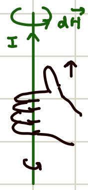
U-3 Magnetostatics

Biot-Savart's law

→ Biot-Savart's Law gives magnetic field intensity $\vec{H}$ at point $\vec{P}$ due to differential current element $I\vec{dl}$. (Analogous to Coulomb's Law)

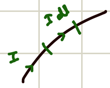
→ $$\boxed{\vec{dH} = \frac{I\vec{dl} \times \hat{a}_R}{4\pi R^2} = \frac{I\vec{dl} \times \vec{R}}{4\pi R^3}}$$



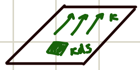
⇒ Right Hand Thumb Rule

→ Biot-Savart's law for various bodies:

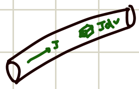
line current ⇒ $$\boxed{\vec{H} = \int_L \frac{I\vec{dl} \times \hat{a}_R}{4\pi R^2}}$$



surface current ⇒ $$\boxed{\vec{H} = \int_S \frac{K\vec{dS} \times \hat{a}_R}{4\pi R^2}}$$



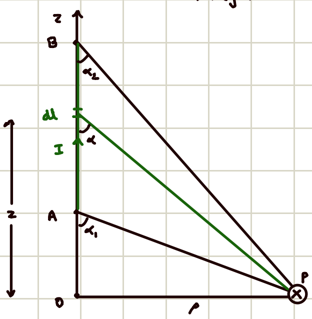
volume current ⇒ $$\boxed{\vec{H} = \int_V \frac{J\vec{dv} \times \hat{a}_R}{4\pi R^2}}$$



$\vec{K}$: surface density

$\vec{J}$: volume density

→ Field due to straight current-carrying conductor of length AB:



$$\vec{dH} = \frac{I\vec{dl} \times \vec{R}}{4\pi R^3} = \frac{I(dz\,\hat{a}_z) \times (\rho\,\hat{a}_\rho - z\,\hat{a}_z)}{4\pi(\rho^2+z^2)^{3/2}} = \frac{I\,\rho\,dz\,\hat{a}_\phi}{4\pi[\rho^2+z^2]^{3/2}}$$

$(z = \rho\cot\alpha \Rightarrow dz = -\rho\,\mathrm{cosec}^2\alpha\,d\alpha)$ $\quad ([\rho^2+z^2]^{3/2} = \rho^3\mathrm{cosec}^3\alpha)$

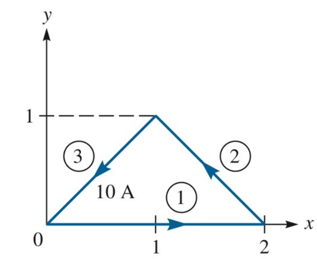
$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I\,\rho\cdot(-\rho\,\mathrm{cosec}^2\alpha)\,d\alpha}{4\pi\rho^3\,\mathrm{cosec}^3\alpha}\,\hat{a}_\phi = -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2}\frac{\sin\alpha}{\rho}\,d\alpha\,\hat{a}_\phi$$

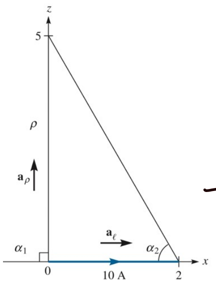
$$\boxed{\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\hat{a}_\phi}$$ Field due to any straight filamentary conductor

→ When wire is semi-infinite $(\alpha_1 = 90^\circ \text{ \& } \alpha_2 = 0^\circ)$ ⇒ $\vec{H} = \frac{I}{4\pi\rho}\hat{a}_\phi$

infinite $(\alpha_1 = 180^\circ \text{ \& } \alpha_2 = 0^\circ)$ ⇒ $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

(To find $\hat{a}_\phi = \hat{a}_\ell \times \hat{a}_\rho$)





Q. The conducting triangular loop in Figure 7.6(a) carries a current of 10 A. Find **H** at $(0, 0, 5)$ due to side 1 of the loop.

A. First find α_1, α_2, ρ and $\hat{a}_\phi$

$\alpha_1 = 90^\circ$

$\alpha_2 = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$

$\rho = 5$

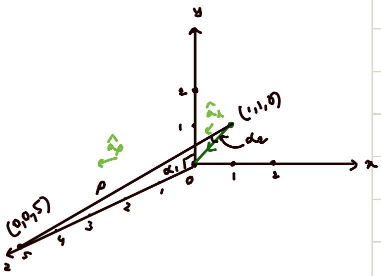
$\hat{a}_\phi = \hat{a}_\ell \times \hat{a}_\rho$

$= \hat{a}_x \times \hat{a}_z = -\hat{a}_y$

$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\hat{a}_\phi$

$= \frac{10}{4\pi \times 5}\left(\frac{2}{\sqrt{29}} - 0\right)(-\hat{a}_y) = -59.1\,\hat{a}_y$ mA/m

Q. Find **H** at $(0, 0, 5)$ due to side 3 of the triangular loop in Figure 7.6(a).



A. $\alpha_1 = 90$

$\alpha_2 = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{27}}\right)$

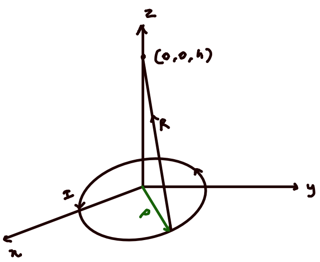
$\rho = 5$

$\hat{a}_\rho = \hat{a}_\ell \times \hat{a}_\rho$

$= \left(\frac{-\hat{a}_x - \hat{a}_y}{\sqrt{2}}\right) \times \hat{a}_z = \frac{\hat{a}_y - \hat{a}_x}{\sqrt{2}}$

$H = \frac{10}{4\pi \times 5}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{\hat{a}_y - \hat{a}_x}{\sqrt{2}}\right) = -30.6\,\hat{a}_x + 30.6\,\hat{a}_y$ mA/m

Q. A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10A along $\hat{a}_\phi$. Determine $\vec{H}$ at $(0,0,4)$ & $(0,0,-4)$



A. By biot-savart's law, $d\vec{H} = \frac{I\,d\vec{l} \times \vec{R}}{4\pi R^3}$

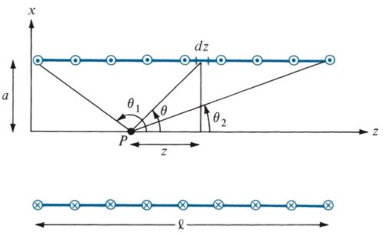
$dl = \rho\, d\rho\, \hat{a}_\phi$, $\vec{R} = (0,0,h) - (x,y,0) = -(x\hat{a}_x + y\hat{a}_y) + h\hat{a}_z = -\rho\hat{a}_\rho + h\hat{a}_z$

$$d\vec{l} \times \vec{R} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \hat{a}_\rho + \rho^2 d\phi\, \hat{a}_z$$

$$d\vec{H} = \frac{I}{4\pi(\rho^2+h^2)^{3/2}}\left(\rho h\, d\phi\, \hat{a}_\rho + \rho^2 d\phi\, \hat{a}_z\right) = \cancelto{0}{dH_\rho \hat{a}_\rho} + dH_z \hat{a}_z$$

$$\vec{H} = \int_0^{2\pi} \frac{I\rho^2 d\phi}{4\pi(\rho^2+h^2)^{3/2}}\,\hat{a}_z = \frac{I\rho^2\, 2\pi}{4\pi(\rho^2+h^2)^{3/2}}\,\hat{a}_z = \frac{I\rho^2}{2(\rho^2+h^2)^{3/2}}\,\hat{a}_z$$

a) $h = 4 \Rightarrow \vec{H} = 0.36\,\hat{a}_z$ A/m b) $h = -4 \Rightarrow \vec{H} = 0.36\,\hat{a}_z$ A/m



