Unit-4 Time varying fields & Wave Theory

Faraday's Law

→ The emf induced in a closed circuit = negative time rate of change of magnetic flux linkage through the circuit

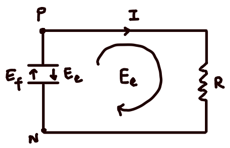
$$\boxed{V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\phi}{dt}}$$

V_{emf} : Induced Voltage
$\lambda = N\phi$: Flux Linkage
ϕ : Magnetic flux through 1 loop

-ve indicates that induced voltage acts in a way as to oppose flux producing it

→ Lenz's law : Direction of induced current is such that magnetic flux created by current opposes change in orignal magnetic flux

→ Lets consider Electric fields that aren't generated by charges, which are called emf-produced fields (sources like batteries, generators, fuel cells etc.,)



E_e : Electrostatic field

E_f : EMF produced field

Because of accumulation of charges E_e exists & $\boxed{\vec{E} = \vec{E_f} + \vec{E_e}}$

Now integrate over closed circuit,

$$\oint_L \vec{E}\cdot d\vec{l} = \oint_L \vec{E_f}\cdot d\vec{l} + \oint_L \vec{E_e}\cdot d\vec{l}$$

$$= \oint_L \vec{E_f}\cdot d\vec{l} + 0 \quad (E_e \text{ is conservative})$$

$$= \int_N^P \vec{E_f}\cdot d\vec{l} = -\int_N^P \vec{E_e}\cdot d\vec{l} = IR$$

→ E_e can't maintain steady current in closed circuit because $\oint_L \vec{E_e}\cdot d\vec{l} = 0 = IR$

→ E_f is non-conservative

→ Except electrostatics, usually voltage ≠ potential difference

Transformer and motional electromotive forces

→ $\boxed{V_{emf} = -\frac{d\Psi}{dt}}$ when $N = 1$

$$= \oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} \quad (\text{Faraday's law in integral form})$$

Variation of flux with time can be caused in 3 ways :

1) By having stationary loop in a time varying $\vec{B}$ field
2) By having time-varying loop area in a static $\vec{B}$ field
3) By having time-varying loop area in a time varying $\vec{B}$ field

1) Stationary loop in Time-varying B field

→ $\boxed{V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \, d\vec{S}}$

By stokes theorem, $\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

For both integrals to be equal, $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

2) Moving loop in static B field

→ $\vec{F}_m = Q \vec{u} \times \vec{B}$

motional electric field $\vec{E}_m = \frac{\vec{F}_m}{Q} = \vec{u} \times \vec{B}$

Then, $\boxed{V_{emf} = \oint_L \vec{E}_m \cdot d\vec{l} = \oint_L (\vec{u} \times \vec{B}) \, d\vec{l}}$

→ Motional emf or flux-cutting emf

→ $\vec{F}_m = I \vec{l} \times \vec{B}$ (or) $F_m = IlB$

$V_{emf} = uBl$

and by stoke's theorem, $\int_S (\vec{\nabla} \times \vec{E}_m) \cdot d\vec{S} = \int_S \vec{\nabla} \times (\vec{u} \times \vec{B}) \cdot d\vec{S}$

$$\vec{\nabla} \times \vec{E}_m = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

3) Moving loop in Time Varying B field

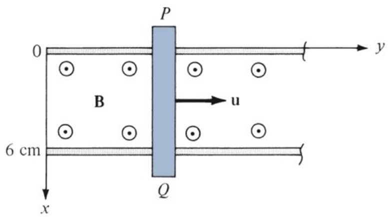
→ $\boxed{V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_L (\vec{u} \times \vec{B}) \cdot d\vec{l}}$

Then, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{u} \times \vec{B})$

Stationary loop & Time varying Field ⇒ $V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$

Motional loop & Static Field ⇒ $V_{emf} = \int_L (\vec{u} \times \vec{B}) \, d\vec{l}$

Motional loop & Time varying Field ⇒ $V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} \, d\vec{S} + \int_L (\vec{u} \times \vec{B}) \, d\vec{l}$



Q. A conducting bar can slide freely over 2 conducting rails as shown. Calculate induced voltage in the bar.

a) If bar is stationed at $y = 8\text{cm}$ & $\vec{B} = 4\cos 10^6 t\,\hat{a}_z$ mWb/m^2

b) If bar slides at velocity $\vec{u} = 20\,\hat{a}_y$ m/s & $\vec{B} = 4\hat{a}_z$ mWb/m^2

c) If bar slides at velocity $\vec{u} = 20\,\hat{a}_y$ m/s & $\vec{B} = 4\cos(10^6 t - y)\,\hat{a}_z$ mWb/m^2

A. a) Case 1 $\Rightarrow$

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = \int_{y=0}^{0.08}\int_{x=0}^{0.06} 4\times 10^{-3}\times 10^{6}\sin 10^6 t\,dx\,dy$$

$$= 4\times 10^{3}\times 0.08\times 0.06\,\sin 10^6 t$$

$$= 19.2\sin 10^6 t \quad V$$

b) Case 2 $\Rightarrow$

$$V_{emf} = \int_L (\vec{u}\times\vec{B})\,d\vec{l} = \int_{x=l}^{0} (u\hat{a}_y \times B\hat{a}_z)\,dx\,\hat{a}_x$$

$$= -uBl = -20\times(4\times 10^{-3})\times 0.06$$

$$= -4.8 \text{ mV}$$

c) case 3 $\Rightarrow$

Method 1:

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\,d\vec{S} + \int_L (\vec{u}\times\vec{B})\,d\vec{l}$$

$$= \int_{x=0}^{0.06}\int_{0}^{y} 4\times 10^{-3}\times 10^{6}\sin[10^6 t - y)\,dy\,dx$$

$$+ \int_{0.06}^{0} 20\hat{a}_y \times 4\times 10^{-3}\cos(10^6 t - y)\hat{a}_z]\,dx\,\hat{a}_x$$

$$= 240\cos(10^6 t - y)\Big|_0^y - 80\times 10^{-3}\times 0.06\cos(10^6 t - y)$$

$$= 240\cos(10^6 t - y) - 240\cos 10^6 t$$

$$= -480\sin\left(10^6 t - \frac{y}{2}\right)\sin\left(\frac{y}{2}\right) \quad V$$

$$\left(\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right)$$

Method 2:

$$V_{emf} = -\frac{\partial \Psi}{\partial t}$$

$$\Psi = \int \vec{B}\cdot d\vec{S}$$

$$= \int_{y=0}^{y}\int_{x=0}^{0.06} 4\cos(10^6 t - y)\,dx\,dy$$

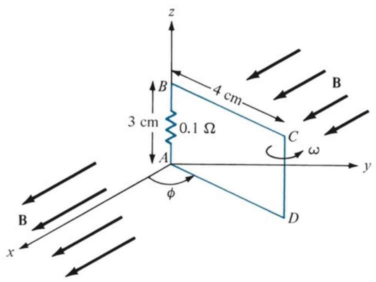
$$= -4(0.06)\sin(10^6 t - y)\Big|_{y=0}^{y}$$

$$= -0.24\sin(10^6 t - y) + 0.24\sin 10^6 t \quad \text{mWb}$$

Hence, $\Psi = -0.24\sin(10^6 t - 20t) + 0.24\sin 10^6 t$ mWb

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20)\cos(10^6 t - 20t) - 0.24(10^6)\cos 10^6 t \text{ mV}$$

$$\simeq 240\cos(10^6 t - y) - 240\cos 10^6 t \text{ V}$$



Q. The loop in the figure is inside a uniform magnetic field $\vec{B} = 50\hat{a}_x \ mWb/m^2$. If side DC of the loop cuts the flux lines at the frequency of 50 Hz and loop lies in yz-plane at time $t=0$, find a) V_{emf} at $t=1s$ b) i at $t=3s$

A. a) Since static field,

$$V_{emf} = \int_L (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

We must take in cylindrical co-ordinates

$\vec{dl} = dz\,\hat{a}_z$ (Direction of DC along z-axis)

$\vec{u} = \frac{\vec{dl}'}{dt} = \frac{\rho\, d\phi}{dt}\hat{a}_\phi = \rho\omega\,\hat{a}_\phi$

$\rho = 4cm$

$\omega = 2\pi f = 100\pi$

$\vec{B} = 0.05\,\hat{a}_x$

$= 0.05(\cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi)$

$$\vec{u} \times \vec{B} = \begin{vmatrix} a_\rho & a_\phi & a_z \\ 0 & \rho\omega & 0 \\ 0.05\cos\phi & -0.05\sin\phi & 0 \end{vmatrix} = -0.05\rho\omega\cos\phi\,\hat{a}_z$$

$(\vec{u} \times \vec{B}) \cdot \vec{dl} = -0.05 \times 0.04 \times 100\pi \times \cos\phi \ dz = -0.2\pi\cos\phi\ dz$

$$V_{emf} = \int_{z=0}^{0.03} -0.2\pi\cos\phi\ dz = -0.2\pi\cos\phi\,(0.03-0) = -6\pi\cos\phi \ mV$$

We know, $\omega = \frac{d\phi}{dt} \longrightarrow \phi = \omega t + c$

At $t=0$, $\phi = \pi/2 \Rightarrow c = \pi/2$

$\phi = \omega t + \frac{\pi}{2}$

$$V_{emf} = -6\pi\cos\left(\omega t + \frac{\pi}{2}\right) mV = 6\pi\sin(100\pi t)\ mV$$

$t = 1ms$, $V_{emf} = 6\pi\sin(0.1\pi) = 5.825\,mV$

b) $i = \frac{V_{emf}}{R} = \frac{6\pi\cos(100\pi t)}{0.1} = 60\pi\sin(100\pi t)\ mA$

$t = 3ms$, $i = 60\pi\sin(0.3\pi)\ mA = 0.1525\ A$

Q. **PRACTICE EXERCISE 9.1**

Consider the loop of Figure 9.5. If $\mathbf{B} = 0.5\mathbf{a}_z$ Wb/m^2, $R = 20\ \Omega$, $\ell = 10$ cm, and the rod is moving with a constant velocity of $8\mathbf{a}_x$ m/s, find

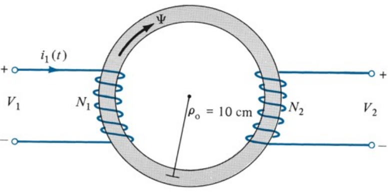
(a) The induced emf in the rod
(b) The current through the resistor
(c) The motional force on the rod
(d) The power dissipated by the resistor.

