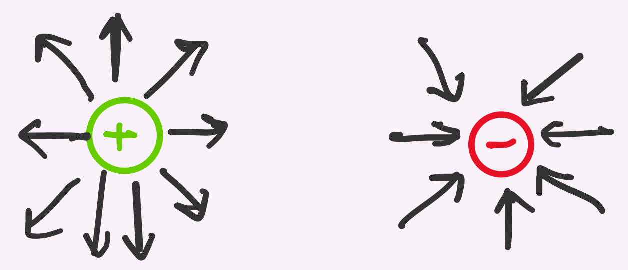
Electric and Magnetic fields

Electric Fields

→ Visualized by Electric Flux lines



→ Charges can be isolated

→ Potential $V_x = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{x}$

→ Electric Field $E_x = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{x^2}$

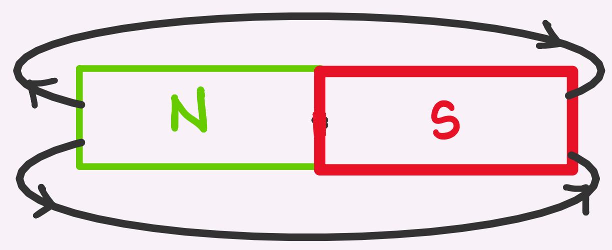
So, $E_x = -\frac{dV_x}{dx}$

Magnetic Fields

→ No monopoles

→ Can be expressed in terms of flux lines

→ Flux lines are continuous



Maxwells Equations

→ Gauss' Law $\nabla \vec{E} = \frac{\rho}{\varepsilon_0}$

→ Gauss' Law for magnetism $\nabla \vec{B} = 0$

→ Faraday's Law $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

→ Ampere-Maxwell Law $\nabla \times \vec{B} = \underbrace{\mu_0 \vec{J}}_{\to\ 0 \text{ in free space}} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Note: $\frac{\partial \vec{\phi}}{\partial t} = \frac{\partial B . \text{Area}}{\partial t} = \frac{\partial \vec{B}}{\partial t}$

Nabla / Del Operator

→ $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ (Del operator)

→ $\vec{\nabla}\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Laplacian operator)

So, $\nabla V = \frac{\partial V_x}{\partial x}\hat{i} + \frac{\partial V_y}{\partial y}\hat{j} + \frac{\partial V_z}{\partial z}\hat{k}$ (grad V)

$\vec{\nabla} . V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ (Divergence V)

$\vec{\nabla} \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$ (curl A)

→ $\vec{\nabla} \times (\vec{\nabla} \times A) = \vec{\nabla}(\vec{\nabla}.A) - \vec{\nabla}^2 A$ (Identity)

→ $\nabla^2 \vec{A} = \left(\frac{1}{\underbrace{v^2}_{\text{velocity}}} \cdot \frac{\partial^2 \vec{A}}{\partial t^2}\right)$ (General wave eq.)

★ Electric waves in free space

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$

$\vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2 E = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$ (Using Identity)

$= -\left(\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t}\right)$

For free space, $\vec{\nabla}.\vec{E} = 0$

So, $-\nabla^2 E = -\left(\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t}\right)$

$\nabla^2 E = \left(\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t}\right)$

$\nabla^2 E = \mu_0 \varepsilon_0 \cdot \frac{\partial^2 E}{\partial t^2}$ (Ampere - Maxwell Law)

$\nabla^2 E = \frac{1}{c^2} \cdot \frac{\partial^2 E}{\partial t^2}$ $\left(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$ (wave eq. for Electric wave)

★ Magnetic waves

$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere - Maxwell Law)

$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t}\right)$

$\vec{\nabla}(\vec{\nabla}.\vec{B}) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \cdot \left(\frac{\partial \vec{\nabla} \times \vec{E}}{\partial t}\right)$

$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \left(\frac{\partial \vec{\nabla} \times \vec{E}}{\partial t}\right)$

$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \left(-\frac{\partial^2 B}{\partial t^2}\right)$ (Faraday's Law)

$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$ $\left(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$

Mutually perpendicularity between B and E

Consider equation $E_x = E_0 \cos(\omega t + kz)\,\hat{i}$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} \quad \hat{i}(0) - \hat{j}\left(-\frac{\partial E_x}{\partial z}\right) + \hat{k}\left(-\frac{\partial E_x}{\partial y}\right)$$

$$= \frac{\partial}{\partial z}\left(E_0 \cos(\omega t + kz)\right)\hat{j} - \frac{\partial}{\partial y}\left(E_0 \cos(\omega t + kz)\right)\hat{k}$$

$$\nabla \times \vec{E} = -kE_0 \sin(\omega t + kz)\,\hat{j}$$

$$\Rightarrow -\frac{\partial \vec{B}}{\partial t} = -kE_0 \sin(\omega t + kz)\,\hat{j} \quad \text{(Faraday's Law)}$$

$$\int \partial B = kE_0 \int \sin(\omega t + kx)\, dt\, \hat{j}$$

$$B = \frac{kE}{\omega} \cos(\omega t + kx)\,\hat{j}$$

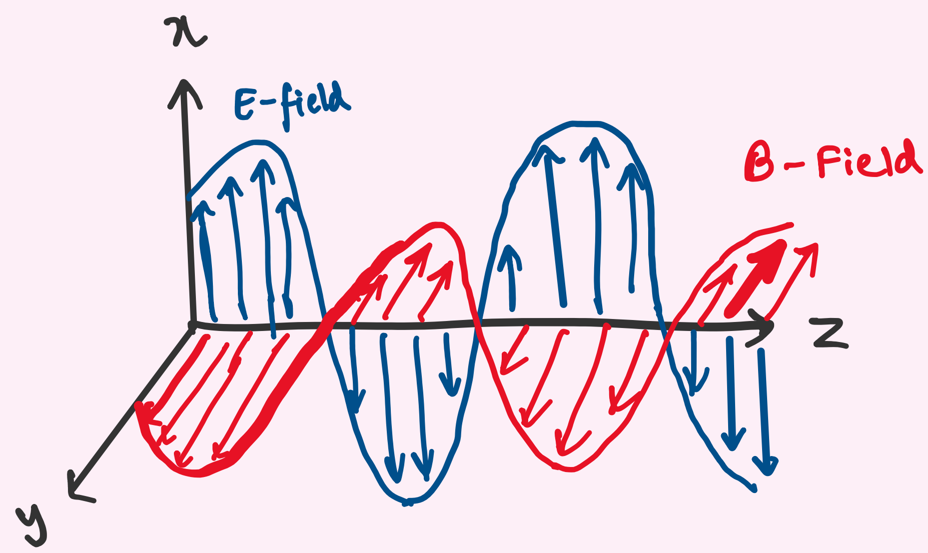
$$B = \frac{E_0}{c} \cos(\omega t + kx)\,\hat{j} \quad \left(c = \frac{\omega}{k}\right)$$

$$B = B_0 \cos(\omega t + kx)\,\hat{j} \quad \left(B_0 = \frac{E_0}{c}\right)$$

So, B is in y component

& E is in x component

Hence, B & E are mutually perpendicular



3. Energy Transportation by EMW

3. Energy Transporta...

Energy Transportation by EMW

For Electromagnetic waves

→ v (velocity) $= c = 3 \times 10^8$

→ λ (wavelength) $= \dfrac{2\pi}{K}$

→ f (frequency) $= \dfrac{v}{\lambda}$

→ T (Period) $= \dfrac{1}{f}$

→ $B(x,t) = \dfrac{E(x,t)}{c}$

→ A (Amplitude) $= \dfrac{B_0}{E_0}$

Energy content of electric component

$$E_E = \frac{1}{2}\varepsilon_0 E_x^2$$

Energy content of magnetic component

$$E_M = \frac{1}{2}\frac{B_y^2}{\mu_0}$$

Total energy content of EM wave

$$E_T = E_E + E_M = \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\frac{B_x^2}{\mu_0}$$

$$= \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\frac{E_x^2}{c^2\mu_0} \qquad \left(B_0 = \frac{E_0}{c}\right)$$

$$= \frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\varepsilon_0 E_x^2 \qquad \left(\frac{1}{c^2} = \mu_0\varepsilon_0\right)$$

$$E_T = \varepsilon_0 E_x^2 \quad \text{(in z direction)}$$

→ Power dissipitated
$= \text{Energy Flux} \times \text{Area}$
$= \varepsilon_0 E^2 c \times A$

