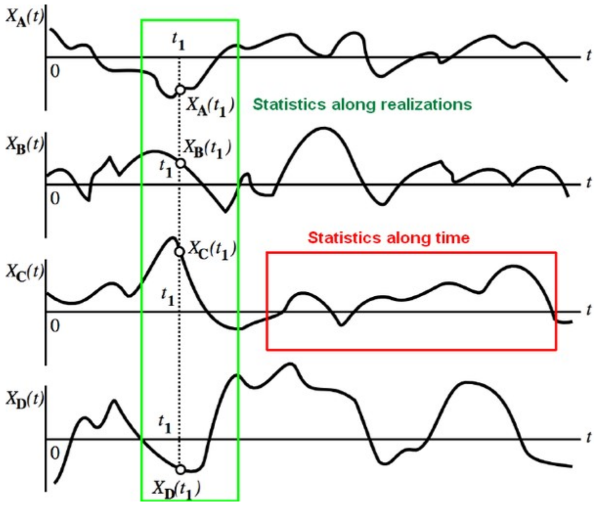
Random-Process

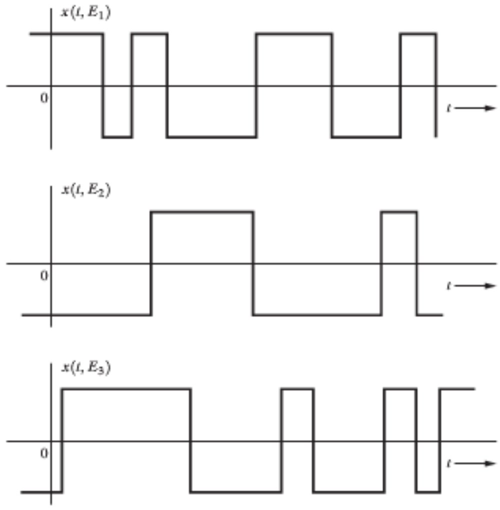
1.1 Introduction

Random Processes:

- Any physical quantity that varies with time is a signal.
- Examples of signals are an electrocardiogram (ECG), and an electroencephalogram (EEG)signals.
- There are two types signals which are
  1. Continuous time signals
  2. Discrete time signals
- Continuous time signals have the time variable t takes values from $-\infty$ to ∞ or in an interval between t_1 to t_2.
- Continuous time signals are indicated as $x(t),\ y(t),\ z(t)$ and so on.
- A discrete-time signal is a set of measurements taken sequentially in time (e.g., at every millisecond).
- Each measurement point is usually called a sample, and a discrete-time signal is indicated by by $x(n),\ y(n),\ z(n)$, where the index n is an integer that points to the order of the measurements in the sequence.
- A random process is a time-varying function that contains the outcome of a random experiment for each time instant, $X(t)$.
- A random process is a time varying function, e.g., a signal.
- A random process consists of infinite number of random variables.
- Random Process are of two types
  1. Continuous random process
  2. Discrete random process
- Real random process also called stochastic process
- The collection of all possible sample functions of $X(t)$ is called an ensemble of $X(t)$.



(a) Ensemble of continuous signals



(b) Ensemble of discrete signals

Figure 1.1: An Ensemble of Signals

1.2 Probability Distribution and Density Functions

The cdf of a random process is defined as

$$F_X(x,t) = P\{x(t) \le x\}$$

The pdf of a random process is defined as

$$f_X(x,t) = \frac{dF_X(x,t)}{dx}$$

The bivariate cdf of a random process is defined as

$$F_{X(t_1)X(t_2)(x,1x_2)} = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

The bivariate pdf of a random process is defined as

$$f_{X(x_1,t_1)(x_2,t_2)}(x,1x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X(t_1)X(t_2)}$$

The bivariate cdf of a nth order multivariate random process is defined as

$$F_{X(t_1)X(t_2)(x,1x_2)...(x_n,t_n)} = P\{X(t_1) \le x_1, X(t_2) \le x_2...X(t_n) \le x_n\}$$

Similarly nth order multivariate density functions of order n exist

$$f_{X(x_1,t_1)(x_2,t_2)...(x_n,t_n)}(x,1x_2 \quad x_n) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X(t_1)X(t_2)}$$

The mean of a random process is defined as

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x,t) dx$$

The mean square μ_X of a random process is defined as

$$\overline{X^2(t)} = E[X^2(t) = \int_{-\infty}^{\infty} x^2 f_X(x,t) dx$$

The variance σ_X^2 of a random process is defined as

$$\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] \\
&= E[X^2 - 2\mu_X X + \mu_X^2] \\
&= E[X^2] - 2\mu_X E[X] + \mu_X^2 \\
&= E[X^2] - 2\mu_X^2 + \mu_X^2 \\
&= E[X^2] - \mu_X^2
\end{aligned}$$

1.3 Stationary

A random process $\{X(t)\}$ is stationary if the pdfs and its statistical properties are invariant with changes in time. For example, for a stationary process, $X(t)$ and $X(t+\Delta)$ have the same probability distributions.

$$F_{X(t)}(x) = F_{X(t+\Delta)}(x) \quad for\ all\ t,\ t+\Delta$$

Stationary of order one: Consider a pdf of random process $X(t)$ is

$$f_{X(t+t_s)}(x) = f_{X(t)}(x) = f_X(x)$$

The pdf is independent of any time shift hence its mean of the random process $X(t)$ is also a constant

$$\mu_{X(t)}(x) = \mu_X$$

The variance of the random process $X(t)$ is also a constant

$$\sigma_{X(t)}^2 = \sigma_X^2$$

Stationary of order two: Consider a pdf of random process $X(t)$ is

$$f_{X(t_1+t_s)X(t_2+t_s)}(x_1,x_2) = f_{X(t_1)X(t_2)}(x_1,x_2)$$

Let

$$\begin{aligned}
t_1 + t_s &= t \\
t_s &= t - t_1 \\
t_2 + t_s &= t_2 + t - t_1 \\
t_2 + t_s &= t + (t_2 - t_1) \\
&= t + \tau
\end{aligned}$$

The mean of the random process $X(t)$ is also a constant

$$\begin{aligned}
E[X(t)] &= \mu_X \\
E[X(t_1)X(t_2)] &= E[X(t)X(t+\tau)]
\end{aligned}$$

The correlation of a wide sense stationer is independent of time, it depends upon τ. When the time difference is 0 i.e., $\tau = t_2 - t_1 = 0 \Rightarrow t_1 = t_2 = t$

$$E[X(t)X(t)] = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

The above relation is a autocorrelation $R_X(\tau)$ of a random process $X(t)$

$$R_X(\tau) = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

If a random process $X(t)$ has first and second order stationary, then it called Wide-Sense Stationary Random Processes

Wide-Sense Stationary Random Processes A continuous-time random process $X(t)$ is wide-sense stationary (WSS) if it follows the following properties

1. The mean is independent of time t

$$E[X(t)] = \mu_{X(t)} = \mu_X = constant$$

   The variance of the random process $X(t)$ is also a constant

$$\sigma^2_{X(t)} = \sigma^2_X$$

2. The autocorrelation function only depends on time difference

$$R_X(\tau) = E[X(t)X(t)] = E[X^2(t)] = \sigma^2_X + \mu^2_X$$

A discrete-time random process $\{X(n), n \in Z\}$ is weak-sense stationary or wide-sense stationary (WSS) if

1. $\mu_X(n) = \mu_X \quad for\ all\ n \in Z$
2. $R_X(n_1, n_2) = R_X(n_1 - n_2) \quad for\ all\ n_1, n_2 \ \in Z$

1.4 Correlation Functions:

The correlation functions between two random variables is a measure of the similarity between the variables.

There are two types of correlation functions

1. Auto-Correlation Functions
2. Cross-Correlation Functions

1.4.1 Auto-Correlation Functions:

The correlation of a signal $X(t)$ with itself is called as autocorrelation. This is denoted as

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

where t and τ are arbitrary. When $\tau = 0$ the autocorrelation function is the average of the random process squared. It is also called average power.

$$R_X(0) = E[X(t)X(t)] = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

For $-\tau$

$$R_X(-\tau) = E[X(t)X(t-\tau)]$$

Let $t' = t - \tau$

$$\begin{aligned} R_X(-\tau) &= E[X(t'+\tau)X(t')] \\ &= E[X(t')X(t'+\tau)] \end{aligned}$$

$$R_X(-\tau) = R_X(\tau)$$

The autocorrelation function is an even function of τ.

$$E[X_iX_j] = \begin{cases} E[X_i^2] = E[X^2] = \mu_X^2 + \sigma_X^2 & j = i \\ E[X_iX_i] = \mu_X^2 & j \neq i \end{cases}$$

The bounds on auto-correlation function is

$$\begin{aligned} E[\{X(t) \pm X(t+\tau)\}^2] &\geq = 0 \\ E[\{X^2(t) \pm 2X(t)X(t+\tau) + X^2(t+\tau)\}^2] &\geq = 0 \end{aligned}$$

By performing expectations

$$R_X(0) \pm 2R_X(\tau) + R_X(0) \geq 0$$

$$|R_X(\tau)| \leq R_X(0)$$

If $X(t)$ is a random process with non zero mean, then it is defined as ***auto-covariance function*** $C_X(\tau)$

$$\begin{aligned} C_X(\tau)R_X(0) &= E[\{X(t) - \mu_X\}\{X(t+\tau) - \mu_X\}] \\ &= R_X(\tau) - \mu_X^2 \end{aligned}$$

The power spectral density of is defined as

$$\int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau \geq 0 \; for \; all \; \omega$$

Properties of Auto Correlation Functions

- $R_X(\tau)$ is bounded as

$$|R_X(\tau)| \leq R_X(0)$$

- Auto Corrrelation function is even symmetry

$$R_X(-\tau) = R_X(\tau)$$

- The mean value of the random process is obtained using autocorrelation using the following relation

$$E[X^2(t)] = \overline{X^2(t)} = \sigma_X^2 + \mu_X^2 = R_X(0)$$

1.4.2 Cross-Correlation Functions:

Consider a two random process $X(t)$ and $Y(t)$ are wide sense stationary. When we consider both the random process it is called as jointly wide sense stationary. Then their cross-correlation function is defined as

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ R_{YX}(\tau) &= E[Y(t)X(t+\tau)] \end{aligned}$$

where t and τ are arbitrary. The order of the subscript is

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

If $Y(t)$ is periodic with period T, then

$$R_{XY}(\tau + T) = R_{XY}(\tau)$$

If $X(t)$ is periodic with T, then

$$R_{YX}(\tau + T) = R_{YX}(\tau)$$

The bounds on cross-correlation function is

$$\begin{aligned} E[\{X(t) \pm kY(t+\tau)\}^2] &\geq = 0 \\ E[\{X^2(t) \pm 2kX(t)Y(t+\tau) + k^2Y^2(t+\tau)\}^2] &\geq = 0 \end{aligned}$$

By performing expectations

$$R_X(0) \pm 2kR_{XY}(\tau) + k^2R_Y(0) \geq 0$$

If $k = 1$

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

Then it becomes

$$|R_{XY}(\tau)| \leq = \frac{1}{2}[R_X(0) + R_Y(0)]$$

If k is a positive and real constant then

$$k^2R_Y(0) + 2kR_{XY}(\tau) + R_X(0) \geq 0$$

The quadratic will be never negative if it does not have real roots. If its discriminant is

$$4R_{XY}^2(\tau) - 4R_X(0)R_Y(0) \leq 0$$

$$|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$$

For example for a quadratic equation

$$ax^2 + bx + c^2 = = 0$$

then its discriminant is $b^2 - 4ac$

The geometric mean is

$$\sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

A cross correlation function is

$$\begin{aligned} C_{XY}(\tau) &= E[(X(t) - \mu_X)(Y(t+\tau) - \mu_Y)] \\ &= R_{XY}(\tau) - \mu_X\mu_Y \end{aligned}$$

$$\begin{aligned} C_{YX}(\tau) &= E[(Y(t) - \mu_Y)(X(t+\tau) - \mu_X)] \\ &= R_{YX}(\tau) - \mu_Y\mu_X \end{aligned}$$

Addition and Subtraction

$$\begin{aligned} sin(A+B) &= sinAcosB - cosAsinB \\ sin(A-B) &= sinAcosB + cosAsinB \\ cos(A+B) &= cosAcosB - sinAsinB \\ cos(A-B) &= cosAcosB + sinAsinB \\ tan(A+B) &= \frac{tanA + tanB}{1 - tanAtanB} \\ tan(A-B) &= \frac{tanA - tanB}{1 + tanAtanB} \end{aligned}$$

Product Identities

$$\begin{aligned} sinAcosB &= \frac{1}{2}\left(sin(A+B) + sin(A-B)\right) \\ cosAsinB &= \frac{1}{2}\left(sin(A+B) - sin(A-B)\right) \\ cosAcosB &= \frac{1}{2}\left(cos(A+B) + cos(A-B)\right) \\ sinAsinB &= \frac{1}{2}\left(cos(A-B) - cos(A+B)\right) \end{aligned}$$

Example 4.2 The random process described by

$$Y(t) = Acos(\omega_c t + \Theta)$$

where A and $\omega_c t$ are constants and Θ is a random variable distributed uniformly between $\pm\pi$ Find the pdf of random variable Θ, mean and autocorrelation function of Y