A. a) $V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l} = uB\ell = 8 \times 0.5 \times (0.1) = 0.4\ V$

b) $I = \frac{V_{emf}}{R} = \frac{0.4}{20} = 0.02\ A$

c) $F = \vec{B} \times (I\vec{\ell}) = 0.5\hat{a}_z \times (0.02 \times 10 \times 10^{-2})\hat{a}_y = -\hat{a}_x$ mN

d) $P = \frac{V_{emf}^2}{R} = \frac{(0.4)^2}{20} = 8\ mW$

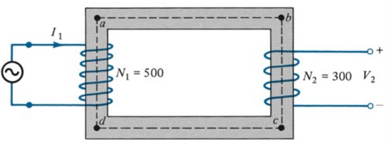


Q. The magnetic circuit has a uniform cross section of 10^{-3} m^2. If the circuit is energized by current $i_1(t) = 3 \sin 100\pi t$ A in the coil of $N_1 = 200$ turns, find V_{emf} in the coil of $N_2 = 100$ turns. Assume that $\mu = 500\mu_0$

A. $\Psi = \frac{F}{R} = \frac{N_1 i_1}{\ell/\mu S} = \frac{N_1 i_1 \mu S}{2\pi\rho_0}$

$V_2 = -N_2 \frac{d\Psi}{dt} = -100 \times \frac{d}{dt}\left(\frac{(200) \times (3 \sin 100\pi t) \times (500 \times 4\pi \times 10^{-7}) \times 10^{-3}}{2\pi \times 10 \times 10^{-2}}\right)$

$= -6\pi \cos 100\pi t \quad V$



Q. **PRACTICE EXERCISE 9.3**

A magnetic core of uniform cross section 4 cm^2 is connected to a 120 V, 60 Hz generator as shown in Figure 9.9. Calculate the induced emf V_2 in the secondary coil.

A. $V_1 = -N_1 \frac{d\Psi}{dt}$ & $V_2 = -N_2 \frac{d\Psi}{dt}$

$\frac{V_2}{V_1} = \frac{N_2}{N_1} \Rightarrow V_2 = \frac{300 \times 120}{500} = 72V$

Displacement Current

→ Lets consider Maxwell's curl equation for time varying magnetic fields

So, we know $\vec{\nabla} \times \vec{H} = \vec{J}$

But divergence of curl of any vector is 0

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0$$
$$= \vec{\nabla} \cdot \vec{J}$$

But continuity equation requires $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$

This is an incompatible equation, so we add another term

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$$
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{J}_d$$
$$\Rightarrow \vec{\nabla} \cdot \vec{J}_d = -\vec{\nabla} \cdot \vec{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D}) = \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t}$$

So, $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$

$$\boxed{\vec{\nabla} \times \vec{H} = \underbrace{\vec{J}}_{\text{Conduction current}} + \overbrace{\frac{\partial \vec{D}}{\partial t}}^{\text{Displacement Current}}}$$

→ Maxwell's equation for time-varying field

Q. A parallel plate capacitor with plate area 5 cm^2 & plate seperation of 3 mm has a voltage 50 sin $10^3 t$ V applied to plates. Calculate I_d assuming $\varepsilon = 2\varepsilon_0$

A. $D = \varepsilon E = \varepsilon \frac{V}{d}$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$

$$I_d = J_d S = \frac{\varepsilon S}{d} \cdot \frac{dV}{dt} = 2\varepsilon_0 \times \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \cdot \frac{d}{dt}(50 \sin 10^3 t) = 2.95 \times 10^{-12} \times 50 \times 10^3 \cos 10^3 t$$
$$= 147.57 \cos 10^3 t \text{ nA}$$

Q. In free space, $\vec{E} = 20 \cos(\omega t - 50x)\, \hat{a}_y$ V/m. Calculate

a) $\vec{J}_d$ b) $\vec{H}$ c) ω

A. In free space $\varepsilon = \varepsilon_0$

a) $\vec{J}_d = \frac{d\vec{D}}{dt} = \frac{d(\varepsilon \vec{E})}{dt} = -\varepsilon_0 \times 20\omega \sin(\omega t - 50x)\, \hat{a}_y$ A/m^2

b) $\nabla \times \vec{H} = \vec{J}_d \Rightarrow -\frac{\partial \vec{H}}{\partial x} \hat{a}_y = -20\omega\varepsilon_0 \sin(\omega t - 50x)\hat{a}_y \Rightarrow \vec{H} = -\frac{20\, \omega \varepsilon_0 \cos(\omega t - 50x)}{-50} \hat{a}_z$

$$= 0.4\, \omega \varepsilon_0 \cos(\omega t - 50x)\, \hat{a}_z \text{ A/m}$$

c) $\nabla \times \vec{E} = -\mu_0 \frac{d\vec{H}}{dt} \Rightarrow \frac{-\partial \hat{E}_z}{\partial x} \hat{a}_z = 0.4 \mu_0 \omega \varepsilon_0 \sin(\omega t - 50x)\, \hat{a}_z$

$$1000 = 0.4 \mu_0 \varepsilon_0 \omega^2$$
$$1000 = 0.4 \frac{\omega^2}{c^2}$$
$$\omega = 1.5 \times 10^{10} \text{ rad/s}$$

Maxwell's Equations in final forms

→ Maxwell's eq for static electric & magnetic fields

Later on it become more generalized for time varying conditions

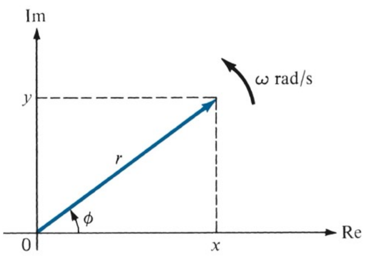
Differential Form	Integral Form	Remarks
$\vec{\nabla}\cdot\vec{D} = \rho_v$	$\oint_s \vec{D}\cdot d\vec{s} = \int_v \rho_v\, dv$	Gauss' Law
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_s \vec{B}\cdot d\vec{s} = 0$	Non existence of magnetic monopole
$\vec{\nabla}\times\vec{E} = 0$	$\oint_L \vec{E}\cdot d\vec{l} = 0$	Conservative nature of electrostatic field
$\vec{\nabla}\times\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot d\vec{l} = \int_s \vec{J}\cdot d\vec{s}$	Ampere's Law

Generalisation →

Differential Form	Integral Form	Remarks
$\vec{\nabla}\cdot\vec{D} = \rho_v$	$\oint_s \vec{D}\cdot d\vec{s} = \int_v \rho_v\, dv$	Gauss' Law
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_s \vec{B}\cdot d\vec{s} = 0$	Non existence of magnetic monopole
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\oint_L \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_s \vec{B}\cdot d\vec{s}$	Faraday's Law
$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$	$\oint_L \vec{H}\cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s}$	Ampere's Circuit Law

Changes from static to time-varying field

Time harmonic fields



→ Fields that vary sinusoidally with time

$A(t) = A_0 \cos(\omega t + \phi)$

Instead of working with sinusoidal functions, we use phasors for simplification

$A(t) = R(A_s e^{j\omega t})$

where $A_s = A_0 e^{j\phi}$

→ Key Phasor Properties

1) Addition / subtraction

$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$

2) Multiplication

$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$

3) Division

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\phi_1 - \phi_2)$

4) Time derivative

$\frac{dA(t)}{dt} = j\omega A_s$

5) Square root

$\sqrt{z} = \sqrt{r} \angle \phi/2$

6) Complex Conjugate

$z^* = x - jy = r \angle -\phi$

Point Form	Integral Form
$\vec{\nabla}\cdot\vec{D}_s = \rho_{vs}$	$\oint \vec{D}_s\cdot d\vec{s} = \int \rho_{vs}\, dv$
$\vec{\nabla}\cdot\vec{B}_s = 0$	$\oint \vec{B}_s\cdot d\vec{s} = 0$
$\vec{\nabla}\times\vec{E}_s = -j\omega\vec{B}_s$	$\oint \vec{E}_s\cdot d\vec{l} = -j\omega\int \vec{B}_s\cdot d\vec{s}$
$\vec{\nabla}\times\vec{H}_s = \vec{J}_s + j\omega\vec{D}_s$	$\oint \vec{H}_s\cdot d\vec{l} = \int(\vec{J}_s + j\omega\vec{D}_s)\, d\vec{s}$

Q. Evaluate the complex numbers

a) $z_1 = \dfrac{j(3-j4)^*}{(-1+j6)(2+j)^2}$ b) $z_2 = \left[\dfrac{1+j}{4-j8}\right]^{1/2}$

A. a) $z_1 = \dfrac{j(3+j4)}{(-1+j6)(2+j)^2}$

$$= \frac{-4+j3}{-27+j14} = \frac{-4+j3}{(-27+j14)} \times \frac{(-27-j14)}{(-27-j14)} = \frac{150-j25}{27^2+14^2} = 0.1622 - j0.027$$

b) $z_2 = \left[\dfrac{1+j}{4-j8}\right]^{1/2}$

$$= \left[\frac{\sqrt{2}\angle 45}{4\sqrt{5}\angle -63.4}\right]^{1/2} = \left[0.1581 \angle 45+63.4\right]^{1/2} = \sqrt{0.1581}\angle \frac{108.4}{2} = 0.3976 \angle 54.2$$

