Unit 3&4 Formulas

Unit - 3

- **Method of Variation of Parameters to find PI**
  - Working:
    - Consider $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X$ (where a_1 & a_2 are constants)
    - Find $CF = c_1 y_1 + c_2 y_2$
    - Find $PI = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$

      where, $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$
    - $y = CF + PI$

- **Legendre's LDE**
  - If LDE of the form,

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y = X(x)$$

Then, $ax+b = e^t \Rightarrow t = \log(ax+b)$

$$\frac{dt}{dx} = \frac{a}{ax+b} \quad \& \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b}\cdot\frac{dy}{dt}$$

$$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$$

Similarly, $(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

Then find CF & PI

- Application of L.D.E
  - LCR Circuit
    - $V = iR$
    - $V_R = R\dfrac{dq}{dt}$; $V_L = L\dfrac{di}{dt}$; $V_C = \dfrac{q}{C}$
    - $E = L\dfrac{di}{dt} + iR + \dfrac{q}{C}$

$$= L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

$$= LD^2q + RDq + \frac{q}{C}$$

$$\boxed{E = \left(LD^2 + RD + \frac{1}{C}\right)q}$$

- SHM
  - $mg - Ks = F_R = 0$ (at eq)
  - $mg - K(s+x) = ma$

$$0 - Kx = ma \Rightarrow ma + Kx = 0$$

$$a + \frac{K}{m}x = 0 \Rightarrow \frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

$$D^2x + \frac{K}{m}x = 0$$

$$D^2x + \omega^2 x = 0$$

also, $m^2 + \omega^2 = 0 \Rightarrow m = \pm \omega i$

$$x = c_1\cos\omega t + c_2\sin\omega t$$

assume $c_1 = A\cos\phi$ & $c_2 = -A\sin\phi$

Then, $x = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t$

$$= A\cos(\phi + \omega t)$$

Where, $A = \sqrt{c_1^2 + c_2^2}$ & $\phi = \tan^{-1}\left(\dfrac{c_2}{c_1}\right)$

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}}$$

$$\boxed{\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{\omega}{2\pi}}$$

$$\boxed{V_{max} = A\omega}$$

- Partial Differential Equation
  - Involves 2 or more independant variables x, y & 1 independant variable & its partial derivatives

    $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}$ etc.,

ex: $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial x}$

- Notations :
  - $\dfrac{\partial z}{\partial x} = p$; $\dfrac{\partial z}{\partial y} = q$; $\dfrac{\partial^2 z}{\partial x^2} = r$; $\dfrac{\partial^2 z}{\partial x \partial y} = s$; $\dfrac{\partial^2 z}{\partial y^2} = t$

General form of 1st order PDE

- $f(x, y, z, p, q) = 0 \longrightarrow (1)$
- PDE can have many solutions but,
  ODE can have only 1 solution

- Complete Solution :
  - $f(x, y, z, a, b) = 0$ having 2 arbitrary constants, satisfying PDE (1)

ex: $z = (x+a)(y+b)$

$\frac{\partial z}{\partial x} = y + b$ & $\frac{\partial z}{\partial y} = x + a$

$p = y + b$ $\quad q = x + a$

$pq = (y+b)(x+a)$

$= z$

$z = pq$ is the complete sol^n

- Particular Sol^n :
  - A Solution obtained by giving particular values in place of arbitrary constants

ex: $z = (x-a)^2 + (y-b)^2$

$p = 2(x-a)$ & $q = 2(y-b)$

$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

$4z = p^2 + q^2$

$4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

Lagrange's Linear PDE

- $Pp + Qq = R$ where P, Q, R are functions of x, y, z
  and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

- WORKING :
  - Compare $Pp + Qq = R$ and find P, Q, R
  - Find the auxillary
    $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
  - Solve auxillary eq^n by method of grouping / multipliers / both
  - $\phi(u, v) = 0$ is the general solution of $Pp + Qq = R$