Solution:

The pdf of random variable Θ

$$\begin{aligned} f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi-(-\pi)} \\ &= \frac{1}{2\pi} \end{aligned}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ \\ 0 & otherwise \end{cases}$$

The mean of random variable Y

$$\begin{aligned} \mu_Y &= \int_{-\pi}^{\pi} Acos(\omega_c t + \theta)\frac{1}{2\pi}d\theta \\ &= \frac{A}{2\pi}\int_{-\pi}^{\pi} cos\omega_c t cos\theta - sin\omega_c t sin\theta d\theta \\ &= \frac{A}{2\pi}[cos\omega_c t sin\theta + sin\omega_c t cos\theta]_{-\pi}^{\pi} \\ &= \frac{A}{2\pi}[0 + sin\omega_c t(-1-(-1))] \\ &= 0 \end{aligned}$$

Autocorrelation function $R_Y(\tau)$

$$\begin{aligned} E[Y(t)Y(t+\tau)] &= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}Acos(\omega_c t+\theta)\right)\left(\frac{1}{2\pi}Acos(\omega_c(t+\tau)+\theta)\right)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} cos(\omega_c t+\theta)cos(\omega_c(t+\tau)+\theta)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}[cos(2\omega_c t+\omega_c\tau+2\theta)+cos\omega_c\tau]d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}[cos(2\omega_c t+\omega_c\tau+2\theta)+cos\omega_c\tau]d\theta \end{aligned}$$

$$\begin{aligned} x &= 2\omega_c t + \omega_c\tau + 2\theta \\ \frac{dx}{d\theta} &= 0+0+2 \\ d\theta &= \frac{dx}{2} \end{aligned}$$

Limits, when $\theta = \pi$

$$x = 2\omega_c t + \omega_c\tau + 2\pi$$

when $\theta = -\pi$

$$x = 2\omega_c t + \omega_c\tau - 2\pi$$

$$\begin{aligned} \int_{-\pi}^{\pi}[cos(2\omega_c t+\omega_c\tau+2\theta)]d\theta &= \int\limits_{2\omega_c t+\omega_c\tau+2\pi}^{2\omega_c t+\omega_c\tau-2\pi}(cosx)\frac{dx}{2} \\ &= \frac{1}{2}\,[sinx]_{2\omega_c t+\omega_c\tau+2\pi}^{2\omega_c t+\omega_c\tau-2\pi} \\ &= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)] \\ &= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)] \\ &= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau)-sin(2\omega_c t+\omega_c\tau)] \\ &= 0 \end{aligned}$$

$$\int_{-\pi}^{\pi}[cos\omega_c\tau]d\theta = cos\omega_c\tau[\theta]_{-\pi}^{\pi}$$
$$= 2\pi cos\omega_c\tau$$

$$E[Y(t)Y(t+\tau)] = \frac{A^2}{2\pi}2\pi cos\omega_c\tau$$
$$= \frac{A^2}{2}cos\omega_c\tau$$

The ensemble variance is constant

$$\sigma_Y^2 = R_Y(0) = \frac{A^2}{2}$$

From the above discussions it is observed that

1. $E[Y(t)] = 0$=Constant and
2. $E[Y(t)Y(t+\tau)] = R_Y(\tau)$, it is independent of absolute time hence

$Y(t)$ is a wide sense stationary.
Example 4.6 Consider two random process $X(t)$ and $Y(t)$ which are independent, jointly wide sense stationary random process described by

$$X(t) = Acos(\omega_1 t + \Theta_1)$$
$$Y(t) = Bcos(\omega_2 t + \Theta_2)$$

where A, B and $\omega_1 t$ $\omega_2 t$are constants and Θ_1 and Θ_2 are random variable distributed uniformly between $\pm\pi$ Let $X(t)$ and $Y(t)$ are related by

$$W(t)X(t) = X(t)Y(t)$$

Find the autocorrelation function of W

Solution:

The autocorrelation function of $X(t)$ and $Y(t)$ are

$$R_X(\tau) = \frac{A^2}{2}cos\omega_1\tau$$
$$R_Y(\tau) = \frac{B^2}{2}cos\omega_2\tau$$

$$R_W(\tau) = \frac{A^2}{2}cos\omega_1\tau\frac{B^2}{2}cos\omega_2\tau$$
$$= \frac{A^2B^2}{4}cos\omega_1\tau cos\omega_2\tau$$
$$= \frac{A^2B^2}{8}[cos(\omega_1+\omega_2)\tau + (\omega_1-\omega_2)\tau]$$

6. The random process described by

$$X(t) = Asin(\omega_c t + \Theta)$$

where A and ω_c are constants and Θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned} f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi-(-\pi)} \\ &= \frac{1}{2\pi} \end{aligned}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= \int_{-\pi}^{\pi} Asin(\omega_c t + \theta)\frac{1}{2\pi}d\theta \\ &= \frac{A}{2\pi}\int_{-\pi}^{\pi} sin\omega_c t\ cos\theta - cos\omega_c t\ sin\theta d\theta \\ &= \frac{A}{2\pi}[sin\omega_c t sin\theta + cos\omega_c t cos\theta]_{-\pi}^{\pi} \\ &= \frac{A}{2\pi}[0 + sin\omega_c t(-1-(-1))] \\ &= 0 \end{aligned}$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\left(Asin(\omega_c t + \theta)\right)\left(Asin(\omega_c(t+\tau)+\theta)\right)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} sin(\omega_c t+\theta)sin(\omega_c(t+\tau)+\theta)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}[cos\omega_c\tau - cos(2\omega_c t + \omega_c\tau + 2\theta)]d\theta \\ &= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}[cos\omega_c\tau - cos(2\omega_c t + \omega_c\tau + 2\theta)]d\theta \end{aligned}$$

$x = 2\omega_c t + \omega_c\tau + 2\theta$
$\frac{dx}{d\theta} = 0 + 0 + 2$
$d\theta = \frac{dx}{2}$
Limits, when $\theta = \pi$
$x = 2\omega_c t + \omega_c\tau + 2\pi$
when $\theta = -\pi$
$x = 2\omega_c t + \omega_c\tau - 2\pi$

$$\begin{aligned}
\int_{-\pi}^{\pi}[cos(2\omega_c t+\omega_c\tau+2\theta)]d\theta &= \int\limits_{2\omega_c t+\omega_c\tau-2\pi}^{2\omega_c t+\omega_c\tau+2\pi}(cosx)\frac{dx}{2}\\
&= \frac{1}{2}\left[sinx\right]_{2\omega_c t+\omega_c\tau-2\pi}^{2\omega_c t+\omega_c\tau+2\pi}\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)]\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)]\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau)-sin(2\omega_c t+\omega_c\tau)]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi}[cos\omega_c\tau]d\theta &= cos\omega_c\tau[\theta]_{-\pi}^{\pi}\\
&= 2\pi cos\omega_c\tau
\end{aligned}$$

$$\begin{aligned}
E[X(t)X(t+\tau)] &= \frac{A^2}{4\pi}2\pi cos\omega_c\tau\\
R_X(\tau) &= \frac{A^2}{2}cos\omega_c\tau
\end{aligned}$$

From the above discussions it is observed that

1. $E[X(t)]=0$=Constant and
2. $E[X(t)X(t+\tau)]=R_X(\tau)$, it is independent of absolute time hence

$X(t)$ is a wide sense stationary.

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7. The random process described by

$$X(t) = Acos(\omega_c t+\phi+\Theta)$$

where A ω_c and ϕ are constants and Θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned}
f_\Theta(\theta) &= \frac{1}{b-a}=\frac{1}{\pi-(-\pi)}\\
&= \frac{1}{2\pi}
\end{aligned}$$

$$f_\Theta(\theta)=\begin{cases}\frac{1}{2\pi} & -\pi<\theta<\pi\\ 0 & otherwise\end{cases}$$

The mean of random variable X

$$\mu_X = \int_{-\pi}^{\pi} cos(\omega_c t + \phi + \Theta)\frac{1}{2\pi}d\theta$$

$$\begin{aligned} x &= \omega_c t + \phi + \theta \\ \frac{dx}{d\theta} &= 0 + 0 + d\theta \\ d\theta &= dx \end{aligned}$$

Limits, when $\theta = \pi$

$$x = \omega_c t + \phi + \pi$$

when $\theta = -\pi$

$$x = \omega_c t + \phi - \pi$$

$$\begin{aligned} \mu_X &= \int_{-\pi}^{\pi} cos(\omega_c t + \phi + \Theta)\frac{1}{2\pi}d\theta \\ &= \frac{A}{2\pi}\int_{\omega_c t+\phi-\pi}^{\omega_c t+\phi+\pi} cosxdx \\ &= \frac{A}{2\pi}[sinx]_{\omega_c t+\phi-\pi}^{\omega_c t+\phi+\pi} \\ &= \frac{A}{2\pi}[sin(\omega_c t + \phi + \pi) - sin(\omega_c t + \phi - \pi)] \\ &= \frac{A}{2\pi}[-sin(\omega_c t + \phi) + sin(\omega_c t + \phi)] \\ &= 0 \end{aligned}$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_{-\pi}^{\pi}\frac{1}{2\pi}\left(Acos(\omega_c t + \phi + \theta)\right)\left(Acos(\omega_c(t+\tau) + \phi + \theta)\right)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} cos(\omega_c t + \phi + \theta)cos(\omega_c(t+\tau) + \phi + \theta)d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}[cos\omega_c\tau - cos(2\omega_c t + \omega_c\tau + 2\theta)]d\theta \\ &= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}[cos(2\omega_c t + \omega_c\tau + 2\phi + 2\theta) + cos\omega_c\tau]d\theta \end{aligned}$$

$$\begin{aligned} x &= 2\omega_c t + \omega_c\tau + 2\theta \\ \frac{dx}{d\theta} &= 0 + 0 + 2 \\ d\theta &= \frac{dx}{2} \end{aligned}$$

Limits, when $\theta = \pi$

$$x = 2\omega_c t + \omega_c\tau + 2\pi$$

when $\theta = -\pi$

$$x = 2\omega_c t + \omega_c\tau - 2\pi$$

$$\begin{aligned}
\int_{-\pi}^{\pi}[cos(2\omega_c t+\omega_c\tau+2\theta)]d\theta &= \int\limits_{2\omega_c t+\omega_c\tau-2\pi}^{2\omega_c t+\omega_c\tau+2\pi}(cosx)\frac{dx}{2}\\
&= \frac{1}{2}\,[sinx]_{2\omega_c t+\omega_c\tau-2\pi}^{2\omega_c t+\omega_c\tau+2\pi}\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)]\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau+2\pi)-sin(2\omega_c t+\omega_c\tau-2\pi)]\\
&= \frac{1}{2}[sin(2\omega_c t+\omega_c\tau)-sin(2\omega_c t+\omega_c\tau)]\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{\pi}[cos\omega_c\tau]d\theta &= cos\omega_c\tau[\theta]_{-\pi}^{\pi}\\
&= 2\pi cos\omega_c\tau
\end{aligned}$$

$$E[X(t)X(t+\tau)] = R_X(\tau)=\frac{A^2}{4\pi}2\pi cos\omega_c\tau=\frac{A^2}{2}cos\omega_c\tau$$

From the above discussions it is observed that

1. $E[X(t)]=0$=Constant and
2. $E[X(t)X(t+\tau)]=R_X(\tau)$, it is independent of absolute time hence

$X(t)$ is a wide sense stationary.

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8. The random process described by

$$X(t) = Acos(\omega_c t+\Theta)+B$$

where A, B and ω_c are constants and Θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned}
f_\Theta(\theta) &= \frac{1}{b-a}=\frac{1}{\pi-(-\pi)}\\
&= \frac{1}{2\pi}
\end{aligned}$$

$$f_\Theta(\theta)=\begin{cases}\frac{1}{2\pi} & -\pi<\theta<\pi\\ 0 & otherwise\end{cases}$$

The mean of random variable X

$$\begin{aligned}
\mu_X &= E[X(t)]=E[Acos(\omega_c t+\Theta)+B]\\
&= \int_{-\pi}^{\pi}Acos(\omega_c t+\theta)\frac{}{2\pi}d\theta+B\int_{-\pi}^{\pi}\frac{1}{2\pi}d\theta
\end{aligned}$$

$$\begin{aligned} x &= \omega_c t + \theta \\ dx &= 0 + d\theta \\ d\theta &= dx \end{aligned}$$

Limits, when $\theta = \pi$

$$x = \omega_c t + \pi$$

when $\theta = -\pi$

$$x = \omega_c t - \pi$$

$$\begin{aligned} \mu_X &= \int_{-\pi}^{\pi} Acos(\omega_c t + \theta)\frac{1}{2\pi}d\theta \\ &= \frac{A}{2\pi}\int_{\omega_c t-\pi}^{\omega_c t+\pi} cosxdx \\ &= \frac{A}{2\pi}[sinx]_{\omega_c t-\pi}^{\omega_c t+\pi} \\ &= \frac{A}{2\pi}[sin(\omega_c t + \pi) - sin(\omega_c t - \pi)] \\ &= \frac{A}{2\pi}[-sin(\omega_c t) + sin(\omega_c t)] \\ &= 0 \end{aligned}$$

$$\begin{aligned} B\int_{-\pi}^{\pi}\frac{1}{2\pi}d\theta &= \frac{B}{2\pi}[\theta]_{-\pi}^{\pi} \\ &= \frac{B}{2\pi}[2\pi] \\ &= B \end{aligned}$$

Autocorrelation function $R_X(\tau)$

$$cosAcosB = \frac{1}{2}\left(cos(A+B) + cos(A-B)\right)$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= E[(Acos(\omega_c t + \phi + B))\,(Acos(\omega_c(t+\tau) + B))] \\ &= E[\left(A^2cos(\omega_c t + \phi)\right)(cos(\omega_c(t+\tau)+\phi)) + AB\,(cos(\omega_c(t+\tau)+\phi)) + AB\,(cos(\omega_c t+\phi)) + B \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left[cos(2\omega_c t + \omega_c\tau + 2\theta) + cos(\omega_c\tau)\right]d\theta + 0 + 0 + B^2 \\ &= \frac{A^2}{2}cos(\omega_c\tau) + B^2 \end{aligned}$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= \frac{A^2}{2}cos(\omega_c\tau) + B^2 \\ R_X(\tau) &= \frac{A^2}{2}cos(\omega_c\tau) + B^2 \end{aligned}$$

From the above discussions it is observed that

1. $E[X(t)] = B$=Constant and
2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence

$X(t)$ is a wide sense stationary.