Q. Given $\vec{A} = 10\cos(10^8 t - 10x + 60^\circ)\hat{a}_z$ Express $\vec{A}$ in phasor form

$\vec{B}_s = \left(\dfrac{20}{j}\right)\hat{a}_x + 10e^{\frac{j2\pi x}{3}}\hat{a}_y$ $\vec{B}_s$ in instantaneous form

A. $\vec{A} = Re\left[10e^{j(\omega t - 10x + 60^\circ)}\hat{a}_z\right]$

$\omega = 10^8$ rad/s

$\vec{A} = Re\left[10e^{j(60^\circ - 10x)}\hat{a}_z e^{j\omega t}\right] = Re\left(\vec{A}_s e^{j\omega t}\right)$

$\vec{A}_s = 10e^{j(60^\circ - 10x)}\hat{a}_z$

and

$\vec{B}_s = \dfrac{20}{j}\hat{a}_x + 10e^{\frac{j2\pi x}{3}}\hat{a}_y$

$= -20j\,\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y$

$= 20e^{-j\pi/2}\hat{a}_x + 10e^{j2\pi x/3}\hat{a}_y$

$\vec{B} = Re\left(\vec{B}_s e^{j\omega t}\right)$

$= Re\left(20e^{j(\omega t - \pi/2)}\hat{a}_x + 10e^{j(\omega t + \frac{2\pi x}{3})}\hat{a}_y\right]$

$= 20\cos\left(\omega t - \frac{\pi}{2}\right)\hat{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\hat{a}_y$

$= 20\sin(\omega t)\hat{a}_x + 10\cos\left(\omega t + \frac{2\pi x}{3}\right)\hat{a}_y$

Q. Given $\vec{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\,\hat{a}_\phi$ V/m & $\vec{H} = \frac{H_0}{\rho}\cos(10^6 t + \beta z)\,\hat{a}_\rho$ A/m

Express in phasor form & determine H_0 & β such that they satisfy maxwell's equations

A. $\vec{E} = \mathrm{Re}\left(\vec{E}_s e^{j\omega t}\right)$ $\quad \vec{H} = \mathrm{Re}\left(\vec{H}_s e^{j\omega t}\right)$

$\omega = 10^6$

$\vec{E}_s = \frac{50}{\rho} e^{j\beta z}\hat{a}_\phi$ $\quad \vec{H}_s = \frac{H_0}{\rho} e^{j\beta z}\hat{a}_\rho$

For free space, $\rho_v = 0$, $\sigma = 0$, $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$

$\vec{\nabla}\cdot\vec{E}_s = 0 \longrightarrow ①$

$\vec{\nabla}\cdot\vec{H}_s = 0 \longrightarrow ②$

$\vec{\nabla}\times\vec{H}_s = j\omega\varepsilon_0\vec{E}_s \longrightarrow ③$

$\vec{\nabla}\times\vec{E}_s = -j\omega\mu_0\vec{H}_s \longrightarrow ④$

For ①, $\vec{\nabla}\cdot\vec{E}_s = \frac{1}{\rho}\cdot\frac{\partial \vec{E}_{\phi s}}{\partial\phi} = 0$

For ②, $\vec{\nabla}\cdot\vec{H}_s = \frac{1}{\rho}\cdot\frac{\partial(\rho\vec{H}_{\rho s})}{\partial\rho} = 0$

For ③, $\vec{\nabla}\times\vec{H}_s = \vec{\nabla}\times\left(\frac{H_0}{\rho}e^{j\beta z}\hat{a}_\rho\right) = \frac{\partial}{\partial z}\left(\frac{H_0}{\rho}e^{j\beta z}\hat{a}_z\cdot\hat{a}_\rho\right) = \frac{jH_0\beta}{\rho}e^{j\beta z}\hat{a}_\phi$

Comparing to $j\omega\varepsilon_0\vec{E}_s = j\omega\varepsilon_0\frac{50}{\rho}e^{j\beta z}\hat{a}_\phi$

Then $H_0\beta = 50\omega\varepsilon_0 \longrightarrow ⑤$

For ④, $\vec{\nabla}\times\vec{E}_s = \vec{\nabla}\times\left(\frac{50}{\rho}e^{j\beta z}\hat{a}_\phi\right) = -j\beta\frac{50}{\rho}e^{j\beta z}\hat{a}_\rho$

Comparing to $-j\omega\mu_0\vec{H}_s = -j\omega\mu_0\frac{H_0}{\rho}e^{j\beta z}\hat{a}_\rho$

Then $50\beta = H_0\mu_0\omega \Rightarrow \frac{H_0}{\beta} = \frac{50}{\omega\mu_0} \longrightarrow ⑥$

⑤ $\times$ ⑥ $\Rightarrow H_0^2 = (50)^2\frac{\varepsilon_0}{\mu_0}$

$H_0 = \pm 50\sqrt{\frac{\varepsilon_0}{\mu_0}} = \pm\frac{50}{120\pi} = \pm 0.1326$

⑤ $\div$ ⑥ $\Rightarrow \beta^2 = \omega^2\mu_0\varepsilon_0$

$\beta = \pm\omega\sqrt{\mu_0\varepsilon_0}$

$= \pm\frac{\omega}{c} = \pm\frac{10^6}{3\times10^8} = \pm 3.33\times10^{-3}$

Waves

→ It is a function of both space and time.

It represents how disturbance at one point in space-time in space-time influences another point later

→ General form of scalar wave equation

$$\boxed{\frac{\partial^2 E}{\partial t^2} = u^2 \frac{\partial^2 E}{\partial z^2}}$$

E : Electric field

u : wave velocity

→ Solution of wave equation

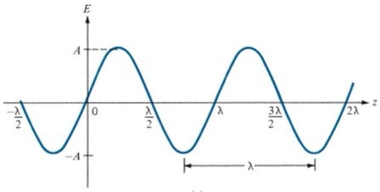
$E(z,t) = f(z - ut) + g(z + ut)$

f : forward travelling wave (+ve z-direction) ⇒ E^+

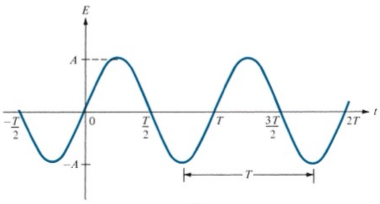
g : backward travelling wave (−ve z-direction) ⇒ E^-

$E(z,t) = A\sin(\omega t - \beta z)$

t is constant



(a)



(b)

$E(z,t) = A\sin(\omega t - \beta z)$

z is constant

→ For sinusoidal fields (Time harmonic) of time dependance $e^{j\omega t}$

The general form becomes $\boxed{\frac{d^2 E_s}{dz^2} + \beta^2 E_s = 0}$

$\left(\boxed{\beta = \frac{\omega}{u}}\right)$ Phase constant

$$E(z,t) = \underbrace{Ae^{j(\omega t - \beta z)}}_{\text{+ve z-travel}} + \underbrace{Be^{j(\omega t + \beta z)}}_{\text{-ve z-travel}}$$

where A, B are real constants

→ Take the imaginary part of $E^+(z,t)$, $E = A\sin(\omega t - \beta z)$

then, wavelength $\boxed{\lambda = \frac{2\pi}{\beta}}$

frequency $\boxed{f = \frac{1}{T}}$

angular frequency $\boxed{\omega = 2\pi f}$

wave velocity $\boxed{u = \frac{\omega}{\beta} = f\lambda}$

direction of propagation ⇒ $\omega t - \beta z$: wave travels in +ve z

$\omega t + \beta z$: wave travels in −ve z

Important Relations

$\sin(-\theta) = -\sin\theta = \sin(\theta \pm \pi)$ $(\theta = \omega t \pm \beta z)$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \pi/2) = \pm\cos\theta$

$\sin(\theta \pm \pi) = -\sin\theta$

$\cos(\theta \pm \pi/2) = \mp\sin\theta$

$\cos(\theta \pm \pi) = -\cos\theta$

TABLE 10.1 Electromagnetic Spectrum

EM Phenomena	Examples of Uses	Approximate Frequency Range
Cosmic rays	Physics, astronomy	10^{14} GHz and above
Gamma rays	Cancer therapy	10^{10}–10^{13} GHz
X-rays	X-ray examination	10^{8}–10^{9} GHz
Ultraviolet radiation	Sterilization	10^{6}–10^{8} GHz
Visible light	Human vision	10^{5}–10^{6} GHz
Infrared radiation	Photography	10^{3}–10^{4} GHz
Microwave waves	Radar, microwave relays, satellite communication	3–300 GHz
Radio waves	UHF television	470–806 MHz
	VHF television, FM radio	54–216 MHz
	Short-wave radio	3–26 MHz
	AM radio	535–1605 kHz

