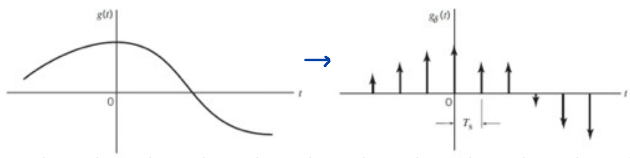
UNIT-2 DIGITAL SIGNAL GENERATION

Sampling Theory

→ Converting continuous time signal $x(t)$ into discrete time signal by taking values at regular intervals

→ sampling frequency, (or) sampling rate $f_s = \frac{1}{T_s}$ → Sampling Period

Sample values : $g(nT_s)$, $n = 0, \pm 1, \pm 2, \ldots$



Ideal sampling : Signal is multiplied with an impulse train

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\, \delta(t - nT_s)$$ → Instantaneous / Impulse Sampling

Here, each delta function captures the value of $g(t)$ at that instant

(If sampling isn't done properly, original signal can't be recovered)

Frequency - domain description of sampling

→ When you sample in time, spectrum repeats in frequency

So apply fourier series,

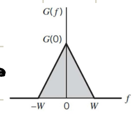
$$G_\delta(f) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} G(f - m f_s)$$ (Derivation is optional!)

→ If $f_s \geq 2W$, replicas don't overlap

→ From this, spectrum of sampled signal is copies of $G(f)$ shifted by multiples of f_s (periodic replication) and if f_s is too small, these shifted copies overlap which causes aliasing

→ irreversible

Sampling Theorem



→ A signal $g(t)$ that is band-limited can be completely represented by samples if $f_s \geq 2W$

→ No freq. components beyond 'W' Hz

$f_s = 2W$ ⇒ Nyquist rate (Min. rate needed)

$T_s = \frac{1}{2W}$ ⇒ Nyquist interval (time spacing b/w samples)

→ We can conclude that sampling below Nyquist rate ⇒ Distortion, Aliasing

above Nyquist rate ⇒ Safe, Easier filter design

Reconstruction

→ After sampling, the spectrum is $G_\delta(f) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} G(f - mf_s)$

→ If $f_s \geq 2W$, no overlap b/w replicas

→ The main goal is to get back $g(t)$ from its samples

→ To achieve this, use low-pass reconstruction filter with cutoff $= W$

→ Whittaker - Shannon Interpolation formula

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}\,(2Wt - n)$$

$$= \sum_{n=-\infty}^{\infty} g(nT_s)\, \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

$\left(\mathrm{sinc}\, x = \frac{\sin \pi x}{\pi x}\right)$

Derivation!

$$G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$$

$$= f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - mf_s)$$

We know $G(f) = 0$ for $|f| \geq W$, $f_s = 2W$

Then, $G(f) = \frac{1}{2W} G_\delta(f)$, $-W < f < W$ → ①

So, $G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\, e^{-j2\pi n f T_s}$ → ②

Put ② in ① & $T_s = 1/2W$

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{-\frac{j\pi n f}{W}}$$

Take inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

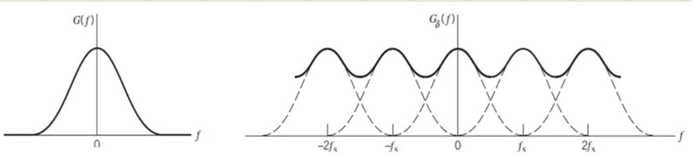
$$= \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \int_{-W}^{W} e^{j2\pi f\left(t - \frac{n}{2W}\right)} df$$

$$= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}\,(2Wt - n)$$

Aliasing

→ If $f_s < 2W$, spectral replicas of $G(f)$ overlap in $G_\delta(f)$

This phenomenon of high frequency components to appear as false-low frequency components



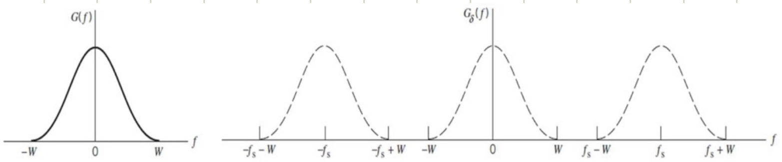
For example, Let $W = 4\text{ KHz}$

Nyquist Rate = 8 KHz

If sampled at 6KHz, 4KHz appears as $|f_s - f| = |6-4| = 2$ KHz in spectrum ✗

→ To prevent this, we use an anti-aliasing filter before sampling, to attenuate high frequency components of signal that aren't essential to information being conveyed by message signal $g(t)$

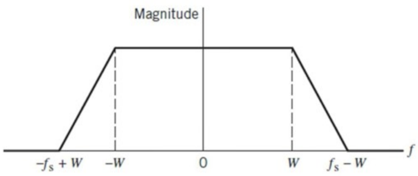
This filtered signal is sampled at a rate higher than nyquist rate



→ At the receiver side, reconstruction (LPF) is used to select baseband replica of sampled spectrum and filter removes all high-frequency shifted copies at multiples of f.

So if no aliasing at transmitter, LPF at receiver recovers original spectrum

Reconstruction filter



Q. Find Nyquist Frequency for the following :

a) $g(t) = \text{sinc}(100t)$

b) $g(t) = \text{sinc}^2(100t)$

c) $g(t) = \text{sinc}(100t) + \text{sinc}(200t)$

d) $g(t) = \text{sinc}(100t) \cdot \text{sinc}(200t)$

e) $g(t) = \text{sinc}(100t) * \text{sinc}(200t)$

f) $g(t) = \text{sinc}(100t)\,\text{sinc}^2(100t)$

A. a) $g(t) = \text{sinc}(100t) = \dfrac{\sin 100\pi t}{100\pi t} \Rightarrow W = 50\text{ Hz}, \; f_s = 100\text{ Hz}$

b) $g(t) = \text{sinc}^2(100t) = \dfrac{\sin^2 100\pi t}{(100\pi t)^2} = \dfrac{1 - \cos 200\pi t}{(100\pi t)^2} \Rightarrow W = 100\text{ Hz}, \; f_s = 200\text{ Hz}$

c) $g(t) = \text{sinc}(100t) + \text{sinc}(200t) = \dfrac{\sin(100\pi t)}{100\pi t} + \dfrac{\sin(200\pi t)}{200\pi t} \Rightarrow W = 100\text{ Hz}, \; f_s = 200\text{ Hz}$

d) $g(t) = \text{sinc}(100t) \cdot \text{sinc}(200t) = \left(\dfrac{\sin 100\pi t}{100\pi t}\right)\left(\dfrac{\sin 200\pi t}{200\pi t}\right) = \dfrac{1}{2}\left(\dfrac{\cos 100\pi t - \cos 300\pi t}{20000\pi^2 t^2}\right) \Rightarrow W = 150\text{ Hz}, \; f_s = 300\text{ Hz}$

e) $g(t) = \text{sinc}(100t) * \text{sinc}(200t) \rightleftharpoons \dfrac{\text{rect}\left(\frac{f}{100}\right)}{100} \cdot \dfrac{\text{rect}\left(\frac{f}{200}\right)}{200} \Rightarrow W = 50\text{ Hz}, \; f_s = 100\text{ Hz}$

↳ $\text{rect}\left(\frac{f}{100}\right)$ spans $[-50, 50]$, $\text{rect}\left(\frac{f}{200}\right)$ spans $[-100, 100]$ → product = $[-50, 50]$

f) $g(t) = \text{sinc}(100t)\,\text{sinc}^2(100t) = \dfrac{\sin^3 100\pi t}{(100\pi t)^3} = \dfrac{\sin 100\pi t - \sin 100\pi t \cos 200\pi t}{(100\pi t)^3} = \dfrac{\sin 100\pi t - 0.5\sin 300\pi t + 0.5 \sin 100\pi t}{(100\pi t)^3} \Rightarrow W = 150\text{ Hz}$

$f_s = 300\text{ Hz}$

Q. Let $g(t)$ be band limited signal with highest frequency of 100 Hz. Find Nyquist frequency for the following :

i) $g^2(t)$

ii) $g(t-3)$

iii) $g(t).g(2t)$

iv) $g(t).\cos(50\pi t)$

Assume $g(t) = \cos 200\pi t$

A. i) $g^2(t) = \cos^2(200\pi t) = \dfrac{1+\cos 400\pi t}{2} \Rightarrow W = 200\text{Hz}, \; f_s = 400\text{ Hz}$

ii) $g(t-3) = \cos(200\pi(t-3)) \Rightarrow W = 100\text{Hz}, \; f_s = 200\text{ Hz}$

iii) $g(t).g(2t) = \cos(200\pi t).\cos(400\pi t)$

$= \dfrac{\cos(200\pi t) + \cos(600\pi t)}{2} \Rightarrow W = 300\text{Hz}, \; f_s = 600\text{ Hz}$

iv) $g(t)\cos(50\pi t) = \cos(200\pi t).\cos(50\pi t)$

$= \dfrac{\cos(150\pi t) + \cos(250\pi t)}{2} \Rightarrow W = 125\text{Hz}, \; f_s = 250\text{ Hz}$

Q. Let $g(t) = 10\cos(200\pi t)\cos(20\pi t)$ ideally be sampled at $f_s = 220\text{Hz}$

Find spectral components that occur at $n = 0, \pm1, \pm2$ of the frequency f_s. Did aliasing occur?

A. $g(t) = 10\cos(200\pi t)\cos(20\pi t) = 5(\cos(220\pi t) + \cos(180\pi t))$ $\left(\cos(2\pi f_0 t) \xleftrightarrow{FT} \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)] \right)$

$G(f) = 2.5(\delta(f-90) + \delta(f+90) + \delta(f-110) + \delta(f+110))$ → Highest freq. component $\Rightarrow W = 110\text{Hz}$

$G_s(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)$

$W = 110\text{ Hz} \Rightarrow f_s = 220\text{ Hz}$

Now, $f = f_0 - nf_s \Rightarrow f_0 = \pm 90, \pm 110$

n	$f_0 - nf_s$	
-2	530, 550, 350, 330	90 + 2(220), 110 + 2(220) -90 + 2(220), -110 + 2(220)
-1	310, 330, 130, 110	90 + 1(220), 110 + 1(220) -90 + 1(220), -110 + 1(220)
0	90, 110, -90, -110	90 + 0(220), 110 + 0(220) -90 + 0(220), -110 + 0(220)
1	-130, -110, -310, -330	90 - 1(220), 110 - 1(220) -90 - 1(220), -110 - 1(220)
2	-350, -330, -530, -550	90 - 2(220), 110 - 2(220) -90 - 2(220), -110 - 2(220)

→ Aliasing occurs because repeated occurance

Q. Let $g(t) = 10\cos(200\pi t)\cos(20\pi t)$ ideally be sampled at $f_s = 250\text{Hz}$

Find spectral components that occur at $n = 0, \pm1, \pm2$ of the frequency f_s. Did aliasing occur?