Average energy of EMW

→ Avg. Energy of EMW = Energy transported in 1cycle

$$= \frac{\text{Total energy}}{\text{cycle}} = \langle \text{Energy} \rangle = \frac{c\,\varepsilon_0}{T}\int_0^T E_x^2\, dt$$

$$= \frac{c\,\varepsilon_0}{T}\int_0^T E_{0x}^2 \cos^2(\omega t + kx)\, dt$$

$$= \frac{1}{2}\frac{c\,\varepsilon_0}{\cancel{T}} \cdot E_{0x}^2 \cdot \cancel{T} = \frac{1}{2} c \cdot \frac{B_{0y}^2}{\mu_0}$$

$$= \frac{1}{2}\frac{E_{0x} B_{0y}}{\mu_0}$$

Intensity $\propto |\text{Amplitude}|^2$

Poynting Vector ($\vec{S}$)

→ $\vec{S}$ describes EM energy transported per unit time per unit volume

$$\vec{S} = \frac{1}{\mu_0} E \times B = c^2 \varepsilon_0 E \times B = c\,\varepsilon_0 E^2$$

→ Direction of $\vec{S}$ is direction of propagation of waves

→ $\vec{S}$ is time dependant

$$\text{flux} = \frac{E_T \times C\cancel{\Delta t}}{\cancel{\Delta t}}$$

$$= c \times \varepsilon_0 E^2$$

$$= c \times \varepsilon_0 E\,(Bc)$$

$$= c^2 \times \varepsilon_0\, E \times B$$

$$\vec{S} = \frac{1}{\mu_0} E \times B$$

? $E(x,t) = 10^3 \cos(\omega t - 3\pi \times 10^6 z)$

Find $v, \lambda, f, T, B(x,t), A$

A. $v = 3 \times 10^8\ ms^{-1}$

$\lambda = \dfrac{2\pi}{K} = \dfrac{2\pi}{3\pi \times 10^6} = 6.67 \times 10^{-7}\ m$

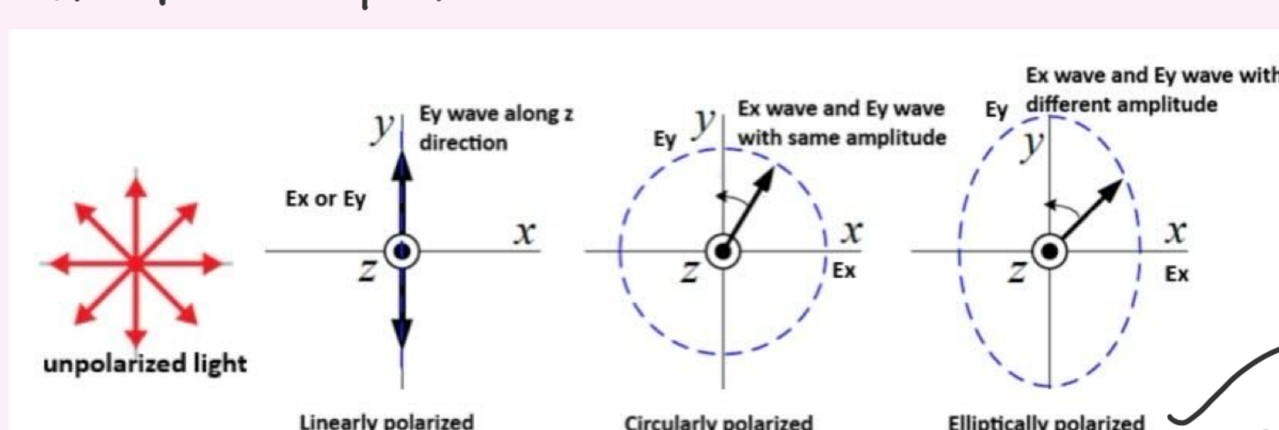
$f = \dfrac{v}{\lambda} = 4.5 \times 10^{14}\ s^{-1}$

$T = 2.22 \times 10^{-16}\ s$

$B(x,t) = \dfrac{E(x,t)}{c} = \dfrac{10^3}{3 \times 10^8} = 3.33 \times 10^{-6}$

Polarization of EMW

→ Natural light is unpolarized
All planes of propagation being equally probe



- Plane wave is called linearly polarized
- Addition of horizontally & vertically linearly polarized waves of same amp in same phase results in linearly polarized wave at 45°

- Light is composed of 2 plane waves of 2 equal amplitude but differ in phase by 90° are circularly polarized

- 2 plane waves of differing amplitude are related in phase by 90° (or) relative phase other than 90° are elliptically polarized

4. Failure of classical...

4. Failure of classical EMW theory and blackbody radiation

- EM Radiation ⇒ Mutual perpendicularity b/w E & B waves
- Classical wave theory ⇒ Energy content of wave $\propto$ (Amplitude of waves)^2
- Wave theory ⇒ Explains reflection, refraction, interference, diffraction & polarization of light

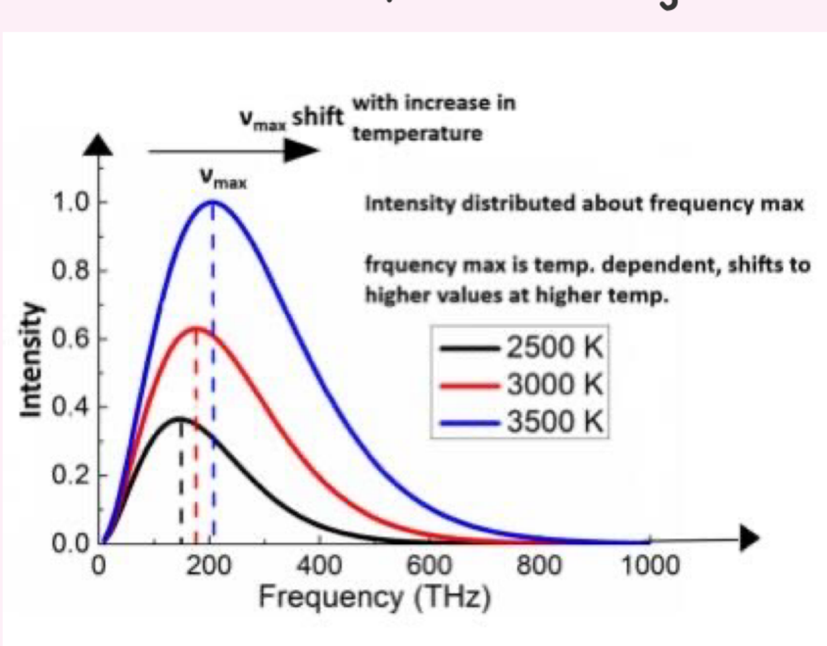
Couldn't explain -
- Photo-electric effect
- Spectrum of Hydrogen emissions (Atomic Spectra)
- Black-body Radiation Spectrum
- Compton Scattering

Blackbody Radiation (Hypothetical Body)

- Black body ⇒ Materials which absorb all incident rays. On heating, it will emit all wavelengths of radiation that was absorbed

ex: sun, incandescent bulb etc,

⇒ Emissivity = 1



⇒ Radiation only depends on temperature. Not material of object

In Intensity vs Wavelength graph, The maximum intensity of prominent wavelength is shifting towards shorter wavelength as temperature increases

There are distinct curves for different temperatures

Rayleigh - Jeans Law

- Assumed black body as cavity oscillators
- No of oscillators b/w ν & $\nu + \delta\nu$ is

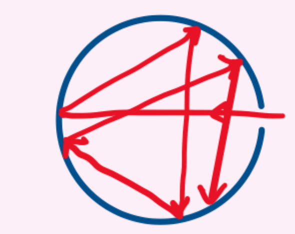
$$dN = \frac{8\pi}{c^3}\nu^2 d\nu$$

(No of modes $\propto \nu^2$)