Solenoidal Field

→ To prove $\boxed{\vec{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\hat{a}_z}$ & if $l >> a$ then $\vec{H} = nI\hat{a}_z$

→ Let length of solenoid $= l$

radius $= a$

no. of turns $= N$

current $= I$

using $dH_z = \frac{I\,dl\,a^2}{2(a^2+z^2)^{3/2}} = \frac{I a^2 n\, dz}{2(a^2+z^2)^{3/2}}$ $(dl = n\,dz)$

also $dz = -a\,\mathrm{cosec}^2\theta\, d\theta = -\frac{(z^2+a^2)^{3/2}}{a^2}\sin\theta\, d\theta$

$$dH_z = \frac{-nI}{2}\sin\theta\, d\theta$$

$$H_z = -\int_{\theta_1}^{\theta_2} \frac{nI}{2}\sin\theta\, d\theta = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\hat{a}_z = \frac{NI}{2l}(\cos\theta_2 - \cos\theta_1)\hat{a}_z \qquad \left(n = \frac{N}{l}\right)$$

at center of solenoid, $\cos\theta_2 = \frac{l/2}{(a^2 + \frac{l^2}{4})^{1/2}} = -\cos\theta_1$

$$\vec{H} = \frac{I n l}{2(a^2 + \frac{l^2}{4})^{1/2}}\hat{a}_z$$

if $l >> a$, $\theta_2 = 0^\circ$ & $\theta_1 = 180^\circ$ ⇒ $\boxed{\vec{H} = nI\hat{a}_z = \frac{NI}{l}\hat{a}_z}$

Q. The solenoid of Figure 7.9 has 2000 turns, a length of 75 cm, and a radius of 5 cm. If it carries a current of 50 mA along $\mathbf{a}_\phi$, find **H** at

(a) (0, 0, 0)
(b) (0, 0, 75 cm)
(c) (0, 0, 50 cm)

A. $N = 2000$ $\quad l = 75\times10^{-2}\,m$ $\quad r = 5\times10^{-2}\,m$ $\quad \vec{I} = 50\times10^{-3}\,\hat{a}_\phi$ A

$$\vec{H} = \frac{NI}{2l}(\cos\theta_2 - \cos\theta_1)\hat{a}_z = \frac{2000\times50\times10^{-3}}{2\times75\times10^{-2}}(\cos\theta_2 - \cos\theta_1)\hat{a}_z$$

a) at (0, 0, 0)

$\theta_1 = 90^\circ$, $\cos\theta_2 = \frac{0.75}{\sqrt{0.75^2+0.05^2}} = 0.9978$ ⇒ $\vec{H} = \frac{100}{1.5}(0.9978 - 0)\hat{a}_z = 66.52\,\hat{a}_z$ A/m

b) at (0, 0, 0.75)

$\cos\theta_1 = \frac{-0.75}{\sqrt{0.75^2+0.05^2}} = -0.9978$, $\theta_2 = 90^\circ$ ⇒ $\vec{H} = \frac{100}{1.5}(0 - (-0.9978))\hat{a}_z = 66.52\,\hat{a}_z$ A/m

c) at (0, 0, 0.5),

$\cos\theta_1 = \frac{-0.5}{\sqrt{0.5^2+0.05^2}} = -0.995$, $\cos\theta_2 = \frac{0.25}{\sqrt{0.25^2+0.05^2}} = 0.9806$ ⇒ $\vec{H} = \frac{100}{1.5}(0.9806 + 0.995)\hat{a}_z = 131.7\,\hat{a}_z$ A/m

Ampere's Circuit Law

→ The line integral of H around a closed path is same as net current I_{enc} enclosed around the path. Analogous to Gauss' Law

$$\oint_L \vec{H}\cdot d\vec{l} = I_{enc} \Rightarrow \int_S (\vec{\nabla}\times\vec{H})\, d\vec{l} = I_{enc}$$

Integral form of Maxwell's Third equation

$$\int_S \vec{J}\cdot d\vec{S} = I_{enc}$$

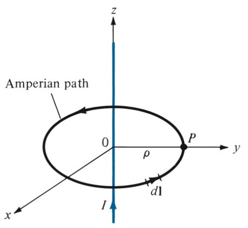
Which shows $\vec{\nabla}\times\vec{H} = \vec{J} \neq 0$ (Magnetostatic energy isn't conservative)

Differential form (Stoke's Theorem)

Applications of Ampere's Law

→ Ampere's Law is used when there's symmetry in current distribution

1) Infinite Line Current



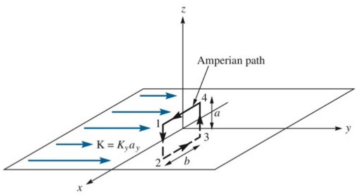
$$I = \int_L H_\phi \hat{a}_\phi \cdot \rho\, d\phi$$

$$= H_\phi \int_L \rho\, d\phi$$

$$= H_\phi 2\pi\rho$$

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

2) Infinite Sheet of Current



Assume sheet has uniform current density $\vec{K} = K_y \hat{a}_y$ A/m

Then, $\oint \vec{H}\cdot d\vec{l} = I_{enc} = K_y b$

$$\vec{H} = \begin{cases} H_0 \hat{a}_x, & z > 0 \\ -H_0 \hat{a}_x, & z < 0 \end{cases}$$

Now $\oint \vec{H}\cdot d\vec{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right) \vec{H}\cdot d\vec{l}$

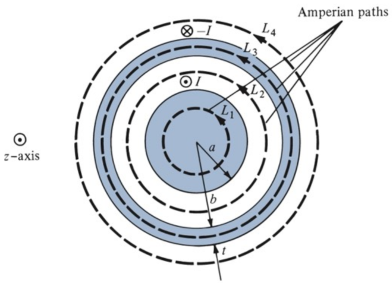
$$= 0(-a) + (-H_0)(-b) + 0(a) + (H_0)(b)$$

$$= 2H_0 b$$

$2H_0 b = K_y b \Rightarrow H_0 = \frac{1}{2}K_y$

Now, $\vec{H} = \begin{cases} \frac{1}{2}K_y \hat{a}_x, & z > 0 \\ -\frac{1}{2}K_y \hat{a}_x, & z < 0 \end{cases}$

