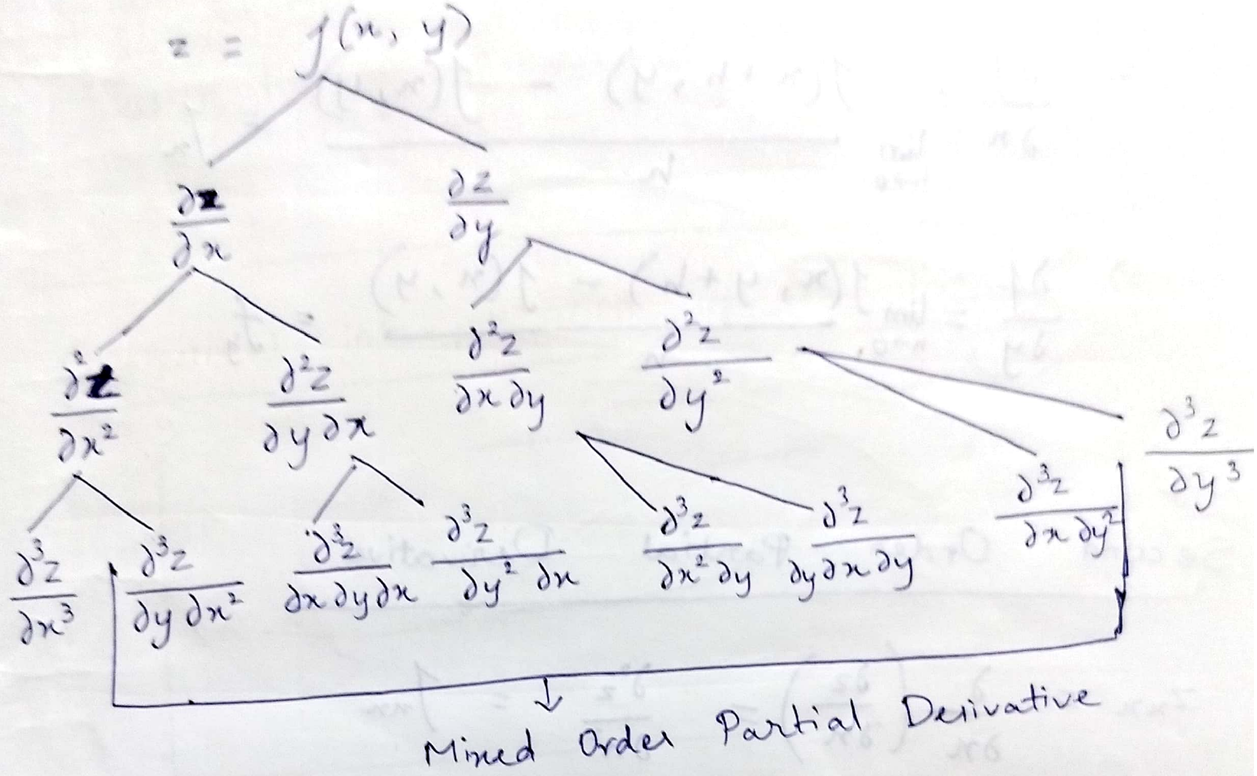
Partial Differentiation

→ Derivative wrt one of the variables, keeping the others held constant



→ Usually in 2nd order $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

↳ Clairut's Theorem

Q. $u = x^2y + x^3yz + xz$

$$\left(\frac{\partial u}{\partial x}\right) = 2xy + 3x^2yz + z$$

$$\left(\frac{\partial u}{\partial y}\right) = x^2 + x^3z + 0$$

$$\left(\frac{\partial u}{\partial z}\right) = 0 + x^3y + x$$

First Order Partial Derivatives

→ $z = f(x, y)$ is a function of 2 variables x & y.

$$\Rightarrow \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

$$\Rightarrow \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = f_y$$

Second Order Partial Derivative

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$$

$$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

Mixed derivative ⇒

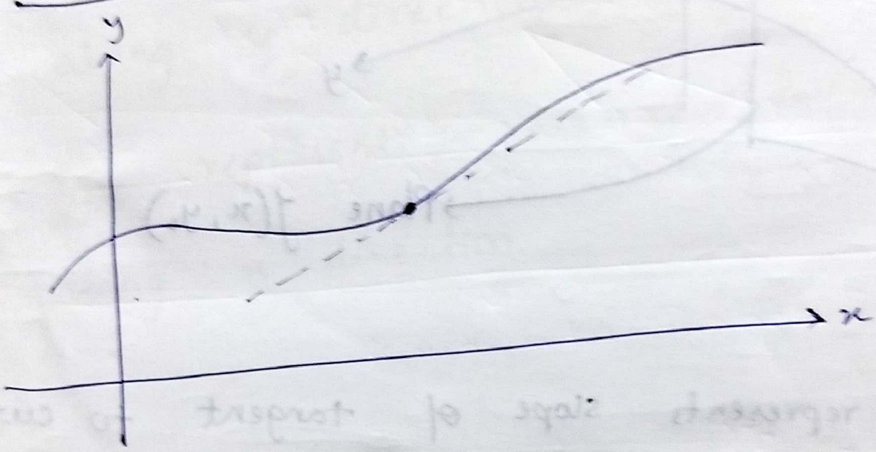
$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} = f_{xy}$$

$$z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y} = f_{yx}$$

→ A function of 2 variables has 2 first order derivatives, 4 second order derivatives and $\boxed{2^n \text{ of } n\text{th}}$ order derivatives

A function of m independant variables will have m^n derivatives of order n

Geometrical Interpretation of derivative



$\frac{dy}{dx} \Rightarrow$ slope of curve

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ minima on graph

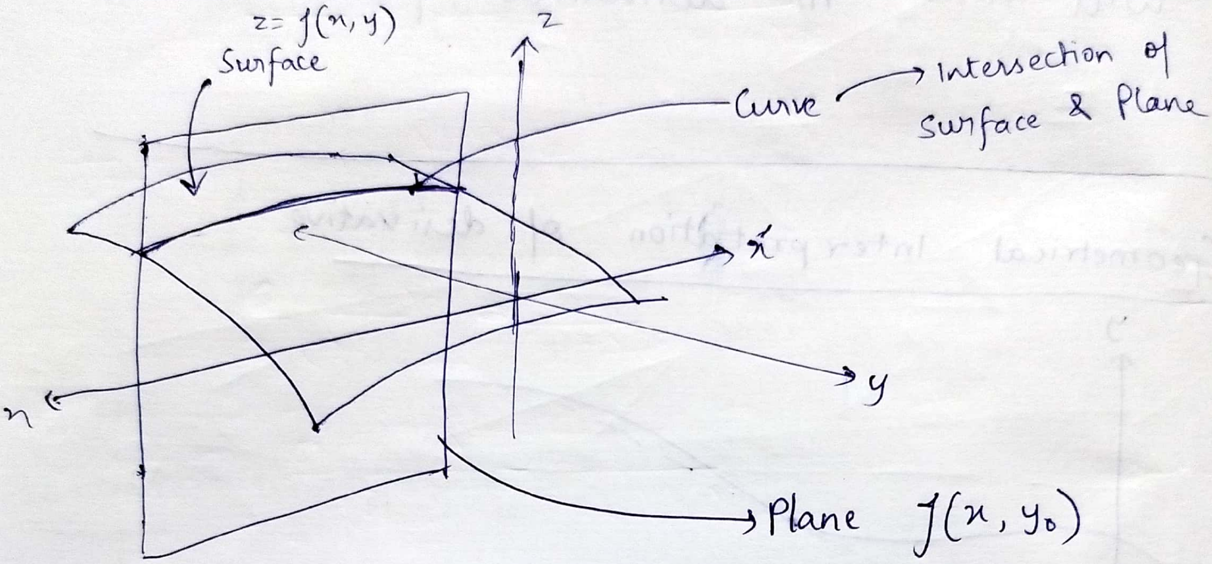
$\frac{d^2y}{dx^2} < 0 \Rightarrow$ maxima on graph

→ Gives nature of function

$y = f(x) \rightarrow$ Curve

$z = f(x, y) \rightarrow$ Surface

Geometrical Interpretation of Partial Derivative



1) $\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}$ represents slope of tangent to curve formed by intersection of plane $y = y_0$ & surface at any point (x_0, y_0)

2) $\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}$ represents slope of tangent to the curve formed by intersection of plane $x = x_0$ & surface at any point (x_0, y_0)

→ If $f_{xx} > 0$, then $f(x, y)$ is concave up in x direction

→ If $f_{yy} > 0$, then $f(x, y)$ is concave up in y direction

→ f_{xy} represents rate of change of f_x along y direction.