- **Type 1** :
  - Solution is o tained by taking 2 members of auxillary eq^n $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ at a time & then integrating those 2 to get 2 independent sol^{n}s

- **Type 2** :
  - Solution is obtained by taking 2 taking 2 members of auxillary eq^n & integrate to have an eq^n (1 independant sol^n) in the variables whose differentials are involved & another independant sol^n obtained by making use of first solution
  - Note :

    by property of proportion, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{Ka_1 + Ka_2 + Ka_3}{Kb_1 + Kb_2 + Kb_3}$

- **Type 3** :
  - Find P, Q, R & auxillary
  - Find multipliers K_1, K_2, K_3 and K_1', K_2', K_3' such that, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{K_1 dx + K_2 dy + K_3 dz}{K_1 P + K_2 Q + K_3 R} = \frac{K_1' dx + K_2' dy + K_3' dz}{K_1' P + K_2' Q + K_3' R}$
  - Integrate 2 new expressions & find complete sol^n
  - Set of multipliers :

x	y	z	$-x$	$-y$	$-z$
x	$-y$	$-z$	1	-1	-1
$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{z}$	1	-1	0
$\frac{1}{x^2}$	$\frac{1}{y^2}$	$\frac{1}{z^2}$	1	0	-1
1	1	1	0	1	-1
l	m	n	x	$-y$	0

Type 1

ex: $p\cot x + q\cot y = \cot z$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$\int \tan x\, dx = \int \tan y\, dy$

$\log \sec x = \log \sec y + \log c_1$

$c_1 = \sec x / \sec y$

$\int \tan y\, dy = \int \tan z\, dz$

$c_2 = \frac{\sec z}{\sec y}$

General Soln, $\phi\left(\frac{\sec x}{\sec y}, \frac{\sec z}{\sec y}\right)$

Type 2

ex: $2p + q = \sin(x - 2y)$

$$\frac{dx}{2} = \frac{dy}{1} = \frac{dz}{\sin(x-2y)}$$

$\int dx = \int 2dy$

$x - 2y = c_1$

$dz = \sin(x-2y)\, dy$

$\int dz = \int \sin c_1\, dy$

$z = y \sin c_1 + c_2$

$z - y\sin(x-2y) = c_2$

$\phi\left(x - 2y,\ z - y\sin(x-2y)\right)$

Type 3

ex:

$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{(z^2-x^2)y} = \frac{dz}{z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x^2y^2} - \cancel{x^2z^2} + \cancel{y^2z^2} - \cancel{y^2x^2} + \cancel{x^2z^2} - \cancel{y^2z^2}}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\cancel{y^2} - \cancel{z^2} + \cancel{z^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}$$

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\frac{x^2 + y^2 + z^2}{2} = c_1$

$x^2 + y^2 + z^2 = 2c_1$

$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$\log x + \log y + \log z = \log c_2$

$c_2 = xyz$

$\phi(x^2 + y^2 + z^2,\ xyz) = 0$

- Solution of PDE using method of seperation of variables
  - ex →

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$$

Assume u as a function of x & y

So, $u = XY$

X is a function of x

Y is a function of y

$$\frac{\partial(XY)}{\partial x} = Y \frac{\partial X}{\partial x} = Y \frac{dX}{dx}$$

$$\frac{\partial(XY)}{\partial y} = X \frac{\partial Y}{\partial y} = X \frac{dY}{dy}$$

$$\frac{\partial^2(XY)}{\partial x^2} = Y \frac{d^2X}{dx^2}$$

$$\frac{\partial^2(XY)}{\partial y^2} = X \frac{d^2Y}{dy^2}$$

$$\rightarrow x^2 \cdot Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

divide by XY

$$\frac{x^2}{X} \cdot \frac{dX}{dx} + \frac{y^2}{Y} \cdot \frac{dY}{dy} = 0$$

$$\frac{x^2}{X} \cdot \frac{dX}{dx} = -\frac{y^2}{Y} \cdot \frac{dY}{dy} = k \text{ (assume)}$$

