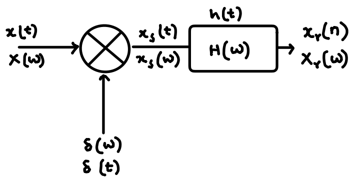
Digital Signal Processing

(UE23EC251B)

—Hitesh

Unit-1 Discrete Fourier Transform



Sampling Theorem

→ A Band limited signal of finite energy having no frequency component greater than ω_h / f_h can be reconstructed when sampling frequency $f_s \geqslant 2f_h$ or $\omega_s \geqslant 2\omega_h$

Nyquist Rate

→ $f_s = 2f_h$

$t_s = \frac{1}{f_s} = \frac{1}{2f_h}$

Q. $x(t) = \sin(200\pi t)$. Find Nyquist rate

A. $\omega_1 = 200\pi$

$f_1 = 100$

$f_s = 2f_1 = 200Hz$

Q. $x(t) = \text{sinc}(200t)$. Find Nyquist rate

A. $x(t) = \frac{\sin(200\pi t)}{200\pi}$

$f_s = 200Hz$

Q. $x(t) = \sin^2(200\pi t)$. Find Nyquist rate

A. $x(t) = \frac{1 - \cos 400\pi t}{2}$

$f_h = 200Hz$

$f_s = 400Hz$

Q. $x(t) = 1 + \cos 200\pi t + \sin 400\pi t$

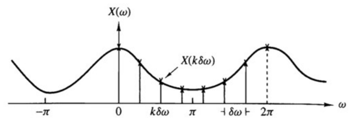
A. $f_h = 200\ Hz$

$f_s = 400Hz$

→ $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \longrightarrow s(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$

Discrete Fourier Transform and its properties

Frequency Domain Sampling & Reconstructing discrete-time signal



Frequency-domain sampling of the Fourier Transform

→ Consider a finite energy aperiodic signal having continuous spectra

Now, we find its Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

→ Since $X(\omega)$ is periodic with period 2π, we consider N equidistant samples between the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$

Take $\omega = \frac{2\pi k}{N}$,

$$X(\omega) = X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad (k = 0, 1, \ldots N-1)$$

$$= \ldots + \sum_{n=-N}^{-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \sum_{n=N}^{2N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} + \ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{\frac{-j2\pi kn}{N}} \longrightarrow 1$$

(Change the inner summation from n to n−lN & interchange the summation)

$$= \sum_{n=0}^{N-1} x_p(n)\, e^{\frac{-j2\pi kn}{N}}$$

$\left(x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \right)$

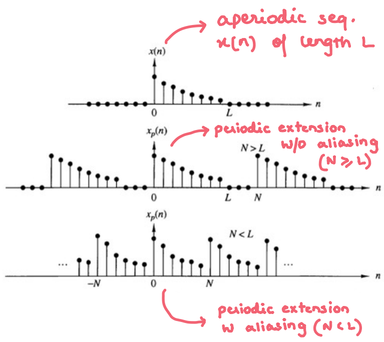
Since $x_p(n)$ is periodic, it can be expanded in Fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{jk2\pi n}{N}}$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{\frac{-jk2\pi n}{N}} \longrightarrow 2$$

Comparing 1 and 2, $\quad c_k = \frac{1}{N} X\left(\frac{2\pi k}{N}\right)$

Therefore, $\quad x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi kn}{N}}$

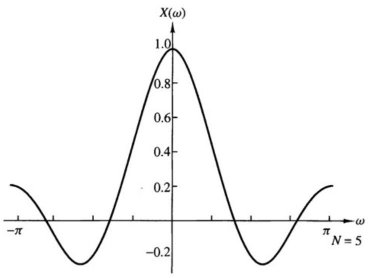


In absence of time-aliasing, $\quad x_p(n) = \begin{cases} x(n) & (0 \le n \le N-1) \\ 0 & \text{elsewhere} \end{cases}$

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) e^{\frac{j2\pi K n}{N}} \right] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} X\left(\frac{2\pi K}{N}\right) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi K}{N}\right)n} \right]}_{\text{Basic Interpolation Function, shifted by } \frac{2\pi K}{N}} \longrightarrow 3$$



$$\frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)}$$

$$P(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \cdot \left(\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right)$$

$$= \frac{1}{N} \left(\frac{e^{-j\omega N/2} \cdot e^{j\omega N/2} - e^{-j\omega N/2} \cdot e^{-j\omega N/2}}{e^{-j\omega/2} \cdot e^{j\omega/2} - e^{-j\omega/2} \cdot e^{-j\omega/2}} \right)$$

$$= \frac{e^{-j\omega\frac{(N-1)}{2}}}{N} \left(\frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right)$$

$$= \frac{e^{\frac{-j\omega(N-1)}{2}}}{N} \cdot \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \longrightarrow 4$$

So, using 4 in 3

$$X(\omega) = \sum_{K=0}^{N-1} X\left(\frac{2\pi K}{N}\right) P\left(\omega - \frac{2\pi K}{N}\right) \quad , \quad N \geq L$$

This means $x(n)$ is causal, finite - duration sequence of length N

$P(\omega) \neq \frac{\sin\theta}{\theta}$ but it is a periodic counterpart

$$P\left(\frac{2\pi K}{N}\right) = \begin{cases} 1, & K = 0 \\ 0, & K = 1, 2, \ldots N-1 \end{cases}$$

Then, DFT :

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)Kn} \quad , \quad K = 0, 1, 2, \ldots N-1$$

$$X(K) = X\left(\frac{2\pi K}{N}\right)$$

IDFT :

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{j\left(\frac{2\pi}{N}\right)Kn} \quad , \quad n = 0, 1, 2, \ldots N-1$$

No of Complex Additions & Multiplications

1) C.A $\Rightarrow N(N-1)$

2) C.M $\Rightarrow N^2$

ex :	1	8	16	256
CA	0	56	240	65536
CM	1	64	256	65280

Q. Let $x[n] = 0, 1, 2, 3$

Find the 4 point DFT

A. $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi K n}{N}}$

$= \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi K n}{4}}$

$= x(0)e^{0} + x(1)e^{\frac{-j2\pi K}{4}} + x(2)e^{\frac{-j4\pi K}{4}} + x(3)e^{\frac{-j6\pi K}{4}}$

$= 0 + e^{\frac{-jK\pi}{2}} + 2e^{-jK\pi} + 3e^{-j\frac{3K\pi}{2}}$

$K=0 \Rightarrow 0 + 1(1) + 2(1) + 3(1)$

$= 6$

$K=1 \Rightarrow 0 + e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2}$

$= 0 + \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) + 2\cos(-\pi) + j2\sin(-\pi) + 3\cos\left(\frac{-3\pi}{2}\right) + j3\cos\left(\frac{-3\pi}{2}\right)$

$= 0 - j - 2 + 3j = -2 + 2j$

$K=2 \Rightarrow 0 + e^{-j\pi} + 2e^{-2j\pi} + 3e^{-j3\pi}$

$= 0 - 1 + 2 - 3 = -2$

$K=3 \Rightarrow 0 + e^{-j3\pi/2} + 2e^{-j3\pi} + 3e^{-j9\pi/2}$

$= 0 + j - 2 - 3j = -2 - 2j$

$x(n) = \{0, 1, 2, 3\}$ $\qquad$ $X(K) = \{6, -2+2j, -2, -2-2j\}$

$x(n) \xleftrightarrow[4]{DFT} X(K)$

Q. $x(n) = \{1, 1, 1, 1\}$. Find the 4 point DFT

A. $X(K) = \sum_{n=0}^{3} x(n)\, e^{\frac{-jK2\pi n}{N}}$

$= x(0)e^{0} + x(1)e^{\frac{-j2\pi K}{4}} + x(2)e^{\frac{-j4\pi K}{4}} + x(3)e^{\frac{-j6\pi K}{4}}$

$= 1 + e^{\frac{-jK\pi}{2}} + e^{-jK\pi} + e^{-j\frac{3K\pi}{2}}$

$K=0 \Rightarrow 1 + e^{0} + e^{0} + e^{0}$

$= 1 + 1 + 1 + 1 = 4$

$K=1 \Rightarrow 1 + e^{-j\pi/2} + e^{-j\pi} + e^{-j\frac{3\pi}{2}}$

$= 1 - 1 + 1 - 1 = 0$

$K=2 \Rightarrow 1 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$

$= 0$

$K=3 \Rightarrow 1 + e^{-j3\pi/2} + e^{-j3\pi} + e^{-j9\pi/2}$

$= 0$

$X(K) = \{4, 0, 0, 0\}$

Q. $x(n) = \begin{cases} 1 & , n = 0 \text{ to } N-1 \\ 0 & , \text{elsewhere} \end{cases}$

A. $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-jK2\pi n}{N}}$

$= \sum_{n=0}^{N-1} \omega_N^{Kn} \qquad \left(\omega = e^{\frac{-j2\pi}{N}}\right)$

$X(K) = \sum_{n=0}^{N-1} (\omega_N^K)^N = \begin{cases} N & , K = 0 \\ \dfrac{1-\omega_N^{KN}}{1-\omega_N^K} & , K = 1, 2, \ldots N-1 \end{cases}$

$= \begin{cases} N & , K = 0 \\ 0 & , K = 1, 2, \ldots N-1 \end{cases}$

$= N\,\delta(K)$

Q. $x_1(n) = \{1, 1, 1, 1\} \xleftrightarrow[4]{\text{DTFT}} x_1(K) = \{4, 0, 0, 0\}$

$x_2(n) = \{1, 1, 1, 1, 0, 0, 0, 0\} \xleftrightarrow[8]{\text{DTFT}} x_2(K) = \ ?$

Recover $X_1(K)$ from $X_2(K)$

Twiddle Factors

$\omega_8^0 = 1$	$\omega_8^4 = -1$
$\omega_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$\omega_8^5 = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$
$\omega_8^2 = -j$	$\omega_8^6 = j$
$\omega_8^3 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$\omega_8^7 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

A. $X_2(K) = \sum_{n=0}^{7} x_2(n)\, \omega_8^{Kn}$

$= 1 + \omega_8^K + \omega_8^{2K} + \omega_8^{3K} + 0 + 0 + 0 + 0$

$= 1 + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^K + (-j)^K + \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^K$

$K = 0 \Rightarrow 1 + 1 + 1 + 1 = 4$

$K = 1 \Rightarrow 1 + \cancel{\frac{1}{\sqrt{2}}} - \frac{j}{\sqrt{2}} - j - \cancel{\frac{1}{\sqrt{2}}} - \frac{j}{\sqrt{2}} = 1 - j(1+\sqrt{2})$

$K = 2 \Rightarrow 1 + (-i) - 1 + j = 0$

$K = 3 \Rightarrow 1 + \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + j + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) = 1 + j(1-\sqrt{2})$

$K = 4 \Rightarrow 1 + (-1) + 1 + (-1) = 0$

$K = 5 \Rightarrow 1 + \left(\cancel{\frac{-1}{\sqrt{2}}} + \frac{j}{\sqrt{2}}\right) - j + \left(\cancel{\frac{1}{\sqrt{2}}} + \frac{j}{\sqrt{2}}\right) = 1 + j(-1+\sqrt{2})$

$K = 6 \Rightarrow 1 + i - 1 - i = 0$

$K = 7 \Rightarrow 1 + \left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + j + \left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) = 1 + j(1+\sqrt{2})$

$x_2(n) = \{1, 1, 1, 1, 0, 0, 0, 0\} \qquad X_2(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$

$x_1(n) = \{1, 1, 1, 1\}$

$X_1(K) = \{4, 0, 0, 0\} = X_2(2K)$

Zero padding $\Rightarrow$ Better display of spectrum

Twiddle factors

$W_4^{-0} = 1$

$W_4^{-1} = j$

$W_4^{-2} = -1$

$W_4^{-3} = -j$

Q. Find inverse DFT of

$X(K) = \{10, -2-2j, -2, -2+2j\}$

A. $x(n) = IDFT\{X(K)\}$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi Kn}{N}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$$

$$N = 4 \Rightarrow x(n) = \frac{1}{4}\sum_{K=0}^{3} X(K)\, W_4^{-Kn}$$

$$= \frac{1}{4}\left(10\, W_4^{0} + (-2-2j)\, W_4^{-n} + (-2)\, W_4^{-2n} + (-2+2j)\, W_4^{-3n}\right)$$

$$= \frac{1}{4}\left(10 - 2(1+j)(j)^n - 2(-1)^n - 2(1-j)(-j)^n\right)$$

$$n = 0 \Rightarrow \frac{1}{4}\left(10 - 2(1+j) - 2 - 2(1-j)\right) = \frac{1}{4} \times 4 = 1$$

$$n = 1 \Rightarrow \frac{1}{4}\left(10 - 2(1+j)(j) - 2(-1) - 2(1-j)(-j)\right)$$

$$= \frac{1}{4}\left(10 - 2j + 2 + 2 + 2j + 2\right) = 4$$

$$n = 2 \Rightarrow \frac{1}{4}\left(10 - 2(1+j)(-1) - 2(1) - 2(1-j)(-1)\right)$$

$$= \frac{1}{4}\left(10 + 2 + 2j - 2 + 2 - 2j\right) = 3$$

$$n = 3 \Rightarrow \frac{1}{4}\left(10 - 2(1+j)(-j) - 2(-1) - 2(1-j)(j)\right)$$

$$= \frac{1}{4}\left(10 + 2j - 2 + 2 - 2j - 2\right) = 2$$

$$x(n) = \{1, 4, 3, 2\}$$

Q. Find DFT of $x(n) = \{1, 4, 3, 2\}$

A. $x(k) = 1 + 4\omega_4^k + 3\omega_4^{2k} + 2\omega_4^{3k}$

$\omega_4^0 = 1$, $\omega_4^1 = -j$, $\omega_4^2 = -1$, $\omega_4^3 = j$

$x(0) = 10$

$x(1) = 1 - j4 - 3 + j2 = -2 - 2j$

$x(2) = 1 + 4(-1) + 3 + 2(-1) = -2$

$x(3) = 1 + j4 - 3 - j2 = -2 + 2j$

DFT as Linear Transformation

→ We know,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, \omega_N^{kn} \quad , k = 0, 1, \ldots N-1 \qquad \left(W_N = e^{-j2\pi/N}\right)$$

and
$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k)\, W_N^{-kn} \quad , n = 0, 1, \ldots N-1$$

→ We can view DFT & IDFT as linear transformation on sequences $x(n)$ & $X(k)$

So we define N-point vector x_N of the signal sequence $x(n)$ where $n = 0, 1, \ldots, N-1$

an N-point vector X_N of frequency samples and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdot & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Periodicity Property

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

then $X_N = W_N x_N$

$x_N = W_N^{-1} X_N$

$x_N = \dfrac{W_N^*}{N} X_N \qquad \left(W_N^{-1} = \dfrac{W_N^*}{N}\right)$

which implies $W_N W_N^* = N I_N$

Q. $x(n) = (0, 1, 2, 3)$

A. $$W_4 = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^0 & w_4^2 \\ 1 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4\, x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$|X(k)| = \sqrt{X_R^2(k) + X_I^2(k)}$$

$$\angle X(k) = \tan^{-1}\left(\frac{X_I(k)}{X_R(k)}\right)$$

Q. $x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

Compute $|X(k)|$ and $\angle X(k)$

A. We know,

$$X(k) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

Then,

$$|X(k)| = \{4,\ 2.613,\ 0,\ 1.082,\ 0,\ 1.082,\ 0,\ 2.613\}$$

and,

$$\angle X(k) = \{0,\ -67.5,\ 0,\ -22.5,\ 0,\ 22.5,\ 0,\ 67.5\}$$

Q. $x(n) = \{1, -1, 1, -1\}$. Compute 4 point DFT

A. $$X_4 = W_4\, x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1+1-1 \\ 1+j-1-j \\ 1+1+1+1 \\ 1-j-1+j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

Q. $x(n) = \delta(n)$

$X(K) = ?$ (N point DFT)

A. $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$

$= \sum_{n=0}^{N-1} \delta(n)\, W_N^{Kn}$

$= W_N^{K(0)} = 1$

Q. $x(n) = \delta(n-n_0)$

$X(K) = ?$ (N point DFT)

A. $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$

$= \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{Kn}$

$= W_N^{Kn_0}$

Q. $x(n) = \begin{cases} 1, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$ let N be even

Then compute N-point DFT

A. $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$

$= W_N^{0} + W_N^{2K} + W_N^{4K} + \dots + W_N^{(N-2)K}$

$= \sum_{n=0}^{\frac{N}{2}-1} W_N^{K2n} = \sum_{n=0}^{\frac{N}{2}-1} e^{\left(-j\frac{2\pi K}{N/2}\right)n}$

$$\begin{cases} \dfrac{1 - e^{\left(-j\frac{2\pi K}{N/2}\right)\frac{N}{2}}}{1 - e^{-j\frac{2\pi K}{N/2}}} = \dfrac{1 - e^{-j2\pi K}}{1 - e^{-j\frac{2\pi K}{N/2}}} = \dfrac{1-1}{1-e^{-j\frac{2\pi K}{N/2}}} = 0 & , K \neq 0 \\ \dfrac{N}{2} & , K = 0 \end{cases}$$

$$= \begin{cases} N/2 & , K = 0 \\ 0 & , K \neq 0 \end{cases}$$

Q. $x(n) = \begin{cases} 1, & n = 0 \text{ to } \frac{N}{2}-1 \\ 0, & n = \frac{N}{2} \text{ to } N-1 \end{cases}$ where N is even

Compute N-point DFT

A. $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} (1)\, e^{-j\frac{2\pi k n}{N}} + \sum_{n=\frac{N}{2}}^{N-1} (0)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} e^{-j\frac{2\pi k n}{N}}$$

$$= \begin{cases} N/2, & k = 0 \\ \dfrac{1 - e^{\left(-j\frac{2\pi k}{N}\right)\frac{N}{2}}}{1 - e^{-j\frac{2\pi k}{N}}} = \dfrac{1 - e^{-j\pi k}}{1 - e^{-j\frac{2\pi k}{N}}} = \dfrac{1 - (-1)^k}{1 - e^{-j\frac{2\pi k}{N}}}, & k \neq 0 \end{cases}$$