So, $\vec{H} = \frac{1}{2}\vec{K}\times\hat{a}_n$



3) Infinitely Long Co-axial Transmission Line

→ Consider inner conductor of radius 'a' carries current 'I' and outer conductor of inner radius 'b' & thickness 't' carries current '-I'

Then we have 4 regions

i) Region 1 $(0 \le \rho \le a)$

$$\oint_L \vec{H}\cdot d\vec{l} = I_{enc} = \int_s \vec{J}\cdot d\vec{S} \qquad \left(\begin{array}{l}\vec{J} = \frac{I}{\pi a^2}\hat{a}_z \\ d\vec{S} = \rho d\phi d\rho\, \hat{a}_z\end{array}\right)$$

$$= \int_s \left(\frac{I}{\pi a^2}\hat{a}_z\right)\cdot\left(\rho d\phi d\rho\, \hat{a}_z\right)$$

$$= \frac{I}{\pi a^2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \rho\, d\phi\, d\rho = \frac{I\pi\rho^2}{\pi a^2} = \frac{I\rho^2}{a^2}$$

$$H_\phi \int_0^{2\pi\rho} dl = \frac{I\rho^2}{a^2} \Rightarrow H_\phi\, 2\pi\rho = \frac{I\rho^2}{a^2} \Rightarrow H_\phi = \frac{I\rho}{2\pi a^2}$$

ii) Region 2 $(a \le \rho \le b)$

$$\oint_{L_2} \vec{H}\cdot d\vec{l} = I_{enc} = I$$

$$H_\phi\, 2\pi\rho = I \Rightarrow H_\phi = \frac{I}{2\pi\rho}$$

iii) Region 3 $(b \le \rho \le b+t)$

$$\oint_{L_3} \vec{H}\cdot d\vec{l} = I_{enc} = I + \int \vec{J}\cdot d\vec{S}$$

$$= I - \int\left(\frac{I}{\pi\left[(b+t)^2 - b^2\right]}\hat{a}_z\right)\cdot\left(\rho d\rho d\phi\, \hat{a}_z\right)$$

$$= I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]}\int_{\phi=0}^{2\pi}\int_{\rho=b}^{\rho} \rho\, d\rho\, d\phi$$

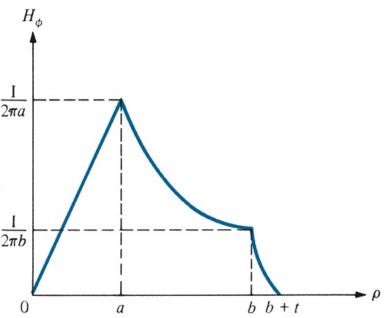
$$H_\phi\, 2\pi\rho = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

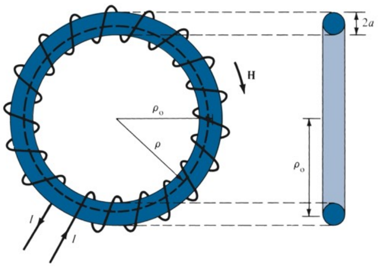
$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

iv) Region 4 $(\rho \geq b+t)$

$$\oint_{L_4} \vec{H}\cdot d\vec{l} = I - I = 0 \Rightarrow H_\phi = 0$$

So,
$$\vec{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\hat{a}_\phi & , \ 0 \le \rho \le a \\[2ex] \dfrac{I}{2\pi\rho}\hat{a}_\phi & , \ a \le \rho \le b \\[2ex] \dfrac{I}{2\pi\rho}\left[\dfrac{\rho^2 - b^2}{t^2 + 2bt}\right] & , \ b \le \rho \le b+t \\[2ex] 0 & , \ \rho \geq b+t \end{cases}$$





4) Toroid

→ Toroid with N turns & carries current I

→ net current enclosed by amperian path is NI

$$\oint \vec{H} \cdot \vec{dl} = I_{enc}$$

$$H \cdot 2\pi\rho = NI$$

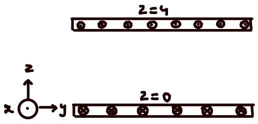
$$H = \frac{NI}{2\pi\rho} \quad \text{for } \rho_o - a < \rho < \rho_o + a$$

ρ_o → mean radius of toroid

$$H_{approx} = \frac{NI}{2\pi\rho_o} = \frac{NI}{l}$$

Q. Plane $z=0$ & $z=4$ carry current $\vec{K} = -10\,\hat{a}_x$ A/m and $\vec{K} = 10\,\hat{a}_x$ A/m respectively

Determine $\vec{H}$ at a) $(1,1,1)$ b) $(0,-3,10)$



A. $\vec{H} = \vec{H_0} + \vec{H_4}$

a) at $(1,1,1)$, Point is between the planes

$$\vec{H_0} = \frac{1}{2} \times \vec{K} \times \hat{a}_n = \frac{1}{2} \times (-10\,\hat{a}_x) \times \hat{a}_z = 5\,\hat{a}_y \text{ A/m}$$

$$\vec{H_4} = \frac{1}{2} \times \vec{K} \times \hat{a}_n = \frac{1}{2} \times (10\,\hat{a}_x) \times (-\hat{a}_z) = 5\,\hat{a}_y \text{ A/m}$$

$$\vec{H} = \vec{H_0} + \vec{H_4} = 10\,\hat{a}_y \text{ A/m}$$

b) At $(0,-3,10)$, point is above the 2 sheets

$$\vec{H_0} = \frac{1}{2}(-10\,\hat{a}_x) \times \hat{a}_z = 5\,\hat{a}_y \text{ A/m}$$

$$\vec{H_4} = \frac{1}{2}(10\,\hat{a}_x) \times \hat{a}_z = -5\,\hat{a}_y \text{ A/m}$$

$$\vec{H} = 0 \text{ A/m}$$

Q. A toroid of circular cross section whose center is at the origin and axis the same as the z-axis has 1000 turns with $\rho_o = 10$ cm, $a = 1$ cm. If the toroid carries a 100 mA current, find $|H|$ at

(a) $(3\text{ cm}, -4\text{ cm}, 0)$
(b) $(6\text{ cm}, 9\text{ cm}, 0)$

A.
$$|H| = \begin{cases} \frac{NI}{2\pi\rho}, & \rho_o - a < \rho < \rho_o + a \Rightarrow 9 < \rho < 11 \\ 0, & \text{otherwise} \end{cases}$$

a) $\rho = \sqrt{3^2 + 4^2} = 5 < 9 \Rightarrow |H| = 0$

b) $\rho = \sqrt{6^2 + 9^2} = \sqrt{117} < 11 \Rightarrow |H| = \dfrac{1000 \times 100 \times 10^{-3}}{2\pi \times \sqrt{117} \times 10^{-2}} = 147.1$ A/m