→ f_{yx} represents rate of change of f_y along x direction

Q. $f(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1,2)$

$$\frac{\partial f}{\partial x} = \frac{-1}{x^2} \times \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} =$$

$$= \frac{yx}{y-x} \times \frac{-1}{x^2} = \frac{-y}{(y-x)x} = -y\left[\frac{1}{x} \times \frac{1}{y-x}\right]$$

$$\frac{\partial^2 f}{\partial x^2} = -y\left[\frac{1}{x}\left(\frac{1}{y-x}\right)^2(-1)\right] + \frac{1}{y-x}\left(\frac{-1}{x^2}\right)$$

$$= \frac{y}{x}\left(\frac{1}{y-x}\right)^2 - \frac{1}{(y-x)x^2}$$

$$\frac{y}{x} \Rightarrow y \times \frac{1}{2}$$

$$(f_{xx})_{(1,2)} = 2\left[\frac{2}{1}\left(\frac{1}{2-1}\right)^2 - \frac{1}{(2-1)(1)^2}\right]$$

$$= 2\left[1 - 1\right] = 0$$

$$f_x = \frac{-y}{(y-x)x} = \frac{1}{x}\left[-y \times \frac{1}{y-x}\right]$$

$$f_{xy} = \frac{1}{x}\left[-1 \times \frac{1}{y-x} + (+y)\frac{1}{(y-x)^2}\right]$$

$$= \frac{1}{x}\left(\frac{y}{(y-x)^2} - \frac{1}{y-x}\right)$$

$$(f_{xy})_{(1,2)} = \frac{1}{1}\left(\frac{2}{1} - \frac{1}{2-1}\right) = 1\left[2 - 1\right] = 1$$

$$f(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$$

$$f_y = \frac{1}{\frac{1}{x} - \frac{1}{y}} \times \left(\frac{1}{y^2}\right) = \frac{xy}{y-x} \times \frac{1}{y^2} = \frac{x}{(y-x)y}$$

$$f_{yy} = x \times \left[\frac{1}{y-x} \times \frac{1}{y}\right]$$

$$f_{yy} = x\left[\frac{1}{y} \times \frac{-1}{(y-x)^2} + \frac{-1}{y^2} \times \frac{1}{y-x}\right]$$

$$(f_{yy})_{(1,2)} = 1\left[\frac{1}{2} \times \frac{-1}{1} + \frac{-1}{4} \times \frac{1}{1}\right]$$

$$= 1\left[\frac{-1}{2} - \frac{1}{4}\right] = -3/4$$

Q. $f(x,y,z) = e^{x^2+y^2+z^2}$ at $(-1, 1, -1)$

$\boxed{\text{Symmetric function}}$

$$\frac{\partial f}{\partial x} = 2x \times e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^{x^2+y^2+z^2} + 4x^2 e^{x^2+y^2+z^2}$$

$$= 2e^3 + 4e^3 = 6e^3$$

$$\frac{\partial f}{\partial y} = 2y \times e^{x^2+y^2+z^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2e^{x^2+y^2+z^2} + 4y^2 e^{x^2+y^2+z^2}$$

$$= 2e^3 + 4e^3 = 6e^3$$

$$\frac{\partial^2 f}{\partial z^2} = 6e^3$$

Q. $f(x, y) = e^x \log y + \cos y \log x$ at $\left(1, \frac{\pi}{2}\right)$

$$f_x = \log y \cdot e^x + \frac{\cos y}{x}$$

$$f_{xx} = \log y \cdot e^x - \frac{\cos y}{x^2}$$

$$f_{xxx} = \log y \cdot e^x + \frac{\cos y}{x^3}$$

$$\boxed{f_{xxx} = \log \frac{\pi}{2} \times e} + 0 =$$

$$\boxed{f_{xxy} = \frac{e^x}{y} + \frac{\sin y}{x^2} = \frac{2e}{\pi}} + 1$$

$$f_{xy} = \frac{e^x}{y} - \frac{\sin y}{x}$$

$$\boxed{f_{xyy} = \frac{-e^x}{y^2} - \frac{\cos y}{x} = \frac{-4e}{\pi^2}}$$

$$f_y = \frac{e^x}{y} - \sin y \log x$$

$$f_{yy} = \frac{-e^x}{y^2} - \cos y \log x$$

$$\boxed{f_{yyy} = \frac{+e^x}{y^3} + \sin y \log x = \frac{16e}{\pi^3}}$$

Find

Q. $v = r^n(3\cos^2\theta - 1)$ satisfies

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - 1 = -\sin^2\theta$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0$$

$$\frac{\partial}{\partial r}\left(n r^{n+1}(3\cos^2\theta - 1)\right) + \frac{1}{\sin\theta} \times \frac{\partial}{\partial\theta}\left(-\sin\theta \times r^n(6\cos\theta\sin\theta)\right)$$

$$\Rightarrow n(n+1)r^n(3\cos^2\theta - 1) + \frac{1}{\sin\theta} \times \left(\frac{\partial}{\partial\theta}\left(r^n \times -6\cos\theta\sin^2\theta\right)\right)$$

$$\Rightarrow n(n+1)r^n(3\cos^2\theta - 1) - \frac{6r^n}{\sin\theta}\left(-\sin^3\theta + 2\cos^2\theta\sin\theta\right)$$

$$\Rightarrow n(n+1)r^n(3\cos^2\theta - 1) - 6r^n(-\sin^2\theta + 2\cos^2\theta)$$

$$\Rightarrow n(n+1)r^n(3\cos^2\theta - 1) - 6r^n\left(\cancel{\cos^2\theta - 1 + 2\cos^2\theta}\right) = 0$$

$$3\cos^2\theta - 1$$

$$\left(n(n+1)r^n - 6r^n\right) = 0$$

$$n(n+1) = 6$$

$$n = 2, -3$$

Q. $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

Then show $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{x^2}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} + \frac{y^2}{1+\left(\frac{x}{y}\right)^2} \times \frac{x}{y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= x\left(\cancel{\frac{x^2+y^2}{x^2+y^2}}\right) - 2y\tan^{-1}\left(\frac{x}{y}\right) = x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{2y}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}}$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2}{1+\left(\frac{y}{x}\right)^2} \times \frac{y}{x^2} - \frac{y^2}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y\left(\cancel{\frac{x^2+y^2}{x^2+y^2}}\right) = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{2x}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} - 1 = \frac{2x^2}{x^2+y^2} - 1$$

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}}$$

Q. $x^x y^y z^z = c$ where $x = y = z$

Then show, $\frac{\partial^2 z}{\partial x \partial y} = -[x\log(ex)]^{-1}$

$$\log(x^x y^y z^z) = \log c$$

$$x\log x + y\log y + z\log z = \log c$$

$$1\cdot\cancel{\frac{x}{x}} + \log x + 0 + \frac{1}{z}\times z\,\frac{\partial z}{\partial x} + \log z \times \frac{\partial z}{\partial x} = 0$$

$$\log x + 1 + \frac{\partial z}{\partial x}(1 + \log z) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{1+\log y}{1+\log z}\right)$$

$$= -(1+\log y)\left(\frac{-1}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x}\right)$$

$$= -\left(\cancel{\frac{1+\log y}{\log ex}}\right)\left(\frac{-1}{(\log ex)^2}\times\frac{1}{x}\times\frac{-\cancel{1+\log x}}{\cancel{1+\log z}}\right)$$

$$= -[x\log ex]^{-1}$$

→ If $z = f(x,y)$, then,

$$\boxed{dz = \frac{\partial f}{\partial x} \times dx + \frac{\partial f}{\partial y} \times dy}$$

Q. $z = x^4 e^{3y}$

$$dz = (4x^3 e^{3y})\,dx + (3 \cdot x^4 e^{3y})\,dy$$

Q. $u = xy$

$$du = y.dx + xdy$$

Q. $u = \frac{x}{y}$

$$du = \cancel{\frac{xdy - y.dx}{y^2}} \quad \frac{ydx - xdy}{y^2}$$

Differentiation of Composite functions

1) Total Derivative

2) Chain Rule

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}}$$

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Q. $u = x^2 - y^2$ $\quad x = e^t \cos t \quad y = e^t \sin t \quad$ at $t = 0$

find $\frac{du}{dt}$

$$\frac{du}{dt} = 2x \times \frac{dx}{dt} - 2y \times \frac{dy}{dt}$$

$$= 2(e^t \cos t) \times [e^t \cos t - e^t \sin t] - 2(e^t \sin t) \times [e^t \sin t + e^t \cos t]$$

$$= 2e^{2t} (\underset{1}{\cancel{\cos^2 t}} - \underset{0}{\cancel{\cos t \sin t}} - \underset{0}{\cancel{\sin^2 t}} - \underset{0}{\cancel{\sin t \cos t}})$$

$$= 2e^{2t} = 2e^0 = 2$$

Q. $\frac{df}{dt}$ at $t = 0$

i) $f(x, y) = x\cos y + e^x \sin y$

$x = t^2 + 1, \quad y = t^3 + t$

$$\Rightarrow \frac{df}{dt} = (\cos y + e^x \sin y)\frac{dx}{dt} - (x \sin y + e^x \cos y)\frac{dy}{dt}$$

$$= \cancel{\cos(t^2+1) + e^{t^2+1} \sin(t^3+t)(2t)} \quad (x \sin y - e^x \cos y)(3t^2+1)$$

$$= 0$$

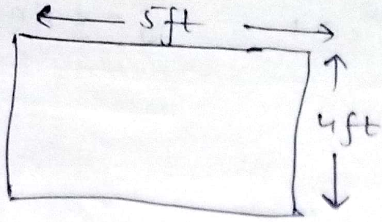
$[t^2+1 \sin(t^3+t) - e^{t^2+1} \cos(t^3+t)]$

