DSP Formulas & Derivations

Unit - 1

Nyquist Rate

→ $f_s = 2f_n$

→ $t_s = \frac{1}{f_s} = \frac{1}{2f_n}$

DFT & IDFT equations using frequency domain sampling methods

→ Consider aperiodic discrete time-signal $x(n)$ & apply FT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Let $\omega = 2\pi K/N$

$$X\left(\frac{2\pi K}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi K n}{N}}$$

$$= \dots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi K n}{N}} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi K n}{N}} + \dots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi K n}{N}}$$

$$= \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN) \right] e^{-j\frac{2\pi K n}{N}}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi K n}{N}}$$

$$x_p(n) = \sum_{K=0}^{N-1} c_K e^{j\frac{2\pi K n}{N}}$$

$$c_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi K n}{N}}$$

$$c_K = \frac{1}{N} X\left(\frac{2\pi K}{N}\right)$$

$$x_p(n) = \frac{1}{N} \sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) e^{j\frac{2\pi K n}{N}}$$

Then, $x_p(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

$$x(\omega) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) e^{\frac{j2\pi K n}{N}} \right] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} X\left(\frac{2\pi K}{N}\right) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi K}{N}\right) n} \right]$$

$$P(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \left(\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right)$$

$$= \frac{1}{N} \left(\frac{e^{-j\omega \frac{N}{2}} \cdot e^{j\omega \frac{N}{2}} - e^{-j\omega \frac{N}{2}} e^{-j\omega \frac{N}{2}}}{e^{-j\omega/2} \cdot e^{j\omega/2} - e^{-j\omega/2} \cdot e^{-j\omega/2}} \right)$$

$$= \frac{e^{-j\omega \frac{N}{2}}}{N e^{-j\omega/2}} \left(\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right) = \frac{e^{-j\omega\frac{(N-1)}{2}}}{N} \left(\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right)$$

$$X(\omega) = \sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) P\left(\omega - \frac{2\pi K}{N}\right)$$

$$P\left(\frac{2\pi K}{N}\right) = \begin{cases} 1, & K = 0 \\ 0, & K = 1, 2, \dots, N-1 \end{cases}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}\right) K n}, \quad K = 0, 1, 2, \dots, N-1$$

$$= \sum_{n=0}^{N-1} x(n) \omega_N^{Kn}$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\left(\frac{2\pi}{N}\right) K n}, \quad n = 0, 1, 2, \dots, N-1$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \omega_N^{-Kn}$$

Twiddle factors

→ $\omega_2^0 = \omega_4^0 = \omega_8^0 = 1$

$\omega_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$\omega_4^1 = \omega_8^2 = -j$

$\omega_8^3 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$\omega_2^1 = \omega_4^2 = \omega_8^4 = -1$

$\omega_8^5 = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$\omega_8^6 = j$

$\omega_8^7 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

DFT as linear transformation

→ $X_N = W_N x_N$

→ $x_N = W_N^{-1} X_N$

$= \frac{W_N^*}{N} X_N$

$|X(K)| = \sqrt{X_R^2(K) + X_I^2(K)}$

$\angle X(K) = \tan^{-1}\left(\frac{X_I(K)}{X_R(K)}\right)$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & \\ 1 & W_N^{N-1} & \cdots & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Relations of DFT w other Transforms

i) Relationship to the Fourier series coefficients of a periodic sequence

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi nk}{N}} \quad , -\infty < n < \infty$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{\frac{-j2\pi nk}{N}} \quad , k = 0, 1, \ldots, N-1$$

Comparing with DFT & IDFT, $X(k) = Nc_k$

ii) Relationship to the Fourier Transform of an aperiodic sequence

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}} = \sum_{n=-\infty}^{\infty} x(n) e^{\frac{-j2\pi nk}{N}} \quad , k = 0, 1, 2, \ldots N-1$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

$x_p(n)$ is determined by aliasing $\{x(n)\}$ over $0 \leq n \leq N-1$

$$\hat{x}(n) = \begin{cases} x_p(n) & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

iii) Relationship to the Z - Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(k) = X(z)\Big|_{z = e^{\frac{j2\pi k}{N}}} \quad , k = 0, 1, 2, \ldots N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{\frac{-j2\pi nk}{N}}$$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}} z^{-1} \right)^n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left(\frac{1 - e^{j2\pi k} z^{-N}}{1 - e^{\frac{j2\pi k}{N}} z^{-1}} \right)$$