A. $g(t) = 10\cos(200\pi t)\cos(20\pi t) = 5(\cos(220\pi t) + \cos(180\pi t))$ $\left(\cos(2\pi f_0 t) \xleftrightarrow{FT} \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)] \right)$

$G(f) = 2.5(\delta(f-90) + \delta(f+90) + \delta(f-110) + \delta(f+110))$

$G_s(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)$

$W = 110\text{ Hz} \Rightarrow f_s = 250\text{ Hz}$

Now, $f = f_0 - nf_s \Rightarrow f_0 = \pm 90, \pm 110$

n	$f_0 - nf_s$	
-2	590, 610, 410, 390	90 + 2(250), 110 + 2(250) -90 + 2(250), -110 + 2(250)
-1	340, 360, 160, 140	90 + 1(250), 110 + 1(250) -90 + 1(250), -110 + 1(250)
0	90, 110, -90, -110	90 + 0(250), 110 + 0(250) -90 + 0(250), -110 + 0(250)
1	-160, -140, -340, -360	90 - 1(250), 110 - 1(250) -90 - 1(250), -110 - 1(250)
2	-410, -390, -590, -610	90 - 2(250), 110 - 2(250) -90 - 2(250), -110 - 2(250)

→ No Aliasing occurs because no repeated occurance

Pulse Amplitude Modulation

→ In PAM, amplitude of regularly spaced pulses is varied in proportion to instantaneous value of message signal $m(t)$

→ It is the most basic pulse modulation technique & basis for PCM, DPCM, DM etc.,

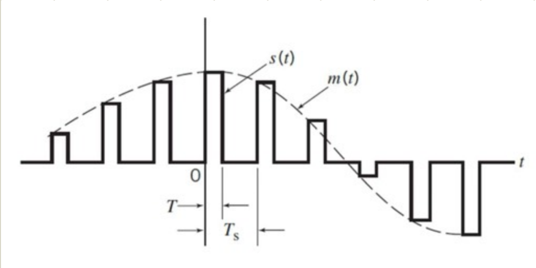
→ There are 2 operations involved in generation of PAM signal :

i) Instantaneous Sampling : Message signal is sampled every T_s seconds

ii) Lengthening : Duration of each sample is T seconds

For flat top modelling, we use rectangular form pulses!

→



$$\longrightarrow \quad s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, h(t - nT_s)$$

Flat-top PAM Signal produced using m(t) & h(t)

message signal

Fourier transformable pulse which is modulated

The delayed version of $h(t - nT_s) = \int_{-\infty}^{\infty} h(t-\tau)\, \delta(\tau - nT_s)\, d\tau$

which means $s(t) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} m(nT_s)\, \delta(\tau - nT_s) \right] h(t-\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} m_\delta(\tau)\, h(t-\tau)\, d\tau$$

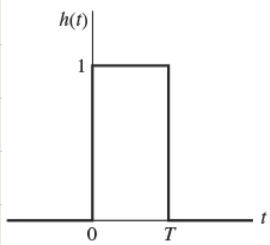
ideal sample signal

$$s(t) = m_\delta(t) * h(t)$$

apply FT, $S(f) = M_\delta(f) H(f) = f_s \sum_{n=-\infty}^{\infty} M(f - nf_s)\, H(f)$

Aperture Effect

→ Consider special rectangular pulse $h(t) = \begin{cases} 1, & 0 < t < T \\ 1/2, & t = 0 \,;\, t = T \\ 0, & \text{otherwise} \end{cases} \Rightarrow \text{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$



The fourier transform of $h(t) \Rightarrow H(f) = T\text{sinc}(fT)\, e^{-j\pi f T}$

introduces 2 issues

→ issues :

Aperture Effect

i) Amplitude distortion due to sinc

ii) Delay of $T/2$ due to exponential function

Equalizer

→ To compensate for sinc attenuation, we cascade a circuit called equalizer with LPF

→ Magnitude response of equalizer, $\frac{1}{|H(f)|} = \frac{1}{\text{sinc}(fT)} = \frac{\pi f T}{\sin(\pi f T)}$

→ It is placed after reconstruction low-pass filter at receiver

→ usually amount of equalization needed in practise is usually small

If duty cycle $\frac{T}{T_s} \le 0.1$, amplitude distortion $< 0.5\%$

Practical Considerations

→ Due to short duration of PAM, it imposes strict requirements on frequency response of channel

→ Poor noise performance of PAM because all information is carried in amplitude

→ Better representation is needed, so we use 3 alternatives

i) PCM (Pulse Code Modulation)

ii) DPCM (Differential Code Modulation)

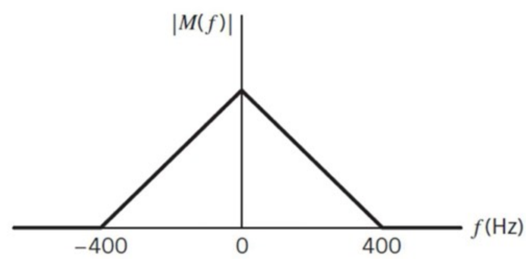
iii) DM (Delta Modulation)

→ They represent analog message in discrete form (time & magnitude)

i) Discrete time by PAM

ii) Discrete amplitude by Quantization

Q. The idealized spectrum of message signal m(t) is given below. Signal is sampled at a rate equal to 1 kHz using flat top pulses, with each pulse being of unit amplitude & duration 0.1ms. Determine & sketch spectrum of resulting PAM signal



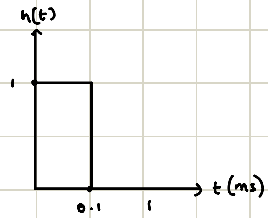
A. Given f_s = 1 kHz, T = 0.1 ms, A = 1

W = 400 Hz

$$|S(f)| = f_s \sum_{n=-\infty}^{\infty} M(f-nf_s) \cdot T\,\text{sinc}(fT)$$

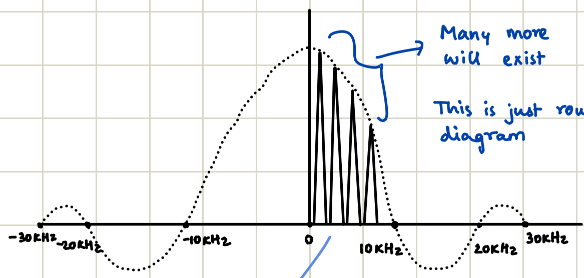
↳ only magnitude

So, no $e^{-j\pi f T}$



Peak & Zero crossings of replicas :

n	Peak (Hz) (centre)	zero-crossings (Hz)
...	...	...
−3	−3000	−3400, −2600
−2	−2000	−2400, −1600
−1	−1000	−1400, −600
0	0	−400, 400
1	1000	600, 1400
2	2000	1600, 2400
3	3000	2600, 3400



Many more will exist

This is just rough diagram

Replicas must be centred at 0, $\pm f_s$, $\pm 2f_s$, $\pm 3f_s$, ...

Now for sinc(fT),

$$\text{sinc}(fT) = \text{sinc}(f \times 0.1 \times 10^{-3}) = \text{sinc}\left(\frac{f}{10 \times 10^{3}}\right)$$

Q. Consider the signal $g(t) = 5\cos(2000\pi t) + 10\cos(6000\pi t)$. Draw the spectrum of the ideally sampled signal, if it is sampled at $f_s = 6\ kHz$. Let the components be in the range of [−18,18] kHz.

A. Given, $g(t) = 5\cos(2000\pi t) + 10\cos(6000\pi t)$

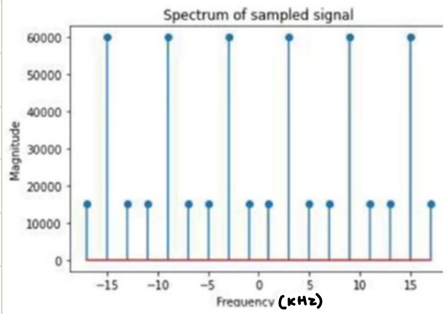
$f_s = 6\text{KHz}$

Then, $G(f) = 2.5\left(\delta(f-1) + \delta(f+1)\right) + 5\left(\delta(f-3) + \delta(f+3)\right)$

$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)$$

$$= 6000 \sum_{n=-\infty}^{\infty} 2.5\left(\delta(f-1-6n) + \delta(f+1-6n)\right) + 5\left(\delta(f-3-6n) + \delta(f+3-6n)\right)$$

n	$f - nf_s = f - 6n$	
−3	15, 17, 19, 21	−3+18, −1+18, 1+18, 3+18
−2	9, 11, 13, 15	−3+12, −1+12, 1+12, 3+12
−1	3, 5, 7, 9	−3+6, −1+6, 1+6, 3+6
0	−3, −1, 1, 3	−3+0, −1+0, 1+0, 3+0
1	−9, −7, −5, −3	−3−6, −1−6, 1−6, 3−6
2	−15, −13, −11, −9	−3−12, −1−12, 1−12, 3−12
3	−21, −19, −17, −15	−3−18, −1−18, 1−18, 3−18



→ Aliasing occurs because repeated occurance

Q. The signal $g(t) = 10\sin(20\pi t) + 4$ is sampled using a periodic sequence of rectangular pulses, with a fundamental frequency of 50 Hz. The pulses are of width 10 ms and have unit amplitude. Draw the spectrum of the sampled signal between 0 Hz and 200 Hz.

A. Given $g(t) = 10\sin(20\pi t) + 4$

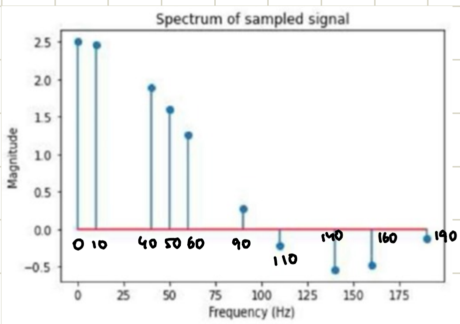
Spectral components at −10, 0, 10 Hz

$f_s = 50\text{Hz}$ $\quad$ $T = 10\text{ms}$ $\Rightarrow$ $f_s \geq 20\text{Hz}$ ✓ Nyquist Condⁿ

$$S(f) = f_s T \sum_{n=-\infty}^{\infty} G(f - nf_s)\ \text{sinc}(fT)\ e^{-j\pi f T}$$

$$|S(f)| = 0.5 \sum_{n=-\infty}^{\infty} \left(5\left[\delta(f-10-nf_s) + \delta(f+10-nf_s)\right] + 4\delta(f-nf_s)\ \text{sinc}\left(\frac{f}{100}\right)\right]\right)$$

n	$f - nf_s = f - 50n$	
0	−10, 0, 10	−10 − 0(50), 0 − 0(50), 10 − 0(50)
−1	40, 50, 60	−10 + 1(50), 0 + 1(50), 10 + 1(50)
−2	90, 100, 110	−10 + 2(50), 0 + 2(50), 10 + 2(50)
−3	140, 150, 160	−10 + 3(50), 0 + 3(50), 10 + 3(50)
−4	190, 200, 210	−10 + 4(50), 0 + 4(50), 10 + 4(50)



Whenever f is an integral multiple of 100, $\text{sinc}(f/100) = 0$

& when $f = 0$, $\text{sinc}(f/100) = 1$

So b/w [0, 200] only consider 0, 10, 40, 50, 60, 90, 110, 140, 150, 160, 190

Q. Twenty-four voice signals are sampled uniformly and then time-division multiplexed (TDM). The sampling operation uses flat-top samples with 1 μs duration. The multiplexing operation includes provision for synchronization by adding an extra pulse of sufficient amplitude and also 1 μs duration. The highest frequency component of each voice signal is 3.4 kHz.

a. Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulses of the multiplexed signal.
b. Repeat your calculation assuming the use of Nyquist rate sampling.