$= (\sin(0) - e^1 \cos 0)$

$= e$

ii) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$

Q.



$$\frac{dx}{dt} = 1.5\ ft/sec \qquad \frac{dy}{dt} = 0.5\ ft/sec$$

$$u = xy$$

$$\cancel{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \partial}$$

$$\frac{du}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$= 5\times 0.5 + 4\times 1.5$$

$$= 2.5 + 6 = 8.5\ ft^2/sec$$

Q. $r = 10cm$ $\qquad h = 15cm$

$$\frac{dr}{dt} = -0.3\ cm/s \qquad \frac{dh}{dt} = 0.2\ cm/s$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= \frac{1}{3}\pi\left[2rh(-0.3) + r^2\cdot 0.2\right]$$

$$= \frac{\pi}{3}\left[2\times 150\times -0.3 + 100\times 0.2\right]$$

$$= \frac{\pi}{3}\left[-90 + 20\right] = \frac{-70\pi}{3}$$

Q. $z = \log(u^2+v)$ $\quad u = e^{x+y^2}$ $\quad v = x+y^2$

Find $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial x}$$

$$= \frac{2u}{u^2+v} \times e^{x+y^2} + \frac{1}{u^2+v} \times 1$$

$$= \frac{2u \times e^{x+y^2} + 1}{u^2+v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial y}$$

$$= \frac{2u}{u^2+v} \times 2y\, e^{x+y^2} + \frac{1}{u^2+v} + 2y$$

$$= 2y\left(\frac{2ue^{x+y^2}}{u^2+v} + \frac{1}{u^2+v}\right)$$

$$\Rightarrow 2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2y\left(\frac{2ue^{x+y^2}}{u^2+v} + \frac{1}{u^2+v}\right) - 2y\left(\frac{2ue^{x+y^2}}{u^2+v} + \frac{1}{u^2+1}\right) = 0$$

Q. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

$u = f(x, y)$; $x = r\cos\theta$; $y = r\sin\theta$

A. $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \times \cos\theta + \frac{\partial u}{\partial y} \times \sin\theta \longrightarrow (1)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \times (-r\sin\theta) + \frac{\partial u}{\partial y} \times r\cos\theta$$

$$\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) = -\frac{\partial u}{\partial x} \times \sin\theta + \frac{\partial u}{\partial y} \times \cos\theta \longrightarrow (2)$$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\cos\theta\right)^2 + \left(\frac{\partial u}{\partial y}\sin\theta\right)^2 + \cancel{2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}}\cdot\cos\theta\sin\theta$$

$$+ \left(\frac{\partial u}{\partial x}\sin\theta\right)^2 + \left(\frac{\partial u}{\partial y}\cos\theta\right)^2 - \cancel{2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}\cos\theta\sin\theta$$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

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Q. $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

To show, $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$

assume $p = \frac{y-x}{xy}$ & $q = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

$= \frac{1}{x} - \frac{1}{y}$

So, $f(p,q) = u$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \times \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \times \frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p} \times \left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial q} \times \left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \times \left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial q} \times 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \times 0 + \frac{\partial u}{\partial q} \times \frac{1}{z^2}$$

$\Rightarrow x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z}$

$$= \cancel{x^2}\left(\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}\right) \times \left(\frac{-1}{\cancel{x^2}}\right) + \frac{\partial u}{\partial p} \times \frac{\cancel{y^2}}{\cancel{y^2}} + \frac{\partial u}{\partial q} \times \frac{\cancel{z^2}}{\cancel{z^2}}$$

$$= -\cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial q}} = 0$$

Implicit Functions

$f(x, y) = c$

$f \to (x, y)$ & $(y \to x) \Rightarrow f \to x$

$$\cancel{\frac{df}{dx}}^{\to 0} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

because it is const.

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y}$$

→ Any function of type $f(x,y) = c$, is called implicit function where y is a function of x & c is a constant.

Q. $\frac{dy}{dx}$ at $(1, 1)$ for $e^y - e^x + xy = 1$

$$\frac{dy}{dx} = \frac{-(0 - e^x + y)}{e^y + x} = \frac{-(-e + 1)}{e + 1} = \frac{e - 1}{e + 1}$$

Q. $u = \tan(x^2+y^2)$ $\quad x^2 - y^2 = 2, \Rightarrow y^2 = x^2 - 2$

$$\cancel{\frac{du}{dx} = \frac{\sec^2(x^2+y^2) \times 2x}{\sec^2(x^2+y^2) \times 2y}}$$

$$y = \sqrt{x^2-2} = (x^2-2)^{+1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2-2)^{-1/2} \times 2x$$

$$= \frac{x}{\sqrt{x^2-2}} = \frac{x}{y}$$

$$u \to (x,y) \to (y \to x) \Rightarrow \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \times \frac{dy}{dx}$$

$$= \sec^2(x^2+y^2) \times 2x + \sec^2(x^2+y^2) \times 2y \times \frac{dy}{dx} \frac{x}{y}$$

$$= 4x \sec^2(x^2+y^2)$$

$$= 4x \sec^2(x^2+x^2-2) = \boxed{4x\sec^2(2x^2-2)}$$

Q. $u = x\log(xy)$; $x^3 + y^3 - 3xy - 1 = 0$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \times \frac{dy}{dx}$$

$$= \left(\frac{x}{xy} \times y + \log(xy)\right) + \frac{x}{xy} \times x \times \frac{dy}{dx}$$

$$\frac{du}{dx} = (1 + \log(xy)) + \frac{x}{y}\frac{dy}{dx}$$

$$x^3 + y^3 - 3xy - 1 = 0$$

$$\frac{dy}{dx} = \frac{3x^2 - 3y}{3y^2 - 3x} = -\left(\frac{x^2-y}{y^2-x}\right)$$

$$\boxed{\frac{du}{dx} = 1 + \log xy + \frac{x}{y}\left(\frac{-x^2+y}{y^2-x}\right)}$$

Q. $u = x^2 y$ $\qquad x^2 + xy + y^2 = 1$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 2xy + x^2 \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\left(\frac{2x + y}{x + 2y}\right)$$

$$= 2xy - x^2 \left(\frac{2x + y}{x + 2y}\right)$$

Q. $xe^y + ye^x = 0$

$$\frac{dy}{dx} = -\left(\frac{e^y + ye^x}{xe^y + e^x}\right)$$

$$\left(\frac{dy}{dx}\right)_{0,0} = -\left(\frac{1+0}{0+1}\right) = -1 \rightarrow m$$

eq of tangent $\Rightarrow$ ~~$y = -\left(\frac{e^y + ye^x}{xe^x + e^x}\right)$~~

at origin $\Rightarrow$ $y = -x$

Homogeneous functions & Euler's Theorem

→ Function is homogeneous function of degree n if it can be expressed in $x^n g\left(\frac{y}{x}\right)$ (or) $y^n g\left(\frac{x}{y}\right)$

where g is an arbitrary function.

ex: $u = 3x + 4y$

$$u = x^1\left(3 + 4\frac{y}{x}\right)$$

→ homogeneous function of degree 1.

ex: $u = x^2 y + xy^2$

$$= x^3\left(\frac{y}{x} + \frac{y^2}{x^2}\right)$$

↳ degree 3

ex: $u = x^3 y \tan^{-1}\left(\frac{x}{y}\right) + xy^3 \sec^{-1}\left(\frac{x}{y}\right)$

$$\cancel{x^4\left(\frac{y}{x}\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^3}{x^3}\sec^{-1}\left(\frac{x}{y}\right)\right)}$$

~~↳ degree 4~~

$$= y^4\left(\frac{x^3}{y^3}\tan^{-1}\left(\frac{x}{y}\right) + \frac{x}{y}\sec^{-1}\left(\frac{x}{y}\right)\right)$$