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$$

$$X(z) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$$

iv) Relationship to the Fourier series coefficients of a continuous - time signal

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0}$$

If we sample $x_a(t)$ at $F_s = \frac{N}{T_p} = \frac{1}{T}$

$$x(n) = x_a(nT) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kn}{N}} = \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j\frac{2\pi kn}{N}}$$

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

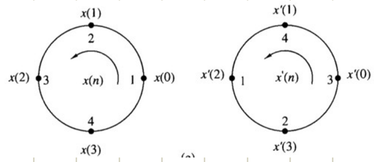
Properties of DFT

i) Periodicity $\Rightarrow$ $x(n+N) = x(n)$

$X(k+N) = X(k)$

ii) Linearity $\Rightarrow$ $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(k) + a_2 X_2(k)$

iii) Circular Symmetry $\Rightarrow$ $x(n) \xRightarrow{\text{shift}} x'(n) = x((n-k))_N$



Circular even : $x(N-n) = x(n)$

Circular odd : $x(N-n) = -x(n)$

Time reversal : $x((-n))_N = x(N-n)$

iv) Symmetric Properties of DFT :

$$X_R(k) = \sum_{n=0}^{N-1}\left(x_R \cos\frac{2\pi kn}{N} + x_I \sin\frac{2\pi kn}{N}\right)$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left(x_R \sin\frac{2\pi kn}{N} - x_I \cos\frac{2\pi kn}{N}\right)$$

$$x_R(n) = \frac{1}{N}\sum_{n=0}^{N-1}\left(x_R \cos\frac{2\pi kn}{N} - x_I \sin\frac{2\pi kn}{N}\right)$$

$$x_I(n) = \frac{1}{N}\sum_{n=0}^{N-1}\left(x_R \sin\frac{2\pi kn}{N} + x_I \cos\frac{2\pi kn}{N}\right)$$

$x(n) \xleftrightarrow[N]{DFT} X(k)$

$x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k)$

$x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$

$x_R(n) \xleftrightarrow[N]{DFT} X_{ce}(k) = \frac{1}{2}\left(X(k) + X^*(N-k)\right)$

$x(n) = x_R^e + x_R^o + j x_I^e + j x_I^o$

$X(k) = X_R^e + X_R^o + j X_I^e + j X_I^o$

$x(n) = \text{real} \Rightarrow X(k) = X^*(N-k)$ $\Rightarrow$ conjugate symmetry

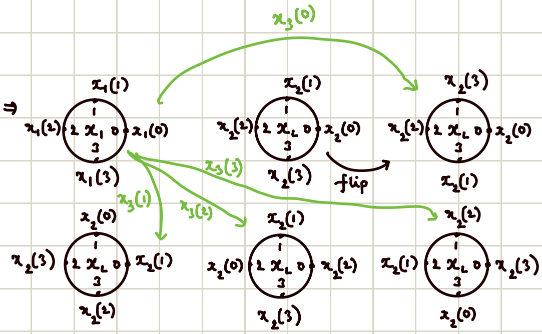
$x(n) = \text{imag.} \Rightarrow X(k) = -X^*(N-k)$ $\Rightarrow$ conjugate anti symmetry

DFT Multiplication & Circular Convolution

$\rightarrow$ $x_3(m) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2((m-n))_N$

$X_3(k) = X_1(k)\, X_2(k)$

$\rightarrow$ Graphical Method $\Rightarrow$



$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))_4$$

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4$$

$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$$

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4$$

Additional Properties of DFT

1) Time Reversal : $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

2) Circular Time Shift : $x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j\frac{2\pi kl}{N}} = X(k)\, W_N^{kl}$

3) Circular Frequency Shift : $x(n)\, e^{j\frac{2\pi ln}{N}} \xleftrightarrow[N]{DFT} X((k-l))_N$

4) Circular Convolution : $x(n) \circledN h(n) \xleftrightarrow{DFT} X(k)\, H(k)$

5) Multiplication : $x(n) \cdot h(n) \xleftrightarrow{DFT} \left(X(k) \circledN H(k)\right)/N$

6) Parseval's Identity : $\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, Y(k)$