- $\langle E \rangle = K_B T$ (Maxwell - Boltzmann distribution law)
- $\rho(\nu)d\nu = \langle E \rangle dN$ (Energy density, according to Rayleigh - Jeans law)

$$= K_B T \times \frac{8\pi}{c^3}\nu^2 d\nu$$

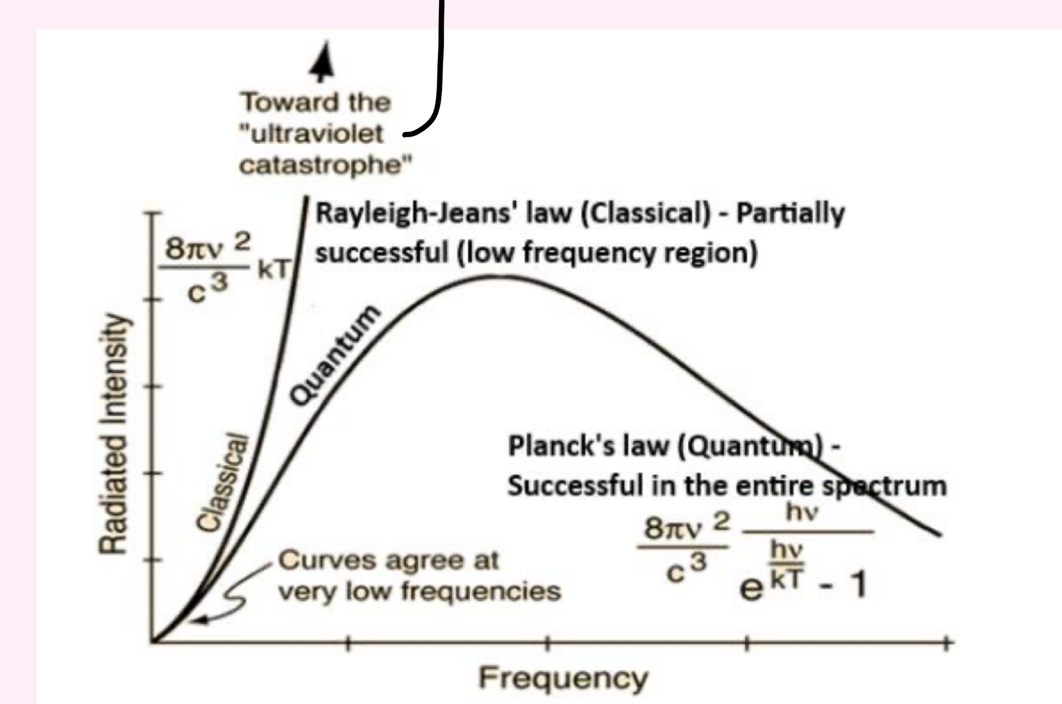
But it contradicts with experimental observations



Model that doesn't allow incident radiation to escape because of inner multiple reflections

This cavity when heated emit radiations of every possible frequency & rate of emission ↑ as Temperature ↑

Breakdown of Rayleigh Jeans at higher frequencies



Max - Planck hypothesis

- Solution for R-JL / Ultraviolet Catastrophe
- Energy of oscillator model of black body must be restricted to finite multiples of fundamental natural frequency times constant ($E = nh\nu$)
- $$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

and, $\rho(\nu)d\nu = \langle E \rangle dN$

$$= \frac{h\nu}{e^{h\nu/kT} - 1} \cdot \frac{8\pi\nu^2}{c^3}d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/kT} - 1} \cdot d\nu$$

$$P(E) = \frac{e^{-E/kT}}{kT}$$

Probability function of oscillators with energy E in thermal equilibrium

$$\langle E \rangle = \frac{\int E \cdot P(E)\, dE}{\int P(E)\, dE} = K_B T$$

$$\rho(\nu)d\nu = K_B T \times \frac{8\pi\nu^2 d\nu}{c^3}$$

? Frequency of harmonic oscillator at 50°c is $6.2 \times 10^{12}\ s^{-1}$. Estimate average energy of oscillator as per planck's idea of cavity oscillator, also compare the same with classical energy & average energy by R-J Law

A. As per planck's idea,

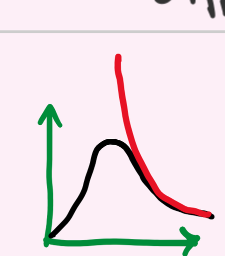
$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1} = \frac{6.63 \times 10^{-34} \times 6.2 \times 10^{12}}{e^{\frac{6.63 \times 6.2 \times 10^{-22}}{1.38 \times 10^{-23} \times 323}} - 1}$$

$$= 2.713 \times 10^{-21}\ J$$

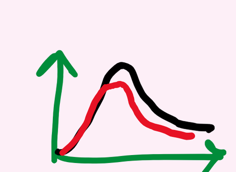
As per classical analysis, $\langle E \rangle = K_B T = 1.38 \times 10^{-23} \times 323 = 4.457 \times 10^{-21}\ J$

Difference $= 1.7444 \times 10^{-21}\ J$

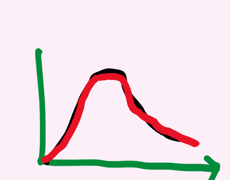
Rayleigh Jeans ⇒ $u(\lambda)\, d\lambda = \frac{8\pi KT}{\lambda^4} d\lambda$



Wiens ⇒ $u(\lambda)\, d\lambda = \frac{c_1}{\lambda^5} e^{-\frac{c_2}{\lambda}} d\lambda$ (Exponential decay)



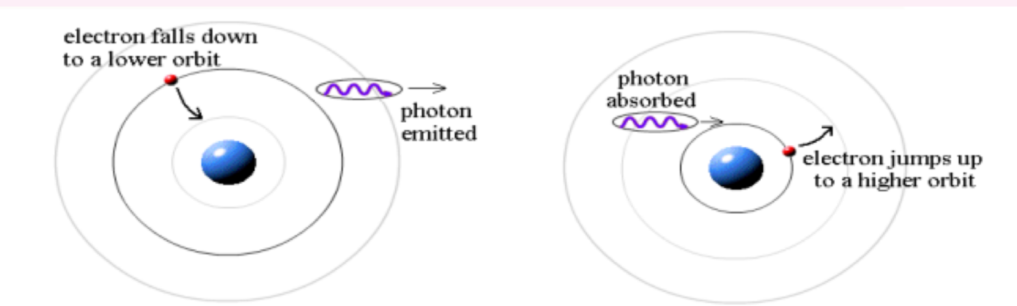
Plancks ⇒ $u(\lambda)\, d\lambda = \frac{8\pi hc}{\lambda^5}\left[\frac{1}{e^{hc/\lambda KT} - 1}\right]$



5. Atomic Spectra, Photoelectric Effect and Compton Shift

Atomic Spectra

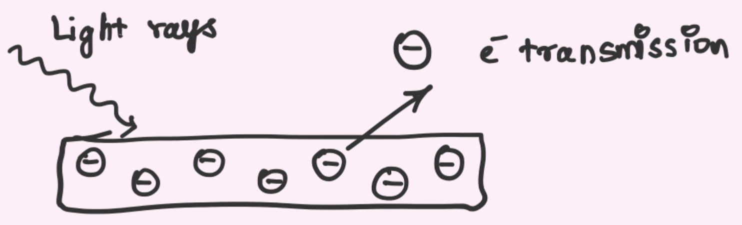
- Different elements have different spectra, So, atomic spectroscopy allows identification of a sample's elemental composition
- Atomic absorption lines in solar spectrum are called Fraunhofer lines
  - ⇒ Discrete lines ⇒ Emission Spectra
  - ⇒ No discrete lines ⇒ Absorption Spectra
- Based on Max Planck, Energy comes in quanta & absorbed/emitted in terms of quanta



Predictions	v/s	Observations
→ e^- should be emitted with time lag		→ No time lag
→ e^- should be emitted irrespective of frequency/wavelength of incident of light		→ There is a min. freq below which there are no photo electrons
→ More intense the light, greater energy of e^-		→ Intensity increases the no. of photo electrons

Photoelectric Effect

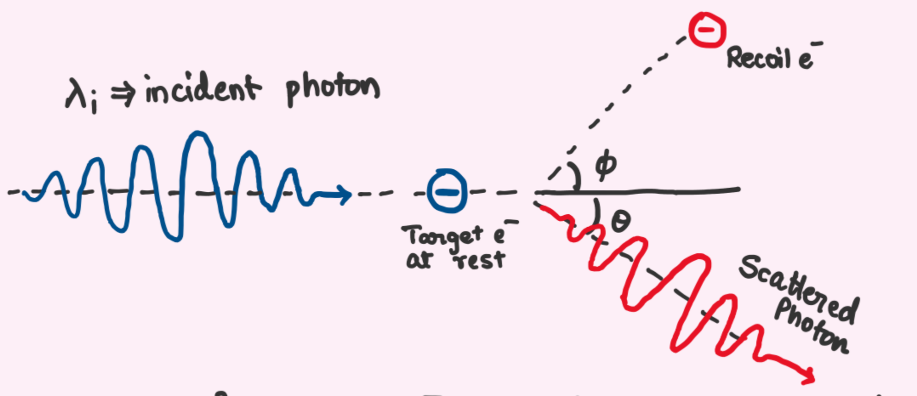
- e^- emission from metals under irradiation