Q. $\vec{E} = 50\cos(10^8 t + \beta x)\,\hat{a}_y$ V/m

a) Find direction of wave propagation

b) Calculate β & time taken to travel distance of $\lambda/2$

c) Sketch wave at $t = 0,\ T/4,\ T/2$

A. a) Since we have eq^n in the form of $\omega t + \beta x \Rightarrow$ -ve x direction $(-\hat{a}_x)$

b) In free space, $u = c$

$$\beta = \frac{\omega}{c} = \frac{10^8}{3\times 10^8} = 0.33 \text{ rad/m}$$

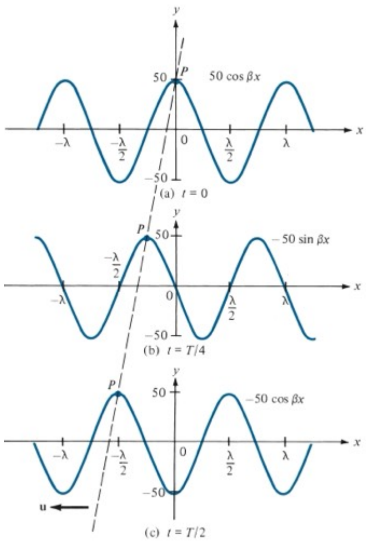
Since it takes T seconds to travel λ

then for $\frac{T}{2} = \frac{\frac{2\pi}{\omega}}{2} = \frac{\pi}{10^8} = 3.142\times 10^{-8} = 31.42$ ns

c) At $t = 0$, $E_y = 50\cos\beta x$

$t = \frac{T}{4}$, $E_y = 50\cos\left(\omega\cdot\frac{T}{4} + \beta x\right) = 50\cos\left(\omega\cdot\frac{2\pi}{4\omega} + \beta x\right) = 50\cos\left(\beta x + \frac{\pi}{2}\right) = -50\sin\beta x$

$t = \frac{T}{2}$, $E_y = 50\cos\left(\omega\cdot\frac{T}{2} + \beta x\right) = 50\cos\left(\omega\cdot\frac{2\pi}{2\omega} + \beta x\right) = 50\cos(\beta x + \pi) = -50\cos\beta x$



Q. In free space $\vec{H} = 0.1\cos(2\times 10^8 t - kx)\,\hat{a}_y$ A/m

a) Calculate k, λ, T

b) Calculate time t_1 to travel $\lambda/8$

c) Plot wave at t_1

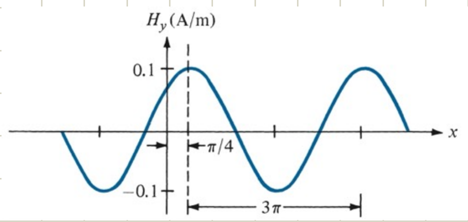
A. a) $T = \frac{2\pi}{\omega} = \frac{2\pi}{2\times 10^8} = 31.42$ ns

$\lambda = uT = 3\times 10^8 \times \pi\times 10^{-8} = 3\pi = 9.426$ m

$k = \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{3\pi} = 0.67$ rad/m

b) $t_1 = T/8 = \frac{\pi}{8}\times 10^{-8} = 3.927$ ns

c) $H\left(t = \frac{T}{8}\right) = 0.1\cos\left(2\times 10^8 \times \frac{\pi}{8}\times 10^{-8} - \frac{2x}{3}\right)\hat{a}_y = 0.1\cos\left(\frac{2x}{3} - \frac{\pi}{4}\right)\hat{a}_y$



Wave propagation in lossy dielectrics

→ Lossy dielectric is a medium in which EM wave, as it propagates, loses power owing to imperfect dielectric

→ A lossy dielectric is a partially conducting medium with $\sigma \neq 0$, $\varepsilon > 0$ & $\mu > 0$

Non-zero conductivity

→ Consider linear, isotropic, homogeneous, lossy dielectric which is charge free & supressing time factor $e^{j\omega t}$ and maxwell's eqn' in phasor form,

$$\vec{\nabla} \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}_s = -j\omega\mu (\vec{\nabla} \times \vec{H}_s)$$

$$\vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}_s}^{\,0}) - \vec{\nabla}^2 \vec{E}_s = -j\omega\mu (\vec{\nabla} \times \vec{H}_s)$$

$$\vec{\nabla}^2 \vec{E}_s = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_s$$

$$\left(\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \right) \qquad \alpha + j\beta = \gamma$$

Propagation Constant

$$\boxed{\vec{\nabla}^2 \vec{E}_s - \gamma^2 \vec{E}_s = 0}$$

Helmholtz's equations

also $\quad \nabla^2 \vec{H}_s - \gamma^2 \vec{H}_s = 0$

Now to find α & β,

we know $\gamma = \alpha + j\beta$ & $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

$$\gamma^2 = \alpha^2 - \beta^2 + j2\alpha\beta = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$= -\omega^2\mu\varepsilon + j\sigma\omega\mu$$

Then $\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad \Big| \quad 2\alpha\beta = \sigma\omega\mu$

$$\beta = \frac{\sigma\omega\mu}{2\alpha}$$

$$\alpha^2 - \left(\frac{\sigma\omega\mu}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$\alpha^4 - \left(\frac{\sigma\omega\mu}{2}\right)^2 = -\omega^2\mu\varepsilon\alpha^2 \Rightarrow (\alpha^2)^2 + (\omega^2\mu\varepsilon)\alpha^2 - \left(\frac{\omega\mu\sigma}{2}\right)^2 = 0$$

Solving quadratic equation

$$\alpha^2 = \frac{-\omega^2\mu\varepsilon \pm \sqrt{(\omega^2\mu\varepsilon)^2 + (\omega\mu\sigma)^2}}{2} = \frac{-\omega^2\mu\varepsilon \pm \omega^2\mu\varepsilon\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}}{2}$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]$$

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)}} \quad \& \quad \boxed{\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)}}$$

Attenuation Constant

Phase Shift constant / Wave number

→ α is attenuation constant which is the measure of spatial rate of decay of wave in medium measured in nepers per meter & expressed in dB/m .

Attenuation of 1 neper denotes reduction to e^{-1} of original value & increase of 1 neper indicates increase by factor of e

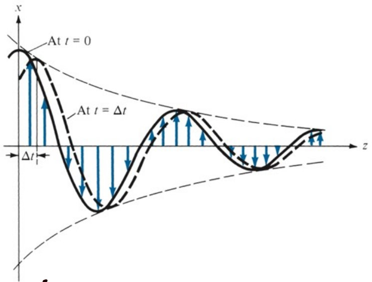
1 Neper = $20 \log_{10} e$ = 8.686 dB

→ $\vec{E}_s = E_0 e^{-\gamma z} + E_0' e^{\gamma z}$ (wave along $\hat{a}_z$)

$$\vec{E} = \mathrm{Re}\left(\vec{E}_0 e^{-\gamma z} e^{j\omega t}\right)\hat{a}_x$$
$$= \mathrm{Re}\left(\vec{E}_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\right)\hat{a}_x$$
$$= \vec{E}_0 e^{-\alpha z}\cos(\omega t - \beta z)\,\hat{a}_x \longrightarrow \text{Wave solution for electric field}$$

Similarly ,

$$\vec{H} = \vec{H}_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta)\,\hat{a}_y \longrightarrow \text{Wave solution for magnetic field}$$



↳ E-field with x-component travelling in z-direction at t=0 to Δt

θ : Phase lag due to complex intrinsic impedance

where $H_0 = \dfrac{E_0}{\eta}$

η : Intrinsic impedance (unit : Ω)

$$\eta = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\, e^{j\theta_\eta} = |\eta| \angle\theta_\eta$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left(1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)^{1/4}} \quad , \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} \quad \left(0 \le \theta_\eta \le 45^\circ\right)$$

↳ $\theta_\eta = \dfrac{1}{2}\tan^{-1}\left(\dfrac{\sigma}{\omega\varepsilon}\right)$

→ At any point of time, $\vec{E}$ leads $\vec{H}$ by θ_η

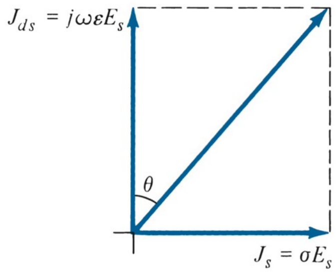
→ Ratio of magnitude of conduction current density to displacement current density in lossy medium,
$$\frac{|J_{cs}|}{|J_{ds}|} = \frac{|\sigma \vec{E}_s|}{|j\omega\varepsilon \vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

θ : loss angle of medium

↳ Loss Tangent ⇒ Explains how lossy medium is

and $\tan\theta = \tan 2\theta_\eta = \dfrac{\sigma}{\omega\varepsilon}$ ⇒ $\theta = 2\theta_\eta$

$\tan\delta$	Medium Type	Behaviour
<< 1	Good dielectric	Very little loss
>> 1	Good Conductor	nearly no $\vec{E}$-field penetrates
≃ 1	Lossy dielectric	Noticeable field loss



→ We know from maxwell's equation in phasor form,

$$\vec{\nabla} \times \vec{H_s} = (\sigma + j\omega\varepsilon)\vec{E_s}$$

$$= j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]\vec{E_s}$$

$$= j\omega\varepsilon_c \vec{E_s}$$

$$\left(\varepsilon_c = \varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right) = \varepsilon(1 - j\tan\theta) = \varepsilon' - j\varepsilon''\right)$$

ε_c : Complex permittivity

$$\tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$$

Q. A lossy dielectric has intrinsic impedance of $200\angle 30^\circ\ \Omega$ at particular angular frequency 'ω'. If, at that frequency, $\vec{H} = 10e^{-\alpha x}\cos\left(\omega t - \frac{x}{2}\right)\hat{a}_y$ A/m, Find $\vec{E}$ and α

A. $\hat{a}_K = \hat{a}_x$ & $\hat{a}_H = \hat{a}_y$

So, $-\hat{a}_E = \hat{a}_K \times \hat{a}_H = \hat{a}_x \times \hat{a}_y = \hat{a}_z$

$\hat{a}_E = -\hat{a}_z$

$H_0 = 10$

$$\frac{E_0}{H_0} = \eta = 200\angle 30 \Rightarrow E_0 = 2000\angle 30$$

$$\vec{E} = \mathrm{Re}\left(2000\, e^{-\alpha x} e^{j\pi/6} e^{-jx/2} e^{j\omega t}\, \hat{a}_E\right)$$

$$= -2000\, e^{-\alpha x}\cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right)\hat{a}_z \ \text{V/m}$$

We know $\beta = 1/2$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)} \quad \text{and} \quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)}$$

$$\frac{\alpha}{\beta} = \left(\frac{\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1}{\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1}\right)^{1/2}$$

$$\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$$

$$2\alpha = \left(\frac{2-1}{2+1}\right)^{1/2} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{1}{2\sqrt{3}} = 0.2887 \ \text{Np/m}$$

Q. A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5\, e^{-z/3}\sin(10^8 t - \beta z)\mathbf{a}_x$ V/m. Determine

(a) β
(b) The loss tangent
(c) Intrinsic impedance
(d) Wave velocity
(e) **H** field

A.

