Transmission Lines, Waveguides & Antennas

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U-1 Transmission Lines & Impedance Matching

Transmission Line (Tx Line)

→ It is the bridge b/w field analysis & basic circuit theory

→ They use specialized construction & impedance matching, to carry electromagnetic signals with minimum reflections & power losses

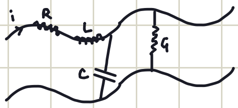
The Lumped-Element Circuit model for a Tx Line

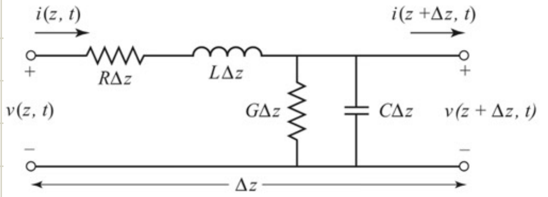
→ R = series resistance (Ω/m)

L = series inductance (H/m)

G = shunt conductance (S/m)

C = shunt capacitance (F/m)





→ Transmission line is a distributed-parameter network, where voltages & current can vary in magnitude & phase over its length while circuit analysis deals with lumped elements where voltage & current don't vary much over physical dimensions of the elements

→ Tx line maybe a considerable fraction of a wavelength, or many wavelength, in size

→ Tx lines uniform cross-sectional dimensions along their wavelength, giving them a uniform impedance, called characteristic impedance Z_0

→ Now apply KVL,

$$v(z,t) - R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z, t) = 0$$

$$v(z+\Delta z, t) - v(z,t) = -\left(R i(z,t) + L\frac{\partial i(z,t)}{\partial t}\right)\Delta z$$

Now take $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z, t) - v(z,t)}{\Delta z} = -\left(R i(z,t) + L\frac{\partial i(z,t)}{\partial t}\right)$$

$$\frac{\partial v(z,t)}{\partial z} = -\left(R i(z,t) + L\frac{\partial i(z,t)}{\partial t}\right) \longrightarrow 1$$

Apply steady state condition,

$$\frac{dV(z)}{dz} = -(R+j\omega L)\, I(z) \longrightarrow 3$$

→ Now apply KCL

$$i(z,t) - G\Delta z\, v(z,t) - C\Delta z \frac{\partial v(z,t)}{\partial t} - i(z+\Delta z, t) = 0$$

$$i(z+\Delta z, t) - i(z,t) = -G\Delta z\, v(z,t) - C\Delta z \frac{\partial v(z,t)}{\partial t}$$

Now take $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z, t) - i(z,t)}{\Delta z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -\left(G v(z,t) + C\frac{\partial v(z,t)}{\partial t}\right) \longrightarrow 2$$

Apply steady state condition,

$$\frac{dI(z,t)}{dz} = -(G+j\omega C)\, V(z) \longrightarrow 4$$

eq 1 & 2 are telegrapher equation

3 & 4 are similar to Maxwell eq's

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

differentiate eq 3,

$$\frac{d^2V(z)}{dz^2} = -(R+j\omega L)\frac{dI(z)}{dz}$$

$$\frac{d^2V(z)}{dz^2} = -(R+j\omega L)\cdot\left(-(G+j\omega C)V(z)\right)$$

$$\frac{d^2V(z)}{dz^2} = (R+j\omega L)(G+j\omega C)\, V(z)$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \text{where} \quad \gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$$

Similarly, $\dfrac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$

Wave propagation on a Transmission Line

→ $\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$ & $\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0$

$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \longrightarrow (5)$

$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \longrightarrow (6)$

These are the travelling wave solutions for the voltage & current eq of Tx lines

$e^{-\gamma z}$: Wave propagation in +ve z direction

$e^{\gamma z}$: Wave propagation in −ve z direction

From eq (3),

$$I(z) = \frac{-1}{(R+j\omega L)} \cdot \frac{dV(z)}{dz}$$

substitute (5)

$$I(z) = \frac{\gamma}{R+j\omega L} \cdot \left(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right)$$

Now compare with (6),

$I_0^+ = \frac{\gamma V_0^+}{R+j\omega L}$ & $I_0^- = \frac{-\gamma V_0^-}{R+j\omega L}$

$$Z_0 = \frac{R+j\omega L}{\gamma} = \frac{R+j\omega L}{\sqrt{R+j\omega L}\sqrt{G+j\omega C}} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$\frac{V_0^+}{I_0^+} = \frac{R+j\omega L}{\gamma} = \frac{-V_0^-}{I_0^-} = Z_0$ (Characteristic Impedance)

So, $I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$

Now convert back to time domain,

$v(z,t) = |V_0^+| \cos(\omega t - \beta z + \phi^+) e^{-\alpha z} + |V_0^-| \cos(\omega t + \beta z + \phi^-) e^{\alpha z}$ → (7)

where $\phi^{\pm}$: phase angle of complex voltage $V_0^{\pm}$

Wavelength of line, $\lambda = \frac{2\pi}{\beta}$

Phase velocity, $v_p = \frac{\omega}{\beta} = \lambda f$

Lossless line

→ (7) is the solution for general transmission line including loss effects ($R = G = 0$ for lossless)

Usually loss is small & negligible

→ Set $R = G = 0$ in γ

$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$ ⇒ $\beta = \omega\sqrt{LC}$

$\alpha = 0$

So attenuation constant $\alpha = 0$ for lossless line

and $Z_0 = \sqrt{\frac{L}{C}}$

Then general solutions for voltage & current on lossless transmission line are

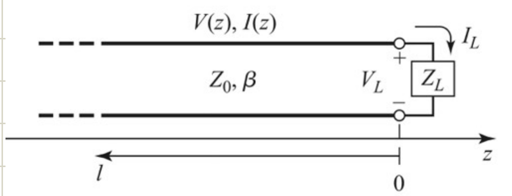
$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$

$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$

Then wavelength, $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$

phase velocity, $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$

Terminated lossless transmission line

→ 

⇒ Lossless transmission line that is connected to impedance Z_L at the end

→ Assume incident wave of the form $V_0^+ e^{-j\beta z}$ is generated from a source at $z < 0$

But when the line is terminated in an arbitrary load $Z_L \neq Z_0$ causing reflections, (→ Source is already & generated ready for transmission.

then ratio of voltage to current at load must be Z_L

Therefore,

total voltage = sum of incident & reflected voltages

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

The total voltage & current at the load are related by the load imp, so at $z=0$

$$Z_L = \frac{V_L}{I_L} = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$

Then,
$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The amplitude of the reflected voltage wave normalized to amplitude of the incident voltage wave normalized at $l=0$ ↙ load

is defined as the voltage reflection coefficient, $\boxed{\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$

In general, $z = -l$

$$\Gamma(0) = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma(l) = \frac{V_0^- e^{j\beta z}}{V_0^+ e^{-j\beta z}} = \frac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}} = \Gamma(0)\frac{e^{-j\beta l}}{e^{j\beta l}} = \Gamma(0)\, e^{-2j\beta l}$$

Standing Waves

→ The voltage & current on the line consist of a superposition of incident & reflected wave and such waves are called standing waves (Both incident wave & reflected wave exist simultaneously on a Tx line)

→ $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$

$\Gamma = \frac{V_0^-}{V_0^+} \Rightarrow V_0^- = \Gamma V_0^+$ ($Z_L \neq Z_0$, so $\Gamma \neq 0$ so both incident & reflected wave exist at same time

$$V(z) = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z}$$
$$= V_0^+ \left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]$$
$$V(z) = V_0^+ e^{-j\beta z}\left[1 + \Gamma e^{2j\beta z}\right]$$
$$I(z) = \frac{V_0^+ e^{-j\beta z}}{Z_0}\left[1 - \Gamma e^{2j\beta z}\right]$$

To obtain $\Gamma = 0$, the condition is $Z_L = Z_0$. Such a load is said to be matched to line since no reflection.

Consider $z = -l$ and $\Gamma = |\Gamma| e^{j\theta}$,

Then $V(z) = V_0^+ e^{j\beta l}\left[1 + |\Gamma| e^{j(\theta - 2\beta l)}\right]$

V_{max} at $e^{j(\theta - 2\beta l)} = 1 \Rightarrow V_{max} = |V_0^+|\, e^{j\beta l}\,(1 + |\Gamma|)$

V_{min} at $e^{j(\theta - 2\beta l)} = -1 \Rightarrow V_{min} = |V_0^+|\, e^{j\beta l}\,(1 - |\Gamma|)$

Standing Wave Ratio (SWR)

→ As $|\Gamma|$ increases, the ratio $\frac{V_{max}}{V_{min}}$ increases, so measure of mismatch of the line is called SWR

$$\boxed{SWR = \frac{V_{max}}{V_{min}}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$1 \leq SWR \leq \infty$

→ The distance between 2 maxima or 2 minima is $l = \frac{2\pi}{2\beta} = \frac{2\pi}{2\times\frac{2\pi}{\lambda}} = \frac{\lambda}{2}$

The distance between a maxima & a minima is $l = \frac{\pi}{2\beta} = \frac{\pi}{2\times\frac{2\pi}{\lambda}} = \frac{\lambda}{4}$

Input Impedance

→ For a matched load, the power flow on the line is constant

For a mismatched load, the power flow on the line is oscillating

→ The real power flow on the line is constant (for lossless line) but voltage amplitude (for mismatched line) is oscillatory with position on the line

→ Basically a Tx line transforms impedance because of wave reflection & phase shift. It's like an impedance transformer

→ $z = -l$, $z_{in} = \frac{V(-l)}{I(-l)} = \frac{V_0^+ e^{j\beta l}(1+\Gamma e^{-j2\beta l})}{\frac{V_0^+ e^{j\beta l}(1-\Gamma e^{-j2\beta l})}{Z_0}} = \boxed{Z_0\left[\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right]}$

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

→ $$Z_{in} = \left(\frac{1+\left(\frac{Z_L-Z_0}{Z_L+Z_0}\right).e^{-j2\beta l}}{1-\left(\frac{Z_L-Z_0}{Z_L+Z_0}\right).e^{-j2\beta l}}\right).Z_0 = \frac{\frac{Z_L+Z_0+(Z_L-Z_0)e^{-j2\beta l}}{\cancel{Z_L+Z_0}}}{\frac{Z_L+Z_0-(Z_L-Z_0)e^{-j2\beta l}}{\cancel{Z_L+Z_0}}}.Z_0$$

$$= \frac{(Z_L+Z_0)e^{j\beta l}+(Z_L-Z_0)e^{-j\beta l}}{(Z_L+Z_0)e^{j\beta l}-(Z_L-Z_0)e^{-j\beta l}}.Z_0 = \frac{Z_Le^{j\beta l}+Z_0e^{j\beta l}+Z_Le^{-j\beta l}-Z_0e^{-j\beta l}}{Z_Le^{j\beta l}+Z_0e^{j\beta l}-Z_Le^{-j\beta l}+Z_0e^{-j\beta l}}.Z_0$$

$$= \frac{Z_L(e^{j\beta l}+e^{-j\beta l})+Z_0(e^{j\beta l}-e^{-j\beta l})}{Z_L(e^{j\beta l}-e^{-j\beta l})+Z_0(e^{j\beta l}+e^{-j\beta l})}.Z_0$$

$$= \frac{2Z_L\cos\beta l+2jZ_0\sin\beta l}{2jZ_L\sin\beta l+2Z_0\cos\beta l}.Z_0$$

$$= \frac{Z_L\cos\beta l+jZ_0\sin\beta l}{jZ_L\sin\beta l+Z_0\cos\beta l}.Z_0 \qquad \left(\begin{array}{l} e^{j\theta}+e^{-j\theta} = 2\cos\theta \\ e^{j\theta}-e^{-j\theta} = 2j\sin\theta \end{array}\right)$$

$$\boxed{Z_{in} = \left(\frac{Z_L+jZ_0\tan\beta l}{Z_0+jZ_L\tan\beta l}\right)Z_0} \longrightarrow \text{Transmission Line Impedance equation}$$

Q. A Generator of 1V, 1KHz supplies power to a 100Km long line terminating in Z_0 and having the characteristic constants given below. Calculate characteristic impedance, propagation constant, wavelength, velocity, voltage and efficiency. $R = 10.4\ \Omega/km$; $L = 0.00367\ H/Km$; $G = 0.8\times10^{-6}\ ℧/Km$; $C = 0.00835\times10^{-6}\ F/m$

A. Given, $Z_L = Z_0$, $V_s = 1V$, $f = 1KHz = 1000\ Hz$

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{482443.7481\angle-23.412} = 694.58\angle-11.706^\circ \qquad \left(\begin{array}{l} \sqrt{r\angle\theta} = r^{0.5}\angle\theta/2 \\ (r\angle\theta)^{1/n} = r^{1/n}\angle\theta/n \end{array}\right)$$

$$Z_0 = \frac{R+j\omega L}{\gamma} \Rightarrow \gamma = \frac{10.4+j(2\pi\times f\times0.00367)}{694.58\angle-11.706} = 0.036\angle77.42^\circ = 7.93\times10^{-3}+j0.0355 = \alpha+j\beta$$