7) Complex Conjugate : $x^*(n) \xleftrightarrow{DFT} X^*(N-k)$

$x^*(N-n) \xleftrightarrow{DFT} X^*(k)$

8) Duality : $X(n) \xleftrightarrow{DFT} N\, x((-k))_N$

$DFT\{DFT\{X(n)\}\} = N\, x((-n))_N$

$DFT\{DFT\{DFT\{DFT\{X(n)\}\}\}\} = N^2 x(n)$

9) Circular Correlation : $r_{xy}(l) = x(l) \circledN y^*(-l)_N$

$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k)$

$R_{xy}(k) = X(k)\, Y^*(k)$

$$DFT\left\{Re\{x(n)\}\right\} = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$DFT\left\{Im\{x(n)\}\right\} = \frac{1}{2j}\left[X(k) - X^*(N-k)\right]$$

$DFT\{\delta(n)\} = 1$; $DFT\{\delta(n-n_0)\} = W_N^{kn_0}$

$$DFT\left\{x(n) = \begin{cases} 1, & n = \text{even} \\ 0, & n = \text{odd}\end{cases}\right\} = \begin{cases} \frac{N}{2}, & k = 0 \\ 0, & k \neq 0\end{cases}$$

$$DFT(a^n) = \frac{1 - a^N}{1 - a W_N^k}$$

$$DFT\left(e^{j\frac{2\pi k_0 n}{N}}\right) = N\,\delta(k-k_0)$$

$$DFT\left(\cos\left(\frac{2\pi k_0 n}{N}\right)\right) = \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta(k+k_0)$$

$$DFT\left(\sin\left(\frac{2\pi k_0 n}{N}\right)\right) = \frac{N}{2j}\delta(k-k_0) - \frac{N}{2j}\delta(k+k_0)$$

Overlap Add Method

1) $N = L + M - 1$

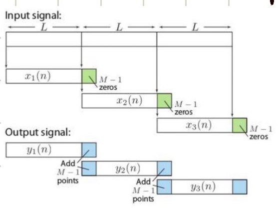
N-point → length of $h(n)$

2) Break $x(n)$ into non-overlapping block $x_m(n)$ of length L
3) Zero pad $x_m(n)$ & $h(n)$ until they are length N
4) $h(n) \xleftrightarrow[N]{DFT} H(k)$
5) For each block,

$x_m(n) \xleftrightarrow[N]{DFT} X_m(k)$

$Y_m(k) = X_m(k) \cdot H(k)$

$Y_m(k) \xleftrightarrow[N]{IDFT} y_m(n)$



6) Form $y(n)$ by overlapping last $M-1$ samples of $y_m(n)$ with first $M-1$ samples of $y_{m+1}(n)$ and add result

no. of DFTs & IDFTs

⇒ If z = length of $x(n)$, M = length of $h(n)$

N point DFT

Then, $N = L + M - 1$

$$\left\lceil \frac{z}{L} \right\rceil = p$$

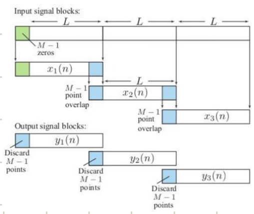
'p+1' DFTs & 'p' IDFTs

Overlap Save Method

1) $N = L + M - 1$
2) Insert $M-1$ zeros at beginning of $x(n)$
3) Break $x(n)$ into $x_m(n)$ of length N with first $M-1$ elements overlapped
4) Zero pad $h(n)$ to be length N
5) $h(n) \xrightarrow[N]{DFT} H(k)$
6) $x_m(n) \xrightarrow{DFT} X_m(k)$

$Y_m(k) = X_m(k) \cdot H(k)$

$Y_m(k) \xrightarrow{IDFT} y_m(n)$



Discard first $M-1$ points of each output block

7) Form $y(n)$ by appending remaining L samples

no. of DFTs & IDFTs

⇒ If z = length of $x(n)$

M = length of $h(n)$, N point DFT

Then, $N = L + M - 1$

$$\left\lceil \frac{z}{L} \right\rceil = p$$

'p+1' DFTs & 'p' IDFTs

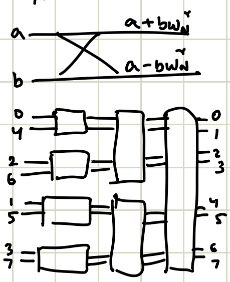
Unit-2

	DFT	FFT
Complex Multiplications ⇒	N^2	$\frac{N}{2}\log_2 N$
Complex Additions ⇒	$N(N-1)$	$N\log_2 N$

