Unit-4 FIR FILTERS

FIR Filter

→ Non-recursive filter (Output depends on present & past input)

ex : $y(n) = x(n) + x(n-1) + x(n-2)$

→ $H(z) = \frac{Y(z)}{X(z)} = \sum_{K=0}^{M-1} b_K z^{-K}$ (Only zeros)

All poles lie on origin

Properties

1) It guarantees stability (Poles at origin)
2) It guarantees linear phase
3) Complexity is more compared to IIR (Takes more space & time because it is non-recursive)

1) Stability

BIBO Stability ⇒ $x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ FIR \end{matrix}} \longrightarrow y(n)$

Differential equation ⇒ $y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots$

$$= \sum_{K=0}^{M-1} b_K x(n-K)$$

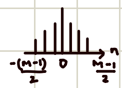
$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$

$$y(n) = \sum_{K=0}^{M-1} h(K)\, x(n-K)$$

Comparing ① & ②, $h(K) = \begin{cases} b_K, & K = 0 \text{ to } M-1 \\ 0, & \text{elsewhere} \end{cases}$ ⇒ b_K bounded, Stability Guaranteed

2) Linearity Property

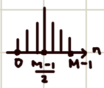
→ $h(n) = h(-n)$ ⇒ Symmetry at $n = 0$



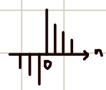
→ If M is length of filter unit sample response,

at $n = \alpha = \frac{M-1}{2}$, M is odd

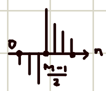
$h(n) = h(M-1-n)$ ⇒ Symmetry at $n = \frac{M-1}{2}$



→ $h(n) = -h(-n)$ ⇒ anti-symmetry at $n = 0$



→ $h(n) = -h(M-1-n)$ ⇒ anti-symmetry at $n = \frac{M-1}{2}$



Type 1 : h(n) is symmetric wrt $\frac{M-1}{2}$ & odd length

$h(n) = h(M-1-n)$ & M is odd

Type 2 : h(n) is symmetric wrt $\frac{M-1}{2}$ & even length

$h(n) = h(M-1-n)$ & M is even

Type 3 : h(n) is anti symmetric wrt $\frac{M-1}{2}$ & odd length

$h(n) = -h(M-1-n)$ & M is odd

Type 4 : h(n) is anti symmetric wrt $\frac{M-1}{2}$ & even length

$h(n) = -h(M-1-n)$ & M is even

Type 1 : $h(n) = h(M-1-n)$ & M is odd

Then, $H(e^{jw}) = \sum_{K=0}^{M-1} h(n)e^{-jwn}$

$$= \sum_{K=0}^{\frac{M-3}{2}} h(n)e^{-jwn} + h\left(\frac{M-1}{2}\right) e^{-jw\left(\frac{M-1}{2}\right)} + \sum_{K=\frac{M+1}{2}}^{M-1} h(n)e^{-jwn} \longrightarrow ②$$

ⓘ ⓘⓘ ⓘⓘⓘ

Consider ⓘⓘⓘ, $\sum_{K=\frac{M+1}{2}}^{M-1} h(n)\, e^{-jwn}$

$$= \sum_{n=\frac{M+1}{2}}^{M-1} h(M-1-n)\, e^{-jwn}$$

Let $K = M-1-n \Rightarrow n = M-1-K$

$$\Rightarrow \sum_{K=\frac{M-3}{2}}^{0} h(K)e^{-jw(M-1-n)}$$

$$H(e^{jw}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-jwn} + h\left(\frac{M-1}{2}\right)e^{-jw\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, e^{-jw(M-1-n)}$$

$$H(e^{jw}) = h\left(\frac{M-1}{2}\right) e^{-jw\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\left(\frac{M-3}{2}\right)} h(n)\left[e^{-jwn} + e^{-jw(M-1-n)}\right]$$

Now, $e^{-jwn} + e^{-jw(M-1-n)}$

$$= e^{-jwn}\, e^{-jw\left(\frac{M-1}{2}\right)}\, e^{jw\left(\frac{M+1}{2}\right)} + e^{-jw\left(\frac{M-1}{2}\right)}\, e^{-jw\left(\frac{M-1}{2}\right)}\, e^{jwn}$$

$$= e^{-jw\left(\frac{M-1}{2}\right)}\left(e^{-jwn}\, e^{jw\left(\frac{M+1}{2}\right)} + e^{-jw\left(\frac{M-1}{2}\right)}\, e^{jwn}\right)$$

$$= e^{-jw\left(\frac{M-1}{2}\right)}\left(e^{jw\left(\frac{M+1}{2}-n\right)} + e^{jw\left(n-\frac{M-1}{2}\right)}\right)$$

$$= 2e^{-jw\left(\frac{M-1}{2}\right)} \cos\left(w\left(\frac{M-1}{2}-n\right)\right)$$

$$H(e^{jw}) = h\left(\frac{M-1}{2}\right)e^{-jw\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left(e^{-jw\left(\frac{M-1}{2}\right)}\, 2\cos\left(w\left(\frac{M-1}{2}-n\right)\right)\right)$$