$\Rightarrow \alpha = 0.00793$ nepers/Km

$\beta = 0.0355$ rad/Km

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.0355} = \frac{2\times3.14}{0.0355} = 176.90\ Km$$

$$v = \frac{\omega}{\beta} = \frac{2\pi\times1000}{0.0355} = 176901.4\ Km/s$$

$$Z_L = Z_0 \Rightarrow I_s = \frac{V_s}{Z_0} = \frac{1\angle0}{694.58\angle-11.706} = 1.41\times10^{-3}+j2.92\times10^{-4} = 1.44\times10^{-3}\angle11.706\ A$$

$P_s = I_s V_s = 1.44 \angle 11.706 \text{ mA} \times 1V = 1.44 \angle 11.706 \text{ mW}$

$I_R = I_s e^{-\gamma l} = 1.44 \angle 11.706 \times e^{-(0.00793 + j0.0355) \times 100}$ mA

$= 1.44 \angle 11.706 \times e^{-0.793} \times e^{-j3.55}$ $\quad$ 1 rad = 57.3°

$= 1.44 \times 0.452 \angle 11.706 \times e^{-j(3.55 \times 57.3)}$

$= 1.44 \times 0.452 \angle 11.706 \times \angle -203.415$

$= 0.65 \angle -191.705$ mA

$V_R = I_R \times Z_L = I_R Z_0 = 0.65 \times 10^{-3} \angle -191.705 \times 694.58 \angle -11.706$

$= 0.451477 \angle -203.411$ V

$P_R = V_R I_R \cos\theta$

$= 0.451 \angle -203.411 \times 0.65 \times 10^{-3} \angle -191.705 \times \cos(-203.411 + 191.705)$

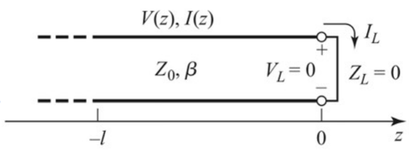
$= 0.287 \times 10^{-3}$

$\eta = \dfrac{P_R}{P_s} = \dfrac{0.287 \times 10^{-3}}{1.44 \times 10^{-3}} = 0.1993$

$\eta\% = 19.93\%$

Short Circuited Line

→ $Z_{in} = Z_0 \left(\dfrac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \right)$ (Tx line whose load impedance, $Z_L = 0$) → 2 Conductors are directly connected together at load end



$= \dfrac{\cancel{Z_0} \times jZ_0 \tan\beta l}{\cancel{Z_0}}$ $\quad (Z_L = 0)$

$Z_{in} = jZ_0 \tan\beta l$

$Z_{sc} = jZ_0 \tan\beta l$ (sc : short circuit)

$\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = -1$

$VSWR = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} = \infty$ (VSWR : Voltage Standing Wave Ratio , VSWR = SWR)

→ A short circuited transmission line is a line whose load head is directly connected to ground so that load is zero,
$Z_L = 0 \Rightarrow$ Voltage $= 0 \Rightarrow$ Max Current

→ The reflection coefficient $\Gamma = -1$ which means the total reflection with 180° phase reversal & standing waves along the line

→ $V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$

$I(z) = \dfrac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z})$

Now consider the short circuited eq, $Z_L = 0$ & $\Gamma = -1$, then,

$V(z) = V_0^+ (e^{-j\beta z} - e^{j\beta z}) = -2j V_0^+ \sin\beta z$

$I(z) = \dfrac{V_0^+}{Z_0} (e^{-j\beta z} + e^{j\beta z}) = \dfrac{2 V_0^+}{Z_0} \cos\beta z$

This shows $V = 0$ at load while current is maximum as mentioned before

→ Now,

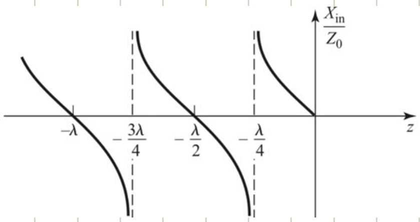
at $l = \lambda/4$, $\beta l = \dfrac{2\pi}{\lambda} \times \dfrac{\lambda}{4} = \pi/2$

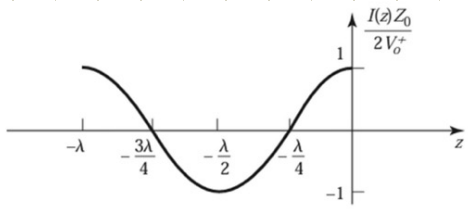
at $l = \lambda/2$, $\beta l = \dfrac{2\pi}{\lambda} \times \dfrac{\lambda}{2} = \pi$

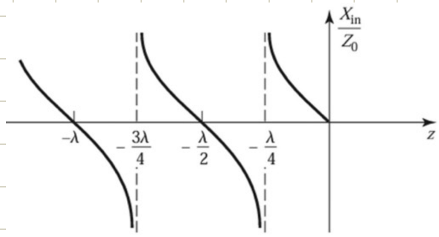
at $l = 3\lambda/2$, $\beta l = \dfrac{2\pi}{\lambda} \times \dfrac{3\lambda}{2} = 3\pi$

So, input impedance ($Z_{in} = jZ_0 \tan\beta l$) which is purely imaginary for any length l & taken on all values b/w $j\infty$ and $-j\infty$

Also impedance is periodic in repeating for multiples of $\lambda/2$





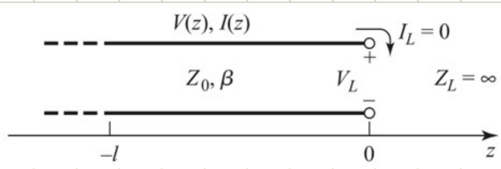


Open Circuited Line

→ Now consider an open circuited line where $Z_L = \infty \Rightarrow \Gamma = 1$

$$V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}) = V_0^+ (e^{-j\beta z} + e^{j\beta z}) = 2V_0^+ \cos\beta z$$

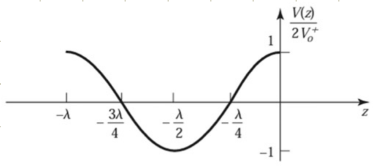
$$I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - e^{j\beta z}) = \frac{-2j V_0^+}{Z_0} \sin\beta z$$



↳ This means the 2 conductors aren't connected at load end

This shows $I = 0$ at load while voltage is maximum

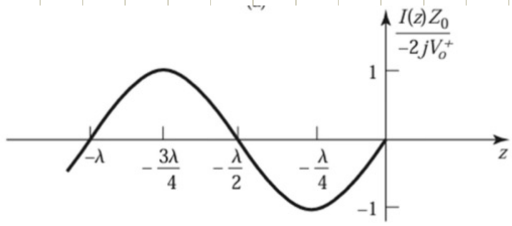
$$Z_{in} = \frac{V(z)}{I(z)} = -jZ_0 \cot\beta l$$ → purely imaginary for any length l.



→ Now consider half wave length $(l = \lambda/2)$

If $l = \frac{\lambda}{2}$, $\quad Z_{in} = \frac{Z_0 (Z_L + jZ_0 \tan\beta l)}{Z_0 + jZ_L \tan\beta l}$

$$\beta l = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi$$



then, $$Z_{in} = \frac{Z_0 (Z_L + jZ_0 \tan\pi)}{Z_0 + jZ_L \tan\pi} = \frac{\cancel{Z_0} \times Z_L}{\cancel{Z_0}} = Z_L$$

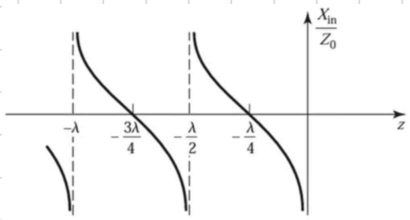
$$Z_{in} = Z_L$$

Quarter Wavelength line $(l = \lambda/4)$

$$\beta l = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$$

$$Z_{in} = Z_0 \frac{\left(\frac{Z_L}{\tan\beta l_1} + jZ_0\right)}{\left(\frac{Z_0}{\tan\beta l_1} + jZ_L\right)}$$

$$= \frac{Z_0 (jZ_0)}{jZ_L} = \frac{Z_0^2}{Z_L}$$



	Property	Short Line	Open Line
1)	Z_L	0	∞
2)	Γ	−1	+1
3)	Voltage at load	0	Max
4)	Current at load	Max	0
5)	Input impedance	$jZ_0 \tan\beta l$	$-jZ_0 \cot\beta l$

Average Power Flow in Tx Line

→ As per poynting theorem, the average power flow Tx line is

$$P_{avg} = \frac{1}{2} \text{Re} \, |E \times H^*|$$

$$= \frac{1}{2} \text{Re} \left(V(z) \times I(z)^* \right)$$

$$V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z})$$

$$I(z)^* = \frac{V_0^+}{Z_0} (e^{j\beta z} - \Gamma^* e^{-j\beta z})$$

$$P_{avg} = \frac{1}{2} \frac{|V_0^+|^2}{Z_0} \text{Re} \left\{ 1 - \Gamma^* e^{-j2\beta z} + \Gamma e^{j2\beta z} - \Gamma \Gamma^* \right\}$$

Consider $\Gamma e^{j2\beta z} = A$

$\Gamma^* e^{-j2\beta z} = A^*$

$A - A^* = 2j \, \text{Im}(A)$

$$P_{avg} = \frac{1}{2} \frac{|V_0^+|^2}{Z_0} \left[1 - |\Gamma|^2 \right]$$

If $\Gamma = 0$, $\quad P_{avg} = \frac{1}{2} \frac{|V_0^+|^2}{Z_0}$

If $\Gamma = 1$, $\quad P_{avg} = 0$

Q. A Lossless line has a characteristic impedance of 400 Ω. Determine SWR if receiving impedance is 800 Ω

A. $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{800 - 400}{800 + 400} = \frac{400}{1200} = \frac{1}{3}$

$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/3}{1 - 1/3} = \frac{4/3}{2/3} = 2$

Q. A 50 Ω Tx line feeds 75 + j20 Ω dipole antenna. Find Γ & VSWR, R_{max} & R_{min}

A. $Z_L = 75 + j20\ \Omega$

$Z_0 = 50\ \Omega$

$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75 + j20 - 50}{75 + j20 + 50} = \frac{25 + j20}{125 + j20} = 0.2529 \angle 29.56$

$VSWR = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.2529}{1 - 0.2529} = 1.677$

$R_{max} = VSWR \cdot Z_0 = 1.677 \times 50 = 83.85$

$R_{min} = \frac{Z_0}{VSWR} = \frac{50}{1.677} = 29.815$

Transmission Coefficient (T)

→ $T = 1 + \Gamma$ (T tells how much of incident wave goes into load)

$= 1 + \frac{Z_L - Z_0}{Z_L + Z_0}$

$= \frac{2Z_L}{Z_L + Z_0}$

Insertion Loss (IL)

→ The transmission coefficient between 2 points in a circuit

$IL = -20 \log |T|$ dB where T is transmission coefficient

Return Loss (RL)

→ When the load is mismatched, not all of the available power from the generator is delivered to the load. This loss is known as return loss

$RL = -20 \log |\Gamma|$ dB

Smith Chart

→ Smith chart is a graphical tool for solving transmission line & impedance matching problems in microwave engg.

→ It is based on a polar plot of the voltage reflection coefficient Γ.

→ Let Γ be expressed in magnitude & phase form as $\Gamma = |\Gamma| e^{j\theta}$

Then magnitude $|\Gamma|$ is plotted as radius from centre of chart

& angle θ is measured counter clockwise from RHS of chart

→ The normalization constant is usually the characteristic impedance of transmission line

so, $z = \frac{Z_L}{Z_0}$ represents normalized version of impedance Z.