---

10. The random process described by

$$X(t) = A^2 cos^2(\omega_c t + \Theta)$$

where A and ω_c are constants and Θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned} f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi-(-\pi)} \\ &= \frac{1}{2\pi} \end{aligned}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[A^2 cos^2(\omega_c t + \Theta)] \\ &= \frac{A^2}{2} E[1 + cos(2\omega_c t + 2\Theta)] \\ &= \frac{A^2}{2} + \frac{A^2}{2}\int_{-\pi}^{\pi} cos(2\omega_c t + 2\theta) d\theta \end{aligned}$$

$$\begin{aligned} x &= 2\omega_c t + 2\theta \\ dx &= 0 + 2d\theta \\ d\theta &= \frac{dx}{2} \end{aligned}$$

Limits, when $\theta = \pi$

$$x = 2\omega_c t + 2\pi$$

when $\theta = -\pi$

$$x = 2\omega_c t - 2\pi$$

$$\begin{aligned} &= \int_{-\pi}^{\pi} cos(2\omega_c t + 2\theta)\frac{1}{2\pi} d\theta \\ &= \frac{A^2}{4\pi}\int_{2\omega_c t - 2\pi}^{2\omega_c t + 2\pi} cosx dx \\ &= \frac{A^2}{4\pi}[sinx]_{2\omega_c t - 2\pi}^{2\omega_c t + 2\pi} \\ &= \frac{A^2}{4\pi}[sin(2\omega_c t + 2\pi) - sin(2\omega_c t - 2\pi)] \\ &= \frac{A^2}{4\pi}[sin(2\omega_c t) - sin(2\omega_c t)] = 0 \\ \mu_X &= \frac{A^2}{2} \end{aligned}$$

Autocorrelation function $R_X(\tau)$

$$
\begin{aligned}
E[X(t)X(t+\tau)] &= E[\left(A^2cos^2(\omega_c t+\Theta)\right)\left(Acos(\omega_c(t+\tau)+B)A^2cos^2(\omega_c t+\Theta)\right)] \\
&= \frac{A^2}{4}E[\{1+cos(2\omega_c t+2\Theta)\}\{1+cos(2\omega_c(t+\tau)+2\Theta)\}] \\
&= \frac{A^2}{4}\{1+\int_{-\pi}^{\pi}cos(2\omega_c t+2\theta)\frac{1}{2\pi}d\theta+\int_{-\pi}^{\pi}cos(2\omega_c(t+\tau)+2\theta)\frac{1}{2\pi}d\theta\} \\
&+ \int_{-\pi}^{\pi}cos(2\omega_c t+2\theta)cos(2\omega_c(t+\tau)+2\theta)\frac{1}{2\pi}d\theta \\
&= \frac{A^2}{4}\{1+0+0+\frac{1}{4\pi}\int_{-\pi}^{\pi}\left[cos(4\omega_c t+2\omega_c\tau+4\theta)+cos(2\omega_c\tau)\right]d\theta\} \\
&= \frac{A^2}{4}\{1+\frac{1}{2}cos(2\omega_c\tau)\}
\end{aligned}
$$

$$
\begin{aligned}
E[X(t)X(t+\tau)] &= \frac{A^2}{4}\{1+\frac{1}{2}cos(2\omega_c\tau)\} \\
R_X(\tau) &= \frac{A^2}{4}\{1+\frac{1}{2}cos(2\omega_c\tau)\}
\end{aligned}
$$

From the above discussions it is observed that

1. $E[X(t)] = \frac{A^2}{2}$=Constant and
2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence

$X(t)$ is a wide sense stationary.

---

12. The random process described by

$$
X(t) = Acos(\omega_c t+\Theta)+Bcos(\omega_s t)
$$

where A, B ω_c, and ω_s $\omega_c \neq \omega_s$ are constants and Θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$
\begin{aligned}
f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi-(-\pi)} \\
&= \frac{1}{2\pi}
\end{aligned}
$$

$$
f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi<\theta<\pi \\ 0 & otherwise \end{cases}
$$

The mean of random variable X

$$
\begin{aligned}
\mu_X &= E[X(t)] = E[Acos(\omega_c t+\Theta)+Bcos(\omega_s t)] \\
&= A\int_{-\pi}^{\pi}cos(\omega_c t+2\theta)\frac{}{2\pi}d\theta+Bcos(\omega_s t) \\
&= Bcos(\omega_s t)
\end{aligned}
$$

From the above discussions it is observed that $X(t)$ is not a wide sense stationary, because its mean is not a constant.

---

13. The random process described by

$$X(t) = Acos(\omega_c t + \Theta)$$

where A and ω_c, are constants and Θ is a random variable uniformly distributed between $\pm\frac{\pi}{2}$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned} f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi/2 - (-\pi/2)} \\ &= \frac{1}{\pi} \end{aligned}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & -\pi/2 < \theta < \pi/2 \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[Acos(\omega_c t + \Theta)\frac{1}{\pi}] \\ &= A\int_{-\pi/2}^{\pi/2} cos(\omega_c t + \theta)\frac{1}{\pi}d\theta \end{aligned}$$

$$\begin{aligned} x &= \omega_c t + \theta \\ d\theta &= dx \end{aligned}$$

Limits, when $\theta = \pi/2$

$$x = \omega_c t + \pi/2$$

when $\theta = -\pi/2$

$$x = \omega_c t - \pi/2$$

$$\begin{aligned} \mu_X &= A\int_{-\pi/2}^{\pi/2} cos(\omega_c t + \theta)\frac{1}{\pi}d\theta \\ &= \frac{A}{\pi}\int_{\omega_c t - \pi/2}^{\omega_c t + \pi/2} cos(x)dx \\ &= \frac{A}{\pi}cos(\omega_c t + \pi/2) - cos(\omega_c t - \pi/2) \\ &= \frac{2A}{\pi}cos(\omega_c t) \end{aligned}$$

From the above discussions it is observed that $X(t)$ is not a wide sense stationary, because its mean is not a constant.

---

14. The random process described by

$$X(t) = Acos(\omega_c t + \Theta)$$

where A and ω_c, are constants and Θ is a random variable uniformly distributed between 0 and π. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$\begin{aligned} f_\Theta(\theta) &= \frac{1}{b-a} = \frac{1}{\pi - (0)} \\ &= \frac{1}{\pi} \end{aligned}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & 0 < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[Acos(\omega_c t + \Theta)\frac{1}{\pi}] \\ &= A\int_0^\pi cos(\omega_c t + \theta)\frac{1}{\pi}d\theta \end{aligned}$$

$$\begin{aligned} x &= \omega_c t + \theta \\ d\theta &= dx \end{aligned}$$

Limits, when $\theta = 0$

$$x = \omega_c t$$

when $\theta = \pi$

$$x = \omega_c t + \pi$$

$$\begin{aligned} \mu_X &= A\int_0^\pi cos(\omega_c t + \theta)\frac{1}{\pi}d\theta \\ &= \frac{A}{\pi}\int_{\omega_c t}^{\omega_c t + \pi} sin(x)dx \\ &= \frac{A}{\pi}sin(\omega_c t + \pi) - sin(\omega_c t) \\ &= \frac{-2A}{\pi}sin(\omega_c t) \end{aligned}$$

From the above discussions it is observed that $X(t)$ is not a wide sense stationary, because its mean is not a constant.

---

15. A random process described by

$$X(t) = V$$

where V is a random variable uniformly distributed between 0 and 4. That means that each realization of $X(t) = V$ is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 4. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$\begin{aligned} f_V(v) &= \frac{1}{b-a} = \frac{1}{4-(0)} \\ &= \frac{1}{4} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{1}{4} & 0 < \theta < 4 \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[V] = \frac{1}{4}\int_0^4 v dv = \frac{1}{4}\left[\frac{}{2}\right]_0^4 = \frac{1}{8}\left[4^2 - 0\right] \\ &= 2 \end{aligned}$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= E[V^2] = \frac{1}{4}\int_0^4 v^2 dv = \frac{1}{4}\left[\frac{}{3}\right]_0^4 \\ &= \frac{1}{12}\left[4^3 - 0\right] \\ &= \frac{16}{3} \end{aligned}$$

From the above discussions it is observed that

1. $E[X(t)] = 2$=Constant and
2. $E[X(t)X(t+\tau)] = R_X(\tau) = \frac{16}{3}$, it is independent of absolute time hence

$X(t)$ is a wide-sense stationary.

---

16. A random process described by

$$X(t) = V$$

where V is a random variable uniformly distributed between 0 and 2. That means that each realization of $X(t) = V$ is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 2. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$\begin{aligned} f_V(v) &= \frac{1}{b-a} = \frac{1}{2-(0)} \\ &= \frac{1}{2} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < \theta < 2 \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[V] = \frac{1}{2}\int_0^2 v dv = \frac{1}{2}\left[\frac{}{2}\right]_0^2 = \frac{1}{4}\left[2^2 - 0\right] \\ &= 1 \end{aligned}$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= E[V^2] = \frac{1}{2}\int_0^4 v^2 dv = \frac{1}{2}\left[\frac{}{3}\right]_0^4 \\ &= \frac{1}{6}\left[2^3 - 0\right] \\ &= \frac{4}{3} \end{aligned}$$

From the above discussions it is observed that

1. $E[X(t)] = 1$=Constant and
2. $E[X(t)X(t+\tau)] = R_X(\tau) = \frac{4}{3}$, it is independent of absolute time hence

$X(t)$ is a wide-sense stationary.

---

17. A random process described by

$$X(t) = V$$

where V is a random variable uniformly distributed between 0 and 3. That means that each realization of $X(t) = V$ is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 3. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$\begin{aligned} f_V(v) &= \frac{1}{b-a} = \frac{1}{3-(0)} \\ &= \frac{1}{2} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < \theta < 3 \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\begin{aligned} \mu_X &= E[X(t)] = E[V] = \frac{1}{3}\int_0^3 v dv = \frac{1}{3}\left[\frac{}{2}\right]_0^3 = \frac{1}{6}\left[3^2 - 0\right] \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} E[X(t)X(t+\tau)] &= E[V^2] = \frac{1}{3}\int_0^3 v^2 dv = \frac{1}{3}\left[\frac{}{3}\right]_0^3 \\ &= \frac{1}{9}\left[3^3 - 0\right] \\ &= 3 \end{aligned}$$

From the above discussions it is observed that

1. $E[X(t)] = \frac{3}{2}$=Constant and
2. $E[X(t)X(t+\tau)] = R_X(\tau) = 3$, it is independent of absolute time hence

$X(t)$ is a wide-sense stationary.