Magnetic Flux Density - Maxwell's Equation

→ It represents how dense the magnetic field lines are in a region

Also called magnetic field / magnetic induction

→ Analogous to Electric Flux Density ($\vec{D} = \varepsilon_0 \vec{E}$)

→ Relationship b/w $\vec{B}$ & $\vec{H}$

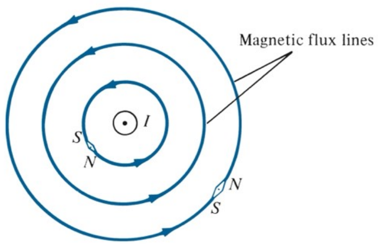
$$\boxed{\vec{B} = \mu_0 \vec{H}}$$

where μ : permeability $= 4\pi \times 10^{-7}$ H/m

unit of $\vec{B} \Rightarrow$ Tesla (T) or Weber/m^2 (Wb/m^2)

unit of $\vec{H} \Rightarrow$ A/m

Magnetic Flux

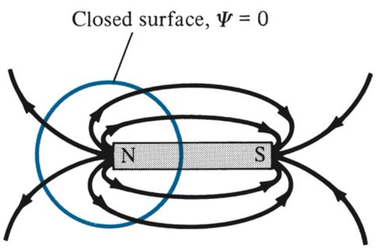


→ The total magnetic field passing through a surface S

$$\Psi = \int_S \vec{B} \cdot d\vec{S}$$

→ Unit of Ψ = Wb

Magnetic Flux Lines



→ Visual representation of magnetic field $\vec{B}$

It is tangent to $\vec{B}$ at every point

→ Always forms closed loops (no beginning or end)

Isolated magnetic charge doesn't exist (No monopoles)

Gauss' Law for magnetostatic fields

→ Analogous to Gauss' law for electrostatic fields ($\oint_S \vec{D} \cdot d\vec{S} = Q$)

→ Even though magnetostatic field isn't conservative, magnetic field is conserved

→ By divergence theorem,

$$\oint_S \vec{B} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{B} \, dv = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

↳ Maxwell's Equations for static magnetic fields

→ From this law, we can infer that magnetic field lines never diverge from or converge to a point

Maxwell's equations for static fields

	Differential/Point Form	Integral Form	Remarks
1)	$\vec{\nabla}\cdot\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot d\vec{S} = \int_v \rho_v \, dv = Q_{enc}$	Gauss' Law
2)	$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{S} = 0$	Magnetic monopoles don't exist
3)	$\vec{\nabla}\times\vec{E} = 0$	$\oint_L \vec{E}\cdot d\vec{l} = 0$	Conservative nature of electrostatic field
4)	$\vec{\nabla}\times\vec{H} = \vec{J}$	$\oint_L \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} = I_{enc}$	Ampere's law

Force due to magnetic fields

→ 3 ways force can be experienced
- i) Moving Charges
- ii) Current elements (Wires)
- iii) Current loops / other current-carrying systems

a) Force on moving charge

→ From electrostatics, we know $\vec{F}_E = Q\vec{E}$

→ So in magnetostatics,

$$\vec{F}_m = Q(\vec{u}\times\vec{B})$$

and Magnetic field can't do work because $\vec{F}_m \cdot \vec{u} = 0$

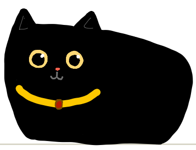
→ For combined electric & magnetic forces (Lorentz Force)

$$\vec{F} = Q(\vec{E} + \vec{u}\times\vec{B})$$

It determines how a charge moves in both fields

It can also be written in terms of mass,

$$m\frac{d\vec{u}}{dt} = Q(\vec{E} + \vec{u}\times\vec{B}) \qquad \left(F = ma = \frac{mdv}{dt}\right)$$



b) Force on a current element

→ From convection current,

$$\vec{J} = \rho_v \vec{u}$$

We also know,

$$I d\vec{l} = \vec{K} ds = \vec{J} dv$$

So,

$$I d\vec{l} = dQ\, \vec{u}$$

Using this in Lorentz Force Law,

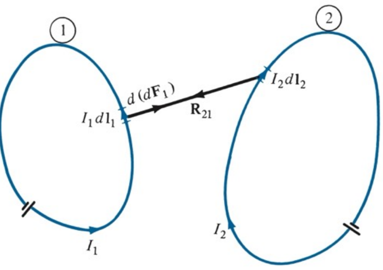
$$d\vec{F} = I d\vec{l} \times \vec{B}$$

$$\boxed{\vec{F} = \oint_L I d\vec{l} \times \vec{B}}$$

or even $\vec{F} = \int_S \vec{K}\, ds \times \vec{B}$ or $\vec{F} = \int_V \vec{J}\, dv \times \vec{B}$

Surface Current   Volume Current

c) Force between 2 current elements



→ Assume 2 current elements $I_1 d\vec{l}_1$ & $I_2 d\vec{l}_2$

Using biot-savort's law,

Field from $I_2 d\vec{l}_2 \Rightarrow d\vec{B} = \dfrac{\mu_0 I_2 (d\vec{l}_2 \times \hat{r}_{21})}{4\pi r_{21}^2}$

and force on $I_1 d\vec{l}_1 \Rightarrow d(d\vec{F}_1) = I_1 d\vec{l}_1 \times d\vec{B}_2$

$$= I_1 d\vec{l}_1 \times \frac{\mu_0 I_2 (d\vec{l}_2 \times \hat{r}_{21})}{4\pi r_{21}^2}$$

Then $\vec{F}_1 = \dfrac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \dfrac{d\vec{l}_1 \times (d\vec{l}_2 \times \hat{r}_{21})}{r_{21}^2}$

Q. A charged particle moves with uniform velocity $4\hat{a}_x$ m/s in a region where $\vec{E} = 20\hat{a}_y$ V/m and $\vec{B} = B_0 \hat{a}_z$ Wb/m². Determine B_0 such that velocity of particle remains constant

A. If velocity = constant ⇒ acceleration = 0

So, $F = 0$ $(F = ma = m \times 0 = 0)$

$F = Q(\vec{E} + \vec{u} \times \vec{B})$

$0 = Q(20\hat{a}_y + 4\hat{a}_x \times B_0 \hat{a}_z)$

$-20\hat{a}_y = -4B_0 \hat{a}_y \Rightarrow B_0 = 5$

Q. Charged particle of mass 2kg & charge 3C starts at point (1, -2, 0) with velocity $4\hat{a}_x + 3\hat{a}_z$ m/s is an electric field $12\hat{a}_x + 10\hat{a}_y$ V/m. At t = 1s, Find a) acceleration b) velocity c) K.E d) Position