→ If a lossless line of characteristic impedance Z_0 is terminated with a load impedance Z_L,

then $\Gamma = \frac{z_L - 1}{z_L + 1} = |\Gamma| e^{j\theta}$

where $z_L = \frac{Z_L}{Z_0}$ is the normalized load impedance

So, $z_L = \frac{1 + |\Gamma| e^{j\theta}}{1 - |\Gamma| e^{j\theta}}$

→ Now, the complex equation can be reduced to 2 real equations by writing Γ & z_L in terms of real & imaginary parts, $\Gamma = \Gamma_r + j\Gamma_i$ & $z_L = r_L + jx_L$

So, $$r_L + jx_L = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i} \times \frac{(1-\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) + j\Gamma_i}$$

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \quad \& \quad x_L = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

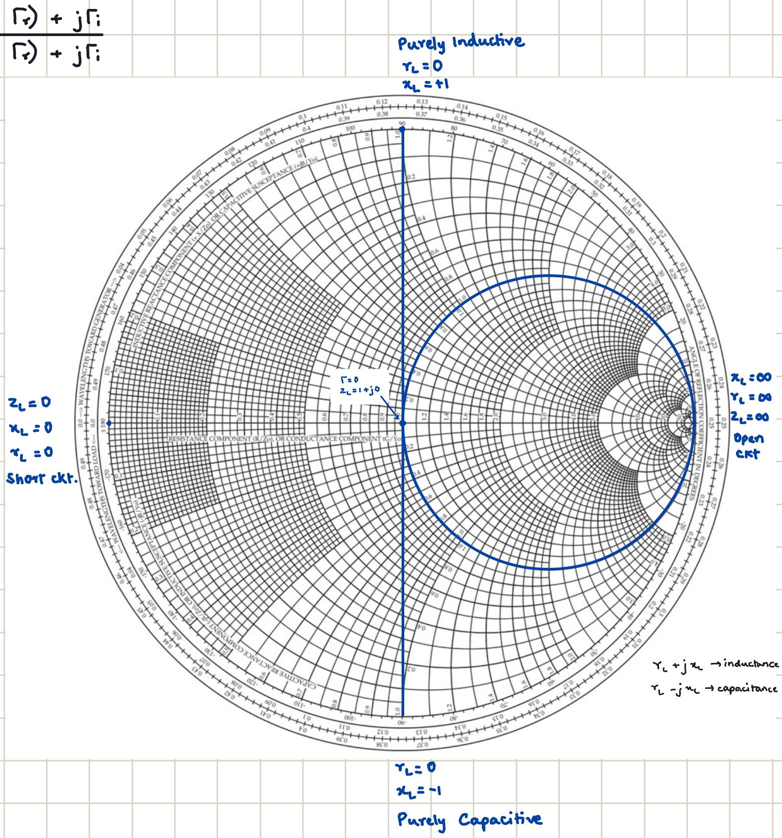
Rearranging, (Circle equation) $$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2$$

So centre $= \left(\frac{r_L}{1+r_L}, 0\right)$ radius $= \frac{1}{1+r_L}$

and, $$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2$$

So centre $= \left(1, \frac{1}{x_L}\right)$ radius $= \frac{1}{x_L}$

→ Total length of smith chart $= \frac{\lambda}{2} = 0.5\lambda$



Q. Determine the input impedance & SWR for a 1.25λ Long transmission line at the sending end with $z_0 = 50\,\Omega$ & the load impedance $z_L = 30 + j40\,\Omega$

A. 1) $z_0 = 50\,\Omega$

$z_L = 30 + j40\,\Omega$

2) $L = 1.25\lambda = 0.5\lambda + 0.5\lambda + 0.25\lambda$

3) $z_L = \frac{Z_L}{Z_0} = \frac{30 + j40}{50} = 0.6 + j0.8$ (normalized)

$R + jx = 0.6 + j0.8$

4) Draw an arc from 0.6 at centre line then draw an arc from 0.8 at upper inner radius. The point of meeting is z_L. Color of Arc

5) Keep centre as midpoint & radius as z_L & draw a circle. Color of Circle

6) Then draw a line from centre to z_L & extend to outer radius to find load admittance of 0.125λ Color of line

7) From L, $0.5\lambda \Rightarrow 1$ Full rotation

After 2 rotations, 0.25λ is left

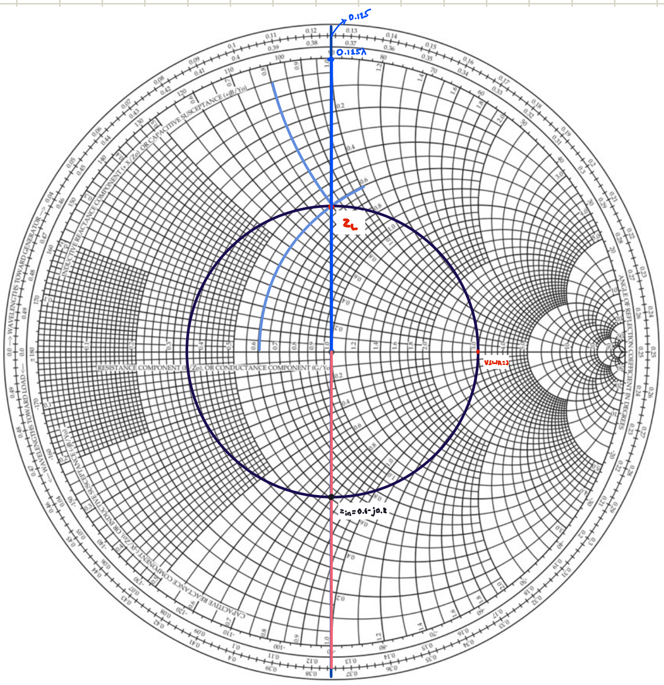
Now, $0.25\lambda + 0.125\lambda = 0.375\lambda$

and draw a line to 0.375λ Color of line

8) The point on our circle that meets the line is z_{in}

9) $z_{in} = \frac{Z_L}{Z_0} \Rightarrow Z_L = Z_0 \times z_i = 50(0.6 - j0.8)$

$= 30 - j40\,\Omega$



①

Q.① Determine the i/p impedance & SWR for a 1.25λ long transmission at the sending end (↳ generator) with $Z_0 = 50\Omega$ and load impedance of $Z_L = 30+j40\,\Omega$

$Z_0 = 50\Omega$
$Z_L = 30+j40\,\Omega$
$L = 1.25\lambda$

Stub Matching

A small length of Tx line → Stub

$50+j40$ $\underline{-j40}$ → open stub (using for load)

$50-j40$ $\underline{+j40}$ → inductive stub (using for load)

* 1 whole Smith chart = λ/2

① $1.25\lambda = 0.5\lambda + 0.5\lambda + \underline{0.25}\lambda$

② Find the normalised impedance, $z_L = \frac{Z_L}{Z_0}$

$z_L = \frac{30+j40}{50} = 0.6 + 0.8j$

③ VSWR.
Draw a circle with radius OA.

④ Draw a straight line passing O & A.
↳ 0.125λ → location of load

⑤ Find generator location
$0.25\lambda + 0.125\lambda$
$= 0.375\lambda$
Draw line from O to 0.375 point

towards generator (CW)
towards load (CCW)

⑥ Take the intersection point of VSWR circle and line joining centre & 0.375 point
Get R & X value
⇒ $0.6 - j0.8$ → Normalised i/p impedance (z_i)

$\boxed{① z_L = \frac{Z_L}{Z_0} \quad ② z_i = \frac{Z_i}{Z_0}}$

⇒ $Z_i = z_i \cdot Z_0 = (0.6-j0.8) * Z_0$

$\boxed{Z_i = 30-j40\,\Omega}$

To find Admittance y_i:-
extend line of O & 0.375λ to up and find intersection of the line & VSWR circle. This gives y_i
Here, $y_i = 0.6+j0.8$ (z_L)

$y_i = \frac{1}{z_i} = \frac{1}{0.6-j0.8} = 0.6+j0.8$ ✓

②
Q. A 50Ω lossless Tx line is terminated in a load impedance of $Z_L = 25+j50\,\Omega$. Use smith chart, to find VSWR, Γ, input admittance & input impedance of the line given that the line is 3.3λ long.

$Z_L = 25+j50\,\Omega$
$L = 3.3\lambda$
$Z_0 = 50\Omega$

* $z_L = \frac{Z_L}{Z_0} = \frac{25+j50}{50} = 0.5+1j\,\Omega$

* VSWR = 4.4 ✓

* line extended crossing O & z_L. outer reading → 0.135λ (WTG)

* $3.3\lambda = [(0.5\times 6) + 0.3]\lambda$
move 0.3λ CW from 0.135λ
$0.135 + 0.3 = 0.435\lambda$

If L = 0.4λ

* For Γ :- (using VSWR)
Take a compass with radius = VSWR
* Make a arc at the bottom scale of Refl.coeff (E or I) (vol)
This gives Γ ↓
Towards generator
Ref coeff → Magnitude |Γ|

②

* 0.28L → matched impedance

* angle of Γ :- (θ) → intersection of angle circle & line

* i/p impedance :-
$z_i = \frac{Z_i}{Z_0}$
$Z_i = (0.29-0.42j) = (0.3-0.4j)\cdot 50$
$\boxed{Z_i = 15-20j}$

* i/p Admittance :- extend line of z_i and find intersecting point of extended line to VSWR circle.

$y_i = 1.2+1.7j$

$\boxed{y_i = \frac{Y_i}{Y_0} = Y_i \cdot Z_0}$

$Y_i = \frac{y_i}{Z_0} = \frac{1.2+1.7j}{50}$ $\boxed{Y_i = 0.024+j0.034}$

* extend line of z_L (load), to find normalised output admittance (y_L)

② Use the smith chart to find the following using the figure given.
Find VSWR, Γ, input admittance (Y_i), Z_{in}, distance from the load to first voltage minima, distance from the load to first voltage maxima.

$Z_0 = 50\Omega$, $Z_L = 60+50j\,\Omega$

* $z_L = \frac{Z_L}{Z_0} = \frac{60+50j}{50} = 1.2+1j$

① VSWR = 2.5 (i)

* 0.42 = 0.25λ + 0.175λ
0.175λ + 0.25λ = 0.42λ
0.42
0.15
0.57λ

* |Γ| = 0.42
θ = 54° (ii)

* $Z_i = z_i \cdot Z_0 = (0.52+0.38j)50$
$Z_{in} = 26+19j$ (iv)

* $y_L = 0.5-j0.42$
$Y_L = \frac{y_L}{Z_0} = \frac{0.5-j0.42}{50} = 0.0096-0.0084j$ (iii)

Short ckt → open ckt
V_{max}

③

(v)
$x + 0.25$ → distance

Distance = 0.325λ Ans.

2nd minima = 0.325 + 0.5 = 0.825λ

(vi) 0.325 − 0.25 = 0.075λ
(or)
0.25 − 0.175 = 0.075λ

Stub Matching → To match with 1+0j

* Smith chart is considered as Admittance chart (Not an impedance smith chart)

* 2 ways in connecting stub
↳ series ①
↳ Parallel → shunt stub ②

1+jb

* Stub is added to nullify the Reactance.

①
distance from load.
To nullify jb at B.

②

Why stub added with some distance from load:-
* To minimize Reflection at load
* In order to avoid the damage to high power generator (microwave gen (μw)). → It is required to nullify the reactance/reflection coming from load.

* Stub → Admittance values are used to nullify.
1+jb −jb
1−jb +jb

Q. A 300Ω Tx line is connected to a load impedance of 450−j600Ω at 10 MHz. Find the position and length of a short circuit stub required to match the line using smith chart.
2 ways
1. Find z_L & then y_L ($1/z_L$)
2. Find y_L directly & mark. $y_L = \frac{Z_0}{Z_L}$

* $z_L = \frac{450-j600}{300} = 1.5-2j$
* y_L
* VSWR circle
* Draw circle passing 1+0j
* Intersection point of VSWR circle & 1+0j passing circle.
* Find reactance of that → +1.7j & −1.7j
↓
stub location.

① $d_1 = (0.052-0.18)\lambda = 0.128\lambda$ Ans.
$l_s = 0.33 - 0.25$

Distance b/w 0.33λ(−j1.7) & 0.25λ (SC) gives length of SC.

⇒ $l_s = 0.33-0.25 = 0.085\lambda$
②

④

Impedance chart
Admittance chart

Q. A 50Ω Tx line is connected to a load impedance of 60+j80Ω. The stub is connected short with the load. Find the position & length of the short ckt stub required to match the line using smith chart.

* $z_L = \frac{60+j80}{50} = 1.2+j1.6$

* $y_L = 0.3-j0.4$

* +j1.5 at 0.175λ
−j1.5 at 0.325λ

* d_1 :- $(0.175\lambda - 0.435\lambda) = 0.16\lambda$
(or)
$0.175\lambda + 0.065\lambda = 0.24\lambda$

* d_2 :- $0.32 - 0.25 = 0.07\lambda$

Microwave Generator
* In the order of 10^9 (GHz)

Reflex klystron

Scattering Matrix :-

S

$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0}$
$S_{12} = \frac{b_1}{a_2}\Big|_{a_1=0}$

→ For 2 port network :- $\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$

→ n port :- $S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1n} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2n} \\ \vdots & & & & \vdots \\ S_{n1} & S_{n2} & & \cdots & S_{nn} \end{bmatrix}$

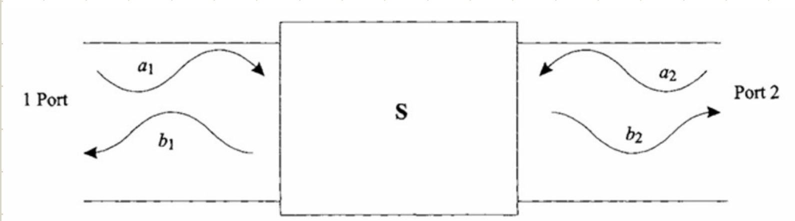
Scattering Matrix

→ Scattering parameters/matrices are essential to analyze microwave devices because :

i) Active devices operates at constant frequencies & it gives instability as frequency changes

ii) z & y parameters are impractical at high frequencies and hard to realize open & short circuit above 1GHz

iii) The measurement of instantaneous voltage & current lose its physical meaning because of its distributed effects



$$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0} \qquad S_{21} = \frac{b_2}{a_1}\Big|_{a_2=0}$$

$$S_{12} = \frac{b_1}{a_2}\Big|_{a_1=0} \qquad S_{22} = \frac{b_2}{a_2}\Big|_{a_1=0}$$