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)} \qquad \left(\alpha = \frac{1}{3}\right)$$

$$= \omega\sqrt{\frac{2\mu_0 \times 8\varepsilon_0}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}$$

$$\alpha = \frac{\omega}{c}\sqrt{\frac{16}{2}}\sqrt{\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)} \qquad \left(c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right)$$

$$\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1} = \frac{\alpha c}{\omega\sqrt{8}} = \frac{1/3 \times 3\times10^8}{10^8\sqrt{8}} = \frac{1}{\sqrt{8}}$$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1+\frac{1}{8} = \frac{9}{8} \Rightarrow 1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2 = \frac{81}{64} \Rightarrow \left(\frac{\sigma}{\omega\varepsilon}\right) = \frac{\sqrt{17}}{8} = 0.5154 = \tan 2\theta_\eta \Rightarrow \theta_\eta = 13.63^\circ$$

a) $$\frac{\beta}{\alpha} = \sqrt{\frac{\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right)}{\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}} = \sqrt{17}$$

$$\beta = \frac{\sqrt{17}}{3} = 1.374 \text{ rad/m}$$

b) $$\frac{\sigma}{\omega\varepsilon} = 0.5154$$

c) $$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\left(1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)^{1/4}} = \frac{120\pi\sqrt{\frac{2}{8}}}{\left(\frac{81}{64}\right)^{1/4}} = 177.72$$

$$\angle\theta_\eta = 13.63$$

$$\eta = 177.72\angle 13.63^\circ\ \Omega$$

d) $$u = \frac{\omega}{\beta} = \frac{10^8}{1.374} = 7.278\times10^7 \text{ m/s}$$

e) $$\hat{a}_H = \hat{a}_k \times \hat{a}_E = \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\vec{H} = \frac{0.5}{177.5} e^{-z/3}\sin(10^8 t - \beta z - 13.63^\circ)\,\hat{a}_y$$

$$= 2.817\, e^{-z/3} \sin(10^8 t - \beta z - 13.63^\circ)\,\hat{a}_y$$

Plane Waves in lossless dielectrics

→ In lossless dielectric, $\frac{\sigma}{\omega \varepsilon} << 1$

$\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$ & $\mu = \mu_0 \mu_r$

Now put in α & β eq^n s

$\alpha = 0$

$\boxed{\beta = \omega \sqrt{\mu \varepsilon}} = \frac{\omega}{u}$

lossless intrinsic impedance

$\boxed{\lambda = \frac{2\pi}{\beta}}$ $\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}}$

Plane waves in free space

→ Special case of wave propagation in lossless media where

$\sigma = 0$, $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$

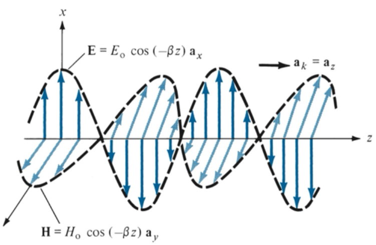
Then, $\alpha = 0$

$\beta = \omega \sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c}$

$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$ & $\lambda = \frac{2\pi}{\beta}$

$\theta_\eta = 0$ and $\eta = \eta_0$ η_0 : intrinsic impedance of free space

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \simeq 377\,\Omega$



$\vec{E} = E_0 \cos(\omega t - \beta z)\ \hat{a}_x$

$\vec{H} = \frac{E_0}{\eta_0} \cos(\omega t - \beta z)\ \hat{a}_y$

→ Transverse electromagnetic wave ⇒ No E_z or H_z

$\hat{a}_k \times \hat{a}_E = \hat{a}_H$

→ The fields $\vec{E}$ and $\vec{H}$ have same magnitude everywhere on any plane $z = c$ (c: constant)

Theoretically extends to ∞

Real waves approximate this at far distances

Plane Waves in Good conductors

→ $\sigma >> \omega \varepsilon$

like Cu, Al,

→ Electric fields can't penetrate deeply - decay rapidly near surface

→ $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma}$

$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$

$\lambda = \frac{2\pi}{\beta}$

$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$= \sqrt{j\omega\mu\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}$

$= \sqrt{j\omega\mu\sigma}\sqrt{1 + \frac{j\omega\varepsilon}{\sigma}}$

→ $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45°$

$\gamma = \alpha + j\beta$

$= \sqrt{\omega\mu\sigma} \angle 45°$

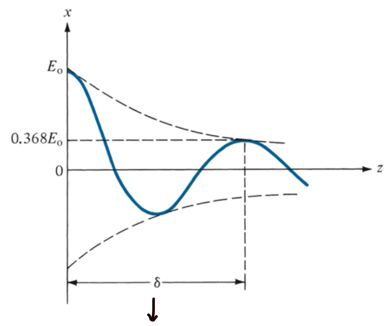
$\vec{E}$ leads $\vec{H}$ by $45°$

$\vec{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{a}_x$

$\vec{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45°)\hat{a}_y$

→ as $\vec{E}$ wave travels in conducting medium, amplitude attenuated by factor $e^{-\alpha z}$.

The distance δ through which wave amplitude decreases to a factor e^{-1} is called skin depth



Skin Depth

→ Skin Depth

→ In simple terms, δ is how deeply wave penetrates before it dies out to $1/e$ of its surface value

$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$

→ The phenomenon whereby the field intensity in a conductor rapidly decreases is known as skin effect

It is the tendency of charges to migrate from the bulk of conducting material to the surface, resulting in higher resistance

TABLE 10.2 Skin Depth in Copper*

Frequency (Hz)	10	60	100	500	10^4	10^8	10^{10}
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

*For copper, $\sigma = 5.8 \times 10^7$ S/m, $\mu = \mu_o$, $\delta = 66.1/\sqrt{f}$ (in mm).

→ Skin effect manifests in different forms

i) Attenuation in wave guides

ii) Effective (or) ac resistance in transmission lines

iii) Electromagnetic shielding

→ It is used to find ac resistance due to skin effect $\left(R_{dc} = \frac{l}{\sigma S}\right)$

$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \Rightarrow R_{ac} = \frac{l}{\sigma\delta w} = \frac{R_s l}{w}$

$(S \simeq \delta w)$

w : width

l : length

$\frac{R_{ac}}{R_{dc}} = \frac{\frac{l}{\sigma 2\pi a \delta}}{\frac{l}{\sigma \pi a^2}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f \mu \sigma}$

→ Summary

	Lossy Medium	Lossless Medium	Free Space	Conductor
α	$\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)}$	0	0	$\sqrt{\pi f \mu \sigma}$
β	$\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu_0\varepsilon_0}$	$\sqrt{\pi f \mu \sigma}$
η	$\sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377$	$(1+j)\frac{\alpha}{\sigma}$
	$v = \frac{\omega}{\beta}$, $\lambda = \frac{2\pi}{\beta}$			

Q. In a lossless for which $\eta = 60\pi$, $\mu_r = 1$ and $\vec{H} = -0.1\cos(\omega t - z)\hat{a}_x + 0.5\sin(\omega t - z)\hat{a}_y$ A/m

Calculate ε_r, ω, $\vec{E}$

A. $\sigma = 0$, $\alpha = 0$, $\beta = 1$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 60\pi \Rightarrow \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

$$\omega = \frac{\beta c}{2} = \frac{1\times 3\times 10^8}{2} = 1.5\times 10^8 \text{ rad/s}$$

Now let $\vec{H} = \vec{H}_1 + \vec{H}_2$

where $\vec{H}_1 = -0.1\cos(\omega t - z)\hat{a}_x$ & $\vec{H}_2 = 0.5\sin(\omega t - z)\hat{a}_y$

and $\vec{E} = \vec{E}_1 + \vec{E}_2$

where $\vec{E}_1 = E_{1_0}\cos(\omega t - z)\hat{a}_{E_1}$ & $\vec{E}_2 = E_{2_0}\sin(\omega t - z)\hat{a}_{E_2}$

$$\hat{a}_{E_1} = -(\hat{a}_k \times \hat{a}_{H_1}) = -(\hat{a}_z \times -\hat{a}_x) = \hat{a}_y$$

$$E_{1_0} = \eta H_{1_0} = 60\pi(0.1) = 6\pi$$

$$\vec{E}_1 = 6\pi\cos(\omega t - z)\,\hat{a}_y$$

$$\hat{a}_{E_2} = -(\hat{a}_k \times \hat{a}_{H_2}) = -(\hat{a}_z \times \hat{a}_y) = \hat{a}_x$$

$$E_{2_0} = \eta H_{2_0} = 60\pi(0.5) = 30\pi$$

$$\vec{E}_2 = 30\pi\sin(\omega t - z)\hat{a}_x$$

$$\vec{E} = 94.25\sin(1.5\times 10^8 t - z)\hat{a}_x + 18.85\cos(1.5\times 10^8 t - z)\hat{a}_y \text{ V/m}$$

Q. A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z}\sin(10^8 t - \beta z)\mathbf{a}_y \text{ V/m}$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ S/m, find α, β, and $\mathbf{H}$.

