Self-Learning Component

UNIT – 1

1. Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$.

Answer:
$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$

A. $u = xyz$, $v = x^2+y^2+z^2$ $w = x+y+z$

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{bmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = yz(2y-2z) - xz(2x-2z) + xy(2x-2y)$$
$$= 2(y^2z - yz^2 - x^2z + xz^2 + x^2y - xy^2)$$
$$= 2(y^2z - yz^2 - x^2z + xz^2 + x^2y - xy^2 + xyz - xyz)$$
$$= 2(x^2y - xyz - x^2z + x^2z - xy^2 + zy^2 + xyz - yz^2)$$
$$= 2(x-y)(xy - zy - xz + z^2)$$
$$= 2(x-y)(x-z)(y-z)$$
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{2(x-y)(y-z)(x-z)}$$
$$\boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}}$$

2. $u = x+y+z$

$v = \frac{y+z}{u^2} = \frac{y+z}{(x+y+z)^2}$

$w = \frac{z}{u^3} = \frac{z}{(x+y+z)^3}$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-2(y+z)}{(x+y+z)^3} & \frac{x-y-z}{(x+y+z)^3} & \frac{x-y-z}{(x+y+z)^3} \\ \frac{-3z}{(x+y+z)^4} & \frac{-3z}{(x+y+z)^4} & \frac{x+y-2z}{(x+y+z)^4} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ \frac{-2(y+z)}{(x+y+z)^3} & \frac{1}{(x+y+z)^2} & \frac{x-y-z}{(x+y+z)^3} \\ \frac{-3z}{(x+y+z)^4} & 0 & \frac{x+y-2z}{(x+y+z)^4} \end{vmatrix} \quad (C_2 = C_2 - C_1)$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-2(y+z)}{(x+y+z)^3} & \frac{1}{(x+y+z)^2} & \frac{1}{(x+y+z)^2} \\ \frac{-3z}{(x+y+z)^4} & 0 & \frac{1}{(x+y+z)^3} \end{vmatrix} \quad (C_3 = C_3 - C_1)$$

$$= 1\left(\frac{1}{(x+y+z)^5} - 0\right) - 0 + 0 = \frac{1}{(x+y+z)^5} = \frac{1}{u^5} = u^{-5}$$

UNIT – 2

1. Solve $\frac{dx}{dt} = 2x - 3y,\quad \frac{dy}{dt} = y - 2x$ given that $x(0) = 8$ and $y(0) = 3$.

Answer:
$x(t) = 3e^{4t} + 5e^{-t},\quad y(t) = 5e^{-t} - 2e^{4t}$.

A. $x'(t) = 2x(t) - 3y(t)$

$x'(t) - 2x(t) + 3y(t) = 0$

$L[x'(t)] - L[2x(t)] + 3L[y(t)] = 0$

$sL[x(t)] - x(0) - 2L[x(t)] + 3L[y(t)] = 0$

$(s-2)L[x(t)] + 3L[y(t)] = 8$

↓ Multiply by $s-1$

$(s-2)(s-1)L[x(t)] + 3(s-1)L[y(t)] = 8(s-1)$ → ①

$y'(t) = y(t) - 2x(t)$

$y'(t) - y(t) + 2x(t) = 0$

$L[y'(t)] - L[y(t)] + 2L[x(t)] = 0$

$2L[x(t)] + sL[y(t)] - y(0) - L[y(t)] = 0$

$2L[x(t)] + (s-1)L[y(t)] = 3$

↓ Multiply by 3

$6L[x(t)] + 3(s-1)L[y(t)] = 9$ → ②

Solving ① & ②

$(s^2 - s - 2s + 2 - 6)L[x(t)] = 8s - 8 - 9$

$(s^2 - 3s - 4)L[x(t)] = 8s - 17$

$L[x(t)] = \frac{8s-17}{s^2-3s-4}$

$x(t) = L^{-1}\left[\frac{8s-17}{(s+1)(s-4)}\right] = L^{-1}\left[\frac{A}{s-4} + \frac{B}{s+1}\right]$

$A = 3,\ B = 5$

$= L^{-1}\left[\frac{3}{s-4}\right] + L^{-1}\left[\frac{5}{s+1}\right]$

$\boxed{x(t) = 3e^{4t} + 5e^{-t}}$

$\frac{dx}{dt} = 12e^{4t} - 5e^{-t}$

& $\frac{dx}{dt} = 2x - 3y$

$y(t) = \frac{1}{3}\left(2x - \frac{dx}{dt}\right)$

$= \frac{1}{3}\left(6e^{4t} + 10e^{-t} - 12e^{4t} + 5e^{-t}\right)$

$\boxed{y(t) = 5e^{-t} - 2e^{4t}}$

2. Solve $\frac{dx}{dt} + y = sint,\quad \frac{dy}{dt} + x = cost$ given that $x = 2,\ y = 0$ when $t = 0$.

Answer:
$x(t) = 2cosht,\quad y(t) = sint - 2sinht$.