$$\int \frac{dX}{X} = \int \frac{k}{x^2} dx$$

$$\log X = -\frac{k}{x} + \log C_1$$

$$X = C_1 e^{-k/x}$$

$$\int \frac{dY}{Y} = -\int \frac{k\,dy}{y^2}$$

$$\log Y = +\frac{k}{y} + \log C_2$$

$$Y = C_2 e^{k/y}$$

$$u = XY = C_1 C_2 e^{-k/x} e^{k/y}$$

$$\boxed{u = c e^{k\left(\frac{1}{y} - \frac{1}{x}\right)}}$$

- **Solution of 1 D Heat equation**

- $U_t = c^2 U_{xx}$

$U = XT \Rightarrow \frac{\partial U}{\partial x} = T\frac{dX}{dx}$; $\frac{\partial^2 U}{\partial x^2} = T\frac{d^2X}{dx^2}$; $\frac{\partial U}{\partial t} = X\frac{dT}{dt}$

$X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2} \Rightarrow \frac{1}{c^2T}\cdot\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = K$

$\frac{dT}{dt} - Kc^2T = 0$ | $\frac{d^2X}{dx^2} - KX = 0$

$DT - Kc^2T = 0$ | $D^2X - KX = 0$

Case(i) :

$K = 0$ | $D^2X = 0$

$\Rightarrow DT = 0$ | $m^2 = 0$

$m = 0$ | $X = (c_1 + c_2x)$

$T = c_3$

$U = (c_1 + c_2x)c_3$

Case (ii) : $K = +ve$ $(K = p^2)$

$(D^2 - p^2)X = 0$ | $(D - p^2c^2)T = 0$

$m^2 - p^2 = 0$ | $m = p^2c^2$

$m = \pm p$

$X = c_1e^{px} + c_2e^{-px}$ | $T = c_3e^{p^2c^2t}$

$U = (c_1e^{px} + c_2e^{-px})c_3e^{p^2c^2t}$

Case(iii) : $K = -ve$ $(K = -p^2)$

$(D^2 + p^2)X = 0$ | $(D + p^2c^2)T = 0$

$m^2 + p^2 = 0$ | $m = -p^2c^2$

$m = \pm ip$ | $T = c_3e^{-p^2c^2t}$

$X = c_1\cos px + c_2\sin px$

$U = (c_1\cos px + c_2\sin px)\cdot c_3e^{-p^2c^2t}$

$\hookrightarrow$ Most ideal Sol^n (As Temp $\downarrow$ as time $\uparrow$)

- Solution of homogeneous linear PDE with constant coefficients

- $F(x,y) = a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + a_{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + a_n \frac{\partial^n z}{\partial y^n}$

$$= \left(a_0 D^n + a_1 D^{n-1} D' + \dots + a_{n-1} D(D')^{n-1} + a_n (D')^n\right) z$$

$$= F(D, D') z$$

In auxillary,

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n$$

where $m = \frac{D}{D'}$

Consider $\left(D^2 + a_1 DD' + a_2 (D')^2\right) z = 0 \Rightarrow m^2 + a_1 m + a_2 = 0$

Case(i) : roots are real & distinct

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

where m_1 & m_2 are roots

Case (ii): roots are equal

$$z = f_1(y + mx) + x f_2(y + mx)$$

where m is the root

- Finding PI

$$\left(D^2 + a_1 DD' + a_2 (D')^2\right) z = F(x,y)$$

$$F(D, D') z = F(x,y)$$

$$PI = \frac{1}{F(D, D')} \cdot F(x,y)$$

Case (i) : $F(x,y) = e^{ax+by}$

$$PI = \frac{1}{F(D, D')} e^{ax+by}$$

$$= \frac{1}{F(a, b)} \cdot e^{ax+by}$$

Case (ii): $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{1}{F(D^2, DD', D'^2)} \cdot F(x,y)$$