Solution:

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \dfrac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi\times 10^{-7}\times 20(10^8)(3)}{2}\right]^{1/2}$$
$$= 61.4$$

$$\alpha = 61.4 \text{ Np/m}, \quad \beta = 61.4 \text{ rad/m}$$

Also

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi\times 10^{-7}\times 20(10^8)}{3}\right]^{1/2}$$
$$= \sqrt{\frac{800\pi}{3}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45° = \frac{\pi}{4}$$

Hence

$$\mathbf{H} = H_o e^{-\alpha z}\sin\left(\omega t - \beta z - \frac{\pi}{4}\right)\mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1\times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1\, e^{-61.4z}\sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right)\mathbf{a}_x \text{ mA/m}$$

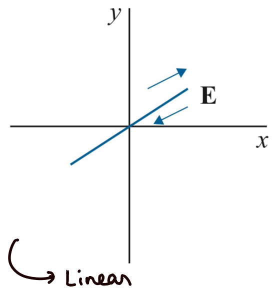
Wave Polarization

→ It is the property of wave describing how electric field vector of EM wave behaves in time at a fixed point in space

→ It is the locus of the tip of electric field (plane $\perp^{ar}$ to direction of propogation) at a given point as function of time

→ There are 3 types :

i) Linear Polarization



Linear

→ Tip of electric field traces a straight line

→ Happens when wave has only 1 field component (or) 2 components in phase

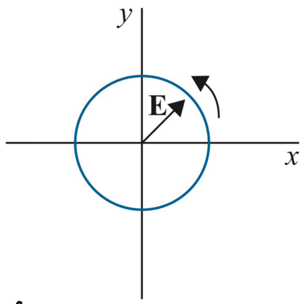
→ Given by, $E_x = E_{0x} \cos(\omega t - \beta z + \phi_x)$

$E_y = E_{0y} \cos(\omega t - \beta z + \phi_y)$

$\vec{E} = E_x \hat{a}_x + E_y \hat{a}_y$

$\Delta\phi = \phi_y - \phi_x = n\pi \qquad (n = 0, 1, 2 \ldots)$

ii) Circular Polarization



Circular

→ Tip of electric field traces a circle

→ Happens when components have equal magnitude & phase difference of $\pm 90°$

→ Given by $E_{0x} = E_{0y} = E_0$

$\Delta\phi = \phi_y - \phi_x = \pm(2n+1)\pi/2 \qquad n = 0, 1, 2, \ldots$

$E_x = E_0 \cos(\omega t - \beta z)$

$E_y = E_0 \cos(\omega t - \beta z + \frac{\pi}{2}) = -E_0 \sin(\omega t - \beta z)$

Combined, $|E|^2 = E_x^2 + E_y^2 = E_0^2$

iii) Elliptical Polarization

→ Tip of electric field traces an ellipse

→ Happens when x & y components aren't equal & phase difference is odd multiple of $\pi/2$

→ Given by, $E_{0x} \neq E_{0y}$

$\Delta\phi = \phi_y - \phi_x = \pm(2n+1)\pi/2 \qquad n = 0, 1, 2, \ldots$

$E_x = E_{0x} \cos(\omega t)$

$E_y = E_{0y} \cos(\omega t + \frac{\pi}{2}) = -E_{0y} \sin(\omega t)$

$\cos^2(\omega t) + \sin^2(\omega t) = 1 = \frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2}$

Q. Determine Polarization of plane wave with

a) $\vec{E}(z,t) = 4e^{-0.25z} \cos(\omega t - 0.8z)\hat{a}_x + 3e^{-0.25z} \sin(\omega t - 0.8z)\hat{a}_y$ V/m

b) $\vec{H}_s(z) = H_0 e^{-j\beta z}\hat{a}_x - 2H_0 e^{-j\beta z}\hat{a}_y$

c) $\vec{E}_s = E_0(\hat{a}_x - j\hat{a}_y)e^{-j\beta z}$

A. a) $E_x = 4e^{-0.25z}\cos(\omega t - 0.8z)$ & $E_y = 3e^{-0.25z}\sin(\omega t - 0.8z)$

In $z=0$ plane, $\frac{1}{4}E_x(0,t) = \cos(\omega t)$ & $\frac{1}{3}E_x(0,t) = \sin(\omega t)$

$\frac{E_x^2}{16} + \frac{E_y^2}{9} = 1 \Rightarrow$ Elliptically polarized

b) Both components are in phase, hence linear

c) $E_{s_x} = E_0 \hat{a}_y$

$E_{s_y} = -jE_0 \hat{a}_y$

$E_x = E_0 \cos(\omega t)$

$E_y = -E_0 \sin(\omega t)$

$E_x^2 + E_y^2 = E_0^2 \Rightarrow$ Circular shift

Power and the poynting vector

→ Energy can be transported from $\underbrace{\text{one point}}_{\text{transmitter}}$ to $\underbrace{\text{another point}}_{\text{receiver}}$ by means of EM waves

The rate of such energy transportation can be obtained from Maxwell's equations :

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Taking , $\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \sigma E^2 + \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t}$

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) + \vec{\nabla} \cdot (\vec{H} \times \vec{E}) = \sigma E^2 + \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \left(\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})\right)$$

also, $\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = \vec{H} \left(-\mu \frac{\partial \vec{H}}{\partial t}\right) = \frac{-\mu}{2} \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t}$

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \sigma E^2 + \frac{\varepsilon}{2} \frac{\partial E^2}{\partial t}$$

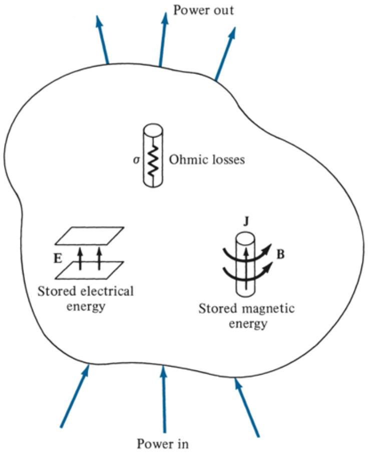
↳ Rearranging terms & volume integral on B.S

$$\int_v \vec{\nabla} \cdot (\vec{E} \times \vec{H})\, dv = -\frac{\partial}{\partial t} \int_v \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right) dv - \int_v \sigma E^2\, dv$$

Apply divergence theorem to LHS

$$\boxed{\oint_s (\vec{E} \times \vec{H}) \cdot d\vec{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv - \int_v \sigma E^2\, dv} \longrightarrow \text{Poynting's Theorem}$$

Total power leaving the volume = rate of decrease in energy stored in electric & magnetic fields − ohmic power dissipated



→ Poynting Vector , $\vec{P} = \vec{E} \times \vec{H}$

Unit : W/m^2

↳ Instantaneous power density vector associated with EM field at a given point

→ The integration of P over any closed surface gives net power flowing out of the surface

Poynting's Theorem - The net power flowing out of a given volume v = time rate of decrease in energy stored within v − ohmic losses

→ $\hat{a}_k = \hat{a}_E \times \hat{a}_H$

$E(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$ & $H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \hat{a}_y$

$$\cos(A)\cos(B) = \frac{\cos(A-B) + \cos(A+B)}{2}$$

Then , $P(z,t) = \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)\, \hat{a}_z = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \left(\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)\right) \hat{a}_z$

↳ 1

Time - Average Power

→ $$\vec{P}_{avg}(z) = \frac{1}{T}\int_0^T \vec{P}(t)\,dt$$

$$= \frac{1}{2}\,\mathrm{Re}\left(\vec{E}_s \times \vec{H}_s^{*}\right)$$

$$= \frac{E_0^2}{2|\eta|}\, e^{-2\alpha z} \cos\theta_\eta\, \hat{a}_z$$

→ Total time-avg power crossing a given surface area S $= \int_S P_{avg} \cdot d\vec{S}$

Q. $\vec{E} = 4\sin(2\pi \times 10^7 t - 0.8x)\,\hat{a}_z$ V/m

Find a) ε_r, η b) P_{avg} c) Total power crossing $100\,cm^2$ of plane $2x + y = 5$

A. a) $\alpha = 0$ & $\beta \neq \frac{\omega}{c}$ ⇒ lossless medium

$\beta = 0.8$ $\qquad \mu = \mu_0$

$\omega = 2\pi \times 10^7$ $\qquad \varepsilon = \varepsilon_0 \varepsilon_r$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r} \Rightarrow \sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8 \times 3 \times 10^8}{2\pi \times 10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = \frac{144}{\pi^2} = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi \times \frac{\pi}{12} = 10\pi^2 = 98.7\,\Omega$$

b) $$\vec{P} = \vec{E} \times \vec{H}$$

$$= \frac{E_0^2}{\eta}\sin^2(\omega t - \beta x)\,\hat{a}_x$$

$$\vec{P}_{avg} = \frac{1}{T}\int_0^T P\,dt = \frac{E_0^2}{2\eta}\,\hat{a}_x = \frac{16}{2 \times 98.7}\,\hat{a}_x = 81\,\hat{a}_x \text{ mW/m}^2$$

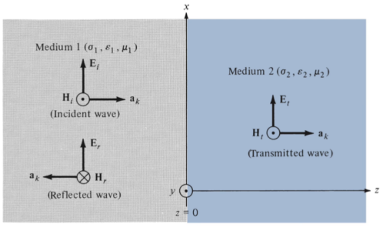
c) On plane $2x + y = 5$,

$$\hat{a}_n = \frac{2\hat{a}_x + \hat{a}_y}{\sqrt{5}}$$

$$P_{ave} = \int \vec{P}_{avg} \cdot d\vec{S} = P_{avg} \cdot S\,\hat{a}_n$$

$$= (81 \times 10^{-3}\,\hat{a}_x)(100 \times 10^{-4})\left(\frac{2\hat{a}_x + \hat{y}}{\sqrt{5}}\right)$$

$$= \frac{162 \times 10^{-5}}{\sqrt{5}} = 724.5\,\mu W$$



Plane wave incident normally on an interface b/w 2 diff. media

Reflection of a plane wave at normal incidence

→ When EM wave travelling in medium 1 hits a boundary with medium 2 perpendicularly