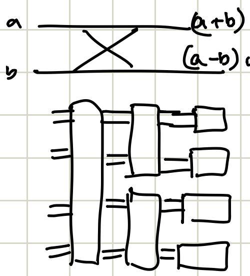
FFT → DIT, DIF

IFFT → DIT, DIF

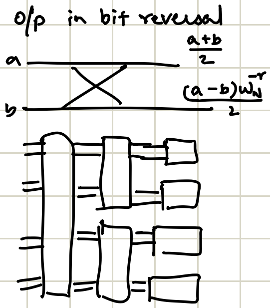
FFT DIT: I/p Bit reversal, o/p in order



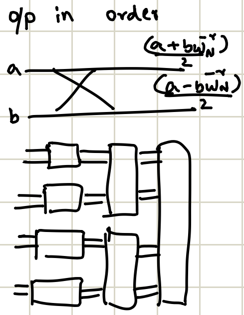
FFT DIF: I/p in order, o/p in bit reversal



IFFT DIT: I/p in order, o/p in bit reversal



IFFT DIF: I/p Bit reversal, o/p in order



$N \to \log_2 N$ stages

$\frac{N}{2}$ butterfly diagrams

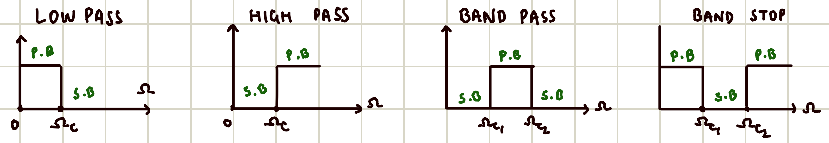
$\frac{N}{2}\log_2 N$ CM

$N\log_2 N$ CA

(2N resistors for in-place computation)

(N resistors for twiddle factors)

UNIT - 3



P.B - Passband

S.B - Stopband

BUTTERWORTH FILTER

$$|H_n(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}} \qquad \left(|H_n(0)|^2 = 1 \ , \ |H_n(\Omega_c)|^2 = 0.5\right)$$

$$H_n(s) = \frac{1}{\prod\limits_{\text{LHP Poles}}(s-s_k)} = \frac{1}{B(s)}$$

n	$B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$

$$K_p = 10\log\left(\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2n}}\right)$$

$$K_s = 10\log\left(\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2n}}\right)$$

$$n = \frac{\log\left(\dfrac{10^{-K_p/10}-1}{10^{-K_s/10}-1}\right)}{2\log\left(\frac{\Omega_p}{\Omega_s}\right)}$$

$$\Omega_c = \frac{1}{\left(10^{-K_p/10}-1\right)^{1/2n}}$$

$\rightarrow$

LP $\rightarrow$ LP $\Rightarrow$ $\Omega_r = \frac{\Omega_s}{\Omega_p}$, $s \rightarrow \frac{s}{\Omega_{c_p}}$

$\Omega_{c_p} = \Omega_p \times \Omega_c$

HP $\rightarrow$ LP $\Rightarrow$ $\Omega_r = \frac{\Omega_p}{\Omega_s}$, $s \rightarrow \frac{\Omega_c}{s}$

$\Omega_c = \Omega_p$

Common for B.W & CHEBY

BP $\rightarrow$ LP $\Rightarrow$ $\Omega_r = \min(|A|, |B|)$

$$A = \frac{-\Omega_1^2 + \Omega_u\Omega_l}{\Omega_1(\Omega_u-\Omega_l)}$$

$$B = \frac{-\Omega_2^2 + \Omega_u\Omega_l}{\Omega_2(\Omega_u-\Omega_l)}$$

$$s \rightarrow \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}$$

BS $\rightarrow$ LP $\Rightarrow$ $\Omega_r = \min(|A|, |B|)$

$$A = \frac{\Omega_1(\Omega_u-\Omega_l)}{-\Omega_1^2+\Omega_u\Omega_l}$$

$$B = \frac{\Omega_2(\Omega_u-\Omega_l)}{-\Omega_2^2+\Omega_u\Omega_l}$$

$$s \rightarrow \frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}$$

CHEBYSHEV TYPE-1 FILTER

$$|H_n(j\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2\left(\frac{\Omega}{\Omega_p}\right)}$$