$$= \frac{1}{F(-a^2, -ab, -b^2)}$$

$D^2 = -a^2$
$DD' = -ab$
$D'^2 = -b^2$

Case (iii): $F(x,y) = x^m y^n$

$$PI = \frac{x^m y^n}{F(D, D')}$$

ex: $(D_x^3 - 2D_x^2 D_y)z = 3x^2y$

$(D_x^2 (D_x - 2D_y))z = 3x^2y$

$\Rightarrow$

$$\begin{array}{r|l} & x^3y + \frac{x^4}{2} \\ \hline D_x - 2D_y & 3x^2y \\ & \underline{3x^2y - 2x^3} \\ & 2x^3 \\ & \underline{2x^3} \\ & 0 \end{array}$$

$$PI = \frac{1}{D^2}\left(x^3y + \frac{x^4}{2}\right) = \frac{1}{D}\left(\frac{x^4y}{4} + \frac{x^5}{10}\right)$$

$$= \left(\frac{x^5y}{20} + \frac{x^6}{60}\right)$$

Case (iv):

$$F(x,y) = e^{ax+by}\, V(x,y)$$

where $V = x^m y^n$ (or) $\sin(ax+by)$ (or) $\cos(ax+by)$

$$PI = \frac{1}{F(D, D')}\, e^{ax+by} \cdot V(x,y)$$

$D \to D+a,\ D' \to D'+b$

$$= e^{ax+by} \frac{1}{F(D+a, D'+b)} \cdot V(x,y)$$

UNIT-4

- Non-homogeneous PDE
  - Linear PDE but order of all P.D aren't equal

ex: $(D^2 - (D')^2 + D - D')z = 0$

- Working :
  - Finding CF, convert $F(D, D')$ into linear factors of the form
$$((D - m_1 D' - a_1).(D - m_2 D' - a_2) \dots (D - m_n D' - a_n))z = 0$$

$$CF = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \dots + e^{a_n x} f_n(y + m_n x)$$

$a_1, a_2, \dots a_n$ & $m_1, m_2 \dots m_n$ are constant

- $(D - mD' - a)z = 0$

$p - mq - az = 0$

$Pp + Qq = R$

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

$-m\int dx = \int dy$

$y + mx = c_1$

$\int \frac{dz}{z} = \int c\,dx$

$\log z = cx + \log c_2$

$z = c_2 \cdot e^{cx}$

$c_2 = f(c_1) = f(y + mx)$

$$\boxed{z = f(y + mx) \cdot e^{cx}}$$

After converting into linear factors directly write in this form

- Easy way to factorize non-homogeneous PDE

Assume $(a_1 D + b_1 D' + c_1)(a_2 D + b_2 D' + c_2)$

coeff of $D^2 = a_1 a_2$

coeff of $D'^2 = b_1 b_2$

constant $= c_1 c_2$

coeff of $D = a_1 c_2 + a_2 c_1$

coeff of $D' = b_1 c_2 + b_2 c_1$

ex: $(2D^2 - DD' - D'^2 + D - D')z = e^{2x+3y}$

$a_1 a_2 = 2$

$b_1 b_2 = -1$

$c_1 c_2 = 0$

$a_1 c_2 + a_2 c_1 = 1$

$b_1 c_2 + b_2 c_1 = -1$

$a_1 : 1$	$a_2 : 2$
$b_1 : -1$	$b_2 : 1$
$c_1 : 0$	$c_2 : 1$

Then, $(D - D')(2D - D' + 1)$

$= (D - D')(D - \frac{D'}{2} + \frac{1}{2})$

- General Method to find PI of homogeneous PDE

Let $F(D, D')z = F(x,y)$

$$PI = \frac{1}{F(D,D')} \cdot F(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')(D-m_3D')\dots(D-m_nD')}$$

