UNIT-4

Non-homogeneous PDE

$\Rightarrow$ Linear PDE, order of all P.D are not equa[l]

ex: $(D^2-(D')^2+D-D')z=0$

$(DD'+D-D'-1)z = xy$

$(D^3-D'^3+D+1)z = (y-1)e^x$

$\Rightarrow$ $F(D,D')z = F(x,y)$

$z = CF + PI$

Find CF $\Rightarrow$ $F(D,D')$ into linear factors of the form

$$(D-m_1D'-a_1)(D-m_2D'-a_2)\cdots(D-m_nD'-a_n)$$

The CF $\Rightarrow$

$$z = e^{a_1x}f_1(y+m_1x) + e^{a_2x}f_2(y+m_2x) + \cdots + e^{a_nx}f_n(y+m_nx)$$

$m_1, m_2, \ldots, m_n$ & $a_1, a_2, \ldots a_n$ are constan[ts]

$(D - mD' - c)z = 0$

$p - mq - cz = 0$

$p - mq = cz$

$\hookrightarrow Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

$dy + m\,dx = 0$

$\boxed{y + mx = a}$

$\frac{dz}{z} = c\,dx$

$\log z = cx + \log b$

$\boxed{z = b.e^{cx}}$

$b = f(a) = f(y+mx)$

$$z = f(y+mx).e^{cx}$$

Q. $[(D + D' - 2)(D + 4D' - 3)]z = 0$

$(D - mD' - c)$

$z = f(y + x)e^{+2x}$ $\quad m = -1 \quad c = 2$

$z = f(y + 4x)e^{+3x}$ $\quad m = -4 \quad c = 3$

$\rightarrow z = f(y+mx)e^{cx}$

Therefore,

$$z = e^{2x} f(y-x) + e^{3x} f(y-4x)$$

Q. $[D^2 + 2DD' + (D')^2 + 2D + 2D' + 1]z = 0$

~~$[(D+D')^2 + 2D + 2D' + 1] z = 0$~~

$[(D+D')^2 + 2(D+D') + 1]z = 0$

$[(D + D' + 1)^2] = 0$

$z = f_1(y-x)e^{-x} + xe^{-x} f_2(y-x)$

Q. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$

$\cancel{((D+D')^2 - 2(D+D'))z = \sin(x+2y)}$

$D^2 + DD' + DD' + D'^2 - 2D - 2D'$

$D(D+D') + D'(D+D') - 2(D+D')$

$(D+D')(D+D'-2) = 0$

$c = 0 \qquad c = 2$

$m = -1 \qquad m = -1$

$CF = f(x-y) + e^{2x} f(x-y)$

$a_1 a_2 = 1$

$b_1 b_2 = 1$

$c_1 c_2 = 0$

$a_1 c_2 + a_2 c_1 = -2$

$b_1 c_2 + b_2 c_1 = -2$

$$PI = \frac{\sin(x+2y)}{\cancel{-1-4-4} - 2D - 2D'}$$

-9

$$= \cancel{\frac{\sin(x+2y)(2D-2D'-9)}{(2D+2D'+9)(2D-2D'-9)}}$$

$$= \cancel{\frac{\sin(x+2y)(2D-2D'-9)}{4D^2-(2D'+9)^2}}$$

$$= \cancel{\frac{-\sin(x+2y)(2D-2D'-9)}{4D^2-(4D'^2+81+36D')}}$$

$$= \cancel{\frac{\sin(x+2y)(2D-2D'-9)}{-4-(-16+81+36D')}}$$

$\cancel{65}$

$-69 - 36D'$

$$= \frac{\sin(x+2y)(2D-2D'-9)(12D'-23)}{\cancel{3(36D'+69)(36}\ \cancel{(12D'+23)(12D'-23)}}$$

$141 \qquad \cancel{144D'^2 - 529}$

$-4 - (4D'^2 + 81 + 36D')$

$-4 - (-16 + 81 + 36D')$

$-4 - (65 + 36D')$

$-4 - (65 + 36D')$

81

$-69 - 36D'$

-16

65

$(23 + 12D')(23 - $

$-529 - 144(-4)$

23
23
69
$46\times$
529

144
4
576
529
4

$$= \frac{-\sin(x+2y)\,(2D+2D'-9)}{(2D+2D'+9)(2D+2D'-9)}$$

$$= \frac{-\sin(x+2y)\,(2D+2D'-9)}{4D^2+4D'^2+8DD'-81}$$

$-4 \quad -16$

$$= \frac{-\sin(x+2y)\,(2D+2D'-9)}{\cancel{-4-16-16-81}}$$

117

$$= \frac{6\cos(x+2y)-9\sin(x+2y)}{117}$$

Q. $(D^2-DD'+D'-1)\,z = \cos(x+2y)$

$((D+1)(D-1)-D'(D-1))\,z = \cos(x+2y)$

$(D-D'+1)(D-0-1)\,z = \cos(x+2y)$

$z = f(x+y)\cdot e^{-x} + e^{x}(y)$

$$PI = \frac{1}{D^2-DD'+D'-1}\cos(x+2y) = \frac{\cos(x+2y)}{\cancel{-1+2-1}+D'}$$

$$= \frac{D'\cos(x+2y)}{-4}$$

$$= \frac{-2\sin(x+2y)}{\cancel{-4}\ 2}$$

$$z = f(x+y)e^{-x} + e^{x}y + \frac{\sin(x+2y)}{2}$$

Q. $(D^2 - D'^2 + D + 3D' - 2) z = x^2 y$

$\cancel{(D^2 - D'^2 + D + 2D' + 2D' - 2) z = x^2 y}$

$\cancel{D(D+1) - D'(D-1) - 2(D-1)}$

$\cancel{(D^2 - D'^2 + D + 2D' + D' - 2)}$

$\cancel{D^2 - D'^2 + 2D - D + 3D' - 2}$

$\cancel{D^2 - D'^2 + D - 2}$

$(D + D')(D - D') + D + 3D' - 2$

$(D + D') + (2D' - 2)$

$(D + D')(D - D') + (D + D') + 2(D' - 1)$

$(D + D')(D - D' + 1) - 2(-D' + 1)$

Q. $(D^2 - D'^2 + D + 3D' - 2)z = x^2y$

$(D^2 - D'^2 + D + 3D' - 2) = 0$

$(D^2 - D'^2 + 2D - D + 2D' + D' - 2) = 0$

$(D^2 - D'^2 + 2D - D + 2D' + D' - 2 + DD' - DD') = 0$

$D(D + D' - 1) - D'(D + D' - 1) + 2(D + D' - 1) = 0$

$((D + D' - 1)(D - D' + 2))z = 0$

CF $= e^{x}(y - x) + e^{-2x}(y + x)$

PI $= \dfrac{x^2y}{\cancel{D^2 - D'^2 + D + 3D'} - 2 \quad D^2 - D'^2 + D + 3D' - 2}$

$= \dfrac{x^2y}{-2\left(1 - \frac{D^2}{2} + \frac{D'^2}{2} - \frac{D}{2} - \frac{3D'}{2}\right)}$

$= \dfrac{x^2y}{-2}\left(1 - \left(\frac{D^2}{2} - \frac{D'^2}{2} + \frac{D}{2} + \frac{3D'}{2}\right)\right)^{-1}$

$\cancel{= \frac{x^2y}{-2}\left(1 + \frac{D^2}{2} - \frac{D'^2}{2} + \frac{D}{2} + \frac{3D'}{2}\right)}$

$\cancel{= \frac{x^2y + y - 0 + xy + \frac{3x^2}{2}}{-2}}$

$\cancel{= -(x}$

$(1 - x)^{-1}$

$= 1 + x + x^2 + x^3 + \dots$

3

$$\frac{x^2y}{-2}\left(1-\frac{1}{2}\left(D^2-D'^2+D+3D'\right)\right)^{-1}$$

$$= \frac{x^2y}{-2}\left(1+\frac{1}{2}\left(D^2-D'^2+D+3D'\right)+\frac{1}{4}\left(D^2-D'^2+D+3D'\right)^2 + \frac{1}{8}\left(D^2-D'^2+D+3D'\right)^3\right)$$

$$= \frac{x^2y}{-2}\left(1+\frac{1}{2}\left(D^2-D'^2+D+3D'\right)+\frac{1}{4}\left(6D^2D'+6DD'+D^2\right) + \frac{1}{8}\left(9D^2D'\right)\right)$$

$$= \frac{x^2y}{-2}\left(1+\frac{D^2}{2}-\frac{D'^2}{2}+\frac{D}{2}+\frac{3D'}{2}+\frac{3D^2D'}{2}+\frac{3DD'}{2}+\frac{D^2}{4}+\frac{9D^2D'}{8}\right)$$

$$= \frac{x^2y}{-2}\left(1+\frac{9D^2}{4}+\frac{D}{2}-\frac{D'^2}{2}+\frac{21D^2D'}{8}+\frac{3DD'}{2}\right)$$

$$= \frac{x^2y}{-2} \text{ [illegible] } \frac{9}{4} \text{ [illegible] } \frac{xy}{2} \text{ [illegible] } \frac{21}{8} \text{ [illegible] } \frac{3x}{2} +$$

$$z = e^x(y+x) + e$$

$$PI = -\frac{1}{2}\left(x^2y + xy + \frac{3x^2}{2} + 3x + \frac{3y}{2} + \frac{21}{4}\right)$$

