

Dimension	Geometric Description	No. of Solutions	Matrix Condition
1) 2D Row picture	Lines parallel & no intersection	No solution	$ A = 0$
2) 2D Row picture	Lines coincident (lines lie on top of each other)	∞	$ A = 0$
3) 3D Row picture	Each pair intersect in a line but no common intersection	No solution	$ A = 0$
4) 3D Row picture	2 planes intersect along a line, 3rd plane \parallel to line	No solution	$ A = 0$
5) 3D Row picture	All 3 planes \parallel , no intersection	No solution	$ A = 0$
6) 3D Row picture	All 3 planes overlap perfectly	∞	$ A = 0$
7) 3D Row picture	All 3 planes intersect along single line	∞	$ A = 0$
8) 3D Row picture	2 planes are \parallel , 3rd intersects them	No solution	$ A = 0$
9) 3D Column picture	\vec{b} is not in the plane formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$	No solution	$ A = 0$
10) 3D Column picture	\vec{b} is in the plane formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$	∞	$ A = 0$

Matrix

Consistency

$$\begin{aligned} \text{rank}(A) = \text{rank}(A:b) &< n \Rightarrow \infty \text{ sol}^n \\ &= n \Rightarrow \text{unique sol}^n \\ &\neq \Rightarrow \text{No sol}^n \end{aligned}$$

Breakdown of Elim^n

Non-singular & curable $\Rightarrow |A| \neq 0$

Singular & non-curable $\Rightarrow |A| = 0$

Singular $\Rightarrow |A| = 0$

Triangular Factors

LU

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -E_{21} & 1 & 0 \\ -E_{31} - E_{32} & 1 \end{bmatrix} \quad U = \text{Gaussian elim}^n$$

LDU

$$A = LDU$$

$$L = \text{same } L$$

$$D = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix} \quad U = \begin{bmatrix} 1 & \frac{U_{12}}{P_1} & \frac{U_{13}}{P_1} \\ 0 & 1 & \frac{U_{23}}{P_2} \\ 0 & 0 & 1 \end{bmatrix}$$

Permutation Matrix

Row exchanges $\Rightarrow PA = LDU$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

P₁₃

P₃₁

P₂₃

P₃₂

Inverses

$$(ABCD)^{-1} = D^{-1} C^{-1} B^{-1} A^{-1}$$

$$Ax = b \Rightarrow x = A^{-1}b$$

Gauss-Jordan

$$[A : I] \rightarrow [U : c] \rightarrow [I : A^{-1}]$$

Transpose

$$\begin{aligned} A &\rightarrow A^T \\ (A^T)^T &= A \\ (AB)^T &= B^T A^T \end{aligned} \quad \left| \begin{array}{l} (A^{-1})^T = (A^T)^{-1} \\ (A \pm B)^T = A^T \pm B^T \end{array} \right| \quad \left| \begin{array}{l} (A^{-1})^T A^T = (BA^{-1})^T = I \\ (A^T)^T A^T = (AA^{-1})^T = I \end{array} \right.$$

Symmetric

$$\begin{aligned} A^T &= A \\ (\bar{A}^T)^T &= A^{-1} \end{aligned}$$

$$\text{and } A = A^T = LDL^T$$

Vector space \Rightarrow If $v, w \in V \Rightarrow v+w \in V \Rightarrow v$ closed under vector addition
If $c \in \mathbb{R}, v \in V \Rightarrow cv \in V \Rightarrow v$ closed under scalar multiplication

Subspace: Non-empty subset of vector space

$$\rightarrow \text{Commutative: } v+w = w+v$$

$$\text{Associative: } v+(w+u) = (v+w)+u$$

$$r(sv) = s(rv)$$

$$\text{Additive: } 0+v = v+0 = v$$

$$\text{Inverse: } v+(-v) = (-v)+v = 0$$

$$\text{Distributive: } c_1(v+w) = c_1v + c_1w$$

$$(c_1+c_2)v = c_1v + c_2v$$

$$\text{Multiplicative: } 1.v = v$$

$$Rx = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot v Free v.

Unit-2

Linear Independence

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

Only if $c_1 = c_2 = c_3 = 0 \Rightarrow$ linearly independent $\Rightarrow \rho(A) = \text{no. of columns}$

else dep. $\Rightarrow \rho(A) < \text{no. of col}^n$

Basis: Subset $S = \{v_1, v_2, \dots, v_n\}$ of vector space if S is linearly indep, S spans vector space V

\hookrightarrow Maximal independent set & Min. spanning set

$$\mathbb{R}^2: \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad \mathbb{R}^3: \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{Z}^{n \times 2}: \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

n-degree polynomial: $\{1, t, t^2, \dots, t^n\}$

4 Fundamental Subspaces

$$1) \text{ Column Space} \Rightarrow \rho(A) = k = \dim(C(A))$$

\hookrightarrow Columns having pivot in echelon form of A

$$2) \text{ Row Space} \Rightarrow \rho(A) = k = \dim(C(A^T))$$

\hookrightarrow Row vectors in A corresponding to pivots in echelon form

$$3) \text{ Null Space} \Rightarrow n-k = \dim(N(A))$$

\hookrightarrow All sol's of $Ax = 0$

$$4) \text{ Left Null Space} \Rightarrow m-k = \dim(N(A^T))$$

\hookrightarrow All sol's of $A^T y = 0$

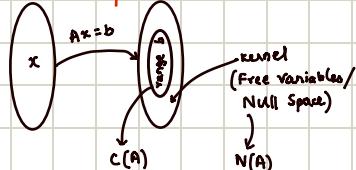
Rectangular Matrix Inverses

For $m \times n$ matrix

$$a) \rho(A) = m, \text{ Right inverse} = A^T(AA^T)^{-1}$$

$$b) \rho(A) = n, \text{ Left inverse} = (A^T A)^{-1} A^T$$

Linear Transformations



General Matrix to rotate

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Polynomial Space

$$\hookrightarrow P_n = c_0 + c_1t + c_2t^2 + \dots$$

$$\text{Basis: } (1 + t + t^2 + \dots + t^n)$$

$$\text{Dimension: } n+1$$

$$\text{Differentiation} \Rightarrow P_3 = 1 + t + t^2 + t^3$$

$$P_2 = 1 + t + t^2$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad 3 \times 4$$

$$\text{Integ} \Rightarrow P_2 = 1 + t + t^2$$

$$P_3 = 1 + t + t^2 + t^3$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad 4 \times 3$$

Rank-Nullity Theorem

\hookrightarrow Aman

$$\dim(C(A)) + \dim(N(A)) = r + (n-r) = n$$

$$\dim(C(A^T)) + \dim(N(A^T)) = r + (m-r) = m$$

Rank 1 Matrices

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad \rightarrow v^T$$

$$A = u \times v^T$$

Rotation

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad Q_\theta Q_\theta = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$Q_\theta Q_\phi = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

Projection

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = P \cdot P = P \cdot P \cdot P = P^3 \quad \hookrightarrow \text{symmetric}$$

Reflection

$$R = 2P - I = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H \cdot H = H^{2n} = I$$