→ Properties :

i) s matrix is a square matrix of nxn where n is no. of ports of a given device

$$[S] = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$$

ii) Identity property : $[S][S]^* = I$

iii) Symmetric property : $S_{ij} = S_{ji}$ (Only for symmetric ports)

iv) Reciprocal matrix : $[S] = [S]^T$

v) Lossless network : $[S]^T[S]^* = I$

$$\left.\begin{aligned} &|S_{11}|^2 + |S_{12}|^2 = 1 && \text{(For } 2\times2) \\ &|S_{11}|^2 + |S_{12}|^2 + \cdots + |S_{1n}|^2 = 1 && \text{(For } n\times n) \end{aligned}\right\} \rightarrow \text{Row}$$

$$\left.\begin{aligned} &|S_{11}|^2 + |S_{21}|^2 = 1 && \text{(For } 2\times2) \\ &|S_{11}|^2 + |S_{21}|^2 + \cdots + |S_{n1}|^2 = 1 && \text{(For } n\times n) \end{aligned}\right\} \rightarrow \text{Column}$$

Q. Let incident voltage $= V_0^+$ & Reflected voltage $= V_0^-$, current $= i_0^+$ in a Tx line . Normalize this by characteristic impedance z_0

A. $V = V_0^+ + V_0^-$

$I = \frac{1}{z_0}[V_0^+ - V_0^-]$

$V_0^+ = \frac{V + IZ_0}{2}$

$V_0^- = \frac{V - IZ_0}{2}$

The normalized incident vector $a = \frac{V_0^+}{\sqrt{Z_0}} = \frac{V + IZ_0}{2\sqrt{Z_0}} = I_0^+\sqrt{Z_0}$

The normalized reflected vector $b = \frac{V_0^-}{\sqrt{Z_0}} = \frac{V - IZ_0}{2\sqrt{Z_0}} = I_0^-\sqrt{Z_0}$

Now in terms of a & b, the incident power $P^+ = |a|^2$

the reflected power $P^- = |b|^2$

the reflected coefficient $\Gamma = \frac{b}{a}$

The actual power delivered to the load $P = |a|^2 - |b|^2$

VSWR of the termination $\rho = \frac{|a| + |b|}{|a| - |b|}$

For n-port network , $a_i = \frac{1}{2}\left(\frac{V_i + Z_0 I_i}{\sqrt{Z_0}}\right)$

$b_j = \frac{1}{2}\left(\frac{V_j - Z_0 I_j}{\sqrt{Z_0}}\right)$

Q. Consider 75 Ω Tx line that is terminated with a 35 + j120 Ω load. Find the reflection coefficient at load & VSWR using wave vectors.

A. Normalizing the voltage source to 1, $V_s = 1$

then $V_0^+ = \frac{V_s}{2} = \frac{1}{2}$

$$a = \frac{V_0^+}{\sqrt{Z_0}} = \frac{1}{2\sqrt{75}} = 0.0577$$

$$\frac{V_0^-}{V_0^+} = \Gamma_L = \left(\frac{Z_0 - Z_L}{Z_0 + Z_L}\right) = \frac{40 - j120}{40 + j120} = -0.377 - j0.679$$

$$V_0^- = V_0^+ (\Gamma_L) = \frac{-0.377 - j0.679}{2} = -0.188 - j0.34$$

$$b = \frac{V_0^-}{\sqrt{Z_0}} = \frac{-0.188 - j0.34}{\sqrt{75}} = -0.0217 - j0.039$$

VSWR, $\rho = \frac{|a| + |b|}{|a| - |b|} = 7.97$

Q. Given, $Z_0 = 50\,\Omega$, $Z_L = 25 + j25\,\Omega$, $f = 1\,GHz$. Calculate normalized load impedance, admittance, admittance at a distance d stub susceptance and stub length for a simple short circuited shunt stub.

A. $Z_0 = 50\,\Omega$

$Z_L = 25 + j25\,\Omega$

$f = 1\,GHz$

$$z_L = \frac{Z_L}{Z_0} = \frac{25 + j25}{50} = 0.5 + j0.5$$

$$y_L = \frac{1}{z_L} = 1 - j1$$

$$z_{in} = \frac{z_0(z_L + jz_0 \tan\beta l)}{(z_0 + jz_L \tan\beta l)} = 1$$

So,

$\beta l = 45° = \pi/4$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi \times 1\times10^9}{3\times10^8} = \frac{20\pi}{3} \Rightarrow l = \frac{\pi/4}{\beta} = \frac{\pi/4}{20\pi/3} = \frac{3}{80} = 0.0375\,m$$

Q. Find voltage & current at $z = 0.25\lambda$, when the characteristic impedance is 50 Ω & incident voltage is 10 V with reflection coefficient of −0.5

A. $V(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z})$

$$\beta z = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

$$e^{-j\pi/2} = \cos(\pi/2) - j\sin(\pi/2)$$
$$= -j$$

U2 Waveguides, cavities, and radiation from point source

Maxwell Equations

→ $\nabla \times E = -\frac{\partial B}{\partial t} \longrightarrow ①$

$\nabla \times H = J + \frac{\partial D}{\partial t} \longrightarrow ②$

$\nabla \cdot D = \rho_v \longrightarrow ③$

$\nabla \cdot B = 0 \longrightarrow ④$

→ If a sinusoidal function is assumed in the form of $e^{j\omega t}$ Then $\frac{\partial}{\partial t}$ can be replaced with assuming waveguide region is source free, eq ① & ② become

$\nabla \times \vec{E} = -j\omega\mu H \longrightarrow ⑤$

$\nabla \times \vec{H} = J + j\omega\varepsilon\vec{E}$

$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \longrightarrow ⑥$

$\nabla \cdot D = \rho_v \longrightarrow ⑦$

$\nabla \cdot B = 0$

→ In free space, space charge density is 0 & in a perfect conductor, time varying static field doesn't exist

$\nabla \cdot D = 0 \longrightarrow ⑨$

$\nabla \cdot E = 0 \longrightarrow ⑩$

→ Now take curl on ⑤,

$\nabla \times \nabla \times \vec{E} = -j\omega\mu(\nabla \times H)$

$-\nabla^2 E + \nabla(\nabla \cdot E) = -j\omega\mu(j\omega\varepsilon\vec{E})$

$\nabla^2 E = j^2\omega^2\mu\varepsilon\vec{E}$

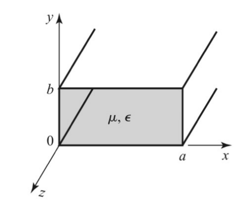
$\nabla^2 E = \gamma^2 E$ where $\gamma = j\omega\sqrt{\mu\varepsilon}$

$\gamma = \alpha + j\beta$

Waveguides

→ When the wavelength λ travels longitudanally down the waveguide & if it is in the direction of propagation of wave, then 2 components will be generated : λ_n (normal to reflecting surface)

λ_p (parallel to reflecting surface)



→ In lossless waveguide we have only 2 modes :

i) transverse electric

ii) transverse magnetic

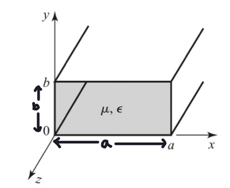
Solutions for wave eqs in rectangular coordinate system

→ Consider a sinusoidal steady state frequency domain solution to the wave eq.

$\nabla^2 E = \gamma^2 E$

$\nabla^2 H = \gamma^2 H$

$\gamma = j\omega\sqrt{\mu\varepsilon} = \alpha + j\beta$



→ The rectangular components of E & H satisfy Helmholtz equation

WKT ∇^2 is a second order operator $\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \longrightarrow ③$

The solution for this Helmholtz eq^n which is an inhomogeneous eq^n in 3D can be calculated

Now, let $\psi = X(x)\,Y(y)\,Z(z) \longrightarrow ④$

Substitute ④ in ③ & divide the resultant with ④

$$\gamma^2 = \frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z} \longrightarrow ⑤$$

Sum of 3 terms is a constant, which means each term is constant

Let, $\frac{1}{X}\frac{d^2X}{dx^2} = -K_x^2$ $\qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -K_y^2$ $\qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -K_z^2$

and substitute in (5)

$$\gamma^2 = -K_x^2 - K_y^2 - K_z^2 \longrightarrow (6)$$

So, $\frac{d^2X}{dx^2} = -XK_x^2$ $\qquad \frac{d^2Y}{dy^2} = -YK_y^2$ $\qquad \frac{d^2Z}{dz^2} = -ZK_z^2$

Then $X = A\sin(K_x x) + B\cos(K_x x)$

$Y = C\sin(K_y y) + D\cos(K_y y)$

$Z = E\sin(K_z z) + F\cos(K_z z)$

$\psi = (A\sin(K_x x) + B\cos(K_x x))(C\sin(K_y y) + D\cos(K_y y))(E\sin(K_z z) + F\cos(K_z z))$

The direction of propagation of wave is assumed to be +z, So the propagation constant γ_g in the guide differs from the intrinsic propagation γ

Let $\gamma_g^2 = \gamma^2 + K_x^2 + K_y^2$

$= \gamma^2 + K_c^2$ where K_c: cut-off wave number

$\gamma_g^2 = -\omega^2\mu\varepsilon + K_c^2$ (or) $K_c^2 = \gamma_g^2 + \omega^2\mu\varepsilon$

Case (i) : No propagation in the waveguide

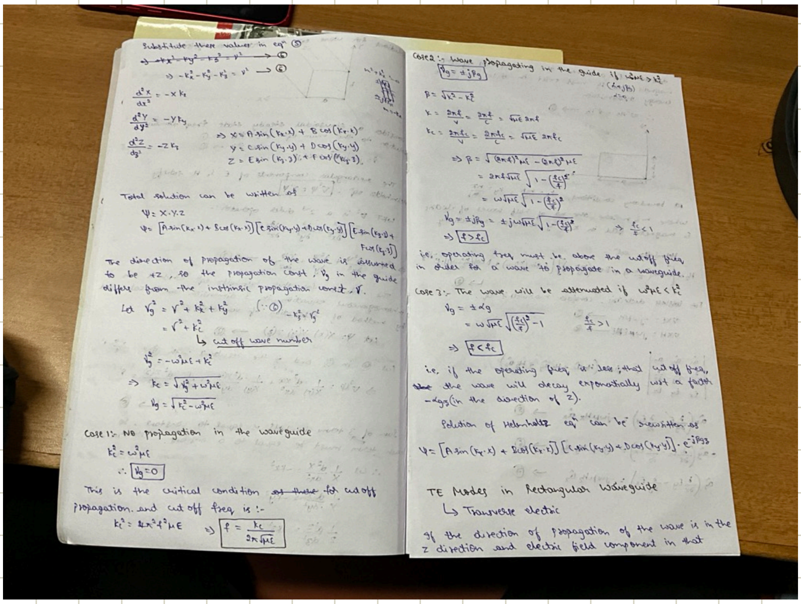
$K_c^2 = \omega^2\mu\varepsilon$ therefore $\gamma_g = 0$

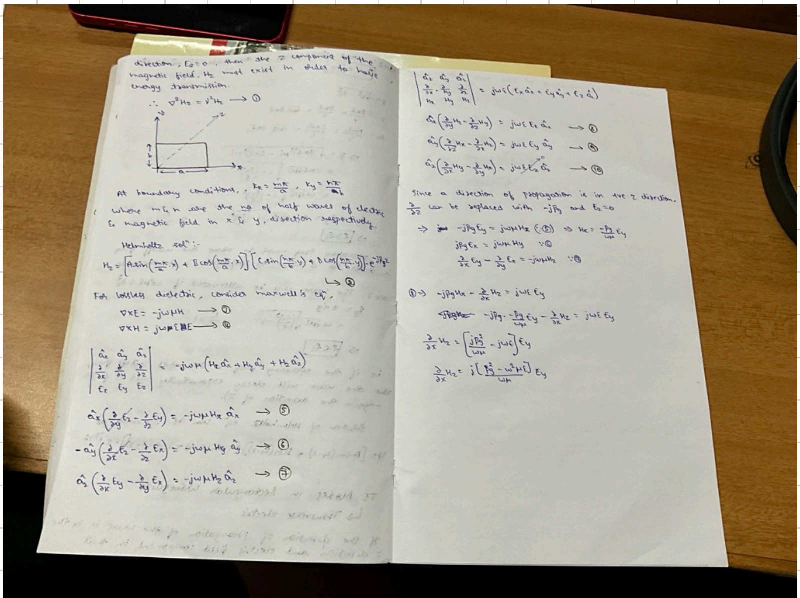
This is a critical condition (No propagation in waveguide)

A cutoff frequency is $f = \frac{1}{2\pi}\frac{K_c}{\sqrt{\mu\varepsilon}}$

Case(ii) : The wave will be propagating in the guide if $\omega^2\mu\varepsilon > K_c^2$