Q. $x(n) = a^n$

Compute N-point DFT

A. $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} \left(a e^{-j\frac{2\pi k}{N}}\right)^n$$

$$= \begin{cases} N, & k = 0 \\ \dfrac{1 - \left(a e^{-j\frac{2\pi k}{N}}\right)^N}{1 - a e^{-j\frac{2\pi k}{N}}} = \dfrac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j\frac{2\pi k}{N}}} = \dfrac{1 - a^N}{1 - a e^{-j\frac{2\pi k}{N}}} = \dfrac{1 - a^N}{1 - a W_N^k}, & k \neq 0 \end{cases}$$

Q. $x(n) = e^{j\frac{2\pi k_0 n}{N}}$

Compute $X(k)$

A. $$x(n) = W_N^{-k_0 n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{N-1} W_N^{-k_0 n} W_N^{kn} = \sum_{n=0}^{N-1} W_N^{(k-k_0)n}$$

$$= \begin{cases} \dfrac{1 - W_N^{(k-k_0)N}}{1 - W_N^{(k-k_0)}} = \dfrac{1-1}{1 - W_N^{(k-k_0)}} = 0, & k \neq k_0 \\ N, & k = k_0 \end{cases} = \begin{cases} N, & k = k_0 \\ 0, & k \neq k_0 \end{cases}$$

$$X(k) = N\,\delta(k-k_0)$$

$\delta(k+k_0)$
$= \delta(k+k_0-N)$
$= \delta(k-(N-k_0))$

Q. $x(n) = \cos\left(\frac{2\pi k_0 n}{N}\right)$

Find the N - point DFT

A. $x(n) = \frac{e^{\frac{2\pi k_0 n}{N}} + e^{\frac{-2\pi k_0 n}{N}}}{2} = \frac{W_N^{-k_0 n} + W_N^{k_0 n}}{2}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left(\frac{W_N^{-k_0 n} + W_N^{k_0 n}}{2} \right) W_N^{kn}$$

$$= \frac{N}{2}\delta(k-k_0) + \frac{N}{2}\delta(k+k_0)$$

Q. $x(n) = \sin\left(\frac{2\pi k_0 n}{N}\right)$

A. Find the N - point DFT

$x(n) = \frac{e^{\frac{2\pi k_0 n}{N}} - e^{\frac{-2\pi k_0 n}{N}}}{j2} = \frac{W_N^{-k_0 n} - W_N^{k_0 n}}{j2}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left(\frac{W_N^{-k_0 n} - W_N^{k_0 n}}{j2} \right) W_N^{kn}$$

$$= \frac{N}{j2}\delta(k-k_0) - \frac{N}{j2}\delta(k+k_0)$$

Q. $x(t) = \cos(2\pi(50)t)$

Compute the 4 point DFT of resulting sequence when $x(t)$ is sampled at $f_s = 200\,Hz$ using linear transformation equation

A. $x(t) \xrightarrow[200Hz]{t=NT} x(n) \xrightarrow{DFT} X(k)$

$x(n) = x(t)\big|_{t=NT}$ $\qquad T = \frac{1}{f_s} = \frac{1}{200} = 5mHz$

$= \cos 2\pi(50)nt = \cos 2\pi\, 50 \frac{n}{200}$

$= \cos\frac{n\pi}{2}$

$= \left\{ \cos(0),\ \cos\frac{\pi}{2},\ \cos\pi,\ \cos\frac{3\pi}{2} \right\} = \{1, 0, -1, 0\}$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0-1+0 \\ 1+0+1+0 \\ 1+0-1+0 \\ 1+0+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

We don't take negative values of n, so we take the last index as N−1 instead of −1 as they are periodic

Q. $x(n) = \{\underset{\uparrow -1}{1/2} \quad \underset{\uparrow 0}{1/2} \quad \underset{\uparrow 1}{1/2}\}$

Compute N-point DFT

A. $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$= \{\underset{\uparrow 0}{1/2} \quad \underset{\uparrow 1}{1/2} \quad \underset{\uparrow 2}{0} \quad 0 \; \ldots\ldots 0 \quad \underset{\uparrow N-1}{1/2}\}$$

$$X(k) = \frac{1}{2} W_N^0 + \frac{1}{2} W_N^k + \ldots + \frac{1}{2} W_N^{(N-1)k}$$

$$= \frac{1}{2} + \frac{1}{2} W_N^k + \frac{1}{2} W_N^{(N-1)k}$$

$$= \frac{1}{2}\left(1 + W_N^k + W_N^{(N-1)k}\right)$$

$W^{Nk} = 1$

$$= \frac{1}{2}\left(1 + W_N^k + W_N^{Nk} \cdot W_N^{-k}\right) = \frac{1}{2}\left(1 + W_N^k + W_N^{-k}\right)$$

$$= \frac{1}{2} + \cos\left(\frac{2\pi k}{N}\right)$$

Q. Find the IDFT of the sequence $X(k) = \{13, 14, 13, 14\}$ using IDFT eq^n

A. $x_N = \frac{W_N^* X_N}{N}$

$$= \frac{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 13 \\ 14 \\ 13 \\ 14 \end{bmatrix}}{4}$$

$$= \left[\frac{13+14+13+14}{4}, \quad 13 - 14j - 13 + 14j, \quad 13 - 14 + 13 - 14, \quad 13 + 14j - 13 - 14j\right]$$

Relationship of DFT to other transforms

i) Relationship to the Fourier series coefficients of a periodic sequence

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{j\frac{2\pi nk}{N}}, \quad -\infty < n < \infty$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi nk}{N}}, \quad k = 0, 1, \ldots, N-1$$

Comparing with DFT & IDFT, $X(k) = N c_k$

ii) Relationship to the Fourier Transform of an aperiodic sequence

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi nk}{N}} \quad , k=0,1,2,\dots N-1$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$x_p(n)$ is determined by aliasing $\{x(n)\}$ over $0 \leq n \leq N-1$

$$\hat{x}(n) = \begin{cases} x_p(n) & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

iii) Relationship to the Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$X(k) = X(z)\Big|_{z=e^{j\frac{2\pi nk}{N}}} \quad , k=0,1,2,\dots N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi nk}{N}}$$

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi k}{N}} z^{-1} \right)^{n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left(\frac{1 - e^{j2\pi k} z^{-N}}{1 - e^{j\frac{2\pi k}{N}} z^{-1}} \right)$$

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}$$

$$X(z) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}$$

iv) Relationship to the Fourier series coefficients of a continuous-time signal

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi k t F_0}$$

If we sample $x_a(t)$ at $F_s = \frac{N}{T_p} = \frac{1}{T}$

$$x(n) = x_a(nT) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2\pi kn}{N}} = \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j\frac{2\pi kn}{N}}$$

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}$$

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

Properties of DFT

i) Periodicity

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

Then $x(n+N) = x(n) \quad \forall n$

$X(K+N) = X(K) \quad \forall K$

ii) Linearity

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

Then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$

iii) Circular Symmetries of a sequence

N point DFT of $\underset{\text{finite}}{x(n)}$ of length $L \le N$ = N-point DFT of $\underset{\text{periodic}}{x_p(n)}$ of period N when

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$$x_p'(n) = x_p(n-K) = \sum_{l=-\infty}^{\infty} x(n-K-lN)$$

$$x'(n) = \begin{cases} x_p'(n) & , \quad 0 \le n \le N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$x'(n)$ is related to $x(n)$ by a circular shift

$x'(n) = x(n-K, \text{ modulo } N)$

$= x((n-K))_N$

→ Circularly even : if N- point sequence is symmetric about point '0' on the circle

$x(N-n) = x(n) \qquad 1 \le n \le N-1$

→ Circularly odd : if N- point sequence is anti-symmetric about point '0' on the circle

$x(N-n) = -x(n) \qquad 1 \le n \le N-1$

→ Time reversal : it is attained by reversing its samples about point zero on the circle

$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1$

→ even : $x_p(n) = x_p(-n) = x_p(N-n)$

odd : $x_p(n) = -x_p(-n) = -x_p(N-n)$

→ If $x_p(n)$ is complex,

conjugate even : $x_p(n) = x_p^*(N-n)$

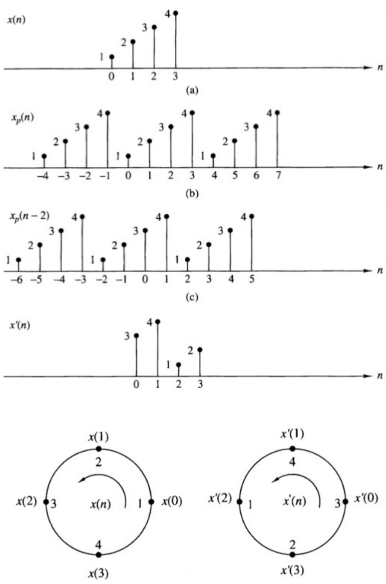
conjugate odd : $x_p(n) = -x_p^*(N-n)$

So, $x_p(n) = x_{pe}(n) + x_{po}(n)$

where $x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$

$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$

iii ex -



Circular Shift

Q. Let $x_p(n)$ be a periodic sequence & 1N point DFT of $x_p(n)$ is $X_1(K)$
3N point DFT of $x_p(n)$ is $X_3(K)$

Write $X_3(K)$ in terms of $X_1(K)$

A. $X_1(K) = \sum_{n=0}^{N-1} x_p(n)\, \omega_N^{Kn}$

$$X_3(K) = \sum_{n=0}^{3N-1} x_p(n)\, \omega_{3N}^{Kn}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, \omega_{3N}^{Kn} + \sum_{n=N}^{2N-1} x_p(n)\, \omega_{3N}^{Kn} + \sum_{n=2N}^{3N-1} x_p(n)\, \omega_{3N}^{Kn}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, \omega_{3N}^{Kn} + \sum_{n=0}^{N-1} x_p(n+N)\, \omega_{3N}^{K(n+N)} + \sum_{n=0}^{N-1} x_p(n+2N)\, \omega_{3N}^{K(n+2N)}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, \omega_N^{\frac{K}{3}n} + \sum_{n=0}^{N-1} x_p(n)\, \omega_N^{\frac{K}{3}n}\, \omega_3^{K} + \sum_{n=0}^{N-1} x_p(n)\, \omega_N^{\frac{K}{3}n}\, \omega_3^{2K}$$

$$= (1 + \omega_3^K + \omega_3^{2K}) \sum_{n=0}^{N-1} x_p(n)\, \omega_N^{\frac{K}{3}n}$$

$$= (1 + \omega_3^K + \omega_3^{2K})\; X_1(K/3)$$

$\Rightarrow \omega_{3N}^{Kn} = \omega_N^{\frac{K}{3}n}$

$\left(e^{-j\frac{2\pi}{3N}}\right)^{Kn} = \left(e^{-j\frac{2\pi}{N}}\right)^{\frac{K}{3}n}$

$\Rightarrow x_p(n) = x_p(n+N) = x_p(n+2N)$

$\Rightarrow \omega_{3N}^{K(n+N)} = \omega_{3N}^{Kn}\, \omega_{3N}^{KN}$

Q. $x(n) = \{1, 2, 3, 4\}$ $\quad K = 2$

$x'(n) = ?$

A. $n=0, \quad x'(n) = x'(0) = (x(0-2))_4 = x(4-2) = x(2) = 3$

$n=1, \quad x'(n) = x'(1) = (x(1-2))_4 = x(4-1) = x(3) = 4$

$n=2, \quad x'(n) = x'(2) = (x(2-2))_4 = x(0) = x(0) = 1$

$n=3, \quad x'(n) = x'(3) = (x(3-2))_4 = x(0+1) = x(1) = 2$

Q. $x(n) = \{\underset{\underset{n=0}{\uparrow}}{2}, -2, 3, 0, 2, -1\}$

Find $x((-n))_6$

A. $x((-n))_6 = \{2, -1, 2, 0, 3, -2\}$

Q. $x(n) = \{1, 2, 3, 4\}$. Find even & odd parts of sequence

A. $x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$

$\frac{1}{2}[x(n) + x(N-n)]$

$x(N-n) = x((-n))_N = \{1, 4, 3, 2\}$

$x_{ce}(n) = \frac{1}{2}\{1+1, 2+4, 3+3, 4+2\}$

$= \{1, 3, 3, 3\}$

$x_{co} = \frac{1}{2}[x(n) - x^*(N-n)]$

$= \frac{1}{2}[x(n) - x(N-n)]$

$x_{co} = \frac{1}{2}\{1-1, 2-4, 3-3, 4-2\}$

$= \{0, -1, 0, 1\}$

iv) Symmetric properties of DFT

Let $x(n) = x_R(n) + jx_I(n)$, $0 \le n \le N-1$

$X(k) = X_R(k) + jX_I(k)$, $0 \le k \le N-1$

Also, $X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$

$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]$

and, $x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi kn}{N} - X_I(k) \sin \frac{2\pi kn}{N} \right]$

$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi kn}{N} + X_I(k) \cos \frac{2\pi kn}{N} \right]$

case (i) : Real valued sequence

$X(N-k) = X^*(k) = X(-k)$

$|X(N-k)| = |X(k)|$

$\angle X(N-k) = -\angle X(k)$

case (ii) : Real & Even sequence

$x(n) = x(N-n)$, $0 \le n \le N-1$

$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N}$, $0 \le k \le N-1$

$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) \cos \frac{2\pi kn}{N}$, $0 \le n \le N-1$

case (iii) : Real & Odd sequence $x_I(n)=0$

$x(n) = -x(N-n)$, $0 \le n \le N-1$

$X(k) = -j\sum_{n=0}^{N-1} x(n) \sin \frac{2\pi kn}{N}$, $0 \le k \le N-1$ (purely imaginary and odd)

$x(n) = \frac{j}{N} \sum_{n=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}$, $0 \le n \le N-1$

case(iv) : $x(n)$ is purely imaginary and even

$x_R(n)=0$

- $X(k)$ is purely imaginary and even

$X_R(k)=0$

- $x(n)$ is imaginary and even.

case (v): $x(n)$ is imaginary and odd $\quad x_R(n)=0$

- $X(k)$ is real & odd

$$X(k) = X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N}$$

- $x(n)$ is imaginary and odd

case (vi): $x(n)$ is purely imaginary

- $x(k)$ exhibits conjugate anti-symmetry
- $x(k) = -x^*(N-k)$

Symmetry Properties

Time domain,		Frequency domain,
$x(n)$	$\xleftrightarrow[N]{DFT}$	$x(k)$
$x^*(n)$	$\xleftrightarrow[N]{DFT}$	$x^*(N-k)$
$x^*(N-n)$	$\xleftrightarrow[N]{DFT}$	$x^*(k)$
$x_R(n)$	$\xleftrightarrow[N]{DFT}$	$X_{ce}(k) = \frac{1}{2}[X(k) + x^*(N-k)]$

Summary:

$$x(n) = \overset{2}{x_R^e(n)} + \overset{3}{x_R^o(n)} + j\,\overset{4}{x_R^e(n)} + j\,\overset{5}{x_I^o(n)}$$

$$x(k) = X_R^e(k) + X_R^o(k) + j\,X_I^e(k) + j\,X_I^o(k)$$

1 $x(n)$ is real $\longrightarrow$ $x(k) = x^*(N-k)$
conjugate symmetry

6 $x(n)$ is imaginary $\longrightarrow$ $x(k) = -x^*(N-k)$
conjugate anti symmetry

$$j\,x_I(n) \xleftrightarrow[N]{DFT} X_{co}(k) = \frac{1}{2}\left[X(k) - X^*(N-k)\right]$$

$$x_{ce} = \frac{1}{2}\left[x(n) + x^*(N-n)\right] \xleftrightarrow[N]{DFT} X_R(k)$$

$$x_{co} = \frac{1}{2}\left[x(n) - x^*(N-n)\right] \xleftrightarrow[N]{DFT} jX_I(k)$$

$x(n)$ is real

$$x(k) = x^*(N-k)$$
$$x_R(k) = x_R(N-k)$$
$$x_I(k) = -x_I(N-k)$$
$$|x(k)| = |x(N-k)|$$
$$\angle x(k) = -\angle x(N-k)$$

$$|X(k)| = \sqrt{X_R^2(k) + X_I^2(k)}$$

$$\angle X(k) = \tan^{-1}\left[\frac{X_I(k)}{X_R(k)}\right]$$

Multiplication of 2 DFT's & circular Convolution

- Linear convolution $x_3(m) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(m-n)$
- For, circular convolution,

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \qquad X_3(k) = X_1(k)\, X_2(k)$$
$$x_2(n) \xleftrightarrow[N]{DFT} X_2(k) \qquad x_3(m) = IDFT\{X_3(k)\}$$

Circular convolution

Time domain eq^n for Circular Convolution

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \qquad X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, W_N^{kn} \quad \text{— (1)}$$

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(k) \qquad X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, W_N^{kn} \quad \text{— (2)}$$

$$X_3(k) = X_1(k) \cdot X_2(k) \quad \text{—— (3)}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, W_N^{-km} \quad \text{—— (4)}$$

using (3) in (4)

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) \cdot W_N^{-km}$$

use (1) & (2)

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n)\, W_N^{kn} \right] \left[\sum_{l=0}^{N-1} x_2(l)\, W_N^{kl} \right] W_N^{-km}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} W_N^{kn}\, W_N^{kl}\, W_N^{-km}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} \underbrace{W_N^{(l+n-m)k}}$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \dfrac{1-a^n}{1-a} & a \neq 1 \end{cases}$$

$$\sum_{k=0}^{N-1} \left(W_N^{(l+n-m)} \right)^k = \begin{cases} N & l = m-n+pN = (m-n)_N \\ \dfrac{1 - W_N^{(l+n-m)N}}{1 - W_N^{(l+n-m)}} = 0, & \text{otherwise} \end{cases}$$

$$W_N^{(l+n-m)N} = e^{-j\frac{2\pi}{N}(l+n-m)} = e^{-j2\pi(r)} = 1$$

$$\therefore \frac{1-1}{1-W_N^{(l+n-m)}} = 0$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} a^k$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