ε : ripple factor

$T_n(x) = \cos nT\big|_{x=\cos t}$ (Chebyshev Polynomial)

n	$T_n(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$
6	$32x^6-48x^4+18x^2-1$
7	$64x^7-112x^5+56x^3-7x$

$s_k = \sigma_k + j\Omega_k$

$$\sigma_k = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

$$\Omega_k = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

$\rightarrow$ vary from 0 to n-1

$$H_n(s) = \frac{K}{\prod\limits_{\text{LHS}}(s-s_k)}$$

$K \rightarrow$ Normalising factor

$\prod(s-s_k) \rightarrow$ Poles on LHS

$$K = \begin{cases} b_0 & , \ n = \text{odd} \\ \dfrac{b_0}{\sqrt{1+\varepsilon^2}} & , \ n = \text{even} \end{cases}$$

$$n = \left[\frac{\log_{10}\left(g+(g^2-1)^{1/2}\right)}{\log_{10}\left(\Omega_r+(\Omega_r^2-1)^{1/2}\right)}\right]$$

$A = 10^{-K_s/20}$ $\qquad \varepsilon = \sqrt{10^{-K_p/10}-1}$ $\qquad g = \sqrt{\frac{A^2-1}{\varepsilon}}$

IIR FILTER DESIGN

i) BACKWARD DIFFERENCE

$y(t) \rightarrow \boxed{H(s) = s} \rightarrow \frac{dy(t)}{dt}$

$y(n) \rightarrow \boxed{H(z) = \frac{1-z^{-1}}{T}} \rightarrow \frac{y(n)-y(n-1)}{T}$

$\rightarrow$ $H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$

ex : $H(s) = \frac{1}{s^2+16}$ $\qquad (T=1)$

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{(1-z^{-1})^2+16} = \frac{1}{z^{-2}-2z^{-1}+17}$$

ii) IMPULSE INVARIANCE

Steps

i) Given $H(s)$

ii) Split into partial fractions

iii) Find numerators of partial fractions

iv) Once in the form $\frac{1}{s - s_K} \xrightarrow{\text{IIT}} \frac{1}{1 - e^{s_K T} z^{-1}}$

v) After this, multiply them back & simplify

iii) BILINEAR TRANSFORM

Steps

i) Convert given digital filter to analog filter

If ω is given, find $\Omega = \frac{2}{T} \tan\left(\frac{\omega_c}{2}\right)$

ii) Now design analog filter (BW or cheby)

iii) Convert analog to digital, $H(z) = H(s)\Big|_{s \to \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$

iv) MATCHED TRANSFORM

→ Directly apply Z.T

ex: $H(s) = \frac{4s(s+1)}{(s+2)(s+3)}$ $\quad T = \frac{1}{4}$

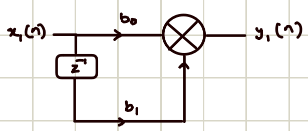
$$= \frac{4(1 - z^{-1})(1 - e^{-T} z^{-1})}{(1 - e^{-2T} z^{-1})(1 - e^{-3T} z^{-1})}$$

$$= \frac{4(1 - z^{-1})(1 - e^{-1/4} z^{-1})}{(1 - e^{-1/2} z^{-1})(1 - e^{-3/4} z^{-1})}$$

IIR FILTER REALISATION

i) DIRECT FORM - I

ex: $H(z) = \frac{\sum_{K=0}^{M} b_K z^{-K}}{1 + \sum_{K=1}^{N} a_K z^{-K}} = \underbrace{(b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots)}_{\frac{Y_1(z)}{X_1(z)}} \cdot \underbrace{\left(\frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots}\right)}_{\frac{Y_2(z)}{X_2(z)}}$

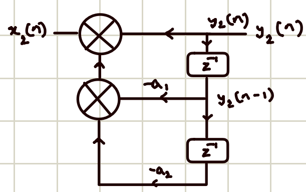


Now apply Inverse Z-Transform, $y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + \dots$