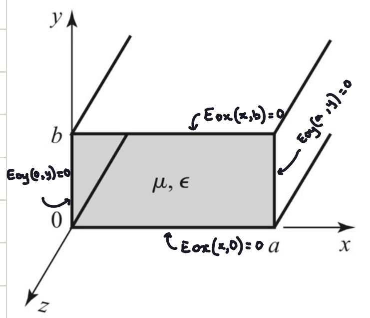
$\gamma_g = \pm j\beta_g$





$$E_x = \frac{-j\omega\mu}{K_c^2} \cdot \frac{\partial H_z}{\partial y}$$

$$E_y = \frac{-j\omega\mu}{K_c^2} \cdot \frac{\partial H_z}{\partial x}$$



$$E_z = 0$$

$$H_x = \frac{-j\beta_g}{K_c^2} \cdot \frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\beta_g}{K_c^2} \cdot \frac{\partial H_z}{\partial y}$$

$$H(z) = \left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right] e^{-j\beta z}$$

→ The tangent E field & the normal H field vanishes at the surface. The boundary conditions require that the tangential component of electrical field be zero with a perfect conductor, therefore, $H(z) = H_{0z}\, e^{-j\beta z}$
where H_{0z} is a function of $X(x)$ & $Y(y)$
Therefore $E_x, E_y, E_z, H_z, H_y, H_z$ can be re-written as

$$E_{0x} = \frac{-j\omega\mu}{K_c^2} \cdot \frac{\partial H_{0z}}{\partial y} = \frac{-j\omega\mu}{K_c^2}\left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\cos\left(\frac{n\pi}{b}\right)y - D\sin\left(\frac{n\pi}{b}\right)y\right] \times \frac{n\pi}{b}$$

$$E_{0y} = \frac{-j\omega\mu}{K_c^2} \cdot \frac{\partial H_{0z}}{\partial x} = \frac{-j\omega\mu}{K_c^2} \cdot \left[A\cos\left(\frac{m\pi}{a}\right)x - B\sin\left(\frac{m\pi}{a}\right)x\right] \cdot \frac{m\pi}{a} \cdot \left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

$$E_{0z} = 0$$

$$H_{0x} = \frac{-j\beta_g}{K_c^2} \cdot \frac{\partial H_{0z}}{\partial x} = \frac{-j\beta_g}{K_c^2} \cdot \left[A\cos\left(\frac{m\pi}{a}\right)x - B\sin\left(\frac{m\pi}{a}\right)x\right] \cdot \frac{m\pi}{a} \cdot \left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

$$H_{0y} = \frac{-j\beta_g}{K_c^2} \cdot \frac{\partial H_{0z}}{\partial y} = \frac{-j\beta_g}{K_c^2}\left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\cos\left(\frac{n\pi}{b}\right)y - D\sin\left(\frac{n\pi}{b}\right)y\right] \times \frac{n\pi}{b}$$

$$H_{0z} = \left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

At $x=0$,

$$E_{0y} = \frac{j\omega\mu}{K_c^2} \cdot \frac{m\pi}{a} \cdot A \cdot \left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

but $E_{0y} = 0$ @ $x=0$ (Boundary Condition), So $A = 0$

At $x=a$,

$$E_{0y} = \frac{-j\omega\mu}{K_c^2} \cdot \frac{m\pi}{a}\left[A\cos m\pi - \cancel{B\sin m\pi}^{\,0}\right] \cdot \left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

$$= \frac{-j\omega\mu}{K_c^2} \cdot \frac{m\pi}{a}\left[A\cos m\pi\right] \cdot \left[C\sin\left(\frac{n\pi}{b}\right)y + D\cos\left(\frac{n\pi}{b}\right)y\right]$$

$K_x = \frac{m\pi}{a}$

$\sin K_x \cdot a = 0$

$K_x \cdot a = m\pi$

At $y=0$,

$$E_{0x} = \frac{-j\omega\mu}{K_c^2}\left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right] \cdot \frac{Cn\pi}{b}$$

but $E_{0x} = 0$ @ $y=0$ (Boundary Condition), So, $C = 0$

At $y=b$,

$$E_{0x} = \frac{-j\omega\mu}{K_c^2} \cdot \frac{n\pi}{b}\left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\cos n\pi - \cancel{D\sin n\pi}^{\,0}\right]$$

$$= \frac{-j\omega\mu}{K_c^2} \cdot \frac{n\pi}{b}\left[A\sin\left(\frac{m\pi}{a}\right)x + B\cos\left(\frac{m\pi}{a}\right)x\right]\left[C\cos n\pi\right]$$

$\sin n\pi = \sin K_y \cdot b = 0$

$K_y \cdot b = n\pi \Rightarrow K_y = \frac{n\pi}{b}$

$$H_{0z} = B\cos\left(\frac{m\pi}{a}\right)x \cdot D\cos\left(\frac{n\pi}{b}\right)y = A_{mn}\left[\cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\right]$$

$$H_z = A_{mn}\left[\cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y\right] \cdot e^{-j\beta z}$$

→ So, $E_x = E_{ox}\, e^{-j\beta z}$

$E_y = E_{oy}\, e^{-j\beta z}$

$E_z = 0$

$H_x = H_{ox}\, e^{-j\beta z}$

$H_y = H_{oy}\, e^{-j\beta z}$

→ Cutoff wave no. $K_c^2 = K_x^2 + K_y^2$

$$= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\omega^2\mu\varepsilon - \beta^2 = K_c^2$$

$$\beta^2 = \omega^2\mu\varepsilon - K_c^2$$

$$= K^2 - K_c^2$$

$$\beta^2 = K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

→ The phase constant must be a real no. for a guided mode. This requires $K > K_c$

At any time $K < K_c$, the mode is cutoff & not supported by waveguide.

$\omega\sqrt{\mu\varepsilon} > K_c$

$2\pi f_c\sqrt{\mu\varepsilon} = K_c$

$$f_c = \frac{K_c}{2\pi\sqrt{\mu\varepsilon}}$$

Characteristic Impedance of TE Mode (Z_{TE})

→ $Z_{TE} = \dfrac{E_x}{H_y}$

$$E_x = \frac{j\omega\mu n\pi}{K_c^2 b} A_m \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$E_y = \frac{-j\omega\mu n\pi}{K_c^2 a} A_{mn} \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$E_z = 0$

$$H_{ox} = \frac{-j\beta}{K_c^2} \cdot \frac{\partial(H_{oz})}{\partial z}$$

$$H_{oz} = A_m \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y$$

So, $H_{ox} = \dfrac{j\beta}{K_c^2} \cdot \dfrac{m\pi}{a} \cdot \left(\sin\left(\dfrac{m\pi}{a}\right)x\right) \cdot \cos\left(\dfrac{n\pi}{b}\right)y = \dfrac{j\beta m\pi}{K_c^2 a} A_m \sin\left(\dfrac{m\pi}{a}\right)x \cos\left(\dfrac{n\pi}{b}\right)y$

$$H_x = \frac{j\beta m\pi}{K_c^2 a} \cdot \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$H_{oy} = \frac{-j\beta}{K_c^2} \cdot \frac{\partial(H_{oz})}{\partial y}$$

$$= \frac{j\beta}{K_c^2} A_m \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \left(\frac{n\pi}{b}\right)$$

$$H_y = \frac{j\beta n\pi}{K_c^2 b} \cdot A_m \cdot \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$Z_{TE} = \frac{\dfrac{j\omega\mu n\pi}{K_c^2 b} A_m \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}}{\dfrac{j\beta n\pi}{K_c^2 b} \cdot A_m \cdot \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}} = \frac{\omega\mu}{\beta}$$

→ Velocity of wave by which phase changes.

Phase velocity in +ve direction of z for TE_{mn} mode is $v_p = \dfrac{\omega}{\beta} = \dfrac{c}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$

Degenerate Modes

→ Whenever 2 or more modes have the same cutoff frequency but different field configuration, they are degenerate modes

→ In a rectangular waveguide, the corresponding TE_{mn} & TM_{mn} nodes are always degenerate

Rectangular waveguides normally have dimension of $a = 2b$ ratio

Dominant Mode

→ The mode with lowest cutoff frequency (highest λ_c) in a particular waveguide

$$\lambda_{c_{mn}} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}} \quad \text{(from } k_c \text{ \& } f_c \text{ equations)}$$

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}$$

$$k = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\frac{f_c}{c} = \lambda$$

$TE_{00} \Rightarrow m = 0$ & $n = 0$

then, $$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = 0$$

$TE_{10} \Rightarrow m = 1$ & $n = 0$

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c}{2a}$$

$TE_{01} \Rightarrow m = 0$ & $n = 1$

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{2b\sqrt{\mu\varepsilon}} = \frac{c}{2b}$$

if $a > b$, $\frac{c}{2a} < \frac{c}{2b}$

Q. An air filled rectangular waveguide of inside dimension 7×3.5 cm operates in a dominant mode. Find f_c, determine phase velocity of the wave in a waveguide operates at a frequency 3.5 GHz. Determine guide wavelength at same frequency

A. $a = 7$ cm, $b = 3.5$ cm

$$TE_{01} = \frac{c}{2a} = \frac{3 \times 10^8}{2 \times 7 \times 10^{-2}} = 0.214 \times 10^{10}$$

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - \left(\frac{f_c}{F}\right)^2}} = \frac{3 \times 10^8}{\sqrt{1 - \frac{0.214 \times 10^{10}}{3.5 \times 10^9}}} = 0.481 \times 10^9 \text{ m/s}$$

Q. When a dominant node is propagated in an airfilled rectangular waveguide, the guide wavelength for a freq of 9000 MHz is 4cm. Calculate breadth of the wave.

A. $TE_{10} \Rightarrow m = 1, n = 0$

$\lambda_g = 4\text{cm} = 0.04$ m

$$\lambda_0 = \frac{c}{f} = \frac{3 \times 10^8}{9 \times 10^9} = 0.033 \text{ m}$$

$$\lambda_g = \frac{\lambda_0}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2}} = \frac{0.033}{\sqrt{1 - \left(\frac{0.033}{\lambda_c}\right)^2}} = 0.04 \Rightarrow \lambda_c = 0.0584 \text{ m}$$

$\lambda_c = 2a \Rightarrow a = \frac{\lambda_c}{2} = 0.0292$ m

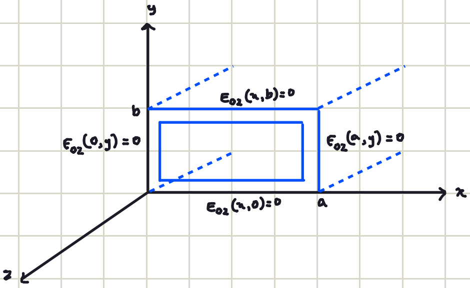
$a = 2b \Rightarrow b = \frac{a}{2} = 0.0146$ m $= 1.46$ cm

TM Mode

- If the direction of propagation of wave is considered wrt +ve z direction, $H_z = 0$, since the transverse magnetic field is considered, so, the z component of Electric field in order to have energy transmission in the guide
- Boundary conditions require that tangential component of electric field be $E = 0$ for a perfect conductor
- Consider Helmholtz equation,

$$\nabla^2 E_z = \gamma^2 E_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_z = \gamma^2 [E_z]$$



$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \gamma^2 [E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z]$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon [E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z]$$

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$$E_z = E_{0z} e^{-j\beta z}$$

$$= B_{mn} \sin\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \, e^{-j\beta z} \quad -(8)$$

Consider the maxwells equation

$$\nabla \times H = j\omega\epsilon E \quad -(9)$$

Expanding $\nabla \times H$

$$\begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon (E_x \hat{a_x} + E_y \hat{a_y} + E_z \hat{a_z})$$

Here $\frac{\partial}{\partial z} = -j\beta$.

$$\begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -j\beta \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon (E_x \hat{a_x} + E_y \hat{a_y} + E_z \hat{a_z})$$

$$\frac{\partial}{\partial y} H_z + j\beta H_y = j\omega\epsilon E_x \quad -(10)$$

$$-j\beta H_x - \frac{\partial}{\partial x} H_z = j\omega\epsilon E_y \quad -(11)$$

$$\frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x = j\omega\epsilon E_z \quad -(12)$$

Similarly $\nabla \times E = -j\omega\mu H \quad -(13)$

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Expanding $\nabla \times E$

$$\begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu (H_x \hat{a_x} + H_y \hat{a_y} \ldots$$

$$\frac{\partial}{\partial y} E_z + j\beta E_y = -j\omega\mu H_x \quad -(14)$$

$$-j\beta E_x - \frac{\partial}{\partial x} E_z = -j\omega\mu H_y \quad -(15)$$

$$\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = -j\omega\mu H_z \quad -(16)$$

We know that $H_z = 0$ in TM mode
eq(10) becomes

$$j\beta H_y = j\omega\epsilon E_x$$

$$H_y = \frac{\omega\epsilon}{\beta} E_x \quad -(17)$$

Substitute in Eq(15)

$$-j\beta E_x - \frac{\partial}{\partial x} E_z = -j\omega\mu \frac{\omega\epsilon}{\beta} E_x$$

$$j\beta E_x + \frac{\partial}{\partial x} E_z = j\frac{\omega^2\mu\epsilon}{\beta} E_x$$

$$\left[j\beta - j\frac{\omega^2\mu\epsilon}{\beta}\right] E_x = -\frac{\partial}{\partial x} E_z$$