1.4.3 Addition of Random Processes:

Consider the addition of two random independent, jointly wide-sense stationary random precess $X(t)$ and $Y(t)$ is

$$W(t) = X(t) + Y(t)$$

The autocorrelation function of the sum $W(t)$ is

$$\begin{aligned} R_W(\tau) &= E[\{X(t)+Y(t)\}\{X(t+\tau)+Y(t+\tau)\}] \\ &= E[X(t)X(t+\tau)+X(t)Y(t+\tau)+Y(t)X(t+\tau)+Y(t)Y(t+\tau)\}] \\ &= R_X(\tau)+R_{XY}(\tau)+R_{YX}(\tau)+R_Y(\tau) \end{aligned}$$

If $X(t)$ and $Y(t)$ are independent, then the cross-correlation function is

$$R_{XY}(\tau) = R_{YX}(\tau) = \mu_X \mu_Y$$

If $X(t)$ or $Y(t)$ has a zero mean then

$$R_W(\tau) = R_X(\tau) + R_Y(\tau)$$

18. The autocorrelation function for random process $Z(t)$ is

$$R_Z(\tau) = \begin{cases} 50(1-|\tau|/T) & -T \le t \le T \\ 0 & otherwise \end{cases}$$

A random process $X(t)$ is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 4$, a variance $\sigma_U^2 = 25$, and is independent of $Z(t)$. Find the autocorrelation function of $X(t)$ [?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} R_X(\tau) &= E[\{Z(t)+U\}\{Z(t+\tau)+U\}] \\ &= E[Z(t)Z(t+\tau)+UZ(t+\tau)+UZ(t)+U^2] \\ &= R_Z(\tau)+0+0+E[U^2] \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+25+4^2 \\ &= R_Z(\tau)+41 \end{aligned}$$

---

19. The autocorrelation function for random process $Z(t)$ is

$$R_Z(\tau) = \begin{cases} 40(1-|\tau|/T) & -T \le t \le T \\ 0 & otherwise \end{cases}$$

A random process $X(t)$ is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 5$, a variance $\sigma_U^2 = 15$, and is independent of $Z(t)$. Find the autocorrelation function of $X(t)$ [?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} R_X(\tau) &= E[\{Z(t)+U\}\{Z(t+\tau)+U\}] \\ &= E[Z(t)Z(t+\tau)+UZ(t+\tau)+UZ(t)+U^2] \\ &= R_Z(\tau)+0+0+E[U^2] \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+15+5^2 \\ &= R_Z(\tau)+40 \end{aligned}$$

---

20. The autocorrelation function for random process $Z(t)$ is

$$R_Z(\tau) = \begin{cases} 60(1-|\tau|/T) & -T \le t \le T \\ 0 & otherwise \end{cases}$$

A random process $X(t)$ is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 4$, a variance $\sigma_U^2 = 20$, and is independent of $Z(t)$. Find the autocorrelation function of $X(t)$ [?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} R_X(\tau) &= E[\{Z(t)+U\}\{Z(t+\tau)+U\}] \\ &= E[Z(t)Z(t+\tau)+UZ(t+\tau)+UZ(t)+U^2] \\ &= R_Z(\tau)+0+0+E[U^2] \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+\sigma_U^2+\mu_U^2 \\ &= R_Z(\tau)+20+4^2 \\ &= R_Z(\tau)+36 \end{aligned}$$

---

21. The random process $X(t)$ has the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) & -\tau_N \leq \tau \leq \tau_N \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is independent of $X(t)$ and has the autocorrelation function

$$R_Y(\tau) = \begin{cases} 15(1 - |\tau|/T) & -T \leq t \leq T \\ 0 & otherwise \end{cases}$$

$$X(t) \quad = \quad Z(t) + U$$

where $T \gg \tau_N$. The random process $Z(t) = X(t) + Y(t)$. For $Z(t)$, find the autocorrelation function, its total power, its dc power, and its ac power. Is $Z(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 15, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_Z(\tau) \quad = \quad R_X(\tau) + R_Y(\tau)$$

It is mentioned that $T \gg \tau_N$, $R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we have to consider the limits for $R_Z(\tau)$ from $0 \leq |\tau| \leq \tau_N$ and $\tau_N \leq |\tau| \leq T$
From $0 \leq |\tau| \leq \tau_N$ the $R_Z(\tau)$ is

$$R_Z(\tau) \quad = \quad 15(1 - |\tau|/T) + 10(1 - |\tau|/\tau_N)$$

From $\tau_N \leq |\tau| \leq T$ the $R_Z(\tau)$ is

$$R_Z(\tau) \quad = \quad 15(1 - |\tau|/T)$$

$$R_Z(\tau) = \begin{cases} 15(1 - |\tau|/T) + 10(1 - |\tau|/\tau_N) & 0 \leq |\tau| \leq \tau_N \\ 15(1 - |\tau|/T) & \tau_N \leq |\tau| \leq T \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$\begin{aligned} &= \sigma_X^2 + \sigma_Y^2 = 10 + 15 \\ &= 25 \end{aligned}$$

22. The random process $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) + Y(t)$ and

$$R_X(\tau) = 10\ exp\left(-\frac{|\tau|}{3}\right)$$

$$R_Y(\tau) = \begin{cases} 10\left(\frac{3-|\tau|}{3}\right) & -3 \le \tau \le 3 \\ 0 & otherwise \end{cases}$$

For $W(t)$, find its autocorrelation function, its total power, its dc power, and its ac power. Is $W(t)$ wide-sense stationary? [?]

Solution:

It is mentioned that $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 10, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_W(\tau) \quad = \quad R_X(\tau) + R_Y(\tau)$$

$R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we to consider the limits for $R_Z(\tau)$ from $0 \le |\tau| \le 3$ and other than this limits
From $0 \le |\tau| \le 3$ the $R_Z(\tau)$ is

$$R_Z(\tau) \quad = \quad 10\ exp\left(-\frac{|\tau|}{3}\right) + 10\left(\frac{3-|\tau|}{3}\right)$$

Otherwise

$$R_Z(\tau) \quad = \quad 10\left(\frac{3-|\tau|}{3}\right)$$

$$R_Z(\tau) = \begin{cases} 10\ exp\left(-\frac{|\tau|}{3}\right) + 10\left(\frac{3-|\tau|}{3}\right) & 0 \le |\tau| \le 3 \\ \\ 10\left(\frac{3-|\tau|}{3}\right) & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$\begin{aligned} &= \sigma_X^2 + \sigma_Y^2 = 10 + 10 \\ &= 20 \end{aligned}$$

23. The random process $X(t)$ has the autocorrelation function

$$R_X(\tau) = \begin{cases} 10\ exp(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is independent of $X(t)$ and has the autocorrelation function

$$R_Y(\tau) = 13\left(\frac{}{\omega_{B\tau}}\right)$$

where $(2\pi)/\omega_B \gg \tau_N$. The random process $Z(t) = X(t)+Y(t)$. For $Z(t)$, find the autocorrelation function, its total power, its dc power, and its ac power. Is $Z(t)$ wide-sense stationary? [?]

Solution:

It is mentioned that $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 13, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_W(\tau) = R_X(\tau) + R_Y(\tau)$$

$R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we have to consider the limits for $R_Z(\tau)$ from $0 \le |\tau| \le \tau_N$ and other than this limits From $0 \le |\tau| \le \tau_N$ the $R_Z(\tau)$ is

$$R_Z(\tau) = 10\ exp(1 - |\tau|/\tau_N) + 13\left(\frac{}{\omega_{B\tau}}\right)$$

Otherwise

$$R_Z(\tau) = 13\left(\frac{}{\omega_{B\tau}}\right)$$

$$R_Z(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) + 13\left(\frac{sin(\omega_{B\tau})}{\omega_{B\tau}}\right) & 0 \le |\tau| \le 3 \\ 13\left(\frac{sin(\omega_{B\tau})}{\omega_{B\tau}}\right) & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$\begin{aligned} &= \sigma_X^2 + \sigma_Y^2 = 10 + 13 \\ &= 23 \end{aligned}$$

24. A random process $X(t)$ has the autocorrelation function

$$R_X(\tau) \quad = \quad 15exp\left(-2|\tau|\right) \qquad -\infty < \tau < \infty$$

The random process $Y(t)$ is $Y(t) = X(t) - 3$

a. What is the autocorrelation function of $Y(t)$?

b. What are the total power, the dc power, and the ac power of $Y(t)$?

c. What is the cross-correlation $R_{XY}(\tau)$? [?]

Solution:

It is given that $R_X(\tau)$ has $\sigma_X^2 = 15$ and $\mu_X = 0$.
It is given that $Y(t) = X(t) - 3$ $\sigma_Y^2 = 15$ and $\mu_X = -3$.

a. The autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t)-3\}\{X(t+\tau)-3\}] \\ &= E[X(t)X(t+\tau) - 3X(t+\tau) - 3X(t) + 9] \\ &= R_{XY}(\tau) + 9 \end{aligned}$$

b. The total power, the dc power, and the ac power of $Y(t)$ are
The total power

$$E[Y^2(t)] \quad = \quad \sigma_Y^2 + \mu_Y^2 = 15 + 9 = 24$$

The DC power is

$$\mu_Y^2 \quad = \quad 9$$

The AC power is

$$\sigma_Y^2 \quad = \quad 15$$

c. The cross-correlation $R_{XY}(\tau)$ is

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau)-3\}] \\ &= E[X(t)X(t+\tau) - 3X(t)] \\ &= R_X(\tau) \end{aligned}$$

25. A random process $X(t)$ has the autocorrelation function

$$R_X(\tau) \quad = \quad 10cos(\omega_c\tau) \qquad -\infty < \tau < \infty$$

The random process $Y(t)$ is $Y(t) = X(t) - 4$

a. What is the autocorrelation function of $Y(t)$?

b. What are the total power, the dc power, and the ac power of $Y(t)$?

c. What is the cross-correlation $R_{XY}(\tau)$? [?]

Solution:

It is given that $R_X(\tau)$ has $\sigma_X^2 = 10 \; and \; \mu_X = 0$.
It is given that $Y(t) = X(t) - 4 \; \sigma_Y^2 = 10 \; and \; \mu_X = -4$.

a. The autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t)-4\}\{X(t+\tau)-4\}] \\ &= E[X(t)X(t+\tau) - 4X(t+\tau) - 4X(t) + 16] \\ &= R_{XY}(\tau) + 16 \end{aligned}$$

b. The total power, the dc power, and the ac power of $Y(t)$ are
The total power

$$E[Y^2(t)] = \sigma_Y^2 + \mu_Y^2 = 10 + 16 = 26$$

The DC power is

$$\mu_Y^2 = 16$$

The AC power is

$$\sigma_Y^2 = 10$$

c. The cross-correlation $R_{XY}(\tau)$ is

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau)-4\}] \\ &= E[X(t)X(t+\tau) - 4X(t)] \\ &= R_X(\tau) \end{aligned}$$

26. A random process $X(t)$ has the autocorrelation function

$$R_X(\tau) = \begin{cases} 13(1-|\tau|/T) & -T \le \tau \le T \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is $Y(t) = X(t) - 2$

a. What is the autocorrelation function of $Y(t)$?

b. What are the total power, the dc power, and the ac power of $Y(t)$?

c. What is the cross-correlation $R_{XY}(\tau)$? [?]

Solution:

It is given that $R_X(\tau)$ has $\sigma_X^2 = 13 \; and \; \mu_X = 0$.
It is given that $Y(t) = X(t) - 2 \; \sigma_Y^2 = 13 \; and \; \mu_X = -2$.

a. The autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t)-2\}\{X(t+\tau)-2\}] \\ &= E[X(t)X(t+\tau) - 2X(t+\tau) - 2X(t) + 4] \\ &= R_{XY}(\tau) + 4 \end{aligned}$$

b. The total power, the dc power, and the ac power of $Y(t)$ are

The total power

$$E[Y^2(t)] = \sigma_Y^2 + \mu_Y^2 = 13 + 4 = 17$$

The DC power is

$$\mu_Y^2 = 4$$

The AC power is

$$\sigma_Y^2 = 13$$

c. The cross-correlation $R_{XY}(\tau)$ is

$$\begin{aligned}
R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau)-2\}] \\
&= E[X(t)X(t+\tau) - 2X(t)] \\
&= R_X(\tau)
\end{aligned}$$

27. $X(t)$ and $Y(t)$ are zero mean jointly wide-sense stationary random process. The random process $Z(t)$ is

$$Z(t) = 3X(t) + Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ *and* $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{aligned}
R_Z(\tau) &= E[Z(t)Z(t+\tau)] = E[\{3X(t)+Y(t)\}\{3X(t+\tau)+Y(t+\tau)\}] \\
&= E[\{9X(t)X(t+\tau)\} + \{3X(t)Y(t+\tau)\} + \{3X(t+\tau)Y(t)\} + \{Y(t)Y(t+\tau)\}] \\
&= 9R_X(\tau) + 3R_{XY}(\tau) + 3R_{YX}(\tau) + R_Y(\tau)
\end{aligned}$$

$$\begin{aligned}
R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{3X(t)+Y(t)\}X(t+\tau)] \\
&= E[\{3X(t)X(t+\tau)\} + \{Y(t)X(t+\tau)\}] \\
&= 3R_X(\tau) + R_{YX}(\tau)
\end{aligned}$$

$$\begin{aligned}
R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{3X(t+\tau)+Y(t+\tau)\}] \\
&= E[\{3X(t)X(t+\tau)\} + \{X(t)Y(t+\tau)\}] \\
&= 3R_X(\tau) + R_{XY}(\tau)
\end{aligned}$$

$$\begin{aligned}
R_{ZY}(\tau) &= E[Z(t)Y(t+\tau)] = E[\{3X(t)+Y(t)\}Y(t+\tau)] \\
&= E[\{3X(t)Y(t+\tau)\} + \{Y(t)Y(t+\tau)\}] \\
&= 3R_{XY}(\tau) + R_Y(\tau)
\end{aligned}$$

$$\begin{aligned}
R_{YZ}(\tau) &= E[Y(t)Z(t+\tau)] = E[Y(t)\{3X(t+\tau)+Y(t+\tau)\}] \\
&= E[\{3Y(t)X(t+\tau)\} + \{Y(t)Y(t+\tau)\}] \\
&= 3R_{YX}(\tau) + R_Y(\tau)
\end{aligned}$$