Part of the wave is

i) Reflected (Back to medium 1)

ii) Transmitted (continues to medium 2)

→ Wave description in phasor form :

Let wave travel in +ve z-direction & boundary be at z = 0

i) Incident Wave

$$\vec{E}_i(z) = E_{i_0}\, e^{-\gamma_1 z}\, \hat{a}_x$$

$$\vec{H}_i(z) = \frac{E_{i_0}}{\eta}\, e^{-\gamma_1 z}\, \hat{a}_y = H_{i_0}\, e^{-\gamma_1 z}\, \hat{a}_y$$

ii) Reflected Wave

$$\vec{E}_r(z) = E_{r_0}\, e^{\gamma_1 z}\, \hat{a}_x$$

$$\vec{H}_r(z) = -\frac{E_{r_0}}{\eta_1}\, e^{\gamma_1 z}\, \hat{a}_y = H_{r_0}\, e^{\gamma_1 z}\,(-\hat{a}_y)$$

iii) Transmitted Wave

$$\vec{E}_t(z) = E_{t_0}\, e^{-\gamma_2 z}\, \hat{a}_x$$

$$\vec{H}_t(z) = \frac{E_{t_0}}{\eta_2}\, e^{-\gamma_2 z}\, \hat{a}_y = H_{t_0}\, e^{-\gamma_2 z}\, \hat{a}_y$$

→ From this diagram,

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r \qquad \vec{E}_2 = \vec{E}_t$$

$$\vec{H}_1 = \vec{H}_i + \vec{H}_r \qquad \vec{H}_2 = \vec{H}_t$$

At interface z = 0, boundary conditions require tangential components of $\vec{E}$ & $\vec{H}$ must be continuous

So, $\vec{E}_{1tan} = \vec{E}_{2tan}$

$\vec{H}_{1tan} = \vec{H}_{2tan}$

$$\Rightarrow \quad E_i(0) + E_r(0) = E_t(0) \longrightarrow E_{i_0} + E_{r_0} = E_{t_0}$$

$$H_i(0) + H_r(0) = H_t(0) \longrightarrow \frac{1}{\eta_1}(E_{i_0} - E_{r_0}) = \frac{E_{t_0}}{\eta_2}$$

$$\left.\right\} \quad E_{r_0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i_0}, \qquad E_{t_0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i_0}$$

→ Reflection coefficient $\Gamma = \dfrac{E_{r_0}}{E_{i_0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient $\tau = \dfrac{E_{t_0}}{E_{i_0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

→ Remarks :

i) $1 + \Gamma = \tau$

ii) Both τ & Γ are dimensionless & can be complex

iii) $0 \leq |\Gamma| \leq 1$

i) **Special case 1** (Perfect Conductor)

→ If medium 1 is perfect dielectric (lossless, $\sigma_1 = 0$)
and medium 2 is perfect conductor ($\sigma_2 \simeq \infty$)

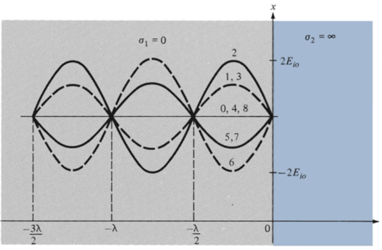
Then, $\left.\begin{array}{l}\eta_2 = 0 \\ \Gamma = -1 \\ \tau = 0\end{array}\right\} \rightarrow$ Wave is totally reflected

The totally reflected wave combines with incident wave to form standing wave

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$
$$= (E_{i_0} e^{-\gamma_1 z} + E_{r_0} e^{\gamma_1 z})\,\hat{a}_x$$

$$\Gamma = \frac{E_{r_0}}{E_{i_0}} = -1 \quad , \quad \sigma_1 = 0 \; , \; \alpha_1 = 0 \; , \; \gamma_1 = j\beta_1$$

$$\vec{E}_1 = -E_{i_0}(e^{j\beta_1 z} - e^{-j\beta_1 z})\,\hat{a}_x$$
$$= -2j\,E_{i_0} \sin\beta_1 z \;\hat{a}_x$$
$$= \mathrm{Re}\,(\vec{E}_1 e^{j\omega t})$$
$$= 2E_{i_0} \sin\beta_1 z \,\sin\omega t \;\hat{a}_x$$
$$\vec{H}_1 = \frac{2E_{i_0}}{\eta_1} \cos\beta_1 z \cos\omega t \;\hat{a}_y$$



↳ Standing waves $E = 2E_{i_0} \sin\beta_1 z \sin\omega t\, \hat{a}_x$
curves at 0, 1, 2, 3, 4 …
respectively at $t = 0, \frac{T}{8}, \frac{T}{4}, \frac{3T}{8}, \frac{T}{2} \ldots$
$\lambda = \frac{2\pi}{\beta_1}$

ii) **Special case 2** (Both Lossless media)

$\sigma_1 = \sigma_2 = 0$, η_1 & η_2 are real & so are Γ and τ

case i : $\eta_2 > \eta_1$ & $\Gamma > 0$

There is standing wave in medium 1 but there is also transmitted wave in medium 2

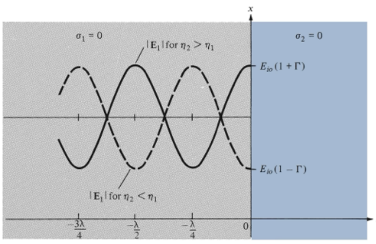
Both incident & reflected waves have amplitudes of unequal magnitudes

$$-\beta_1 z_{max} = n\pi$$
$$z_{max} = \frac{-n\pi}{\beta_1} = \frac{-n\lambda_1}{2} \quad , \quad n = 0, 1, 2, \ldots$$

and minimum value of $|E_1|$ occur at

$$-\beta_1 z_{min} = (2n+1)\frac{\pi}{2}$$

$$z_{min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\lambda_1}{4} \quad , \quad n = 0, 1, 2 \ldots$$



↳ Standing waves due to reflection at an inteface b/w 2 lossless media
$\lambda = \frac{2\pi}{\beta_1}$

case ii : $\eta_2 < \eta_1$, $\Gamma < 0$

$|E_1|$ max given by z_{min} & $|E_1|$ min given by z_{max}

1) $|H_1|$ min occurs whenever there is $|E_1|$ max & vice versa

2) Transmitted wave in medium 2 is purely traveling wave , and there is no minima & maxima in this region

Standing Wave Ratio $s = \frac{E_{max}}{E_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$ also $|\Gamma| = \frac{s-1}{s+1}$ (Unit : dimensionless)

Since $|\Gamma| \leq 1$, it follows $1 \leq s \leq \infty$ & $s\,\text{dB} = 20\log_{10} s$

Q. In free space $(z \leq 0)$, a plane wave with $\vec{H_i} = 10\cos(10^8 t - \beta z)\,\hat{a}_x$ mA/m is incident normally on a lossless medium $(\varepsilon = 2\varepsilon_0,\ \mu = 8\mu_0)$ in region $z \geq 0$. Determine $\vec{H_r}$, $\vec{E_r}$ and $\vec{H_t}$, $\vec{E_t}$

A. $\beta_1 = \frac{\omega}{c} = \frac{10^8}{3\times10^8} = \frac{1}{3}$ $\qquad \eta_1 = \eta_0 = 120\pi$

For lossless dielectric medium,

$$\beta_2 = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{2\times8} = 4\beta_1 = \frac{4}{3}$$

$$\eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = 2\eta_0$$

Given $\vec{H_i} = 10\cos(10^8 t - \beta_1 z)\,\hat{a}_x$ mA/m

Then $\vec{E_i} = E_{i_0}\cos(10^8 t - \beta_1 z)\,\hat{a}_{E_i}$

$$\hat{a}_{E_i} = \hat{a}_{H_i} \times \hat{a}_{K_i} = \hat{a}_x \times \hat{a}_z = -\hat{a}_y$$

$$E_{i_0} = \eta_1 H_{i_0} = 10\eta_0$$

$$\vec{E_i} = -10\eta_0\cos(10^8 t - \beta_1 z)\,\hat{a}_y \text{ mV/m}$$

$$\frac{E_{r_0}}{E_{i_0}} = \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_0 - \eta_0}{2\eta_0 + \eta_0} = \frac{1}{3}$$

$$E_{r_0} = \frac{E_{i_0}}{3}$$

$$\vec{E_r} = \frac{-10}{3}\eta_0\cos\left(10^8 t + \frac{z}{3}\right)\hat{a}_y \text{ mV/m}$$

$$\vec{H_r} = \frac{-10}{3}\cos\left(10^8 t + \frac{z}{3}\right)\hat{a}_x \text{ mV/m}$$

$$\frac{E_{t_0}}{E_{i_0}} = \tau = 1 + \Gamma = 1 + \frac{1}{3} = \frac{4}{3} \Rightarrow E_{t_0} = \frac{4}{3}E_{i_0}$$

$$\hat{a}_{E_t} = \hat{a}_{E_i} = -\hat{a}_y,$$

$$\vec{E_t} = \frac{-40}{3}\eta_0\cos\left(10^8 t - \frac{4z}{3}\right)\hat{a}_y \text{ mV/m}$$

$$\vec{H_t} = \frac{20}{3}\cos\left(10^8 t - \frac{4z}{3}\right)\hat{a}_x \text{ mV/m}$$

Q. A 5 GHz uniform plane wave $\mathbf{E}_{is} = 10\, e^{-j\beta z}\, \mathbf{a}_x$ V/m in free space is incident normally on a large, plane, lossless dielectric slab $(z > 0)$ having $\varepsilon = 4\varepsilon_o$, $\mu = \mu_o$. Find the reflected wave $\mathbf{E}_{rs}$ and the transmitted wave $\mathbf{E}_{ts}$.