- Instantaneous emission of e^- with KE dependant on wavelength of radiation
- Energy of e^- independant of intensity of radiation
- $h\nu = W + KE$
- Waves can be of dual nature

Compton Shift

⇒ Process of scattering of radiation from loosely bound e^-, which is released from atom & remainder of energy is rereleased as electromagnetic radiation



$E = m_0c^2$ ⟶ Energy of rest mass particle

$E = \sqrt{p^2c^2 + m_0^2c^4}$

pc → KE of particle with 'p' momentum

$\Delta\lambda = \lambda_f - \lambda_i = \frac{h}{m_e c}(1-\cos\theta)$ $\left[\frac{h}{m_0 c} = 2.45\text{ pm}\right]$

$\frac{h}{m_e c} = \lambda_c$ (Compton wavelength)

$\Delta\lambda$ is independant of incident wavelength

$\Delta\lambda$ dependant on scattering angle

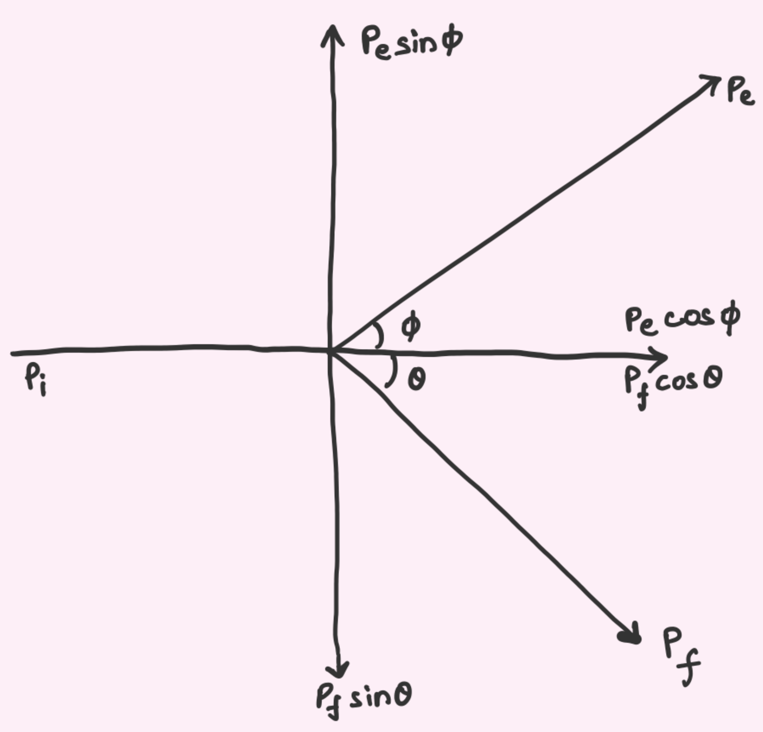
For e^-, $\lambda_c = 2.42 \times 10^{-12}$ m

If $\theta = 180°$, max shift

$P_i = \frac{E_i}{c} = \frac{h\nu_i}{c} = \frac{h}{\lambda_i}$

Compton Shift Derivation

- Along incident direction ⇒ $P_i + 0 = P_f\cos\theta + P_e\cos\phi$
- Along perpendicular direction ⇒ $0 = P_f\sin\theta - P_e\sin\phi$



- Before & After collision conservation,

  $P_e^2 = P_i^2 + P_f^2 - 2P_iP_f\cos\theta$ ⟶ ①
- Before & After energy conservation

  $P_ic + m_0c^2 = P_fc + \sqrt{P_e^2c^2 + m_0^2c^4}$

  ⇒ Square on Both sides & divide by c^2

  $P_e^2 = P_i^2 + P_f^2 - 2P_iP_f - 2m_0c(P_i - P_f)$ ⟶ ②

From ① & ②

$\cancel{P_i^2} + \cancel{P_f^2} - 2P_iP_f\cos\theta = \cancel{P_i^2} + \cancel{P_f^2} - 2P_iP_f + 2m_0c(P_i - P_f)$

Also, substitute $P_i = \frac{h}{\lambda_i}$ & $P_f = \frac{h}{\lambda_f}$

$\cancel{2}P_iP_f(1-\cos\theta) = \cancel{2}m_0c(P_i - P_f)$

$(1-\cos\theta) = m_0c\frac{(P_i - P_f)}{P_iP_f}$

$1-\cos\theta = m_0c\left(\frac{1}{P_f} - \frac{1}{P_i}\right)$

$1-\cos\theta = m_0c\left(\frac{\lambda_f - \lambda_i}{h}\right)$

$\lambda_f - \lambda_i = \frac{h}{m_0c}(1-\cos\theta)$ ⇒ Proved particle nature of EM radiation

Q. X-rays of wavelength 0.112 nm is scattered from a carbon target. Calculate wavelength of X-rays scattered at an angle 90° wrt original direction. Whats the energy lost by X-ray photons? Whats the energy gained by e^-? If incident X-ray retraces back what will be the shift?

A. $\Delta\lambda = \frac{h}{m_ec}(1-\cos\theta)$

$= 2.42\times10^{-12}(1-0)$

$= 0.0242$ Å

Wavelength of scattered X-rays, $\lambda' = \lambda + \Delta\lambda$

$= 1.12 + 0.0242$

$= 1.1442$ Å

Energy lost by X-ray photons $= h\nu - h\nu'$

$= \frac{hc}{\lambda} - \frac{hc}{\lambda'}$

$= 1.77\times10^{-15} - 1.74\times10^{-15}$

$= 0.03\times10^{-15}$

$= 3\times10^{-17}$ J

$= 187.2$ eV $\quad (1\text{J} = 6.242\times10^{18}\text{eV})$

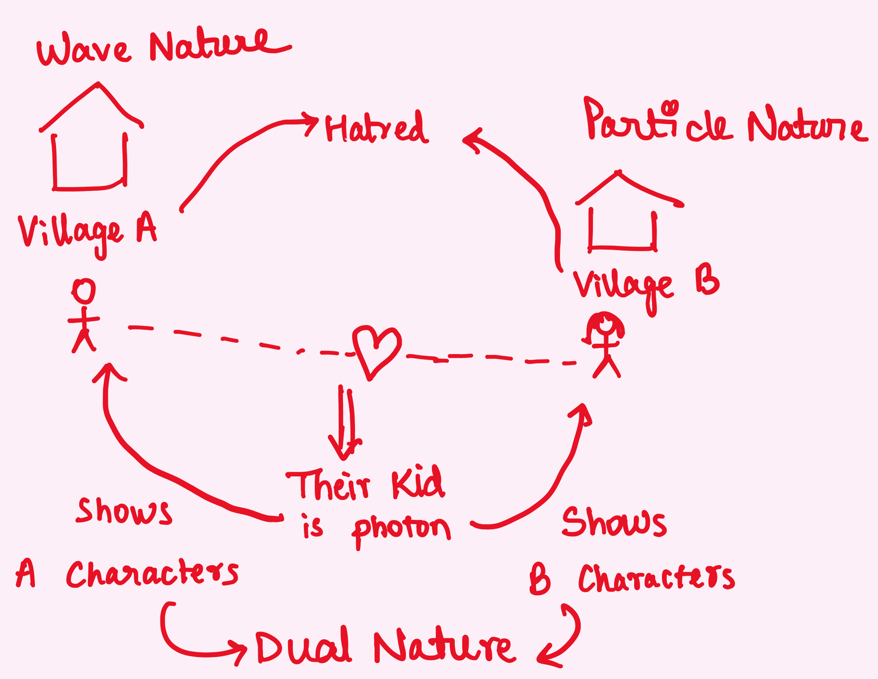
$\theta = 180°$, Max shift

$E = pc$

$E = \frac{hc}{\lambda}$ ⇒ $pc = \frac{hc}{\lambda}$

$\lambda = \frac{h}{p}$ — wave property (λ), Particle property (p)

$\lambda = \frac{h}{mv}$ — particle's speed (must be moving)



6. Dual Nature of Radiation

Electromagnetic Radiation

Wave Nature
- Reflection, Refraction, Interference, Diffraction, Polarization
- Energy depends on intensity

Particle Nature
- Blackbody radiation, Atomic spectra, Photoelectric effect, Compton Effect
- Energy depends on wavelength & frequency

