- form a commutative ring for addition
- with a multiplicative identity
- note some peculiarities
  - if $(a+b)\equiv(a+c) \bmod n$ then $b\equiv c \bmod n$
  - but $(ab)\equiv(ac) \bmod n$ then $b\equiv c \bmod n$ only if a is relatively prime to n

Modulo 8 Example



+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(a) Addition modulo 8

Greatest Common Divisor (GCD)

- a common problem in number theory
- GCD (a,b) of a and b is the largest number that divides evenly into both a and b
  - eg GCD(60,24) = 12
- often want **no common factors** (except 1) and hence numbers are **relatively prime**
  - eg GCD(8,15) = 1
  - hence 8 & 15 are relatively prime

Euclid's GCD Algorithm

- an efficient way to find the GCD(a,b)
- uses theorem that:
  - `GCD(a,b) = GCD(b, a mod b)`
- **Euclid's Algorithm** to compute GCD(a,b):
  - `A=a, B=b`
  - `while B>0`
    - `R = A mod B`
    - `A = B, B = R`
  - `return A`

Example GCD(1970,1066)

1970 = 1 x 1066 + 904	gcd(1066, 904)
1066 = 1 x 904 + 162	gcd(904, 162)
904 = 5 x 162 + 94	gcd(162, 94)
162 = 1 x 94 + 68	gcd(94, 68)
94 = 1 x 68 + 26	gcd(68, 26)
68 = 2 x 26 + 16	gcd(26, 16)
26 = 1 x 16 + 10	gcd(16, 10)
16 = 1 x 10 + 6	gcd(10, 6)
10 = 1 x 6 + 4	gcd(6, 4)
6 = 1 x 4 + 2	gcd(4, 2)
4 = 2 x 2 + 0	gcd(2, 0)

Polynomial Arithmetic

- can compute using polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

- several alternatives available
  - ordinary polynomial arithmetic
  - poly arithmetic with coords mod p
  - poly arithmetic with coords mod p and polynomials mod M(x)

Ordinary Polynomial Arithmetic

- add or subtract corresponding coefficients
- multiply all terms by each other
- eg
  - let $f(x) = x^3 + x^2 + 2$ and $g(x) = x^2 – x + 1$

    $f(x) + g(x) = x^3 + 2x^2 – x + 3$

    $f(x) – g(x) = x^3 + x + 1$

    $f(x)$ x $g(x) = x^5 + 3x^2 – 2x + 2$

Polynomial Arithmetic with Modulo Coefficients

- when computing value of each coefficient do calculation modulo some value
- could be modulo any prime
- but we are most interested in mod 2
  - ie all coefficients are 0 or 1
  - eg. let $f(x) = x^3 + x^2$ and $g(x) = x^2 + x + 1$

  $f(x) + g(x) = x^3 + x + 1$

  $f(x) \text{ x } g(x) = x^5 + x^2$

Modular Polynomial Arithmetic

- can write any polynomial in the form:
  - $f(x) = q(x)\, g(x) + r(x)$
  - can interpret $r(x)$ as being a remainder
  - $r(x) = f(x) \bmod g(x)$
- if have no remainder say $g(x)$ divides $f(x)$
- if $g(x)$ has no divisors other than itself & 1 say it is **irreducible** (or prime) polynomial
- arithmetic modulo an irreducible polynomial forms a field

Polynomial GCD

- can find greatest common divisor for polys
  - *c(x)* = GCD(*a(x), b(x)*) if *c(x)* is the poly of greatest degree which divides both *a(x), b(x)*
  - can adapt Euclid’s Algorithm to find it:
  - EUCLID[*a*(*x*)*, b*(*x*)]

```
1. $A(x) = a(x)$; $B(x) = b(x)$
2. 2. if $B(x) = 0$ return $A(x) = \gcd[a(x), b(x)]$
3. $R(x) = A(x) \bmod B(x)$
4. $A(x)$ ¨ $B(x)$
5. $B(x)$ ¨ $R(x)$
6. goto 2
```

Computational Considerations

- since coefficients are 0 or 1, can represent any such polynomial as a bit string
- addition becomes XOR of these bit strings
- multiplication is shift & XOR
  - cf long-hand multiplication
- modulo reduction done by repeatedly substituting highest power with remainder of irreducible poly (also shift & XOR)