Q. $(D+D'-1)(D+2D'-3) = 4+3x+6y$

$\downarrow$

$CF = f(y-x)e^{x} + e^{3x} f(y-2x)$

$$PI = \frac{4+3x+6y}{-(1-(D+D'))\cdot(-3)\left(1-\frac{1}{3}(D+2D')\right)}$$

$$= \frac{4+3x+6y}{3\left(1+(D+D')\right)\left(1-\frac{1}{3}(D+2D')\right)}$$

$$= \frac{4+3x+6y}{3}\cdot\left((1+(D+D'))\right)^{-1}\left(1-\frac{1}{3}(D+2D')\right)^{-1}$$

$$= \frac{4+3x+6y}{3}\left(1+D+D'\right)\left(1+\frac{D}{3}+\frac{2D'}{3}\right)$$

$$= \frac{4+3x+6y}{3}\left(1+\frac{D}{3}+\frac{2D'}{3}+D+\frac{2DD'}{3}+D'+\frac{DD'}{3}\right) \quad DD' \quad +DD'$$

$$\cancel{= \frac{4+3x+6y}{3}\left(1+\frac{D}{3}+\frac{2D'}{3}-\frac{2D}{3}-\frac{D'}{3}-DD'\right)}$$

$$\cancel{= \frac{4+3x+6y}{3} - \frac{2}{3}(1) - \frac{1}{3}(2) - 0}$$

$$\frac{4+3x+6y}{3}\left(1+\frac{4D}{3}+\frac{5D'}{3}+DD'\right)$$

$$\cancel{= \frac{4+3x+6y}{3} - \frac{2}{3} - \frac{2}{3}}$$

$$\frac{4+3x+6y}{3}+\frac{4}{3}+\frac{5}{3}(2)$$

$$\cancel{= \frac{3x+6y}{3}} = x+2y$$

$$= \frac{18}{3} + x+2y$$

Q. $(D - 3D' - 2)^2 z = 2e^{2x}\sin(y+3x)$

$m = 3$, $c = 2$

$CF = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x)$

$$PI = 2 \cdot \frac{e^{2x}\sin(y+3x)}{(D-3D'-2)^2}$$

$$= 2 \cdot \frac{e^{2x} \cdot \sin(y+3x)}{(D-3D')^2}$$

$$= 2 \cdot \frac{e^{2x} \cdot \sin(y+3x)}{D^2 + 9D'^2 - 6DD'}$$

$$= 2e^{2x} \frac{x\sin(y+3x)}{\cancel{-9-9-6(-3)}}(2D+6D') \quad (2D-6D')(2D+6D')$$

$-18 \quad +18$

$12\cos(y+3x)$

$$= 2e^{2x}x^2 \frac{(\cancel{6}\cos(y+3x) + 6\cancel{\cos}(y+3x))}{\cancel{4D^2} - 36D'^2} (D+9D') \quad \cancel{8D - 72D'}$$

$4(-9) - 36(-1) \quad 0 \quad 8(D-9D')(D+$

$$= \cancel{\frac{2e^{2x}x^2(-36\sin(y+3x) - 108\sin(y+3x))}{8(D^2 - 9D'^2)}}$$

4

$$= \cancel{\frac{e^{2x}x^2}{4} \cdot \frac{(-144\sin(y+3x))}{-9}}$$

$$= \frac{2e^{2x}x^2(12\cos(y+3x))}{\cancel{4-8D^2}(-9) \atop -36} \times D = \frac{e^{2x}x^2(36\sin(y+3x))}{36}$$

$= x^2e^{2x}\sin(y+3x)$

Q. $r - 3s + 2t - p + 2q = (2+4x)e^{-y}$

$$\frac{\partial^2 z}{\partial x^2} - \frac{3\partial^2 z}{\partial x \partial y} + \frac{2\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} + \frac{2\cdot\partial z}{\partial y} = (2+4x)e^{-y}$$

CF $\Rightarrow$ $(D^2 - 3DD' + 2D'^2 - D + 2D')z = 0$

$(D^2 - 2DD' - DD' + 2D'^2 - D + 2D')z = 0$

$D(D-2D') - D'(D-2D') - (D-2D') = 0$

$(D - D' - 1)(D - 2D' + 0) = 0$

$m = +1 \quad c = 1$ $\qquad$ $m = 2 \quad c = 0$

CF $\Rightarrow$ $e^{x} f_1(y+x) + f_2(y+2x)$

$$PI = \frac{(2+4x)e^{-y}}{D^2 - 3DD' + 2D'^2 - D + 2D'}$$

$$= \frac{e^{-y}(2+4x)}{D^2 - 3D(D'-1) + 2(D'-1)^2 - D + 2(D'-1)}$$

$$= \frac{e^{-y}(2+4x)}{D^2 - 3DD' + 3D + 2(D'^2 + 1 - 2D') - D + (2(D'-1))}$$

$$= \frac{e^{-y}(2+4x)}{D^2 - 3DD' + \underset{2D}{\cancel{3D}} + 2D'^2 + \cancel{2} - \underset{-2D'}{\cancel{4D'}} - \cancel{D} + \cancel{2D'} - \cancel{2}}$$

$$= \frac{e^{-y}(2+4x)}{D^2 - 3DD' + 2D'^2 + 2D - 2D'}$$

$$= \frac{e^{-y}(2+4x)}{D^2-3DD'+2D'^2+2D-2D'}$$

$$= \frac{e^{-y}(2+4x)}{D^2-2DD'-DD'+2D'^2+2D-2D'}$$

$$= \frac{e^{-y}(2+4x)}{D(D-D')-2D'(D-D')+2(D-D')}$$

$$= \frac{e^{-y}(2+4x)}{(D-2D'+2)(D-D')}$$

$$= \frac{e^{-y}(2+4x)}{2\left(1+\frac{D}{2}-D'\right)D\left(1-\frac{D'}{D}\right)}$$

$$= \frac{e^{-y}}{2}\cdot\frac{1}{D}\left[1+\frac{D}{2}-D'\right]^{-1}\left(1-\frac{D'}{D}\right)^{-1}$$

$$= \frac{e^{-y}}{2}\cdot\frac{1}{D}\left[1-\left(\frac{D}{2}-D'\right)\right]\left[1+\frac{D'}{D}\right][4x+2]$$

$$= \frac{e^{-y}}{2}\cdot\frac{1}{D}\left[1-\left(\frac{D}{2}-D'\right)\right][4x+2+0]$$

$$= \frac{e^{-y}}{2}\cdot\frac{1}{D}\left[4x+2-\left(\frac{4}{2}-0\right)\right]$$

$$= \frac{e^{-y}}{2}\cdot\frac{1}{D}\left[4x+\cancel{2}-\cancel{2}\right]$$

$$= \frac{e^{-y}}{\cancel{2}}\cdot\frac{\cancel{4}x^2}{\cancel{2}} = e^{-y}\cdot x^2$$

$$z = e^{x}f_1(y+x)+f_2(y+2x)+e^{-y}x^2$$

2.

$$-2+D+3D'+D^2-D'^2 \;\big)\; x^2y$$

$$-\frac{x^2y}{2}-\frac{xy}{2}-\frac{3x^2}{4}-\frac{3y}{4}-3x/2-21/8$$

$$-2+D+3D'+D^2-D'^2 \big)\; x^2y$$

$$x^2y - xy - \frac{3x^2}{2} - y$$

$$- \quad + \quad + \quad +$$

$$xy + \frac{3x^2}{2} + y$$

$$xy - 0 - \frac{y}{2} - \frac{3x}{2}$$

$$- \quad + \quad +$$

$$0 + \frac{3x^2}{2} + \frac{3y}{2} + \frac{3x}{2}$$

$$+\frac{3x^2}{2} \qquad -\frac{3x}{2} - \frac{3}{2}$$

$$0 + \frac{3y}{2} + 3x + \frac{3}{2}$$

$$\frac{3y}{2} + 0 - \frac{9}{4}$$

$$- \qquad +$$

$$3x + \frac{15}{4}$$

$$\frac{9}{4}+\frac{6}{4} = \frac{15}{4}$$

$$3x - \frac{3}{2}$$

$$- \quad +$$

$$\frac{21}{4}$$

$$\frac{21}{4}$$

$$0$$

$$PI = -\frac{x^2y}{2} - \frac{xy}{2} - \frac{3x^2}{4} - \frac{3y}{4} - \frac{3x}{2} - \frac{21}{8}$$

3

- Easy way to factorize non-homogeneous PDE

assume in the form

$$(a_1 D + b_1 D' + c_1)(a_2 D + b_2 D' + c_2)$$

coeff of $D^2 = a_1 a_2$

coeff of $D'^2 = b_1 b_2$

const $= c_1 c_2$

coeff of $D = a_1 c_2 + a_2 c_1$

coeff of $D' = b_1 c_2 + b_2 c_1$

Q. $(D^2 - D'^2 + D + 3D' - 2) = x^2 y$

$a_1 a_2 = 1$, $b_1 b_2 = -1$

4 $c_1 c_2 = -2$

$a_1 c_2 + a_2 c_1 = 1$

$b_1 c_2 + b_2 c_1 = 3$

a_1	a_2	$\rightarrow$	1	1
b_1	b_2		b_1	b_2
c_1	c_2		c_1	c_2

$a_1 c_2 + a_2 c_1 = 1$

$c_2 + c_1 = 1$

$c_1 c_2 = -2$

$c_1 = 2 \quad c_2 = -1$

$b_1 = -1 \quad b_2 = 1$

$$(D - D' + 2)(D + D' - 1)$$

Q. $(2D^2 - DD' - D'^2 + D + D')z = e^{2x+3y}$

$a_1 a_2 = 2$

$b_1 b_2 = -1$

$c_1 c_2 = 0$

$a_1 c_2 + a_2 c_1 = 1$

$b_1 c_2 + b_2 c_1 = 1$

a_1	a_2	1	2
b_1	b_2	-1	$+1$
c_1	c_2	0	1

$$(D - D')(2D + D' + 1)$$

$$= (D - D')\left(D + \frac{D'}{2} + \frac{1}{2}\right)$$

$$CF = f_1(y + x) + e^{-x/2} f_2\left(y - \frac{x}{2}\right)$$

General Method to find PI of homogeneous PDE

- $F(D, D')z = F(x, y)$

$$PI = \frac{1}{F(D, D')} \cdot F(x, y)$$

$$= \frac{1}{(D - m_1 D')(D - m_2 D') \cdots (D - m_n D')} \cdot F(x, y) \rightarrow ①$$

To find PI given by ①, we find solution of eq $(D - mD')z = F(x, y)$ given by

$$\boxed{z = \frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,dx}$$

where after integration, the constant c must be replaced by $y + mx$.