A. a) Since it is initial value problem

$$\vec{F} = m\vec{a} = Q\vec{E}$$

$$\vec{a} = \frac{Q\vec{E}}{m} = \frac{3}{2}\left(12\hat{a}_x + 10\hat{a}_y\right) = 18\hat{a}_x + 15\hat{a}_y \quad m/s^2$$

and, $\vec{a} = \frac{d\vec{u}}{dt} = \frac{d}{dt}\left(u_x, u_y, u_z\right) = 18\hat{a}_x + 15\hat{a}_y$

b) $\vec{u} = \int \frac{d\vec{u}}{dt} \cdot dt \Rightarrow \frac{du_x}{dt} = 18 \rightarrow u_x = 18t + A$

$$\frac{du_y}{dt} = 15 \rightarrow u_y = 15t + B$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = c$$

Given $\vec{u}(t=0) = 4\hat{a}_x + 3\hat{a}_z$

$4 = 0 + A \Rightarrow A = 4$; $0 = 0 + B \Rightarrow B = 0$; $3 = c$

Then $u(t) = (18t + 4)\hat{a}_x + (15t)\hat{a}_y + 3\hat{a}_z$ m/s

$u(1) = 22\hat{a}_x + 15\hat{a}_y + 3\hat{a}_z$

c) $KE = \frac{1}{2}m|u|^2 = \frac{1}{2} \times 2 \times \left(22^2 + 15^2 + 3^2\right) = 718\ J$

d) $\vec{u} = \frac{dl}{dt} = \frac{d}{dt}(x, y, z) = (18t + 4, 15t, 3)$

$\frac{dx}{dt} = 18t + 4 \Rightarrow x = 9t^2 + 4t + A \Rightarrow x(0) = 1 \Rightarrow 0 + 0 + A = 1 \Rightarrow A = 1$

$\frac{dy}{dt} = 15t \Rightarrow y = \frac{15t^2}{2} + B \Rightarrow y(0) = -2 \Rightarrow 0 + B = -2 \Rightarrow B = -2$

$\frac{dz}{dt} = 3 \Rightarrow z = 3t + C \Rightarrow z(0) = 0 \Rightarrow 0 + C = 0 \Rightarrow C = 0$

$(x, y, z) = \left(9t^2 + 4t + 1,\ 7.5t^2 - 2,\ 3t\right)$

(x, y, z) at $t = 1 \Rightarrow (9 + 4 + 1,\ 7.5 - 2,\ 3)$

$= (14, 5.5, 3)$

Q. A charged particle of mass 2 Kg & charge 1 C starts at origin with velocity $3\hat{a}_y$ m/s and travels in a region of uniform magnetic field $\vec{B} = 10\hat{a}_z$ Wb/m². At t = 4s, Calculate a) velocity and acceleration b) Magnetic force c) KE

A. a) $\vec{F} = m\vec{a} = m\frac{d\vec{u}}{dt} = Q\vec{u}\times\vec{B}$

$$\vec{a} = \frac{d\vec{u}}{dt} = \frac{Q\vec{u}\times\vec{B}}{m} \qquad \text{where } \vec{u} \text{ is unknown}$$

$$\Rightarrow \frac{d}{dt}\left(u_x\hat{a}_x + u_y\hat{a}_y + u_z\hat{a}_z\right) = \frac{1}{2}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ u_x & u_y & u_z \\ 0 & 0 & 10 \end{vmatrix} = 5\left(u_y\hat{a}_x - u_x\hat{a}_y\right)$$

So, $\underbrace{\frac{du_x}{dt} = 5u_y}_{①}$, $\underbrace{\frac{du_y}{dt} = -5u_x}_{②}$ $\frac{du_z}{dt} = 0 \Rightarrow u_z = C_0$

Differentiate ① & substitute ② in it

$$\frac{d^2u_x}{dt^2} = \frac{5du_y}{dt} = 5(-5u_x) \Rightarrow \frac{d^2u_x}{dt^2} + 25u_x = 0 \Rightarrow \underbrace{u_x = C_1\cos 5t + C_2\sin 5t}_{③}$$

$$5u_y = \underbrace{\frac{du_x}{dt}}_{\text{substitute ③}} = \frac{d}{dt}\left(C_1\cos 5t + C_2\sin 5t\right) = -5C_1\sin 5t + 5C_2\cos 5t = 5u_y$$

$$u_y = -C_1\sin 5t + C_2\cos 5t$$

Then at $t = 0$, $\vec{u} = 3\hat{a}_y$

$$\left.\begin{aligned} u_x(t=0) &= C_1(1) + C_2(0) = 0 &&\Rightarrow C_1 = 0 \\ u_y(t=0) &= -C_1(0) + C_2(1) = 3 &&\Rightarrow C_2 = 3 \\ u_z(t=0) &= C_0 = 0 &&\Rightarrow C_0 = 0 \end{aligned}\right\} \rightarrow \vec{u} = 3\sin 5t\,\hat{a}_x + 3\cos 5t\,\hat{a}_y$$

At $t = 4$, $\vec{u}(t=4) = 3\sin 20\,\hat{a}_x + 3\cos 20\,\hat{a}_y = 2.739\,\hat{a}_x + 1.224\,\hat{a}_y$ m/s

and $\vec{a} = \frac{d\vec{u}}{dt} = 15\cos 5t\,\hat{a}_x - 15\sin 5t\,\hat{a}_y$

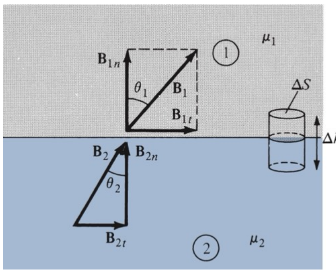
at $t = 4$, $\vec{a}(t=4) = 15\cos 20\,\hat{a}_x - 15\sin 20\,\hat{a}_y = 6.121\hat{a}_x - 13.694\,\hat{a}_y$ m/s²

b) $\vec{F} = m\vec{a}$ at $t=4$

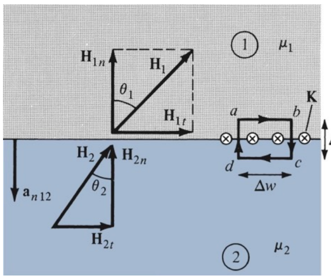
$$= 2\left(6.121\hat{a}_x - 13.694\,\hat{a}_y\right) = 12.27\,\hat{a}_x - 27.4\,\hat{a}_y \quad N$$

c) $\underbrace{KE}_{\text{at } t=4s} = \frac{1}{2}mu^2 = \frac{1}{2}\times 2\times\left(2.739^2 + 1.224^2\right) = 9J$

Magnetic boundary conditions



↳ Boundary conditions wrt $\vec{B}$



↳ Boundary conditions wrt $\vec{H}$

→ Conditions that magnetic field vectors $\vec{B}$ and $\vec{H}$ must satisfy at boundary between 2 magnetic materials with different permeabilities μ_1 and μ_2

→ We can use Gauss' Law and Ampere's Circuit Law

$$\oint_S \vec{B}\cdot d\vec{S} = 0$$

$$\oint_L \vec{H}\cdot d\vec{l} = I$$

Then $B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0 \Rightarrow$ $\vec{B}_{1n} = \vec{B}_{2n}$ or $\mu_1\vec{H}_{1n} = \mu_2\vec{H}_{2n}$ $\quad(\vec{B} = \mu\vec{H})$