A. a) $T_s = \frac{1}{8000} = 125\mu s$

Given, 24 voice pulses + 1 sync pulse = 25 pulses

So, Time b/w adjacent samples, $T_x = \frac{T_s}{25} = \frac{125\mu}{25} = 5\mu s$

Pulse duration $T_p = 1\mu s$

Spacing b/w 2 successive pulses, $T_g = T_x - T_p = 5 - 1 = 4\mu s$

b) Now $W = 3400\,Hz$, $f_s = 6800\,Hz$

$T_s = \frac{1}{6800} = 147\mu s$

$T_x = \frac{T_s}{25} = 5.58\mu s$

$T_p = 1\mu s$

$T_g = T_x - T_p = 4.58\mu s$

Q. Twelve different message signals, each with a bandwidth of 10 kHz, are to be multiplexed and transmitted. Determine the minimum bandwidth required if the multiplexing/modulation method used is time-division multiplexing (TDM)

A. Given $N = 12$, $W = 10\,kHz$

Min. bandwidth is achieved at nyquist frequency

(baud : samples/s)

TDM signal carries $f_s = N \times 2W = 12 \times 20\,KHz = 240$ kilo baud

Bandwidth required $= f_s/2 = 120\,KHz$

Quantization

→ A sampled signal has ∞ possibilities for amplitude values within its range but it isn't necessary to transmit exact amplitudes

For ex, Human senses can only detect finite intensity differences (> 3dB)

→ So we can approximate continuous signal with discrete amplitude levels

This leads to amplitude quantization

→ Quantization is the process of transforming sample amplitude $m(nT_s)$ of message signal $m(t)$ at time $t = nT_s$ into discrete amplitude $v(nT_s)$, taken from finite set of possible amplitudes

↳ So quantizer is memoryless & instantaneous (scalar quantization)

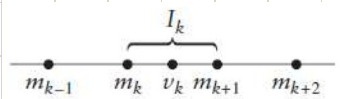
→ $I_K : \{m_K < m < m_{K+1}\}$, $K = 1, 2, \ldots, L$

Continuous sample m → $\boxed{\text{Quantizer } g(\cdot)}$ → Discrete sample v ⇒ $v = g(m)$

L : Total no. of amplitude levels in quantizer

m_K : Decision levels / threshold

v_K : Representation level

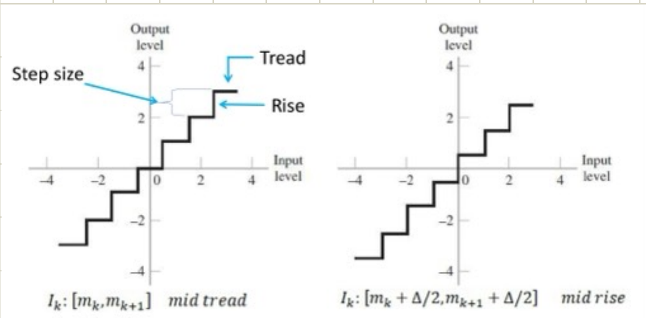


↳ Difference b/w representation levels is called quantum or step

→ The quantizer can be uniform or non-uniform based on uniformly spaced representation levels

→ $v = g(m)$ ⇒ Quantizer Characteristic ⟶ represented by staircase function

↳ 2 Types



(only rises are considered as decision thresholds)

↓ Origin at middle of flat tread

↳ always even no. of levels

↓ Origin at middle of rising step

↳ always odd no. of levels

→ There are 2 Types of Quantization :

i) Uniform Quantization

→ Representations are equally spaced

→ Step size is constant

↳ $\Delta = \dfrac{2m_{max}}{L}$ m_{max} : max amplitude

L : no. of levels $= 2^R$ (R : No. of bits per sample)

ii) Non-uniform Quantization

→ Levels aren't equally spaced

→ Used for speech / audio

Q. Suppose $m(t)$ varies b/w $[-1, 1]$ volts, $L = 4$ representation levels

A. $L = 4 \Rightarrow R = \log_2 4 = 2$ bits

$\Delta = \frac{2m_{max}}{L} = \frac{2}{4} = 0.5$

k	1	2	3	4
I_k	[-1, -0.5]	[-0.5, 0]	[0, 0.5]	[0.5, 1]
Code	00	01	10	11

Statistical Characterization given uniform quantizer

→ Quantization Error or Noise is the difference between actual sample & quantized sample

$q = m - v$ ⟶ Ideally $E(M) = 0$, so $E(V) = 0$ & $E(Q) = 0$

For uniform quantizer,

Quantization error is bounded $-\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2}$

Often assumed as random variable uniformly distributed in this interval

For sufficiently small Δ, (Large L)

we can assume Q follows uniform distribution

$$f_Q(q) = \begin{cases} 1/\Delta, & -\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

implication is that quantization noise is similar to white noise

→ Average Noise Power (variance of Q)

$$\sigma_Q^2 = E[Q^2] - E[Q]^2$$

$$= E[Q^2] - 0$$

$$= \int_{-\Delta/2}^{\Delta/2} q^2 f_Q(q)\,dq = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} q^2\,dq = \frac{1}{\Delta}\left[\frac{q^3}{3}\right]_{-\Delta/2}^{\Delta/2} = \frac{\Delta^3 - (-\Delta^3)}{3\Delta \times 8} = \frac{\Delta^2}{12}$$

$$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{(2m_{max}/2^R)^2}{12} = \frac{m_{max}^2\, 2^{-2R}}{3}$$

→ $(SNR)_o$ of uniform quantizer (SNR - Signal to (Quantisation) Noise Ratio)

$$(SNR)_o = \frac{\text{Average power of } m(t)}{\sigma_Q^2} = \left(\frac{3E^2\{m^2(t)\}}{m_{max}^2}\right)2^{2R} = \frac{3P \times 2^{2R}}{m_{max}^2}$$

It increases exponentially with R

This increases transmission bandwidth

Q. A speech signal has a total duration of 10 s. It is sampled at the rate of 8 kHz and then encoded. The signal-to-(quantization) noise ratio is required to be 40 dB. Calculate the minimum storage capacity needed to accommodate this digitized speech signal.

A. Avg. power of sinusoid $m(t) = A_m \cos(2\pi f_m t)$

$$E(m^2(t)) = \frac{A_m^2}{2}$$

$$(SNR)_o = \frac{3E(m^2(t))}{m_{max}^2} \cdot 2^{2R}$$

$$= \frac{3 \times (A_m^2/2)}{A_m^2} \cdot 2^{2R} = \frac{3}{2} \times 2^{2R}$$

$$(SNR)_o \text{ in dB} = 10\log_{10}\left(\frac{3}{2} \times 2^{2R}\right) = 10\log_{10} 3/2 + 10\log_{10} 2^{2R}$$

$$= 1.76 + 20R\log_{10} 2$$

$$= 1.76 + 6R$$

Given $(SNR)_o = 40\,dB = 1.76 + 6R \Rightarrow R = 6.37 \simeq 7$ bits per sample

As signal is sampled at 8KHz, $f_s = 8000$ samples per second

$R_b = R \times f_s = 7 \times 8000 = 56$ kbps

In 10s, data produced is over 560 kb $\longrightarrow$ Minimum storage capacity

Q. • A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to 50 x 10^6 bits/s.

a. What is the maximum message bandwidth for which the system operates satisfactorily?

b. Determine the output signal-to-(quantization) noise when a full-load sinusoidal modulating wave of frequency 1 MHz is applied to the input.

A. $R = 7$ & $R_b = 50 \times 10^6$ bits/s

a) $$B = \frac{f_s}{2} = \frac{R_b/R}{2} = \frac{50 \times 10^6/7}{2} = 3.57 \times 10^6 \text{ Hz} = 3.57 \text{ MHz}$$

b) $(SNR)_o = 1.76 + 6R$ $\longrightarrow$ Uniform Quantizer like last Q. The modulating wave isn't required

$= 1.76 + 6(7) = 43.76$ dB

Q. Show that with a nonuniform quantizer the mean-square value of the quantization error is approximately equal to $\left(\frac{1}{12}\right)\sum_k \Delta_k^2 p_k$, $\longrightarrow \sigma_Q^2$
where Δ_k is the k-th step size and p_k is the probability that the input signal amplitude lies within the k-th interval. Assume that the step size Δ_k is small compared with the excursion of the input signal

A. For non-uniform quantizer, $e = x - q(x)$

x → input; $q(x)$ → quantized o/p; e → quantized error in k^{th} interval

Suppose input sample x lies in the interval $I_k = (x_k, x_{k+1}]$ & step interval is Δk $(k = 1, 2, \ldots L)$

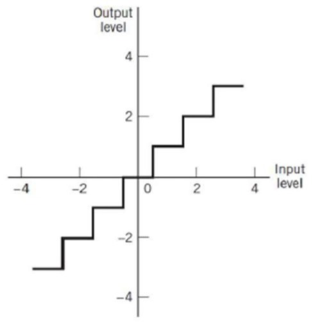
Suppose quantizer outputs midpoints $y_k = \frac{x_k + x_{k+1}}{2}$ & $\Delta_k = x_{k+1} - x_k$ $\in k = 1, 2, \ldots L$

The PDF of x is symmetric & equal in each interval, $f_x(x) = f_x(y_k)$ for $x_k < X \leq x_{k+1}$

Then $p_k = P(x_k < X \leq x_{k+1}) = f_x(y_k)\,\Delta_k$ $\in k = 1, 2, \ldots L$

$$E[Q^2] = \sigma_Q^2 = E[(x - y_k)^2] = \int_{-x_{max}}^{x_{max}} (x - y_k)^2 f_x(x)\,dx \Rightarrow \sigma_Q^2 = \sum_{k=1}^{L} \int_{x_k}^{x_{k+1}} \left(x - \frac{x_k + x_{k+1}}{2}\right)^2 \left(\frac{p_k}{\Delta_k}\right) dx = \sum_{k=1}^{L} \frac{p_k}{\Delta_k} \frac{(x_{k+1} - x_k)^3}{12} = \sum_{k=1}^{L} p_k \frac{\Delta_k^2}{12}$$

Q. A sinusoidal signal with an amplitude of 3.25 V is applied to a uniform quantizer of the midtread type whose output takes on the values 0, ±1, ±2, ±3 V. Sketch the waveform of the resulting quantizer output for one complete cycle of the input.



A. Assume $m(t) = 3.25\cos(20\pi t)$

$W = 10\,Hz$, Time period $= 0.1s$

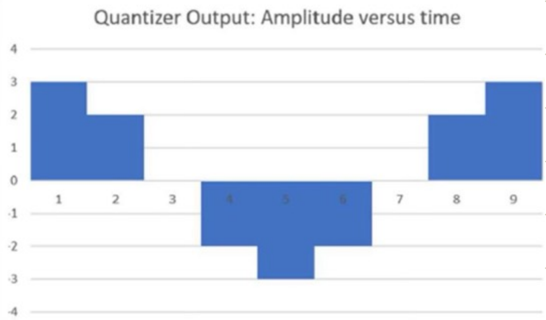
Selecting sampling rate as $f_s = 80\,Hz > 2W$

then we get 8 samples in 0.1s

So, $m(nT_s) = 3.25\cos(20\pi t)$

$= 3.25\cos\left(\frac{20\pi n}{80}\right) = 3.25\cos(\pi n/4)$

Sample n	Input $m(nT_s)$	Output $v(nT_s)$
0	3.25	3
1	2.298	2
2	0	0
3	-2.298	-2
4	-3.25	-3
5	-2.298	-2
6	0	0
7	2.298	2
8	3.25	3



Q. The signal $x(t) = 5\cos(10^5\pi t)$ is sampled at twice the Nyquist rate and quantized with R bits/sample. What is the minimum bit rate that ensures an SNR of at least 43 dB? Also find the corresponding value of the quantization noise power.