$\frac{dx}{dt} + y = \sin t \qquad \frac{dy}{dt} + x = \cos t$

$x'(t) + y(t) - \sin t = 0$

$L[x'(t)] + L[y(t)] - L[\sin t] = 0$

$sL[x(t)] - x(0) + L[y(t)] - \frac{1}{s^2+1} = 0$

$sL[x(t)] + L[y(t)] = 2 + \frac{1}{s^2+1} = \frac{2s^2+3}{s^2+1}$ → ①

$y'(t) + x(t) - \cos t = 0$

$L[y'(t)] + L[x(t)] - L[\cos t] = 0$

$sL[y(t)] - y(0) + L[x(t)] = \frac{s}{s^2+1}$

$sL[y(t)] + L[x(t)] = \frac{s}{s^2+1}$

↓ Multiply by s

$s^2L[y(t)] + sL[x(t)] = \frac{s^2}{s^2+1}$ → ②

Solving ① & ②

$(1-s^2)L[y(t)] = \frac{s^2+3}{s^2+1}$

$L[y(t)] = \frac{s^2+3}{(s^2+1)(1-s)(1+s)}$

$= \frac{As+B}{s^2+1} + \frac{C}{1-s} + \frac{D}{1+s}$

$= \frac{(As+B)(1-s^2) + C\overbrace{(1+s)(s^2+1)}^{s^2+s^3+s+1} + D\overbrace{(1-s)(1+s^2)}^{1-s+s^2-s^3}}{(s^2+1)(s^2-1)}$

$= \frac{As + B - As^3 - Bs^2 + Cs^3 + Cs^2 + Cs + C + D - Ds + Ds^2 - Ds^3}{(s^2+1)(s^2-1)}$

$\Rightarrow$ $-A + C - D = 0$, $-B + C + D = 1$, $A + C - D = 0$, $B + C + D = 3$; $A = 0$, $B = 1$, $C = 1$, $D = 1$

$y(t) = L^{-1}\left[\frac{1}{s^2+1} + \underbrace{\left(\frac{1}{1-s} + \frac{1}{1+s}\right)}_{\frac{2}{1-s^2}}\right] = \boxed{\sin t - 2\sinh t}$

$= \sin t - e^{t} + e^{-t}$

$\frac{dy}{dt} = \cos t - e^{t} - e^{-t}$

$= \cos t - \cosh t$

also, $\frac{dy}{dt} = \cos t - x$

$\Rightarrow \boxed{x = \cosh t}$

UNIT – 3

1. Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval [0, 4].

(a) What is the probability density function?

Answer: $f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$

(b) What is the probability that any given conference lasts at least 3 hours?

Answer: $P(X \ge 3) = \frac{1}{4}$.

a) $f(x) = \frac{1}{b-a} = \frac{1}{4-0} = \frac{1}{4}$

$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

b) $P(X \ge 3) = \int_3^4 \frac{1}{4}\,dx = \left[\frac{x}{4}\right]_3^4$

$= \frac{4-3}{4} = \frac{1}{4}$

2. The daily amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random variable X having a continuous uniform distribution with A = 7 and B = 10. Find the probability that on a given day the amount of coffee dispensed by this machine will be

(a) at most 8.8 liters. **Answer:** $P(x \le 8.8) = 0.6$.

(b) more than 7.4 liters but less than 9.5 liters.
Answer: $P(7.4 \le x \le 9.5) = 0.7$.

(c) at least 8.5 liters. **Answer:** $P(x \ge 8.5) = 0.5$.