Then, we realize normal component of $\vec{B}$ is continuous at boundary
and normal component of $\vec{H}$ is discontinuous at boundary

Also, $\oint_C \vec{H}\cdot d\vec{l} = I_{enc} = K\cdot\Delta w$ where K : Surface Current density along the boundary

$$K\cdot\Delta w = H_{1t}\cdot\Delta w + H_{1n}\cdot\frac{\Delta h}{2} + H_{2n}\frac{\Delta h}{2} - H_{2t}\cdot\Delta w - H_{2n}\frac{\Delta h}{2} - H_{1n}\frac{\Delta h}{2}$$

$$K\cdot\Delta w = (H_{1t} - H_{2t})\,\Delta w \Rightarrow \frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K$$

$$(\vec{H}_1 - \vec{H}_2)\times\hat{a}_{12} = \vec{K}$$

If the boundary is free of current (or) media aren't conductors, $\vec{K} = 0$

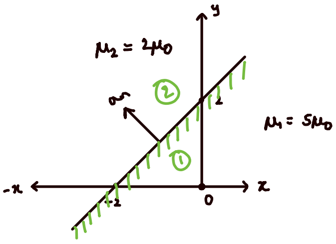
$$\vec{H}_{1t} = \vec{H}_{2t} \Rightarrow \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$$

Which ultimately results in,

$$B_1\cos\theta_1 = B_{1n} = B_{2n} = B_2\cos\theta_2$$

$$\frac{B_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$



Q. Given that $\vec{H}_1 = -2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z$ A/m in region $y - x - 2 \le 0$ where $\mu_1 = 5\mu_0$

Calculate a) M_1 & B_1 b) H_2 & B in region $y - x - 2 \ge 0$ where $\mu_2 = 2\mu_0$

A. Let $f(x,y) = y - x - 2$

Then $\hat{a}_n = \dfrac{\nabla f}{|\nabla f|} = \dfrac{\hat{a}_y - \hat{a}_x}{\sqrt{1^2+1^2}} = \dfrac{\hat{a}_y - \hat{a}_x}{\sqrt{2}}$

a) $\vec{M}_1 = \chi_{m_1}\vec{H}_1 = (\mu_{r_1} - 1)\vec{H}_1 = (5-1)\left[-2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z\right] = \left(-8\hat{a}_x + 24\hat{a}_y + 16\hat{a}_z\right)$ A/m

$\vec{B}_1 = \mu_1\vec{H}_1 = 5\mu_0\vec{H}_1 = 5 \times 4\pi \times 10^{-7} \times \left(-2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z\right) = \left(-12.57\,\hat{a}_x + 37.7\hat{a}_y + 25.13\hat{a}_z\right)\ \mu\text{Wb/m}^2$

b) $\vec{H}_{1n} = (\vec{H}_1 \cdot \hat{a}_n)\,\hat{a}_n = \left[\left(-2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z\right)\dfrac{(-\hat{a}_x + \hat{a}_y)}{\sqrt{2}}\right]\left[\dfrac{(-\hat{a}_x + \hat{a}_y)}{\sqrt{2}}\right]$

$= \left(\dfrac{2+6}{\sqrt{2}}\right)\left(\dfrac{-\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right) = -4\hat{a}_x + 4\hat{a}_y$

But $\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t} \Rightarrow \vec{H}_{1t} = \left(-2\hat{a}_x + 6\hat{a}_y + 4\hat{a}_z\right) - \left(-4\hat{a}_x + 4\hat{a}_y\right)$

$\vec{H}_{1t} = \left(2\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z\right)$

Using boundary conditions,

$\vec{H}_{2t} = \vec{H}_{1t} = 2\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z$

$\vec{B}_{2n} = \vec{B}_{1n} \Rightarrow \mu_2\vec{H}_{2n} = \mu_1\vec{H}_{1n} \Rightarrow \vec{H}_{2n} = \dfrac{\mu_1\vec{H}_{1n}}{\mu_2} = \dfrac{5}{2}\left(-4\hat{a}_x + 4\hat{a}_y\right) = -10\hat{a}_x + 10\hat{a}_y$

$\vec{H}_2 = \vec{H}_{2n} + \vec{H}_{2t} = \left(-10\hat{a}_x + 10\hat{a}_y\right) + \left(2\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z\right) = -8\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$ A/m

$\vec{B}_2 = \mu_2 H_2 = 2\mu_0\left(-8\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z\right)$

$= -20.11\hat{a}_x + 30.16\hat{a}_y + 10.05\,\hat{a}_z$ Wb/m²

Q.

PRACTICE EXERCISE 8.8

Region 1, described by $3x + 4y \ge 10$, is free space, whereas region 2, described by $3x + 4y \le 10$, is a magnetic material for which $\mu \simeq 10\mu_o$. Assuming that the boundary between the material and free space is current free, find $\mathbf{B}_2$ if $\mathbf{B}_1 = 0.1\mathbf{a}_x + 0.4\mathbf{a}_y + 0.2\mathbf{a}_z$ Wb/m².

Answer: $-1.052\mathbf{a}_x + 1.264\mathbf{a}_y + 2\mathbf{a}_z$ Wb/m².

$$\vec{a}_n = \frac{3\vec{a}_x + 4\vec{a}_y}{5}$$

$$\vec{B}_{1n} = (\vec{B}_1 \bullet \vec{a}_n)\vec{a}_n = \frac{(6+32)(6\vec{a}_x + 8\vec{a}_y)}{1000}$$

$$= 0.228\,\vec{a}_x + 0.304\,\vec{a}_y = B_{2n}$$

$$\vec{B}_{1t} = (\vec{B}_1 \bullet \vec{B}_{1n}) = -0.128\vec{a}_x + 0.096\vec{a}_y + 0.2\vec{a}_z$$

$$\vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t} = 10\vec{B}_{1t} = -1.28\vec{a}_x + 0.96\vec{a}_y + 2\vec{a}_z$$

Q. The xy-plane serves as interface between 2 different media. Medium 1 $(z<0)$ is filled with $\mu_r = 6$ and medium 2 $(z>0)$ is filled with material whose $\mu_r = 4$. If interface carries current of $\frac{1}{\mu_0}\hat{a}_y$ mA/m and $\vec{B}_2 = 5\hat{a}_x + 8\hat{a}_z$ Wb/m², Find H_1 & B_1

A. Here $\vec{K} = \frac{1}{\mu_0}\hat{a}_y$ mA/m

In the prev ex, $\vec{K} = 0$

Now,

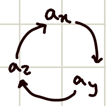
$$\hat{a}_n = \frac{\hat{a}_z}{1}$$

$$\vec{B}_{2n} = (\vec{B}_2 \cdot \hat{a}_n)\hat{a}_n = [(5\hat{a}_x + 8\hat{a}_z)\cdot\hat{a}_z]\hat{a}_z = (8)\hat{a}_z$$

$$\vec{B}_{1n} = \vec{B}_{2n} = 8\hat{a}_z \quad \Rightarrow \quad B_z = 8$$

$$\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} = \left(\frac{5a_x + 8a_z}{4\mu_0}\right) \text{ mA/m}$$

$$\vec{H}_1 = \frac{\vec{B}_1}{\mu_1} = \left(\frac{B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z}{6\mu_0}\right) \text{ mA/m}$$