Q. Let $x_1(n) = \{2, 1, 2, 1\}$ and $\{1, 2, 3, 4\}$

A. $x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$

a) Graphical Method

$x_1(n) \Rightarrow$

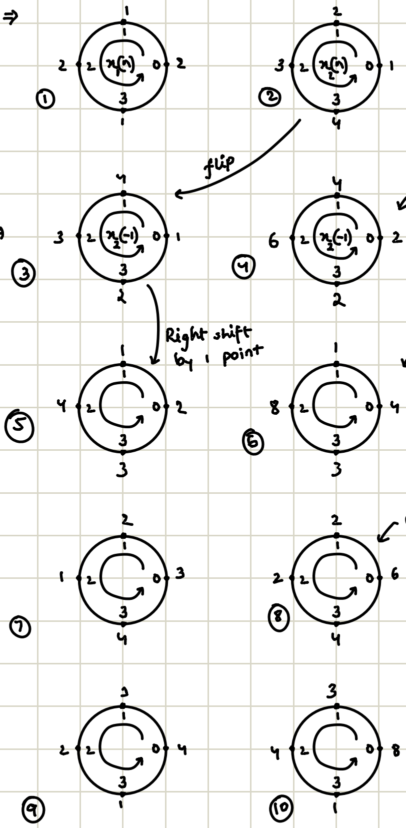
$N = 4$

$m=0 \Rightarrow$

$m=1 \Rightarrow$

$m=2$

$m=3$



① × ③ $= x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))_4 \Rightarrow 2+4+6+2 = 14$

① × ⑤ $= x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4 \Rightarrow 4+1+8+3 = 16$

① × ⑦ $= x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4 \Rightarrow 6+2+2+4 = 14$

$= x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4 \Rightarrow 8+3+4+1 = 16$

$\Rightarrow x_3(m) = \{14, 16, 14, 16\}$

b) Frequency Domain Method

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) W_N^{Kn}$$

$$X_1(K) = 2 + W_4^K + 2W_4^{3K} + W_4^{3K}$$

$$X_2(K) = 1 + 2W_4^K + 3W_4^{2K} + 4W_4^{3K}$$

$$X_3(K) = X_1(K)\, X_2(K)$$

$$= \left(2 + W_4^K + 2W_4^{3K} + W_4^{3K}\right)\left(1 + 2W_4^K + 3W_4^{2K} + 4W_4^{3K}\right)$$

$$= 2 + 4W_4^K + 6W_4^{2K} + 8W_4^{3K} + W_4^K + 2W_4^{2K} + 3W_4^{3K} + 4W_4^{4K}$$
$$+ 2W_4^{2K} + 4W_4^{3K} + 6W_4^{4K} + 8W_4^{5K} + W_4^{3K} + 2W_4^{4K} + 3W_4^{5K} + 4W_4^{6K}$$

$$\left(\begin{matrix} W_4^{5K} = W_4^K \\ W_4^{6K} = W_4^{2K} \end{matrix}\right)$$

$$= (2+4+6+2) + (4+1+8+3)W_4^K + (6+2+2+4)W_4^{2K} + (8+3+4+1)W_4^{3K}$$

$$\delta(n-n_0) \xleftrightarrow{DFT} W_4^{Kn_0}$$

$$X_3(K) = 14 + 16W_4^K + 14W_4^{2K} + 16W_4^{3K}$$

$$X_3(K) \xleftrightarrow{IDFT} x_3(m)$$

$$x_3(m) = 14\,\delta(n) + 16\,\delta(n-1) + 14\,\delta(n-2) + 16\,\delta(n-3$$

$$= \{14, 16, 14, 16\}$$

c) Matrix Approach

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+2+6+4 \\ 1+4+3+8 \\ 2+2+6+4 \\ 1+4+3+8 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

d) Linear Convolution

$$x_1 = \{2, 1, 2, 1\}$$

$$x_2 = \{1, 2, 3, 4\}$$

$$x_3 = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(m-n)$$

	2	1	2	1
1	2	1	2	1
2	4	2	4	2
3	6	3	6	3
4	8	4	8	4

$$\Rightarrow \{\underset{\substack{\uparrow \\ n=0}}{2}, 5, 10, 16, 12, 11, \underset{\substack{\uparrow \\ n=6}}{4}\}$$

Note : Circular convolution is an aliased version of linear convolution

Relation b/w linear & circular ⇒

2	5	10	16
12	11	4	0
{14	16	14	16}

Q. Let $x_1(n) = \{0, 1, 2, 3\}$

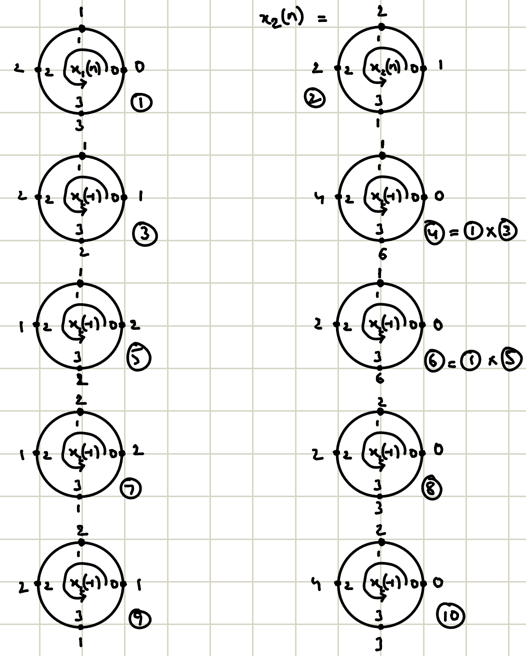
$x_2(n) = \{1, 2, 2, 1\}$

Compute 4-Point Circular convolution of the 2 sequences

$x_3(n) = x_1(n) \ (N)_4 \ x_2(n)$ (or) $x_1(n) \ (C)_4 \ x_2(n)$

a) Graphical b) Frequency c) Matrix

A. a) $x_1(n) =$ $x_2(n) =$



$= 0 + 1 + 4 + 6 = 11$

$= 0 + 1 + 2 + 6 = 9$

$= 0 + 2 + 2 + 3 = 7$

$= 0 + 2 + 4 + 3 = 9$

b)
$$X_1(K) = 0 + W_4^{K} + 2W_4^{2K} + 3W_4^{3K}$$
$$X_2(K) = 1 + 2W_4^{K} + 2W_4^{2K} + W_4^{3K}$$
$$X_3(K) = X_1(K)\,X_2(K)$$
$$= 0+0+0+0 + W_4^{K} + 2W_4^{2K} + 2W_4^{3K} + W_4^{4K} + 2W_4^{2K} + 4W_4^{3K} + 4W_4^{4K} + 2W_4^{5K} + 3W_4^{3K} + 6W_4^{4K} + 6W_4^{5K} + 3W_4^{6K}$$
$$= (1+4+6) + (1+2+6)W_4^{K} + (2+2+3)W_4^{2K} + (2+4+3)W_4^{3K}$$
$$= 11 + 9W_4^{K} + 7W_4^{2K} + 9W_4^{3K}$$
$$x_3(n) = 11\,\delta(n) + 9\,\delta(n-1) + 7\,\delta(n-2) + 9\,\delta(n-3)$$
$$= \{11, 9, 7, 9\}$$

c)
$$\begin{bmatrix} 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+6+4+1 \\ 1+0+6+2 \\ 2+2+0+6 \\ 3+4+2+0 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 7 \\ 9 \end{bmatrix}$$

Additional Properties of DFT

i) Time Reversal

If $x(n) \xleftrightarrow[n]{DFT} X(k)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} x((-k))_N = x(N-k)$

Proof :

$$DFT(x(N-n)) = \sum_{n=0}^{N-1} x(N-n)\, e^{-j\frac{2\pi kn}{N}}$$

Let $m = N-n$

$$DFT(x(N-n)) = \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi k(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi k} \cdot e^{j\frac{2\pi km}{N}} \qquad \left(e^{-j\frac{2\pi kN}{N}} = e^{-j2\pi k} = 1\right)$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi km}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi km}{N}} \cdot e^{-j\frac{2\pi kN}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi m(N-k)}{N}}$$

$$= X(N-k)$$

ii) Circular Time Shift

If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j\frac{2\pi kl}{N}}$

Proof :

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N\, e^{-j\frac{2\pi kn}{N}} + \sum_{n=l}^{N-1} x(n-l)\, e^{-j\frac{2\pi kn}{N}}$$

$$x((n-l))_N = x(N-l+n)$$

$$\sum_{n=0}^{l-1} x((n-l))_N\, e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{l-1} x(N-n+l)\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{m=N-l}^{N-1} x(m)\, e^{-j\frac{2\pi k(m+l)n}{N}}$$

$$\sum_{n=l}^{N-1} x(n-l)\, e^{-j\frac{2\pi kn}{N}} = \sum_{m=0}^{N-1-l} x(m)\, e^{-j\frac{2\pi k(m+l)n}{N}}$$

$$DFT\{x((n-l))\} = \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi k(m+l)n}{N}} = X(k)\, e^{-j\frac{2\pi kln}{N}}$$

iii) Circular Frequency Shift

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x(n)\, e^{\frac{j2\pi l n}{N}} \xleftrightarrow[N]{DFT} X((K-l))_N$

Proof : $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi l n}{N}} . e^{\frac{-j2\pi K n}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi (K-l) n}{N}} = X((K-l))_N$$

iv) Circular Convolution

$x(n) \, (N) \, h(n) \xleftrightarrow{DFT} X(K) . H(K)$

Proof : $DFT\{x(n) \, (N) \, h(n)\} = DFT\left\{\sum_{l=0}^{N-1} x(l)\, h((n-l))_N\right\}$

$$= \sum_{l=0}^{N-1} x(l)\; DFT\{h((n-l))_N\} = \sum_{l=0}^{N-1} x(l)\; W_N^{Kl}\; H(K)$$

$$= X(K)\; H(K)$$

$h((n-l))_N = W_N^{Kl}\, H(K)$

v) Multiplication in time

$x(n) . y(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X(K) \, (N) \, Y(K)\right]$

Proof : $DFT\{x(n).y(n)\} = DFT\left\{\frac{1}{N}\sum_{l=0}^{N-1} X(l)\, W_N^{-ln} . y(n)\right\}$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X(l)\; DFT\{y(n)\, W_N^{-ln}\} = \frac{1}{N}\sum_{l=0}^{N-1} X(l)\; Y((K-l))_N$$

$$= \frac{1}{N}\, X(K) \, (N) \, Y(K)$$

vi) Parseval Identity

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

$y(n) \xleftrightarrow[N]{DFT} Y(K)$

Then $\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N}\sum_{K=0}^{N-1} X^*(K)\, Y(K)$

Proof : $\sum_{n=0}^{N-1} x^*(n)\, y(n) = \sum_{n=0}^{N-1} x^*(n)\left(\frac{1}{N}\sum_{K=0}^{N-1} Y(K)\, e^{\frac{j2\pi K n}{N}}\right) = \sum_{n=0}^{N-1}\sum_{K=0}^{N-1}\frac{1}{N}\, Y(K)\, x^*(n)\, e^{\frac{j2\pi K n}{N}} = \frac{1}{N}\sum_{K=0}^{N-1} X^*(K)\; Y(K)$

vii) Complex - Conjugate

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

Then $x^*(n) \xleftrightarrow[N]{DFT} X^*(N-K)$

Proof : $X(K) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{Kn} = \left[\sum_{n=0}^{N-1} x(n)\, W_N^{-Kn}\right]^*$

$$= \sum_{n=0}^{N-1}\left(x(n)\; W_N^{(N-K)n}\right)^* = X^*(N-K)$$

$$x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$$

Proof: $\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{-kn}$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{kn}\right]^*$$

$$= \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{(N-n)k}\right]^*$$

$$= x^*(N-n)$$

viii) Duality of DFT

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$X(n) \xleftrightarrow[N]{DFT} N\, x((-k))_N$$

Proof: $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn}$

$$DFT\{X(n)\} = \sum_{n=0}^{N-1} X(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} x(m)\, W_N^{mn}\right) W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} W_N^{(m+k)n}$$

$$= \begin{cases} \sum_{m=0}^{N-1} x(m)\, N & , m=-k \\ \sum_{m=0}^{N-1} x(m)\left(\frac{1-\cancel{W_N^{(m+k)N}}^{\,1}}{1-W_N^{m+k}}\right) & , m \neq -k \end{cases}$$

$$= N\, x((-k))_N$$

$$DFT\{DFT\{x(n)\}\} = N\, x((-n))_N$$

$$DFT\{DFT\{DFT\{DFT\{x(n)\}\}\}\} = N^2 x(n)$$

Summary :

1) Time Reversal : $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

2) Circular Time Shift : $x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{\frac{-j2\pi k l}{N}}$

3) Circular Frequency Shift : $x(n)\, e^{\frac{j2\pi l n}{N}} \xleftrightarrow[N]{DFT} X((k-l))_N$

4) Circular Convolution : $x(n) \circledN h(n) \xleftrightarrow{DFT} X(k).H(k)$

5) Multiplication in time : $x(n).h(n) \xleftrightarrow{DFT} X(k) \circledN H(k)$

6) Parseval's Identity : $\sum_{n=0}^{N-1} x^*(n)\, y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, Y(k)$

7) Complex Conjugate : $x^*(n) \xleftrightarrow{DFT} X^*(N-k)$

$$x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$$

8) Duality : $X(n) \xleftrightarrow[N]{DFT} N\, x((-k))_N$

9) Circular Correlation : $r_{xy}(l) = x(l) \circledN y^*(-l)_N$

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k)$$

$$R_{xy}(k) = X(k)\, Y^*(k)$$

Q. Let $X(K)$ be a 4 point DFT of $x(n)$ and $X(K) = \{0, 1+j, -1, 1-j\}$

Find DFT

A. $x((n-1))_4 \xleftrightarrow[4]{DFT} W_4^{K(1)} X(K)$

$= W_4^K X(K)$

$= \{ W_4^0 X(0), W_4^1 X(1), W_4^2 X(2), W_4^3 X(3) \}$

$= \{ (1)(0), (-j)(1+j), (-1)(-1), (j)(1-j) \}$

$= \{ 0, 1-j, +1, 1+j \}$

Q. $x((-n))_4 \longleftrightarrow X(-K)_4$

A. $= [0, 1-j, -1, 1+j]$

Q. $x_1(n) = \{(0,0,1,0) \text{ N}_4\, x(n)\}$

$x_2(n) = e^{\frac{j\pi n}{2}} x(n)$

$x_3(n) = \cos\left(\frac{\pi n}{2}\right) x(n)$

$X_1(K) = ?$ $X_2(K) = ?$ $X_3(K) = ?$

A. $x_1(n) = \{0,0,1,0 \text{ N}_4\, x(n)\}$

$= \{ \delta(n-2) \text{ N}_4\, x(n) \}$

$= x((n-2))_4$

$x_1(n) \xleftrightarrow{DFT} X_1(K) = W_4^{2K} X(K)$

$= [W_4^0, W_4^2, W_4^4, W_4^6]\, X(K)$

$= [1, -1, 1, -1][0, 1+j, -1, 1-j]$

$= [0, -1-j, -1, -1+j]$

$x_2(n) = e^{j\frac{\pi}{2}n} x(n)$

$= e^{j\frac{2\pi}{4}(1)n} x(n) \xleftrightarrow[4]{DFT} X((K-1))_4$ → Circular frequency shift

$X(K) = [0, 1+j, -1, 1-j]$

$X((K-1))_4 = [1-j, 0, 1+j, -1]$

$x_3(n) = \cos\left(\frac{\pi}{2}n\right) x(n) = \dfrac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi n}{2}}}{2} x(n) = \frac{1}{2}\left(e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi n}{4}}\right) x(n)$

$x_3(n) \xleftrightarrow{DFT} \frac{1}{2}\left[X((K-1))_4 + X((K+1))_4\right]$ → Circular frequency shift

$= \frac{1}{2}\left[(1-j, 0, 1+j, -1) + (1+j, -1, 1-j, 0)\right]$

$= \frac{1}{2}\left[2, -1, 2, -1\right] = \left[1, \frac{-1}{2}, 1, \frac{-1}{2}\right]$

Q. Let $x(n)$ & $h(n)$ be 2 4-point sequence (Convolution)

$$x(n) \xleftrightarrow[4]{DFT} X(K) = \{1, -2, 1, -2\}$$

$$h(n) \xleftrightarrow[4]{DFT} H(K) = [1, j, 1, -j]$$

Find DFT of $y(n) = x(n) \; Ⓝ_4 \; h(n)$

A. $y(n) \longleftrightarrow Y(K) = X(K)\,H(K)$

$$= [1, -2, 1, -2][1, j, 1, -j]$$

$$= [1, -2j, 1, +2j]$$

Q. Let $x(n)$ be a 8 point sequence $\{1,1,1,1,1,1,1,1\}$ and $h(n) = \cos(0.25\pi n)$.

Compute DFT of $y(n) = x(n) \cdot h(n)$

A. $y(n) \xleftrightarrow{DFT} Y(K)$

$$x(n) \cdot h(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X(K) \; Ⓝ_4 \; H(K)\right]$$

$$\cos\frac{2\pi K_0 n}{N} \xrightarrow{DFT} \frac{N}{2}\left[\delta(K-K_0) + \delta(K+K_0)\right]$$

$$X(K) = \{8, 0, 0, 0, 0, 0, 0, 0\}$$

$$H(K) = \sum_{n=0}^{7} h(n)\, W_8^{Kn}$$

$$= \sum_{n=0}^{7} \left(\frac{e^{j0.25\pi n} + e^{-j0.25\pi n}}{2}\right) e^{\frac{-j2\pi Kn}{8}}$$

$$= \frac{1}{2}\sum_{n=0}^{7} \left(e^{-j\frac{\pi n}{4}(K-1)} + e^{-j\frac{\pi n}{4}(K+1)}\right)$$

$$= \frac{1}{2}\left(8\,\delta(K-1) + 8\,\delta(K+1)\right)$$

$$= \left(4\,\delta(K-1) + 4\,\delta(K+1)\right)$$

$$= \{0, 4, 0, 0, 0, 0, 0, 4\}$$

$$Y(K) = \frac{X(K) \; Ⓝ \; H(K)}{N} = \frac{1}{8}\begin{bmatrix} 8&0&0&0&0&0&0&0 \\ 0&8&0&0&0&0&0&0 \\ 0&0&8&0&0&0&0&0 \\ 0&0&0&8&0&0&0&0 \\ 0&0&0&0&8&0&0&0 \\ 0&0&0&0&0&8&0&0 \\ 0&0&0&0&0&0&8&0 \\ 0&0&0&0&0&0&0&8 \end{bmatrix}\begin{bmatrix} 0\\4\\0\\0\\0\\0\\0\\4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 0\\32\\0\\0\\0\\0\\0\\32 \end{bmatrix} = \begin{bmatrix} 0\\4\\0\\0\\0\\0\\0\\4 \end{bmatrix}$$

or $$\frac{X(K) \; Ⓝ \; H(K)}{N} = \frac{\cancel{8}\,\delta(K) \; Ⓝ_8 \; H(K)}{\cancel{8}} = \delta(K) \; Ⓝ_8 \; H(K) = H(K)$$