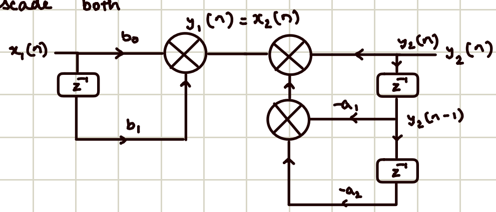
↳ all zero

$y_2(n) = x_2(n) - a_1 y_2(n-1) - a_2 y_2(n-2) - \dots$

↳ all poles



Then cascade both



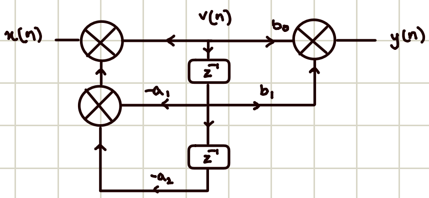
Here we use N+M delay elements

ii) DIRECT FORM - II

Similar to DF-I, instead take $\frac{V(z)}{X(z)}$ & $\frac{Y(z)}{V(z)}$

All poles ↑ All zeros ↑

Same ex as prev,

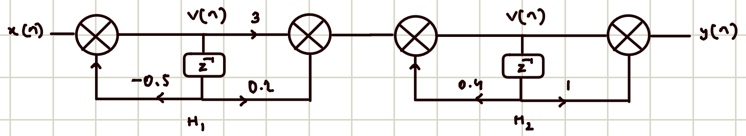


III) CASCADED REALISATION

→ If $H(z) = H_1(z) \cdot H_2(z) \ldots$

Then use DF-II and cascade each individual function

ex: $H(z) = \dfrac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} = \dfrac{3(1+0.2z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})} = \dfrac{3(1+0.2z^{-1})}{1+0.5z^{-1}} \times \dfrac{1+z^{-1}}{1-0.4z^{-1}}$



IV) PARALLEL REALISATION

Steps

i) Given some $H(z)$ in terms of z^{-1}

ii) Split into partial fractions and ensure only constants in numerator

iii) Now Take $H_1(z)$ & $H_2(z)$ and seperate denominator as $\dfrac{V(z)}{X(z)}$ and numerator as $\dfrac{Y(z)}{V(z)}$

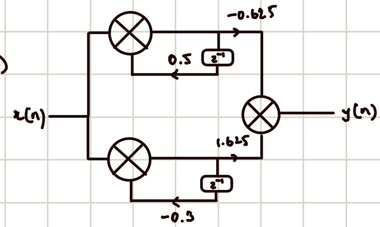
ex: $H(z) = \dfrac{1-z^{-1}}{1-0.2z^{-1}-0.15z^{-1}} = \dfrac{A}{1-0.5z^{-1}} + \dfrac{B}{1+0.3z^{-1}} \Rightarrow$ Solving, $\underbrace{\dfrac{-0.625}{1-0.5z^{-1}}}_{H_1} + \underbrace{\dfrac{1.625}{1+0.3z^{-1}}}_{H_2}$

① $\dfrac{V(z)}{X(z)} = \dfrac{1}{1-0.5z^{-1}} \Rightarrow v(n) = x(n) - 0.5x(n-1)$

$\dfrac{Y(z)}{V(z)} = -0.625 \Rightarrow y(n) = -0.625\, v(n)$

② $\dfrac{V(z)}{X(z)} = \dfrac{1}{1+0.3z^{-1}} \Rightarrow v(n) = x(n) + 0.3x(n-1)$

$\dfrac{Y(z)}{V(z)} = 1.625 \Rightarrow y(n) = 1.625\, v(n)$



V) LATTICE REALISATION

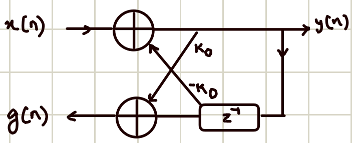
Systems only with poles. Both poles & zeros aren't tested

Steps

i) Given $H(z) = \dfrac{1}{A_m(z)}$, where $A_m(z) = a_m(0) + a_m(1)z^{-1} + \ldots$

ii) $K_m = \dfrac{a_{m+1}(m+1)}{a_{m+1}(0)} \Rightarrow A_m = \dfrac{A_{m+1} - K_m \bar{A}^{-1}_{m+1}}{1-K_m^2}$