A. Given $x(t) = 5\cos(10^5\pi t)$

$\Rightarrow m_{max} = 5\,V$

$W = 50\,kHz$ $\quad f_s = 2W = 100\,kHz$

So, it is sampled at $2f_s = 200\,kHz$

$SNR_o = 1.76 + 6R = 43 \Rightarrow R = 6.87 \approx 7$ bits

$R_b = R \times f_s = 7 \times 200 \times 10^3 = 1.4\,Mbps$

$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{\left(\frac{2m_{max}}{2^R}\right)^2}{12} = \frac{\left(\frac{10}{2^7}\right)^2}{128} = 508\,\mu W$

Pulse Code Modulation

→ It is a discrete time, discrete amplitude waveform coding process by means of which an analog signal is directly represented by a sequence of coded pulses

→ It is the most widely used digital representation of analog signals

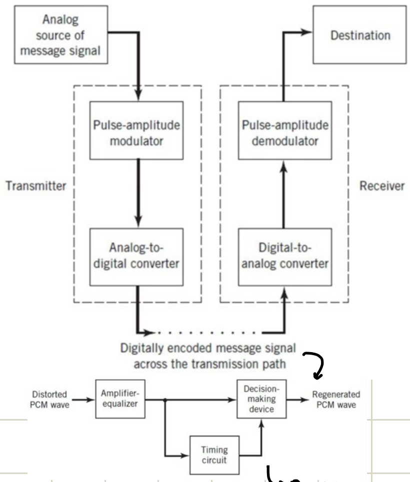
→ 2 sides of operations

i) Transmitter side operations → Pulse amplitude modulator, Analog-to-Digital convertor

ii) Receiver side operations → Digital-to-Analog convertor, Pulse amplitude demodulator

→ Regenerative circuits recover pulses from channel induced distortion

→ Quantization noise produced at transmitter is unavoidable



Transmitter side operations

i) Sampling (Pulse Amplitude Modulator)

→ Low pass anti-aliasing filter is used to restrict frequency to W Hz

→ Sampling frequency > 2W Hz is chosen

→ Flat top sampling is applied (preffered)

ii) Quantizer (ADC)

→ Either uniform or non-uniform quantizer is used

→ Non-uniform quantizer is used for speech signals to deal with non-uniform distribution of amplitudes

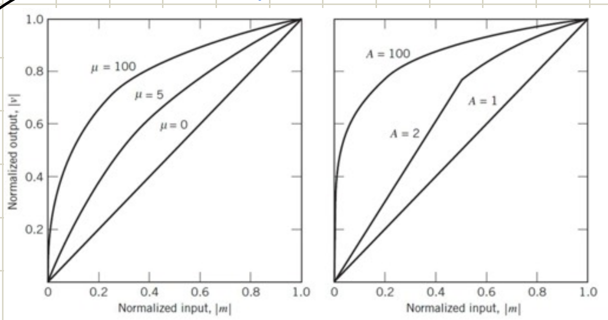
→ Non-uniform quantizer = Passing signal through compressor & sending comp. signal to uniform quantizer

→ There are 2 possible compressor characteristics

μ law ($\mu = 255$) — Europe

A-law ($A = 87.6$) — America

$$|v| = \frac{\ln(1+\mu|m|)}{\ln(1+\mu)}$$



→ Uniform Quantizer

$\mu = 0$ or $A = 1$

$$|v| = \begin{cases} \dfrac{A|m|}{1+\ln A}, & 0 \leq |m| \leq \dfrac{1}{A} \\ \dfrac{1+\ln(A|m|)}{1+\ln A}, & \dfrac{1}{A} \leq |m| \leq 1 \end{cases}$$

Receiver side operations

→ After compressing the signal, it is later expanded using expander

Sometimes, compressor + expander = compander

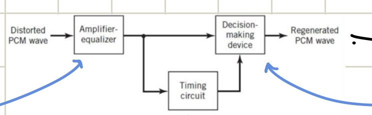
→ Encoding

→ Line coding represents each binary codeword by sequence of pulses

→ Binary pulses are easy to generate & robust to noise

→ Regenerative Circuits

→ First step in receiver is to regenerate received pulse



Equalizer reshapes the received amplitude & phase distorted pulse

Extracts & Restores timing information

Emits pulse if amplitude of restored pulse is above threshold

→ Clean pulses are regrouped into codewords & decoded into quantized pulse-amplitude modulated signal

→ Decoding process involves generating a pulse whose amplitude is the linear sum of all the pulses in the codeword

→ Final operation in receiver is to reconstruct using low pass reconstruction filter

Prediction error filtering

→ The reason why we use this is because :

redundancies

voice/video signals that are sampled at higher rate than Nyquist rate have high correlation

But removing redundancies reduces bandwidth for PCM

Instead of encoding actual samples, we encode variation in samples

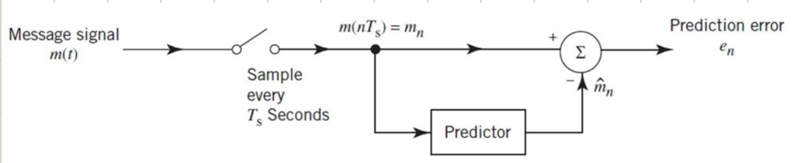
↳ Have small dynamic range compared to actual samples

→ The predictor predicts the n^{th} sample m_n using $m_{n-1}, m_{n-2}, \dots m_{n-p}$ & output = $\hat{m}_n$

So, prediction error ⇒ $e_n = m_n - \hat{m}_n$

and objective is to minimize variance of prediction error e_n

→ The bandwidth required for transmitting e_n < bandwidth required for transmitting m_n



→ Prediction error filter operates on sample-by-sample basis

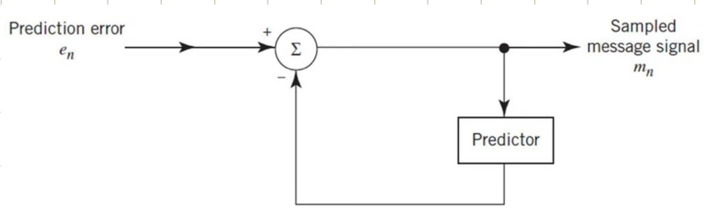
→ Assume the predictor uses some linear operation L,

then, $\hat{m}_n = L[m_n]$

$e_n = m_n - \hat{m}_n = m_n - L[m_n] = (1-L)[m_n]$

so, $m_n = \dfrac{[e_n]}{1-L}$

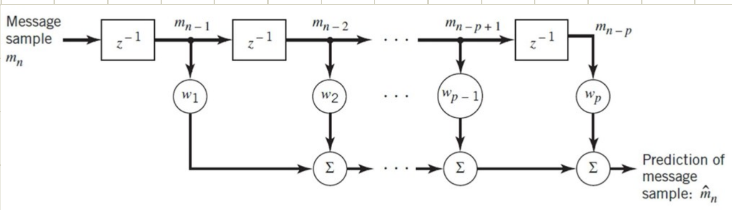
Now to get back the samples, we can create $\underbrace{\text{inverse of prediction error}}$



Discrete time structure for prediction

→ Assume a finite time impulse response filter of order p (Predictor)

→ Prediction sample $\hat{m}_n = \sum_{k=1}^{p} w_k m_{n-k}$



→ Weights are arbitrary, So we can't be sure than e_n will be small.

→ Assume $E[m_n] = 0$ makes variance of e_n = $E[e_n^2]$

The objective function to be minimized, is given by $J(w) = E[e_n^2]$ where $w = [w_1, w_2, \ldots, w_P]^T$

→ Since $e = m_n - \hat{m}_n$,

$$J(w) = E[m_n^2] - 2\sum_{k=1}^{P} w_k E[m_n m_{n-k}] + \sum_{j=1}^{P}\sum_{k=1}^{P} w_j w_k E[m_{n-j} m_{n-k}]$$

$$= \sigma_M^2 - 2\sum_{k=1}^{P} w_k R_{M,k} + \sum_{j=1}^{P}\sum_{k=1}^{P} w_j w_k R_{M,k-j}$$

→ Now differentiate $J(w)$ wrt w_j & equate to 0

$$\sum_{j=1}^{P} w_{o,j} R_{M,k-j} = R_{M,k} \quad \text{where } k = 1, 2, \ldots, P$$

In matrix form,

$$R_M w_o = r_M$$ (r_m: autocorrelations from $\overset{\text{lag } 1}{R_{M,1}}$ to $\overset{\text{lag } P}{R_{M,P}}$)

Wiener-Hopf equations

$$w_o = R_M^{-1} r_m$$

$$w_o = [w_{o,1}, w_{o,2}, \ldots, w_{o,P}]^T$$

$$r_M = [R_{M,1}, R_{M,2}, \ldots, R_{M,P}]^T$$

So, $P \times P$ auto-correlation matrix R_M =

$$\begin{bmatrix} R_{M,0} & R_{M,1} & \cdots & R_{M,P-1} \\ R_{M,1} & R_{M,0} & \cdots & R_{M,P-2} \\ \vdots & \vdots & & \vdots \\ R_{M,P-1} & R_{M,P-2} & \cdots & R_{M,0} \end{bmatrix}$$

Using optimal filter coefficients, $w_o = R_M^{-1} r_m$,

$$J_{min} = J(w_o) = \sigma_M^2 - 2 w_o r_M + w_o^T R_M w_o$$

$$= \sigma_M^2 - r_M^T R_M^{-1} r_m$$

Differential Pulse Code Modulation

→ Similar to PCM, we use a predictor but instead of input as message samples, but quantised message samples

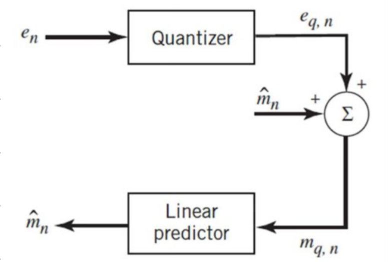
$$\hat{m}_n = L[m_{q,n}]$$

Waveform encoding requires quantization

Waveform decoding requires processing of quantized signal

→ Transmitter Side

→ Differential Quantization



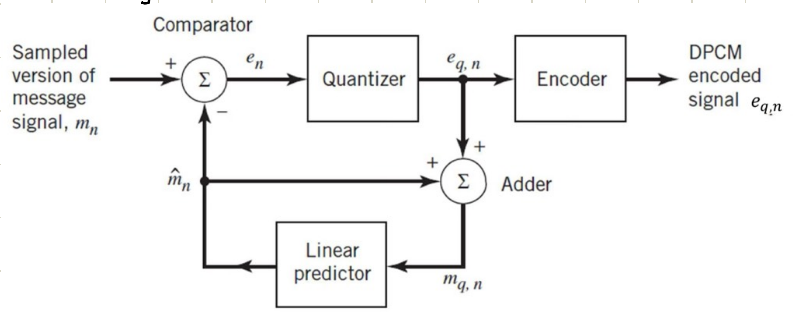
Differential quantization

Difference equation : $e_n = m_n - \hat{m}_n$

Quantized samples : $\hat{m}_{q,n} = \hat{m}_n + e_{q,n}$

Linear prediction : $\hat{m}_n = L[m_{q,n}]$

→ Block diagram



Quantization error : $q_n = e_n - e_{q,n}$

Quantized sample : $m_{q,n} = \hat{m}_n + e_{q,n}$

$= (\hat{m}_n + e_n) + q_n$

$= m_n + q_n$

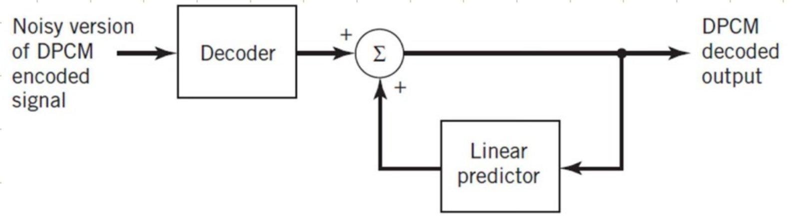
→ Receiver Side

→ Similar structure to linear predictor is repeated, so, predictor output $\hat{m}_n = L[m_{q,n}]$