Q. $X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, \_\_, \_\_, \_\_\}$

Compute the remaining terms using complex conjugate property

Given $x(n)$ is real

A. $\underset{\uparrow \text{ real}}{x(n)} \xleftrightarrow[8]{DFT} X(K)$

Then, $X(K) = X^*(N-K)$

$= X^*(8-K)$

$X(5) = X^*(8-5) = X^*(3) = 1+j0.414$

$X(6) = X^*(2) = 0$

$X(7) = X^*(1) = 1+j2.414$

Q. Find the even samples of a 5-point DFT

$X(K) = \{3, \_\_\_, 0.5+j1.538, \_\_\_\_, 0.5-j0.364\}$. $x(n)$ is real

A. $X(K) = X^*(N-K)$

$X(K) = X^*(5-K)$

$X(1) = X^*(4)$

$= 0.5+j0.364$

$X(3) = X^*(2)$

$= 0.5-j1.538$

$X(K) = \{3, 0.5+j0.364, 0.5+j1.538, 0.5-j1.538, 0.5-j0.364\}$

Q. Find the quantity $\sum_{n=0}^{N-1} x_1(n) \cdot x_2^*(n)$

a) $x_1(n) = x_2(n) = \cos \frac{2\pi n}{N}$

b) $x_1(n) = \cos \frac{2\pi n}{N}$ $\quad x_2(n) = \sin \frac{2\pi n}{N}$

c) $x_1(n) = \delta(n) + \delta(n-8)$ $\quad x_2(n) = u(n) - u(n-N)$ $\quad$ where $N > 8$

A. a) $\sum_{n=0}^{N-1} \left(\frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}}{2} \right)\left(\frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}}{2} \right) = \sum_{n=0}^{N-1} \left(\frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}}{2} \right)^2 = \frac{1}{4} \sum_{n=0}^{N-1} \left[e^{j\frac{4\pi n}{N}} + e^{-j\frac{4\pi n}{N}} + 2 \right]$

$= \frac{1}{4} \left[\sum_{n=0}^{N-1} e^{j\frac{4\pi n}{N}} + \sum_{n=0}^{N-1} e^{-j\frac{4\pi n}{N}} + \sum_{n=0}^{N-1} 2 \right] = \frac{1}{4} \left(\left[\cancelto{0}{\frac{1-e^{j\frac{4\pi N}{N}}}{1-e^{j\frac{4\pi}{N}}}} \right] + \left[\cancelto{0}{\frac{1-e^{-j\frac{4\pi N}{N}}}{1-e^{-j\frac{4\pi}{N}}}} \right] + 2N \right) = \frac{N}{2}$

b) $\sum_{n=0}^{N-1} \left(\frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}}{2} \right)\left(\frac{e^{j\frac{2\pi n}{N}} - e^{-j\frac{2\pi n}{N}}}{-2j} \right) = \frac{1}{4j} \sum_{n=0}^{N-1} \left(\left(e^{-j\frac{2\pi n}{N}}\right)^2 - \left(e^{j\frac{2\pi n}{N}}\right)^2 \right)$

$= \frac{1}{4j} \left(\frac{1-\cancelto{1}{e^{-j\frac{4\pi N}{N}}}}{1-e^{-j\frac{4\pi}{N}}} - \frac{1-\cancelto{1}{e^{j\frac{4\pi N}{N}}}}{1-e^{j\frac{4\pi}{N}}} \right) = \frac{1}{4j}(0-0) = 0$

c) $x_2(n)$ is real $\Rightarrow x_2(n) = x_2^*(n)$

$\sum_{n=0}^{N-1} x_1(n)\, x_2(n) = 1+1 = 2$

Q. Compute the DFT of $x_1(n) = \cos\left(\frac{2\pi K_0 n}{N}\right)x(n)$, $x_2(n) = \sin\left(\frac{2\pi K_0 n}{N}\right)x(n)$

Write the answer in terms of $X(K)$

A. $$X_1(K) = \sum_{n=0}^{N-1}\left(\frac{e^{j\frac{2\pi K_0 n}{N}} + e^{-j\frac{2\pi K_0 n}{N}}}{2}\right)x(n)$$

take as $X_{1a}(K)$ and $X_{1b}(K)$

$$X_{1a}(K) = \frac{N}{2}\left[\delta(K-K_0)+\delta(K+K_0)\right]$$

$$X_{1b}(K) = X(K)$$

$$X_1(K) = \frac{1}{2}\left[X((K-K_0))_N + X(K+K_0)\right]$$

$$X_2(K) = \sum_{n=0}^{N-1}\left(\frac{e^{j\frac{2\pi K_0 n}{N}} - e^{-j\frac{2\pi K_0 n}{N}}}{2j}\right)x(n)$$

take as $X_{2a}(K)$ and $X_{2b}(K)$

$$X_{2a}(K) = \frac{N}{2j}\left[\delta(K-K_0)-\delta(K+K_0)\right]$$

$$X_{2b}(K) = X(K)$$

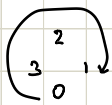
$$X_2(K) = \frac{1}{2j}\left[X((K-K_0))_N - X(K+K_0)\right]$$

Q. Let $x_1(n) = \{0,1,2,3\}$ & $x_2(n) = \{1,-1,1,1\}$

Compute DFT of $x_1(n)$ & $x_2(n)$ using a single 4 point DFT

A. $$x(n) = x_1(n) + jx_2(n)$$
$$= [0+j,\ 1-j,\ 2+j,\ 3+j]$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} j \\ 1-j \\ 2+j \\ 3+j \end{bmatrix} = \begin{bmatrix} 6+2j \\ j-j-1-2-j+3j-1 \\ j-1+j+2+j-3-j \\ j+j+1-2-j-3j+1 \end{bmatrix} = \begin{bmatrix} 6+2j \\ -4+2j \\ -2+2j \\ -2j \end{bmatrix}$$



$$x_1(n) \xleftrightarrow[4]{DFT} X_1(K)$$
$$x_2(n) \xleftrightarrow[4]{DFT} X_2(K)$$

$$X_1(K) = \frac{1}{2}\left[X(K) + X^*(N-K)_N\right] = \frac{1}{2}\left[6+2j+6-2j,\ -4+2j+2j,\ -2+2j-2-2j,\ -2j-4-2j\right]$$
$$= [6,\ -2+2j,\ -2,\ -2-2j]$$

$$X_2(K) = \frac{1}{2j}\left[X(K) - X^*(N-K)_N\right] = \frac{1}{2j}\left[6+2j-6+2j,\ -4+2j-2j,\ -2+2j+2+2j,\ -2j+4+2j\right]$$
$$= \frac{1}{j}\left[2j,\ -2,\ 2j,\ 2\right] = [2,\ 2j,\ 2,\ -2j]$$

Q. Let $x_1(n)$ & $x_2(n)$ be an N-point sequence. Find N point DFT of the 2 sequences using a single N-point DFT

A.

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$$

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

$$x(n) = x_1(n) + j\,x_2(n)$$

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$X_1(k) = \frac{1}{2}\left[X(k) + X^*(N-k)_N\right]$$

$$X_2(k) = \frac{1}{2j}\left[X(k) - X^*(N-k)_N\right]$$

Correlation (Lab & Assignment Only)

→ $x(n)$ & $y(n)$ → Cross Correlation ↓ $\text{Conv}(x[n], \text{flip}(y[n]))$

$x(n)$ & $z(n)$ → Circular Correlation ↓ $\text{Conv}(x[n], \text{flip}(z[n]))$

Circular Correlation

$x(n)$ & $y(n)$ are complex

→ $r_{xy}(l) = x(l) \, \text{(N)} \, y^*(-l)_N$

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k)$$

$$\left(x^*(-n)_N = x^*(N-n) \xleftrightarrow[N]{DFT} X(\ ) \right)$$

$$R_{xy}(k) = X(k)\, Y^*(k)$$

Q. $x_1(n) = (1, 2, 3)$ $\quad x_2(n) = (3, 2, 1)$

Find cross-relation

A. $l = 0, 1, 2, -1, -2$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l)$$

$l = 0 \Rightarrow \sum_{n=-\infty}^{\infty} x(n)\, y(n)$

1	2	3
3	2	1

$3 + 4 + 3 = 10$

$l = 1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n)\, y(n-1)$

1	2	3	0
0	3	2	1

$0 + 6 + 6 + 0 = 12$

$l = 1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n)\, y(n-2)$

1	2	3	0	0
0	0	3	2	1

$0 + 0 + 9 + 0 + 0 = 9$

$l = -1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n)\, y(n+1)$

0	1	2	3
3	2	1	0

$0 + 2 + 2 + 0 = 4$

$l = -1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n)\, y(n+2)$

0	0	1	2	3
3	2	1	0	0

$0 + 0 + 1 + 0 + 0 = 1$

$$r_{xy}(l) = \{\ldots, 1, 4, 10, 12, 9 \ldots\}$$

Q. Find autocorrelation for $x(n) = (1, 2, 3)$

A. $r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l)$

$l = 0 \Rightarrow \sum_{n=-\infty}^{\infty} x(n) x(n)$

1	2	3
1	2	3

$1 + 4 + 9 = 14$

$l = 1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n) x(n-1)$

1	2	3	0
0	1	2	3

$0 + 2 + 6 + 0 = 8$

$l = 1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n) x(n-2)$

1	2	3	0	0
0	0	1	2	3

$0 + 0 + 3 + 0 + 0 = 3$

$l = -1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n) x(n+1)$

0	1	2	3
1	2	3	0

$0 + 2 + 6 + 0 = 8$

$l = -1 \Rightarrow \sum_{n=-\infty}^{\infty} x(n) x(n+2)$

0	0	1	2	3
1	2	3	0	0

$0 + 0 + 3 + 0 + 0 = 3$

$r_{xy}(l) = \{\ldots 3, 8, 14, 8, 3 \ldots\}$

Q. Let $x(n) = \cos \frac{2\pi n}{N}$ & $y(n) = \sin \frac{2\pi n}{N}$

i) Find N-point circular convolution of $x(n)$ & $y(n)$
ii) Find N-point circular correlation of $x(n)$ & $y(n)$
iii) Find N-point circular auto correlation of $x(n)$
iv) Find N-point circular auto correlation of $y(n)$

A. $x(n) \xleftrightarrow[N]{DFT} X(k) = \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]$

$y(n) \xleftrightarrow[N]{DFT} Y(k) = \frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]$

i) $v(n) = x(n) \circledN y(n) \xleftrightarrow[N]{DFT} X(k)\, Y(k)$

$v(n) = IDFT\{X(k) Y(k)\}$

$= IDFT\left\{ \frac{N}{2}\left(\delta(k-1) + \delta(k+1)\right) \cdot \frac{N}{2j}\left(\delta(k-1) - \delta(k+1)\right) \right\}$

$= IDFT\left\{ \frac{N^2}{4j}\left(\delta(k-1) - \delta(k+1)\right) \right\}$

$= \frac{N}{2}\; IDFT\left\{ \frac{N}{2j}\left(\delta(k-1) - \delta(k+1)\right) \right\}$

$v(n) = \frac{N}{2} \sin \frac{2\pi n}{N}$

$(\delta(k-1))^2 = \delta(k-1)$

$(\delta(k+1))^2 = \delta(k+1)$

$(a+b)(a-b) = a^2 - b^2$

ii) $r_{xy} = x(n) \text{ corr } y(n)$

$$r_{xy} \xleftrightarrow[N]{DFT} R_{xy} = X(k)\,Y^*(k)$$

$$v(n) = IDFT\{X(k)\,Y^*(k)\}$$

$$= IDFT\left\{\frac{N}{2}\left[\delta(k-1)+\delta(k+1)\right] \cdot \frac{-N}{2j}\left[\delta(k-1)-\delta(k+1)\right]\right\}$$

$$= IDFT\left\{\frac{-N^2}{4j}\left[\delta(k-1)-\delta(k+1)\right]\right\}$$

$$= \frac{-N}{2}\; IDFT\left\{\frac{N}{2j}\left[\delta(k-1)-\delta(k+1)\right]\right\}$$

$$r_{xy}(l) = \frac{-N}{2}\sin\frac{2\pi l}{N}$$

iii) $r_{xx}(l) = x(l) \; \textcircled{N} \; x^*(-l)_N$

$$r_{xx}(l) \xleftrightarrow{DFT} R_{xx}(l) = \{X(k)\,X^*(k)\}$$

$$r_{xx}(l) = IDFT\left\{\frac{N}{2}\left[\delta(k-1)+\delta(k+1)\right] \cdot \frac{N}{2}\left[\delta(k-1)+\delta(k+1)\right]\right\}$$

$$= IDFT\left\{\frac{N^2}{4}\left[\delta(k-1) + \cancelto{0}{2\delta(k-1)\delta(k+1)} + \delta(k+1)\right]\right\}$$

$$= \frac{N}{2}\cos\frac{2\pi l}{N}$$

iv) $r_{yy}(l) = y(l) \; \textcircled{N} \; y^*(-l)_N$

$$r_{yy}(l) \xleftrightarrow{DFT} R_{yy}(l) = \{Y(k)\,Y^*(k)\}$$

$$r_{yy}(l) = IDFT\left\{\frac{N}{2j}\left[\delta(k-1)-\delta(k+1)\right] \cdot \frac{-N}{2j}\left[\delta(k-1)-\delta(k+1)\right]\right\}$$

$$= IDFT\left\{\frac{N^2}{4}\left[\delta(k-1)+\delta(k+1)\right]\right\}$$

$$= \frac{N}{2}\cos\frac{2\pi l}{N}$$

Q. Compute the energy of $x(n) = \cos\frac{2\pi k_0 n}{N}$ where $n = 0$ to $N-1$

A. $E = \sum_{n=0}^{N-1} x(n)\,x^*(n)$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right] \cdot \frac{1}{2}\left[e^{j\frac{2\pi k_0 n}{N}} + e^{-j\frac{2\pi k_0 n}{N}}\right]\right]$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{4}\left(e^{j\frac{4\pi k_0 n}{N}} + e^{-j\frac{4\pi k_0 n}{N}} + 2\right)\right]$$

$$= \frac{N}{2}$$ (refer pg 29)

Q. Let $X(K)$ be a 4 point DFT of $x(n) = \{1, -j, 2, 3j\}$. Find the DFT of

a) $x^*(n)$

b) $x(-n)_N$

c) $\mathrm{Re}\{x(n)\}$

d) $\mathrm{Im}\{x(n)\}$

A. a) $\mathrm{DFT}[x^*(n)] = X^*(N-K) = (1, -3j, 2+j)$

b) $\mathrm{DFT}[x(-n)_N] = X(-K)_N = [1, 3j, 2, -j]$

c) $= \frac{1}{2}[X(K) + X^*(N-K)] = \frac{1}{2}(1+1, -3j-j, 2+2, j+3j) = (1, -2j, 2, 2j)$

d) $= \frac{1}{2j}[X(K) - X^*(N-K)] = \frac{1}{2j}(1-1, 3j \quad = (0, 1, 0, 1)$

Q. Let $x(n)$ be a 4 point IDFT of $X(K)$ where $x(n) = [1, -2j, j, -4j]$. Compute IDFT of

a) $X^*(K)$

b) $X(-K)_N$

c) $\mathrm{Re}\{X(K)\}$

d) $\mathrm{Im}\{X(K)\}$

A. a) $\mathrm{IDFT}\{X^*(K)\} = x^*(N-n) = [1, 4j, -j, 2j]$

b) $\mathrm{IDFT}\{X(-K)_N\} = x((-n))_N = x(N-n) = [1, -4j, j, -2j]$

c) $\mathrm{IDFT}\{\mathrm{Re}\{X(K)\}\} = \frac{1}{2}[x(n) + x^*(N-n)] = \frac{1}{2}[1+1, -2j+4j, j-j, -4j+2j] = [1, j, 0, -j]$

d) $\mathrm{IDFT}\{\mathrm{Im}\{X(K)\}\} = \frac{1}{2j}[x(n) - x^*(N-n)] = \frac{1}{2j}[1-1, -2j-4j, j+j, -4j-2j] = [0, -3, 1, -3]$

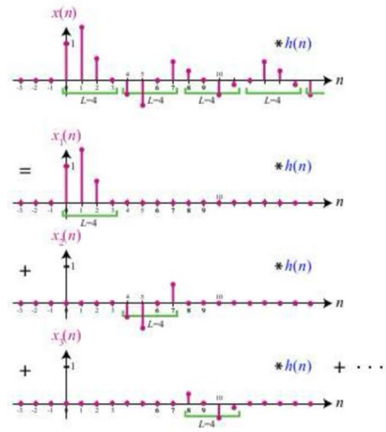
Filtering of Long Data Sequences

→ The technique of modifying a signal in a way that unwanted components are removed (or) desired components are enhanced

→ For this process, we use DFT using 2 methods:

a) Overlap Add Method (Output sequence is overlapped & added)

b) Overlap Save Method (Input sequence is overlapped, Output sequence is added)



Overlap Add Method

→ Deals with the following signal processing techniques:

i) Linear Convolution of discrete time signal of length L & discrete time signal of length M produces discrete time convolved result of length L+M-1

ii) Additivity : $(x_1(n) + x_2(n)) * h(n) = x_1(n) * h(n) + x_2(n) * h(n)$

→ $x(n)$ is divided into non-overlapping blocks $x_m(n)$ each of length L and each block is filtered individually to produce $y_m(n)$

→ Steps :

1) Break $x(n)$ into non-overlapping blocks $x_m(n)$ of length L

2) Zero pad $h(n)$ to be length $N = L-M+1$

3) $h(n) \xleftrightarrow[N]{DFT} H(K)$, $K = 0, 1, \dots, N-1$

4) For each block :

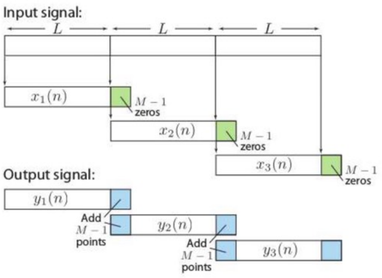
Zero pad $x_m(n)$ to be length $N = L+M-1$