↳ degree 4

→ Also, $u = f(x, y, z)$ is homogeneous if it can be expressed in the form,

$$x^n g\left(\frac{y}{x}, \frac{z}{x}\right) \text{ (or) } y^n g\left(\frac{x}{y}, \frac{z}{y}\right) \text{ (or) } z^n g\left(\frac{x}{z}, \frac{y}{z}\right)$$

ex: $u = x^3 + y^3 + z^3 + 3xyz$

$$u = x^3\left(1 + \frac{y^3}{z^3} + \frac{z^3}{x^3} + 3 \times \frac{y}{x} \times \frac{z}{x}\right)$$

$$= y^3\left(\frac{x^3}{y^3} + 1 + \left(\frac{z}{y}\right)^3 + 3 \times \frac{x}{y} \times \frac{z}{y}\right)$$

$$= z^3\left(\left(\frac{x}{z}\right)^3 + \left(\frac{y}{z}\right)^3 + 1 + 3 \times \frac{x}{z} \times \frac{y}{z}\right)$$

↳ degree - 3

→ A function is homogeneous of degree n

if $f(xt, yt) = t^n f(x, y)$

ex: $3x + 4y$

$= 3xt + 4yt$

$u = t(3x + 4y)$

↳ degree 1

Euler's Theorem

Statement 1 : if $f(x,y)$ is homogeneous funᶜ of degree n,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Corollary 1 : If $u = f(x,y)$ is homo funcⁿ

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Statement 2 : $u = f(x,y,z)$ is homogeneous of degree n,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

→ **Deductions from Euler's Theorem**

1) if $z = f(u)$, is a homo eq of degree n then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

2) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,(g'(u) - 1)$$

where $g(u) = n\frac{f(u)}{f'(u)}$

Q. $u = \dfrac{y^{1/3} - x^{1/3}}{x+y}$

i) Find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

$$u = \frac{y^{1/3} - x^{1/3}}{x+y} = \frac{x^{1/3}\left(\frac{y^{1/3}}{x^{1/3}} - 1\right)}{x\left(\frac{y}{x}+1\right)} = x^{-2/3}\frac{\left(\left(\frac{y}{x}\right)^{1/3} - 1\right)}{\left(\frac{y}{x}+1\right)}$$

$$= x^{-2/3}\, g\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -\frac{2}{3}u$$

ii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-2}{3}\times\frac{-5}{3}u$$

$$= \frac{10}{9}u$$

Q. $u = \sqrt{y^2 - x^2}\,\sin^{-1}\left(\frac{x}{y}\right)$

$$= y\left(\sqrt{1-\left(\frac{x}{y}\right)^2}\right)\times\sin^{-1}\left(\frac{x}{y}\right)$$

$$= y\left(g\left(\frac{x}{y}\right)\right)$$

$\hookrightarrow n = 1$

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = nu = v$$

Q. $u = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{x}{y}\right)$

$= e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{1}{\frac{x}{y}}} \cos\left(\frac{x}{y}\right)$

$= y^0 \left[g\left(\frac{x}{y}\right)\right]$

$\hookrightarrow n = 0$

$x u_x + y u_y = 0$

Q. $\cancel{u = \tan^{-1}\left(\frac{x^4+y^4}{xy}\right)}$

$\cancel{= \tan^{-1}\left(\frac{x^4\left(1+\left(\frac{y}{x}\right)^4\right)}{x^2(y/x)}\right)}$

$\cancel{= \tan^{-1}\left(x^2\left(\frac{1+\left(\frac{y}{x}\right)^4}{\frac{y}{x}}\right)\right)}$

$\cancel{= \tan^{-1}\left(x^2 g\left(\frac{y}{x}\right)\right)}$

$u = \tan^{-1}\left(\frac{x^4+y^4}{xy}\right)$

$\tan u = \frac{x^4+y^4}{xy}$

$= \frac{x^{\cancel{4}\,2}\left(1+\left(\frac{y}{x}\right)^4\right)}{x^{\cancel{2}}\left(\frac{y}{x}\right)}$

$\tan u = x^2 \cdot g\left(\frac{y}{x}\right)$

$\hookrightarrow n = 2$

$\rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$

$= 2 \times \frac{\tan u}{\sec^2 u} = \frac{2 \times \cancel{\sin u}}{2 \times \cos u \times \sin u}$

$= \sin 2u$

$\rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\,(2\cos 2u - 1)$

$= \sin 2u\left[2(1-2\sin^2 u) - 1\right]$

$= \sin 2u\left[2 - 4\sin^2 u - 1\right]$

$= \sin 2u\left[1 - 4\sin^2 u\right]$

$$\left(\sqrt{x}+\sqrt{y}\right)$$

Prove $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{-\sin u \cos 2u}{4\cos^3 u}$

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$= \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \sqrt{x}\left(\frac{1+y/x}{1+\sqrt{y/x}}\right) = x^{1/2} g(x)$$

↳ Homo. eq of order $1/2$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)} = \cancel{\frac{1}{2}\frac{x+y}{\sqrt{x}+\sqrt{y}}}\ \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{\tan u}{2} = g(u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u).(g'(u)-1)$$

$$= \frac{\tan u}{2}\times\left(\frac{\sec^2 u}{2} - \cancel{2}\right)$$

$$= \cancel{\frac{\tan u.\sec^2 u - 2}{4}}$$

$$= \cancel{\frac{\frac{\sin u}{\cos u}\times\frac{1}{\cos^2 u} - 4}{4}}$$

$$= \cancel{\frac{\frac{\sin u}{\cos^3 u} - 4}{4}} = \cancel{\frac{\sin u - 4\cos u}{4\cos u}}$$

$$\Rightarrow \frac{\sec^2 u\tan u - 2\tan u}{4}$$

$$= \frac{\frac{\sin u}{\cos^3 u} - \frac{2\sin u}{\cos u}}{4}$$

$$= \frac{\sin u - 2\sin u\cos^2 u}{4\cos^3 u}$$

$$= \frac{\sin u\left(1-2\cos^2 u\right)}{4\cos^3 u}$$

$$\sin u\left(\sin^2 u + \cos^2 u - 2\cos^2 u\right) \quad \left(\sin^2 u - \cos^2 u\right) \quad \left(\cos 2u\right)$$

Q. $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \sqrt{x}\, g\left(\frac{y}{x}\right)$$

$\hookrightarrow$ degree $= 1/2$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{1}{2}u$$

$$\cancel{x\frac{\partial u}{\partial x} = x\,\sin u}$$

$$\cancel{y\frac{\partial u}{\partial y} =}$$

$$x\frac{\partial u}{\partial x} = \frac{x\sqrt{x}+2x\sqrt{y}-y\sqrt{x}}{2(\sqrt{x}+\sqrt{y})^2}$$

$$y\frac{\partial u}{\partial y} = \frac{2y\sqrt{x}+y\sqrt{y}-x\sqrt{y}}{2(\sqrt{x}+\sqrt{y})^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{u}{2}$$

Q. $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$

$$\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} = \frac{x\left(1+\frac{2y}{x}+\frac{3z}{x}\right)}{x^4\sqrt{1+\frac{y^8}{x^8}+\left(\frac{z}{x}\right)^8}}$$

$$= x^{-3}\, g\left(\frac{y}{x},\frac{z}{x}\right)$$

By applying Euler's Theorem,

$$xu_x + yu_y + zu_z = -3\times\frac{\sin u}{\cos u} = -3\tan u$$

$$xu_x + yu_y + zu_z + 3\tan u = 0$$

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Q. $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{y^3}\tan^{-1}\left(\frac{x}{y}\right)$

$$\cancel{= x^3 \operatorname{cosec}^{-1}\left(\frac{x}{y}\right) + \frac{1}{y^3}\tan^{-1}\left(\frac{x}{y}\right)}$$

$$\cancel{= y^3\left(\frac{x^3}{y^3}\operatorname{cosec}^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right)}$$

$$\cancel{u = y^3 g\left(\frac{x}{y}\right)}$$

$v = x^3 \sin^{-1}\left(\frac{y}{x}\right)$; $w = \frac{1}{y^3}\tan^{-1}\left(\frac{x}{y}\right)$

$\to n = 3$ $\qquad\to n = -3$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v \to (1) \qquad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -3w \to (2)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 6v \to (3) \qquad x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 12w \to (4)$$

$9v + 9w$

$$\Rightarrow x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial (v+w)}{\partial y}\right) + x^2\left(\frac{\partial^2 (v+w)}{\partial x^2}\right) + y^2\left(\frac{\partial^2 (v+w)}{\partial y^2}\right) + 2xy\left(\frac{\partial^2 (v+w)}{\partial x \partial y}\right) = 9v+$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) + x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) = 9u$$

Taylor's & Maclaurin's Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

For single variable, **Taylor Series**

$$\boxed{f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots}$$

Maclaurin's Series

$$f(x) = f(0) + (x) f'(0) + \frac{x^2}{2} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

Q. Find the taylor's series for e^x at $x=0$ upto 4th degree terms

A. $\cancel{e^x = f(0) + \frac{(x-0)}{1!} f'(0) + \frac{(x-0)^2}{2!} f''(0) + \frac{(x-0)^3}{3!} f'''(0) + \frac{1}{4} f''''(0)}$