→ Ideally, decoder reconstructs quantized error $e_{q,n}$

→ Accumulator o/p is given by $m_{q,n} = \hat{m}_n + e_{q,n}$

→ DPCM receiver produces $m_{q,n}$ & not m_n



→ $(SNR)_o = \dfrac{\sigma_M^2}{\sigma_Q^2} = \underbrace{\dfrac{\sigma_M^2}{\sigma_E^2}}_{\text{Processing gain } G_p} \cdot \underbrace{\dfrac{\sigma_E^2}{\sigma_Q^2}}_{\text{o/p SNR of uniform quantizer } SNR_Q}$

$Var(m_n) = \sigma_M^2$

$Var(q_n) = \sigma_Q^2$

$Var(e_n) = \sigma_E^2$

So, $SNR_o = G_p \cdot SNR_Q$

$= 10\log_2 G_p + SNR_Q$ (in dB)

Trade-off for DPCM is increase in hardware complexity for improved performance than PCM

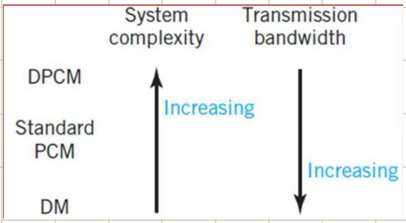
→ For voice signals, DPCM provides 4–11 dB improvement over PCM

& optimal predictor order $p = 4$ or 5

Delta Modulation

→ PCM transmits n-bit words per sample which gives large bit rate.

Unlike PCM, DM simplifies this by transmitting only 1 bit per sample (Bandwidth transmission is traded off for design simplicity)

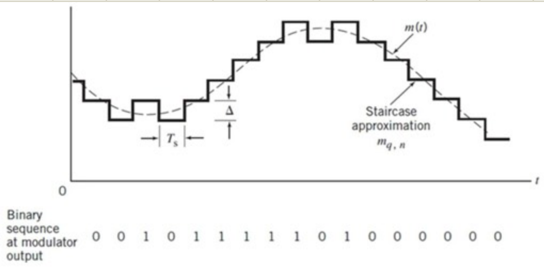


→ DM sends difference b/w preset sample & previous sample approximation instead of actual amplitude

$e_n = m_n - \hat{m}_n = m_n - m_{q,n-1}$

$e_{q,n} = \Delta \operatorname{sgn}(e_n)$

$m_{q,n} = m_{q,n-1} + e_{q,n}$ ⟶ Difference eqⁿ of order one



→ DM tells receiver:

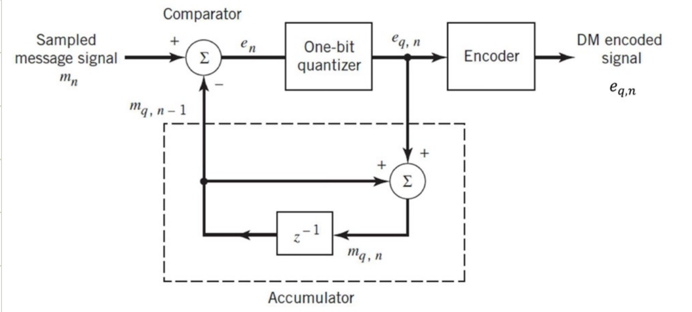
GO UP if input > predicted value → transmits 1

GO DOWN if input < predicted value → transmits 0

→ Transmitter Side

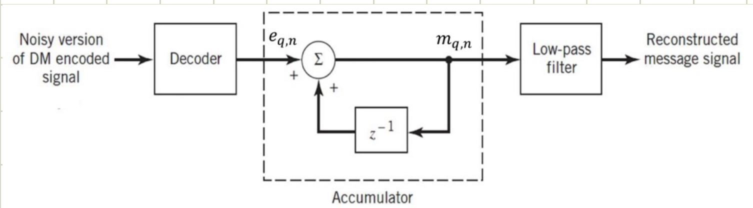
→ In DM, sampling rate (= 1 bit/sample) = bit rate

→ Accumulator is digital equivalent of integrator



→ Receiver side

→ Receiver side accumulator produces staircase waveform $m_{q,n}$

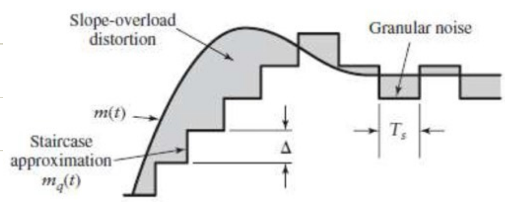


→ Quantization Noise

→ Quantization noise q_n produced in transmitter side is 2 Types:

i) Slope Overload distortion

ii) Granular Noise



→ Only one of these noises can be prevented if signal has variations & flat portions

→ Condition for avoiding slope overload distortion ⇒ $\frac{\Delta}{T_s} \geq \max \left| \frac{dm(t)}{dt} \right|$

Q. Let X be a uniform random variable over the range $[-3, 3]$ V be given as an input to the DPCM system. Suppose the quantizer gives an output SNR of 30 dB based on the prediction error. Suppose the SNR of the DPCM signal is 40 dB. Find the variance of the difference signal $e(n) = x(n) - \hat{x}(n)$.

A. SNR_Q (dB) = 30 dB

SNR_o (dB) = 40 dB

X is uniformly distributed over $[-3, 3]$, $\sigma_X^2 = \frac{(2m_{max})^2}{12} = \frac{6 \times 6}{12} = 3$ W

For DPCM, $SNR_o = SNR_Q \, G_p$

$$G_p = \frac{10^4}{10^3} = 10$$

& $G_p = \frac{\sigma_X^2}{\sigma_E^2} \Rightarrow \sigma_E^2 = \frac{\sigma_X^2}{G_P} = \frac{3}{10} = 0.3$ W

Q. The Gaussian random process $X(n)$ with mean 0 and variance 4 is encoded as a DPCM signal. Suppose the peak value of the prediction error $e(n)$ is assumed to be ± 0.5 and the uniform quantizer uses $R = 4$ bits/sample. If the output SNR of the uniform quantizer is 36 dB, find the following: i) the variance of the error signal $e(n) = x(n) - \hat{x}(n)$ and ii) the processing gain of the predictor in dB.

A. Given $\sigma_M^2 = 4$ $\quad m_{max} = 0.5$

$R = 4$ bits, $SNR_Q = 36$ dB

$$SNR_o = G_P \cdot SNR_Q = \frac{\sigma_M^2}{\sigma_E^2} \cdot \frac{\sigma_E^2}{\sigma_Q^2}$$

$$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{1}{3} m_{max}^2 \, 2^{-2R} = \frac{1}{3} \times 0.25 \times 2^{-8} \approx 0.00033$$

$$SNR_Q = \frac{\sigma_E^2}{\sigma_Q^2} \Rightarrow \sigma_E^2 = 10^{36/10} \times 0.00033 = 1.296$$

$$G_p = \frac{\sigma_M^2}{\sigma_E^2} = \frac{4}{1.296} \simeq 3.086 = 4.89 \text{ dB}$$

Q. A DPCM system uses a linear predictor with a single tap. The normalized autocorrelation function of the input signal for a lag of one sampling interval is 0.75. The predictor is designed to minimize the prediction-error variance. Determine the processing gain attained by the use of this predictor.

A. One tap $\Rightarrow p = 1$

Normalized auto-correlation for one lag $\frac{R_{M,1}}{R_{M,0}} = 0.75$

$R_M = R_{M,0}$ & $r_M = R_{M,1}$

$$w_o = R_M^{-1} r_m$$

$$w_{o,1} = \frac{R_{M,1}}{R_{M,0}} = 0.75$$

$$e_n = x_n - \hat{x}_n = x_n - w_{o,1} x_{n-1}$$

Assuming input to linear predictor is wide sense stationary process $\quad Var(x_n) = Var(x_{n-1}) = \sigma_M^2$

$$\sigma_E^2 = Var(x_n) - Var(w_{o,1} \cdot x_{n-1}) = \sigma_M^2 - w_{o,1}^2 \sigma_M^2 = \sigma_M^2(1 - 0.75^2) = 0.4375\,\sigma_M^2$$

$$G_p = \frac{\sigma_M^2}{\sigma_E^2} = \frac{\sigma_M^2}{0.4375\,\sigma_M^2} = 2.286 = 3.59 \text{ dB}$$

Line Codes

→ Quantizer produces PCM, DPCM, DM as waveform coding scheme

→ For transmission, electrical representation of encoded binary streams is required which is fulfilled by line codes

→ 5 types of line codes :

i) Unipolar nonreturn-to-zero signaling

ii) Polar NRZ signaling

iii) Unipolar return-to-zero signaling

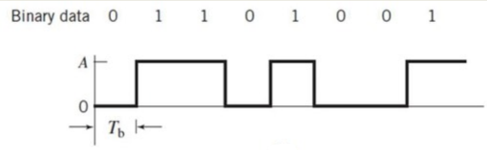
iv) Bipolar RZ signaling

v) Split phase or Manchester Code

→ **Properties**

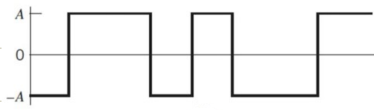
i) Let T & T_b denote pulse duration & bit duration

Unipolar NRZ ⇒ $A_k = \begin{cases} A , & b_k = 1 \\ 0 , & b_k = 0 \end{cases}$ $T = T_b$



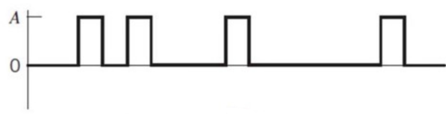
The disadvantage is waste of power due to transmitted DC level

Polar NRZ ⇒ $A_k = \begin{cases} A , & b_k = 1 \\ -A , & b_k = 0 \end{cases}$



It is easy to generate but power spectrum of signal is large at zero frequency

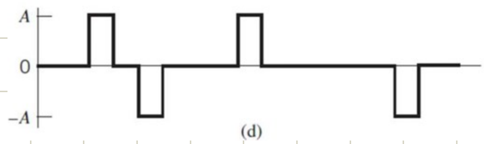
Unipolar RZ ⇒ $A_k = \begin{cases} A , & b_k = 1 \\ 0 , & b_k = 0 \end{cases}$ $T = T_b/2$



Delta functions at $f = 0, \pm \frac{1}{T_b}$ is used for bit-timing recovery

The disadvantage is it requires 3dB more power than polar RZ signaling

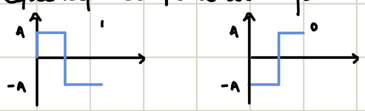
Bipolar RZ ⇒ $\begin{cases} A ; -A , & b_k = 1 \text{ (alternating)} \\ 0 , & b_k = 0 \end{cases}$ $T = T_b/2$

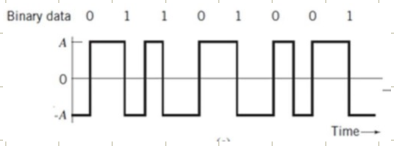


Power spectrum of transmitted has no DC component

Relatively insignificant low-frequency components for the case when 1 & 0 occur with same prob.