$x_m(n) \xleftrightarrow[N]{DFT} X_m(K)$, $K = 0, 1, \dots, N-1$

$Y_m(K) = X_m(K) \cdot H(K)$, $K = 0, 1, \dots, N-1$

$Y_m(K) \xleftrightarrow[N]{IDFT} y_m(n)$, $n = 0, 1, \dots, N-1$

5) Form $y(n)$ by overlapping the last M-1 samples of $y_m(n)$ with the first M-1 samples of $y_{m+1}(n)$ and adding the result



NOTE: If overlap & add isn't done, and only appended, then true y(n) sequence won't be received

Q. $x(n) = n+1 \qquad n = 0 \text{ to } 9$

$= \{1,2,3,4,5,6,7,8,9,10\}$

$h(n) = \{1,0,-1\}$ $\qquad$ 6 point DFT

A. $N = L+M-1$

$6 = L+3-1 \Rightarrow L = 4$

$x_1(n) = \{1,2,3,4,0,0\}$

$x_2(n) = \{5,6,7,8,0,0\}$

$x_3(n) = \{9,10,0,0,0,0\}$

$h(n) = \{1,0,-1,0,0,0\}$

$y_1(n) = x_1(n) \, Ⓝ_6 \, h(n)$

$= \text{IDFT}\{X_1(k).H(k)\}$

$= \text{IDFT}\{[1+2W_6^k+3W_6^{2k}+4W_6^{3k}][1-W_6^{2k}]\}$

$= \text{IDFT}\{1+2W_6^k+3W_6^{2k}+4W_6^{3k}-W_6^{2k}-2W_6^{3k}-3W_6^{4k}-4W_6^{5k}]$

$= \text{IDFT}\{1+2W_6^k+2W_6^{2k}+2W_6^{3k}-3W_6^{4k}-4W_6^{5k}\}$

$= \{1,2,2,2,-3,-4\}$

$y_2(n) = \text{IDFT}\{X_2(k).H(k)\}$

$= \text{IDFT}\{[5+6W_6^k+7W_6^{2k}+8W_6^{3k}][1-W_6^{2k}]\}$

$= \text{IDFT}\{5+6W_6^k+7W_6^{2k}+8W_6^{3k}-5W_6^{2k}-6W_6^{3k}-7W_6^{4k}-8W_6^{5k}]$

$= \text{IDFT}\{5+6W_6^k+2W_6^{2k}+2W_6^{3k}-7W_6^{4k}-8W_6^{5k}\}$

$= \{5,6,2,2,-7,-8\}$

$y_3(n) = \text{IDFT}\{X(k).H(k)\}$

$= \text{IDFT}\{[9+10W_6^k][1-W_6^{2k}]\}$

$= \text{IDFT}\{9+10W_6^k-9W_6^{2k}-10W_6^{3k}]$

$= \{9,10,-9,-10,0,0\}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y_1(n)$	1	2	2	2	-3	-4								
$y_2(n)$					5	6	2	2	-7	-8				
$y_3(n)$									9	10	-9	-10	0	0
	1	2	2	2	2	2	2	2	2	2	-9	-10	0	0

Q. The linear convolution of length 55 sequence with a length 1100 is to be computed using 64 point DFT & IDFT. Find the smallest no. of DFT & IDFT needed to compute using overlap add method

A. $64 = L+55-1 \Rightarrow L = 10$

$\left\lceil \frac{1100}{L} \right\rceil = \left\lceil \frac{1100}{10} \right\rceil = 110$

and 1 DFT of $h(n)$, So 111 DFTs & 110 IDFTs

Overlap Save Method

→ Deals with the following signal processing techniques :

i) N - circular convolution of DT signals of length N & M using N-DFT & N-IDFT

ii) Time - domain aliasing

$$x_c(n) = \sum_{l=-\infty}^{\infty} x_L(n-lN) \quad , n = 0,1,\dots N-1$$

→ 1) $N = L + M - 1$

2) Let $x_m(n)$ have support $n = 0,1,\dots N-1$

$h(n)$ have support $n = 0,1,\dots M-1$

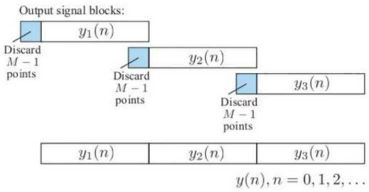
3) Zero pad $h(n)$ to have support $n = 0,1,\dots N-1$

4) $x_m(n) \xrightarrow[N]{DFT} X_m(K)$

$h(n) \xrightarrow[N]{DFT} H(K)$

$Y_m(K) = X_m(K) \cdot H(K)$

$Y_m(K) \xrightarrow[N]{IDFT} y_{c,m}(n)$



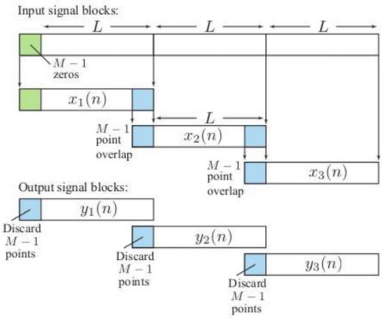
5) $y_{c,m}(n) = \begin{cases} \text{aliasing} & , n = 0,1,\dots,M-2 \\ y_{L,m}(n) & , M-1, M, \dots, N-1 \end{cases}$

where $y_{L,m}(n) = x_m(n) * h(n)$ is desired output

→ Steps :

1) insert $M-1$ zeros at beginning of $x(n)$

2) Break padded input signal into overlapping blocks $x_m(n)$ of length $N = L+M-1$ (overlap length is $M-1$)



3) Zero pad $h(n)$ to be length $N = L+M-1$

4) $h(n) \xrightarrow[N]{DFT} H(K)$, $K = 0,1,\dots,N-1$

5) For each block m :

i) $x_m(n) \xrightarrow[N]{DFT} X_m(K)$, $K = 0,1,\dots,N-1$

ii) $Y_m(K) = X_m(K) \cdot H(K)$, $K = 0,1,\dots,N-1$

iii) $Y_m(K) \xrightarrow[N]{IDFT} y_{c,m}(n)$, $n = 0,1,\dots,N-1$

iv) Discard first $M-1$ points of each output block $y_m(n)$

6) Form $y(n)$ by appending remaining L samples of each block $y_m(n)$

Q. $x(n) = \{1, -1, 1, -1, 0, 0, -1, -2\}$

$h(n) = \{1, -1\}$ Find linear convolution of $x(n)$ & $h(n)$ / Perform Linear Filtering using overlap save method. Assume 4 point DFT & IDFT

A. $N = 4$ $M = 2$ $8 + 2 - 1 = 9$

$N = L + M - 1 \Rightarrow L = 4 - 2 + 1 = 3$

$\left\lceil \frac{9}{N} \right\rceil = \left\lceil \frac{9}{3} \right\rceil = 3 \Rightarrow 3 + 1 = 4$ DFTs

3 IDFTs

i) $x_0(n) = \{0, 1, -1, 1\}$

$h(n) = \{1, -1, 0, 0\}$

$y_0(n) = \text{IDFT}\left(X_0(k) \cdot H(k)\right)$

$= \text{IDFT}\left((W_4^k - W_4^{2k} + W_4^{3k})(1 - W_4^k)\right)$

$= \text{IDFT}\left(W_4^k - W_4^{2k} + W_4^{3k} - W_4^{2k} + W_4^{3k} - W_4^{4k}\right)$

$= \text{IDFT}\left(-1 + W_4^k - 2W_4^{2k} + 2W_4^{3k}\right)$

$= \{\cancel{-1}, 1, -2, 2\}$

$y_1(n) = \text{IDFT}\left(X_1(k) \cdot H(k)\right)$

$X_1(k) = \{1, -1, 0, 0\}$

$y_1(n) = \text{IDFT}\left((1 - W_4^k)(1 - W_4^k)\right)$

$= \text{IDFT}\left(1 + W_4^{2k} - 2W_4^k\right)$

$= \{\cancel{1}, -2, 1, 0\}$

$y_2(n) = \text{IDFT}\left(X_2(k) \cdot H(k)\right)$

$X_2(k) = \{0, -1, -2, 0\}$

$y_2(n) = \text{IDFT}\left((-W_4^k - 2W_4^{2k})(1 - W_4^k)\right)$

$= \text{IDFT}\left(-W_4^k - 2W_4^{2k} + W_4^{2k} + 2W_4^{3k}\right)$

$= \{\cancel{0}, -1, -1, 2\}$

$y(n) = (1, -2, 2, -2, 1, 0, -1, -1, 2)$

$$y_0(n) = x_0(n) \text{ Ⓝ}_4\, h(n) = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{-1} \\ 1 \\ -2 \\ 2 \end{bmatrix}$$

$$y_1(n) = x_1(n) \text{ Ⓝ}_4\, h(n) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{1} \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$y_2(n) = x_2(n) \text{ Ⓝ}_4\, h(n) = \begin{bmatrix} 0 & 0 & -2 & -1 \\ -1 & 0 & 0 & -2 \\ -2 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

$\downarrow$

$\{1, -2, 2, -2, 1, 0, -1, -1, 2\}$

Q. The linear convolution of length 55 sequence with a length 1100 is to be computed using 64 point DFT & IDFT. Find the smallest no. of DFT & IDFT needed to compute using overlap save method

A. $M = 55$, $N = 64$

$L = N - M + 1 = 64 - 55 + 1 = 10$

$1100 + 55 - 1 = 1154$

$\left\lceil \frac{1154}{N} \right\rceil = \left\lceil \frac{1154}{10} \right\rceil = 116$

So, $116 + 1 = 117$ DFTs

116 IDFTs

Q. Consider the twiddle factor $w_N = e^{-j2\pi/N}$

Compute magnitude and phase

A. Magnitude : 1

Phase : $2\pi/N$

Q. Match the appropriate twiddle factors

a) $W_N^{N/4}$	1) 1
b) W_N^{K}	2) W_N^{-1}
c) W_N^{K+N}	3) $-j$
d) $W_N^{K+N/2}$	4) -1
e) W_N^{2K}	5) j
f) W_N^{*}	6) W_N^{K}
g) $W_N^{N/2}$	7) $W_{N/2}^{K}$
h) $W_N^{3N/4}$	8) $-W_N^{K}$

A.

a - 3	e - 7
b - 1	f - 2
c - 6	g - 4
d - 8	h - 5

$\left(W_N = e^{-j\frac{2\pi}{N}} \right)$

Q. For $N = a$, find the total no. of real multiplications & real additions required to compute N - point DFT

A. Suppose $(a+ib)$ & $(c+id)$

Real Additions → $(a+c)$ & $(b+d)$

Real Multiplications → (ac), (bc), (ad), (bd)

So for N-DFT ⇒ $4N^2$ real multiplications

$2N^2 + 2N(N-1)$ real additions

and $N = a$ ⇒ $4a^2$ real multiplications

$2a^2 + 2a(a-1)$ real additions

↳ $4a^2 - 2a$

Unit - 2 Fast Fourier Transform

→ DFT is really important but it involves lot of complex & real additions & multiplications making it inefficient as they don't exploit symmetry $\left(W_N^{K+\frac{N}{2}} = -W_N^K\right)$ & periodicity $\left(W_N^{K+N} = W_N^K\right)$

Fast Fourier Transform

→ It is used over DFT because the exploit symmetry & periodicity properties

→ It uses a Divide & Conquer Approach

→ Basic Steps : (Cooley - Tukey Algorithm)

i) Divide : If N is composite, divide DFT into smaller DFTs

ii) Compute Smaller DFTs : Recursively divide the smaller DFTs until you get size 2

iii) Combine : Use butterfly operations to combine results of smaller DFTs.

→ For N - length sequence,

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^K = e^{-j\frac{2\pi K}{N}}$$

→ Dividing Sequence ⇒ Split sequence into even & odd

even : x_{2m}

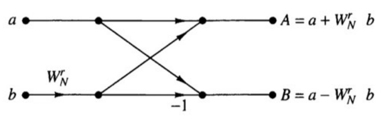
odd : x_{2m+1}

→ Recursive DFT ⇒ For even indices : $X_K^e = \sum_{m=0}^{\frac{N}{2}-1} x_{2m} \cdot e^{-j\frac{2\pi Km}{N/2}}$

For odd indices : $X_K^o = \sum_{m=0}^{\frac{N}{2}-1} x_{2m+1} \cdot e^{-j\frac{2\pi Km}{N/2}}$

→ Combine Results ⇒ $X_K = X_K^e + W_N^K \cdot X_K^o$

$X_{K+\frac{N}{2}} = X_K^e - W_N^K \cdot X_K^o$



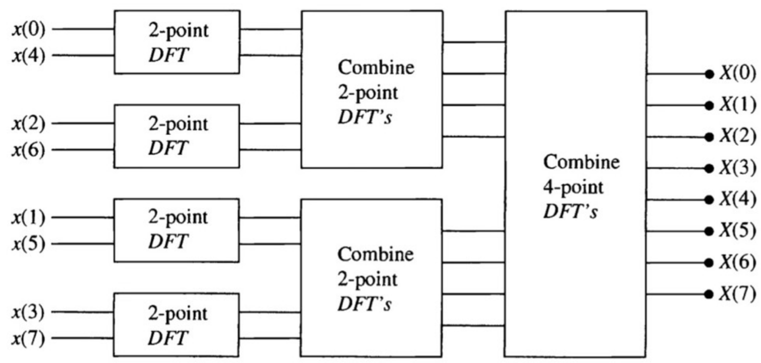
Basic Butterfly Computation in the decimation-in-time FFT

→ Butterfly Operations ⇒ even path : $X_{K,new} = X_{K,old} + W_N^K \cdot X_{K+\frac{N}{2},odd}$

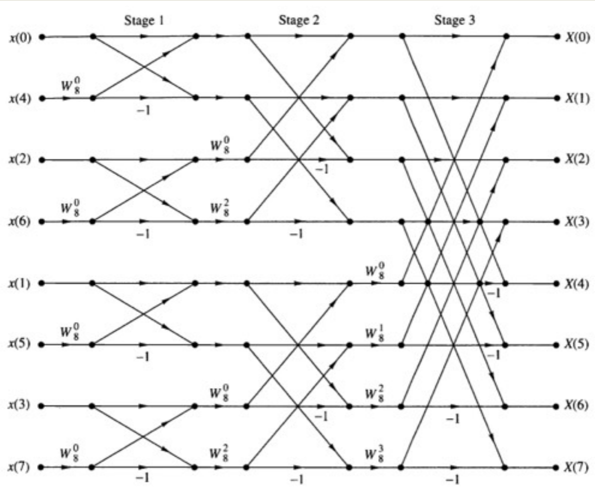
$X_{K+\frac{N}{2},new} = X_{K,old} - W_N^K \cdot X_{K+\frac{N}{2},even}$

DIT : decimation-in-time

DIF : decimation-in-frequency



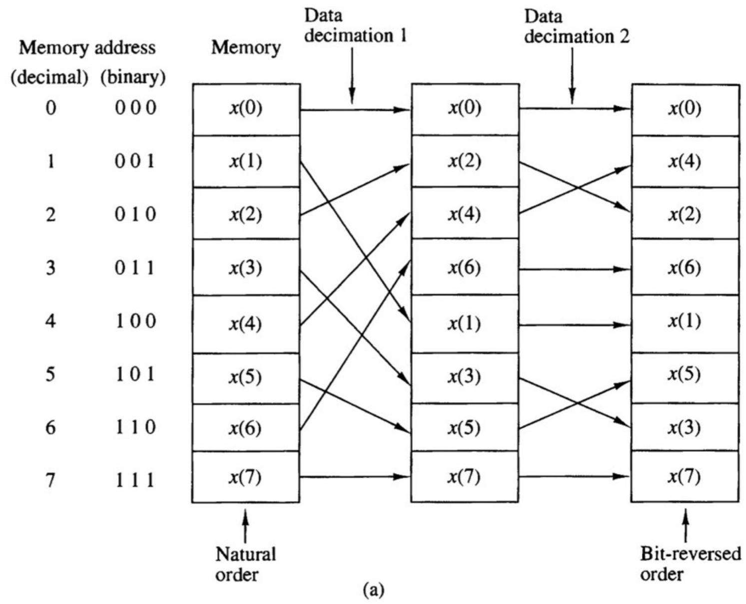
3 Stages in computation of 8 point DFT



8 - point DIT FFT Signal Flow Graph

(or)

Butterfly Diagram



(a)

$(n_2n_1n_0)$	$\rightarrow$	$(n_1n_0n_2)$	$\rightarrow$	$(n_0n_1n_2)$
(0 0 0)	$\rightarrow$	(0 0 0)	$\rightarrow$	(0 0 0)
(0 0 1)	$\rightarrow$	(0 1 0)	$\rightarrow$	(1 0 0)
(0 1 0)	$\rightarrow$	(1 0 0)	$\rightarrow$	(0 1 0)
(0 1 1)	$\rightarrow$	(1 1 0)	$\rightarrow$	(1 1 0)
(1 0 0)	$\rightarrow$	(0 0 1)	$\rightarrow$	(0 0 1)
(1 0 1)	$\rightarrow$	(0 1 1)	$\rightarrow$	(1 0 1)
(1 1 0)	$\rightarrow$	(1 0 1)	$\rightarrow$	(0 1 1)
(1 1 1)	$\rightarrow$	(1 1 1)	$\rightarrow$	(1 1 1)

(b)

→ No. of stages $= \log_2 N$

No. of butterfly diagrams per stage $= \frac{N}{2}$

Total no. of butterfly diagrams $= \frac{N}{2}\log_2 N$

→ FFT reduces computational complexity from N^2 to $\frac{N}{2}\log N$ operation

→ No. of complex multiplications $= \frac{N}{2}\log_2 N$

No. of complex additions $= 2 \times \frac{N}{2}\log_2 N = N\log_2 N$

Q. If N = 8, 16, 256, 512, Find no. of CM & CA in DFT & FFT

A.