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$$E_x \left(\frac{j\beta^2 - j\omega^2\mu\epsilon}{\beta}\right) = -\frac{\partial}{\partial x} E_z$$

$$\frac{-jk_c^2}{\beta} E_x = -\frac{\partial}{\partial x} E_z$$

$$E_x = \frac{-j\beta}{k_c^2} \cdot \frac{\partial}{\partial x} E_z \quad -(18)$$

$$\text{i.e. } E_{0x} = \frac{-j\beta}{k_c^2} \frac{\partial}{\partial x} E_{0z} \quad -(19)$$

Eq (11) becomes

$$-j\beta H_x = j\omega\epsilon E_y$$

$$H_x = \frac{-\omega\epsilon}{\beta} E_y \quad -(20)$$

Substitute in Eq(14)

$$\frac{\partial}{\partial y} E_z + j\beta E_y = +j\omega\mu \frac{\omega\epsilon}{\beta} E_y$$

$$\frac{\partial}{\partial y} E_z = \left(+j\frac{\omega^2\mu\epsilon}{\beta} - j\beta\right) E_y$$

$$= +j\left(\frac{\omega^2\mu\epsilon - \beta^2}{\beta}\right) E_y$$

$$= j\frac{k_c^2}{\beta} E_y$$

$$E_y = \frac{-j\beta}{k_c^2} \cdot \frac{\partial}{\partial y} E_z \quad -(21)$$

$$\text{i.e. } E_{0y} = \frac{-j\beta}{k_c^2} \cdot \frac{\partial}{\partial y} E_{0z} \quad -(22)$$

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From eq(11)

$$-j\beta H_x = j\omega\epsilon E_y$$

$$= j\omega\epsilon \frac{-j\beta}{k_c^2} \frac{\partial}{\partial y} E_z$$

$$= -j^2 \frac{\omega\epsilon\beta}{k_c^2} \frac{\partial}{\partial y} E_z$$

$$H_x = \frac{j\omega\epsilon}{k_c^2} \frac{\partial}{\partial y} E_z \quad -(23)$$

$$\text{i.e. } H_{0x} = \frac{j\omega\epsilon}{k_c^2} \frac{\partial}{\partial y} E_{0z} \quad -(24)$$

From eq(10)

$$H_y = \frac{\omega\epsilon}{\beta} E_x$$

$$= \frac{\omega\epsilon}{\beta} \frac{-j\beta}{k_c^2} \cdot \frac{\partial}{\partial x} E_z$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial}{\partial x} E_z \quad -(25)$$

$$\text{i.e. } H_{0y} = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial}{\partial x} E_{0z} \quad -(26)$$

Substituting E_{0z} value in 19, 22, 24

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$$E_{0x} = \frac{-j\beta B_{mn}}{k_c^2}\left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi}{b}\right)y$$

$$E_{0y} = \frac{-j\beta B_{mn}}{k_c^2}\left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0x} = \frac{j\omega\epsilon}{k_c^2} \cdot \frac{n\pi}{b} \cdot B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right)$$

$$H_{0y} = \frac{-j\omega\epsilon}{k_c^2}\left(\frac{m\pi}{a}\right) \cdot B_{mn} \cos\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right)$$

$$H_{0z} = 0.$$

$\rightarrow \quad \nabla^2 E_z = \gamma^2 E_z$

The solution for the equation, $E_z = \left(A_m \sin\left(\frac{m\pi}{a}\right)x + B_m \cos\left(\frac{m\pi}{a}\right)x\right) \cdot \left(C_n \sin\left(\frac{n\pi}{b}\right)y + D_n \cos\left(\frac{n\pi}{b}\right)y\right) \cdot e^{-j\beta z}$

$= E_{0z} \cdot e^{-j\beta z}$

At $x = 0$, $E_{0z} = 0$

$E_z = B_m \cdot \left(C_n \sin\left(\frac{n\pi}{b}\right)y + D_n \cos\left(\frac{n\pi}{b}\right)y\right) = 0 \quad \Rightarrow \quad B_m = 0$

At $x = a$, $E_{0z} = 0$

$E_z = \left(A_m \sin m\pi + B_m \cos m\pi\right) \cdot \left(C_n \sin\left(\frac{n\pi}{b}\right)y + D_n \cos\left(\frac{n\pi}{b}\right)y\right) = 0$

so, $a \cdot K_x = m\pi \Rightarrow K_x = \frac{m\pi}{a}$

At $y = 0$, $E_{0z} = 0$

$E_z = \left(A_m \sin\left(\frac{m\pi}{a}\right)x + B_m \cos\left(\frac{m\pi}{a}\right)x\right)\left(D_n \cos 0\right) \quad \Rightarrow \quad D_n = 0$

At $y = b$, $E_{0z} = 0$

$E_z = \left(A_m \sin\left(\frac{m\pi}{a}\right)x + B_m \cos\left(\frac{m\pi}{a}\right)x\right)\left(C_n \sin n\pi + D_n \cos n\pi\right)$

So, $b \cdot K_y = n\pi \Rightarrow K_y = \frac{n\pi}{b}$

$E_{0z} = A_m \sin\left(\frac{m\pi}{a}\right)x \cdot C_n \sin\left(\frac{n\pi}{b}\right)y = B_{mn} \sin\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y$

$E_z = B_{mn} \sin\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$

$Z_{TE} = \frac{E_x}{H_y}$

$Z_{TM} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \frac{\beta\eta}{k}$

$\frac{\beta\eta}{k} = \frac{\beta \times \sqrt{\frac{\mu}{\varepsilon}}}{\omega\sqrt{\mu\varepsilon}}$

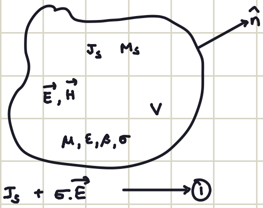
TM_{mn}, $m = 0, n = 0 \Rightarrow f_c = 0$

TM_{01} $m = 0, n = 1 \Rightarrow f_c = \frac{c}{2b}$

Poynting Theorem

→ The rate of degrees of electromagnetic energy in a volume equals to the power flowing out through the surface + power dissipated as losses

→ Volume v is enclosed by a surface S containing fields $\vec{E}$ & $\vec{H}$ and current sources J_s & M_s



→ If we have an electric current source J_s & conduction current σE & $J = J_s + \sigma.\vec{E}$ ⟶ ①

→ Maxwell's equation considering electric current & magnetic current

$$\nabla \times E = -j\omega\mu H - M_s \longrightarrow ②$$

$$\nabla \times H = J + j\omega\varepsilon E \longrightarrow ③$$

Multiply ② & conjugate of H

$$H^* \times ② \Rightarrow H^*(\nabla \times E) = -j\omega\mu |H|^2 - H^* M_s \longrightarrow ④$$

Multiply conjugate of ③ & E

$$E(\nabla \times H^*) = J^* E - j\omega\varepsilon^* |E|^2$$

where, ε^* is considered since the medium is lossy

$$J^* = J_s^* + \sigma E^*$$

So, $E(\nabla \times H^*) = (J_s^* + \sigma E^*) E - j\omega\varepsilon^* |E|^2$

$$= J_s^* E + \sigma |E|^2 - j\omega\varepsilon^* |E|^2 \longrightarrow ⑤$$

We know, $\nabla.(E \times H^*) = H^*(\nabla \times E) - \vec{E}(\nabla \times H^*)$

$$= -j\omega\mu |H|^2 - H^* M_s - (E.J_s^* + \sigma |E|^2 - j\omega\varepsilon^* |E|^2)$$

$$= \sigma |E|^2 - (E.J_s^* + H^* M_s) + j\omega(\varepsilon^* |E|^2 - \mu |H|^2)$$

Integrate over the volume v & use divergence theorem $\left(\iiint_V \nabla.A\,dv = \iint_S A.ds \right)$

$$\iiint_V \nabla(E \times H^*)\,dv = \iint_S (E \times H^*)\,ds$$

$$= \int_V -\sigma |E|^2 dv - \int_V (E.J_s^* + H^* M_s)\,dv + j\omega \int_V (\varepsilon^* |E|^2 - \mu |H|^2) dv$$

Let $\varepsilon = \varepsilon' - j\varepsilon''$ & $\mu = \mu' - j\mu''$ (Because it is lossy environment)

$\varepsilon = \varepsilon' + j\varepsilon''$

So, $$\iint_S (E \times H^*)\,ds + \int_V (E.J_s^* + H^* M_s)\,dv = \int_V -\sigma |E|^2 dv + j\omega \int_V (\varepsilon^* |E|^2 - \mu |H|^2) dv$$

$$-\frac{1}{2}\int_V (E.J_s^* + H^* M_s)\,dv = \int_V \frac{\sigma}{2}|E|^2 dv + \frac{1}{2}\oint_S (E \times H^*)\,ds + \frac{\omega}{2}\int_V (\varepsilon'' |E|^2 + \mu'' |H|^2) dv + \frac{j\omega}{2}\int_V (\mu' |H|^2 - \varepsilon' |E|^2) dv$$

This is called Poynting theorem, that is $P_s = P_o + P_l + 2j\omega(w_m - w_e)$

where P_s is the power delivered by the sources J_s & M_s

$$P_s = \int_V \frac{\sigma}{2}|E|^2 dv + \frac{1}{2}\oint_S (E \times H^*)\,ds + \frac{\omega}{2}\int_V (\varepsilon'' |E|^2 + \mu'' |H|^2) dv + \frac{j\omega}{2}\int_V (\mu' |H|^2 - \varepsilon' |E|^2) dv$$

P_o : Pout

$P_o = \frac{1}{2}\oint (E \times H^*)\,ds$ where $E \times H^* = S$ which is called poynting vector

P_o is defined as the power flow out of the closed surface S

$$P_l = \frac{\sigma}{2}\int |E|^2 dv + \frac{\omega}{2}\int_V (\varepsilon'' |E|^2 + \mu'' |H|^2) dv$$

P_l is defined as the average power dissipated or the loss due to conductivity dielectric & magnetic loss

$2j\omega$ is the net reactive energy stored in the volume where $w_m = \frac{1}{4}\mu |H|^2$ & $w_e = \frac{1}{4}\varepsilon |E|^2$

Power Transmission in Rectangular Waveguide

→ For TE Mode, TE_{10} is dominant mode, that is $m=1, n=0$

$$E_x = \frac{j\omega\mu n\pi}{K_c^2 b} \cdot A_{mn} \cdot \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$E_y = \frac{-j\omega\mu m\pi}{K_c^2 a} \cdot A_{mn} \cdot \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$H_x = \frac{j\beta m\pi}{K_c^2 a} \cdot A_{mn} \sin\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$H_y = \frac{j\beta n\pi}{K_c^2 b} A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$H_z = A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

Now substitute $m=1, n=0$

$$E_x = 0$$

$$E_y = -\frac{j\omega\mu a}{\pi} \cdot A_{10} \cdot \sin\left(\frac{\pi x}{a}\right) \cdot e^{-j\beta z}$$

$$E_z = 0$$

$$H_x = \frac{j\beta a}{\pi} \cdot A_{10} \cdot \sin\left(\frac{\pi x}{a}\right) \cdot e^{-j\beta z}$$

$$H_y = 0$$

$$H_z = A_{10} \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z}$$

→ The power flow down the guide for TE_{10} mode, $P_{10} = \frac{1}{2} \mathrm{Re} \int_{x=0}^{a} \int_{y=0}^{b} (E \times H^*)\, \hat{z}\, dy\, dx$

$$= \frac{1}{2} \frac{\omega\mu a^2}{\pi^2} |A_{10}|^2 \mathrm{Re}(\beta) \cdot \int_{x=0}^{a} \int_{y=0}^{b} E \cdot H^* \, dy\, dz$$

$$= \frac{1}{2} \frac{\omega\mu a^2}{\pi^2} |A_{10}|^2 \mathrm{Re}(\beta) \cdot \int_{x=0}^{a} \int_{y=0}^{b} \sin^2\left(\frac{\pi x}{a}\right) dy\, dx$$

$$= \frac{1}{2} \frac{\omega\mu a^2}{\pi^2} |A_{10}|^2 \mathrm{Re}(\beta) \cdot \int_{y=0}^{b} dy \cdot \frac{1}{2} \int_{x=0}^{a} \left(1 - \cos\left(\frac{2\pi x}{a}\right)\right) dx$$

$$= \frac{1}{2} \frac{\omega\mu a^2}{\pi^2} |A_{10}|^2 \mathrm{Re}(\beta) \cdot b \cdot \frac{a}{2}$$

$$P_{10} = \frac{\omega\mu a^3 b}{4\pi^2} |A_{10}|^2 \mathrm{Re}(\beta)$$

Power loss in Rectangular Waveguide in TE Mode

→ The Attenuation in Rectangular Waveguide may occur due to dielectric loss / conductor loss. The power lost per unit length due to finite wall conductivity

$$P_\ell = \frac{R_s}{2} \int_s |J_s|^2 dl$$ where R_s : Wall surface resistance

There are surface current on all 4 walls & due to symmetry, currents on opposite sides are equal, so, compute the power in walls at $x=0$ & $y=0$ & double their sum to obtain total power loss