A.

$$\eta_1 = \eta_o = 120\pi, \eta_2 = \sqrt{\frac{\eta}{\varepsilon}} = \frac{\eta_o}{2}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 2/3, \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3$$

$$E_{ro} = \Gamma E_{io} = -\frac{10}{3}$$

$$E_{rs} = -\frac{10}{3} e^{j\beta_1 z} a_x \text{ v/m}$$

where $\beta_1 = \omega / c = 100\pi / 3$.

$$E_{to} = \tau E_{io} = \frac{20}{3}$$

$$E_{ts} = \frac{20}{3} e^{-j\beta_2 z} a_x \text{ V/m}$$

where $\beta_2 = \omega \sqrt{\varepsilon_r} / c = 2\beta_1 = 200\pi / 3$.

Q. Given a uniform plane wave in air as

$$\mathbf{E}_i = 40 \cos(\omega t - \beta z)\mathbf{a}_x + 30 \sin(\omega t - \beta z)\mathbf{a}_y \text{ V/m}$$

(a) Find $\mathbf{H}_i$.
(b) If the wave encounters a perfectly conducting plate normal to the z-axis at $z = 0$, find the reflected wave $\mathbf{E}_r$ and $\mathbf{H}_r$.
(c) What are the total $\mathbf{E}$ and $\mathbf{H}$ fields for $z \leq 0$?
(d) Calculate the time-average Poynting vectors for $z \leq 0$ and $z \geq 0$.

A.

(a) This is similar to the problem in Example 10.3. We may treat the wave as consisting of two waves $\mathbf{E}_{i1}$ and $\mathbf{E}_{i2}$, where

$$\mathbf{E}_{i1} = 40 \cos(\omega t - \beta z)\mathbf{a}_x, \quad \mathbf{E}_{i2} = 30 \sin(\omega t - \beta z)\mathbf{a}_y$$

At atmospheric pressure, air has $\varepsilon_r = 1.0006 \simeq 1$. Thus air may be regarded as free space. Let $\mathbf{H}_i = \mathbf{H}_{i1} + \mathbf{H}_{i2}$.

$$\mathbf{H}_{i1} = H_{i1o} \cos(\omega t - \beta z)\mathbf{a}_{H_1}$$

where

$$H_{i1o} = \frac{E_{i1o}}{\eta_o} = \frac{40}{120\pi} = \frac{1}{3\pi}$$

$$\mathbf{a}_{H_1} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Hence

$$\mathbf{H}_{i1} = \frac{1}{3\pi} \cos(\omega t - \beta z)\mathbf{a}_y$$

Similarly,

$$\mathbf{H}_{i2} = H_{i2o} \sin(\omega t - \beta z)\mathbf{a}_{H_2}$$

where

$$H_{i2o} = \frac{E_{i2o}}{\eta_o} = \frac{30}{120\pi} = \frac{1}{4\pi}$$

$$\mathbf{a}_{H_2} = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

Hence

$$\mathbf{H}_{i2} = -\frac{1}{4\pi} \sin(\omega t - \beta z)\mathbf{a}_x$$

and

$$\begin{aligned}\mathbf{H}_i &= \mathbf{H}_{i1} + \mathbf{H}_{i2} \\ &= -\frac{1}{4\pi} \sin(\omega t - \beta z)\mathbf{a}_x + \frac{1}{3\pi} \cos(\omega t - \beta z)\mathbf{a}_y \text{ mA/m}\end{aligned}$$

(b) Since medium 2 is perfectly conducting,

$$\frac{\sigma_2}{\omega \varepsilon_2} \gg 1 \rightarrow \eta_2 \ll \eta_1$$

that is,

$$\Gamma \simeq -1, \quad \tau = 0$$

showing that the incident $\mathbf{E}$ and $\mathbf{H}$ fields are totally reflected:

$$E_{ro} = \Gamma E_{io} = -E_{io}$$

Hence,

$$\mathbf{E}_r = -40 \cos(\omega t + \beta z)\mathbf{a}_x - 30 \sin(\omega t + \beta z)\mathbf{a}_y \text{ V/m}$$

We can find $\mathbf{H}_r$ from $\mathbf{E}_r$ just as we did in part (a) of this example or by using Method 2 of Example 10.9, starting with $\mathbf{H}_i$. Whichever approach is taken, we obtain

$$\mathbf{H}_r = \frac{1}{3\pi} \cos(\omega t + \beta z)\mathbf{a}_y - \frac{1}{4\pi} \sin(\omega t + \beta z)\mathbf{a}_x \text{ A/m}$$

(c) The total fields in air

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r \quad \text{and} \quad \mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r$$

can be shown to be standing waves. The total fields in the conductor are

$$\mathbf{E}_2 = \mathbf{E}_t = \mathbf{0}, \quad \mathbf{H}_2 = \mathbf{H}_t = \mathbf{0}$$

(d) For $z \leq 0$,

$$\begin{aligned}\mathcal{P}_{1ave} &= \frac{|\mathbf{E}_{1s}|^2}{2\eta_1}\mathbf{a}_k = \frac{1}{2\eta_o}[E_{io}^2\mathbf{a}_z - E_{ro}^2\mathbf{a}_z] \\ &= \frac{1}{240\pi}[(40^2 + 30^2)\mathbf{a}_z - (40^2 + 30^2)\mathbf{a}_z] \\ &= \mathbf{0}\end{aligned}$$

For $z \geq 0$,

$$\mathcal{P}_{2ave} = \frac{|\mathbf{E}_{2s}|^2}{2\eta_2}\mathbf{a}_k = \frac{E_{to}^2}{2\eta_2}\mathbf{a}_z = \mathbf{0}$$

Q. The plane wave $\mathbf{E} = 50\sin(\omega t - 5x)\mathbf{a}_y$ V/m in a lossless medium $(\mu = 4\mu_o, \varepsilon = \varepsilon_o)$ encounters a lossy medium $(\mu = \mu_o, \varepsilon = 4\varepsilon_o, \sigma = 0.1$ S/m$)$ normal to the x-axis at $x = 0$. Find

(a) Γ, τ, and s
(b) $\mathbf{E}_r$ and $\mathbf{H}_r$
(c) $\mathbf{E}_t$ and $\mathbf{H}_t$
(d) The time-average Poynting vectors in both regions

A. $\alpha_1 = 0,\ \beta_1 = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{2\omega}{c} = 5 \longrightarrow \omega = 5c/2 = 7.5x10^8$

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{0.1}{7.5x10^8 x4x\frac{10^{-9}}{36\pi}} = 1.2\pi$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} - 1\right]} = 6.021$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} + 1\right]} =$$ [redacted] 7,826

$$|\eta_2| = \frac{60\pi}{\sqrt[4]{1+1.44\pi^2}} = 95.445, \eta_1 = 120\pi\sqrt{\varepsilon_{r1}} = 754$$

$$\tan 2\theta_{\eta_2} = 1.2\pi \longrightarrow \theta_{\eta_2} = 37.57^o$$

$$\eta_2 = 95.445\angle 37.57^o$$

(a)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{95.445\angle 37.57^o - 754}{95.445\angle 37.57^o + 754} = \underline{0.8186\angle 171.08^o}$$

$$\tau = 1 + \Gamma = \underline{0.2295\angle 33.56^o}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.8186}{1-0.8186} = \underline{10.025}$$

(b) $E_i = 50\sin(\omega t - 5x)a_y = \text{Im}(E_{is}e^{j\omega t})$, where $E_{is} = 50e^{-j5x}a_y$.

$$E_{ro} = \Gamma E_{io} = 0.8186e^{j171.08^o}(50) = 40.93e^{j171.08^o}$$

$$E_{rs} = 40.93e^{j5x+j171.08^o}a_y$$

$$E_r = \text{Im}(E_{rs}e^{j\omega t}) = \underline{40.93\sin(\omega t + 5x + 171.1^o)a_y} \text{ V/m}$$

$$a_H = a_k x a_E = -a_x x a_y = -a_z$$

$$H_r = -\frac{40.93}{754}\sin(\omega t + 5x + 171.1^o)a_z = \underline{-0.0543\sin(\omega t + 5x + 171.1^o)a_z} \text{ A/m}$$

(c)

$$E_{to} = \tau E_{io} = 0.229e^{j33.56^o}(50) = 11.475e^{j33.56^o}$$

$$E_{ts} = 11.475e^{-j\beta_2 x + j33.56^o}e^{-\alpha_2 x}a_y$$

$$E_t = \text{Im}(E_{ts}e^{j\omega t}) = \underline{11.475e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o)a_y} \text{ V/m}$$

$$a_H = a_k x a_E = a_x x a_y = a_z$$

$$H_t = \frac{11.495}{95.445}e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o - 37.57^o)a_z$$

$$= \underline{0.1202e^{-6.021x}\sin(\omega t - 7.826x - 4.01^o)a_z} \text{ A/m}$$

(d)

$$\mathcal{P}_{1ave} = \frac{E_{io}^2}{2\eta_1}a_x + \frac{E_{ro}^2}{2\eta_1}(-a_x) = \frac{1}{2(240\pi)}[50^2 a_x - 40.93^2 a_x] = \underline{0.5469a_x} \text{ W/m}^2$$

$$\mathcal{P}_{2ave} = \frac{E_{to}^2}{2|\eta_2|}e^{-2\alpha_2 x}\cos\theta_{\eta_2}a_x = \frac{(11.475)^2}{2(95.445)}\cos 37.57^o e^{-2(6.021)x}a_x = \underline{0.5469e^{-12.04x}a_x} \text{ W/m}^2$$