$\cancel{= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}}$

$\cancel{= \frac{2 + 12 + 4 + 1}{24} = 2 + \frac{17}{24} = \frac{65}{24}}$

A. $e^x = f(0) + \frac{(x-0)}{1!} f'(0) + \frac{(x-0)^2}{2!} f''(0) + \frac{(x-0)^3}{3!} f'''(0) + \frac{(x-0)^4}{4!} f''''(0)$

$= e^0 + x e^0 + \frac{x^2 e^0}{2!} + \frac{x^3 e^0}{3!} + \frac{x^4 e^0}{4!}$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Q. Find taylor's series for $f(x) = \sin x$ at $x = \frac{5\pi}{6}$ upto 3rd degree

A. $$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

$$= \sin\frac{5\pi}{6} + \left(x - \frac{5\pi}{6}\right)\cos\frac{5\pi}{6} + \frac{\left(x-\frac{5\pi}{6}\right)^2}{2!}f''\left(\frac{5\pi}{6}\right) + \frac{\left(x-\frac{5\pi}{6}\right)^3}{3!}f'''\left(\frac{5\pi}{6}\right)$$

$$= \sin\frac{5\pi}{6} + \left(x - \frac{5\pi}{6}\right)\cos\frac{5\pi}{6} - \frac{\left(x-\frac{5\pi}{6}\right)^2}{2!}\sin\frac{5\pi}{6} - \frac{\left(x-\frac{5\pi}{6}\right)^3}{3!}\cos\frac{5\pi}{6}$$

$$\boxed{f(x) = +\frac{1}{2} - \left(x-\frac{5\pi}{6}\right)\frac{\sqrt{3}}{2} - \frac{\left(x-\frac{5\pi}{6}\right)^2}{4} + \frac{\left(x-\frac{5\pi}{6}\right)^3}{12} \times \sqrt{3}}$$

Taylor Series expansion for a function of 2 variables

$$\rightarrow f(x,y) = f(a,b) + \frac{1}{1!}\{(x-a)f_x(a,b) + (y-b)f_y(a,b)\}$$
$$+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)$$
$$+ (y-b)^2 f_{yy}(a,b)\}$$

Maclaurin's series expansion for a function of 2 variables

$$\rightarrow f(x,y) = f(0,0) + \frac{1}{1!}\{x f_x(0,0) + y f_y(0,0)\}$$
$$+ \frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\} + \cdots$$

Q. Expand $f(x,y) = \sin x \sin y$ upto 2nd order for $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

A. $$f(x,y) = f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \frac{1}{1!}\left\{\left(x - \frac{\pi}{4}\right) f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right\}$$

$$+ \frac{1}{2!}\left\{\left(x - \frac{\pi}{4}\right)^2 \cdot f_{xx}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) f_{xy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2 f_{yy}\left(\frac{\pi}{4}, \frac{\pi}{4}\right)\right\}$$

$$= \frac{1}{2} + \left\{\left(x - \frac{\pi}{4}\right) \cdot \frac{1}{2} + \left(y - \frac{\pi}{4}\right) \cdot \frac{1}{2}\right\} + \frac{1}{2}\left\{\left(x - \frac{\pi}{4}\right)^2 \cdot \frac{1}{2} + \left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2 \cdot \frac{1}{2}\right\}$$

$$= \frac{1}{2}\left\{1 + \left(x - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) - \frac{1}{2}\left(y - \frac{\pi}{4}\right)^2\right\}$$

Q. $f(x,y) = \sin(x + 2y)$, upto 3rd degree for $(0,0)$

A.
$$f(x,y) = f(0,0) + \frac{1}{1!}\left\{\cancel{x \cdot \cos(x+2y)} \cdot 1 + \cancel{y \cdot \cos(x+2y)} \cdot 2\right\}$$

$$+ \frac{1}{2!}\left\{x^2\left(-\cancel{\sin(x+2y)}\right) + 2xy \cdot \left(-\cancel{\sin(x+2y)}\right) \cdot 2 + y^2\left(-\cancel{\sin(x+2y)}\right) \cdot 4\right\}$$

$$+ \frac{1}{3!}\left\{x^3\left(-\cancel{\cos(x+2y)}\right) + 3x^2y\left(-2\cos(x+2y)\right) + 3xy^2\left(-4\cos(x+2y)\right) + y^3\left(-8\cos(x+2y)\right)\right\}$$

$$= 0 + \left\{x \cancel{\cos(x+2y)} + 2y \cancel{\cos(x+2y)}\right\}$$

$$+ \cancel{\frac{1}{2}\{0\}} + \frac{1}{3!}\left\{x^3 - 6x^2y - 12xy^2 - 8y^3\right\}$$

$$= \{x + 2y\} + \left\{\frac{x^3 - 6x^2y - 12xy^2 - 8y^3}{6}\right\}$$

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Q. $\sqrt{1+x+y^2}$ & $(x-1), (y-0)$ & 2nd degree

A. $f(x,y) = f(1,0) + \cancel{\frac{(x-1)(y-0)}{1!} f(1,0) + \frac{1}{2!}\{(x-1)^2 f(1,0)}$

$+\frac{1}{1!}\{(x-a) f_x(a,b) + (y-b) f_y(a,b)\}$

$+\frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,$

$\cancel{= \sqrt{2} + \left\{(x-1)\cdot\frac{1}{2}\times\frac{1}{\sqrt{1+1+0^2}} + y\cdot\frac{2}{2}\times\frac{1}{\sqrt{1+1+0}}\right\}}$

$\cancel{+\frac{1}{2}\left\{(x-1)^2\cdot\left(-\frac{1}{4}(2)^{-3/2}\right) + 2(x-1)y\cdot\left(\frac{2}{4}(2)^{-3/2}\right) + y^2\left(\frac{-1}{2}(2)^{-3}\right.\right.}$

$\frac{\partial}{\partial x}\{1+x+y^2\}^{1/2} = \frac{1}{2}(1+x+y^2)^{-1/2}$

$f_y(x,y) = \frac{1}{2}(1+x+y^2)^{-1/2}\times 2y$

$f_{xx}(x,y) = \frac{-1}{4}(1+x+y^2)$

$f_{xy}(x,y) = -\frac{2}{4}(1+x+y^2)^{-3/}$

$f_{yy}(x,y) = \frac{-1}{2}(1+x+y^2)^{-3/}$

$\cancel{\Rightarrow \sqrt{2} + \left\{\frac{x-1}{2\sqrt{2}} + \frac{y}{\sqrt{2}}\right\} + 2^{-5/2}\left\{-\frac{(x-1)^2}{4} - (x-1)y - \frac{y^2}{2}\right\}}$

$= \sqrt{2} + \left\{(x-1)\cdot\frac{1}{2}\frac{1}{\sqrt{1+1+0}} + y\cdot 0\right\}$

$+\frac{1}{2}\left\{(x-1)^2\cdot\left(\frac{-1}{4}\right)(2)^{-3/2} + 0\right\} + 0\}$

$= \sqrt{2} + \frac{(x-1)}{2\sqrt{2}} - 2^{-9/2}(x-1)^2$

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Q. $\sqrt{1+x+y^2}$ $x-1, y-0$ for 2 order

$$f(x,y) = f(a,b) + \frac{1}{1!}\{(x-a)f_x(a,b) + (y-b)f_y(a,b)\}$$
$$+ \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)$$
$$+ (y-b)^2 f_{yy}(a,b)\}$$

$$= \sqrt{1+1+0} + \{(x-1)\left(\frac{1}{2}\times(1+1+0)^{-1/2}\right) + 0\}$$

$$+\frac{1}{2}\{(x-1)^2\left(\frac{-1}{4}(1+1+0)^{-3/2}\right) + 2(x-1)y \times 0 + y^2\left(\frac{-2y}{4}\times(1+x+y^2)^{-3/2}\times 2y + \frac{1}{2}(1+x+y^2)^{-1/2}\right)\}$$

$$= \sqrt{2} + \frac{x-1}{2\sqrt{2}} + \frac{1}{2}\left((x-1)^2\left(-(2)^{-7/2}\right) + 0 + y^2\left(0 + (2^{-1/2})\right)\right)$$

$$= \sqrt{2} + \frac{x-1}{2\sqrt{2}} + \frac{1}{2}\left(-\frac{(x-1)^2}{2^{7/2}} + \frac{y^2}{\sqrt{2}}\right)$$

Q. $\sqrt{1+x+y^2}$ $\quad x=1,\ y=0$ $\quad$ 2nd order.

A. $f(x,y) = \sqrt{1+x+y^2} = (1+x+y^2)^{1/2} \Rightarrow \sqrt{2}$

$f_x(x,y) = \frac{1}{2}(1+x+y^2)^{-1/2} \Rightarrow \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)$

$f_y(x,y) = \frac{1}{2}(1+x+y^2)^{-1/2} \times 2y \Rightarrow 0$

$f_{xx}(x,y) = \frac{-1}{4}(1+x+y^2)^{-3/2} \Rightarrow -\frac{1}{4} \times \frac{1}{\sqrt{8}} = \frac{-1}{8\sqrt{2}}$