At $x=0$,

$$J_s = \hat{n} \times H \big|_{x=0}$$

$$= \hat{x} \times \hat{z}\, H_z$$

$$= -\hat{y}\, H_z$$

$$= -\hat{y}\, A_{10} e^{-j\beta z}$$

Surface current on $y=0$, $\vec{J_s} = \hat{n} \times H \big|_{y=0} = \hat{y} \times \left(\hat{x} H_x \big|_{y=0} + \hat{z} H_z \big|_{y=0}\right)$

$$= -\hat{z} \frac{j\beta a}{\pi} A_{10} \sin\frac{\pi x}{a} e^{-j\beta z} + \hat{x} A_{10} \cos\frac{\pi x}{a} e^{-j\beta z}$$

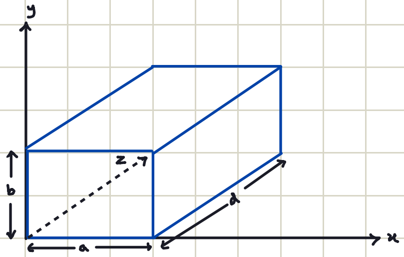
So, $P_l = R_s \int_{y=0}^{b} |J_{sy}|^2 dy + R_s \int_{x=0}^{a} \left[|J_{sx}|^2 + |J_{sz}|^2 \right] dx$

$$= R_s |A_{10}|^2 \left(b + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2} \right)$$

→ The Power handling capacity of an air filled rectangular waveguide is usually limited by voltage breakdown which occurs at a field strength of about $E_d = 3 \times 10^6$ V/m for room temperature air at sea level pressure. In an air filled rectangular waveguide, the electric field varies as $E_y = E_0 \sin\left(\frac{\pi}{a}\right)x$ which has a maximum value of E_0 at $x = \frac{a}{2}$ (middle of guide), therefore maximum power capacity before breakdown, $P_{max} = \frac{abE_0^2}{4Z}$ $(a = 2b)$

→ A Cavity resonator is a metallic enclosure that confines electromagnetic energy. Stored electric & magnetic energies inside the cavity determines its inductance & capacitance. The energy dissipated by the finite conductivity of cavity walls determines its equivalent resistivity.

→ When frequency of impressed signal is equal to resonant frequency, a maximum amplitude of standing wave occurs & peak energies stored in $\vec{E}$ & $\vec{H}$ field are equal. The mode having lowest resonant frequency is called as Dominant Mode.



→ TE Mode $E_z = 0$, $H_z \neq 0$

$$H_z = A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

H_z for resonator,

$$H_z = A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot \left(e^{-j\beta z} + e^{j\beta z}\right)$$

$$= \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \left(A_{mn} e^{-j\beta z} + A'_{mn} e^{j\beta z}\right)$$

$$H_z\big|_{z=0} = 0$$

$$A_{mn} + A'_{mn} = 0 \Rightarrow A_{mn} = -A'_{mn}$$

$$H_z = \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}\right)y \left(A_{mn} e^{-j\beta z} - A_{mn} e^{j\beta z}\right)$$

$$= A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \left(e^{-j\beta z} - e^{j\beta z}\right)$$

$$= -2j A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot \sin(\beta z)$$

$$H_z\big|_{z=d} = 0 \Rightarrow \sin\beta d = 0$$

$$\beta d = p\pi$$

$$\beta = \frac{p\pi}{d}$$

$$H_z = -A_{mn} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot \sin\left(\frac{p\pi}{d}\right)z$$

$$E_x = \frac{-j\omega\mu}{K_c^2} \cdot \frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{K_c^2} \cdot \left(-A_{mn}\left(\frac{n\pi}{b}\right) \cdot \cos\left(\frac{m\pi}{a}\right)x \cdot \sin\left(\frac{n\pi}{b}\right)y \cdot \sin\left(\frac{p\pi}{d}\right)y\right)$$

$$E_y = \frac{j\omega\mu}{K_c^2} \cdot \frac{\partial H_y}{\partial x}$$

$$H_x = \frac{-j\beta}{K_c^2} \cdot \frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\beta}{K_c^2} \cdot \frac{\partial H_z}{\partial y}$$

$$K_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$f_r = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

Condition: $a > b < d$

Q. Calculate the resonant frequency of a rectangular cavity resonator of dimensions $3 \times 2 \times 4$ in cm when filled with air & mode of operation is TE_{101}

A. $m=1, \quad n=0, \quad p=1$

Vector potential, 'A', from electric current J

→ The Vector potential, 'A' is a magnetic vector potential is used for solving Electromagnetic field generated by a given harmonic electric current source, 'J'

→ The magnetic flux density 'B' is always solenoidal (closed loop), so, $\nabla \cdot B = 0$

$\nabla \cdot B_A = 0$ (For magnetic vector potential) $\longrightarrow ①$

→ From maxwell's eqⁿ, $\nabla \times \vec{E}_A = -j\omega\mu H_A \longrightarrow ②$

The magnetic flux density can be represented as a curl of another vector which obeys $\nabla \cdot (\nabla \times A) = 0$

So, $B_A = \nabla \times A = \mu H_A \longrightarrow ③$

Substitute ③ in ②,

$$\nabla \times E = -j\omega(\nabla \times A) \longrightarrow ④$$

$$\nabla \times (E + j\omega A) = 0 \longrightarrow ⑤$$

$$\nabla \times (-\nabla\phi_e) = 0$$

ϕ_e : arbitrary electric scalar which is a function of possesion

$$\nabla \times \nabla \times A = \nabla \times \mu H_A$$

$$= \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\mu(\nabla \times H_A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

We know, $\nabla \times H_A = J + j\omega\varepsilon E_A$ (Maxwell's eqⁿ)

So, $\mu(J + j\omega\varepsilon E_A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$$\mu J + j\omega\mu\varepsilon E_A = \nabla(\nabla \cdot A) - \nabla^2 A \longrightarrow ⑥$$

$$E_A + j\omega A = -\nabla\phi_e$$

$$E_A = -\nabla\phi_e - j\omega A \longrightarrow ⑦$$

$$\mu J + j\omega\mu\varepsilon(-\nabla\phi_e - j\omega A) = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\mu J - j\omega\mu\varepsilon(\nabla\phi_e) + \omega^2\mu\varepsilon A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\mu J - \nabla(j\omega\mu\varepsilon\phi_e) + k^2 A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\nabla^2 A + k^2 A = -\mu J + \nabla(j\omega\mu\varepsilon\phi_e + \nabla \cdot A)$$

Let, $\nabla \cdot A = -j\omega\mu\varepsilon\phi_e \longrightarrow ⑧$

Then, $\nabla^2 A + k^2 A = -\mu J \longrightarrow ⑨$

$\phi_e = \dfrac{-\nabla \cdot A}{j\omega\mu\varepsilon}$ ⇒ Lorentz Equation

↳ Relationship b/w ϕ_e & A

→ Once A is known, then E_A & H_A can be found

(a) $\nabla \cdot (\nabla T) = \nabla^2 T$ = a scalar field

(b) $\nabla \times (\nabla T) = 0$

(c) $\nabla(\nabla \cdot h)$ = a vector field

(d) $\nabla \cdot (\nabla \times h) = 0$

(e) $\nabla \times (\nabla \times h) = \nabla(\nabla \cdot h) - \nabla^2 h$

(f) $(\nabla \cdot \nabla)h = \nabla^2 h$ = a vector field

$$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$$
$$\nabla(\mathbf{A} + \mathbf{B}) = \nabla\mathbf{A} + \nabla\mathbf{B}$$
$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$
$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$A \times (B \times C) = (C \times B) \times A = B(A \cdot C) - C(A \cdot B)$

$\nabla(uv) = u\nabla v + v\nabla u$

$\nabla(A \cdot B) = A \times (\nabla \times B) + (A \cdot \nabla)B + B \times (\nabla \times A) + (B \cdot \nabla)A$

$\nabla \cdot uA = u\nabla \cdot A + A \cdot \nabla u$

$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B$

$\nabla \times (uA) = u\nabla \times A - A \times \nabla u$

$\nabla \times (A \times B) = (B \cdot \nabla)A + A(\nabla \cdot B) - (A \cdot \nabla)B - B(\nabla \cdot A)$

$(\overline{\nabla \cdot A)B} = (A \cdot \nabla)B + B(\nabla \cdot A)$

$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - (\nabla \cdot \nabla)A$

Vector potential, 'F', for magnetic current M

→ The field generated by magnetic current M in a homogeneous medium with $J=0$ & $M \neq 0$ must satisfy $\nabla . D = 0$

$$E_f = \frac{-1}{\varepsilon}(\nabla \times F)$$

From Maxwell eqn,

$$\nabla \times H = j\omega\varepsilon E_F$$
$$= j\omega\varepsilon \times \frac{-1}{\varepsilon}(\nabla \times F)$$
$$= -j\omega(\nabla \times F)$$
$$\nabla(H_F + j\omega F) = 0$$

Also, $\nabla \times (-\nabla\phi_m) = 0$

So, $\nabla\phi_m = H_F + j\omega F$

$$H_F = -\nabla\phi_m - j\omega F$$

From Maxwell eq,

$$\nabla \times E_F = -M - j\omega\mu H_F$$
$$= -M - j\omega\mu(-\nabla\phi_m - j\omega F)$$
$$= -M + j\omega\mu\nabla\phi_m - \omega^2\mu F$$
$$\nabla \times -\frac{1}{\varepsilon}\nabla \times F = -M - \omega^2\mu F + j\omega\mu\nabla\phi_m$$
$$-\frac{1}{\varepsilon}\left[\nabla(\nabla . F) - \nabla^2 F\right] = -M - \omega^2\mu F + \nabla(j\omega\mu\phi_m)$$

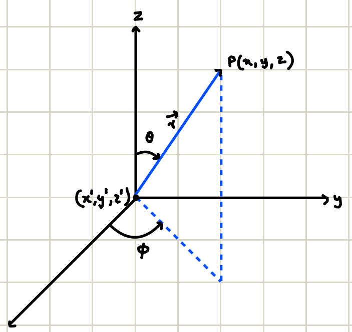
$$\nabla(\nabla . F) - \nabla^2 F = M\varepsilon + \omega^2\mu\varepsilon F - \nabla(j\omega\mu\varepsilon\phi_m)$$
$$= M\varepsilon + K^2 F - \nabla(j\omega\mu\varepsilon\phi_m)$$
$$\nabla^2 F + K^2 F = M\varepsilon + \nabla(j\omega\mu\varepsilon\phi_m + \nabla . F)$$
$$\nabla . F = -j\omega\mu\varepsilon\phi_m$$
$$\phi_m = \frac{-1}{j\omega\mu\varepsilon}(\nabla . F)$$

→ $\nabla^2 A + K^2 A = -\mu J$

$\nabla^2 F + K^2 F = -\varepsilon M$

→ $(x', y', z') = (0, 0, 0)$

$f(r, \theta, \phi)$



→ Source at the origin means the radiating source is located at coordinates of origin.

Source at $(x', y', z') = (0, 0, 0)$

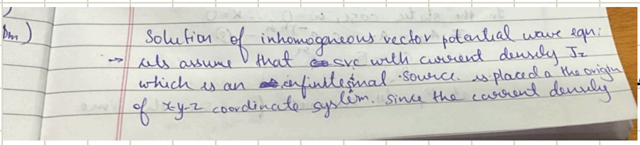
The observation point or field point is x, y, z

The position vector $\vec{r}$, directly points from source to observation point

θ is a polar angle measured from +ve z-axis

ϕ is the azimuthal angle measured in xy plane from +ve x-axis

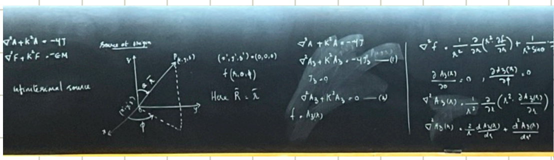
Let $\vec{R}$ be the vector from source point to observation point, therefore here $\vec{R} = \vec{r}$

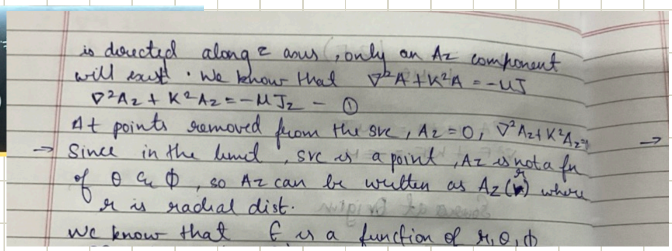
→ 

Solution of inhomogeneous vector potential wave eqn:

→ lets assume that src with current density J_z which is an infinitesimal source is placed a the origin of x-y-z coordinate system. Since the current density

ensity J_z which





is directed along z axis, only an A_z component will exist. We know that $\nabla^2 A + K^2 A = -\mu J$

$\nabla^2 A_z + K^2 A_z = -\mu J_z$ — ①

At points removed from the src, $A_z = 0$, $\nabla^2 A_z + K^2 A_z = 0$

→ Since in the limit, src is a point, A_z is not a fn of θ & ϕ, so A_z can be written as $A_z(r)$ where r is radial dist.