Hence, PI given by ① can be obtained by applying operation ② by the factors in succession starting from right.

<u>Note</u>: While using general method of finding PI of $F(D, D')z = F(x, y)$

i) $\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\,dx$ where $c = y + mx$

ii) $\frac{1}{D + mD'} F(x, y) = \int F(x, c + mx)\,dx$ where $c = y - mx$

Q. $4D^2z - 4DD'z + D'^2z = 16\log(x+2y)$

$z = CF + PI$

$CF \Rightarrow 4D^2 - 4DD' + D'^2$

$4m^2 - 4m + 1 = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$CF = f_1\left(y+\frac{x}{2}\right) + x f_2\left(y+\frac{x}{2}\right)$

$PI \Rightarrow \dfrac{16\log(x+2y)}{(2D-D')^2}$

$= \dfrac{4\log(x+2y)}{\left(D-\frac{D'}{2}\right)^2} = \dfrac{4\log(x+2y)}{\left(D-\frac{D'}{2}\right)\left(D-\frac{D'}{2}\right)}$

$= \dfrac{4}{D-\frac{D'}{2}} \cdot \int \log\left(x+2\left(c-\frac{x}{2}\right)\right)dx$

$= \dfrac{4}{D-\frac{D'}{2}} \int \log(\cancel{x}+2c-\cancel{x})\,dx$

$= \dfrac{4}{D-\frac{D'}{2}} (x.\log 2c_1)$

$\cancel{= 4\int x\log 2c_1}$

$\cancel{= 24\frac{x^2}{2}\log 2c_1}$

$\cancel{= 2x^2.c}$

$= 4\int x\log 2\left(y+\frac{x}{2}\right)$

$= 4\int x\log(2y+x)$

$= 4\int x\log\left(x+2\left(c-\frac{x}{2}\right)\right)$

$= 4\int x\log 2c$

$= \frac{\cancel{4}^2 x^2}{\cancel{2}}\log 2\left(y+\frac{x}{2}\right)$

$= 2x^2\log(2y+x)$

Q. $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

$m^2 + m - 6 = 0$

$m = 2, -3$

$CF = f_1(y+2x) + f_2(y-3x)$

$$PI = \frac{1}{(D-2D')(D+3D')} \cos(2x+y)$$

$$= \frac{1}{(D+3D')} \int \cos(2x + c - 2x) \qquad c = y+2x$$

$$= \frac{1}{(D+3D')} \int \cos(c)$$

$$= \frac{1}{D+3D'} \cdot x\cos(c)$$

$$= \frac{1}{D+3D'} x\cos(y+2x) \qquad c = y-3x$$

$$= \int x\cos(2x+c+3x)\,dx$$

$$= \int x\cos(5x+c)\,dx$$

$$\cancel{= \frac{x^2}{2}\cos(5x+c) - \int \frac{x^2}{2}\sin(5x+c)}$$

$\int (u.v)' = \int u'v + \int uv'$

$$= \frac{x\sin(5x+c)}{5} - \int \frac{\sin(5x+c)}{5}$$

$uv - \int uv' = \int u'v$

$$= \frac{x\sin(5x+c)}{5} - \frac{\cos(5x+c)}{25}$$

$\int uv' = uv - \int u'v$

$$= \frac{x\sin(5x+y-3x)}{5} - \frac{\cos(5x+y-3x)}{25}$$

$$= \frac{x\sin(2x+y)}{5} - \frac{\cos(2x+y)}{25}$$

SPECIAL FUNCTIONS

Algebraic

→ Functions obtained by algebraic ~~operations~~ operations of

$+, -, \div, \times, \sqrt{\ }$ of polynomials & rational func

ex: $x^3 - 3x^2 + 2x + 1$

Transcendental Function

→ involves log, trig func, hyperbolic func, exp func,

ex: $xe^x - 1$

$\sin x - x\cos x$

Elementary Functions

Beta, Gamma ⇒ Expressed as integrals

Bessel Function ⇒ Expressed as solutions of ODE's

→ Helps to find Definite Integral easily

Gamma Function : $\int_0^\infty e^{-x} \cdot x^{n-1}\,dx$ → gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\,dx \quad (n > 0)$$

Note :

$$\Gamma(1) = 1$$

Now put $x = t^2$

$dx = 2t\,dt$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

so, $\int_0^\infty e^{-t^2} (t^2)^{n-1} \cdot 2t\,dt = \boxed{2\int_0^\infty e^{-t^2} t^{2n-1}\,dt}$

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Q. $\int_0^\infty e^{-t} t^3 dt = 0$

$n = 4$

$\Gamma(4) = 3!$

Q. $\int_0^\infty e^{-x} x^7 dx = \Gamma(8) = 7!$

Q. $\int_0^\infty e^{-x} x^{2/3} dx = \Gamma\left(\frac{5}{3}\right) = \frac{2}{3}\Gamma\frac{2}{3}$

$\Gamma\overline{n+1} = n\Gamma n$

Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad , (m>0, n>0)$$

put $x = \sin^2\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$dx = 2\sin\theta\cos\theta \, d\theta$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} d\theta \cdot \sin\theta\cos\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

Q. $\int_0^1 x^3(1-x)^5 dx$

$\beta(4,6)$

Gamma Function

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \quad ; n > 0$$

$$= 2\int_0^\infty e^{-t^2} x^{2t-1}\, dt \quad ; t > 0.$$

Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \; ; m, n > 0$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Properties of β & Γ functions

1. Recurrence formula of Gamma function

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma n+1 = \int_0^\infty e^{-x} x^{n}\, dx$$

$$= \int_0^\infty \underbrace{x^n}_{u} \underbrace{e^{-x}\, dx}_{dv}$$

$u = x^n$, $du = nx^{n-1}$ | $dv = e^{-x}dx$, $v = -e^{-x}$

$$= \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty -e^{-x}\, n \cdot x^{n-1}\, dx$$

$$= 0 + n\int_0^\infty +e^{-x} x^{n-1}\, dx$$

$$\boxed{\Gamma n+1 = n\,\Gamma n}$$

Q1. $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\frac{1}{2}$

$$\Gamma\frac{5}{2} = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{4}\Gamma\frac{1}{2}$$

$$\Gamma\frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2} = \frac{15}{8}\Gamma\frac{1}{2}$$

$$\Gamma\frac{11}{2} = \frac{9}{2}\times\frac{7}{2}\times\frac{15}{8}\Gamma\frac{1}{2} = \frac{945}{32}\Gamma\frac{1}{2} = \frac{945}{32}\sqrt{\pi}$$

$$\Gamma\frac{9}{2} = \frac{7}{2}\times\frac{15}{8}\Gamma\frac{1}{2} = \frac{105}{16}\sqrt{\pi}$$

Q. 2. $\Gamma(n+1) = n!$

1. $\Gamma 4 = 3! = 6$

$\Gamma 6 = 5! = 120$

$\Gamma 10 = 9! = 362880$

$\Gamma 15 = 14! = 1.3809 \times 10^{15}$

Note: 1. $\Gamma 0$ is not defined & for -ve integer

2. when n is -ve integer,

$$\Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\frac{-3}{2} = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma-\frac{1}{2}}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{+\frac{3}{4}} = \frac{\Gamma\frac{1}{2}}{\frac{3}{4}} = \frac{4\sqrt{\pi}}{3}$$

Q. Prove $\int_0^{\infty} x^p e^{-ax^q}\,dx = \dfrac{\Gamma\left(\frac{p+1}{q}\right)}{q\,a^{\left(\frac{p+1}{q}\right)}}$

$$I = \int_0^{\infty} x^p e^{-ax^q}\,dx$$

Put $ax^q = t$

$$x^q = t/a$$

$$x = (t/a)^{1/q}$$

$$dx = \frac{1}{a^{1/q}} \cdot \frac{1}{q} \cdot t^{\frac{1}{q}-1}\,dt$$

when $x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

Therefore,

$$\int_0^{\infty} \left(\frac{t}{a}\right)^{\frac{p}{q}} \cdot e^{-t} \cdot \frac{1}{a^{1/q}} \cdot \frac{1}{q} \cdot t^{\frac{1}{q}-1}\,dt$$

$$= \frac{1}{a^{\frac{p+1}{q}} \cdot q} \int_0^{\infty} e^{-t}\, t^{\frac{p+1}{q}-1}\,dt$$

$$= \cancel{\frac{1}{a^{\frac{p+1}{q}} \cdot q}\,\Gamma\left(\frac{p+1}{q}\right)} = \frac{\Gamma\left(\frac{p+1}{q}\right)}{q \cdot a^{\frac{p+1}{q}}}$$