$f_{xy}(x,y) = \frac{-1}{4}(1+x+y^2)^{-3/2} \times 2y \Rightarrow 0$

$f_{yy}(x,y) = \frac{-1}{4}(1+x+y^2)^{-3/2} \times 4y^2 + \frac{1}{2}(1+x+y^2)^{-1/2} \times 2$

$= 0 + \frac{1}{\sqrt{2}}$

$\Rightarrow \sqrt{1+x+y^2} = \sqrt{2} + \frac{(x-1)}{2\sqrt{2}} + \frac{1}{2}\left\{(x-1)^2 \times \frac{-1}{8\sqrt{2}} + \frac{y^2}{\sqrt{2}}\right\}$

Q. Compute $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately $\quad (0.9 \to 1,\ 1.1 \to 1)$

A. $f(x,y) = f(1,1) + \frac{1}{1!}\left\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\right\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1)$
$+ 2(x-1)(y-1)f_{xy}$
$+ (y-1)^2 f_{yy}(1,1)$

$f(x,y) = \tan^{-1}\left(\frac{x}{y}\right) \Rightarrow \frac{\pi}{4}$

$f_x(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{1}{y} \Rightarrow \frac{y}{x^2+y^2} = \frac{1}{2}$

$f_y(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{-x}{y^2} = \frac{-x}{y^2+x^2} = \frac{-1}{2}$

$f_{xx} = \frac{-y}{(x^2+y^2)^2} \times 2x = \frac{-1}{2}$

$f_{xy} = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = 0$

$f_{yy} = \frac{+x}{(x^2+y^2)^2} \times 2y = \frac{1}{2}$

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$$\Rightarrow f(x,y) = \frac{\pi}{4} + \left\{\frac{(x-1)}{2} - \frac{y-1}{2}\right\} + \frac{1}{2}\left\{\frac{(x-1)^2}{2} + \frac{(y-1)^2}{2}\right\}$$

$$= \frac{\pi}{4} + \frac{x-y}{2} + \frac{(y-1)^2-(x-1)^2}{4}$$

$$f(0.9, 1.1) = \frac{\pi}{4} - \cdot 1 = 0.6858$$

Q. $f(x,y) = xy^2 + y\cos(x-y)$ upto 2nd order $(1,1)$

$\cancel{f(x,y)}$ $f(1,1) = 1 + 1\cos(0) = 2$

$f_x(x,y) = y^2 - y\sin(x-y) = 1$

$f_y(x,y) = 2xy + \cos(x-y) + y\sin(x-y) = 3$

$f_{xx}(x,y) = -y\cos(x-y) = -1$

$f_{xy}(x,y) = 2y - \sin(x-y) + y\cos(x-y) = 3$

$f_{yy}(x,y) = 2x + \sin(x-y) - y\cos(x-y) + \sin(x-y) = 1$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left\{(x-a)f_x(a,b) + (y-b)f_y(a,b)\right\} + \frac{1}{2!}\left\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right\}$$

$$= 2 + \left\{(x-1) + 3(y-1)\right\} + \frac{1}{2}\left\{-(x-a)^2 + 6(x-1)(y-1) + (y-1)^2\right\}.$$

$$2 + \left[(x-1) + 3(y-1)\right] + \frac{1}{2}\left[-(x-1)^2 + 6(x-1)(y-1) + (y-1)^2\right]$$

Maxima & Minima for a function of 2 variable

→ A function $f(x,y)$ is said to have relative maximum at a point (a,b) if there exist a neighborhood of the point (a,b) say $(a+h, b+k)$, h & k are small such that

$f(a,b) > f(a+h, b+k)$

→ If $f(a,b) < f(a+h, b+k)$ then $f(x,y)$ is said to have relative minimum at (a,b). Also $f(x,y)$ is said to be an extreme value of $f(a,b)$ if its either a max or a min

→ Necessary cond^n : $rt - s^2 > 0$

where $r = f_{xx}$ $t = f_{yy}$ $s = f_{xy} = f_{yx}$

↳ Hessian matrix

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

else, saddle point

but $rt - s^2 = 0$, then needs further investigation

→ $rt - s^2 > 0$, $r < 0$ at (a,b), $f(a,b)$ is maxima

→ $rt - s^2 > 0$, $r > 0$ at (a,b), $f(a,b)$ is minima

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Q. $f(x,y) = x^3 - 12x + y^3 + 3y^2 - 9y$

$f_x(x,y) = 3x^2 - 12$

$f_{xx}(x,y) = 6x = r$

$f_y(x,y) = 3y^2 + 6y - 9$

$f_{yy}(x,y) = 6y + 6 = t$

$f_{xy}(x,y) = 0 = s$

$\rightarrow \dfrac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 12 = 0$

$x = \pm 2$

$\rightarrow \dfrac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 + 6y - 9 = 0$

$y^2 + 2y - 3 = 0$

$y = 1, -3$

Stationary points are $\Rightarrow (2,1), (2,-3), (-2,1), (-2,-3)$

$rt - s^2 = \cancel{0 - (6y+6)^2} \quad 6x(6y+6) = 36xy + 36x$

$= 36x(y+1)$

(x,y)	$r = 6x$	$s = 0$	$t = 6y+6$	$rt - s^2$
minima $(2,1)$	$r > 0$	0		144 ✓ minima
$(2,-3)$	$r > 0$	0		-144 ✗ saddle point
$(-2,1)$	$r < 0$	0		-144 saddle point
$(-2,-3)$	$r < 0$	0		$+144$ ✓ maxima

$f(2,1) = \cancel{8} - 24 + \cancel{1} + 3 - \cancel{9} = -21 \longrightarrow$ Minima

$f(-2,-3) = -8 + 24 \; \cancel{27} + \cancel{27} + 27 = 43 \longrightarrow$ Maxima

Q. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x(x,y) = 3x^2 + 3y^2 - 30x + 72$

$f_{xx}(x,y) = 6x - 30$

$f_y(x,y) = 6xy - 30y$

$f_{yy}(x,y) = 6x - 30$

$f_{xy}(x,y) = 6y$

$rt - s^2 = (6x-30)^2 - 36y^2$

$f_x = 0 \Rightarrow x^2 + y^2 - 10x + 24 = 0 \Rightarrow$ $25 + y^2 - 50 + 24 =$

$y^2 = 1$

$y = \pm 1$

$f_y = 0 \Rightarrow 6x - 30 \Rightarrow x = 5$; $y = 0$

$x^2 - 10x + 24 = 0$

~~$25 + y^2 - 50 + 24 =$~~

$\frac{10 \pm \sqrt{100-96}}{2} = \frac{10 \pm 2}{2} = 6$ (or) 4

$(6,0)$, $(4,0)$, $(5,1)$, $(5,-1)$

	$r = 6x-30$	$s = 6y$	$t = 6x-30$	$rt - s^2$
6, 0	6	0	6	36 $\rightarrow$ minima
4, 0	-6	0	-6	36 $\rightarrow$ maxima
5, 1	0	6	0	-36 × $\rightarrow$ saddle
5, -1	0	-6	0	-36 × $\rightarrow$ saddle

Q. $f(x,y) = 4x^2 + 2y^2 + 4xy - 10x - 2y - 3$

$f_x(x,y) = 8x + 4y - 10$

$f_{xx}(x,y) = 8 = r$

$f_y(x,y) = 4y + 4x - 2$

$f_{yy}(x,y) = 4$

$f_{xy}(x,y) = 4$

$f_x = 0 \Rightarrow 8x + 4y = 10 \Rightarrow 8x - 10 = -4y$

$f_y = 0 \Rightarrow 4y + 4x = 2 \Rightarrow 4x - 2 = -4y$

$8x - 10 = 4x - 2$

$4x = 8 \Rightarrow \boxed{x = 2}$

$\Rightarrow 16 - 10 = -4y$

$\boxed{y = -6/4}$

	$r = 8$	s	t	$rt - s^2$
$(2, -6/4)$	8	4	4	16

Lagrange's Method of undetermined Multipliers

→ Suppose we need to maximize (or) minimize a function ~~f(x,y)~~ $f(x, y, z)$ subject to the constraint $\phi(x, y, z) = 0$, we introduce λ

1) $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) \longrightarrow (1)$

2) Partially diff. F in (1) wrt x, y, z respectively

3) $F_x = 0$, $F_y = 0$, $F_z = 0$ & constraint for lagrange multiplier λ & x, y, z stationary values

---

Q. $\underbrace{a^3x^2 + b^3y^2 + c^3z^2}_{f(x)}$ & $\underbrace{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1}_{\phi(x)}$ $\cancel{\frac{yz+xz+xy}{xyz}}$

where $a > 0$, $b > 0$, $c > 0$

Let $F(x,y,z) = f(x) + \lambda \phi(x)$

$$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

$$F_x(x,y,z) = 2a^3x - \frac{\lambda}{x^2} = 0 \Rightarrow 2a^3x^3 = \lambda$$

$$F_y(x,y,z) = 2b^3y - \frac{\lambda}{y^2} = 0 \Rightarrow 2b^3y^3 = \lambda$$

$$F_z(x,y,z) = 2c^3z - \frac{\lambda}{z^2} = 0 \Rightarrow 2c^3y^3 = \lambda$$

$ax = by = cz \Rightarrow y = \frac{ax}{b}$ & $z = \frac{ax}{c}$

$$\phi(x) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

$$\Rightarrow yz + xz + xy - xyz = 0$$

$\Rightarrow a^2x^2$ ...