$\frac{1}{b-a} = \frac{1}{10-7} = \frac{1}{3}$

a) atmost 8.8 lit

$P(X \le 8.8) = \int_7^{8.8} \frac{1}{3}\,dx = \frac{1.8}{3} = 0.6$

b) More than 7.4, but less than 9.5

$P(7.4 \le x \le 9.5) = \int_{7.4}^{9.5} \frac{1}{3}\,dx = \frac{9.5-7.4}{3} = \frac{2.1}{3} = 0.7$

c) atleast 8.5 lit

$P(X \ge 8.5) = \int_{8.5}^{10} \frac{1}{3}\,dx = \frac{10-8.5}{3} = 0.5$

UNIT – 4

1. Given

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Obtain the constant term and the co-efficient of first sine and cosine terms. $a0 = 41.66; a1 = -1.154; b1 = -8.33$

Answer: $a_0 = 41.66,\quad a_1 = -1.154,\quad b_1 = -8.33$

$2l = N = 6$

x	y	$\cos\frac{\pi x}{l}$	$y\cos\frac{\pi x}{l}$	$\sin\frac{\pi x}{l}$	$y\sin\frac{\pi x}{l}$
0	9	1	9	0	0
1	18	0.5	9	0.86	15.58
2	24	-0.5	-12	0.86	20.78
3	28	-1	-28	0	0
4	26	-0.5	-13	-0.86	-22.51
5	20	0.5	10	-0.86	-17.32
Total	125		-25		-3.46

$a_0 = \frac{2}{N}\sum_{n=0}^{N-1} y_n = \frac{2}{6}(125) = 41.67$

$a_1 = \frac{2}{N}\sum_{n=0}^{N-1} y\cos\frac{\pi x}{l} = \frac{2}{6} \times -25 = -8.33$

$b_1 = \frac{2}{N}\sum_{n=0}^{N-1} y\sin\frac{\pi x}{l} = \frac{2}{6} \times -3.46 = -1.154$

2. The following table gives the variation of periodic current A over a period T. Show that there is a constant part of 0.75 amperes in the current A and obtain the amplitude of the first harmonic.

x:	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
y:	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Answer: $a_0 = 0.75,\quad a_1 = 0.372,\quad b_1 = 1.005$

Amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2} = 1.0717$.

$2l = T \Rightarrow l = T/2$

x	y	$\cos\frac{\pi x}{l}$	$y\cos\left(\frac{\pi x}{l}\right)$	$\sin\left(\frac{\pi x}{l}\right)$	$y\sin\left(\frac{\pi x}{l}\right)$
0	1.98	1	1.98	0	0
T/6	1.30	0.5	0.65	0.86	1.126
2T/6	1.05	-0.5	-0.525	0.86	0.909
3T/6	1.30	-1	-1.3	0	0
4T/6	-0.88	-0.5	0.44	-0.86	0.762
5T/6	-0.25	0.5	-0.125	-0.86	0.217
Total	4.5		1.12		3.013

$a_0 = \frac{2}{N}\sum_{n=0}^{N-1} y = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{N}\sum_{n=0}^{N-1}\left(y\cos\frac{\pi x}{l}\right) = \frac{2}{6}(1.12) = 0.373$

$b_1 = \frac{2}{N}\sum_{n=0}^{N-1}\left(y\sin\frac{\pi x}{l}\right) = \frac{2}{6}(3.013) = 1.005$

Amplitude $= \sqrt{a_1^2 + b_1^2}$

$= 1.0717$

3. Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & for\ |x| < a \\ 0, & for\ |x| > a \end{cases}$ and hence show that $\int_0^\infty \left(\frac{sint}{t}\right)^4 dt = \frac{\pi}{3}$.

Answer: $F[f(x)] = \frac{4}{s^2}\sin^2\left(\frac{as}{2}\right)$

A. $f(x) = \begin{cases} a+x, & (-a,0) \\ a-x, & (0,a) \\ 0, & \text{rest} \end{cases}$

I

$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx = \int_{-a}^{0}(a+x)e^{-isx}\,dx + \int_0^a (a-x)e^{-isx}\,dx$

$= \left((a+x)\frac{e^{-isx}}{-is}\right)_{-a}^{0} - \left(\frac{e^{-isx}}{(-is)^2}\right)_{-a}^{0} + \left((a-x)\frac{e^{-isx}}{-is}\right)_0^a - \left(\frac{-e^{-isx}}{(-is)^2}\right)_0^a$

$= \cancel{\frac{a}{-is}} - 0 - \frac{1}{(-is)^2} + \frac{e^{+isa}}{(-is)^2} + 0 - \cancel{\frac{a}{is}} + \frac{e^{-isa}}{(-is)^2} - \frac{1}{(-is)^2}$

$= \frac{-2}{i^2s^2} + \frac{e^{+isa}}{(-is)^2} + \frac{e^{-isa}}{(-is)^2}$

$= \frac{2}{s^2} - \frac{1}{s^2}\left(e^{-isa} + e^{+isa}\right) = \frac{2 - 2\cos sa}{s^2} = \frac{2(1-\cos sa)}{s^2}$