Q. $\int_0^\infty e^{-kx} x^{n-1}\, dx$

$t = kx$

$x = \frac{t}{k}$

$dx = \frac{dt}{k}$

$$\Rightarrow \int_0^\infty e^{-t} \cdot x^{n-1} \frac{dt}{k} = \int_0^\infty e^{-t} \cdot \left(\frac{t}{k}\right)^{n-1} \cdot \frac{dt}{k}$$

$$= \frac{\Gamma n}{k^n}$$

Q. $\int_0^1 x^m (\ln x)^n\, dx$

$-t = \ln x$ | $x = 0 \Rightarrow t = \infty$

$e^{-t} = x$ | $x = 1 \Rightarrow t = 0$

$dx = -e^{-t}\, dt$

$$\Rightarrow I = \int_\infty^0 e^{-mt} (-t)^n \cdot -e^{-t}\, dt$$

$$= (-1)^n \int_0^\infty e^{-(1+m)t} (+t)^n\, dt$$

$$= \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}$$

Q. $\int_0^1 x^p (1-x^q)^r \, dx$

$t = x^q$

$x = t^{1/q}$

$dx = \frac{1}{q} t^{1/q - 1}$ $\Rightarrow x = 0 \to t = 0$
$x = 1 \Rightarrow t = 1$

$I = \int_0^1 t^{p/q} (1-t)^r \frac{1}{q} \cdot t^{\frac{1}{q}-1} \, dt$

$= \frac{1}{q} \int_0^1 t^{\frac{p+1}{q}-1} (1-t)^r \, dt$

$= \frac{1}{q} \beta\left(\frac{p+1}{q}, r+1\right)$

Q. $\int_0^\infty \sqrt{y} \, e^{-y^2} dy$

$= \int_0^\infty y^{1/2} e^{-y^2} dy$

Method 1

$= \dfrac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2 \cdot (1)} = \dfrac{\Gamma\left(\frac{3}{4}\right)}{2} =$

Method 2 : $y^2 = t \Rightarrow y = t^{1/2} \Rightarrow$

$2y \, dy = dt$

$dy = \dfrac{dt}{2t^{1/2}}$ $\to y = 0 \Rightarrow t = 0$
$y = \infty \Rightarrow t = \infty$

$I = \int_0^\infty t^{1/4} e^{-t} \dfrac{dt}{2t^{1/2}} = \frac{1}{2} \int_0^\infty e^{-t} \cdot t^{1/4 - 1/2} dt$

$= \frac{1}{2} \int_0^\infty e^{-t} t^{-1/4} dt = \frac{1}{2} \Gamma\left(\frac{3}{4}\right)$

Q. $\int_0^\infty e^{-x^4}\,dx$

$p=0 \quad a=1 \quad q=4$

$$I = \frac{\Gamma\frac{p+1}{q}}{q\cdot a^{\frac{p+1}{q}}} = \frac{\Gamma\frac{1}{4}}{4\cdot(1)^{\frac{1}{4}}} = \frac{1}{4}\cdot\Gamma\frac{1}{4}$$

Q. $\int_0^\infty 3^{-4x^2}\,dx$

$$= \int_0^\infty e^{\log(3^{-4x^2})}\,dx = \int_0^\infty e^{-4x^2\log 3} = \int_0^\infty e^{-4\log 3\cdot x^2}$$

$p=0 \quad a=4\log 3 \quad q=2$

$$I = \frac{\Gamma\frac{1}{2}}{2\cdot(4\log 3)^{1/2}} = \frac{1}{4}\cdot\sqrt{\frac{\pi}{\log 3}} = \frac{1}{4}\sqrt{\frac{\pi}{\log 3}}$$

Q. $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_0^1 \frac{dx}{\sqrt{\log x}} =$

$t = -\log x \Rightarrow x = e^{-t}$ $\quad \Rightarrow x=0 \Rightarrow t=\infty$

$dx = -e^{-t}\,dt$ $\quad x=1 \Rightarrow t=0$

$$= \int_0^\infty e^{-t}\cdot t^{-1/2}\,dt$$

$$= \Gamma\frac{1}{2} = \sqrt{\pi}$$

Q. $\int_0^\infty (x^2+4)\, e^{-2x^2}\, dx$

$\int x^2 e^{-2x^2}\, dx + \int 4 e^{-2x^2}\, dx$

$p=2 \quad a=2 \quad q=2$

$p=0 \quad a=2 \quad q=2$

$\frac{\Gamma\frac{p+1}{q}}{q\, a^{\frac{p+1}{q}}} = \frac{\Gamma\frac{3}{2}}{2 \cdot 2^{3/2}}$

$4 \cdot \frac{\Gamma\frac{1}{2}}{2 \cdot 2^{1/2}} = \frac{4\sqrt{\pi}}{2^{3/2}}$

$= \frac{\frac{\sqrt{\pi}}{2}}{2^{5/2}} = \frac{\sqrt{\pi}}{2^{7/2}}$

$\frac{\sqrt{\pi}}{8\sqrt{2}} + \frac{4\sqrt{\pi}}{\sqrt{2}\, 2\sqrt{2}}$

$\cancel{\frac{\sqrt{\pi}}{2^{3/2}}\left(\frac{1}{4}+1\right)} =$

$= \sqrt{\frac{\pi}{2}}\left(\frac{1}{8}+\frac{1}{2}\right)$

$= \sqrt{\frac{\pi}{2}}\left(\cancel{\frac{5}{8}}\right)\left(\frac{5}{8}\right)$

Q. $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx = \frac{\Gamma(n)}{m^n}$

$= \int_0^1 x^{m-1} (\log x^{-1})^{n-1}\, dx$

$= \int_0^1 x^{m-1} (-\log x)^{n-1}\, dx$

$= (-1)^{n-1} \int_0^1 x^{m-1} (\log x)^{n-1}\, dx$

$= (-1)^{n-1} \cdot \frac{(-1)^{n-1}\Gamma n}{(m)^n} = \frac{(-1)^{2n-2}\,\Gamma n}{m^n}$

$= \frac{\cancel{(-1)^{2n-2}}\,\Gamma n}{m^n}$

Q. $\int_0^{\pi/2} \frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}}\, dx$

$= \int_0^{\pi/2} \sin^{8/3} x \cos^{-1/2} x \, dx$

$=$

$2m - 1 = 8/3 \Rightarrow m = 11/6$

$2n - 1 = -1/2 \Rightarrow n = 1/4$

$= \frac{1}{2} \beta\left(\frac{11}{6}, \frac{1}{4}\right)$

$\frac{22+3}{12}$

$= \frac{1}{2} \frac{\Gamma\frac{11}{6}\,\Gamma\frac{1}{4}}{\Gamma\frac{25}{12}}$

$= \frac{\frac{1}{\cancel{2}} \times \frac{5}{\cancel{6}} \times \Gamma\frac{5}{6} \cdot \Gamma\frac{1}{4}}{\frac{13}{\cancel{12}} \times \frac{1}{12}\, \Gamma\frac{1}{12}} = \frac{60}{13} \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{4}}{\Gamma\frac{1}{12}}$

---

8. $\int_0^1 x^4 (1 - x)^3 \, dx$

$\beta(5, 4) = \frac{\Gamma 5\, \Gamma 4}{\Gamma 9} = \frac{4!\, 3!}{8!} = \frac{\cancel{3} \times 2}{8 \times 7 \times \cancel{6} \times 5} = \frac{1}{8 \times 7 \times 5} = \frac{1}{280}$

35×72

$= \begin{array}{r} 72 \\ 35 \\ \hline 360 \\ 216\times \\ \hline 2520 \end{array}$

Q. $\int_0^1 x^2 (1-x^5)^{-1/2}\, dx$

$= \frac{1}{5} \beta\left(\frac{2+1}{5}, \frac{-1}{2}+1\right)$

$= \frac{1}{5} \beta\left(\frac{3}{5}, \frac{1}{2}\right)$

$= \frac{1}{5} \frac{\Gamma\frac{3}{5}\,\Gamma\frac{1}{2}}{\Gamma\frac{11}{10}}$ $\qquad \frac{6+5}{10} = \frac{11}{10}$

$= \frac{2\cancel{1}}{\cancel{5}} \times \frac{\Gamma\frac{3}{5} \times \sqrt{\pi}}{\Gamma\frac{1}{10} \cdot \frac{\cancel{1}}{\cancel{10}}2} = \frac{2\sqrt{\pi}\cdot\Gamma\frac{3}{5}}{\Gamma\frac{1}{10}}$

Q. $\int_0^1 x^2(1-x^3)^4\, dx$

$= \frac{1}{3} \beta\left(\frac{2+1}{3}, 4+1\right)$

$= \frac{1}{3} \beta(1,5) = \frac{1}{3} \cdot \frac{\Gamma 1\,\Gamma 5}{\Gamma 6} = \frac{1}{3} \times \frac{0!\,4!}{5!}$

$= \frac{1}{3\times 5} = \frac{1}{15}$

Result: Some integrals involves algebraic func. can be converted to trig form by applying subs^n

1. $a^2 - x^2$ — $x^2 = a^2 \sin^2\theta$
2. $1 - x^4$ — $x^4 = \sin^2\theta$
3. $1 + x^2$ — $x^2 = \tan^2\theta$
4. $1 + x^6$ — $x^6 = \tan^2\theta$
5. $1 - x^n$ — $x^n = \sin^2\theta$
6. $a - x$ — $x = a\sin^2\theta$
7. $1 - x^{1/4}$ — $x^{1/4} = \sin^2\theta$