Q. $x^3 + 8y^3 + 64z^3$; $xyz = 1$

A. $F(x) = x^3 + 8y^3 + 64z^3 + \lambda(xyz - 1)$

$F_x(x) = 3x^2 + \lambda yz = 0 \Rightarrow \lambda = \frac{-3x^2}{yz}$

$F_y(x) = 24y^2 + \lambda xz = 0 \Rightarrow \lambda = \frac{-24y^2}{xz}$

$F_z(x) = 192z^2 + \lambda xy = 0 \Rightarrow \lambda = \frac{-192z^2}{xy}$

$\frac{3x^2}{y\cancel{z}} = \frac{24y^2}{x\cancel{z}}$

$\cancel{3}x^3 = \frac{\cancel{24}}{8} y^3$

$\boxed{x = 2y}$

$\boxed{x = 4z}$

$\frac{24y^2}{\cancel{x}z} = \frac{192z^2}{\cancel{x}y}$

$\frac{\cancel{24}}{\cancel{12}}y^3 = \frac{\cancel{192}}{\cancel{96}\,8}z^3$

$y = 2z$

$xyz - 1 = 0$

$x \times \frac{x}{2} \times \frac{x}{4} = 1$

$x^3 = 8$

$\boxed{x = 2 \quad y = 1 \quad z = 1/2}$

$f(2, 1, \frac{1}{2}) = 8 + 8 + 8 = 24$

Q. Find max & min distance of point $(3, 4, 12)$ from unit sphere with centre as origin

A. sphere eq $\Rightarrow (x-0)^2 + (y-0)^2 + (z-0)^2 = 1$

$x^2 + y^2 + z^2 = 1 \Rightarrow \phi(x, y, z)$

$f(x, y, z) = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$

$F(x, y, z) = \cancel{\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}} + \lambda(x^2 + y^2 + z^2 - 1)$

$((x-3)^2 + (y-4)^2 + (z-12)^2)^{1/2}$

$F_x = \frac{1}{\cancel{2}}((x-3)^2 + (y-4)^2 + (z-12)^2)^{-1/2} \times \cancel{2}(x-3) + 2\lambda x = 0 \Rightarrow \lambda = \frac{-P}{2(x-3)x}$

$\hookrightarrow P$

$F_y = ((x-3)^2 + (y-4)^2 + (z-12)^2)^{-1/2} \times (y-4) + 2\lambda y \Rightarrow \lambda = \frac{-P}{2(y-4)y}$

$F_z = ((x-3)^2 + (y-4)^2 + (z-12)^2)^{-1/2} \times (z-12) + 2\lambda z \Rightarrow \lambda = \frac{-P}{2(z-12)z}$

$\cancel{\frac{1}{x-3} = \frac{1}{y-4}} \Rightarrow \cancel{y-4 = x-3}$ | $\cancel{y-4 = z-12}$ | $\cancel{z-12 = x-3}$

$\cancel{y = x+1}$ | $\cancel{y = z-8}$ | $\cancel{z = x+9}$

$\cancel{x^2 + x^2 + 2x + 1 + x^2 + 18x + 81 = 1}$

$\cancel{3x^2 + 20x + 81 = 0}$

$-20 \pm \sqrt{400 -}$

$$\cancel{\frac{-P}{x(x-3)} = \frac{-P}{y(y-4)} = \frac{P}{z(z-12)}}$$

$$\cancel{x(x-3) = y(y-4) = z(z-12)}$$

$$\cancel{x^2 - 3x = y^2 - 4y}$$

$$\cancel{xy - 3y =}$$

$$\frac{x-3}{x} = \frac{y-4}{y} = \frac{z-12}{z}$$

$$xy - 3y = xy - 4x$$

$$4x = 3y$$

$$zy - 4z = yz - 12y$$

$$z = 3y$$

$$4x = 3y = z$$

$$x^2 + y^2 + z^2 = 1$$

$$\frac{z^2}{16} + \frac{z^2}{9} + z^2 = 1$$

$$9z^2 + 16z^2 + 144z^2 = 144$$

$$z = \pm\frac{12}{13}$$

$$z = 3y \quad = \boxed{\pm\frac{12}{39} = y}$$

$$x = \frac{z}{4} = \boxed{\pm\frac{3}{13} = x}$$

$$\boxed{z = \pm\frac{12}{13}}$$

Distance PA $= \sqrt{\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2} = 12$

QA $= \sqrt{\left(\frac{-3}{13}-3\right)^2 + \left(\frac{-4}{13}-4\right)^2 + \left(\frac{-12}{13}-12\right)^2} = 14$

Min distance $= 12$

Max distance $= 14$

Q. A rectangular box without top is to have a given volume. How should the box be made so as to use least material

A. x, y, z → width
x → length, y → breadth

$V = xyz = a$

Required Surface Area $= xy + 2yz + 2zx$

$\phi = xyz - a = 0$

$F = f + \lambda\phi$

$= xy + 2yz + 2zx + \lambda(xyz - a) \longrightarrow (1)$

$\Rightarrow F_x = y + 2z + \lambda yz = 0 \Rightarrow \lambda = \frac{-(y+2z)}{yz} \longrightarrow (2)$

$\Rightarrow F_y = x + 2z + \lambda xz = 0 \Rightarrow \lambda = \frac{-(x+2z)}{xz} \longrightarrow (3)$

$\Rightarrow F_z = 2y + 2x + \lambda xy = 0 \Rightarrow \lambda = \frac{-2(x+y)}{xy} \longrightarrow (4)$

$xyz - a = 0$

$\frac{y+2z}{y\cancel{z}} = \frac{x+2z}{x\cancel{z}} \Rightarrow \cancel{xy} + \cancel{2}xz = \cancel{xy} + \cancel{2}yz$

$\boxed{x = y} \longrightarrow (5)$

$\frac{x+2z}{\cancel{x}z} = \frac{2(x+y)}{\cancel{x}y} \Rightarrow xy + \cancel{2yz} = 2xz + \cancel{2yz}$

$\boxed{y = 2z} \longrightarrow (6)$

$xyz = a$

$4z^3 = a \Rightarrow z = \left(\frac{a}{4}\right)^{1/3}$

$\frac{x^3}{2} = a \Rightarrow x = (2a)^{1/3} = y$

Linear Differential Equations

→ ~~the equation sho~~

→ An equation is said to be linear D.E if

i) y & all of its derivatives is in 1st degree

ii) y & all of its derivatives shouldn't be multiplied together ~~with~~

iii) Transcendantal functions shouldn't be present
↳ Trig, Log, e^y

ex : $\frac{dy}{dx} = kx^2$

$y\frac{d^2y}{dx^2} + \sin x = 0$ Not linear

$\left(\frac{d^3y}{dx^3}\right)^2 - 6x^2\left(\frac{dy}{dx}\right)^8 + e^y = \sin xy$

Standard form of linear diff. eq & solⁿ

→ $\frac{dy}{dx} + Py = Q$

Integrating factor $= e^{\int P dx}$

$y(IF) = \int Q . IF\, dx + c$

(or)

→ $\frac{dy}{dx} + Px = Q$

Integrating factor $= e^{\int P dy}$

$y(IF) = \int Q . IF\, dy + c$

[redacted]

ex: $\frac{dy}{dx} + y\cot x = \cos x$

$Q = \cos x$

$P = \cot x$

$e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

soln $\Rightarrow$ $y \cdot \sin x = \int \cos x \cdot \sin x \cdot dx + c$

$= \frac{1}{2} \int \sin 2x + c$

$y\sin x = -\frac{1}{2} \cdot \frac{\cos 2x}{2} + c$

(or) $t = \sin x$

$dt = \cos x\, dx$

$\Rightarrow y\sin x = \int t\, dt + c$

$= \frac{t^2}{2} + c$

$y\sin x = \frac{\sin^2 x}{2} + c = \frac{1-\cos^2 x}{2} + c$

$= \frac{1}{2} - \left(\frac{1+\cos 2x}{4}\right) + c$

$= \frac{1}{2} - \frac{1}{4} \cdot \frac{\cos 2x}{4} + c$

$= \frac{1}{4} - \frac{\cos 2x}{4} + c$

$y\sin x = -\frac{\cos 2x}{4} + c'$

$y\sin x + \frac{\cos 2x}{4} = c'$

Bernoulli's Differential eq.