Split Phase (Manchester Code)





It suppresses DC component & has relatively insignificant low-frequency components

Power spectra of line codes

→ Line codes $x(t)$ can be viewed as convolution of impulse response $v(t)$ of an LTI filter by a weakly stationary process $A(t)$

→ $A(t) = \{\ldots, A_k, \ldots\}$ containing a sequence of random amplitudes

i) Unipolar $A_k = \begin{cases} a, & b_k = 1 \\ 0, & b_k = 0 \end{cases}$

ii) Polar/Manchester $A_k = \begin{cases} a, & b_k = 1 \\ -a, & b_k = 0 \end{cases}$

Bipolar $A_k = \begin{cases} a ; -a, & b_k = 1 \\ 0, & b_k = 0 \end{cases}$

→ Power spectra of line codes

i) NRZ codes $\Rightarrow$ $v(t) = \text{rect}\left(\frac{t}{T_b} - \frac{1}{2}\right) = \begin{cases} 1, & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases}$

ii) RZ codes $\Rightarrow$ $v(t) = \text{rect}\left(\frac{2t}{T_b} - \frac{1}{2}\right) = \begin{cases} 1, & 0 \leq t \leq T_b/2 \\ 0, & \text{elsewhere} \end{cases}$

iii) Manchester $\Rightarrow$ $v(t) = \text{rect}\left(\frac{2t}{T_b} - \frac{1}{2}\right) - \text{rect}\left(\frac{2t}{T_b} - \frac{3}{2}\right) = \begin{cases} 1, & 0 \leq t \leq T_b/2 \\ 0, & T_b/2 \leq t \leq T_b \end{cases}$

Steps to find Power spectra derivation

1) Find F.T of $v(t)$ $\Rightarrow$ $V(f)$

2) Find power spectral density of $v(t)$ $\Rightarrow$ $\frac{|V(f)|^2}{T_b}$

3) Find autocorrelation $R_A(n) = E[A(t)\,A(t+n)]$ for different values of n

4) Find power spectral density $S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n)\, e^{-j2\pi n T_b}$

5) Express power spectral density of $x(t)$ $\Rightarrow$ $S_x(f) = \frac{|V(f)|^2}{T_b} S_A(f)$

Power spectra of unipolar NRZ

1. Given $v(t) = rect\left(\frac{t}{T_b} - \frac{1}{2}\right) \Rightarrow V(f) = T_b sinc(fT_b)\exp(-j\pi f T_b)$
2. $\frac{|V(f)|^2}{T_b} = \frac{T_b^2 sinc^2(fT_b)}{T_b} = T_b sinc^2(fT_b)$
3. To find $R_A(n) = \mathbb{E}[A(t)A(t+n)]$ for different values of n

$$\text{Unipolar:} \quad A_k = \begin{cases} a & b_k = 1 \\ 0 & b_k = 0 \end{cases}$$

- For $n = 0$:

$$R_A(0) = a^2 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{a^2}{2}$$

b_k	A_k	$p_k = P(b_k, A_k)$	$p_k A_k^2$
0	0	1/2	0
1	a	1/2	$a^2/2$

Power spectra of line codes

Power spectra of unipolar NRZ

- For $n \neq 0$

b_k	b_{k+n}	A_k	A_{k+n}	$p_k = P(b_k, b_{k+n}, A_k, A_{k+n})$	$p_k A_k A_{k+n}$
0	0	0	0	1/4	0
0	1	0	a	1/4	0
1	0	a	0	1/4	0
1	1	a	a	1/4	$a^2/4$

$$R_A(n) = 0 \cdot 0 \cdot \frac{1}{4} + 0 \cdot a \cdot \frac{1}{4} + a \cdot 0 \cdot \frac{1}{4} + a \cdot a \cdot \frac{1}{4} = \frac{a^2}{4} \quad n = \pm 1, \pm 2, \ldots$$

- The general expression for $R_A(n)$ is given by

$$R_A(n) = \frac{a^2}{4}\left(1 + \delta(n)\right) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Power spectra of unipolar NRZ

4. Power spectra of $S_A(f)$ is given by

$$S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi n f T_b} = \sum_{n=-\infty}^{\infty} \left[\frac{a^2}{4}(1+\delta(n))\right] e^{-j2\pi n f T_b}$$

5. The power spectra of unipolar NRZ $X(t)$ is given by

$$\therefore\ S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = \frac{a^2}{4} T_b sinc^2(fT_b) \sum_{n=-\infty}^{\infty} (1+\delta(n))\, e^{-j2\pi n f T_b}$$

- We can rewrite the PSD $S_X(f)$ as

$$S_X(f) = \frac{a^2}{4} T_b sinc^2(fT_b) + \frac{a^2}{4} T_b sinc^2(fT_b) \sum_{n=-\infty}^{\infty} 1 \cdot\ e^{-j2\pi n f T_b}$$

Power spectra of line codes

Power spectra of unipolar NRZ

- The $\sum_{n=-\infty}^{\infty} 1 \cdot \exp(-j2\pi nfT_b)$ represents an impulse train in the frequency domain (impulses are spaced $1/T_b$ Hertz apart)

$$S_X(f) = \frac{a^2}{4}T_b sinc^2(fT_b) + \frac{a^2}{4}T_b sinc^2(fT_b)\sum_{n=-\infty}^{\infty} 1 \cdot e^{-j2\pi nfT_b}$$

- We can also express the above impulse train as $\frac{1}{\mathrm{T_b}}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right)$

$$S_X(f) = \frac{a^2}{4}T_b \,\mathrm{sinc}^2(fT_b) + \frac{a^2}{4}T_b \,\mathrm{sinc}^2(fT_b)\,\frac{1}{T_b}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right)$$

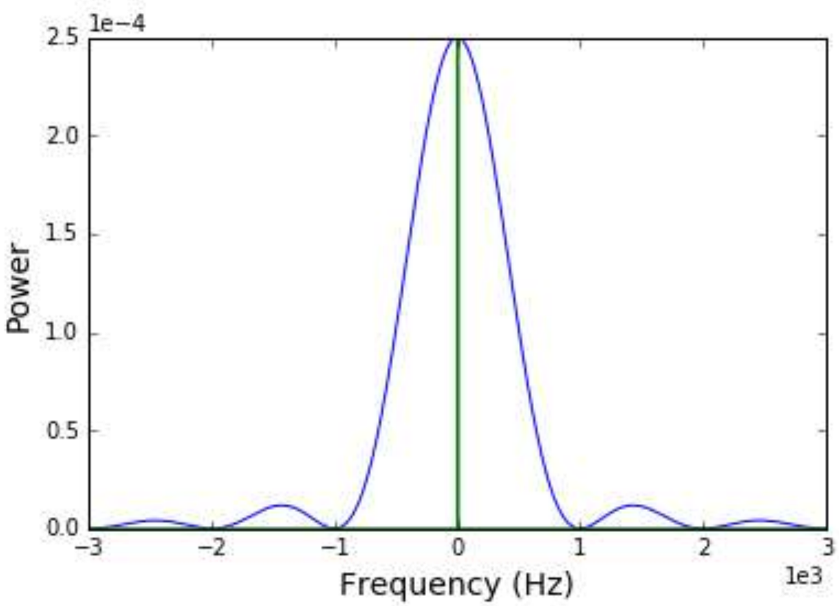
- As $sinc^2(fT_b)$ is 1 at $f = 0$ and 0 at $\pm 1/T_b$, $\pm 2/T_b$, … the PSD of unipolar NRZ becomes

$$\boxed{\therefore\ S_X(f) = \frac{a^2}{4}\delta(f) + \frac{a^2}{4}T_b \,\mathrm{sinc}^2(fT_b)}$$

Power spectra of line codes

Power spectra of unipolar NRZ

$$\boxed{\therefore\ S_X(f) = \frac{a^2}{4}\delta(f) + \frac{a^2}{4} T_b \operatorname{sinc}^2(fT_b)}$$



Bandwidth of the unipolar NRZ (below) is R_b

Example: T_b = 1 ms and a = 1 V

Image cropped to show the sinc^2 shape; DC component dominates the plot. This is a waste of the power!

Power spectra of line codes

Power spectra of polar NRZ

1. Calculation of $\frac{|V(f)|^2}{T_b}$ is the same as the previous example.
2. Next, calculate $R_A(n) = \mathbb{E}[A(t)A(t+n)]$ for different values of n

$$\text{Polar/Manchester:} \quad A_k = \begin{cases} a & b_k = 1 \\ -a & b_k = 0 \end{cases}$$

b_k	A_k	$p_k = P(b_k, A_k)$	$p_k A_k^2$
0	$-a$	1/2	$a^2/2$
1	a	1/2	$a^2/2$

$$R_A(0) = \frac{a^2}{2} + \frac{a^2}{2} = a^2$$

b_k	b_{k+n}	A_k	A_{k+n}	$p_k = P(b_k, b_{k+n}, A_k, A_{k+n})$	$p_k A_k A_{k+n}$
0	0	$-a$	$-a$	1/4	$a^2/4$
0	1	$-a$	a	1/4	$-a^2/4$
1	0	a	$-a$	1/4	$-a^2/4$
1	1	a	a	1/4	$a^2/4$

$$R_A(n) = E[A_k A_{k+n}] = \frac{a^2}{4} - \frac{a^2}{4} - \frac{a^2}{4} + \frac{a^2}{4} = 0 \quad n = \pm 1, \pm 2, \ldots$$

Power spectra of line codes

Power spectra of polar NRZ

- The general expression of $R_A(n)$ is given by

$$R_A(n) = a^2\delta(n) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

3. The PSD of the random process A(t) is given by

$$S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi n f T_b} \Rightarrow S_A(f) = \sum_{n=-\infty}^{\infty} a^2\delta(n) e^{-j2\pi n f T_b} = a^2$$