Q. $$\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$$

$$= \int_0^2 \frac{x^2}{\sqrt{2 - x}}\,dx$$

Put $x = 2\sin^2\theta$

$dx = 4\sin\theta\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{4\sin^4\theta}{\sqrt{2}\cos\theta} \cdot 4\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 8\sqrt{2}\sin^5\theta\,d\theta = 8\sqrt{2}\int_0^{\pi/2} \sin^5\theta\cos^0\theta\,d\theta$$

$$= 4\sqrt{2}\,\beta\left(\frac{5+1}{2}, \frac{0+1}{2}\right) = 4\sqrt{2}\,\beta\left(3, \frac{1}{2}\right)$$

$$4\sqrt{2}\,\frac{\Gamma 3\,\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}}$$

$$= 4\sqrt{2} \times \frac{2 \times \sqrt{\pi}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{64\sqrt{2}}{15}$$

Q. $\int_0^2 (8-x^3)^{-1/3}\,dx$

$x^3 = 8\cos^2\theta$

$\int_0^2 (8 - 8\cos^2 2\theta)^{-1/3}(-2\sin\theta\, d\theta)$

$\int_0^2 \frac{1}{2}(1-$

$\Rightarrow x = 8^{1/3}\sin^{2/3}\theta \qquad x^3 = 8\sin^2\theta$

$dx = 2\cdot\frac{2}{3}\sin^{-1/3}\theta\cdot\cos\theta\cdot d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \pi/2$

$I = \int_0^2 (8-x^3)^{-1/3}\,dx$

$= \int_0^{\pi/2} (8 - 8\sin^2\theta)^{-1/3}\cdot\frac{4}{3}\sin^{-1/3}\theta\cdot\cos\theta\, d\theta$

$= \frac{4}{3}\int (2^{-1})(\cos^2\theta)^{-1/3}\cos\theta\,\sin^{-1/3}\theta\, d\theta$

$= \frac{2}{3}\int \cos^{1/3}\theta\,\sin^{-4/3}\theta\, d\theta$

$= \frac{2}{3}\left[\frac{1}{2}\beta\left(\left(\frac{-\frac{1}{3}+1}{2}\right), \left(\frac{\frac{1}{3}+1}{2}\right)\right)\right]$

$= \frac{1}{3}\left[\beta\left(\frac{2}{6}, \frac{4}{6}\right)\right]$

$= \frac{1}{3}\cdot\frac{\Gamma\frac{2}{6}\,\Gamma\frac{4}{6}}{\Gamma 1} = \frac{1}{3}\Gamma\frac{1}{3}\cdot\Gamma\frac{2}{3} \Rightarrow \frac{\pi}{}$

3. 8. $\int_0^a x^4 \sqrt{a^2-x^2}\, dx$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$\theta = \pi/2 \Rightarrow x = a$

$\theta = 0 \Rightarrow x = 0$

$$\int a^4 \sin^4\theta \sqrt{a^2\cos^2\theta}\; a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} a^6 \sin^4\theta \cdot \cos^2\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta$$

$$= \frac{a^6}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{a^6}{2} \frac{\Gamma\frac{5}{2}\, \Gamma\frac{3}{2}}{\Gamma 4} = \frac{a^6}{2} \frac{\frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{6}$$

$$= \frac{a^6 \times 3 \cdot \pi}{16 \times 6\, 2} = \frac{a^6 \pi}{32}$$

Q. $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$$

$$= \beta\left(\frac{\frac{1}{2}+1}{2}, \frac{1}{2}\right) \times \beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) \times \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\cancel{\Gamma\frac{3}{4}} \times \Gamma\frac{1}{2}}{\Gamma\frac{5}{4}} \times \frac{1}{2} \frac{\Gamma\frac{1}{4} \times \Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}} = \frac{\pi}{4} \times \frac{\cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}} = \pi$$

Q. $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$

$\int_0^\infty x^{1/2} e^{-x^2} dx \times \int_0^\infty e^{-x^2} x^{-1/2} dx$

$p = \frac{1}{2} \quad a = 1 \quad q = 2 \qquad p = -\frac{1}{2} \quad a = 1 \quad q = 2$

$\Rightarrow \frac{\Gamma\frac{\frac{1}{2}+1}{2}}{2 \cdot (1)^{3/4}} \times \frac{\Gamma\frac{1-\frac{1}{2}}{2}}{2}$

$= \frac{\Gamma\frac{3}{4} \times \Gamma\frac{1}{4}}{4} = \frac{\pi}{\frac{\sin\frac{\pi}{4}}{4}} = \frac{\pi}{4 \times \frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}}$

Q. $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}} dx \times \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$ $\cdot\ 4 \times \frac{\cancel{3} \times 2 \times \cancel{\sqrt{\pi}}}{\frac{7}{2} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi}}$

$= \frac{8 \times 16}{35}$

$= \frac{128}{35}$

$I_1 \Rightarrow x = 3\sin^2\theta$

$dx = 6\sin\theta\cos\theta\, d\theta$

$\int_0^{\pi/2} \frac{3^{3/2} \cdot \sin^3\theta\ 6\sin\theta\cos\theta}{\sqrt{3}\ \cancel{\cos\theta}} d\theta = 18\int_0^{\pi/2} \sin^4\theta \cdot \cos^0\theta\, d\theta$

$\boxed{\frac{27\pi}{\cancel{8}} \times \frac{\cancel{128}^{16}}{35} = \frac{432\pi}{35}}$

$= \frac{18}{2}\beta\left(\frac{4+1}{2}, \frac{0+1}{2}\right)$

$= \frac{18}{2}\beta\left(\frac{5}{2}, \frac{1}{2}\right)$

$= 9\frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{\Gamma 3} = \frac{9}{2} \times \frac{3}{2} \cdot \frac{\pi}{2} = \frac{27\pi}{8}$

Q. $x^{1/4} = \sin^2\theta$

$x = \sin^8\theta$

$dx = 8\sin^7\theta \cdot \cos\theta\, d\theta$

$x = 1 \Rightarrow \theta = \pi/2 \qquad x = 0 \Rightarrow \theta = 0$

$\int_0^{\pi/2} \frac{8\sin^7\theta \cdot \cancel{\cos\theta}\, d\theta}{\sqrt{\cancel{\cos\theta}}}$

$= \frac{8}{2}\beta\left(\frac{7+1}{2}, \frac{0+1}{2}\right) = 4\frac{\Gamma 4\ \Gamma\frac{1}{2}}{\Gamma\frac{9}{2}}$

1. $\int_0^\infty \frac{x^2}{(1+x^4)^3}dx$. $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x^2)}$

$\downarrow$ $\downarrow$

$x^4 = \tan^2\theta$ $x = \tan^2\theta$

5. $\int_0^1 (x \log x)^4 dx$

$= \int_0^1 x^4 (\log x)^4 = \frac{(-1)^4 \Gamma 5}{(5)^5} = \frac{4\times3\times2\times1}{5\times5\times5\times5\times5}$

$= \frac{24}{3125}$

125 25

$625 \times 5 = 3125$

Q. $\int_{-1}^{1}(1-x^2)^n \, dx$

$= \int_{-1}^{1}(1+x)^n(1-x)^n \, dx$

$x = \cos 2\theta$

$dx = -2\sin 2\theta \, d\theta$ $\quad x = -1, \; \theta = \frac{\pi}{2}$

$= -4\sin\theta\cos\theta$ $\quad x = 1, \; \theta = 0$

$\int_{\pi/2}^{0}(1+\cos 2\theta)^n(1-\cos 2\theta)^n \, (-4\sin\theta\cos\theta) \, d\theta$

$= 4\int_{0}^{\pi/2}(2\cos^2\theta)^n(2\sin^2\theta)^n \sin\theta\cos\theta \, d\theta$

$= 2^{2+2n}\int_{0}^{\pi/2}\cos^{2n+1}\theta \sin^{2n+1}\theta \, d\theta$

$= \frac{2^{2+2n}}{2}\beta(n+1, n+1)$

3

Bessel's Functions

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^n)y$$

$$y(x) = c_1 J_n(x) + c_2 J_{-n}(x)$$

$$J_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2}\right)^{2m+n} \frac{1}{\Gamma m+n+1}$$

$$J_{-n} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2}\right)^{2m-n} \frac{1}{\Gamma m-n+1}$$

→ Bessel functions

When n is a +ve integer

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2}\right)^{2m+n} \frac{1}{\Gamma m+n+1}$$

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2}\right)^{2m-n} \frac{1}{\Gamma m-n+1}$$

For Γ to exist, $m-n+1>0$

$m > n-1$

$m = 0, 1, 2, \ldots, n-1 \Rightarrow \Gamma$ becomes ∞

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!}\left(\frac{x}{2}\right)^{2m-n} \frac{1}{\Gamma m-n+1}$$