where $m_1, m_2, \dots m_n$ are factors of the auxillary of $F(x,y)$

So, $z = \frac{1}{D-mD'} F(x,y) = \int F(x, c-mx)\,dx$

where c must be replaced $y+mx$

<u>Note</u>: i) $\frac{1}{D-mD'} F(x,y) = \int F(x, c-mx)\,dx$ where $c = y+mx$

ii) $\frac{1}{D+mD'} F(x,y) = \int F(x, c+mx)\,dx$ where $c = y-mx$

ex: $(4D^2 - 4DD' + D'^2)z = 16 \log(x+2y)$

$z = CF + PI$

$CF \Rightarrow 4m^2 - 4m + 1$

$m = 1/2 \,,\, 1/2$

$CF = f_1\left(y+\frac{x}{2}\right) + x f_2\left(y+\frac{x}{2}\right)$

$$PI = \frac{16 \log(x^2+2y)}{(2D-D')^2} = \frac{4\log(x+2y)}{\left(D-\frac{D'}{2}\right)^2}$$

$$= \frac{4}{D-\frac{D'}{2}} \cdot \int \log\left(x+2\left(c-\frac{x}{2}\right)\right)dx$$

$$= \frac{4}{D-\frac{D'}{2}} \int \log(\cancel{x}+2c-\cancel{x})\,dx = \frac{4}{D-\frac{D'}{2}} x\log 2c = 4\int x \log 2\left(y+\frac{x}{2}\right)dx$$

$$= 4\int x\log(2y+x) = 4\int x\log\left(x+2\left(c-\frac{x}{2}\right)\right) = 4\int x \log 2c$$

$$= \frac{4x^2}{2}\log 2c \qquad \boxed{= \frac{4x^2}{2}\log(2y+x)}$$

- Algebraic Functions
  - Functions obtained by algebraic operators $+, -, \div, \times, \sqrt{\ }$ etc.,
- Transcendental Functions
  - Involves log, trig, hyperbolic, exponential etc.,

$\rightarrow$ Elementary Functions

- Special Functions
  - Beta, Gamma $\Rightarrow$ Expressed in integrals
  - Bessel $\Rightarrow$ Expressed as sol^n of ODEs

- Gamma Function
  - $\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1}\, dx$ where $(n > 0)$

    Note : $\Gamma 1 = 1$
  - $\Gamma(n) = 2\int_0^{\infty} e^{-t^2} \cdot t^{2n-1}\, dt$

Put $x = t^2$, $dx = 2t\,dt$; $x = 0 \Rightarrow t = 0$, $x = \infty \Rightarrow t = \infty$

$$\int_0^{\infty} e^{-t^2} (t^2)^{n-1} \cdot 2t\,dt = 2\int_0^{\infty} e^{-t^2} t^{2n-1}\, dt$$

- Beta Function
  - $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$ $(m > 0, n > 0)$

    $\beta = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$

$x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$; $x = 0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \pi/2$

$$\int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$
$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

- Properties of β & Γ functions

1) $\Gamma(n+1) = n\Gamma n$
2) $\Gamma(n+1) = n!$

   Note : $\Gamma 0$ and Γ-ve integers is not defined
3) $\beta(m,n) = \beta(n,m)$
4) $\beta(m,n) = \dfrac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$
5) $\Gamma\frac{1}{2} = \sqrt{\pi}$
6) For -ve fractions,

   $\Gamma n = \dfrac{\Gamma(n+1)}{n}$

$\left(\text{ex} : \Gamma\frac{-3}{2} = \dfrac{\Gamma\frac{-1}{2}}{\frac{-3}{2}} = \dfrac{\Gamma\frac{1}{2}}{\frac{-3}{2} \times \frac{-1}{2}} = \dfrac{4\sqrt{\pi}}{3} \right)$

3) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Let $y = 1 - x \Rightarrow x = 1 - y$; $x = 0 \Rightarrow y = 1$, $x = 1 \Rightarrow y = 0$

$$= \int_1^0 (1-y)^{m-1} (y)^{n-1} (-dy)$$
$$= \int_0^1 (1-y)^{m-1} (y)^{n-1}\, dy = \beta(n,m)$$