De Broglie Hypothesis

→ Moving matter should exhibit wave characteristics

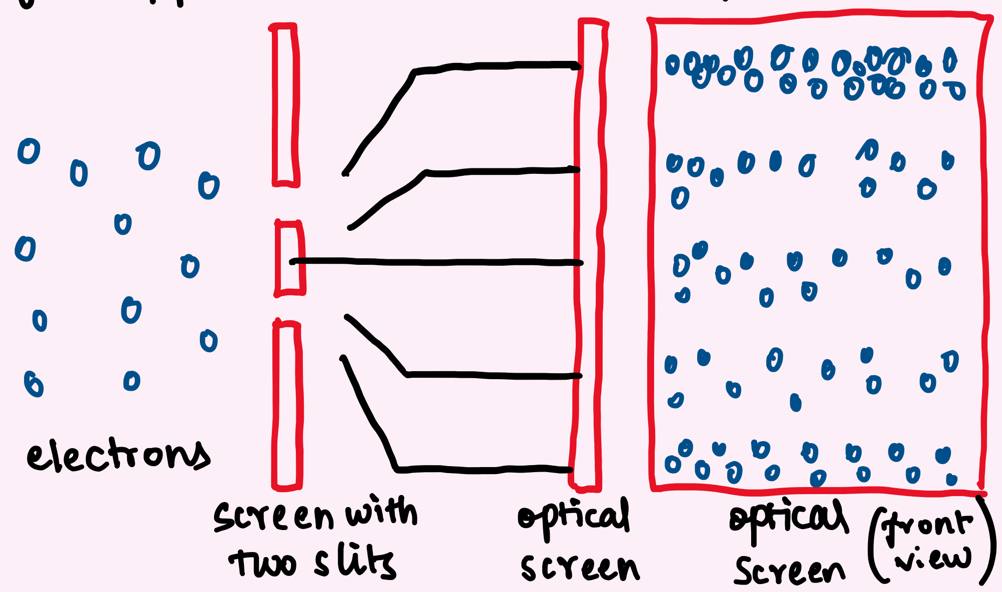
→ $\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$

→ Wavelength of macro particles are too small to be measured
Wavelength of sub-atomic particles are measurable ($\sim 10^{-10}$ m)

Double Slit Experiment

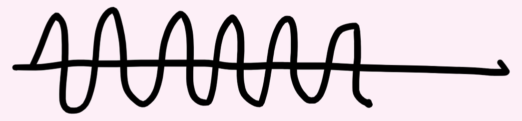
→ Diffraction is a characteristic wave phenomenon

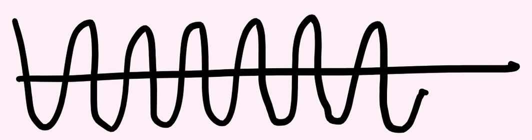
→ When single photon is sent through double-slit, it behaves like a wave by creating an interference pattern on screen
However, when photons are detected individually on the screen, they appear as discrete particles hitting specific points on the screen

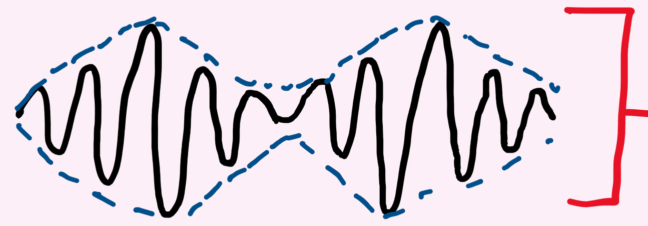


7. Matter Waves

→ Wave packets describe matter waves (Defined wavelength & amplitude maximum)

→  $\Rightarrow y_1 = A\sin(\omega t - Kx)$

 $\Rightarrow y_2 = A\sin\{(\omega+\Delta\omega)t - (K+\Delta K)x\}$

Wave packet $= y_1 + y_2$  → Wave Packet → Matter waves

If we use $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$y_1 + y_2 = 2A\sin\left(\frac{(2\omega+\Delta\omega)t - (2K+\Delta K)x}{2}\right)\cos\left(\frac{\Delta\omega t - \Delta K x}{2}\right)$$

$$\simeq 2A\sin(\omega t - Kx)\cos\left(\frac{\Delta\omega t - \Delta K x}{2}\right)$$

⇒ We start with constant Amplitude & End with time varying Amplitude

→ Spread around central maximum is approximate position of particle

Phase Velocity (V_p)

→ It is the velocity with which a constant phase point marked on the superposed wave moves

$$V_p = \frac{dx}{dt} = \frac{\omega}{K}$$

$Kx - \omega t = C$

$K\frac{dx}{dt} - \omega = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{K}$

Group Velocity

→ It is the velocity of common velocity of the superposed wave

$$V_g = \frac{dx}{dt} = \frac{d\omega}{dK}$$

$\frac{\Delta K x - \Delta\omega t}{2} = c$

$\Delta K\frac{dx}{dt} - \Delta\omega = 0 \Rightarrow \frac{dx}{dt} = \lim_{\Delta K\to 0}\frac{\Delta\omega}{\Delta K} = \frac{d\omega}{dK}$

Relation b/w V_g & V_p

→ $V_g = \frac{d\omega}{dK} = \frac{d(V_pK)}{dK} = V_p + K\cdot\frac{dV_p}{dK}$

Also, $\frac{dV_p}{dK} = \frac{dV_p}{d\lambda}\cdot\frac{d\lambda}{dK} = \frac{dV_p}{d\lambda}\cdot\frac{d\left(\frac{2\pi}{K}\right)}{dK} = \frac{dV_p}{d\lambda}\left(\frac{-2\pi}{K^2}\right)$

$V_g = V_p - K\cdot\frac{dV_p}{d\lambda}\left(\frac{2\pi}{K^2}\right)$

$= V_p - \frac{dV_p}{d\lambda}\left(\frac{2\pi}{K}\right)$

$= V_p - \lambda\frac{dV_p}{d\lambda}$

Case 1 : $V_g = V_p$

→ Only when $\frac{dV_p}{d\lambda} = 0$ (Light is not dispersive)

λ can't be 0

Case 2 : $V_g < V_p$

→ $V_g = V_p/2$

⇒ $V_g = V_p - \lambda\frac{dV_p}{d\lambda}$

$\frac{V_p}{2} = V_p - \lambda\frac{dV_p}{d\lambda}$

$\frac{V_p}{2} = \lambda\frac{dV_p}{d\lambda} \Rightarrow \int\frac{d\lambda}{\lambda} = 2\int\frac{dV_p}{V_p}$

$\ln\lambda \propto 2\ln V_p$

$V_p \propto \sqrt{\lambda}$

Case 3 : $V_g > V_p$

→ $V_g = 2V_p$

⇒ $V_g = V_p - \lambda\frac{dV_p}{d\lambda}$

$V_p = -\lambda\frac{dV_p}{d\lambda}$

$\int\frac{dV_p}{V_p} = -\int\frac{d\lambda}{\lambda}$

$\ln V_p \propto -\ln\lambda$

$V_p \propto \lambda^{-1}$

→ $V_g = \frac{d\omega}{dK}$

$\omega = \frac{E}{\hbar}$ $\left(\hbar = \frac{h}{2\pi}\right)$

$d\omega = \frac{dE}{\hbar}$

$K = \frac{p}{\hbar}$

$dK = \frac{dp}{\hbar}$

$\omega = 2\pi\nu$

$= \frac{2\pi}{h}\times E$ $(E = h\nu)$

$K = \frac{2\pi}{\lambda}$

$= \frac{2\pi}{h}\times p$ $\left(\lambda = \frac{h}{p}\right)$

$V_g = \frac{d\omega}{dK} = \frac{dE}{\cancel{\hbar}}\frac{\cancel{\hbar}}{dp} = \frac{d\left(\frac{p^2}{2m}\right)}{dp}$

$= \frac{1}{\cancel{2}m}\cdot\cancel{2}p = \frac{p}{m} = V_{particle}$

8. Heisenberg's Analysis

→ Wave packets have uncertainities

→ Δx : Spread in estimation of position

Δk : propagation constant

$\Rightarrow \Delta x \cdot \Delta k \geq \frac{1}{2}$

$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$

Heisenberg's Uncertainity Principle

1) Position Momentum Uncertainity

→ Position & momentum can't be determined simultaneously with unlimited precision