Put $m-n = s \Rightarrow m = s+n$

s varies from 0 to ∞

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{(s+n)!}\left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma s+1} = (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+n)!}\left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma s+1}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma s+n+1}$$

$$= (-1)^n J_n(x)$$

prove

1. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

& find $\int_0^{\pi/2} \sqrt{x}\, J_{1/2}(2x)\, dx$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}} \frac{1}{\Gamma\left(m+\frac{3}{2}\right)}$$

$$= \left(\frac{x}{2}\right)^{1/2} \cdot \frac{1}{\Gamma\frac{3}{2}} - \left(\frac{x}{2}\right)^{5/2} \cdot \frac{1}{\Gamma\frac{5}{2}} + \frac{1}{2!}\left(\frac{x}{2}\right)^{9/2} \frac{1}{\Gamma\frac{7}{2}}$$

$$= \left(\frac{x}{2}\right)^{1/2} \cdot \frac{1}{\Gamma\frac{3}{2}} - \left(\frac{x}{2}\right)^{2+\frac{1}{2}} \cdot \frac{1}{\Gamma\frac{5}{2}} + \frac{1}{2!}\left(\frac{x}{2}\right)^{4+\frac{1}{2}} \cdot \frac{1}{\Gamma\frac{7}{2}} - \ldots$$

$$= \left(\frac{x}{2}\right)^{1/2} \left(\frac{1}{\frac{\sqrt{\pi}}{2}} - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{\frac{3\sqrt{\pi}}{4}} + \left(\frac{x}{2}\right)^4 \frac{1}{\frac{30\sqrt{\pi}}{8}} - \ldots \right)$$

$$= \left(\frac{x}{2}\right)^{1/2} \times \frac{1}{\sqrt{\pi}} \left(2 - \left(\frac{x}{2}\right)^2 \cdot \frac{4}{3} + \left(\frac{x}{2}\right)^4 \cdot \frac{\cancel{8}^{4}}{\cancel{30}_{15}} - \ldots \right)$$

$$= \left(\frac{x}{2}\right)^{1/2} \times \frac{1}{\sqrt{\pi}} \left(2 - \frac{x^2}{3} + \frac{x^4}{60} - \ldots \right)$$

$$= \frac{\sqrt{\frac{x}{2\pi}} \left(\frac{2x}{x} - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \right)}{x/2}$$

$$= \sqrt{\frac{2}{\pi x}} (\sin x)$$

$$\int_0^{\pi/2} \sqrt{x} \cdot J_{1/2}(2x)\, dx = \int_0^{\pi/2} \sqrt{x} \cdot \sqrt{\frac{2}{\pi \times 2x}} \sin 2x \, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\pi/2} \sin 2x\, dx = \frac{1}{\sqrt{\pi}} \left[\frac{-\cos 2x}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{\pi}} \left[-\cos\pi + \cos 0 \right]$$

$$= \frac{1}{2\sqrt{\pi}} \left[-(-1) + 1 \right] = \frac{1}{\sqrt{\pi}}$$

3 Recurrence Relations

1. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

2. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

3. $\frac{d}{dx}\left[J_n(x)\right] = J_{n-1}(x) - \frac{n}{x} J_n(x)$

4. $J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

5. $J'_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

6. $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

---

4 Q. Prove $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$n = 1/2$

$$\frac{d}{dx}\left(x^{1/2} J_{1/2}(x)\right) = x^{1/2} J_{-1/2}(x)$$

$$\frac{d}{dx}\left(\sqrt{x} \cdot \sqrt{\frac{2}{\pi x}} \sin x\right)$$

$$\sqrt{\frac{2}{\pi}} \cdot \cos x = x^{1/2} J_{-1/2}(x)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

1. Express $J_5(x)$ in terms of $J_0(x)$ & $J_1(x)$

$$\cancel{J_3(x) + J_5(x) = \frac{8}{x} J_4(x)}$$

$$\cancel{J_5(x) = \frac{8}{x} J_4(x) - J_3(x)}$$

Using 6^{th}

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x)$$

$$J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$$

$$J_2(x) + J_4(x) = \frac{6}{x} J_3(x)$$

$$J_3(x) + J_5(x) = \frac{8}{x} J_4(x)$$

$$J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$$

$$= \frac{8}{x}\left(\frac{6}{x} J_3(x) - J_2(x)\right) - \left(\frac{4}{x} J_2 - J_1(x)\right)$$

$$= \frac{8}{x}\left(\frac{6}{x}\left(\frac{4}{x} J_2 - J_1(x)\right)\right) - \left(\frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x)\right)$$

$-\frac{2}{x} J_1(x) + J_0$

$$= \frac{8}{x}\left(\frac{24}{x^2} J_2 - \frac{6}{x} J_1(x)\right) - \left(\frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)\right)$$

$-\frac{2}{x} J_1(x) + J_0$

$$= \frac{184}{x^3}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - \frac{32}{x^2} J_1(x) + \frac{8 J_0}{x} - \frac{8}{x^2} J_1(x) - \frac{4 J_0(x)}{x} - J_1(x)$$

$$\cancel{= \frac{384}{x^4} J_1 - \frac{184}{x^3} J_0 - \frac{40}{x^2} J_1 + \frac{4 J_0}{x} - J_1(x)}$$

$$= \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1 + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0$$

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Q. Express $J_{\frac{5}{2}}(x)$ & $J_{-\frac{5}{2}}(x)$ in terms of sine & cosx

where $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ & $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$$J_{n+1} = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

A. $J_{\frac{5}{2}} = \frac{\cancel{2}\times 3/\cancel{2}}{x} J_{\frac{3}{2}}(x) - J_{1/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$

$J_{\frac{3}{2}} = \frac{\cancel{2}\times 1/\cancel{2}}{x} J_{\frac{1}{2}}(x) - J_{-1/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

$$J_{5/2} = \frac{3}{x}\left(\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right) - J_{1/2}(x)$$

$$= \frac{3}{x^2} J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x) - J_{1/2}(x)$$

$$J_{5/2} = \left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{\pi x}}\sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{n-1} = \frac{2n}{x} J_n(x) - J_{n+1}(x)$$

$J_{-5/2} = \frac{2\left(-\frac{3}{2}\right)}{x} J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x) = \frac{-3}{x} J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x)$

$J_{-3/2} = \frac{\cancel{2}}{x}\left(\frac{-1}{\cancel{2}}\right) J_{-\frac{1}{2}}(x) - J_{1/2}(x) = -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{1/2}(x)$

$$J_{-5/2}(x) = \frac{-3}{x}\left(\frac{-1}{x} J_{-1/2} - J_{1/2}(x)\right) - J_{-1/2}(x)$$

$$= \frac{3}{x^2} J_{-1/2} + \frac{3}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{\pi x}}\cos x + \frac{3}{x}\sqrt{\frac{2}{\pi x}}\sin x$$

Integrals of Bessel's Function

Result 1

$$\frac{d}{dx}\left[x^{v} J_v(x)\right] = x^{v} J_{v-1}(x)$$

$\int J_{-1}(x)\,dx = J_1(x) + c$

On integrating,

$$\int x^{v} J_{v-1}(x)\,dx = x^{v} J_v(x) + c$$

Put $v = 1, 2, 3 \ldots$

$v = 1$: $\int x J_0(x)\,dx = x J_1(x) + C$

$v = 2$: $\int x^2 J_1(x)\,dx = x^2 J_2(x) + c$

$v = 3$: $\int x^3 J_2(x)\,dx = x^3 J_3(x) + c$

Result 2:

$$\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$$

On integrating, $\int x^{-v} J_{v+1}(x) = -x^{-v} J_v(x) + c$

Put $v = 0, 1, 2, \ldots$

$v = 0$: $\int J_1(x) = -J_0(x) + c$

$v = 1$: $\int x^{-1} J(x)\,dx = -x^{-1} J_1(x) + c$

$v = 2$: $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + c$

Note 1: $\int x^m J_n(x)\,dx$

$\hookrightarrow$ m & n are integers & $m+n \geq 0$

can be ~~written~~ integrated by parts completely when $m+n$ is odd & write in terms of J_0 &

Note 2:

When $m+n$ is even, integral depends on residual integral $\int J_0(x)\,dx$

Q. $\int x^2 J_1(x)\,dx$

$= x^2 J_2(x) + c$

$= x^2\left(\frac{2}{x} J_1(x) - J_0(x)\right) + c$

$= 2x J_1(x) - x^2 J_0(x) + c$

$xy = x\int dy + y\int dx$

$\int u\,dv = uv - \int v\,du$

Q. $\int x^4 J_1(x)\,dx$

$= \int x^2 \cdot x^2 J_1(x)\,dx = x^2 x^2 J_2(x) - \int 2x \cdot x^2 J_2(x)$

$= 2x^3 J$

$= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx + c$

$= x^4 J_2(x) - 2x^3 J_3(x) + c$

prove

Q. $\int J_3(x)\,dx = c - J_2(x) - \frac{2}{x} J_1(x)$

~~$\int x^2 J_3(x)\,dx = x^2\, x^{-2} J_2(x) - 2x^{-3} J(x)$~~

$\int x^2 \cdot x^{-2} J_3(x)\,dx =$ ~~$c - J_2(x) - \frac{2}{x} J_1$~~

$\int x^2\, x^{-2} J_3(x)\,dx = x^2 \cdot (-x^{-2} J_2(x)) - \int 2x \cdot (-x^{-2} J_2(x)) + c$

$= -J_2(x) + 2\int x^{-1} J_2(x) + c$

$= -J_2(x) - 2\, x^{-1} J_1(x) + c$

$$\boxed{\int J_3(x)\,dx = -J_2(x) - \frac{2}{x} J_1(x) + c}$$

HW

1. $\int x^3 J_0(x)\,dx = \int x^2\, x\, J_0(x)\,dx$

$= x^2 . x J_1(x) - \int 2x\, x J_1(x)$

$= x^3 J_1(x) - 2x^2 J_2(x) + c$

2. $\int x^3 J_3(x)\,dx = \int x^5 x^{-2} J_3\,dx$

Generating function for Bessel's function as $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Q. Establish Jacobi Series

a) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \dots$

b) $\sin(x\cos\theta) = 2\left[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \dots\right]$

A. $$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$= J_0(x) + J_1(x)\cdot t + J_2(x)\, t^2 + \dots$$
$$+ J_{-1}(x)\, t^{-1} + J_{-2}(x)\, t^{-2} + J_{-3}(x)\, t^{-3} + \dots$$