4) $\Gamma m = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\, dx$ $\quad \Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$

$$\Gamma m\, \Gamma n = 4 \int_0^{\infty} e^{-x^2} x^{2m-1}\, dx \int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$$
$$= 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\,dy$$

Region bound by $x = 0$ to ∞ & $y = 0$ to ∞

$x = r\cos\theta$, $y = r\sin\theta$ $\quad dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

r varies from 0 to ∞ ; θ varies from 0 to $\pi/2$

$$\Gamma m\, \Gamma n = 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r^{2m-1} \cos^{2m-1}\theta \cdot r^{2n-1} \sin^{2n-1}\theta\, d\theta\, r\,dr\,d\theta$$
$$= 4\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r^{2m+2n-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\,d\theta$$
$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr \times 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$
$$= \Gamma(m+n)\, \beta(m,n) \Rightarrow \beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

5) $\beta = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$

$m = n = 1/2$

$\beta(\frac{1}{2}, \frac{1}{2}) = 2\int_0^{\pi/2} 1 \cdot 1 \cdot d\theta = \pi$

$\beta(\frac{1}{2}, \frac{1}{2}) = \dfrac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 1} = (\Gamma\frac{1}{2})^2$

$(\Gamma\frac{1}{2})^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

Some other results

- $$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

- **Beta Function expressed as improper integral**

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$x = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1 \Rightarrow dx = \frac{-1}{(1+y)^2}\,dy$$

$x = 0 \Rightarrow y = \infty$; $x = 1 \Rightarrow y = 0$

$$\beta(m,n) = \int_1^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1} \cdot \frac{-1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}\,dx$$

(By symmetry & variable change)

- $$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin nx}$$

Legendre's Duplication Formula

- $$\Gamma(2p)\sqrt{\pi} = 2^{2p-1}\,\Gamma(p)\,\Gamma\left(p+\tfrac{1}{2}\right)$$

- $$\beta\left(p, \tfrac{1}{2}\right) = 2^{2p-1}\,\beta(p,p)$$

Important Results

- $$\int_0^\infty x^p e^{-ax^q}\,dx = \frac{\Gamma\left(\frac{p+1}{q}\right)}{q \cdot a^{\left(\frac{p+1}{q}\right)}}$$

- $$\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$$

- $$\int_0^1 x^m (\ln x)^n\,dx = \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}}$$

- $$\int_0^1 x^p(1-x^q)^r\,dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$$

1. $$I = \int_0^\infty x^p e^{-ax^q}\,dx$$

Put $ax^q = t$, $x^q = t/a \Rightarrow x = (t/a)^{1/q}$

$$dx = \frac{1}{a^{1/q}} \cdot \frac{1}{q} \cdot t^{\frac{1}{q}-1}\,dt$$

Therefore, $$\int_0^\infty \left(\frac{t}{a}\right)^{\frac{p}{q}} \cdot e^{-t} \cdot \frac{1}{a^{1/q}} \cdot \frac{1}{q} \cdot t^{\frac{1}{q}-1}\,dt$$

$$= \frac{1}{a^{\frac{p+1}{q}} \cdot q}\int_0^\infty e^{-t}\, t^{\frac{p+1}{q}-1}\,dt = \frac{\Gamma\left(\frac{p+1}{q}\right)}{a^{\frac{p+1}{q}} \cdot q}$$

2. $$\int_0^\infty e^{-kx} x^{n-1}\,dx$$ $t = kx$ $x = t/k$ $dx = dt/k$

$$\Rightarrow \int_0^\infty e^{-t} \cdot \left(\frac{t}{k}\right)^{n-1} \frac{dt}{k} = \frac{\Gamma(n)}{k^n}$$