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \geq \frac{h}{4\pi}$$

2) Energy Time Uncertainity

→ Energy & Time can't be determined simultaneously with unlimited precision

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2} \geq \frac{h}{4\pi}$$

3) Uncertainity Relation for Circular motion

→ Angular position & Angular Momentum can't be determined simultaneously with unlimited precision

$$\Delta \theta \cdot \Delta L \geq \frac{\hbar}{2} \geq \frac{h}{4\pi}$$

Applications of Uncertainity Principle

1) Non-existence of e^- inside nuclei

→ e^- don't exist inside nuclei but are emitted from nucleus during β decay with energies of order of 8 MeV

→ Assuming e^- to be inside nucleus, then $\Delta x \simeq 10^{-14}$ m $\simeq$ nuclear diameter

So, $\Delta x \cdot \Delta p = \frac{\hbar}{2}$

Then, $\Delta p = \frac{6.63 \times 10^{-34}}{4\pi \times 10^{-14}} \simeq 5.28 \times 10^{-21}$ kg m s^{-1}

Hence, $p \simeq \Delta p$

$$KE = \frac{p^2}{2m} = \frac{(\Delta p)^2}{2m} = \frac{(5.28 \times 10^{-21})^2}{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}} = 96 \text{ MeV}$$

↳ e^- energy should be quite high to be inside nuclei

→ Thus, e^- doesn't reside inside nuclei

2) Gamma ray microscope (a thought experiment)

→ Experiment to measure position of electron

→ To observe the electron, wavelength of radiation must be of a comparable size to e^-

So, wavelength of γ rays = 10^{-12} m

Resolution of microscope $\Rightarrow \Delta x = \frac{\lambda}{\sin\theta}$

↳ Min. distance at which 2 distinct points of a specimen can still be seen

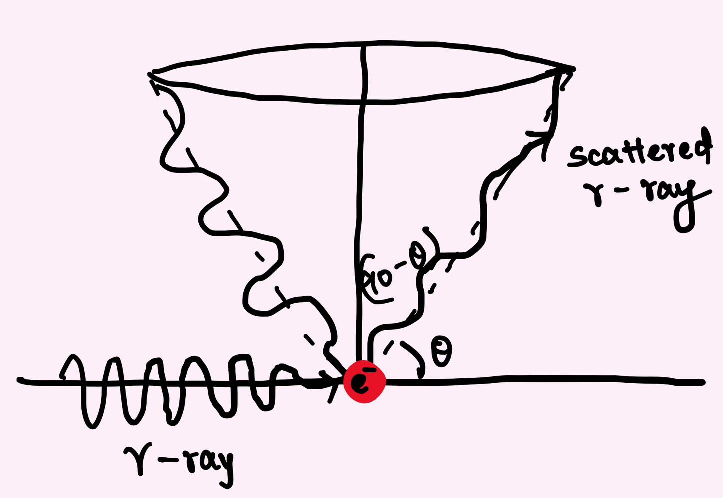
→ High energy γ rays impart momentum to e^- ⇒ Compton Effect

$p_x \simeq \pm \frac{h}{\lambda} \sin\theta$

$\Delta p_x \simeq \frac{2h\sin\theta}{\lambda}$ $(< p_x)$

So, $\Delta x \cdot \Delta p_x = \frac{\lambda}{\sin\theta} \cdot \frac{2h\sin\theta}{\lambda} \simeq 2h > \frac{h}{4\pi}$

Thus, position & momentum can't be found simultaneously



Other forms of Uncertainity Relations

1) Position - Wavelength uncertainity relation

$$\Delta x \cdot \Delta p = \Delta\left(\frac{h}{\lambda}\right) = h \cdot \Delta\left(\frac{1}{\lambda}\right) = h \cdot \left(\frac{-1}{\lambda^2}\Delta\lambda\right)$$

$$\Delta x \cdot h\left(\frac{-1}{\lambda^2}\Delta\lambda\right) \geq \frac{h}{4\pi}$$

$$\Delta x \cdot \Delta\lambda \geq \frac{-\lambda^2}{4\pi} \Rightarrow \Delta x \cdot \Delta\lambda \geq \left|\frac{\lambda^2}{4\pi}\right|$$

2) Uncertainity in terms of position & velocity

$\Delta p = \Delta(mv)$

$\Delta x \cdot \Delta v \geq \frac{h}{4\pi m}$

3) Position - propagation vector uncertainity relation

$\Delta x \cdot \Delta k \geq 1/2$

4) Min. uncertainity in 1 parameter corresponds to Max. uncertainity in other

$\Delta x_{min} \cdot \Delta p_{max} \geq \frac{h}{4\pi}$

5) Accuracy & Uncertainity

Uncertainity in velocity Δv = velocity × accuracy

= velocity × % error

6) Energy - time uncertainity relation (λ & t terms)

$\Delta E \cdot \Delta t \geq \frac{h}{4\pi}$

$$E = \frac{hc}{\lambda} \Rightarrow \Delta E = \Delta\left(\frac{hc}{\lambda}\right) = hc \cdot \Delta\left(\frac{1}{\lambda}\right) = hc\left(\frac{-1}{\lambda^2}\Delta\lambda\right)$$

$$\Delta t \cdot hc\left(\frac{-1}{\lambda^2} \cdot \Delta\lambda\right) \geq \frac{h}{4\pi}$$

$$\Delta t \cdot \Delta\lambda \geq \frac{-\lambda^2}{4\pi c} \geq \left|\frac{\lambda^2}{4\pi c}\right|$$

7) Energy - time uncertainity relation (ν & t terms)

$E = h\nu$

$\Delta E = \Delta h\nu = h \cdot \Delta\nu$

$\Delta E \cdot \Delta\nu \geq \frac{1}{4\pi}$

Q. In the balmer series of the hydrogen spectra, the longest wavelength spectral line with a wavelength 656 nm is emitted when e^- makes a transition from n = 3 to n = 2 state. This wavelength, even though pure is seen to have a small spread around λ of the order of 10^{-5} nm. Estimate the min. time spent by the electron before dexcitation

A. $\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$

$\Delta E = \frac{hc}{\lambda^2} \cdot \Delta\lambda$

$\Delta E \cdot \Delta t \geq \frac{h}{4\pi}$

$$\Delta t \geq \frac{\cancel{h}}{4\pi} \cdot \frac{\lambda^2}{\cancel{h}c \cdot \Delta\lambda} = \frac{(656 \times 10^{-9})^2}{4\pi \times 3 \times 10^8 \times 10^{-5} \times 10^{-9}}$$

$= 1.14 \times 10^{-8}$ s

$= 11.4$ ns

$$\frac{1}{\Delta\lambda} = R\left(\frac{1}{4} - \frac{1}{9}\right) = 1.097 \times 10^7 \left(\frac{5}{36}\right)$$

$$\Delta\lambda = \frac{36}{1.097 \times 10^7 \times 5} = 0.000065 = 6.5 \times 10^{-7}$$

→ Matter waves of moving bodies can be represented by wave function Ψ depends on position & time

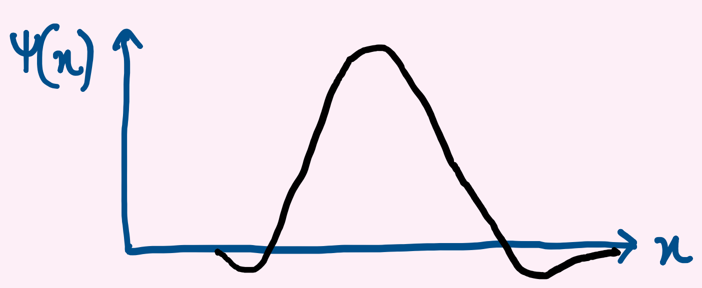
$$\Psi(x,y,z,t) = \psi(x) \cdot \phi(y) \cdot \chi(z) \cdot \varphi(t)$$

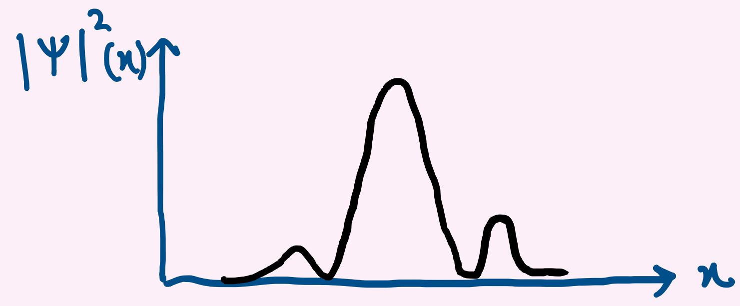
where $\psi(x), \phi(y), \chi(z)$ are orthogonal functions