28. $X(t)$ **and** $Y(t)$ **are zero mean jointly wide-sense stationary random process. The random process** $Z(t)$ **is**

$$Z(t) = 3X(t) + 2Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ *and* $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] = E[\{3X(t)+2Y(t)\}\{3X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{9X(t)X(t+\tau)\} + \{6X(t)Y(t+\tau)\} + \{6X(t+\tau)Y(t)\} + 4\{Y(t)Y(t+\tau)\}] \\ &= 9R_X(\tau) + 6R_{XY}(\tau) + 6R_{YX}(\tau) + 4R_Y(\tau) \end{aligned}$$

$$\begin{aligned} R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{3X(t)+2Y(t)\}X(t+\tau)] \\ &= E[\{3X(t)X(t+\tau)\} + 2\{Y(t)X(t+\tau)\}] \\ &= 3R_X(\tau) + 2R_{YX}(\tau) \end{aligned}$$

$$\begin{aligned} R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{3X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{3X(t)X(t+\tau)\} + 2\{X(t)Y(t+\tau)\}] \\ &= 3R_X(\tau) + 2R_{XY}(\tau) \end{aligned}$$

$$\begin{aligned} R_{ZY}(\tau) &= E[Z(t)Y(t+\tau)] = E[\{3X(t)+2Y(t)\}Y(t+\tau)] \\ &= E[\{3X(t)Y(t+\tau)\} + 2\{Y(t)Y(t+\tau)\}] \\ &= 3R_{XY}(\tau) + 2R_Y(\tau) \end{aligned}$$

$$\begin{aligned} R_{YZ}(\tau) &= E[Y(t)Z(t+\tau)] = E[Y(t)\{3X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{6Y(t)X(t+\tau)\} + 2\{Y(t)Y(t+\tau)\}] \\ &= 6R_{YX}(\tau) + 2R_Y(\tau) \end{aligned}$$

29. $X(t)$ **and** $Y(t)$ **are zero mean jointly wide-sense stationary random process. The random process** $Z(t)$ **is**

$$Z(t) = X(t) + 2Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ *and* $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] = E[\{X(t)+2Y(t)\}\{X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{X(t)X(t+\tau)\} + \{2X(t)Y(t+\tau)\} + \{2X(t+\tau)Y(t)\} + 4\{Y(t)Y(t+\tau)\}] \\ &= R_X(\tau) + 2R_{XY}(\tau) + 2R_{YX}(\tau) + 4R_Y(\tau) \end{aligned}$$

$$\begin{aligned} R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{X(t)+2Y(t)\}X(t+\tau)] \\ &= E[\{X(t)X(t+\tau)\} + 2\{Y(t)X(t+\tau)\}] \\ &= R_X(\tau) + 2R_{YX}(\tau) \end{aligned}$$

$$\begin{aligned} R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{X(t)X(t+\tau)\}+2\{X(t)Y(t+\tau)\}] \\ &= R_X(\tau)+2R_{XY}(\tau) \end{aligned}$$

$$\begin{aligned} R_{ZY}(\tau) &= E[Z(t)Y(t+\tau)] = E[\{X(t)+2Y(t)\}Y(t+\tau)] \\ &= E[\{X(t)Y(t+\tau)\}+2\{Y(t)Y(t+\tau)\}] \\ &= R_{XY}(\tau)+2R_Y(\tau) \end{aligned}$$

$$\begin{aligned} R_{YZ}(\tau) &= E[Y(t)Z(t+\tau)] = E[Y(t)\{X(t+\tau)+2Y(t+\tau)\}] \\ &= E[\{Y(t)X(t+\tau)\}+2\{Y(t)Y(t+\tau)\}] \\ &= R_{YX}(\tau)+2R_Y(\tau) \end{aligned}$$

1.4.4 Multiplication of Random Process:

Consider the multiplication of two random independent, jointly wide-sense stationary random precess $X(t)$ and $Y(t)$ is

$$W(t) = X(t)Y(t)$$

The mean of $W(t)$ is

$$E[W(t)] = \mu_W = E[X(t)]E[Y(t)] = \mu_X\mu_Y$$

The variance of $W(t)$ is

$$\begin{aligned} \sigma_W^2 &= E[X(t)]E[Y(t)-\mu_X\mu_Y]^2 \\ &= (\sigma_X^2+\mu_X^2)(\sigma_Y^2+\mu_Y^2)\mu_X^2\mu_Y^2 \end{aligned}$$

The autocorrelation function of the sum $W(t)$ is

$$\begin{aligned} R_W(\tau) &= E[\{X(t)Y(t)\}\{X(t+\tau)+Y(t+\tau)\}] \\ &= E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)] \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

30. The random process $X(t)$ is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1-|\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is independent of $X(t)$ and has the autocorrelation function

$$R_Y(\tau) = \begin{cases} 15(1-|\tau|/T) & -T \le t \le T \\ 0 & otherwise \end{cases}$$

where $T \gg \tau_N$. The random process $Z(t) = X(t)\times Y(t)$. For $Z(t)$, find the autocorrelation function, its total power, its dc power, and its ac power. Is $Z(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 15, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau)\times R_Y(\tau)$$

$$\begin{aligned} R_Z(\tau) &= 15(1-|\tau|/T) \times 10(1-|\tau|/\tau_N) \\ &= 150(1-|\tau|/T)(1-|\tau|/\tau_N) \end{aligned}$$

$$R_Z(\tau) = \begin{cases} 150(1-|\tau|/T)(1-|\tau|/\tau_N) & |\tau| \le \tau_N \\ \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 150

31. The random process $X(t)$ is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1-|\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is independent of $X(t)$ and has the autocorrelation function

$$R_Y(\tau) = 18cos(\omega_c\tau) \quad -\infty \le \tau \le \infty$$

where $2\pi/\omega_c \gg \tau_N$. The random process $Z(t) = X(t) \times Y(t)$. For $Z(t)$, find the autocorrelation function, its total power, its dc power, and its ac power. Is $Z(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 18, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$\begin{aligned} R_Z(\tau) &= 15(1-|\tau|/T) \times 18cos(\omega_c\tau) \\ &= 180(1-|\tau|/T)(cos(\omega_c\tau)) \end{aligned}$$

$$R_Z(\tau) = \begin{cases} 180(cos(\omega_c\tau))(1-|\tau|/T) & |\tau| \le \tau_N \\ \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 180

32. The random process $X(t)$ is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1-|\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process $Y(t)$ is independent of $X(t)$ and has the autocorrelation function

$$R_Y(\tau) = 13\frac{sin(\omega_B \tau)}{(\omega_B \tau)} \qquad -\infty \le \tau \le \infty$$

where $2\pi/\omega_B \gg \tau_N$. The random process $Z(t) = X(t) \times Y(t)$. For $Z(t)$, find the autocorrelation function, its total power, its dc power, and its ac power. Is $Z(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 13, \mu_Y = 0$, **Autocorrelation function** $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$\begin{aligned} R_Z(\tau) &= 10(1 - |\tau|/T) \times 13\frac{sin(\omega_B \tau)}{(\omega_B \tau)} \\ &= 130(1 - |\tau|/T)\frac{sin(\omega_B \tau)}{(\omega_B \tau)} \end{aligned}$$

$$R_Z(\tau) = \begin{cases} 130\frac{sin(\omega_B \tau)}{(\omega_B \tau)}(1 - |\tau|/T) & |\tau| \le \tau_N \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$
2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 130

33. The random process $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 10exp\left(-\frac{|\tau|}{3}\right) \qquad -\infty \le \tau \le \infty$$

$$R_Y(\tau) = \begin{cases} 11\left(\frac{3-|\tau|}{3}\right) & -3 \le \tau \le 3 \\ 0 & otherwise \end{cases}$$

For $W(t)$ find its autocorrelation function, its total power, its dc power, and its ac power. Is $W(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 11, \mu_Y = 0$, **Autocorrelation function** $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$\begin{aligned} R_W(\tau) &= 10exp\left(-\frac{|\tau|}{3}\right) \times 11\left(\frac{3-|\tau|}{3}\right) \\ &= 110exp\left(-\frac{|\tau|}{3}\right)\left(\frac{3-|\tau|}{3}\right) \end{aligned}$$

$$R_W(\tau) = \begin{cases} 110exp\left(-\frac{|\tau|}{3}\right)\left(\frac{3-|\tau|}{3}\right) & |\tau| \leq \tau_N \\ \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_W(\tau) = 0$
2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 110

34. The random process $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 12exp\left(-\frac{|\tau|}{4}\right) \qquad -\infty \leq \tau \leq \infty$$

$$R_Y(\tau) = \begin{cases} 10\left(\frac{4-|\tau|}{4}\right) & -4 \leq \tau \leq 4 \\ 0 & otherwise \end{cases}$$

For $W(t)$ find its autocorrelation function, its total power, its dc power, and its ac power. Is $W(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 12, \mu_X = 0, \sigma_Y^2 = 10, \mu_Y = 0,$ **Autocorrelation function** $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$\begin{aligned} R_W(\tau) &= 12exp\left(-\frac{|\tau|}{4}\right) \times 10\left(\frac{4-|\tau|}{4}\right) \\ &= 120exp\left(-\frac{|\tau|}{4}\right)\left(\frac{4-|\tau|}{4}\right) \end{aligned}$$

$$R_W(\tau) = \begin{cases} 120exp\left(-\frac{|\tau|}{4}\right)\left(\frac{4-|\tau|}{4}\right) & |\tau| \leq 4 \\ \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_W(\tau) = 0$
2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 120

35. The random process $X(t)$ and $Y(t)$ are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 11exp\left(-\frac{|\tau|}{5}\right) \qquad -\infty \leq \tau \leq \infty$$

$$R_Y(\tau) = \begin{cases} 12\left(\frac{5-|\tau|}{5}\right) & -5 \leq \tau \leq 5 \\ 0 & otherwise \end{cases}$$

For $W(t)$ find its autocorrelation function, its total power, its dc power, and its ac power. Is $W(t)$ wide-sense stationary? [?]

Solution:

$\sigma_X^2 = 11, \mu_X = 0, \sigma_Y^2 = 12, \mu_Y = 0$, **Autocorrelation function** $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$\begin{aligned} R_W(\tau) &= 11exp\left(-\frac{|\tau|}{5}\right) \times 12\left(\frac{5-|\tau|}{5}\right) \\ &= 132exp\left(-\frac{|\tau|}{5}\right)\left(\frac{5-|\tau|}{5}\right) \end{aligned}$$

$$R_W(\tau) = \begin{cases} 132exp\left(-\frac{|\tau|}{5}\right)\left(\frac{5-|\tau|}{5}\right) & |\tau| \leq 5 \\ \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_W(\tau) = 0$
2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 132

1.5 Ergodic Random Processes

Any random process is wide-sense stationarity if it satisfies the conditions like mean and and autocorrelation function are independent of time. To calculate mean and autocorrelation function of a random process, it requires an ensemble of sample functions (data records). It is difficult to collect the data in real time situations. In many real-life applications Its convenient to calculate the averages from a single data record. This is possible in certain random processes called ergodic processes.

A random process is ergodic if time averages with a sample function equal to ensemble. The avaragieng of random process is defined as

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt$$

The autocorrelation function for random process $x(t)$ is defined as

$$\langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

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36. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0,\ 2\ and\ 4\ ms$, where $\Delta t = 2ms$

x(t)	**1.5**	**2.1**	**1.0**	**2.2**	**-1.6**	**-2.0**	**-2.5**	**2.5**	**1.6**	**-1.8**
k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0,\ 2\ and\ 4\ ms$, where $\Delta t = 2ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 2$ and $\Delta t = 2ms$ $i = 2/2 = 1$, $\tau = 4$ and $\Delta t = 2ms$ $i = 4/2 = 2$

$$\begin{aligned}
r_X(i) &= \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i) \\
r_X(0) &= \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n}\left[x(k)^2\right] \\
&= \frac{1}{10}\left[(1.5)^2 + (2.1)^2 + (1.0)^2 + (2.2)^2 + (-1.6)^2 + (-2.0)^2 + (-2.5)^2 + (2.5)^2 + (1.6)^2 + (-1.8)^2\right] \\
&= 3.736 \\
r_X(2\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-2} x(k)x(k+1) \\
&= \frac{1}{10}[(1.5)(2.1) + (2.1)(1.0) + (1.0)(2.2) + (2.2)(-1.6) + (-1.6)(-2.0) + (-2.0)(-2.5) \\
&+ (-2.5)(2.5) + (2.5)(1.6) + (1.6)(-1.8)] \\
&= \frac{1}{10}[3.15 + 2.1 + 2.2 - 3.52 + 3.2 + 5 - 6.25 + 4 - 2.88] \\
&= -0.700
\end{aligned}$$

$$\begin{aligned}
r_X(4\ ms) &= \frac{1}{n}\sum_{k=0}^{k=N-3} x(k)x(k+2) \\
&= \frac{1}{10}[(1.5)(1.0)+(2.1)(2.2)+(1.0)(-1.6)+(2.2)(-2.0)+(-1.6)(-2.5)+(-2.0)(2.5) \\
&+ (-2.5)(1.6)+(2.5)(-1.8)] \\
&= -0.938
\end{aligned}$$

37. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0,\ 3\ and\ 6\ ms$, where $\Delta t = 3ms$

x(t)	1.0	2.2	1.5	-3.0	-0.5	1.7	-3.5	-1.5	1.6	-1.3
k	0	1	2	3	4	5	6	7	8	9

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n}\sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0,\ 3\ and\ 6\ ms$, where $\Delta t = 3ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 3$ and $\Delta t = 3ms$ $i = 3/3 = 1$, $\tau = 6$ and $\Delta t = 23ms$ $i = 6/3 = 2$

$$\begin{aligned}
r_X(i) &= \frac{1}{n}\sum_{k=0}^{k=N-1-i} x(k)x(k+i) \\
r_X(0) &= \frac{1}{n}\sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n}\left[x(k)^2\right] \\
&= \frac{1}{10}\left[(1.0)^2+(2.2)^2+(1.5)^2+(-3.0)^2+(-0.5)^2+(1.7)^2+(-3.5)^2+(-1.5)^2+(1.6)^2+(-1.3)^2\right] \\
&= 3.898
\end{aligned}$$

$$\begin{aligned}
r_X(3\ ms) &= \frac{1}{n}\sum_{k=0}^{k=N-2} x(k)x(k+1) \\
&= \frac{1}{10}[(1.0)(2.2)+(2.2)(1.5)+(1.5)(-3.0)+(-3.0)(-0.5)+(-0.5)(1.7)+(1.7)(-3.5) \\
&+ (-3.5)(-1.5)+(-1.5)(1.6)+(1.6)(-1.3)] \\
&= -0.353
\end{aligned}$$

$$\begin{aligned}
r_X(6\ ms) &= \frac{1}{n}\sum_{k=0}^{k=N-3} x(k)x(k+2) \\
&= \frac{1}{10}[(1.0)(1.5)+(2.2)(-3.0)+(1.5)(-0.5)+(-3.0)(1.7)+(-0.5)(-3.5)+(1.7)(-1.5) \\
&+ (-3.5)(1.6)+(-1.5)(-1.3)] \\
&= -1.540
\end{aligned}$$

38. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0,\ 7\ and\ 14\ ms$, where $\Delta t = 7ms$

x(t)	1.5	0.4	0.8	0.3	-0.4	-1.7	2.0	-2.0	0.8	-0.2
k	0	1	2	3	4	5	6	7	8	9

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0,\ 2\ and\ 4\ ms$, where $\Delta t = 2ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 7$ and $\Delta t = 7ms\ i = 7/7 = 1$, $\tau = 14$ and $\Delta t = 7ms\ i = 14/7 = 2$

$$\begin{aligned}
r_X(i) &= \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i) \\
r_X(0) &= \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n}\left[x(k)^2\right] \\
&= \frac{1}{10}\left[(1.5)^2 + (0.4)^2 + (0.8)^2 + (0.3)^2 + (-0.4)^2 + (-1.7)^2 + (2.0)^2 + (-2.0)^2 + (0.8)^2 + (-0.2)^2\right] \\
&= 1.487
\end{aligned}$$

$$\begin{aligned}
r_X(7\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k+1) \\
&= \frac{1}{10}[(1.5)(0.4) + (0.4)(0.8) + (0.8)(0.3) + (0.3)(-0.4) + (-0.4)(-1.7) + (-1.7)(2.0) \\
&+ (2.0)(-2.0) + (-2.0)(0.8) + (0.8)(-0.2)] \\
&= -0.744 \\
r_X(14\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-2} x(k)x(k+2) \\
&= \frac{1}{10}[(1.5)(0.8) + (0.4)(0.3) + (0.8)(-0.4) + (0.3)(-1.7) + (-0.4)(2.0) + (-1.7)(-2.0) \\
&+ (2.0)(0.8) + (-2.0)(-0.2)] \\
&= 0.509
\end{aligned}$$

1.6 Power Spectral Densities

The power spectral density of a wide-sense stationary random process is the Fourier transform of the autocorrelation function. The distribution of power over range frequencies is described by power power density spectrum.

Fourier transform pair is

$$\begin{aligned} X_T(j\omega) &= \int_{-\infty}^{\infty} x_T(t)e^{-j\omega t}dt \\ x_T(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(j\omega)e^{j\omega t}d\omega \end{aligned}$$

The power of signal is its square. Averaging the power of the signal $x(t)$ within a window of width T is

$$\begin{aligned} P_T &= \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x_T^2(t)dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)\frac{1}{2\pi}\int_{-\infty}^{\infty} x_T(t)e^{j\omega}d\omega dt \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} x_T(j\omega)\frac{1}{2\pi}\int_{-\infty}^{\infty} x_T(t)e^{j\omega t}dtd\omega \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} x_T(j\omega)x_T(-j\omega)d\omega \end{aligned}$$

$$x_T(-j\omega) = X_T^*(j\omega)$$

$$\begin{aligned} P_T &= \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(j\omega)|^2 d\omega \\ \int_{-T/2}^{T/2} X^2(t)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(j\omega)|^2 d\omega \end{aligned}$$

Expectation of the above equation is

$$\begin{aligned} E\left[\int_{-T/2}^{T/2} X^2(t)dt\right] &= \frac{1}{2\pi}E\left[\int_{-\infty}^{\infty} |X_T(j\omega)|^2\right] d\omega \\ \int_{-T/2}^{T/2} E[X^2(t)]dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_T(j\omega)|^2]d\omega \\ TE[X^2(t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_T(j\omega)|^2]d\omega \\ TR_X(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_T(j\omega)|^2]d\omega \\ R_X(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} E[|X_T(j\omega)|^2]d\omega \end{aligned}$$

The above equation can be split into two pars as

$$\begin{aligned} R_X(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\ S_X(\omega) &= \lim_{T\to\infty} E[|X_T(j\omega)|^2] \end{aligned}$$

39. A PSD is as shown in Figure 1.2 where the constants are $a = 55,\ b = 5,\ \omega_o = 1000$ and $\omega_1 = 100$. Calculate values for $E[X^2(t)]$, the σ_X^2 and $|\mu_X|$.

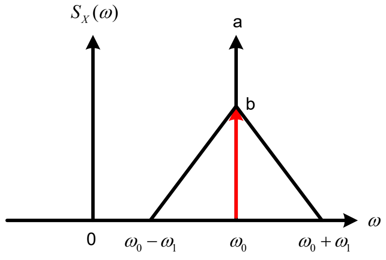

Figure 1.2

Solution:

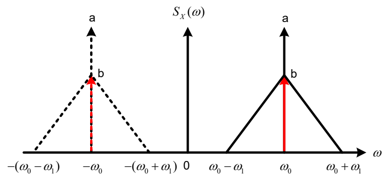

Figure 1.3

$$\begin{aligned}
\omega_o - \omega_1 &= 1000 - 100 = 900 \\
\omega_o + \omega_1 &= 1000 + 100 = 1100 \\
(\omega_o + \omega_1) - (\omega_o - \omega_1) &= 1100 - 1000 = 200
\end{aligned}$$

The given spectrum is in the triangular form, its base is 200 and its height is 5 and it also has $a = 55$. The modified two sided spectrum is redrawn and is as shown in Figure 1.3. The x axis is ω *radians/sec*. The Autocorrelation function is

$$\begin{aligned}
R_X(0) &= E[X^2(t)] = Area\ under\ PSD = Area\ of\ impulse + Area\ of\ Triangle \\
&= 2 \times \frac{1}{2\pi}[Area\ of\ impulse + Area\ of\ Triangle] \\
&= \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{}{2\pi}\left(55 + \frac{1}{2}(5)(200)\right) \\
&= 176.6620 \\
|\mu_X| &= 0
\end{aligned}$$

40. A PSD is as shown in Figure 1.2 where the constants are $a = 450,\ b = 6,\ \omega_o = 10,000$ and $\omega_1 = 1000$. Calculate values for $E[X^2(t)]$, the σ_X^2 and $|\mu_X|$.

Solution:

The given spectrum has the following details constant $a = 450$

$$\begin{aligned}
\omega_o - \omega_1 &= 10,000 - 1000 = 9900 \\
\omega_o + \omega_1 &= 10000 + 1000 = 11000 \\
(\omega_o + \omega_1) - (\omega_o - \omega_1) &= 11000 - 9000 = 2000
\end{aligned}$$

The signal magnitude is $b = 6$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= E[X^2(t(] = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{}{2\pi}\left(450 + \frac{1}{2}(6)(2000)\right) \\
&= 2053.0988 \\
|\mu_X| &= 0
\end{aligned}$$

41. A PSD is as shown in Figure 1.2 where the constants are $a = 72,\ b = 4,\ \omega_o = 1000$ and $\omega_1 = 50$. Calculate values for $E[X^2(t)]$, the σ_X^2 and $|\mu_X|$.

Solution:

The given spectrum has the following details constant $a = 72$

$$\begin{aligned}
\omega_o - \omega_1 &= 1000 - 50 = 950 \\
\omega_o + \omega_1 &= 1000 + 50 = 1050 \\
(\omega_o + \omega_1) - (\omega_o - \omega_1) &= 1050 - 950 = 100
\end{aligned}$$

The signal magnitude is $b = 4$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= E[X^2(t(] = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{}{2\pi}\left(72 + \frac{1}{2}(4)(100)\right) \\
&= 86.5803 \\
|\mu_X| &= 0
\end{aligned}$$

42. A PSD is as shown in Figure 1.4 where the constants are $a = 3,\ b = 5,\ \omega_1 = 8$ and $\omega_2 = 12$. Calculate values for $R_X(0)$, the variance and μ_X^2.

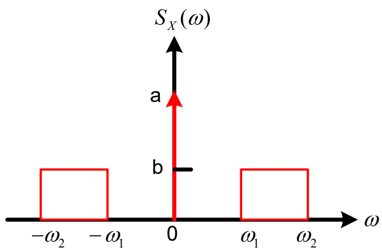

Figure 1.4

Solution:

$$\begin{aligned}
\omega_1 &= 8 \\
\omega_2 &= 12 \\
\omega_2 - \omega_1 &= 12 - 8 = 4
\end{aligned}$$

The given spectrum is in the rectangular form, its base is 4 and its height is 5 and it also has $a = 3$ at $\omega = 0$. The x axis is ω *radians/sec*. The Autocorrelation function is

$$\begin{aligned}
R_X(0) &= E[X^2(t)] = Area\ under\ PSD = Area\ of\ impulse + Area\ of\ Rectangle \\
&= \frac{1}{2\pi}[Area\ of\ impulse + Area\ of\ Rectangle] \\
&= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{1}{2\pi}\left(3 + 2[(5)(12-8)]\right) \\
&= 6.8437 \\
\mu_X^2 &= \frac{3}{2\pi} = 0.4775 \\
\sigma_X^2 &= R_X(0) - \mu_X^2 = 6.8437 - 0.4775 \\
&= 6.3662
\end{aligned}$$

43. A PSD is as shown in Figure 1.4 where the constants are $a = 5,\ b = 3,\ \omega_1 = 7$ and $\omega_2 = 13$. Calculate values for $R_X(0)$, the variance and μ_X^2.

Solution:

The given spectrum has the following details constant $a = 5$

$$\begin{aligned}
\omega_1 &= 7 \\
\omega_2 &= 13 \\
\omega_2 - \omega_1 &= 13 - 7 = 6
\end{aligned}$$

The signal magnitude is $b = 3$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{1}{2\pi}\left(5 + 2(3)(13-7)\right) \\
&= 6.5254 \\
\mu_X^2 &= \frac{3}{2\pi} = 0.7958 \\
\sigma_X^2 &= R_X(0) - \mu_X^2 = 6.5254 - 0.7958 \\
&= 5.7296
\end{aligned}$$

44. A PSD is as shown in Figure 1.4 where the constants are $a = 4,\ b = 3,\ \omega_1 = 9$ and $\omega_2 = 14$. Calculate values for $R_X(0)$, the variance and μ_X^2.