Repeat till m=0



UNIT - 4

LOW PASS DIGITAL USING RELATIVE SPECIFICATIONS

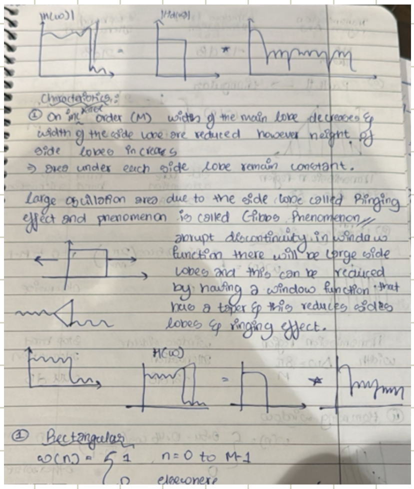
→ $$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} + 1}$$

$$\delta_s = 10^{-A_s/20}(1 + \delta_p)$$

KAISER FORMULA

→ $$M = \frac{-20\log_{10}\left(\sqrt{\delta_p \delta_s}\right) - 13}{14.6\,\frac{\Delta\omega}{2\pi}} + 1$$

Take next odd number, for symmetry we don't use even number



|M(ω)| |Hd(ω)|

Characteristics:

① On incr. order (M) width of the main lobe decreases & width of the side lobe are reduced however height of side lobes increases

⇒ area under each side lobe remain constant.

Large oscillation area due to the side lobe called Ringing effect and phenomenon is called Gibbs Phenomenon

abrupt discontinuity in window function there will be large side lobes and this can be reduced by having a window function that has a taper & this reduces sides lobes & ringing effect.

H(ω)

① Rectangular

$\omega(n) = \begin{cases} 1 & n = 0 \text{ to } M-1 \\ 0 & \text{elsewhere} \end{cases}$

FIR FILTER DESIGN USING WINDOW METHOD

→ $$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ \dfrac{\omega_c}{\pi}, & n = \alpha \end{cases}$$

→ DESIRED UNIT SAMPLE RESPONSES

i) LPF ⇒ $$h_d(n) = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ \dfrac{\omega_c}{\pi}, & n = \alpha \end{cases}$$

ii) HPF ⇒ $$h_d(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ 1 - \dfrac{\omega_c}{\pi}, & n = \alpha \end{cases}$$

iii) BPF ⇒ $$h_d(n) = \begin{cases} \dfrac{\sin\omega_{c_2}(n-\alpha) - \sin\omega_{c_1}(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ \dfrac{\omega_{c_2} - \omega_{c_1}}{\pi}, & n = \alpha \end{cases}$$

iv) BSF ⇒ $$h_d(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) + \sin\omega_{c_1}(n-\alpha) - \sin\omega_{c_2}(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ \dfrac{\pi + \omega_{c_1} - \omega_{c_2}}{\pi}, & n = \alpha \end{cases}$$

→ WINDOW FUNCTIONS

		$\Delta\omega$	Window f^n Atten^n	Filter S.B Att^n
i) Rectangular ⇒	$w(n) = \begin{cases} 1, & n = 0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\Delta\omega = \frac{4\pi}{M}$	$-13\,dB$	$-21\,dB$
ii) Bartlett / Triangular ⇒	$w(n) = \begin{cases} 1 - \dfrac{2\left\lvert n - \frac{M-1}{2}\right\rvert}{M-1}, \\ 0, \end{cases}$	$\frac{8\pi}{M}$	$-25\,dB$	$-25\,dB$
iii) Hanning ⇒	$w(n) = \begin{cases} 0.5 - 0.5\cos\left[\frac{2\pi n}{M-1}\right] \\ 0 \end{cases}$	$\frac{8\pi}{M}$	$-31\,dB$	$-44\,dB$
iv) Hamming ⇒	$w(n) = \begin{cases} 0.54 - 0.46\cos\left[\frac{2\pi n}{M-1}\right] \\ 0 \end{cases}$	$\frac{8\pi}{M}$	$-41\,dB$	$-53\,dB$
v) Blackman ⇒	$w(n) = \begin{cases} 0.42 - 0.5\cos\left[\frac{2\pi n}{M-1}\right] + 0.08\cos\left[\frac{2\pi n}{M-1}\right] \\ 0 \end{cases}$	$\frac{12\pi}{M}$	$-57\,dB$	$-74\,dB$

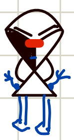
vi) Kaiser ⇒ $$w(n) = \frac{I_0\left[\beta\sqrt{1 - \left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0\,\beta}$$