Wave Function & Probability Density

→ Wave Function can be $+, -$, complex & can change with time

→ $|\psi|^2$ is called probability density which represents probability of finding the particle in unit volume of space





Characteristics of accepted wave functions

→ Finite, Continuous, Single Valued (FCS)

→ Derivatives : Finite, Continuous, Single Valued (dFCS)

→ Normalisable (Normalisable)

Normalization of wave functions

→ Total probability in the range where function is defined has to be unity

$$\int \psi \times \psi \, dx = 1$$

→ Normalisation condition is $\int_{-\infty}^{\infty} \psi \times \psi \, dx = 1$

State Function

→ A Well-behaved function satisfying FCS, dFCS & Normalisable are state functions

$$\psi = A e^{i(kx - \omega t)}$$

$$K = \frac{p}{\hbar} \quad \& \quad \omega = \frac{E}{\hbar}$$

$$\psi = A e^{\frac{i}{\hbar}(px - Et)}$$

Double Slit Experiment

Let, $\psi 1$ & $I1 = |\psi_1|^2$ be wave function & probability of photons in slit 1

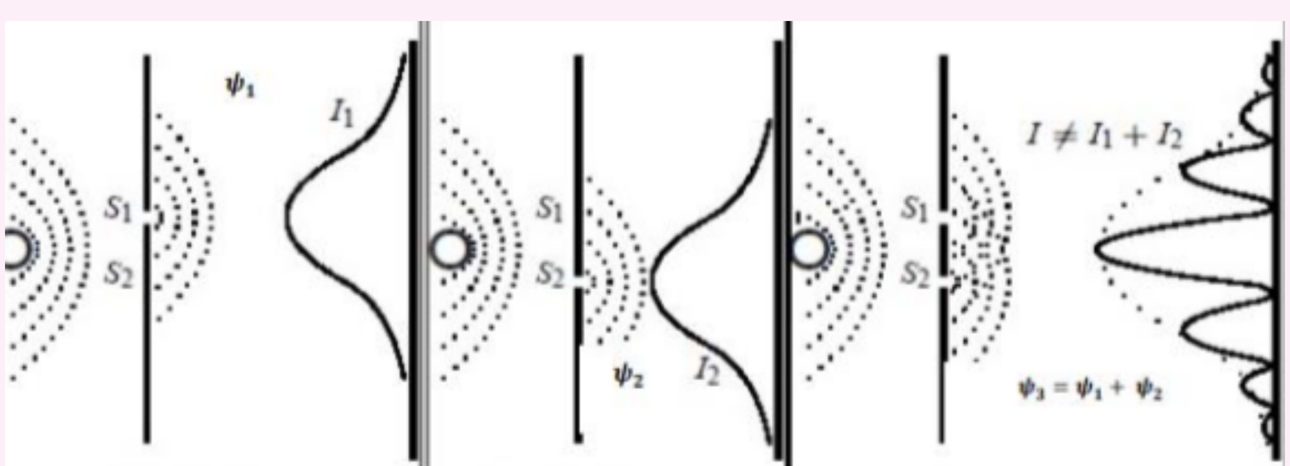
$\psi 2$ & $I2 = |\psi_2|^2$ be wave function & probability of photons in slit 2

$\psi 3 = \psi 1 + \psi 2$ (Superposed wave function for photons from both slits)

But, $I_3 = |\psi_3|^2 \neq I_1 + I_2$

$$I_3 = |\psi_1|^2 + |\psi_2|^2 + \psi_1^* \psi_2 + \psi_1 \psi_2^*$$

So, $|\psi_1 + \psi_2|^2 = (\psi_1^* + \psi_2^*)(\psi_1 + \psi_2)$



Only slit 1 is open   Only slit 2 is open   Both slits are open

10. Operators and Eigen value

Eigen Value

→ A mathematical operator $\hat{G}$ operating on wave function can result in eigen value G of the observable

→ $\hat{G}\psi = G\psi$

→ Operators arise because in quantum mechanics, a system is described with waves & not discrete particles

ex: e^{4x} is eigen function of the operator $\frac{d^2}{dx^2}$,

then, Eigen value equation : $\hat{G}\psi = G\psi$

$\frac{d^2}{dx^2}(e^{4x}) = 16e^{4x}$

Eigen value : 16

ex: $\psi(x) = A\sin kx$ & $\hat{G} = i\hbar\frac{\partial}{\partial x}$

Then, $\hat{G}\psi = G\psi$

$i\hbar\frac{\partial}{\partial x}(A\sin kx) = i\hbar k\, A\cos kx$ ⇒ NOT Eigen Function

Momentum Operator

→ Partial derivative of ψ wrt position

$$\frac{\partial\psi}{\partial x} = \frac{\partial\left(Ae^{\frac{i}{\hbar}(px-Et)}\right)}{\partial x} = \frac{ip}{\hbar}\left(Ae^{\frac{i}{\hbar}(px-Et)}\right) = \left(\frac{ip}{\hbar}\right)\psi$$

$$\Rightarrow \frac{\hbar}{i}\frac{\partial\psi}{\partial x} = p\psi \Rightarrow \left\{-i\hbar\frac{\partial}{\partial x}\right\}\psi = p\psi$$

→ Momentum operator $\hat{p}$ $= -i\hbar\frac{\partial}{\partial x}$

Kinetic Energy Operator

→ 2nd derivative of ψ wrt position

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(Ae^{\frac{i}{\hbar}(px-Et)}\right) = \left(\frac{ip}{\hbar}\right)^2\psi$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{p^2}{2m}\psi = KE\,\psi$$

→ KE operator $\widehat{KE}$ $= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

Total Energy operator

→ Derivative of ψ wrt time

$$\frac{\partial\psi}{\partial t} = \frac{\partial\left(Ae^{\frac{i}{\hbar}(px-Et)}\right)}{\partial t} = \frac{-iE}{\hbar}\psi$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

→ TE operator $\hat{E}$ $= i\hbar\frac{\partial}{\partial t}$ ⇒ Also called Hamiltonian operator $\hat{H}$

Position Operator

→ Corresponds to position observable of a particle

$\hat{x}\psi = x\psi$

Potential Energy Operator

→ It isn't explicitly described.

Eigen value can be expressed difference of T.E & KE

$\hat{V}\psi = V\psi$

Expectation values of observables

→ Expectation values ≡ avg. of repeated measurements of system

→ In general,

operator $\hat{G}$ of observable g gives expectation value of observable $\langle g\rangle = \frac{\int\psi_*G\psi\,dx}{\int\psi_*\psi\,dx}$

$$\Rightarrow \int\psi_*\hat{G}\psi\,dx = \int\psi_*g\psi\,dx = \langle g\rangle\int\psi_*\psi\,dx$$

→ In 3D space, $\langle g\rangle = \frac{\int\psi_*G\psi\,dV}{\int\psi_*\psi\,dV}$

Time independant Schrodinger Equation

$$\rightarrow \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

$$\psi(x,t) = Ae^{i/\hbar(px-Et)} = Ae^{\frac{ipx}{\hbar}}\cdot e^{\frac{-iEt}{\hbar}} = \phi(x)\cdot e^{\frac{-iEt}{\hbar}}$$

Now, $\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\phi(x)\cdot e^{\frac{-iEt}{\hbar}} = i\hbar\frac{\partial}{\partial t}\phi(x)\cdot e^{\frac{-iEt}{\hbar}}$

$= \phi(x)\left\{i\hbar\left(\frac{-i}{\hbar}\right)E\cdot e^{\frac{-it}{\hbar t}}\right.$

So, $\left[\frac{-\hbar^2}{2m^2}\cdot\frac{\partial^2}{\partial x^2} + U(x)\right]\phi(x) = E\phi(x)$

$\hat{H}\phi(x) = E\phi(x)$

U1 problems

? The electric field associated with an EM radiation (light) is given by,
$E(x,t) = 10^3\cos(\omega t - \pi x 3x10^6 z)$
Evaluate
1. Speed of the Electric vector
2. Wavelength
3. Frequency
4. Period of the wave
5. Magnetic field associated with the wave
6. Direction of propagation of the magnetic transverse wave
7. Amplitude of the electric field vector
8. Amplitude and direction of the transverse magnetic wave

A. 1. $V = 3\times10^8\,ms^{-1}$

2. $\lambda = \frac{2\pi}{K} = \frac{2\pi}{3\pi\times10^6} = \frac{3}{3}\times10^{-6}\,m$

3. $\nu = \frac{c}{\lambda} = \frac{3\times10^8}{\frac{2}{3}\times10^{-6}} = 4.5\times10^{14}\,Hz$

4. $T = \frac{1}{\nu} = 2.22\times10^{15}\,s$

5. $B = \frac{E_0}{c} = \frac{10^3}{3\times10^8} = 0.33\times10^{-5}$

6. $\perp^{ar}$ to E (in y-direction)

7. $E_0 = 10^3$

8. $B_0 = 3.33\times10^{-6}$ in y-direction

? The frequency of harmonic oscillator at 50C is 6.2x10^12 per sec. Estimate the average energy of the oscillator as per Planck's idea of cavity oscillator, also compare the same with classical average energy and average energy by R-J law.