We know that f is a function of r, θ, ϕ

→ We know that f is a function of r, θ, ϕ

$$\nabla^2 f = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2 f}{\partial \phi^2}$$

Since $f = A_z(r)$ depends only on r, $\frac{\partial f}{\partial \theta} = \frac{\partial A_z}{\partial \theta} = 0$ & $\frac{\partial^2 f}{\partial \phi^2} = \frac{\partial^2 A_z}{\partial \phi^2} = 0$

Then, $\nabla^2 A_z(r) = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial A_z(r)}{\partial r}\right)$

$$\nabla^2 H_z(r) = \frac{2}{r} \cdot \frac{d A_z(r)}{dr} + \frac{d^2 A_z(r)}{dr^2}$$

$$\frac{d^2 A_z(r)}{dr^2} + \frac{2}{r} \cdot \frac{d A_z(r)}{dr} - \nabla^2 A_z(r) = 0$$

This D.E has 2 solutions

$$A_{z_1} = C_1 \frac{e^{-jkr}}{r} \longrightarrow (6)$$

& $$A_{z_2} = C_2 \frac{e^{+jkr}}{r} \longrightarrow (7)$$

(6) & (7) represents an outwardly travelling wave & inwardly travelling wave respectively.
In this case, the source is at the origin, radiated fields travelling outward therefore solution is

$$A_z = A_{z_1} \longrightarrow (8)$$

In the static case, $w = 0$ that is $k = 0$, So, $A_z = \frac{C_1}{r}$
that is, at points removed from source, the time varying & static solution of (8) & (9) differs only by e^{-jkr}
In the presence of source $J_z \neq 0$ & $k = 0$, $\nabla^2 A_z \neq -\mu J \longrightarrow (10)$
This eq^n is recognised as poisson's eq^n which resembles $\nabla^2 \phi = \frac{-\rho}{\varepsilon}$ where ϕ : Scalar electric potential
ρ : Charge density

whose sol^n is $\phi = \frac{1}{4\pi\varepsilon} \iiint_V \frac{\rho}{r} dv'$

where r is the distance from any point on charge density to any observation point
therefore A_z as a solution for (10) can be written as

So, $A_z = \frac{\mu}{4\pi} \iiint_V \frac{J_z}{r} \cdot dv'$

Therefore for time varying, $A_z = \frac{\mu}{4\pi} \iiint_V J_z \left(\frac{e^{-jkr}}{r}\right) dv'$

Similarly $A_x = \frac{\mu}{4\pi} \iiint_V J_x \left(\frac{e^{-jkr}}{r}\right) dv'$

$$A_y = \frac{\mu}{4\pi} \iiint_V J_y \left(\frac{e^{-jkr}}{r}\right) dv'$$

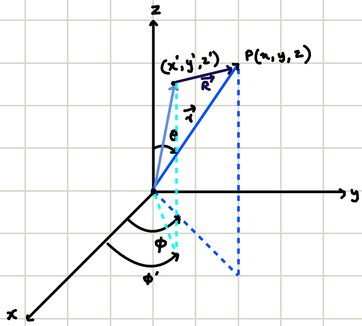
So the final solution $A = \frac{\mu}{4\pi} \iiint_V J \left(\frac{e^{-jkr}}{r}\right) dv'$

Source not at origin

→ prime (x', y', z')

$$A = \frac{\mu}{4\pi}\iiint_v J \cdot \frac{e^{-jkr}}{r} \cdot dv'$$

$$= \frac{\mu}{4\pi}\iiint_v J(x', y', z') \cdot \frac{e^{-jkr}}{r} \cdot dv'$$



If the source is removed from origin & placed at a position represented by prime where prime coordinates represent the source while unprime coordinates represent the observational point

R is the distance from any point from source to observation point

Similarly $F(x, y, z) = \frac{\varepsilon}{4\pi}\iiint_v M(x', y', z') \cdot \frac{e^{-jkR}}{R} \cdot dv'$

If J & M are linear densities, then $A = \frac{\mu}{4\pi}\iint J_s \cdot \frac{e^{-jkR}}{R}\, ds'$

$$F = \frac{\varepsilon}{4\pi}\iint_s M_s \cdot \frac{e^{-jkR}}{R} \cdot ds'$$

For electric & magnetic currents I_E & I_M eq reduce to line integral

$$A = \frac{\mu}{4\pi}\int I_e \cdot \frac{e^{-jkR}}{R} \cdot dl'$$

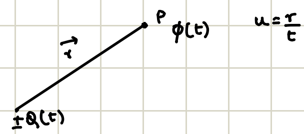
$$F = \frac{\mu}{4\pi}\int I_m \cdot \frac{e^{-jkR}}{R} \cdot dl'$$

Retarded Potential

→ For the electromagnetic field generated by time varying electric current / charge distribution in past

→ Consider charge Q of varying magnitude wrt t in a free space considering P.D b/w point P & location of Q

→ Let distance of seperation be r, then $\vec{v} = \frac{\vec{r}}{t}$.



In free space, $\mu_0 = 4\pi \times 10^{-7}$ H/m

$\varepsilon_0 = 8.85 \times 10^{-12}$ F/m

Here P.D $\phi(t)$ is a function of t & effect of charge $Q(t)$ will be experienced at point t after a delay $\Delta t = r/v$. The field at the field points lag their sources in an amount of propagation delay time Δt. There lagging field are called as retarded fields & time varying potentials produced by those retarded fields are called as retarded potentials.

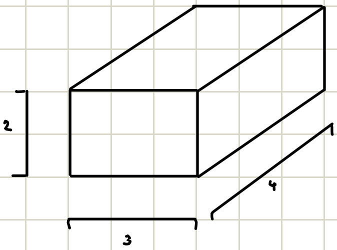
$$\phi = \pm\int \frac{\rho_v\, dv}{4\pi\varepsilon_0 r} \qquad \phi(t) = \frac{\rho_v\left(t - \frac{r}{v}\right) dv}{4\pi\varepsilon_0 r}$$

Q. Calculate resonant frequency of rectangular cavity resonator of dimensions $3 \times 2 \times 4$ cm when filled with air & mode of operation is TE_{101}

A. $d = 4cm$

$a = 3cm$

$b = 2cm$



& since TE_{101}, $m=1$ $n=0$ $p=1$

$$f_r = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{d}\right)^2}$$

$$= \frac{3\times10^8\pi}{2\pi}\sqrt{\left(\frac{1}{3\times10^{-2}}\right)^2+\left(\frac{1}{4\times10^{-2}}\right)^2}$$

$$= 6.25 \text{ GHz}$$

Q. A rectangular metal waveguide filled with dielectric material of relative permittivity $\varepsilon_r = 4$ has inside dimensions 3×1.2 cm Calculate cutoff freq for dominant mode

A. Dominant mode $\Rightarrow TE_{10}$ (2D cuz rectangular metal waveguide)

$a = 3$ cm $b = 1.2$ cm

$$f_c = \frac{c}{\sqrt{\varepsilon_r}\cdot 2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

$$= \frac{3\times10^8}{\sqrt{4}\times 2}\sqrt{\left(\frac{1}{3\times10^{-2}}\right)^{\cancel{2}}} = 2.5\times10^9 \text{ Hz}$$

$$v = \frac{c}{\sqrt{\varepsilon_r}} = 1.5\times10^8 \text{ ms}^{-1}$$

Q. For the dominant mode propagation along +ve z direction in air filled rectangular waveguide, the longitudinal magnetic field $H_z = \cos(100x)\, e^{-j\beta z}$. Determine dimensions of waveguide if the ratio $a:b = 2:1$

A. $$H_z = H_{0z}\cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$$

$$= \cos(100x)\, e^{-j\beta z}$$

So, $H_{0z} = 1$, $\frac{m\pi}{a} = 100$

$TE_{10} \Rightarrow m=1,\ n=0$

So, $a = \frac{\pi}{100}$ Then $\frac{a}{b} = \frac{2}{1} \Rightarrow b = \frac{a}{2} = \frac{\pi}{200}$

Q. The rectangular waveguide with dimensions $a = 1.07$ cm $b = 0.43$cm filled with certain dielectric exhibits $f_c = 31.03$ GHz for TM_{21} mode. Find the relative dielectric constant.

A. $$f_c = \frac{c}{\sqrt{\varepsilon_r}\cdot 2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

$$31.03\times10^9 = \frac{3\times10^8}{\sqrt{\varepsilon_r}\times 2}\sqrt{\left(\frac{2}{1.07\times10^{-2}}\right)^2+\left(\frac{1}{0.43\times10^{-2}}\right)^2}$$

$$\varepsilon_r = \left(\frac{3\times10^8}{31.03\times10^9\times2}\right)^2\times\left(\left(\frac{2}{1.07\times10^{-2}}\right)^2+\left(\frac{1}{0.43\times10^{-2}}\right)^2\right)$$

$$= 2.08$$

Q. The dimensions of waveguide are 2.5×1 cm & frequency is 8.6 GHz, Find the following

i) Possible modes of operation ii) Cutoff freq. iii) Wavelength

A. i) Condition for propagation $\Rightarrow \lambda_c > \lambda_0$ (or) $f_c < f_0$

Here $\lambda_0 = \frac{c}{f_0} = \frac{3\times10^8}{8.6\times10^9} = 0.034$ m

$TE_{01} \Rightarrow m=0,\ n=1$ $\lambda_c = \frac{2ab}{\sqrt{m^2b^2+n^2a^2}} = \frac{5\times10^{-4}}{2.5\times10^{-2}} = 2\times10^{-2} < \lambda_0$ ✗

$TE_{10} \Rightarrow m=1,\ n=0$ $\lambda_c = \frac{5\times10^{-4}}{1\times10^{-2}} = 5\times10^{-2}$ cm $> \lambda_0$ ✓

$TE_{11} \Rightarrow m=1,\ n=1$ $\lambda_c = \frac{5\times10^{-2}}{\sqrt{6.25+1}} = 1.85\times10^{-2}$ cm $< \lambda_0$ ✗

ii) $f_c \text{ (10 Mode)} = \frac{c}{\lambda_c} = \frac{3\times10^8}{5\times10^{-2}} = 6\times10^9 \text{ Hz} = 6 \text{ GHz}$

iii) $\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}} = \frac{0.034}{\sqrt{1-\left(\frac{0.034}{0.05}\right)^2}} = 4.637 \text{ cm}$

Q. An air filled rectangular cavity resonator is operating in dominant mode. Determine resonant freq. of resonator if the field inside the resonator is $H_z = \cos(20\pi x) \cdot \sin(5\pi z)$. Now if the resonator is filled with dielectric $\varepsilon_r = 4$, determine new resonant freq.

A. $H_z = H_{0z} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$

$= \cos(20\pi x) \cdot \sin(5\pi z)$

$m=1, \; n=0, \; p=1$

$\frac{m\pi}{a} = 20\pi \Rightarrow a = \frac{1}{20} = 0.05\text{m} = 5\text{cm}$

$\frac{p\pi}{d} = 5\pi \Rightarrow d = \frac{1}{5}$

$f_r = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2} = \frac{3\times10^8}{2}\sqrt{\left(\frac{1}{0.05}\right)^2+\left(\frac{1}{1/5}\right)^2} = 3.1 \text{ GHz}$

At ε_r, $f_r = \frac{f_c}{\sqrt{\varepsilon_r}} = \frac{3.1}{\sqrt{4}} = 1.55 \text{ GHz}$

Q. For a dominant mode propagation, in an air filled rectangular waveguide, the longitudinal magnetic wavefield is given by $H_z = \cos\left(\frac{100\pi}{3}x\right) \cdot e^{-j\beta z}$. If operating freq. of wave is $f = 8.333$ GHz, determine phase constant β

A. $H_z = H_{0z} \cos\left(\frac{m\pi}{a}\right)x \cdot \cos\left(\frac{n\pi}{b}\right)y \cdot e^{-j\beta z}$

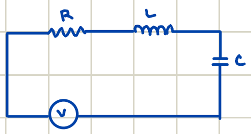
$TE_{10} \Rightarrow m=1, \; n=0$

$\frac{m\pi}{a} = \frac{100\pi}{3} \Rightarrow a = \frac{3}{100}$

$f_c = \frac{c}{2a} = \frac{3\times10^8}{2\times\frac{3}{100}} = 5\times10^9 = 5\text{GHz}$

$\beta = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi f}{c}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 139.54 \text{ rad/m}$

Q. The series RLC circuit has $R = 100\Omega$, $L = 0.5\text{H}$, $C = 0.4\mu\text{F}$. Find resonant freq, half power freq & bandwidth



A. $f_r = \frac{1}{2\pi\sqrt{LC}} = 355.88 \text{ Hz}$

$f_1 = f_r - \frac{R}{4\pi L} = 339.96 \text{ Hz}$

$f_2 = f_r + \frac{R}{4\pi L} = 371.79 \text{ Hz}$

$BW = f_2 - f_1 = 31.83 \text{ Hz}$