Solution:

The given spectrum has the following details constant $a = 4$

$$\begin{aligned}
\omega_1 &= 9 \\
\omega_2 &= 14 \\
\omega_2 - \omega_1 &= 14 - 9 = 5
\end{aligned}$$

The signal magnitude is $b = 3$

Autocorrelation function is

$$\begin{aligned} R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\ &= \frac{1}{2\pi}\left(4 + 2(3)(14-9)\right) \\ &= 5.4113 \\ \mu_X^2 &= \frac{3}{2\pi} = 0.6366 \\ \sigma_X^2 &= R_X(0) - \mu_X^2 = 5.4113 - 0.6366 \\ &= 4.7746 \end{aligned}$$

45. A PSD is as shown in Figure 1.5 where the constants are $a = 5$, $\omega_o = 100$ and $w = 8$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

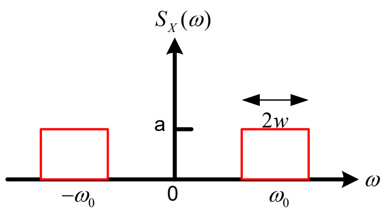

Figure 1.5

Solution:

The given spectrum has the following details

$$\begin{aligned} w &= 8 \\ 2w &= 16 \end{aligned}$$

The signal magnitude is $a = 5$

Autocorrelation function is

$$\begin{aligned} R_X(0) &= \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\ &= \frac{2}{2\pi}\left(5(16)\right) \\ &= 25.4648 \\ \mu_X &= 0 \end{aligned}$$

46. A PSD is as shown in Figure 1.5 where the constants are $a = 3$, $\omega_o = 150$ and $w = 7$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$\begin{aligned} w &= 7 \\ 2w &= 14 \end{aligned}$$

The signal magnitude is $a = 3$

Autocorrelation function is

$$\begin{aligned} R_X(0) &= \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\ &= \frac{2}{2\pi}(3(14)) \\ &= 13.3690 \\ \mu_X &= 0 \end{aligned}$$

47. A PSD is as shown in Figure 1.6 where the constants are $a = 4,\ \omega_o = 125$ and $w = 6$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$\begin{aligned} w &= 6 \\ 2w &= 12 \end{aligned}$$

The signal magnitude is $a = 3$
Autocorrelation function is

$$\begin{aligned} R_X(0) &= \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\ &= \frac{2}{2\pi}(4(12)) \\ &= 15.2789 \\ \mu_X &= 0 \end{aligned}$$

48. A PSD is as shown in Figure 1.6 where the constants are $a = 300,\ b = 10,\ \omega_M = 100$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

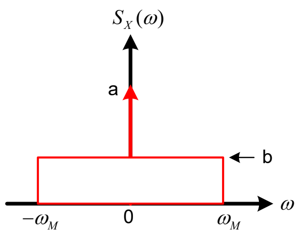

Figure 1.6

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 100 + 100 = 200$$

The signal magnitude is $a = 300,\ b = 10$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{1}{2\pi}\left(300 + (10)(200)\right) \\
\sigma_X^2 &= \frac{2000}{2\pi} = 318.3099 \\
\mu_X^2 &= \frac{300}{2\pi} = 47.7465 \\
|\mu_X| &= 6.9099 \\
\sigma_X^2 &= R_X(0) - \sigma_X^2 = 318.3099 - 47.7465 = 366.0564
\end{aligned}$$

49. A PSD is as shown in Figure 1.6 where the constants are $a = 200$, $b = 20$, $\omega_M = 80$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 80 + 80 = 160$$

The signal magnitude is $a = 200$, $b = 20$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{1}{2\pi}\left(200 + (20)(160)\right) \\
&= 541.1268 \\
\mu_X^2 &= \frac{300}{2\pi} = 47.7465 \\
|\mu_X| &= 6.9099 \\
\sigma_X^2 &= R_X(0) - \mu_X^2 = 541.1268 - 31.8310 = 509.2958
\end{aligned}$$

50. A PSD is as shown in Figure 1.6 where the constants are $a = 300$, $b = 15$, $\omega_M = 75$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 75 + 75 = 150$$

The signal magnitude is $a = 300$, $b = 15$

Autocorrelation function is

$$\begin{aligned}
R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega \\
&= \frac{1}{2\pi}\left(300 + (15)(150)\right) \\
&= 405.8451 \\
\mu_X^2 &= \frac{300}{2\pi} = 47.7465 \\
|\mu_X| &= 6.9099 \\
\sigma_X^2 &= R_X(0) - \mu_X^2 = 405.8451 - 47.7465 = 358.0986
\end{aligned}$$

	$R_X(\tau)$	$S_X(\omega)$
1	$cos\omega_c\tau$	$\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$
2	$\delta(\tau)$	1
3	$exp(-a\|\tau\|)\ a>0$	$\frac{2a}{\omega^2+a^2}$
4	$\begin{cases} T(1-\|\tau\|/T) & -T\le\tau\le T \\ 0 & otherwise \end{cases}$	$T^2\left(\frac{sin(\omega T/2)}{\omega T/2}\right)^2$
5	$(\omega_B/\pi)\frac{sin(\omega_B\tau)}{\omega_B\tau}$	$\begin{cases} 1 & -\omega_B\le\omega\le\omega_B \\ 0 & otherwise \end{cases}$

1.7 Weiner-Khinchin Relations

Weiner-Khinchin Relation states that the PSD and the autocorrelation function for wide-sense stationary random process are Fourier transform pair:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$
$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega$$

Let a rectangular function of time τ be

$$\begin{cases} T(1-|\tau|/T) & -T\le\tau\le T \\ 0 & otherwise \end{cases}$$

51. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 1 in table 4.1.

Solution:

$$S_X(\omega) = \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$$

Show that

$$R_X(\tau) = cos\omega_c\tau$$

It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)dx = g(x_0)$$

Based on the above relation

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega t}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]e^{j\omega\tau}d\omega \\ &= \frac{1}{2}\left[e^{j\omega_c\tau}+e^{-j\omega_c\tau}\right] \\ &= cos\omega_c\tau\end{aligned}$$

52. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 2 in table 4.1.

Solution:

$$S_X(\omega) = 1$$

Show that

$$R_X(\tau) = \delta(\tau)$$

It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0)$$

Based on the above relation

$$\begin{aligned} \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega t}d\tau &= \int_{-\infty}^{\infty} \delta(\tau)e^{-j\omega\tau}d\omega = e^0 \\ &= 1 \end{aligned}$$

53. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 3 in table 4.1.

$$R_X(\tau) = exp(-a|\tau|)\ a > 0$$

Show that

$$S_X(\omega) = \frac{2a}{\omega^2 + a^2}$$

Solution: It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0)$$

Based on the above relation

$$\begin{aligned} S_X(\omega) &= \int_{-\infty}^{0} e^{-a\tau}e^{-j\omega t}d\tau + \int_{-\infty}^{\infty} e^{a\tau}e^{-j\omega t}d\tau \\ &= \int_{-\infty}^{0} e^{(a-j\omega)\tau}d\tau + \int_{0}^{\infty} e^{-(a+j\omega)\tau}d\tau \\ &= \left[\frac{e^{(a-j\omega)\tau}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{(a+j\omega)\tau}}{(a+j\omega)}\right]_{0}^{\infty} \\ &= \frac{1}{(a-j\omega)} - \frac{1}{(a+j\omega)} = \frac{2a}{(a^2+\omega^2)} \end{aligned}$$

Multiplication of Two Random Process

Consider a two random process $X(t)$ and $Y(t)$ have the autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, multiplication of two random process is

$$W(t) = X(t)Y(t)$$

The multiplication of autocorrelation functions is

$$R_W(\tau) = R_X(\tau)R_Y(\tau)$$

$$\begin{aligned}
S_W(\omega) &= \int_{-\infty}^{\infty} R_X(\tau)R_Y(\tau)d\tau \\
&= \int_{-\infty}^{\infty} R_X(\tau)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(u)e^{ju\tau}du\right)e^{-j\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(u)\int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(u)S_X(\omega-u)du = \frac{1}{2\pi}S_X(\omega) * S_Y(\omega)
\end{aligned}$$

56. The random process $X(t)$ and $Y(t)$ have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)} \quad and \quad R_Y(\tau) = 5cos(600\tau)$$

If $Z(t) = X(t)Y(t)$, and if $X(t)$ and $Y(t)$ are independent , what is the PSD for $Z(t)$?

Solution:

$$R_X(\tau) = e^{(-10|\tau|)}$$

$$S_X(\omega) = \frac{20}{\omega^2 + 100}$$

$$R_Y(\tau) = 5cos(600\tau)$$

$$S_Y(\omega) = 5\pi[\delta(\omega - 600) + \delta(\omega + 600)]$$

$$\begin{aligned}
S_Z(\omega) &= \frac{1}{2\pi}S_X(\omega)S_Y(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 5\pi[\delta(\omega - u - 600) + \delta(\omega - u + 600)]\frac{20}{u^2+100}du \\
&= 50\left(\frac{1}{(\omega-600)^2+100} + \frac{1}{(\omega+600)^2+100}\right)
\end{aligned}$$

57. The random process $X(t)$ and $Y(t)$ have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)} \quad and \quad R_Y(\tau) = 6cos(400\tau)$$

If $Z(t) = X(t)Y(t)$, and if $X(t)$ and $Y(t)$ are independent , what is the PSD for $Z(t)$?

Solution:

$$\begin{aligned} R_X(\tau) &= e^{(-10|\tau|)} \\ S_X(\omega) &= \frac{20}{\omega^2+100} \\ R_Y(\tau) &= 6cos(400\tau) \\ S_Y(\omega) &= 6\pi[\delta(\omega-400)+\delta(\omega+400)] \end{aligned}$$

$$\begin{aligned} S_Z(\omega) &= \frac{1}{2\pi} S_X(\omega) S_Y(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 6\pi[\delta(\omega-u-400)+\delta(\omega-u+400)]\frac{20}{u^2+100}du \\ &= 60\left(\frac{1}{(\omega-400)^2+100}+\frac{1}{(\omega+400)^2+100}\right) \end{aligned}$$

58. The random process $X(t)$ and $Y(t)$ have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)} \quad and \quad R_Y(\tau) = 7cos(500\tau)$$

If $Z(t) = X(t)Y(t)$, and if $X(t)$ and $Y(t)$ are independent , what is the PSD for $Z(t)$?

Solution:

$$\begin{aligned} R_X(\tau) &= e^{(-10|\tau|)} \\ S_X(\omega) &= \frac{20}{\omega^2+100} \\ R_Y(\tau) &= 7cos(500\tau) \\ S_Y(\omega) &= 7\pi[\delta(\omega-500)+\delta(\omega+500)] \end{aligned}$$

$$\begin{aligned} S_Z(\omega) &= \frac{1}{2\pi} S_X(\omega) S_Y(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 7\pi[\delta(\omega-u-500)+\delta(\omega-u+500)]\frac{20}{u^2+100}du \\ &= 70\left(\frac{1}{(\omega-500)^2+100}+\frac{1}{(\omega+500)^2+100}\right) \end{aligned}$$

59. A bandlimited wide-sense stationary random process $X(t)$ has the PSD

$$S_X(\omega) = \begin{cases} 7cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process $C(t)$ which is independent of $X(t)$, is

$$C(t) = \sqrt{60}cos(\omega_C t + \Theta)$$

where $\omega_C \gg \omega_M$, and where Θ is a random variole uniformly distributed between $\pm\pi$. If $Y(t) = X(t)C(t)$, what is the PSD for $Y(t)$?

Solution:

$$\begin{aligned}
C(t) &= \sqrt{60}cos(\omega_C t + \Theta) \\
R_C(\tau) &= 30cos(\omega_C \tau) \\
S_C(\omega) &= 30\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)] \\
S_X(\omega) &= 7cos(\pi u\omega/2\omega_M)
\end{aligned}$$

$$\begin{aligned}
S_Y(\omega) &= \frac{1}{2\pi} S_X(\omega) \star S_C(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 7cos(\pi u/2\omega_M)30\pi[\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)]du \\
&= 105cos(\pi(\omega - \omega_C)/2\omega_M) + 105cos(\pi(\omega + \omega_C)/2\omega_M)
\end{aligned}$$

60. A bandlimited wide-sense stationary random process $X(t)$ has the PSD

$$S_X(\omega) = \begin{cases} 5cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process $C(t)$ which is independent of $X(t)$, is

$$C(t) = \sqrt{200}cos(\omega_C t + \Theta)$$

where $\omega_C \gg \omega_M$, and where Θ is a random variole uniformly distributed between $\pm\pi$. If $Y(t) = X(t)C(t)$, what is the PSD for $Y(t)$?

Solution:

$$\begin{aligned}
C(t) &= \sqrt{200}cos(\omega_C t + \Theta) \\
R_C(\tau) &= 100cos(\omega_C \tau) \\
S_C(\omega) &= 100\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)] \\
S_X(\omega) &= 5cos(\pi u\omega/2\omega_M)
\end{aligned}$$

$$\begin{aligned}
S_Y(\omega) &= \frac{1}{2\pi} S_X(\omega) \star S_C(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 5cos(\pi u/2\omega_M)100\pi[\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)]du \\
&= 250cos(\pi(\omega - \omega_C)/2\omega_M) + 250cos(\pi(\omega + \omega_C)/2\omega_M)
\end{aligned}$$

61. A bandlimited wide-sense stationary random process $X(t)$ has the PSD

$$S_X(\omega) = \begin{cases} 6cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process $C(t)$ which is independent of $X(t)$, is

$$C(t) = \sqrt{34}cos(\omega_C t + \Theta)$$

where $\omega_C \gg \omega_M$, and where Θ is a random variole uniformly distributed between $\pm\pi$. If $Y(t) = X(t)C(t)$, what is the PSD for $Y(t)$?