→ $\frac{dy}{dx} + Py = Qy^n$ if in this form,

Step 1 : $\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$

Step 2 : $t = y^{1-n}$

$$\frac{dt}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}$$

$$\frac{dt}{dx} + P't = Q'$$

$P' = P(1-n)$

$Q' = Q(1-n)$

$$IF = e^{\int P' dx}$$

$$t(IF) = \int Q'.(IF)dx + C$$

→ $\frac{dy}{dx} + Px = Qx^n$

Q. $x\frac{dy}{dx} + 2y = x^3y^2$

$$\frac{dy}{dx} + \frac{2y}{x} = x^2y^2$$

In above D.E, (Bernoulli's DE of $\frac{dy}{dx} + Py = Qy^n$)

$P = \frac{2}{x}$ $Q = x^2$ $n = 2$

$\Rightarrow \frac{1}{y^2}\times\frac{dy}{dx} + \frac{2}{x}\cdot\frac{1}{y} = x^2$

$t = 1/y$ $\frac{dt}{dx} = \frac{-1}{y^2}\times\frac{dy}{dx}$

$\Rightarrow -\frac{dt}{dx} + Pt = Q$

$IF = e^{-\int \frac{2}{x}dx}$

$= e^{-2\int\frac{1}{x}dx}$

$= e^{-2\log x} = x^{-2}$

$t \times \frac{1}{x^2} = -\int x^2 \cdot x^{-2} dx + C$

$= -x + C$

$\frac{t}{x^2}$

Q. $\frac{dy}{dx} + 4xy = -xy^3$

$\Rightarrow \frac{1}{y^3} \cdot \frac{dy}{dx} + \frac{4x}{y^2} = -x$

$t = y^{-2}$

$\cancel{\frac{dt}{dx} = 2y \cdot \frac{dy}{dx}}$ $\quad \frac{dt}{dx} = -2 \cdot \frac{1}{y^3}$

$\Rightarrow -\frac{1}{2} \cdot \frac{dt}{dx} + \underbrace{4x}_{P} t = \underbrace{-x}_{Q}$

$\frac{dt}{dx} \underbrace{-8x}_{P'} t = \underbrace{2x}_{Q'}$

$IF = e^{\int P' dx} = e^{-\int 8x\, dx} = e^{-4x^2}$

$t \cdot e^{-4x^2} = \int 2x \cdot e^{-4x^2} dx + C$

$u = -4x^2$

$\frac{du}{dt} = -8x$

$t \cdot e^{-4x^2} = \int \frac{e^u}{-4} \cdot du + c$

$= -\frac{1}{4} \cdot (e^u + c') + c$

$t \cdot e^{-4x} = -\frac{e^u}{4} + c''$

$\boxed{\frac{e^{-4x}}{y^2} = \frac{-e^{-4x^2}}{4} + c''} \Rightarrow \boxed{\frac{e^{-4x}}{y^2} + \frac{e^{-4x^2}}{4} = c''}$

Q. $\frac{dy}{dx} - y = y^2(\sin x + \cos x)$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = (\sin x + \cos x)$$

$$t = \frac{-1}{y} \Rightarrow \frac{dt}{dx} = \frac{1}{y^2} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dt}{dx} + t = \sin x + \cos x$$

$$IF = e^{\int P' dx} = e^{\int dx} = e^x$$

$$t \cdot IF = \int Q' \cdot IF \, dx + c$$

$$te^x = \int e^x(\sin x + \cos x) dx + c$$

$$\int e^x (f(x) + f'(x)) = e^x f(x)$$

$$\cancel{\frac{-e^x}{y} = -\cos x + \sin x + c' + c}$$

$$\cancel{\frac{e^x}{y} = \cos x - \sin x + c''}$$

$$\boxed{\frac{-e^x}{y} = e^x \sin x + c}$$

Q. $y - \cos x \cdot \frac{dy}{dx} = y^2 (1 - \sin x) \cos x$

Given $y = 2,\ x = 0$

$\frac{dy}{dx} - \frac{y}{\cos x} = y^2 (1 - \sin x)$

$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y\cos x} = 1 - \sin x$

$t = -\frac{1}{y} \Rightarrow \frac{dt}{dx} = \frac{1}{y^2} \cdot \frac{dy}{dx}$

$\frac{dt}{dx} - \frac{t}{\cos x} = 1 - \sin x$

$IF = e^{\int (\cos x)^{-1} dx} = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)}$

$= (\sec x + \tan x)$

$t \cdot IF = \int Q' \cdot IF \cdot dx + C$

$\frac{-(\sec x + \tan x)}{y} = \int (1 - \sin x)(\sec x + \tan x) \cdot dx + C$

$\cancel{= \int \left(\sec x - \tan x + \tan x - \frac{\sin^2 x}{\cos x}\right) dx + c}$

$\cancel{-\left(\frac{1+0}{2}\right) = \log(\sec x + \tan x) - \int \frac{\sin^2 x}{\cos x} dx + c}$

$\cancel{\frac{-1}{2} = 0 + \int \cos x \sin x\, dx + c}$

$\cancel{\frac{-1}{2} = \int \frac{\sin 2x \cdot du}{2} + c}$

$\cancel{u = (\cos x)^{-1}}$

$\cancel{\frac{du}{dx} = \frac{-1}{(\cos x)^2} \cdot \sin x}$

$\cancel{du = \frac{-\sin x}{(\cos^2 x)} dx}$

$\cancel{-\cos x\, du = \frac{\sin x}{\cos x} dx}$

$= \int (1 - \sin x)\left(\frac{1 + \sin x}{\cos x}\right) dx$

$\frac{-1}{2} = \int \frac{1 - \sin^2 x}{\cos x} dx = \int \cos x\, dx = \sin x + C$

$\frac{-1}{2} = 0 + C = -1/2$

$\boxed{\frac{1}{y}(\sec x + \tan x) = \sin x + \frac{1}{2}}$

Ans

Exact DE

→ $M(x,y)\,dx + N(x,y)\,dy = 0$

is exact DE if its LHS is exact DE of $u(x,y)$

$du = M(x,y)\,dx + N(x,y)\,dy = 0$

So, $\boxed{u(x,y) = c}$

ex : $y\,dx + x\,dy = 0$

$d(xy) = y\,dx + x\,dy = 0$

$\Rightarrow \boxed{xy = c}$

→ To check if its exact,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Soln $\Rightarrow$ ~~$\int N(y)\,dy = c$~~ ~~$\int M(x)\,dx + \int N(y)$~~

M not containing x

$\int M\,dx + \int N(y)\,dy = c$

(or)

$\int M(x)\,dx + \int N\,dy = c$

N not containing y

Q. $(3x^2 + 2e^y)\,dx + (3y^2 + 2xe^y)\,dy = 0$

$M = 3x^2 + 2e^y \qquad N = 3y^2 + 2xe^y$

$\dfrac{\partial M}{\partial y} = 2e^y \qquad \dfrac{\partial N}{\partial x} = 2e^y$

$\hookrightarrow$ Soln is exact D.E

$\int M\,dx + \int N(y)\,dy = C$

$\Rightarrow \int (3x^2 + 2e^y)\,dx + \int (3y^2)\,dy = C$

$\Rightarrow \left[x^3 + 2e^y x\right] + \left[y^3\right] = C$

$= x^3 + y^3 + 2xe^y$

(or)

$\int M(x)\,dx + \int N\,dy = C$

$\int 3x^2\,dx + \int (3y^2 + 2xe^y)\,dy = C$

$= \left[x^3\right] + y^3 + 2xe^y$

Q. $\{2xy\cos x^2 - 2xy + 1\}\,dx + \{\sin x^2 - x^2 + 3\}\,dy = 0$

$M = 2xy\cos x^2 - 2xy + 1$

$N = \sin x^2 - x^2 + 3$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$

$\frac{\partial N}{\partial x} = 2x \cdot \cos x^2 - 2x$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Now, $\int M\,dx + \int N\,dy = C$

$\int 1 \cdot dx + \int (\sin x^2 - x^2 + 3)\,dy = C$

$x + [y(\sin x^2 - x^2 + 3)] = C$

Q. $(y+x^3)dx + (ax+by^3)dy = 0$

$M = y + x^3$

$N = ax + by^3$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = a$

$\boxed{a=1}$

Now, $\int M dx + \int N(y) dy = c$

$\Rightarrow \int (y+x^3) dx + \int (by^3) dy = c$

$\Rightarrow \left[xy + \frac{x^4}{4}\right] + \frac{by^4}{4} = c \longrightarrow (1)$

and, $\int M(x) dx + \int N dy = c$

$\Rightarrow \int x^3 dx + \int (ax + by^3) dy = c$

$\Rightarrow \frac{x^4}{4} + axy + \frac{by^4}{4} = c$

b & c can be any value