↳ $$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0, & A < 21 \end{cases}$$

& $$M = \frac{A - 8}{2.285\,\Delta\omega} \qquad (\Delta\omega = \omega_s - \omega_p)$$

$$A = -20\log_{10}\delta \qquad (\delta = \min(\delta_p, \delta_s))$$

(If $21 \le A \le 50$, usually $\beta = 3.4$)



FIR FILTER DESIGN USING FREQUENCY SAMPLING METHOD

Steps

i) Given $H_d(e^{j\omega})$

ii) Substitute $H(K) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi K}{M}}$

iii) $h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} \mathrm{Re}\left\{H(K)\, e^{\frac{j2\pi Kn}{M}}\right\}\right]$

PHASE DELAY

→ Delay experienced by single frequency component

$\tau_p(\omega) = \frac{-\theta(\omega)}{\omega}$

GROUP DELAY

→ Delay experienced by multiple frequency components

$\tau_g(\omega) = \frac{-d\theta(\omega)}{d\omega}$

LOCATION OF ZEROS & POLES

Steps

i) Given $H(\omega)$

ii) $H(\omega) = H(z)\Big|_{z=e^{j\omega}}$ ex:

iii) Take out $e^{-j\left(\frac{\text{lowest order}}{2}\right)\omega}$ common

So, $e^{j\theta(\omega)} \cdot H(e^{j\omega})$

iv) Now with $\theta(\omega)$ find τ_p & τ_g

FIR DIFFERENTIATOR

Steps

i) Given some $H_d(\omega)$

Find $h_d(n)$ $(\mathrm{IDFT}\{H_d(\omega)\})$

ii) With $w(n)$, find $h(n) = h_d(n) \times w(n)$

iii) Finally

$|H(\omega)| = \left|2\sum_{n=0}^{\frac{M-1}{2}} h(n) \sin\omega\left(\frac{M-1}{2} - n\right)\right|$

↳ Differentiation found using frequency sampling method

ideal freq. response

↑

If $H_d(e^{j\omega}) = j\omega \Rightarrow h_d(n) = \begin{cases} \frac{\cos n\pi}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$ & $h_d(n-\alpha) = \begin{cases} \frac{\cos \pi(n-\alpha)}{n-\alpha}, & n \neq \alpha \\ 0, & n = \alpha \end{cases}$

HILBERT TRANSFORM

Steps

i) Given some $H_d(\omega)$

Find $h_d(n)$ $(\mathrm{IDFT}\{H_d(\omega)\})$

ii) With $w(n)$, find $h(n) = h_d(n) \times w(n)$

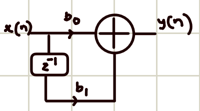
iii) Finally

$|H(\omega)| = \left|2\sum_{n=0}^{\frac{M-1}{2}} h(n) \sin\omega\left(\frac{M-1}{2} - n\right)\right|$ ← Notice

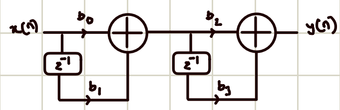
If $H_d(\omega) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases} \Rightarrow$ & $h_d(n-\alpha) = \begin{cases} \frac{2\sin^2\left(\frac{n-\alpha}{2}\pi\right)}{(n-\alpha)\pi}, & n \neq \alpha \\ 0, & n = \alpha \end{cases}$

FIR REALISATION

a) DIRECT $\Rightarrow H(z) = b_0 + b_1 z^{-1} \Rightarrow$



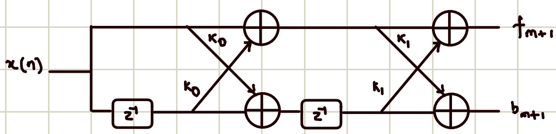
b) CASCADED $\Rightarrow (b_0 + b_1 z^{-1})(b_2 + b_3 z^{-1})$



c) LATTICE $\Rightarrow$ i) Given $H(z) = A_m(z)$. ii) $K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)} \longrightarrow A_m = \frac{A_{m+1} - K_m A^{-1}_{m+1}}{1 - K_m^2}$

repeat till u get K_0

ex



if you're finding equations from reflection coefficients

$A_0(z) = 1$ always

$A_{m+1}(z) = A_m(z) + K_m z^{-1} A_m^{-1}(z)$