$= \frac{2}{s^2}\left(2\sin^2\left(\frac{as}{2}\right)\right) = \frac{4\sin^2\left(\frac{as}{2}\right)}{s^2}$

II

$F^{-1}[F(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{+isx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2\left(\frac{as}{2}\right)}{s^2}e^{+isx}\,ds$

Put $x = 0$ & $a = 2$

$= \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty}\frac{\cancel{4}\sin^2 s}{s^2}\,ds = \cancel{2} \Rightarrow \frac{2}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\,ds \Rightarrow \int_0^\infty \frac{\sin^2 s}{s^2}\,ds = \frac{\pi}{2}$

$\int_0^\infty \frac{\sin^2 t\,dt}{t^2} = \frac{\pi}{2}$

III Also, Parseval's Identity.

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(F(s)\right)^2 ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4\sin^2\left(\frac{as}{2}\right)}{s^2}\right)^2 ds = \int_{-a}^{a}|a-|x||^2\,dx$

$\frac{16}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^4 ds = 2\int_0^a (a-x)^2\,dx$

$t = \frac{as}{2} \Rightarrow dt = \frac{a\,ds}{2} \Rightarrow ds = \frac{2\,dt}{a}$

$\frac{16}{\cancel{2}\pi} \times \frac{\cancel{2}}{a}\int_{-\infty}^{\infty}\left(\frac{\sin t}{\frac{2t}{a}}\right)^4 dt = 2\left(\frac{(a-x)^3}{3}\right)_0^a$

$\frac{\cancel{16}}{\pi a} \times \frac{a^4}{\cancel{16}}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = 2\left(0 - \frac{a^3}{3}\right)$

$\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{-2\pi}{3}$

$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{-\pi}{3}$

4. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & for\ |x| \le a \\ 0, & for\ |x| > a \end{cases}$ and hence show that $\int_0^\infty \frac{(xcosx - sinx)^2}{x^6}dx = \frac{\pi}{15}$.

Answer: $F[f(x)] = \frac{2}{\pi s^3}\sin(as) - as\cos(as)$

A. $f(x) = \begin{cases} a^2 - x^2, & -a \le x \le a \\ 0, & |x| > a \end{cases}$

$F[f(x)] = \int_{-\infty}^{\infty} f(x)\cdot e^{-isx}\,dx = \int_{-a}^{a}(a^2-x^2)\cdot e^{-isx}\,dx = \left[\cancel{(a^2-x^2)}^{\,0}\cdot\frac{e^{-isx}}{-is} - \frac{(-2x)\cdot e^{-isx}}{(-is)^2} + \frac{(-2)e^{-isx}}{(-is)^3}\right]_{-a}^{a}$

$= 2\left[\frac{x\cdot e^{-isx}}{(-is)^2} - \frac{e^{-isx}}{(-is)^3}\right]_{-a}^{a} = 2\left(\frac{ae^{-isa} + ae^{isa}}{(-is)^2} - \left(\frac{e^{-isa} - e^{isa}}{(-is)^3}\right)\right)$

$= 2\left(\frac{-a}{s^2}\left(e^{-isa} + e^{isa}\right) + \frac{e^{isa} - e^{-isa}}{is^3}\right) = 2\left(\frac{-2a\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right)$

$= \frac{4}{s^3}\left(\sin sa - as\cos sa\right)$

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(F(s)\right)^2 ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx \Rightarrow \frac{1}{\cancel{2}\pi}\int_{-\infty}^{\infty}\frac{\cancel{16}\,8}{s^6}\left(\sin sa - as\cos sa\right)^2 ds = \int_{-a}^{a}\left(a^2 - x^2\right)^2 dx$

$\Rightarrow \frac{8}{\pi}\int_{-\infty}^{\infty}\frac{(\sin sa - as\cos sa)^2}{s^6}\,ds = \int_{-a}^{a}\left(x^4 - 2a^2x^2 + a^4\right) \Rightarrow \frac{8\times 2}{\pi}\int_0^\infty \frac{(as\cos sa - \sin sa)^2}{s^6}\,ds = \left(\frac{x^5}{5} - \frac{2a^2x^3}{3} + a^4x\right)_{-a}^{a}$

$\Rightarrow \frac{\cancel{16}}{\pi}\int_0^\infty \frac{(as\cos sa - \sin sa)^2}{s^6}\,ds = \frac{\cancel{16}a^5}{15}$ Take $a = 1$ & $s = x$ $\Rightarrow \int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6} = \frac{\pi}{15}$