

Unit-1

	Dimension	Geometric Description	No. of Solutions	Matrix Condition
1)	2D Row picture	Lines parallel & no intersection	No solution	$ A \neq 0$
2)	2D Row picture	Lines coincident (lines lie on top of each other)	∞	$ A = 0$
3)	3D Row picture	Each pair intersect in a line but no common intersection	No solution	$ A \neq 0$
4)	3D Row picture	2 planes intersect along a line, 3rd plane \parallel to line	No solution	$ A \neq 0$
5)	3D Row picture	All 3 planes \parallel , no intersection	No solution	$ A \neq 0$
6)	3D Row picture	All 3 planes overlap perfectly	∞	$ A = 0$
7)	3D Row picture	All 3 planes intersect along single line	∞	$ A = 0$
8)	3D Row picture	2 planes are \parallel , 3rd intersects them	No solution	$ A \neq 0$
9)	3D Column picture	\vec{b} is not in the plane formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$	No solution	$ A \neq 0$
10)	3D Column picture	\vec{b} is in the plane formed by $\vec{a}_1, \vec{a}_2, \vec{a}_3$	∞	$ A = 0$

Consistency

$$\begin{aligned} \text{RANK}(A) = \text{RANK}(A:b) &< n \Rightarrow \infty \text{ sol}^n \\ &= n \Rightarrow \text{unique sol}^n \\ &\neq \Rightarrow \text{No sol}^n \end{aligned}$$

Breakdown of Elimⁿ

$$\text{Non-singular \& curable} \Rightarrow |A| \neq 0$$

$$\text{Singular \& non-curable} \Rightarrow |A| = 0$$

$$\text{Singular} \Rightarrow |A| = 0$$

Triangular Factors

LU

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -E_{21} & 1 & 0 \\ -E_{31} & -E_{32} & 1 \end{bmatrix} \quad U = \text{Gaussian elim}^n$$

LDU

$$A = LDU$$

$$L = \text{same}$$

$$D = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad U = \begin{bmatrix} 1 & \frac{u_{12}}{p_1} & \frac{u_{13}}{p_1} \\ 0 & 1 & \frac{u_{23}}{p_2} \\ 0 & 0 & 1 \end{bmatrix}$$

Permutation Matrix

Row exchanges $\Rightarrow PA = LDU$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{13} \quad P_{31}$$

$$P_{23} \quad P_{32}$$

Inverses

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

$$Ax = b \Rightarrow x = A^{-1}b$$

Gauss-Jordan

$$[A : I] \rightarrow [U : C] \rightarrow [I : A^{-1}]$$

Transpose

$$A \rightarrow A^T$$

$$(A^T)^T = A$$

$$(AB)^T = B^T A^T$$

$$\left| \begin{array}{l} (\bar{A}^T)^T = (A^T)^T \\ (A \pm B)^T = A^T \pm B^T \end{array} \right| \quad (A^T)^T A^T = (BA)^T = I$$

Symmetric

$$\begin{aligned} &\rightarrow A^T = A \\ &(\bar{A}^T)^T = A^{-1} \\ &\text{and } A = A^T = LDL^T \end{aligned}$$

Vector space \Rightarrow If $u, v \in V \Rightarrow u+v \in V \Rightarrow V$ closed under vector addition
 If $c \in \mathbb{R}, u \in V \Rightarrow cu \in V \Rightarrow V$ closed under scalar multiplication

Subspace: Non-empty subset of vector space

- \rightarrow Commutative: $u+v = v+u$
 Associative: $u+(v+w) = (u+v)+w$
 $r(su) = s(ru)$
 Additive: $0+u = u+0 = u$
 Inverse: $u+(-u) = (-u)+u = 0$
 Distributive: $c_1(u+v) = c_1u + c_1v$
 $(c_1+c_2)u = c_1u + c_2u$
 Multiplicative: $1u = u$

$$Rx = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot v
Free v.

Unit-2

Linear Independence

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

Only if $c_1 = c_2 = c_3 = 0 \Rightarrow$ linearly independent $\Rightarrow \rho(A) = \text{no. of columns}$
 else dep. $\Rightarrow \rho(A) < \text{no. of col}^n$

Basis: Subset $S = \{v_1, v_2, \dots, v_n\}$ of vector space if S is linear indep, S spans vector space V
 \hookrightarrow Maximal independent set & Min. spanning set

$$\mathbb{R}^2: \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \mathbb{R}^3: \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$2 \times 2: \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$n\text{-degree polynomial: } \{1, t, t^2, \dots, t^n\}$$

4 Fundamental Subspaces

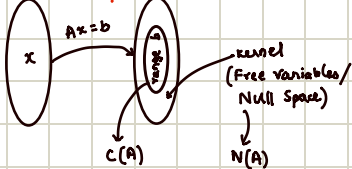
- 1) Column Space $\Rightarrow \rho(A) = K = \dim(C(A))$
 \hookrightarrow Columns having pivot in echelon form of A
- 2) Row Space $\Rightarrow \rho(A) = K = \dim(C(A^T))$
 \hookrightarrow Row vectors in A corresponding to pivots in echelon form
- 3) Null Space $\Rightarrow n - K = \dim(N(A))$
 \hookrightarrow All sol's of $Ax = 0$
- 4) Left Null Space $\Rightarrow m - K = \dim(N(A^T))$
 \hookrightarrow All sol's of $A^T y = 0$

Rectangular Matrix Inverses

For $m \times n$ matrix

- a) $\rho(A) = m$, Right Inverse $= A^T(AA^T)^{-1}$
- b) $\rho(A) = n$, Left Inverse $= (A^T A)^{-1} A^T$

Linear Transformations



$$\text{Stretching} \Rightarrow \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$\text{Rotation} \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Reflection} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Projection} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

General Matrix to rotate

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Polynomial Space

$$\hookrightarrow P_n = c_0 + c_1t + c_2t^2 + \dots$$

$$\text{Basis} = \{1, t, t^2, \dots, t^n\}$$

$$\text{Dimension} = n+1$$

$$\text{Differentiation} \Rightarrow P_3 = 1, t, t^2, t^3$$

$$P_2 = 1, t, t^2$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3 \times 4}$$

$$\text{Integ} \Rightarrow P_2 = 1, t, t^2$$

$$P_3 = 1, t, t^2, t^3$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4 \times 3}$$

Rank-Nullity Theorem

$\hookrightarrow A_{m \times n}$

$$\dim(C(A)) + \dim(N(A)) = r + (n-r) = n$$

$$\dim(C(A^T)) + \dim(N(A^T)) = r + (m-r) = m$$

Rank 1 Matrices

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \rightarrow v^T$$

$$A = u x v^T$$

Rotation

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q_\theta Q_\theta = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$Q_\theta Q_\phi = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

Projection

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = P \cdot P = P \cdot P \cdot P = P^n$$

\hookrightarrow symmetric

Reflection

$$H = 2P - I = \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$H \cdot H = H^2 = I$$