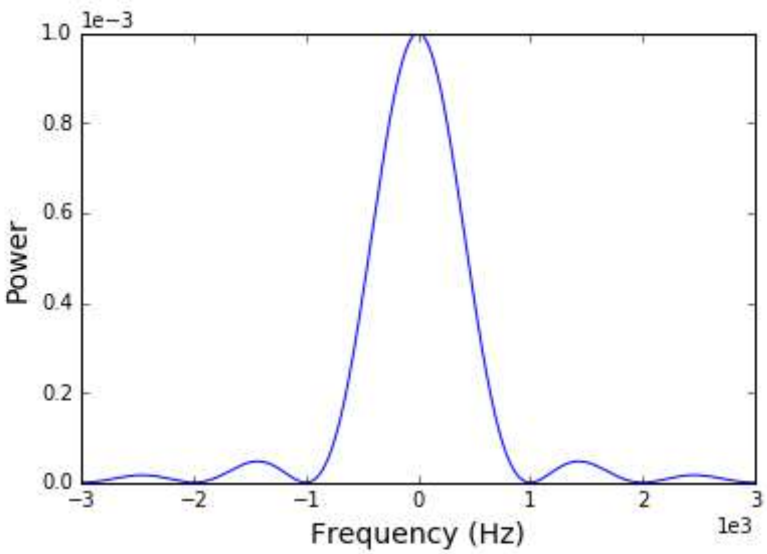
4. The PSD of the polar NRZ format is given by

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = a^2 T_b \operatorname{sinc}^2(f T_b)}$$

Power spectra of line codes

Power spectra of polar NRZ

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = a^2 T_b \operatorname{sinc}^2(f T_b)}$$



Bandwidth of the polar NRZ (left) is R_b

Example: T_b = 1 ms and a = 1 V

Power spectra of bipolar NRZ

1. Calculation of $\frac{|V(f)|^2}{T_b}$ is the same as the previous example.
2. Next, calculate $R_A(n) = \mathbb{E}[A(t)A(t+n)]$ for different values of n

$$\text{Bipolar:} \quad A_k = \begin{cases} a; -a & b_k = 1 \\ 0 & b_k = 0 \end{cases}$$

For $n = 0$:

b_k	A_k	$p_k = P(b_k, A_k)$	$p_k A_k^2$
0	0	1/2	0
1	$a; -a$	1/2	$a^2/2$

For $n = \pm 1$:

b_k	$b_{k\mp 1}$	A_k	$A_{k\mp 1}$	$p_k = P(b_k, b_{k\mp 1}, A_k, A_{k\mp 1})$	$p_k A_k A_{k\mp n}$
0	0	0	0	1/4	0
0	1	0	$a; -a$	1/8; 1/8	0
1	0	$a; -a$	0	1/8; 1/8	0
1	1	$a; -a$	$-a; a$	1/8; 1/8	$-a^2/4$

Power spectra of line codes

Power spectra of bipolar NRZ

$$R_A(n) = 0 \; for \; n = \pm 2, \pm 3, \ldots$$

b_k	b_{k+1}	b_{k+2}	A_k	A_{k+1}	A_{k+2}	$p_k = P(b_k, b_{k+2}, A_k, A_{k+2})$	$p_k A_k A_{k+2}$
0	0	0	0	0	0	1/8	0
0	0	1	0	0	$a; -a$	1/16; 1/16	0
0	1	0	0	$a; -a$	0	1/8	0
0	1	1	0	$-a; a$	$a; -a$	1/16; 1/16	0
1	0	0	$a; -a$	0	0	1/16; 1/16	0
1	0	1	$a; -a$	0	$-a; a$	1/16; 1/16	$-a^2/8$
1	1	0	$a; -a$	$-a; a$	0	1/16; 1/16	0
1	1	1	$a; -a$	$-a; a$	$a; -a$	1/16; 1/16	$a^2/8$

- The general expression for $R_A(n)$ is given by

$$R_A(n) = \frac{a^2}{2}\delta(n) - \frac{a^2}{4}\left(\delta(n+1) + \delta(n-1)\right)$$

Power spectra of bipolar NRZ

3. PSD of $A(t)$ is given by

$$S_A(f) = \sum_{n=-\infty}^{\infty} R_A(n) e^{-j2\pi n f T_b}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{a^2}{2} \delta(n) - \frac{a^2}{4} \left(\delta(n+1) + \delta(n-1) \right) \right) e^{-j2\pi n f T_b}$$

$$= \frac{a^2}{2} - \frac{a^2}{4} \left(e^{j2\pi f T_b} + e^{-j2\pi f T_b} \right) = \frac{a^2}{2} (1 - \cos(2\pi f T_b)) = a^2 \sin^2(\pi f T_b)$$

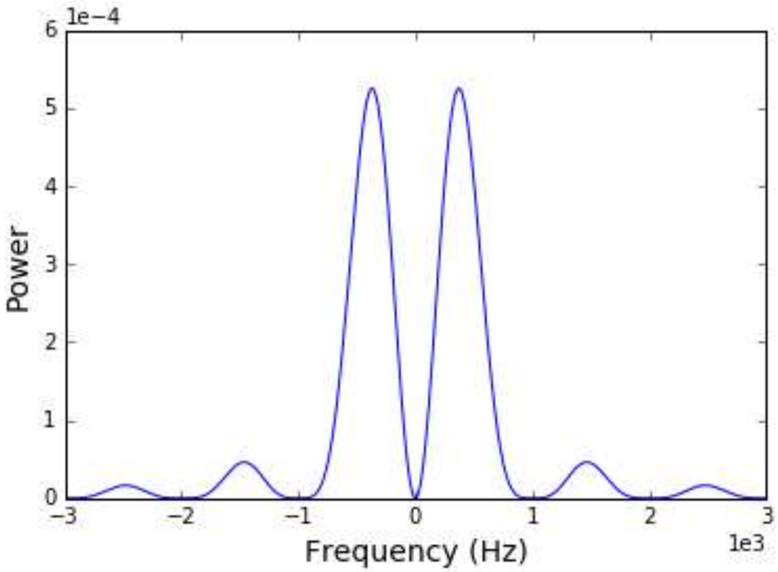
4. The PSD of bipolar NRZ is given by

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = a^2 T_b \sin^2(\pi f T_b) \operatorname{sinc}^2(f T_b)}$$

Power spectra of line codes

Power spectra of bipolar NRZ

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = a^2 T_b \sin^2(\pi f T_b)\,\mathrm{sinc}^2(f T_b)}$$

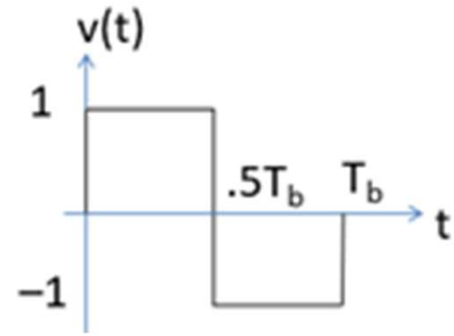


Bandwidth of the bipolar NRZ (left) is R_bExample: T_b = 1 ms and a = 1 V

Power spectra of line codes

Power spectra of Manchester code

1. Given $v(t) = rect\left(\frac{2t}{T_b} - \frac{1}{2}\right) - rect\left(\frac{2t}{T_b} - \frac{3}{2}\right)$



$$\Rightarrow V(f) = 2j\left(\frac{T_b}{2} sinc\left(\frac{fT_b}{2}\right)\right)\exp(-j\pi f T_b)\sin\left(\frac{\pi f T_b}{2}\right)$$

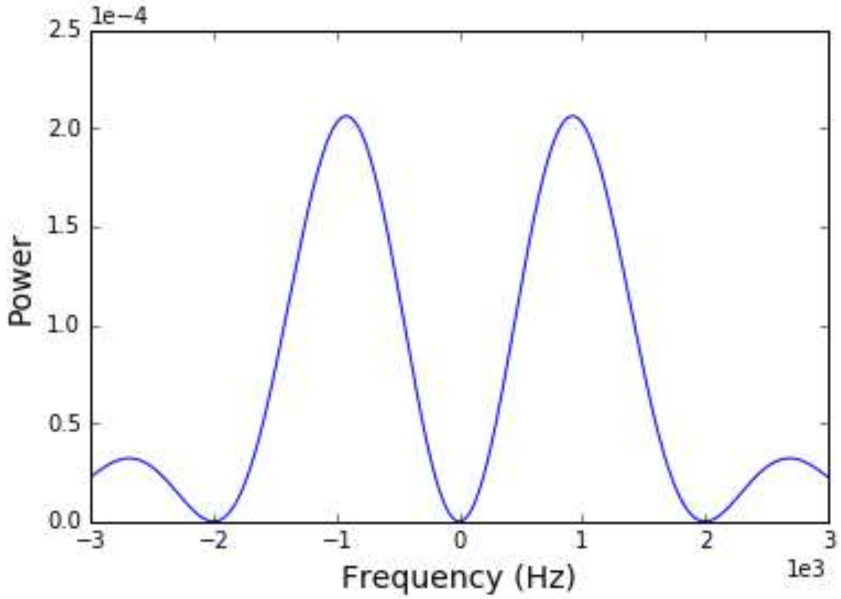
2. $\frac{|V(f)|^2}{T_b} = T_b sinc^2\left(\frac{fT_b}{2}\right)\sin^2\left(\frac{\pi f T_b}{2}\right)$
3. To find $R_A(n) = \mathbb{E}[A(t)A(t+n)]$ for different values of n

$$\text{Polar/Manchester:} \quad A_k = \begin{cases} a & b_k = 1 \\ -a & b_k = 0 \end{cases}$$

4. As polar and Manchester codes use same amplitudes, $R_A(n) = a^2\delta(n)$ and $S_A(f) = a^2$

5. PSD of the Manchester code is given by

$$\boxed{S_X(f) = \frac{|V(f)|^2}{T_b} S_A(f) = a^2 T_b sinc^2\left(\frac{fT_b}{2}\right)\sin^2\left(\frac{\pi f T_b}{2}\right)}$$



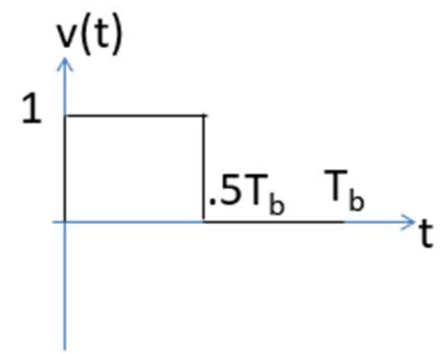
Bandwidth of Manchester is $2R_b$

Example: T_b = 1 ms and a = 1 V

Power spectra of line codes

Power spectra of Unipolar RZ

1. Given $v(t) = rect\left(\frac{2t}{T_b} - \frac{1}{2}\right)$



$$\Rightarrow V(f) = \frac{T_b}{2} sinc\left(\frac{fT_b}{2}\right) \exp\left(-j\pi f \frac{T_b}{2}\right)$$

2. $\frac{|V(f)|^2}{T_b} = \frac{T_b}{4} sinc^2\left(\frac{fT_b}{2}\right)$

3. Calculation of $R_A(n)$ and $S_A(f)$ are same as in slides 6-8 (previous lecture)

$$S_A(f) = \frac{a^2}{4} + \frac{a^2}{4}\frac{1}{T_b}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right)$$

$$\boxed{\therefore\ S_X(f) = \left[\frac{a^2}{16}T_b + \frac{a^2}{16}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right)\right] sinc^2\left(\frac{fT_b}{2}\right)}$$