When $n > 0$, $J_{-n}(x) = (-1)^n J_n(x)$

$J_{-1}(x) = -J_1(x)$

$J_{-2}(x) = J_2(x)$

$J_{-3}(x) = -J_3(x)$

$\vdots$

So, $\Rightarrow J_0(x) + J_1(x)t + J_2(x)t^2 + \dots$

$$- \frac{J_1(x)}{t} + \frac{J_2(x)}{t^2} - \frac{J_3(x)}{t^3} + \dots$$

$\Rightarrow$

$$J_0(x) + J_1(x)\left(t - \frac{1}{t}\right) + J_2(x)\left(t^2 + \frac{1}{t^2}\right) + J_3(x)\left(t^3 - \frac{1}{t^3}\right)$$

$t = \cos\theta + i\sin\theta$ $\qquad t^n - \frac{1}{t^n} = 2i\sin n\theta$

$t^n = \cos n\theta + i\sin n\theta$ $\qquad t^n + \frac{1}{t^n} = 2\cos n\theta$

$\frac{1}{t^n} = \cos n\theta - i\sin n\theta$

$$J_0(x) + J_1(x)(2i\sin\theta) + J_2(x)(2\cos 2\theta)$$
$$+ J_3(x)(2i\sin 3\theta) + \cdots$$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}2i\sin\theta} = e^{ix\sin\theta}$$

$$e^{ix\sin\theta} = J_0(x) + J_1(x)(2i\sin\theta) + J_2(x)\cdot 2\cos 2\theta + J_3(x)(2i\sin 3\theta)$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = \quad ''$$

$$\cos(x\sin\theta) = J_0(x) + J_2(x)\cdot 2\cos 2\theta + \cdots \longrightarrow (1)$$

$$\sin(x\sin\theta) = J_1(x)\, 2\sin\theta + J_3(x)\cdot 2\sin 3\theta + \cdots \longrightarrow (2)$$

$\longrightarrow$ Jacobi Series

$$\theta = \frac{\pi}{2} - \theta$$

$$\cos(x\cos\theta) = J_0(x) + J_2(x)\cdot 2\cos(\pi - 2\theta) + 2J_4(x)(\cos(2\pi - 4\theta)) + \cdots$$

$$\sin(x\cos\theta) = \cancel{2J_1(x) + J_3(x)\, 2\sin 3(\tfrac{\pi}{2}-\theta)}$$

$$2J_1(x)\sin\left(\frac{\pi}{2} - \theta\right) + J_3(x)\, 2\sin 3\left(\frac{\pi}{2} - \theta\right) + \cdots$$

$$= 2\left[J_1(x)\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots\right]$$

3

Q. $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots = 1$

Using Jacobi Series

$J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots = \cos(x\sin\theta)$

$2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \dots = \sin(x\sin\theta)$

Square on B.S ① & ② & integrate wrt θ b/w

$$J_0^2(x)\int_0^{\pi} d\theta + 4J_2^2(x)\int_0^{\pi}\cos^2 2\theta\, d\theta + 4J_4^2(x)\int_0^{\pi}\cos^2 4\theta\, d\theta + \dots = \int_0^{\pi}\cos^2(x\sin\theta)\,d\theta$$

$$4J_1^2(x)\int_0^{\pi}\sin^2\theta\, d\theta + 4J_3^2(x)\int_0^{\pi}\sin^2 3\theta\, d\theta + \dots = \int_0^{\pi}\sin^2(x\sin n\theta)\,d\theta$$

Results

$$\int_0^{\pi}\cos^2 n\theta\, d\theta = \frac{\pi}{2} \text{ is } n \text{ is even}$$

$$\int_0^{\pi}\sin^2 n\theta\, d\theta = \frac{\pi}{2} \text{ if } n \text{ is odd}$$

From eq (3)

$$J_0^2(x).\pi + 4J_2^2(x).\frac{\pi}{2} + 4J_4^2(x).\frac{\pi}{2} + \dots = \int_0^\pi \cos^2(x\sin\theta)\,d\theta$$

$$\pi J_0^2(x) + 2\pi J_2^2(x) + 2\pi J_4^2(x) + \dots = \int_0^\pi \cos^2(x\sin\theta)\,d\theta \rightarrow (5)$$

From eq (4)

$$4J_1^2(x).\frac{\pi}{2} + 4J_3^2(x).\frac{\pi}{2} + \dots = \int_0^\pi \sin^2(x\sin\theta)\,d\theta$$

$$2\pi J_1^2(x) + 2\pi J_3^2(x) \dots = \int_0^\pi \sin^2(x\sin\theta)\,d\theta \rightarrow (6)$$

$$\pi J_0^2(x) + 2\pi J_1^2(x) + 2\pi J_2^2(x) + 2\pi J_3^2(x) + \dots = \int_0^\pi [\sin^2(x\sin\theta) + \cos^2(x\sin\theta)]\,d\theta$$

$$\cancel{\pi}\,[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + \dots] = \int_0^\pi \theta = \cancel{\pi}$$

$$J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + \dots = 1$$

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Orthogonality Property of Bessel Functions

Prove

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{n+1}^2(a\alpha) & \text{if } \alpha = \beta \end{cases}$$

where α & β are roots of $J_n(ax) = 0$

Proof: Consider the DE

$$x^2 u'' + xu' + (\alpha^2 x^2 - n^2) u = 0$$

$$x^2 v'' + xv' + (\beta^2 x^2 - n^2) v = 0$$

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ be the solution of eq ① & ②

$① \times \frac{v}{x}$; $② \times \frac{u}{x}$

$$xvu'' + u'v + \alpha^2 xuv - \frac{n^2 uv}{x} = 0$$

$$xuv'' + uv' + \beta^2 xuv - \frac{n^2 uv}{x} = 0$$

$$x(vu'' - uv'') + (vu' - uv') + (\alpha^2 - \beta^2) xuv = 0$$

$$= d[x(vu' - uv')] + (\alpha^2 - \beta^2) xuv = 0$$

$$\Rightarrow d[x(vu' - uv')] = (\beta^2 - \alpha^2) xuv$$

Integrate B.S

$$[x(vu' - uv')]_0^a = (\beta^2 - \alpha^2) \int^a xuv\,dx$$

$u = J_n(\alpha x)$; $v = J_n(\beta x)$

$u' = \alpha J_n'(\alpha x)$ $v = \beta J_n'(\beta x)$

$$\Rightarrow \left[x\left(J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x)\beta J_n'(\beta x)\right)\right]_0^a = (\beta^2 - \alpha^2)\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx$$

$$\Rightarrow a\left(J_n(a\beta)\alpha J_n'(\alpha a) - J_n(\alpha a)\beta J_n'(\beta a)\right) - 0 = (\beta^2 - \alpha^2)\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx$$

$$\Rightarrow \int_0^a x J_n(\alpha x) J_n(\beta x) = \frac{a}{\beta^2 - \alpha^2}\left[\alpha \underbrace{J_n(a\beta)}_{\to 0} J_n'(a\alpha) - \beta \underbrace{J_n(a\alpha)}_{\to 0} J_n'(a\beta)\right] \quad (3)$$

Since α β are roots of $J_n(ax) = 0$

$J_n(a\alpha) = 0$ & $J_n(a\beta) = 0$

$$\int_0^a x J_n(\alpha x) J_n(\beta x) = 0 \text{ if } \alpha \neq \beta$$

$$\frac{0}{0} \text{ if } \alpha = \beta$$

Now let us consider $\alpha = \beta$ case and take α as a root so, $J_n(\alpha x) = 0$ and β as a variable such that β tends to α, so, $\lim_{\beta \to \alpha}\left[\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx\right]$

$$= \lim_{\beta \to \alpha}\left[\frac{a}{\beta^2 - \alpha^2}\left(\alpha J_n(a\beta) J_n'(a\alpha)\right)\right]$$

Apply L'Hospital Rule to RHS,

$$\int_0^a x J_n^2(\alpha x)\,dx = \lim_{\beta \to \alpha}\left[\frac{a(\alpha J_n'(a\alpha) J_n(a\beta))}{2\beta}\right] = \frac{a}{2\alpha}\alpha J_n'(\alpha a)\, a J_n'(a\alpha)$$

$$= \frac{a^2}{2}\left[J_n'(a\alpha)\right]^2$$

$$\int_0^a x J_n^2(\alpha x)dx = \frac{a^2}{2}\left[\frac{n}{x}J_n(x) - J_{n+1}(x)\right]^2$$

$$= \frac{a^2}{2}\left[\frac{n}{a\alpha}\cancel{J_n(a\alpha)} - J_{n+1}(a\alpha)\right]^2$$

$$= \frac{a^2}{2}\left[-J_{n+1}(a\alpha)\right]^2$$

$$= \frac{a^2}{2}J_{n+1}^2(a\alpha)$$

Bessel's Integral Formula

Prove, $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)d\theta$

By Jacobi series,

$$\cos(x\sin\theta) = J_0(x) + J_2(x).\, 2\cos 2\theta + \dots \quad \rightarrow (1)$$

$$\sin(x\sin\theta) = J_1(x)\, 2\sin\theta + 2J_3(x)\sin 3\theta + \dots \quad \rightarrow (2)$$

Multiply by $\cos n\theta$ on both sides of (1) & integrate wrt θ btn limits 0 to π

$$\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta = J_0(x)\int_0^\pi \cos n\theta d\theta + 2J_2(x)\int_0^\pi \cos 2\theta \cos n\theta\, d\theta + \dots$$