		8	16	256	512
DFT	CM	64	256	65536	262144
	CA	56	240	65280	261632
FFT	CM				
	CA				

	DFT	FFT
C.M	N^2	$\frac{N}{2}\log_2 N$
C.A	$N(N-1)$	$N\log_2 N$

Q. Using DIT FFT compute 4 point DFT of $x[n] = \{1, 0, 1, 0\}$

A. For $N = 4$

even indices : $x[0]$ $x[2]$

odd indices : $x[1]$ $x[3]$

$X_0^e = x_0 + x_2 = 1 + 1 = 2$

$X_1^e = x_0 - x_2 = 1 - 1 = 0$

$X_0^o = x_1 + x_3 = 0 + 0 = 0$

$X_1^o = x_1 - x_3 = 0 - 0 = 0$

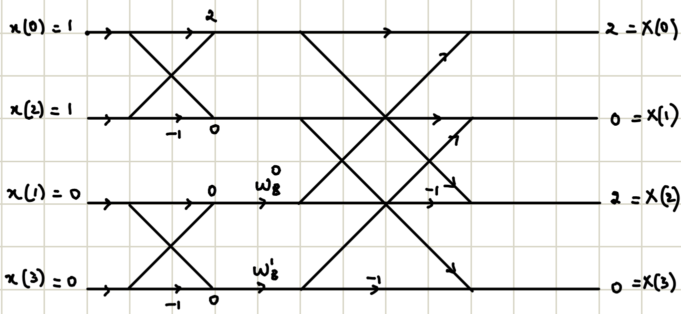
$X_0 = X_0^e + W_4^0 . X_0^o = 2 + 1(0) = 2$

$X_2 = X_0^e - W_4^0 X_0^o = 2 - 1(0) = 2$

$X_1 = X_1^e + W_4^1 X_1^o = 0 + (-j)(0) = 0$

$X_3 = X_1^e - W_4^1 X_1^o = 0 - (-j)(0) = 0$

$X = [2, 0, 2, 0]$



Q. Using DIT FFT compute 4 point DFT of $x[n] = \{1, -1, 2, -2\}$

A. For $N = 4$

even indices : $x[0]$ $x[2]$

odd indices : $x[1]$ $x[3]$

$X_0^e = x_0 + x_2 = 1 + 2 = 3$

$X_1^e = x_0 - x_2 = 1 - 2 = -1$

$X_0^o = x_1 + x_3 = -1 - 2 = -3$

$X_1^o = x_1 - x_3 = -1 + 2 = 1$

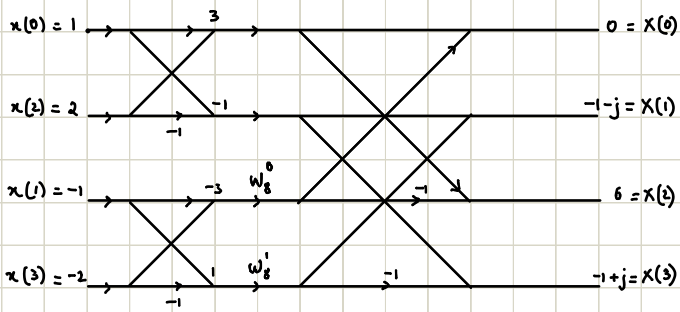
$X_0 = X_0^e + W_4^0 . X_0^o = 3 + (-3) = 0$

$X_2 = X_0^e - W_4^0 X_0^o = 3 - (-3) = 6$

$X_1 = X_1^e + W_4^1 X_1^o = -1 + (-j)(1) = -1-j$

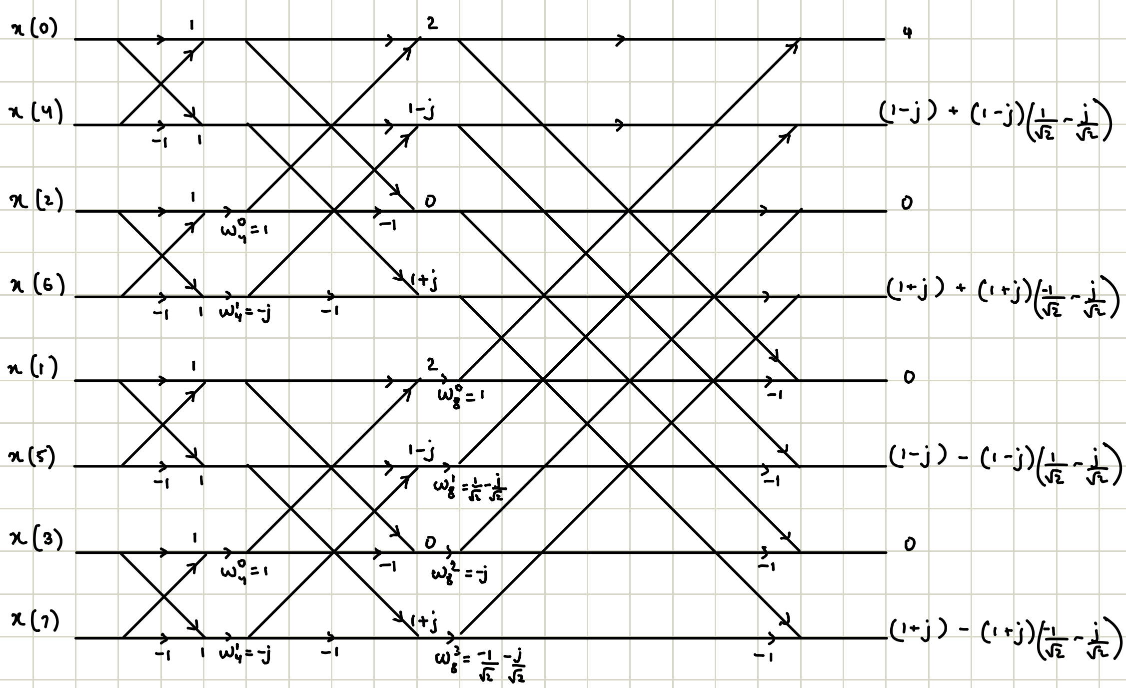
$X_3 = X_1^e - W_4^1 X_1^o = -1 - (-j)(1) = -1+j$

$X = \{0, -1-j, 6, -1+j\}$



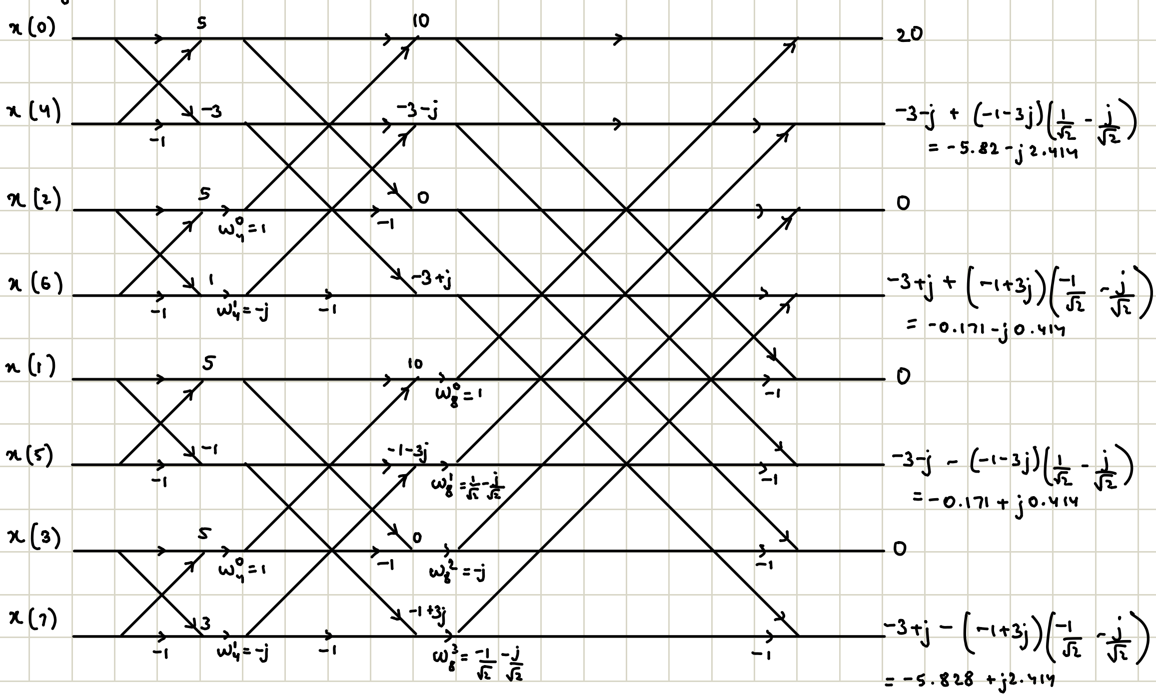
Q. $x(n) = [1, 1, 1, 1, 0, 0, 0, 0]$. Compute 8 point DFT using radix-2 DIT FFT

A.



Q. $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

Signal Flow Graph :



HW

Q. $x(n) = \{1, -1, 1, -1, 1\}$. Find 8 Pnt DFT using DIT-FFT

DIF FFT

→ It is similar to DIT (Divide & Conquer)

→ $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad (k = 0 \text{ to } N-1)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + W_N^{\frac{Nk}{2}} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left(x(n) + (-1)^k\, x\left(n+\frac{N}{2}\right)\right) W_N^{kn} \qquad \left(W_N^{\frac{kN}{2}} = (-1)^k \right)$$

Now split (decimate) $X(k)$ into even & odd numbered samples

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left(x(n) + x\left(n+\frac{N}{2}\right)\right) W_{\frac{N}{2}}^{kn} \quad , \; k = 0, 1, \ldots, \frac{N}{2}-1$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left(\left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_N^{n} \right) W_{\frac{N}{2}}^{kn} \quad , \; k = 0, 1, \ldots, \frac{N}{2}-1$$

Let us define the $\frac{N}{2}$ sequences $e_1(n)$ & $e_2(n)$ as $\qquad \left(W_N^2 = W_{\frac{N}{2}} \right)$

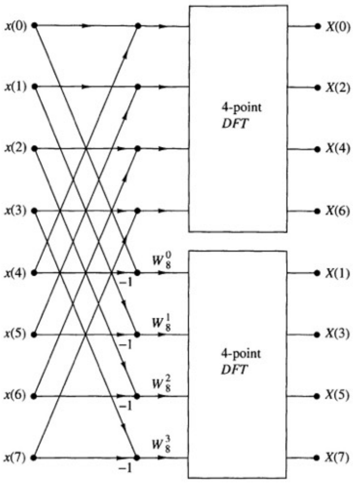
$$e_1(n) = x(n) + x\left(n+\frac{N}{2}\right)$$

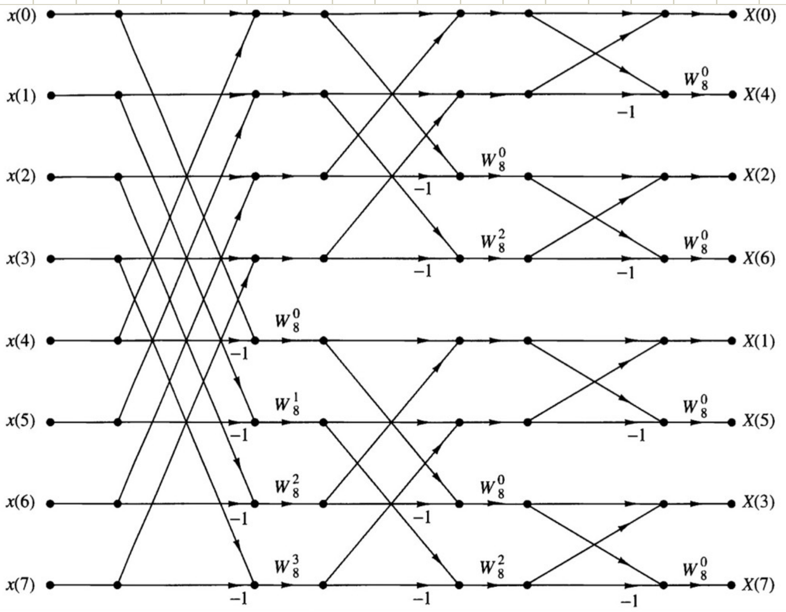
$$e_2(n) = \left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_N^{n} \quad , \; n = 0, 1, 2, \ldots, \frac{N}{2}-1$$

Then,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} e_1(n)\, W_{\frac{N}{2}}^{kn}$$

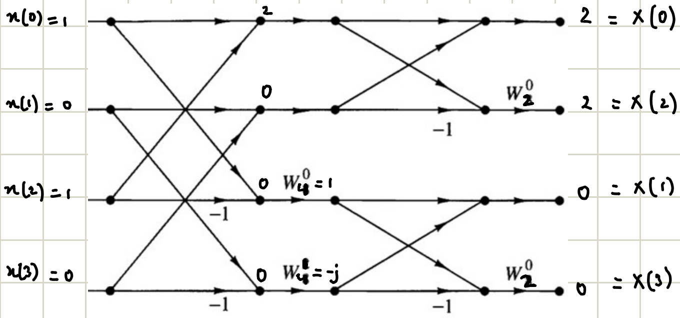
$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} e_2(n)\, W_{\frac{N}{2}}^{kn}$$





Q. $x[n] = [1, 0, 1, 0]$. Find 4 Point DFT using FFT DIF

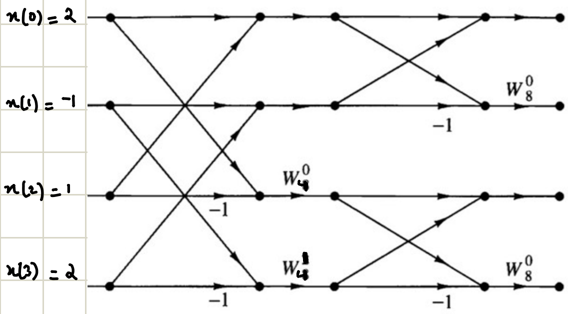
A.



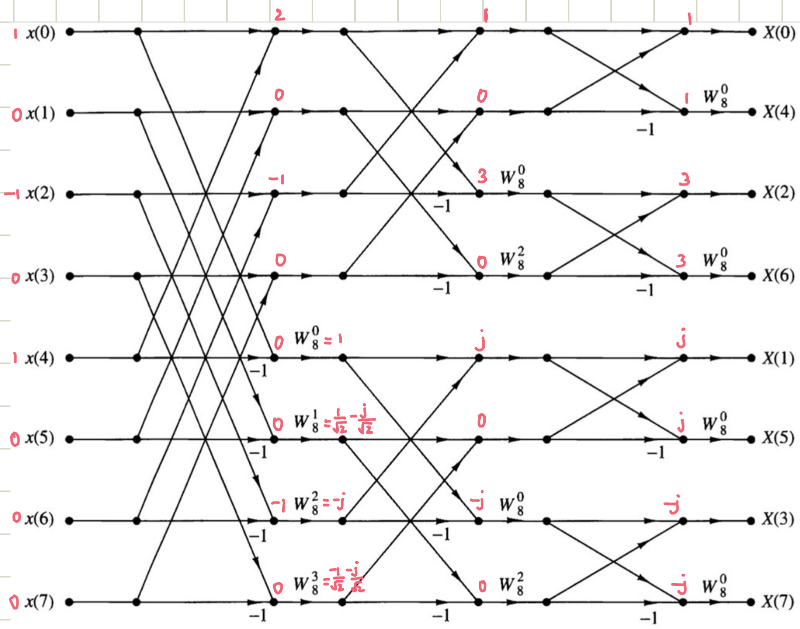
$X(k) = \{2, 0, 2, 0\}$

Q. $x[n] = [2, -1, 1, 2]$. Find 4 Point DFT using FFT DIF

A.



Q. $x[n] = \{1, 0, -1, 0, 1\}$



$X(k) = \{1, j, 3, -j, 1, j, 3, -j\}$

Inverse DIT FFT

$\rightarrow \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad (n = 0 \text{ to } N-1)$

$$= \frac{1}{N}\left[\sum_{k=0}^{\frac{N}{2}-1} X(k)\,W_N^{-kn} + \sum_{k=\frac{N}{2}}^{N-1} X(k)\,W_N^{-kn}\right]$$

Let $\quad l = k - \frac{N}{2} \quad \Rightarrow k = l + \frac{N}{2}$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{\frac{N}{2}-1} X(k)\,W_N^{-kn} + \sum_{l=0}^{\frac{N}{2}-1} X\left(l+\frac{N}{2}\right) W_N^{-(l+\frac{N}{2})n}\right]$$

$$= \frac{1}{N}\left[\sum_{k=0}^{\frac{N}{2}-1} X(k)\,W_N^{-kn} + \sum_{k=0}^{\frac{N}{2}-1} X\left(l+\frac{N}{2}\right) W_N^{-kn}\, W_N^{-\frac{N}{2}n}\right]$$

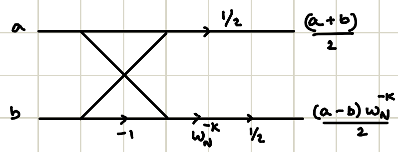
$$= \frac{1}{N}\left[\sum_{k=0}^{\frac{N}{2}-1} \left[X(k) + (-1)^n X\left(k+\frac{N}{2}\right)\right] W_N^{-kn}\right]$$

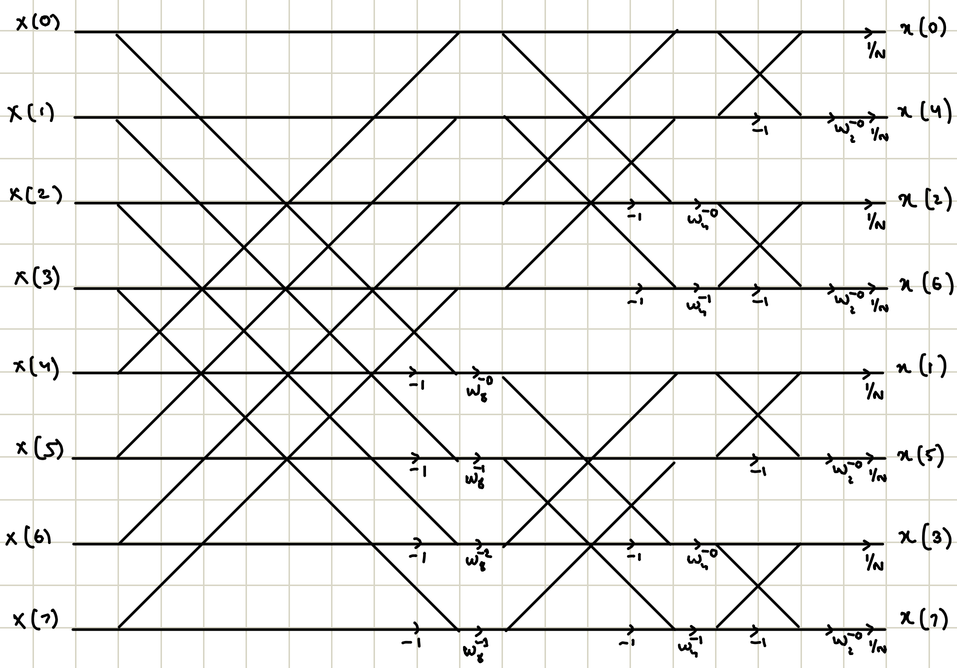
$$x(2n) = \frac{1}{2}\left[\frac{1}{N/2}\sum_{k=0}^{\frac{N}{2}-1}\left[X(k) + (-1)^{2n} X\left(k+\frac{N}{2}\right)\right] W_{N/2}^{-kn}\right]$$