Then $(\vec{H}_1 - \vec{H}_2) \times \hat{a}_{n12} = \vec{K}$

$$\vec{H}_1 \times \hat{a}_{n12} = \vec{H}_2 \times \hat{a}_{n12} + \vec{K}$$

$$\left(\frac{B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z}{6\mu_0}\right) \times \hat{a}_z = \left(\frac{5a_x + 8a_z}{4\mu_0}\right) \times \hat{a}_z + \frac{\hat{a}_y}{\mu_0}$$

$$\frac{-B_x\hat{a}_y + B_y\hat{a}_x + 0}{6\mu_0} = \frac{-5\hat{a}_y + 0}{4\mu_0} + \frac{\hat{a}_y}{\mu_0}$$

Comparing both sides $\Rightarrow$ $\hat{a}_x \rightarrow \frac{B_y}{6\mu_0} = 0 \Rightarrow B_y = 0$

$$\hat{a}_y \rightarrow \frac{-B_x}{6\mu_0} = \left(\frac{-5}{4} + 1\right)\frac{1}{\mu_0} \quad \Rightarrow \quad B_x = \frac{6}{4} = 1.5$$

$$\vec{B}_1 = 1.5\hat{a}_x + 8\hat{a}_z \text{ mWb/m}^2$$

$$\vec{H}_1 = \frac{\vec{B}_1}{\mu_1} = \frac{1}{6\mu_0}\left(1.5\hat{a}_x + 8\hat{a}_z\right) \text{ mA/m}$$

$$\vec{B}_2 = 5\hat{a}_x + 8\hat{a}_z \text{ mWb/m}^2$$

$$\vec{H}_2 = \frac{\vec{B}_2}{\mu_2} = \frac{1}{4\mu_0}\left(5\hat{a}_x + 8\hat{a}_z\right) \text{ mA/m}$$

Q.

PRACTICE EXERCISE 8.9

A unit normal vector from region 2 $(\mu = 2\mu_o)$ to region 1 $(\mu = \mu_o)$ is $\mathbf{a}_{n21} = (6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z)/7$. If $\mathbf{H}_1 = 10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$
(b) The surface current density $\mathbf{K}$ on the interface
(c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface

Answer: (a) 5.833, (b) $4.86\mathbf{a}_x - 8.64\mathbf{a}_y + 3.95\mathbf{a}_z$ A/m, (c) 76.27°, 77.62°.

(a) $\vec{B}_{1n} = \vec{B}_{2n} \rightarrow \mu_1\vec{H}_{1n} = \mu_2\vec{H}_{2n}$

or $\mu_1\vec{H}_1 \bullet \vec{a}_{n21} = \mu_2\vec{H}_2 \bullet \vec{a}_{n21}$

$$\mu_o\frac{(60+2-36)}{7} = 2\mu_o\frac{(6H_{2x}+10-12)}{7}$$

$35 = 6H_{2x}$

$\underline{H_{2x} = 5.833}$

(b) $\vec{K} = (\vec{H}_1 - \vec{H}_2)\times\vec{a}_{n12} = \vec{a}_{n21}\times(\vec{H}_1 - \vec{H}_2)$

$= \vec{a}_{n21}\times[(1..,1,12)-(35/6,-5,4)]$

$$= \frac{1}{7}\begin{vmatrix} 6 & 2 & -3 \\ 25/6 & 6 & 8 \end{vmatrix}$$

$\vec{K} = \underline{4.86\vec{a}_x - 8.64\vec{a}_y + 3.95\vec{a}_z}$ A/m

(c) Since $\vec{B} = \mu\vec{H}$, $\vec{B}_1$ and $\vec{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\vec{H}_1 \bullet \vec{a}_{n21}}{|\vec{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$\underline{\theta_1 = 76.27°}$

$$\cos\theta_2 = \frac{\vec{H}_2 \bullet \vec{a}_{n21}}{|\vec{H}_2|} = \frac{13}{7\sqrt{(5.833)^2+25+16}} = 0.2144$$

$\underline{\theta_2 = 77.62°}$

Inductor

→ It is any circuit/object that carries current & as a result stores magnetic energy

ex: Solenoid, Toroid, co-axial cables, transmission lines

→ **Magnetic flux & flux linkage**

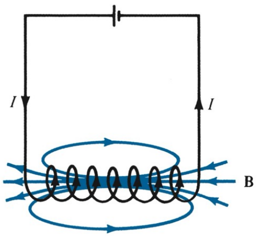
→ When current flows through a circuit, it produces magnetic flux Ψ through the area it encloses

→ For single loop,

$$\Psi = \int_S \vec{B} \cdot d\vec{S}$$

For multi-turn coil,

$$\lambda = N\Psi \qquad (N \text{ turns})$$



Inductance

→ If material is linear $(\vec{B} \propto \vec{H})$, flux linkage is proportional to current

$$\lambda \propto I$$

$$\lambda = LI$$

L : self inductance

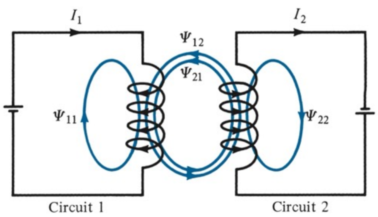
↳ Depends on geometry & material. Not current

$$L = \frac{\lambda}{I} = \frac{N\Psi}{I}$$

→ Inductance represents system's ability to store magnetic energy

→ Magnetic energy stored in inductor

$$\boxed{W_m = \frac{1}{2} L I^2} \Rightarrow \boxed{L = \frac{2W_m}{I^2}}$$



Mutual Inductance b/w 2 circuits

→ If you have 2 circuits each carrying I_1 & I_2 then ② creates flux Ψ_{12} that links with ①,

$$\lambda_{12} = N_1 \Psi_{12} = M_{12} I_2 \Rightarrow \boxed{M_{12} = \frac{N_1 \Psi_{12}}{I_2}} \; \& \; \boxed{M_{21} = \frac{N_1 \Psi_{21}}{I_2}}$$

In linear medium, $M_{12} = M_{21}$

→ Total energy in magnetic field is sum of energies due to $\underbrace{L_1, L_2}_{\text{self-inductance}}$ and $\underbrace{M_{12}}_{\text{Mutual inductance}}$

$$\boxed{W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2}$$

Use + if fields in same direction and - if fields oppose each other

Internal Inductance - Due to field inside conductor (L_{in})

External Inductance - Due to field in space around conductor (L_{ext}) $\quad (L_{ext} C = \mu\varepsilon)$