A. $T = 273 + 50 = 323\,K$

$\nu = 6.2\times10^{12}\,Hz$

According to planck, $E = \frac{h\nu}{e^{\frac{h\nu}{K_BT}} - 1} = 2.71\times10^{-21}\,J$

According to RJ, $E = K_BT = 4.46\times10^{-12}\,J$

Difference $= 1.744\times10^{-21}\,J$

? X-rays of wavelength 0.112 nm is scattered from a carbon target. Calculate the wavelength of X-rays scattered at an angle 90o with respect to the original direction. What is the energy lost by the X-ray photons? What is the energy gained by the electrons? If the incident x-ray retraces back what will be the shift?

$\lambda = 0.112\times10^{-9}\,m$ $\quad \lambda' = ?$ $\quad$ If $\theta = 60^0$

$\Delta\lambda = \frac{h}{m_ec}(1-\cos\theta) \Rightarrow \lambda' = \lambda + \frac{h}{m_ec}\left(1-\frac{1}{2}\right)$

$= 0.112 + 0.0012$

$= 1.1321\,\mathring{A}$

Energy lost $= \frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$

$= 1.89\times10^{-17}\,J$

$= 118.31\,eV$

? Find the de Broglie wavelength of electrons moving with a speed of 10^7 m/s

$\lambda = \frac{h}{p} = \frac{h}{m_eV} = 7.27\times10^{-11}\,m$

? An alpha particle is accelerated through a potential difference of 1 kV. Find its de Broglie wavelength.

$V = 1000\,V$. $\lambda = ?$

$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mKE}} = \frac{h}{\sqrt{2m_\alpha qV}}$

$m_\alpha = 4m_p$ & $q = 2\times e$ $\quad V = 1000\,V$

$\lambda = \frac{h}{\sqrt{2\times4\times2m_pe\times10^3}} = 4.525\times10^{-13}\,m$

? Compare the momenta and energy of an electron and photon whose de Broglie wavelength is 650nm

$\lambda = 650\,nm$

$p = h/\lambda$ $\qquad E = \frac{p^2}{2m}$

$P_e = P_p = \frac{h}{650\times10^{-9}} = 1.02\times10^{-27}\,Kgms^{-1}$

$K_e = \frac{p^2}{2m_e} = 5.71\times10^{-25}\,Kgm^2s^{-2}$

$K_p = \frac{hc}{\lambda_p} = 3.056\times10^{-19}\,Kgm^2s^{-2}$

$\frac{K_e}{K_p} = 1.868\times10^{-6}$

? Calculate the de Broglie wavelength of electrons and protons if their kinetic energies are i) 1% and ii) 5% of their rest mass energies.

A. rest mass energy of $e^- = m_ec^2 = 8.2\times10^{-14}\,J$

rest mass energy of photon $= m_pc^2 = 1.5\times10^{-10}\,J$

i) $KE_e = 1\%$ of $E = \frac{1}{100}\times E = 8.2\times10^{-16}\,J$

$KE_p = 1\%$ of $E = \frac{1}{100}\times E = 1.5\times10^{-12}\,J$

$\lambda_e = \frac{h}{\sqrt{2mKE_e}} = 1.72\times10^{-11}\,m$; $\lambda_p = \frac{h}{\sqrt{2m\,KE_p}} = 9.36\times10^{-15}\,m$

ii) $KE_e = 5\%$ of $E = \frac{5}{100}\times E = 4.1\times10^{-15}\,J$

$KE_p = 5\%$ of $E = \frac{5}{100}\times E = 7.5\times10^{-12}\,J$

$\lambda_e = \frac{h}{\sqrt{2mKE_e}} = 7.68\times10^{-12}\,m$; $\lambda_p = \frac{h}{\sqrt{2m\,KE_p}} = 4.18\times10^{-15}\,m$

? An electron and a photon have a wavelength of 2A. Calculate their momenta and total energies.

$P_e = \frac{h}{\lambda} = \frac{h}{2\times10^{-10}} = 3.315\times10^{-24}\,Kgms^{-1} = P_p$

$TE_e = \underbrace{m_ec^2}_{PE} + \underbrace{\frac{p^2}{2m_e}}_{KE}$

$= 8.2\times10^{-14}\,J$

$TE_p = \frac{hc}{\lambda} = 9.945\times10^{-16}\,Kgm^2s^{-2}$

? What is the wavelength of an hydrogen atom moving with a mean velocity corresponding to the average kinetic energy of hydrogen atoms under thermal equilibrium at 293K?

$\lambda = \frac{h}{\sqrt{2mKE}} = \frac{h}{\sqrt{2meV}} = \frac{h}{\sqrt{\cancel{2}m\times\frac{3}{\cancel{2}}K_bT}}$

$m_H = 1.67\times10^{-27}\,kg$

$\lambda = \frac{6.63\times10^{-34}}{\sqrt{1.67\times10^{-27}\times3\times5.67\times10^{-8}\times293}}$

$= 1.47\times10^{-10}\,m$

? A wave packet is represented as, $y = 10\,sin(30t - 40x)\,.\,cos(0.3t - 0.5\,x)$
Find the phase and group velocities

$V_p = \frac{\omega}{K} = \frac{30}{40} = 0.75$

$V_g = \frac{d\omega}{dK} = \frac{0.3}{0.5} = 0.6$

? The speed of an electron is measured to be 1 km/s with an accuracy of 0.005%. Estimate the uncertainty in the position of the particle.

$\Delta x\,.\,\Delta p \geq \frac{h}{4\pi}$

$\Delta x\,.\,m\Delta v \geq \frac{h}{4\pi}$

$\Delta x\,.\,\Delta v \geq \frac{h}{4\pi m}$

$\Delta v = v \times$ accuracy $= 1000\,ms^{-1}\times\frac{0.005}{100}$

$= 0.05\,ms^{-1}$

$\Delta x = \frac{h}{4\pi m\,\Delta v} = \frac{6.63\times\cancel{10^{-34}}\,10^{-1}}{4\pi\times9.11\times\cancel{10^{-31}}\times\frac{\cancel{0.05}}{5}}$

$= 1.159\times10^{-3}\,m$

? The spectral line of Hg green is 546.1 nm has a width of 10-5 nm. Evaluate the minimum time spent by the electrons in the upper state before de excitation to the lower state.

$\Delta t\,.\,\Delta\lambda \geq \left|\frac{\lambda^2}{4\pi c}\right|$

$\Delta t \geq \left|\frac{\lambda^2}{4\pi c\,\Delta\lambda}\right| = \frac{(546.1\times10^{-9})^2}{4\pi c\times10^{-5}\times10^{-9}}$

$= \frac{546.1\times546.1\times10^{-9}}{4\pi c\times10^{-5}} \simeq 7.91\times10^{-9}\,s$

? The uncertainty in the location of a particle is equal to it's de Broglie wavelength. Show that the corresponding uncertainty in its velocity is approx one tenth of it's velocity.

$\Delta x = \lambda = \frac{h}{p}$

$p = mv$

$\Delta p = m\Delta v$

$\Delta x\,.\,\Delta p \geq \frac{h}{4\pi}$

$\frac{\cancel{h}}{p}\,.\,\Delta p \geq \frac{\cancel{h}}{4\pi} \Rightarrow \frac{\cancel{m}\,\Delta v}{\cancel{m}\,v} \geq \frac{1}{4\pi} \Rightarrow \Delta v = \frac{v}{12.56}$

$\simeq \frac{v}{10}$

? A proton is confined to a box of length 2 nm. What is the minimum uncertainty in its velocity?

$\Delta x\,.\,\Delta p \geq \frac{h}{4\pi}$

$\Delta x\,.\,m_p\Delta v \geq \frac{h}{4\pi}$

$\Delta v \geq \frac{h}{4\pi}\times\frac{1}{2\times10^{-9}\times m_p}$

$\Delta v \geq 15.76$