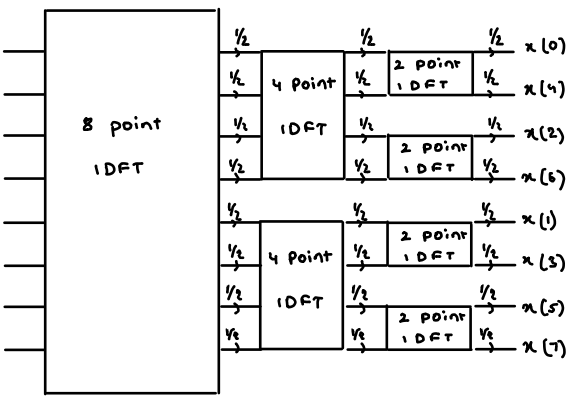
$$= \frac{1}{N}\left[\sum_{k=0}^{\frac{N}{2}-1}\left[X(k) + X\left(k+\frac{N}{2}\right)\right] W_{N/2}^{-kn}\right]$$

$$x(2n+1) = \frac{1}{2}\left\{\frac{1}{N/2}\left[\sum_{k=0}^{\frac{N}{2}-1}\left[X(k) + (-1)^{2n+1} X\left(k+\frac{N}{2}\right)\right] W^{-k(2n+1)}\right]\right\}$$

$$= \frac{1}{N}\left[\sum_{k=0}\left[X(k) - X\left(k+\frac{N}{2}\right)\right] W^{-k}\right] W_{N/2}^{-kn}$$



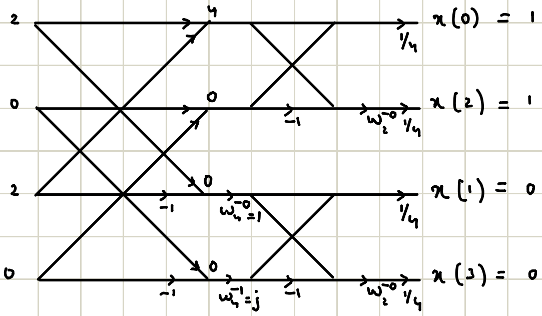




$\Rightarrow \quad \frac{1}{8}$ at end $\quad \left(\frac{1}{N} \text{ for } N \text{ point}\right)$

Q. Given the 4 point DFT of $X(K) = \{2, 0, 2, 0\}$. Find IDFT of $X(K)$ using DIT IFFT

A.

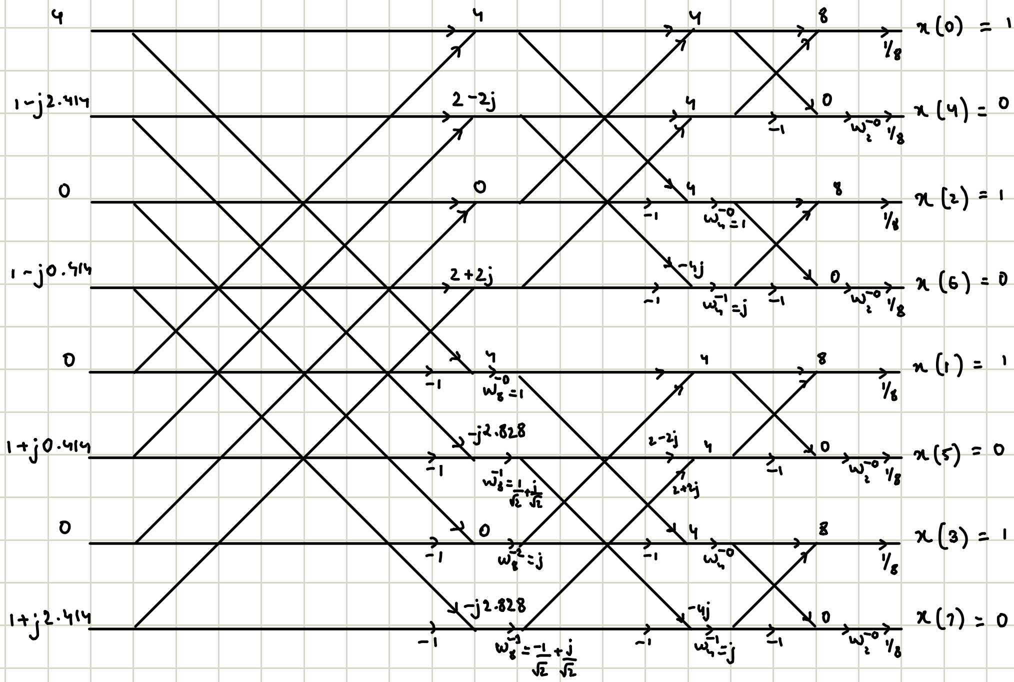


$x(n) = \{1, 0, 1, 0\}$

Q. $X(K) = \{4, 1 - j2.414, 0, 1, 1 - j0.414, 0, 1 + j0.414, 0 + 1 + j2.414\}$

Compute IDFT of $X(K)$ using Inverse DIT FFT

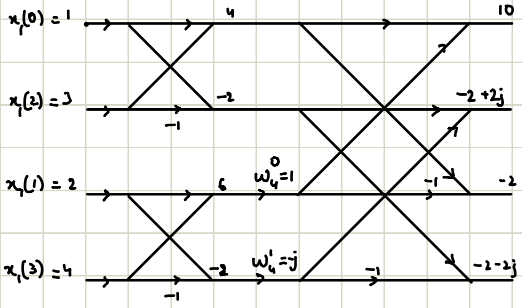
A.

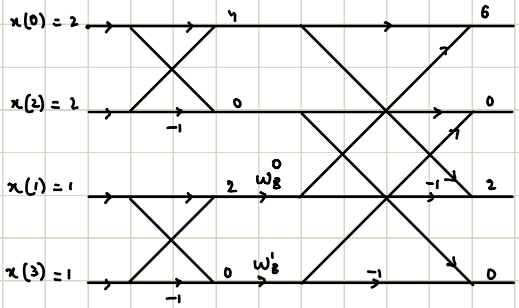


$x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$

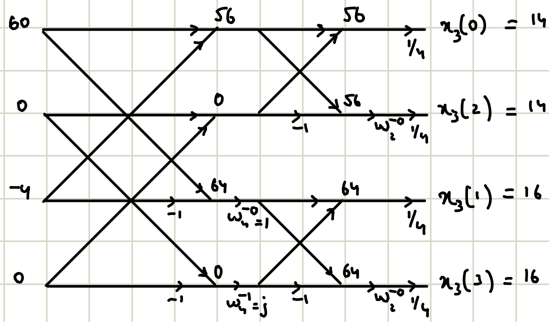
Q. Find the circular convolution of the given 2 sequences $x_1(n) = (1,2,3,4)$ & $x_2(n) = (2,1,2,1)$ using DIT FFT & Inverse DIT FFT. N=4

A.



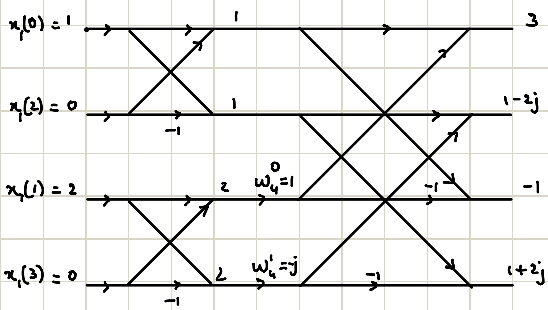


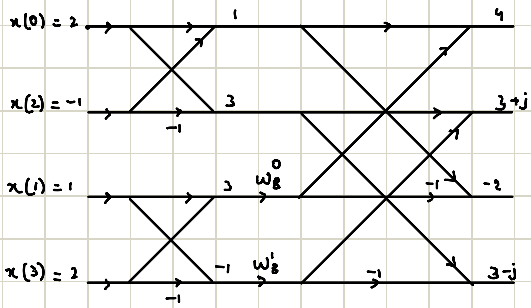
$X_3(k) = X_1(k) \cdot X_2(k) = [60, 0, -4, 0]$



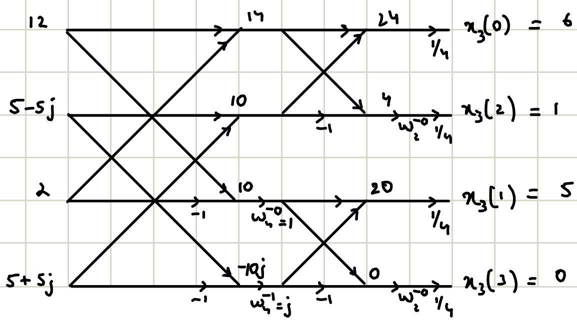
$x_3 = (14, 16, 14, 16)$

Q. Find the circular convolution of the given 2 sequences $x_1(n) = (1,2,0,0)$ & $x_2(n) = (2,1,-1,2)$ using DIT FFT & Inverse DIT FFT. N=4





$X_3(k) = (12, 5-5j, 2, 5+5j)$



HW Q. $X(k) = (20, -5.82 - j2.414, 0, -0.171 - j0.414, 0, -0.171 + j0.414, 0, -5.82 + j2.414)$

Find 8 point inverse DIT FFT

Find no. of CM & CA

Inverse DIF FFT

$$\rightarrow \quad x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn} \qquad n = 0 \text{ to } N-1$$

$$= \frac{1}{N}\left[\underbrace{\sum_{K=0}^{\frac{N}{2}-1} X(2K)\, W_N^{-2Kn}}_{\text{even}} + \underbrace{\sum_{K=0}^{\frac{N}{2}-1} X(2K+1)\, W_N^{-(2K+1)n}}_{\text{odd}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{N/2}\left\{\sum_{K=0}^{\frac{N}{2}-1} F_1(K)\, W_{N/2}^{-Kn} + \sum_{K=0}^{\frac{N}{2}-1} F_2(K)\, W_{N/2}^{-Kn}\, W_N^{-n}\right\}\right]$$

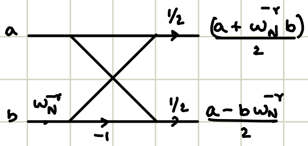
$$= \frac{1}{2}\left[\frac{1}{N/2}\sum_{K=0}^{\frac{N}{2}-1} F_1(K)\, W_{N/2}^{-Kn} + \frac{W_N^{-n}}{N/2}\sum_{K=0}^{\frac{N}{2}-1} F_2(K)\, W_{N/2}^{-Kn}\right]$$

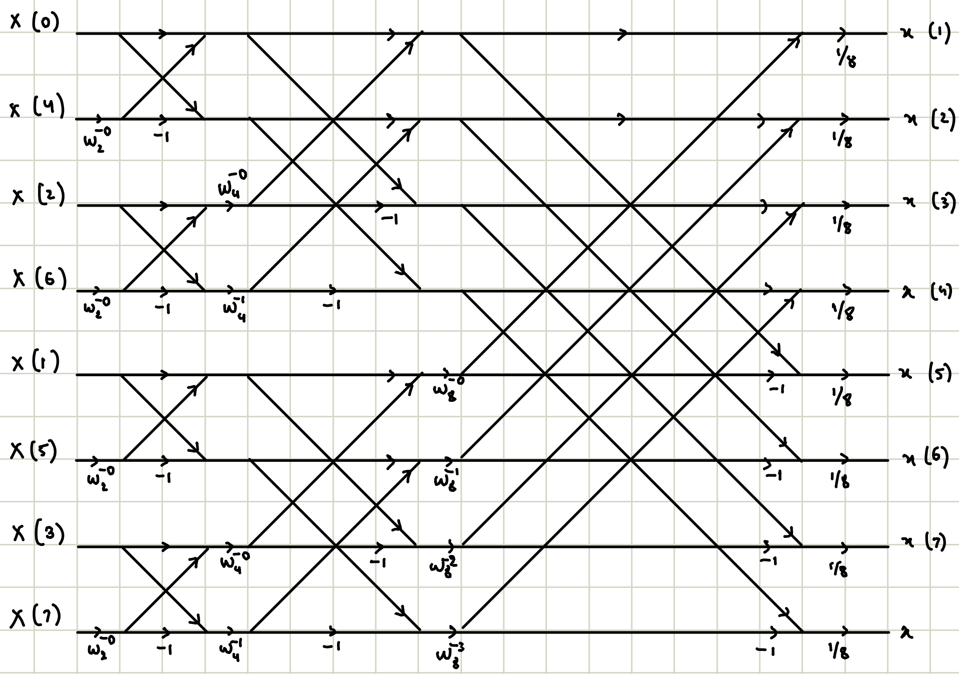
$$x(n) = \frac{1}{2}\left[f_1(n) + W_N^{-n} f_2(n)\right] \longrightarrow ①$$

$$x\left(n+\frac{N}{2}\right) = \frac{1}{2}\left[f_1\left(n+\frac{N}{2}\right) + W_N^{-\left(n+\frac{N}{2}\right)} f_2\left(n+\frac{N}{2}\right)\right]$$

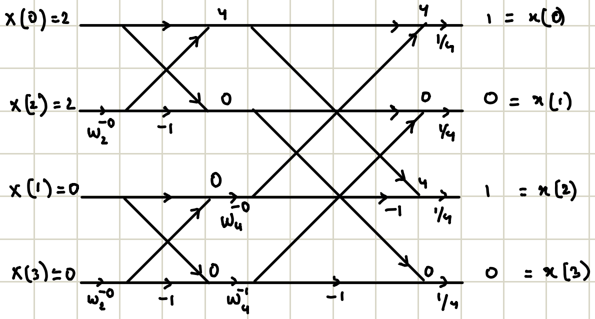
$$x\left(n+\frac{N}{2}\right) = \frac{1}{2}\left[f_1(n) - W_N^{-n} f_2(n)\right] \longrightarrow ②$$

from ① & ②





Q. $X(K) = [2, 0, 2, 0]$. Find inverse DIF FFT (4 point)



Q. $X(K) = [4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414]$

Compute 8 point IDFT using IDIF - FFT

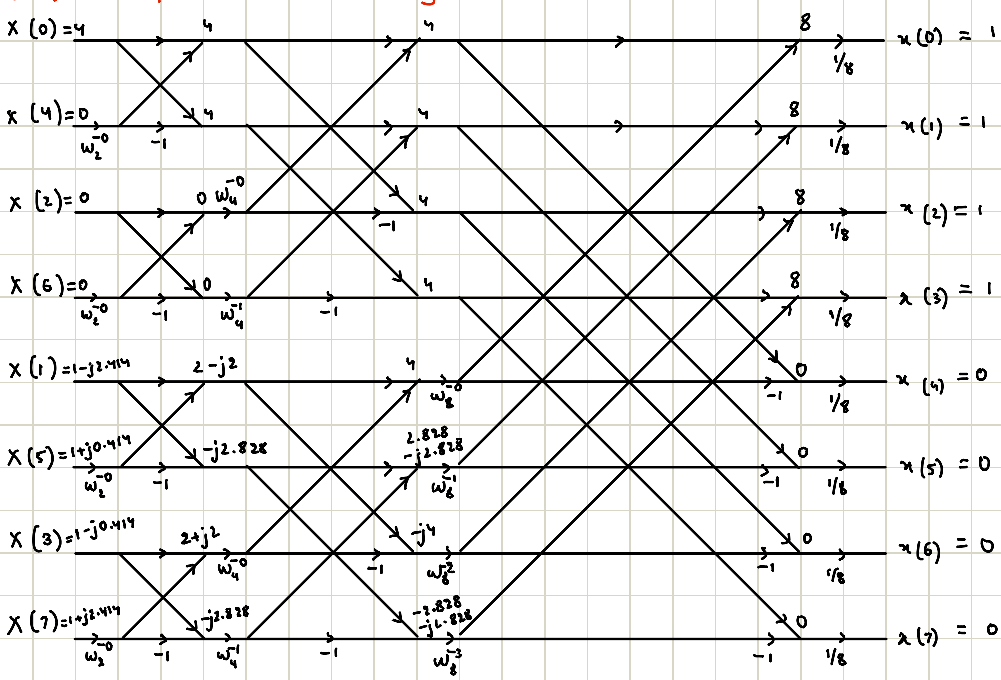
$W_8^{-0} = W_4^{-0} = W_2^{-0} = 1$

$W_8^{-1} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$W_8^{-2} = j = W_4^{-1}$

$W_8^{-3} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

A

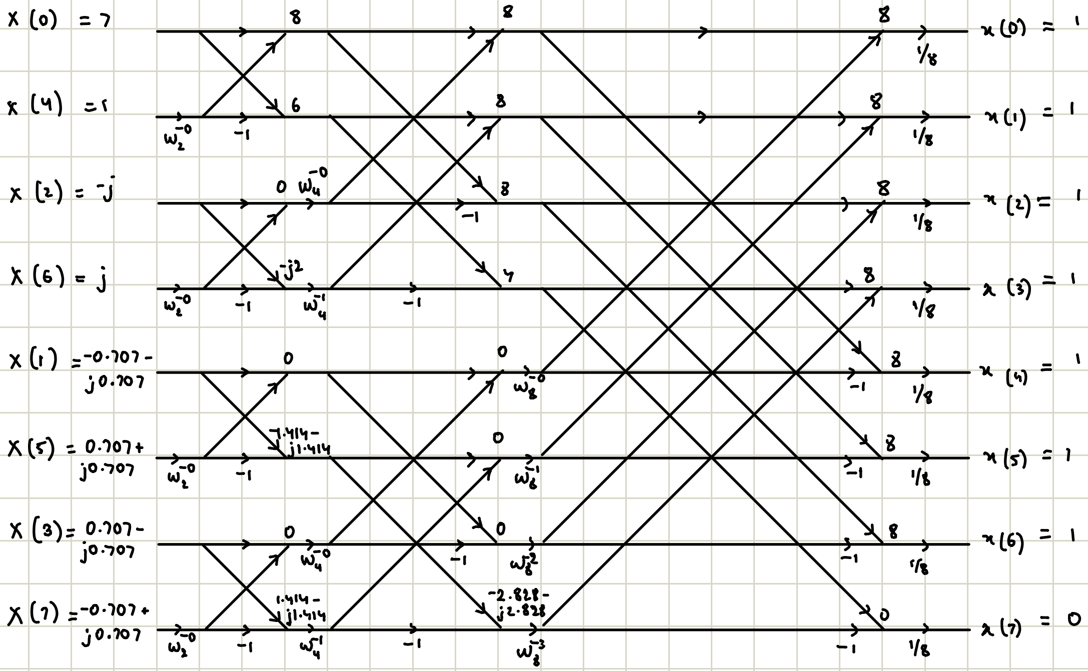


Q. $X(K) = \{7, -0.707-j0.707, -j, 0.707-j0.707, 1\}$

Compute 8 point IDFT of $X(K)$ if $x(n)$ is real.