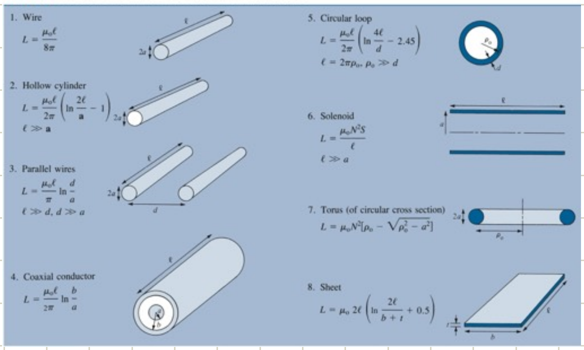
Total Inductance $\quad L = L_{in} + L_{ext}$

Magnetic Energy (W_m)

→ When current flows through an inductor, it sets up magnetic field

$$W_m = \frac{1}{2} L I^2$$

TABLE 8.3 A Collection of Formulas for Inductance of Common Elements



→ Energy in terms of $\vec{B}$ and $\vec{H}$

$$\boxed{dW_m = \frac{1}{2} \vec{B} \cdot \vec{H}\, dv}$$

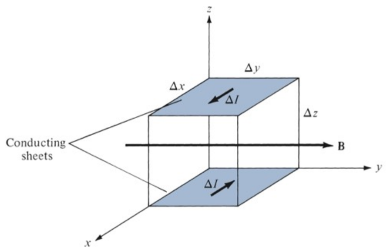
$$\boxed{W_m = \frac{1}{2} \int_v \vec{B} \cdot \vec{H}\, dv}$$

Magnetostatic Energy Density

→ Energy per unit volume

$$w_m = \frac{1}{2} \vec{B} \cdot \vec{H} = \frac{1}{2} \mu H^2 = \frac{B^2}{2\mu}$$

unit : J/m



→ Consider differential volume in magnetic field

$$\Delta L = \frac{\Delta \Psi}{\Delta I} = \frac{\mu H \Delta x \Delta z}{\Delta I} \qquad (\text{where } \Delta I = H \Delta y)$$

Also, $\Delta W_m = \frac{1}{2} \Delta L \Delta I^2 = \frac{1}{2}\left(\frac{\mu H \Delta x \Delta z}{\cancel{\Delta I}}\right) \Delta I^{\cancel{2}}$

$$= \frac{1}{2}(\mu H^2 \Delta x \Delta y \Delta z) = \frac{1}{2} \mu H^2 \Delta v \qquad (\Delta v = \Delta x \Delta y \Delta z)$$

The magnetostatic energy density $w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2 = \frac{1}{2} \vec{B} \cdot \vec{H} = \frac{B^2}{2\mu}$

$$\boxed{W_m = \int_v w_m\, dv = \frac{1}{2} \int_v \vec{B} \cdot \vec{H}\, dv = \frac{1}{2} \int_v \mu H^2\, dv}$$

(Similar to electrostatic field, $W_E = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, dv = \frac{1}{2} \int \varepsilon E^2 dv$)

Q. Calculate self inductance per unit length of an infinitely long solenoid

A. We know, magnetic flux inside solenoid per unit length,

$$B = \mu H$$
$$= \mu I n = \frac{\mu I N}{l}$$

$$\Psi = BS = \mu I n S$$

Linkage per unit length is $\lambda' = \frac{\lambda}{l} = n\Psi = \mu I n^2 S$

Inductance per unit length is $L' = \frac{L}{l} = \frac{\lambda'}{I}$ $\quad \boxed{L' = \mu n^2 S}$ H/m

Q.

PRACTICE EXERCISE 8.10

A very long solenoid with 2 × 2 cm cross section has an iron core ($\mu_r = 1000$) and 4000 turns per meter. It carries a current of 500 mA. Find the following:

(a) Its self-inductance per meter
(b) The energy per meter stored in its field

A. a) $L' = \mu_o \mu_r n^2 S$

$$= 4\pi \times 10^{-7} \times 1000 \times 4000^2 \times (2 \times 2 \times 10^{-4})$$
$$= 8.042 \text{ H/m}$$

b) $W_m' = \frac{1}{2} L' I^2 = \frac{1}{2} \times 8.042 \times (500 \times 10^{-3}) = 1.005$ J/m

Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

Solution:

The self-inductance of the inductor can be found in two different ways: by taking the four steps given in Section 8.8 or by using eqs. (8.54) and (8.66).

Method 1: Consider the cross section of the cable as shown in Figure 8.22. We recall from eq. (7.29) that by applying Ampère's circuit law, we obtained for region 1 ($0 \leq \rho \leq a$),

$$\mathbf{B}_1 = \frac{\mu I \rho}{2\pi a^2} \mathbf{a}_\phi$$

and for region 2 ($a \leq \rho \leq b$),

$$\mathbf{B}_2 = \frac{\mu I}{2\pi\rho} \mathbf{a}_\phi$$

We first find the internal inductance L_{in} by considering the flux linkages due to the inner conductor. From Figure 8.22(a), the flux leaving a differential shell of thickness $d\rho$ is

$$d\Psi_1 = B_1\, d\rho\, dz = \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz$$

The flux linkage is $d\Psi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area, that is,

$$d\lambda_1 = d\Psi_1 \cdot \frac{I_{enc}}{I} = d\Psi_1 \cdot \frac{\pi\rho^2}{\pi a^2}$$

because I is uniformly distributed over the cross section for dc excitation. Thus, the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu I \rho\, d\rho\, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

For length ℓ of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3\, d\rho\, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{in} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi} \tag{8.11.1}$$

The internal inductance per unit length, given by

$$\boxed{L'_{in} = \frac{L_{in}}{\ell} = \frac{\mu}{8\pi} \quad \text{H/m}} \tag{8.11.2}$$

$$d\Psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz$$

In this case, the total current I is enclosed within the path enclosing the flux. Hence,

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I\, d\rho\, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{ext} = \frac{\lambda_2}{I} = \frac{\mu \ell}{2\pi} \ln \frac{b}{a}$$

Thus

$$L = L_{in} + L_{ext} = \frac{\mu \ell}{2\pi} \left[\frac{1}{4} + \ln \frac{b}{a} \right]$$

or the inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi} \left[\frac{1}{4} + \ln \frac{b}{a} \right]} \quad \text{H/m}$$

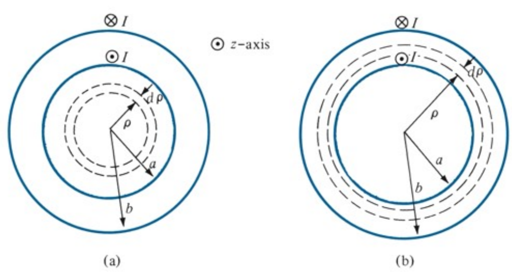


FIGURE 8.22 Cross section of the coaxial cable: (**a**) for region 1, $0 < \rho < a$, (**b**) for region 2, $a < \rho < b$; for Example 8.11.