Power spectra of line codes

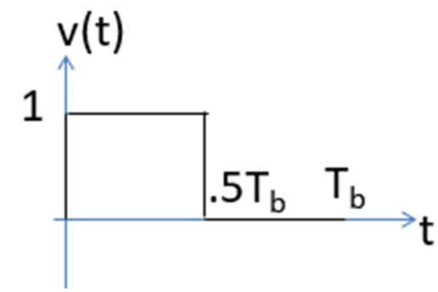
Power spectra of polar RZ

1. Given $v(t) = rect\left(\frac{2t}{T_b} - \frac{1}{2}\right)$

$$\Rightarrow V(f) = \frac{T_b}{2} sinc\left(\frac{fT_b}{2}\right) \exp\left(-j\pi f \frac{T_b}{2}\right)$$

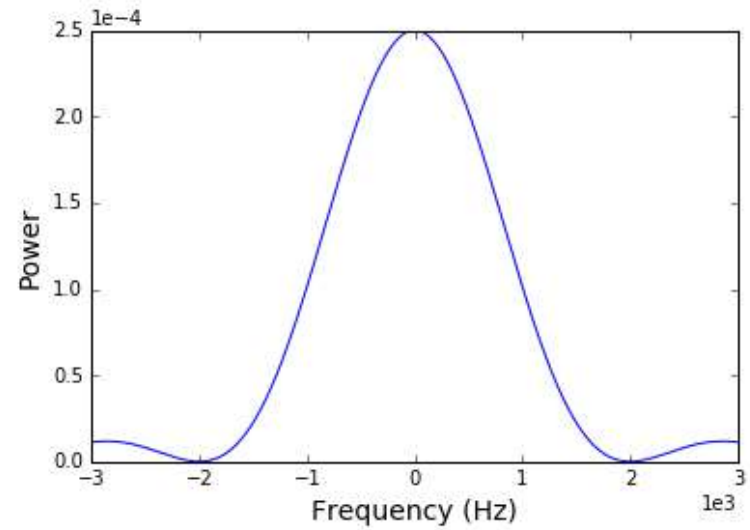
2. $\frac{|V(f)|^2}{T_b} = \frac{T_b}{4} sinc^2\left(\frac{fT_b}{2}\right)$
3. Calculation of $R_A(n)$ and $S_A(f)$ are same as in slides 11-12 in previous lecture

$$\boxed{S_A(f) = a^2 \quad S_X(f) = \frac{a^2 T_b}{4} sinc^2\left(\frac{fT_b}{2}\right)}$$



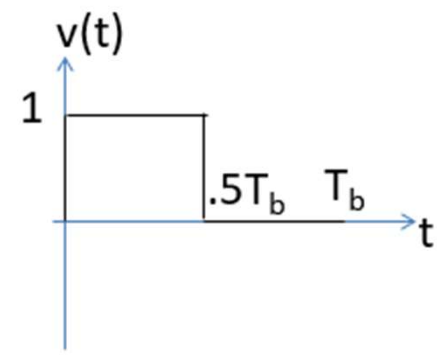
Bandwidth of the polar RZ (right) is $2R_b$

Example: T_b = 1 ms and a = 1 V



Power spectra of line codes

Power spectra of bipolar RZ



1. Given $v(t) = rect\left(\frac{2t}{T_b} - \frac{1}{2}\right)$

$$\Rightarrow V(f) = \frac{T_b}{2} sinc\left(\frac{fT_b}{2}\right)\exp\left(-j\pi f\frac{T_b}{2}\right)$$

2. $\frac{|V(f)|^2}{T_b} = \frac{T_b}{4} sinc^2\left(\frac{fT_b}{2}\right)$

3. Calculation of $R_A(n)$ and $S_A(f)$ are same as in slides 14-16 in previous lecture

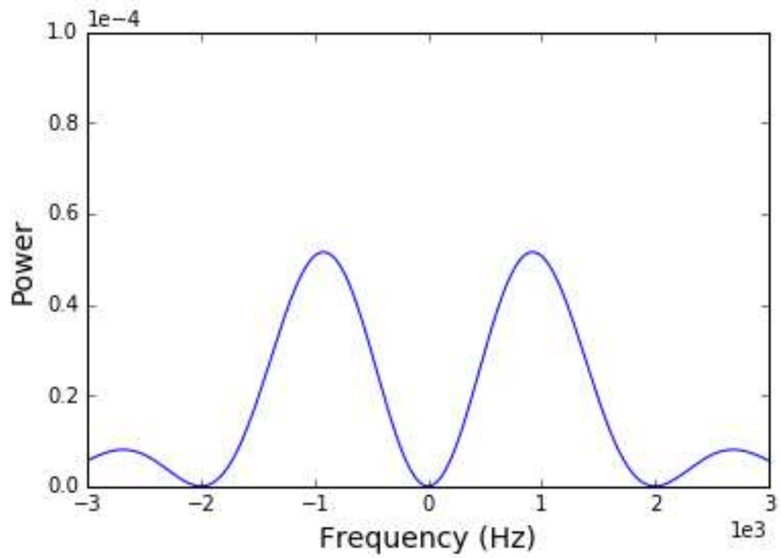
$$S_A(f) = a^2 \sin^2(\pi f T_b) \qquad \boxed{S_X(f) = \frac{a^2 T_b}{4}\sin^2(\pi f T_b)\, sinc^2\left(\frac{fT_b}{2}\right)}$$

Power spectra of line codes

Power spectra of bipolar RZ

Bandwidth of the bipolar RZ (right) is $2R_b$

Example: T_b = 1 ms and a = 1 V



UNIT - 1

MULTIPLE CHOICE QUESTIONS

9.1 A source delivers symbols x_1, x_2, x_3, and x_4 with probabilities 1/2, 1/4, 1/8, and 1/8, respectively. The entropy of the system is
a. 1.75 bits/s
b. 1.75 bits/symbol
c. 1.75 symbols/s
d. 1.75 symbols/bit

[ESE 1999]

9.2 A communication channel has a bandwidth of 100 MHz. The channel is extremely noisy such that the signal power is very much below the noise power. What is the capacity of this channel?
a. 100 Mbps
b. 50 Mbps
c. 2400 bps
d. nearly 0 bps

[ESE 2006]

9.3 A memoryless source emits n symbols each with a probability p. The entropy of the source as a function of n
a. increases as log n
b. decreases as log $(1/n)$
c. increases as n
d. increases as n log n

[GATE 2008]

9.4 A communication channel with AWGN operating at a signal-to-noise ratio SNR >> 1 and bandwidth B has capacity C_1. If the SNR is doubled keeping B constant, the resultant capacity C_2 is given
a. $C_2 \approx 2C_1$
b. $C_2 \approx C_1 + B$
c. $C_2 \approx C_1 + 2B$
d. $C_2 \approx C_1 + 0.3B$

[GATE 2009]

9.5 A binary symmetric channel (BSC) has a transition probability of 1/8. If the channel transmits symbol X such that $P(X = 0) = 9/10$, then the probability of error for an optimum receiver will be
a. 7/80
b. 63/80
c. 9/10
d. 1/10

[GATE 2012]

9.6 Let (X_1, X_2) be independent random variables. X_1 has mean 0 and variance 1, while X_2 has mean 1 and variance 4. The mutual information $I(X_1; X_2)$ between X_1 and X_2 (in bits) is
a. 0
b. 1
c. 0.5
d. 2

[GATE 2017]

9.7 A communication channel disturbed by additive white Gaussian noise has a bandwidth of 4 kHz and SNR of 15. The highest transmission rate that such a channel can support (in kbits/s) is
a. 16
b. 1.6
c. 3.2
d. 60

[ISRO 2006]

9.8 In a binary source, 0s occur three times as often as 1s. What is the information contained in the 1s?
a. 0.415 bit
b. 0.333 bit
c. 3 bits
d. 2 bits

[ISRO 2015]

9.9 A binary source in which 0s occur 3 times as often as 1s. Then its entropy in bits/symbol is
a. 0.75 bits/symbol
b. 0.25 bits/symbol
c. 0.81 bits/symbol
d. 0.85 bits/symbol

[ISRO 2016]

9.10 A source generates one of the five symbols s_1, s_2, s_3, s_4, and s_5 once in every 1/60 second. The symbols are assumed to be independent and occur with probabilities 1/4, 1/4, 1/4, 1/8, and 1/8. The average information rate of the source (in bits/s) is
a. 100
b. 125
c. 135
d. 150

[DRDO 2009]

UNIT - 2

5.1 In a PCM system, each quantization level is encoded into 8 bits. The signal-to-quantization-noise ratio is equal to
a. 256 dB
b. 48 dB
c. $\frac{1}{12}\left(\frac{1}{256}\right)^2$
d. 64 dB

[IES 2000]

5.2 Four signals each band-limited to 5 kHz are sampled twice at the Nyquist rate. The resulting PAM samples are transmitted over a single channel after time division multiplexing. The theoretical minimum transmission bandwidth of the channel should be equal to
a. 5 kHz
b. 20 kHz
c. 40 kHz
d. 80 kHz

[IES 2000]

5.3 A signal $x(t) = 60 \cos(10\pi t)$ is sampled at the rate of 14 Hz. To recover the original signal, the cutoff frequency f_c of the ideal low-pass filter should be
a. 5 Hz < f_c < 9 Hz
b. 9 Hz
c. 10 Hz
d. 14 Hz

[IES 2001]

5.4 Analog data having highest harmonic at 30 kHz generated by a sensor has been digitized using 6-level PCM. What will be the rate of digital signal generated?
a. 120 kbps
b. 200 kbps
c. 240 kbps
d. 180 kbps

[IES 2005]

5.5 The peak-to-peak input to an 8-bit PCM code is 2 V. The signal power-to-quantization-noise power (in dB) for an input of $0.5 \cos(w_m t)$ is
a. 47.8
b. 43.8
c. 96.5
d. 99.6

[GATE 1999]

5.6 The Nyquist sampling interval for the signal $\sin(700t) + \sin(500t)$ is
a. 1/350 s
b. π/350 s
c. 1/700 s
d. π/175 s

[GATE 2001]

5.7 A sinusoidal signal with peak-to-peak amplitude of 1.536 V is quantized to 128 levels using a midrise uniform quantizer. The quantization noise power is
a. 0.768 V
b. 48×10^{-6} V^2
c. 12×10^{-6} V^2
d. 3.072 V

[GATE 2003]

5.8 A signal is sampled at 8 kHZ and is quantized using an 8-bit uniform quantizer. Assuming output signal-to-noise ratio $(SNR)_O$ for a sinusoidal signal, the correct statement for PCM signal with a bit rate is
a. bit rate = 32 kbps, $(SNR)_O$ = 25.8 dB
b. bit rate = 64 kbps, $(SNR)_O$ = 49.8 dB
c. bit rate = 64 kbps, $(SNR)_O$ = 55.8 dB
d. bit rate = 32 kbps, $(SNR)_O$ = 49.8 dB

[GATE 2003]

5.9 A PCM system uses a uniform quantizer which has a range of $-V$ to $+V$ and it is followed by a 7-bit binary encoder. A zero-mean signal applied to the quantizer extends over its entire range and has uniform probability density. The ratio of the signal power to quantization noise power at the output of the quantizer is (use $\log_{10} 2 \approx 0.3$)
a. 14 dB
b. 28 dB
c. 42 dB
d. 56 dB

[DRDO 2009]

5.10 Four voice signals, each limited to 4 kHz and sampled at Nyquist rate, are converted into binary PCM signal using 256 quantization levels. The bit transmission rate for the time division multiplexing signal will be
a. 8 kbps
b. 64 kbps
c. 256 kbps
d. 5126 kbps

[ISRO 2011]