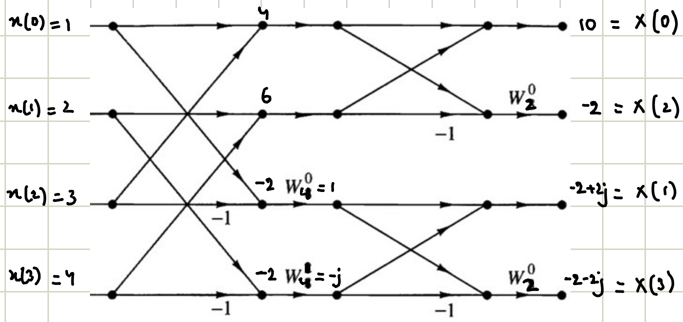
A. Since $x(n)$ is real, $X(K)$ exhibits conjugate symmetry

$X(K) = \{7, -0.707-j0.707, -j, 0.707-j0.707, 1, 0.707+j0.707, j, -0.707+j0.707\}$

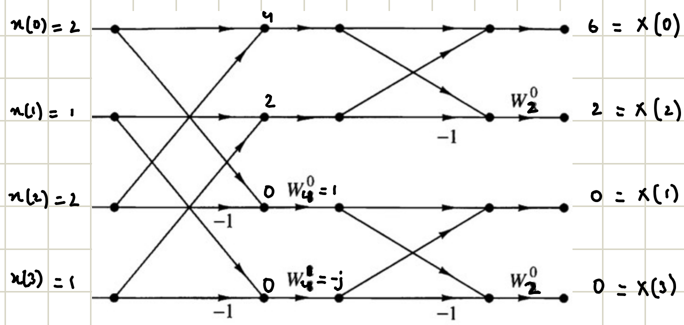


Q. $x_1(n) = (1, 2, 3, 4)$ & $x_2(n) = (2, 1, 2, 1)$. Compute Circular convolution

A.

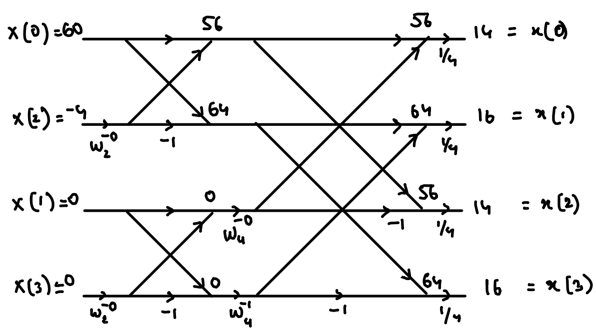


$X_1(K) = \{10, -2+2j, -2, -2-2j\}$



$X_1(K) \cdot X_2(K) = X_3(K)$

$X_3(K) = \{60, 0, -4, 0\}$



$x_3(n) = \{14, 16, 14, 16\}$

Q. Let $x_1(n) = \{2, 3\}$ and $x_2(n) = \{1, 1\}$. Compute linear convolution of 2 sequences using circular convolution. Use DIT-FFT

A. $L = L_1 + L_2 - 1 = 2 + 2 - 1 = 3$

We must do 3 point DFT but since we must use DIT-FFT, we use 4 point

$x(n) = x_1(n) * x_2(n)$

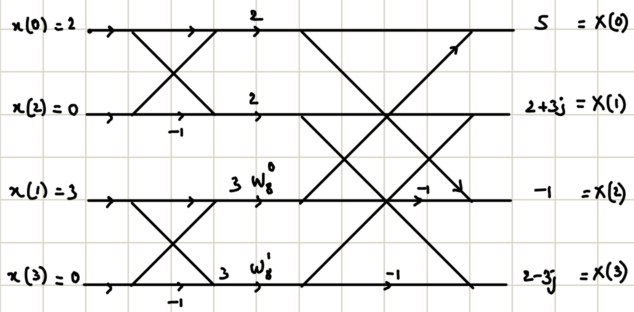
$$\begin{bmatrix} 2 & 0 & 3 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

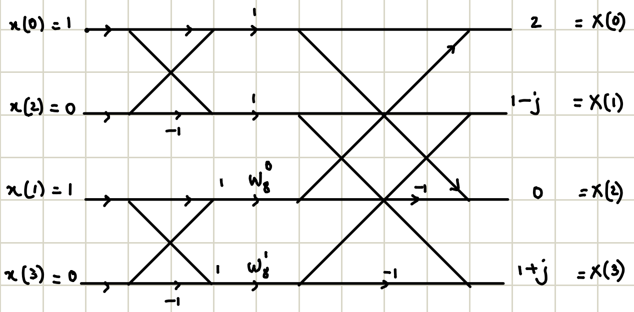
and

	1	1
2	2	2
3	3	3

$\Rightarrow \{2, 5, 3\}$

Linear convolution matches circular convolution

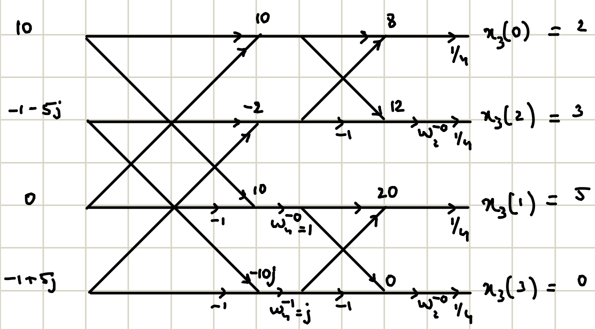




$X(K) = X_1(K) \cdot X_2(K)$

$= \{5, 2+3j, -1, 2-3j\}\{2, 1-j, 0, 1+j\}$

$= \{10, -1-5j, 0, -1+5j\}$

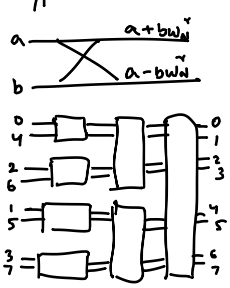


FFT

DIT

I/p Bit reversal

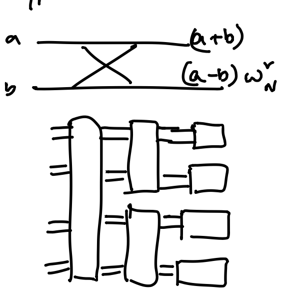
O/p in order



DIF

I/p in order

O/p in bit reversal

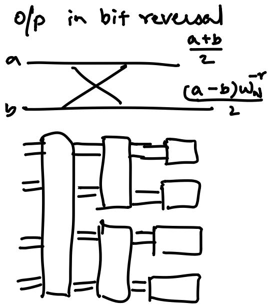


IFFT

DIT

I/p in order

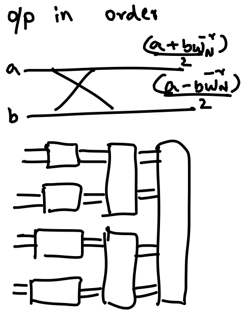
O/p in bit reversal



DIF

I/p Bit reversal

O/p in order



$N \rightarrow \log_2 N$ stages

$\frac{N}{2}$ butterfly diagrams

$\frac{N}{2}\log_2 N$ CM

$N\log_2 N$ CA

(2N resistors for in-place computation)

(N resistors for twiddle factors)

Application of FFT

→ Linear Filtering

Overlap Save & Overlap Add

Q. Let $x(n) = n+1$ for $n = 0$ to 9

Find linear convolution of $x(n)$ with $h(n)$ where $h(n) = \{1, -1\}$ (use DIT-FFT)

a) Overlap save

b) Overlap add

A. a) Assume $N = 4$

$M = 2 \Rightarrow L = N - M + 1 = 3$

$x_0(n) = \{0, 1, 2, 3\}$ $\quad x_1(n) = \{3, 4, 5, 6\}$ $\quad x_2(n) = \{6, 7, 8, 9\}$ $\quad x_3(n) = \{9, 10, 0, 0\}$ ✓

→ easier calculation

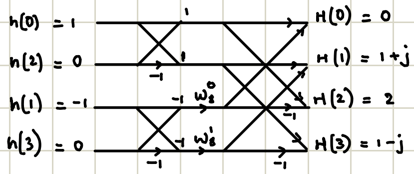
Assume $N = 8$

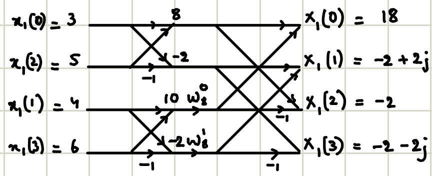
$M = 2 \Rightarrow L = N - M + 1 = 7$

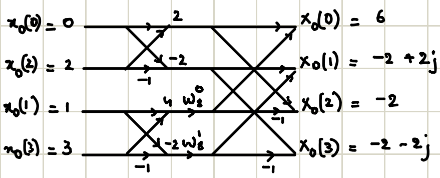
$x_0(n) \Rightarrow \{0, 1, 2, 3, 4, 5, 6, 7\}$ $\quad x_1(n) = \{7, 8, 0, 0, 0, 0, 0, 0\}$

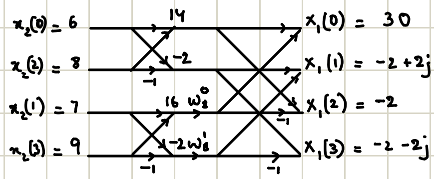
$y_0(n) = IDFT\,(X_0(K) \cdot H(K))$

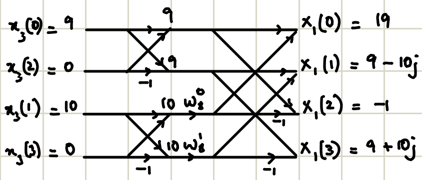
For $H(K)$,









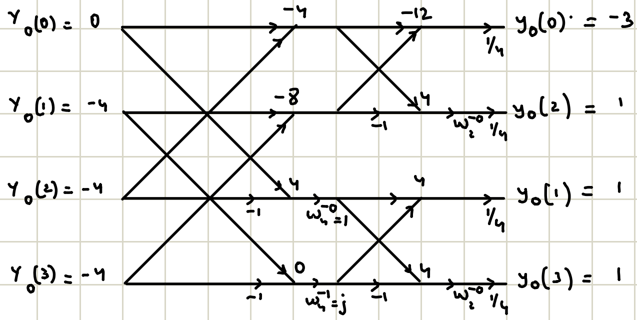


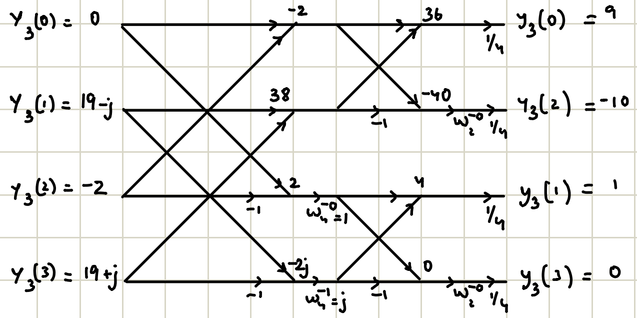
$X_0(K) \cdot H(K) = \{0, -4, -4, -4\}$

$X_1(K) \cdot H(K) = \{0, -4, -4, -4\}$

$X_2(K) \cdot H(K) = \{0, -4, -4, -4\}$

$X_3(K) \cdot H(K) = \{0, 19-j, -2, 19+j\}$





$y_0(n) = y_1(n) = y_2(n) = \{\cancel{-3}, 1, 1, 1\}$

$y_3(n) = \{\cancel{9}, 1, -10, 0\}$

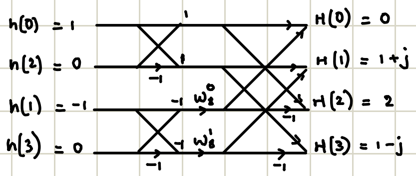
$y(n) = \{1,1,1,1,1,1,1,1,1,1,-10,0\}$

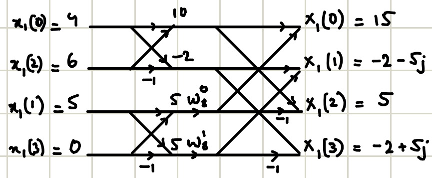
b) $N = 4$, $L = 2$

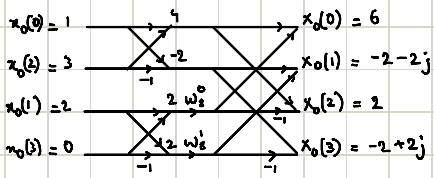
$N = L + M - 1 \Rightarrow M = 3$

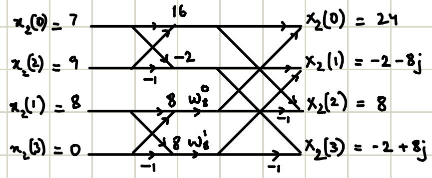
$x_0(n) = \{1, 2, 3, 0\}$, $x_1(n) = \{4, 5, 6, 0\}$, $x_2(n) = \{7, 8, 9, 0\}$, $x_3(n) = \{10, 0, 0, 0\}$

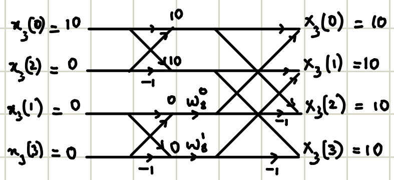
$h(n) = \{1, -1, 0, 0\}$











$X_0(K) \cdot H(K) = \{0, -4j, 4, 4j\} = Y_0(K)$

$X_1(K) \cdot H(K) = \{0, 3-7j, 10, 3+7j\} = Y_1(K)$

$X_2(K) \cdot H(K) = \{0, 6-10j, 16, 6+10j\} = Y_2(K)$

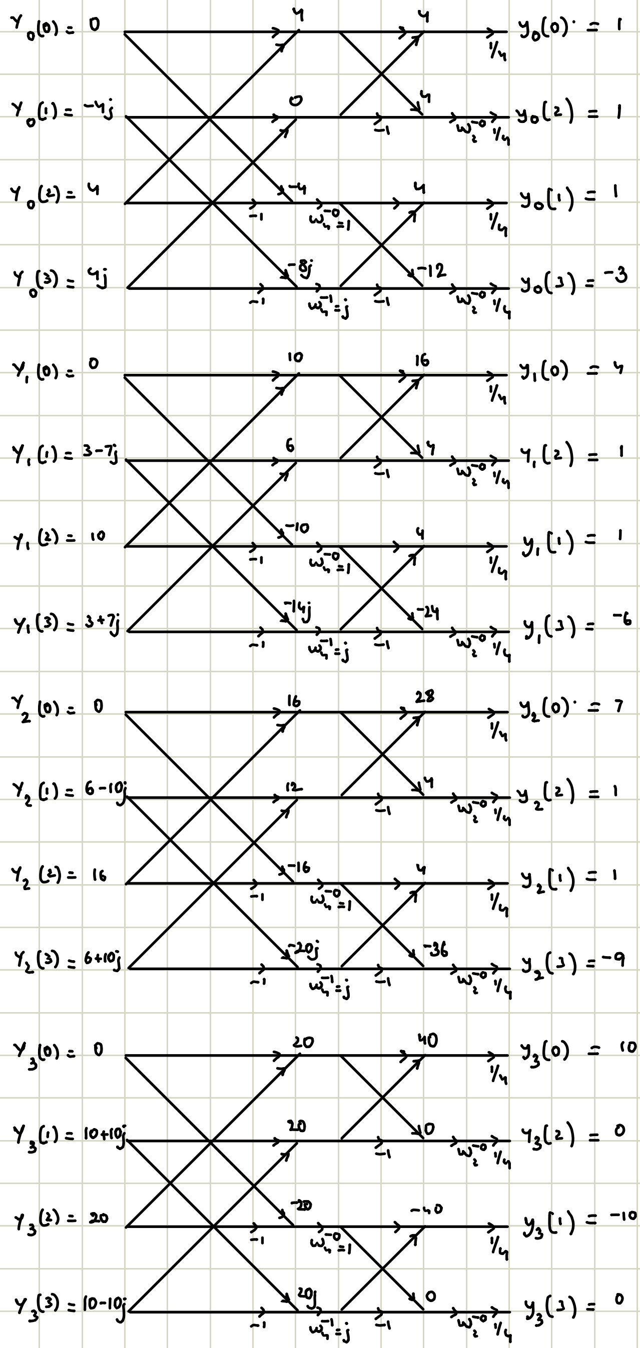
$X_3(K) \cdot H(K) = \{0, 10+10j, 20, 10-10j\} = Y_3(K)$

$\{0, -4j, 4, 4j\} = Y_0(K)$

$\{0, 3-7j, 10, 3+7j\} = Y_1(K)$

$\{0, 6-10j, 16, 6+10j\} = Y_2(K)$

$\{0, 10+10j, 20, 10-10j\} = Y_3(K)$



1	1	1	-3									
			4	1	1	-6						
						7	1	1	-9			
									10	-10	0	0
1	1	1	1	1	1	1	1	1	1	-10	0	0

$y(n) = \{1,1,1,1,1,1,1,1,1,1,-10,0\}$