We know,

$$\int_0^\pi \cos m\theta \cos n\theta = \int_0^\pi \sin m\theta \sin n\theta = 0$$ when $m \neq n$ & m & n are int

$$\int_0^\pi \cos^2 n\theta\, d\theta = \frac{\pi}{2}$$ when n is even

$$\int_0^\pi \sin^2 n\theta d\theta = \pi/2$$ when n is odd

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta = 0 + \underbrace{2J_2(x)\left(\frac{\pi}{2}\right)}^{\text{when } n=2} + \underbrace{2J_4(x)\left(\frac{\pi}{2}\right)}^{\text{when } n=4} + \dots$$

$$= \begin{cases} \pi J_n(x) & \text{when } n = \text{even} \rightarrow (3) \\ 0 & \text{when } n = \text{odd} \rightarrow (4) \end{cases}$$

Multiply by $\sin n\theta$ on b.s of eq (2) & integrate wrt θ b/w 0 & π

$$\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = 2J_1(x)\int_0^{\pi} \sin\theta \sin n\theta \, d\theta + 2J_3(x)\int_0^{\pi} \sin 3\theta \sin n\theta \, d\theta + \dots$$

$$= \underbrace{\cancel{2}J_1(x)\left(\frac{\pi}{\cancel{2}}\right)}_{\text{when } n=1} + \underbrace{\cancel{2}J_3(x)\left(\frac{\pi}{\cancel{2}}\right)}_{\text{when } n=3} + \dots$$

$$= \begin{cases} \pi J_n(x), & \text{when } n = \text{odd} \rightarrow (5) \\ 0, & \text{when } n = \text{even} \rightarrow (6) \end{cases}$$

(3) + (6)

$$\pi J_n(x) + 0 = \int_0^{\pi} \left(\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right) d\theta$$

$$\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$$

$$\frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta) = J_n(x) \quad \text{when } n \text{ is even}$$

6. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1+x^2}$

$x = e^t$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x\frac{dy}{dx} = Dy$

$(D^2 - \cancel{D} + \cancel{D} - 1)\, y = \frac{e^{3t}}{1+e^{2t}}$

$(D^2-1)\, y = \frac{e^{3t}}{1+e^{2t}}$

$m = \pm 1$

$y = \underset{\downarrow}{c_1 e^t}_{\,y_1} + \underset{\downarrow}{c_2 e^{-t}}_{\,y_2} = c_1 x + \frac{c_2}{x}$

$W = y_1y_2' - y_2y_1'$
$= e^t(-e^{-t}) - e^{-t}(e^t)$
$= -2$

$\Rightarrow PI = -e^t\int \frac{e^{-t}\cdot e^{3t}}{-2(1+e^{2t})}dt + e^{-t}\int\frac{e^t\cdot e^{3t}}{-2(1+e^{2t})}dt$

$= e^t\int\frac{e^{2t}}{2(1+e^{2t})}dt - \frac{e^{-t}}{2}\int\frac{e^{4t}dt}{1+e^{2t}}$

$= \frac{e^t}{4}\log(1+e^{2t}) - \frac{e^{-t}}{4}\int\frac{z}{1+z}dz$

$= \frac{x\log(1+x^2)}{4} - \frac{e^{-t}}{4}\int\frac{z+1-1}{1+z}dz$

$= \frac{x\log(1+x^2)}{4} - \left(\frac{e^{-t}}{4}\int dz - \frac{e^{-t}}{4}\int\frac{1}{1+z}dz\right)$

$= \frac{x\log(1+x^2)}{4} - \left[\frac{e^{-t}z}{4}\right] - \frac{e^{-t}}{4}\log(1+z)$

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$$= \frac{x\log(1+x^2)}{4} - \left[\frac{e^{-t}e^{2t}}{4}\right] - \frac{e^{-t}}{4}\log(1+e^{2t})$$

$$= \frac{x\log(1+x^2)}{4} - \left[\frac{e^{t}}{4}\right] - \frac{1}{4x}\log(1+x^2)$$

$$PI = \frac{1}{4}\log(1+x^2)\left(x-\frac{1}{x}\right) - \frac{x}{4}$$

$$y = c_1 x + \frac{c_2}{x} + \frac{1}{4}\log(1+x^2)\left(x-\frac{1}{x}\right) - \frac{x}{4}$$

Q. $x^2y'' + xy' - y = x^2\log x$

$D(D-1)y + Dy - 1 = x^2\log x$

$(D^2-1)y = x^2\log x \left[= e^{2t}.t \right.$

$m = \pm 1$

$CF = c_1e^{t} + c_2e^{-t}$

$= c_1x + \frac{c_2}{x}$

$W = e^t(-e^{-t}) - (e^t)(e^{-t})$

$= -2$

$PI = -y_1\int \frac{y_2 X}{W} + y_2\int \frac{y_1 X}{W}$

$= -e^t\int \frac{e^{-t}.e^{2t}t}{-2} + e^{-t}\int \frac{e^{3t}.t}{-2}$

$= \frac{e^t}{2}\int e^t t\,dt - \frac{e^{-t}}{2}\int e^{3t}.t\,dt$

$= \frac{e^t}{2}\left[e^t t - \int e^t dt\right] - \frac{e^{-t}}{2}\left[\frac{e^{3t}t}{3} - \int \frac{e^{3t}}{3}dt\right]$

$= \frac{e^t}{2}\left[e^t(t-1)\right] - \frac{e^{-t}}{2}\left[\frac{e^{3t}t}{3} - \frac{e^{3t}}{9}\right]$

$= \frac{e^{2t}t}{2} - \frac{e^{2t}}{2} - \frac{e^{2t}t}{6} + \frac{e^{2t}}{18}$

$= e^{2t}t\left[\cancel{\frac{1}{2} - \frac{1}{6}}\right]^{\frac{2}{6}} - e^{2t}\left[\cancel{\frac{1}{2} - \frac{1}{18}}\right]^{\frac{8}{18}}$

$= \frac{te^{2t}}{3t} - \frac{4e^{2t}}{9}$

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$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 2\cos^2(\log x)$$

$e^t = x \qquad \log x = t$

$e^{2t} = x^2$

$D(D-1)y + Dy + 1 = 2\cos^2 t$

$(D^2 - \cancel{D} + \cancel{D} + 1)y = 2\cos^2 t$

$W = 1$

$m = \pm i$

$CF = c_1 \cos t + c_2 \sin t$

$$PI = -y_1 \int \frac{y_2 X}{W} dt + y_2 \int \frac{y_1 X}{W} dt$$

$$= -\cos t \int \frac{\sin t \, 2\cos^2 t \, dt}{1} + 2\sin t \int \cos t \cos^2 t \, dt$$

$$= -\frac{2\cos t}{3}[\cos^3 t] + 2\sin t \int \cos t(1 - \sin^2 t) dt$$

$$= -\frac{2\cos t}{3}[\cos^3 t] + 2\sin t \left[\int \cos t - \sin^2 t \cos t \, dt\right]$$

$$= -\frac{2\cos^4 t}{3} + 2\sin^2 t - \frac{2\sin^4 t}{3}$$

$$= -\frac{2\cos^4(\log x)}{3} + 2\sin^2(\log x) - \frac{2\sin^4(\log x)}{3}$$

1. $L = 0.05\,H \quad R = 20\,\Omega \quad C = 100\,\mu F$

$Q = 0$ & $i = 0$ at $= 0$

$$E = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C}$$

$$= 0.05\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + \frac{Q}{100\times10^{-6}} = 0$$

$$= \frac{d^2Q}{dt^2} + 400\frac{dQ}{dt} + 200000Q = 0$$

$$Q = -200 \pm 400i$$

$$Q = e^{-200t}(c_1\cos 400t + c_2\sin 400t)$$

~~$Q = 0$ at $t = 0$~~

~~$\Rightarrow 0 = e^{-200\times0}(c_1\times1 + c_2\times0)$~~

~~$c_1 = 0$~~

~~$i = 0$ at $t = 0$~~

$$i = \frac{dQ}{dt} = e^{-200t}(-400c_1\sin 400t + 400c_2\cos 400t)$$
$$-200e^{-200t}(c_1\cos 400t + c_2\sin 400t)$$

$i = 0$ at $t = 0$

$$\Rightarrow e^0(\cancel{-400}\ 0 + 400c_2) - 200(c_1)$$

$$\Rightarrow \overset{}{\cancel{400}}_{2}\,c_2 = \cancel{200}_{1}\,c_1$$

$$c_2 = c_1/2$$

$q = 0$ at $t = 0$

$$0 = (400c_2)$$

S. L. C

$u = xyz$

$\log u = \log x + \log y + \log z$

$$\frac{\partial u}{u} = \frac{\partial x}{x} + \frac{\partial y}{y} + \frac{\partial z}{z}$$

$$= \frac{1}{100} + \frac{1}{100} + \frac{1}{100} = \frac{3}{100}$$

$$\frac{\partial v}{36} = \frac{3}{100} \Rightarrow \partial v = 1.08$$

$1.08 \times 450 = 486$

$486 \times \frac{530}{1000} = 257.58$ ₹

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2. $V = \pi r^2 h = \frac{\pi d^2 h}{4}$

$\log V = \log \frac{\pi}{4} + 2\log d + \log h$

$$\frac{\partial V}{V} = 0 + 2\frac{\partial d}{d} + \frac{\partial h}{h}$$

$$= 2 \times \frac{0.1}{5} + \frac{0.1}{8}$$

$$= \frac{0.2}{5} + \frac{0.1}{8}$$

$$\frac{\partial V}{V} = 0.0525$$

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