3. $$\int_0^1 x^m (\ln x)^n\,dx$$

$-t = \ln x$, $e^{-t} = x$, $dx = -e^{-t}\,dt$

$x = 0 \Rightarrow t = \infty$, $x = 1 \Rightarrow t = 0$

$$\int_\infty^0 e^{-mt}(-t)^n(-e^{-t})\,dt = (-1)^n \int_0^\infty e^{-(1+m)t}\, t^n\,dt$$

$$= \frac{(-1)^n\,\Gamma(n+1)}{(m+1)^{n+1}}$$

4. $$\int_0^1 x^p(1-x^q)^r\,dx$$

$t = x^q$, $x = t^{1/q}$, $dx = \frac{1}{q} t^{\frac{1}{q}-1}$ | $x = 0 \Rightarrow t = 0$, $x = 1 \Rightarrow t = 1$

$$= \int_0^1 t^{p/q}(1-t)^r \cdot \frac{1}{q} t^{\frac{1}{q}-1}\,dt$$

$$= \frac{1}{q}\int_0^1 t^{\frac{p+1}{q}-1}(1-t)^r\,dt = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$$

- Assumptions to be taken while solving some of the integrals

IF	THEN
• $a^m - x^n$	$x^n = a^m \sin^2\theta$
• $a^m + x^n$	$x^n = a^m \tan^2\theta$

Where m, n can be any integer / fraction
a can be an whole number

- Bessel's Functions

- $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - n^2)y = 0$

↳ Solution $\Rightarrow y(x) = c_1 J_n(x) + c_2 J_{-n}(x)$

$$\left.\begin{aligned} J_n &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m+n} \cdot \frac{1}{\Gamma(m+n+1)} \\ J_{-n} &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m-n} \cdot \frac{1}{\Gamma(m-n+1)} \end{aligned}\right\} \rightarrow \text{Bessel's Functions}$$

Prove $J_{-n} = (-1)^n J_n(x)$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m+n} \cdot \frac{1}{\Gamma(m+n+1)}$$

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m-n} \cdot \frac{1}{\Gamma(m-n+1)}$$

Also for Γ to exist,
$m - n + 1 > 0$
$m > n - 1$

So, $$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m-n} \frac{1}{\Gamma(m-n+1)}$$

$m - n = s$
$m = s + n$
s varies from 0 to ∞

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{(s+n)!} \left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma(s+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+n)!} \left(\frac{x}{2}\right)^{2s+n} \cdot \frac{1}{\Gamma(s+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma(s+n+1)}$$

$$= (-1)^n J_n(x)$$

- $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$ & $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

- Recurrence Relations

1) $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

2) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

3) $\frac{d}{dx}\left[J_n(x)\right] = J_{n-1}(x) - \frac{n}{x} J_n(x)$

4) $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

5) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

6) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

$\rightarrow$ Proof not needed

- Integrals of Bessel's Functions

1) $\int x^v J_{v-1}(x)\,dx = x^v J_v(x) + C$

2) $\int x^{-v} J_{v+1}(x)\,dx = -x^{-v} J_v(x) + C$

__Note__: $\int x^m J_n(x)\,dx$, where m & n are integers & $m+n \geq 0$ then, they can be integrated by parts and expressed in terms of J_0 & J_1

__Note__: When $m+n$ is even, integral depends on residual $\int J_0(x)\,dx$

- Generating function for Bessel's function as $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

- $$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

- Some proofs

a) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots$

b) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \dots$

c) $\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \dots$

d) $\sin(x\cos\theta) = 2J_1(x)\sin\theta - 2J_3(x)\sin 3\theta + \dots$

e) $J_0^2 + 2J_1^2 + 2J_2^2 + \dots = 1$

- Orthogonality Property of Bessel Functions

  - $$\int_0^a x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{a^2}{2} J_{n+1}^2(a\alpha) & \text{if } \alpha = \beta \end{cases}$$

- Bessel's Integral Formula

  - $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \quad \text{(where, n is even)}$$