Solution:

$$
\begin{aligned}
C(t) &= \sqrt{34}cos(\omega_C t + \Theta) \\
R_C(\tau) &= 17cos(\omega_C \tau) \\
S_C(\omega) &= 17\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)] \\
S_X(\omega) &= 6cos(\pi u\omega/2\omega_M)
\end{aligned}
$$

$$
\begin{aligned}
S_Y(\omega) &= \frac{1}{2\pi} S_X(\omega) \star S_C(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 6cos(\pi u/2\omega_M)17\pi[\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)]du \\
&= 51cos(\pi(\omega - \omega_C)/2\omega_M) + 51cos(\pi(\omega + \omega_C)/2\omega_M)
\end{aligned}
$$

62. The wide sense stationary random process $X(t)$ has a PSD that is a constant $5 \times 10^{-6}V^2$ when $|f| < 1kHz$ and is 0 otherwise. The random process $Y(t) = 10cos(\omega_o t + \Theta)$ V where $f_O = 100KHz$ and Θ is uniformly distributed between $\pm\pi$. $X(t)$ and $Y(t)$ are independent $Z(t) = X(t)Y(t)$. What is the PSD for $Z(t)$?

Solution:

$$
\begin{aligned}
Y(t) &= 10cos(\omega_o t + \Theta) \\
R_Y(\tau) &= 50cos(\omega_o \tau) \\
S_Y(\omega) &= 50\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]
\end{aligned}
$$

$$
\begin{aligned}
S_Z(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u)S_Y(\omega - u)du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 5 \times 10^{-6} \times 50\pi[\delta(\omega - u - \omega_o) + \delta(\omega - u + \omega_o)]du \\
&= 125 \times 10^{-6}V^4 \quad |\pm\omega_o - \omega| < 2\pi \times 10^3
\end{aligned}
$$

63. The wide sense stationary random process $X(t)$ has a PSD that is a constant $7 \times 10^{-6}V^2$ when $|f| < 0.9kHz$ and is 0 otherwise. The random process $Y(t) = 8cos(\omega_o t + \Theta)$ V where $f_O = 100KHz$ and Θ is uniformly distributed between $\pm\pi$. $X(t)$ and $Y(t)$ are independent $Z(t) = X(t)Y(t)$. What is the PSD for $Z(t)$?

Solution:

$$
\begin{aligned}
Y(t) &= 8cos(\omega_o t + \Theta) \\
R_Y(\tau) &= 32cos(\omega_o \tau) \\
S_Y(\omega) &= 32\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]
\end{aligned}
$$

$$
\begin{aligned}
S_Z(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u) S_Y(\omega-u)du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 7\times 10^{-6}\times 32\pi[\delta(\omega-u-\omega_o)+\delta(\omega-u+\omega_o)]du \\
&= 112\times 10^{-6}V^4 \quad |\pm\omega_o-\omega| < 2\pi\times 10^3
\end{aligned}
$$

64. The wide sense stationary random process $X(t)$ has a PSD that is a constant $6\times 10^{-6}V^2$ when $|f| < 1.1kHz$ and is 0 otherwise. The random process $Y(t) = 9cos(\omega_o t+\Theta)$ V where $f_O = 100KHz$ and Θ is uniformly distributed between $\pm\pi$. $X(t)$ and $Y(t)$ are independent $Z(t) = X(t)Y(t)$. What is the PSD for $Z(t)$?

Solution:

$$
\begin{aligned}
Y(t) &= 9cos(\omega_o t+\Theta) \\
R_Y(\tau) &= 40.5cos(\omega_o\tau) \\
S_Y(\omega) &= 40.5\pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]
\end{aligned}
$$

$$
\begin{aligned}
S_Z(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(u) S_Y(\omega-u)du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 6\times 10^{-6}\times 40.5\pi[\delta(\omega-u-\omega_o)+\delta(\omega-u+\omega_o)]du \\
&= 121.5\times 10^{-6}V^4 \quad |\pm\omega_o-\omega| < 2\pi\times 10^3
\end{aligned}
$$

1.8 Linear Systems:

Consider a signal $x(t)$ is passed through a filter with $h(t)$, then the output of the filter $y(t)$ is expressed as

$$y(t) = \int_0^\infty h(u)x(t-u)du$$

where $h(t)$ is the impulse response of the system. Also above mentioned equation represents the convolution operation between input signal $x(t)$ with $h(t)$. The laplace transform the above system is

$$Y(s) = H(s)X(s)$$

where $X(s)$ and $Y(s)$ are the Laplace transforms of the input signal $x(t)$ and output $t(t)$ signals.

The above relation is applied for the random process $X(t)$ and which expressed as

$$y(t) = \int_0^\infty h(u)X(t-u)du$$

1.8.1 The mean of $Y(t)$:

The mean of the output random process is

$$\mu_Y = E[Y(t] = \int_0^\infty h(u)E[X(t-u)]du$$

$$\mu_Y = E[Y(t] = \mu_X \int_0^\infty h(u)du$$

The Laplace transform of $h(t)$

$$H(s) = \int_0^\infty h(t)e^{-st}dt$$

$$\mu_Y = E[Y(t] = \mu_X H(0)$$

1.8.2 Cross-Correlating $Y(t)$ and $X(t)$:

Consider a two random process $X(t)$ and $Y(t)$ are wide sense stationary. When we consider both the random process it is called as jointly wide sense stationary. Then their cross-correlation function is defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int_0^\infty h(u)E[X(t)X(t+\tau-u)]du$$

$$R_{XY}(\tau) = \int_0^\infty h(u)R_X(\tau-u)du$$

$$S_{XY}(j\omega) = \int_0^\infty h(u)\int_{-\infty}^\infty R_X(\tau-u)e^{-j\omega\tau}d\tau du$$

Let $v = \tau - u$

$$\begin{aligned} S_{XY}(j\omega) &= \int_0^\infty h(u)e^{-j\omega u}\int_{-\infty}^\infty R_X(v)e^{-j\omega v}dv \\ &= H(j\omega)S_X(\omega) \end{aligned}$$

1.8.3 Autocorrelation of $Y(t)$:

$$\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\
&= \int_0^\infty h(v) \int_0^\infty h(u) E[X(t-v)X(t+\tau-u)dudv \\
&= \int_0^\infty h(v) \int_0^\infty h(u) R_X(\tau+v-u)dudv
\end{aligned}$$

$$S_Y(\omega) = \int_0^\infty h(v) \int_0^\infty h(u) \int_{-\infty}^\infty R_X(\tau+v-u)e^{-j\omega\tau} d\tau dudv$$

Let $w = \tau + v - u$

$$\begin{aligned}
S_Y(\omega) &= \int_0^\infty h(v)e^{j\omega v}dv \int_0^\infty h(u)e^{-j\omega u} \int_{-\infty}^\infty R_X(w)e^{-j\omega w}dw \\
&= H(-j\omega)H(j\omega)S_X(\omega) \\
&= |H(j\omega)|^2 S_X(\omega)
\end{aligned}$$

65. Suppose in a given application

$$R_{XY}(\tau) = \begin{cases} aKe^{-a\tau} & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}$$

$$S_{XY}(j\omega) = \frac{aK}{j\omega + a}$$

What are $R_{YX}(\tau)$ and $S_{YX}(j\omega)$ in this case? Interpret your results. [?]

Solution:

We Know that

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

$$R_{YX}(\tau) = \begin{cases} aKe^{a\tau} & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}$$

And also

$$S_{YX}(j\omega) = S^*_{YX}(j\omega) = \frac{aK}{-j\omega + a}$$

66. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input $X(t)$ with the output $Y(t)$ of the linear system is found to be

$$R_{XY}(\tau) = K \begin{cases} e^{-\tau} + 3e^{-2\tau} & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}$$

What is the power filter function $|H(j\omega)|^2$? . [?]

Solution:

If $S_X(\omega) = K$ then

$$R_X(\tau) = K\delta(\tau)$$

$$R_{XY}(\tau) = \int_0^\infty h(u)R_X(\tau - u)du$$

$$\begin{aligned} R_{XY}(\tau) &= K\int_0^\infty h(u)\delta(\tau - u)du \\ &= Kh(\tau) \end{aligned}$$

Similarly

$$h(t) = K \begin{cases} e^{-t} + 3e^{-2t} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

$$H(s) = \frac{1}{s+1} + \frac{3}{s+2} = \frac{(s+2) + 3(s+1)}{(s+1)(s+2)} = \frac{4s+5}{s^2+3s+2}$$

$$
\begin{aligned}
H(j\omega) &= \frac{j4\omega + 5}{-\omega^2 + 3j\omega + 2} \\
|H(j\omega)|^2 &= \frac{16\omega^2 + 25}{\omega^4 + 5\omega^2 + 4}
\end{aligned}
$$

67. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input $X(t)$ with the output $Y(t)$ of the linear system is found to be

$$
R_{XY}(\tau) = K \begin{cases} 3e^{-\tau} + e^{-2\tau} & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}
$$

What is the power filter function $|H(j\omega)|^2$? . [?]

Solution:

If $S_X(\omega) = K$ then

$$
R_X(\tau) = K\delta(\tau)
$$

$$
\begin{aligned}
R_{XY}(\tau) &= \int_0^\infty h(u) R_X(\tau - u) du \\
R_{XY}(\tau) &= K \int_0^\infty h(u)\delta(\tau - u) du \\
&= K h(\tau)
\end{aligned}
$$

Similarly

$$
h(t) = K \begin{cases} 3e^{-t} + e^{-2t} & t \geq 0 \\ 0 & t \leq 0 \end{cases}
$$

$$
H(s) = \frac{3}{s+1} + \frac{1}{s+2} = \frac{3(s+2) + (s+1)}{(s+1)(s+2)} = \frac{4s+7}{s^2+3s+2}
$$

$$
\begin{aligned}
H(j\omega) &= \frac{j4\omega + 7}{-\omega^2 + 3j\omega + 2} \\
|H(j\omega)|^2 &= \frac{16\omega^2 + 49}{\omega^4 + 5\omega^2 + 4}
\end{aligned}
$$

68. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input $X(t)$ with the output $Y(t)$ of the linear system is found to be

$$
R_{XY}(\tau) = K \begin{cases} 2e^{-2\tau} + 3e^{-\tau} & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}
$$

What is the power filter function $|H(j\omega)|^2$? . [?]

Solution:

If $S_X(\omega) = K$ then

$$R_X(\tau) = K\delta(\tau)$$

$$R_{XY}(\tau) = \int_0^\infty h(u)R_X(\tau - u)du$$

$$\begin{aligned} R_{XY}(\tau) &= K\int_0^\infty h(u)\delta(\tau - u)du \\ &= Kh(\tau) \end{aligned}$$

Similarly

$$h(t) = K\begin{cases} 2e^{-t} + 3e^{-t} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

$$H(s) = \frac{2}{s+2} + \frac{3}{s+1} = \frac{2(s+1)+3(s+2)}{(s+1)(s+2)} = \frac{5s+8}{s^2+3s+2}$$

$$H(j\omega) = \frac{j5\omega + 8}{-\omega^2 + 3j\omega + 2}$$

$$|H(j\omega)|^2 = \frac{25\omega^2 + 64}{\omega^4 + 5\omega^2 + 4}$$

69. The PSD of the random process $X(t)$ and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (100)^2} \quad and \quad H(s) = \frac{s}{(s+10)(s+9000)}$$

$Y(s) = H(s)X(s)$. **Find μ_Y, $S_{XY}(j\omega)$ and $S_Y(\omega)$.** [?]

Solution:

$S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

$$\mu_X = 0$$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$\begin{aligned} S_{XY}(j\omega) &= H(j\omega)S_X(\omega) \\ &= \frac{j\omega}{(\omega^2 + (100)^2)(j\omega + 10)(j\omega + 9000)} \end{aligned}$$

$$\begin{aligned} S_Y(\omega) &= |H(j\omega)|^2 S_X(\omega) \\ &= \frac{\omega^2}{(\omega^2 + (100)^2)(\omega^2 + (10)^2)(\omega^2 + (9000)^2)} \end{aligned}$$

70. The PSD of the random process $X(t)$ and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (80)^2} \quad and \quad H(s) = \frac{s}{(s+9)(s+10000)}$$

$Y(s) = H(s)X(s)$**. Find μ_Y, $S_{XY}(j\omega)$ and $S_Y(\omega)$.** [?]

Solution:

$S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

$$\mu_X = 0$$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$\begin{aligned} S_{XY}(j\omega) &= H(j\omega)S_X(\omega) \\ &= \frac{j\omega}{(\omega^2 + (80)^2)(j\omega + 9)(j\omega + 10000)} \end{aligned}$$

$$\begin{aligned} S_Y(\omega) &= |H(j\omega)|^2 S_X(\omega) \\ &= \frac{\omega^2}{(\omega^2 + (80)^2)(\omega^2 + (9)^2)(\omega^2 + (10000)^2)} \end{aligned}$$

71. The PSD of the random process $X(t)$ and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (70)^2} \quad and \quad H(s) = \frac{s}{(s+10)(s+11000)}$$

$Y(s) = H(s)X(s)$**. Find μ_Y, $S_{XY}(j\omega)$ and $S_Y(\omega)$.** [?]

Solution:

$S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

$$\mu_X = 0$$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$\begin{aligned} S_{XY}(j\omega) &= H(j\omega)S_X(\omega) \\ &= \frac{j\omega}{(\omega^2 + (70)^2)(j\omega + 10)(j\omega + 11000)} \end{aligned}$$

$$\begin{aligned} S_Y(\omega) &= |H(j\omega)|^2 S_X(\omega) \\ &= \frac{\omega^2}{(\omega^2 + (70)^2)(\omega^2 + (10)^2)(\omega^2 + (11000)^2)} \end{aligned}$$