$$= h\left(\frac{M-1}{2}\right) e^{-jw\left(\frac{M-1}{2}\right)} + 2\sum_{n=0}^{M-3/2} h(n)\cos\left(w\left(\frac{M-1}{2}-n\right)\right) e^{-jw\left(\frac{M-1}{2}\right)}$$

$$= e^{-jw\left(\frac{M-1}{2}\right)}\left(h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\left(w\left(\frac{M-1}{2}-n\right)\right)\right)$$

$$H(e^{jw}) = |H(e^{jw})|\, e^{j\angle H(e^{jw})} = H_r(e^{jw})e^{j\theta w}$$

$$H_r(e^{jw}) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\left(w\left(\frac{M-1}{2}-n\right)\right)$$

$$\theta(w) = \begin{cases} -w\left(\frac{M-1}{2}\right), & H_r(w) \geq 0 \\ -w\left(\frac{M-1}{2}\right)+\pi, & H_r(w) < 0 \end{cases}$$

Phase Shift of π makes H_r -ve

So piecewise linear

Piece wise Linear ⟵ Phase

→ FIR ⇒ linear phase for $h(n) = h(M-1-n)$, M is odd

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right), & H_r(e^{j\omega}) \geq 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi, & H_r(e^{j\omega}) < 0 \end{cases}$$

→ The type of filter designed using type I filter

Let $K = \frac{M-1}{2} - n \Rightarrow \frac{M-1}{2} - K = n$

$n = 0 \Rightarrow K = \frac{M-1}{2}$

$n = \frac{M-3}{2} \Rightarrow K = \frac{M-1}{2} - \frac{M-3}{2} = 1$

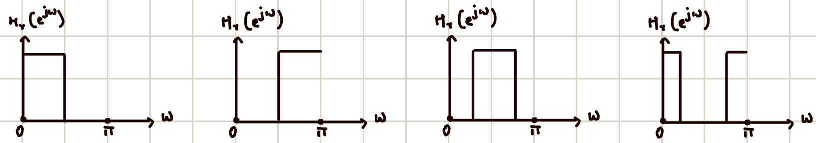
$$H_r(e^{j\omega}) = h\left(\frac{M-1}{2}\right) + 2\sum_{K=\frac{M-1}{2}}^{1} h\left(\frac{M-1}{2}-K\right)\cos\omega K$$

$$= 2\sum_{K=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2}-K\right)\cos\omega K + h\left(\frac{M-1}{2}\right)$$

$$H_r(e^{j\omega}) = \sum_{K=0}^{\frac{M-1}{2}} a(K)\cos\omega K \quad \text{where} \quad a(K) = \begin{cases} h\left(\frac{M-1}{2}\right), & K=0 \\ 2h\left(\frac{M-1}{2}-K\right), & K=1 \text{ to } \frac{M-1}{2} \end{cases}$$

When $\omega = 0$, $H_r(e^{j\omega})$ = Non-zero

$\omega = \pi$, $H_r(e^{j\omega})$ = Non-zero



LP HP BP BS ⇒ All types of filters can be designed

Type 2: $h(n) = h(M-1-n)$ & M is even

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n)e^{-j\omega n} + \sum_{K=\frac{M}{2}}^{M-1} h(n)e^{-j\omega n}$$

(i) (ii)

Consider (ii), $\sum_{K=\frac{M+1}{2}}^{M-1} h(n)e^{-j\omega n} = \sum_{n=\frac{M+1}{2}}^{M+1} h(M-1-n)e^{-j\omega n}$ & $M-1-n = K \Rightarrow n = M-1-K$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{M}{2}-1} h(n)e^{-j\omega(M-1-n)}$$

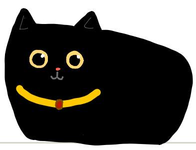
$$= \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(M-1-n)}\right] = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[2e^{-j\omega\left(\frac{M-1}{2}\right)}\cos\left(\omega\left(\frac{M-1}{2}-n\right)\right)\right]$$

$$H_r(e^{j\omega}) = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos\left(\omega\left(\frac{M-1}{2}-n\right)\right)$$ M is even

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right), & H_r(e^{j\omega}) \geq 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi, & H_r(e^{j\omega}) < 0 \end{cases}$$

⇒ Phase $\theta(\omega) \propto \omega$

Linear Phase is guaranteed (Piecewise)

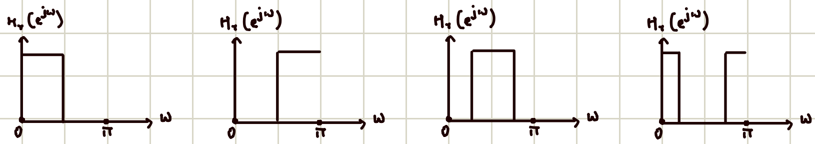


$$H_r(e^{j\omega}) = 2\sum_{k=1}^{M/2} h\left(\frac{M}{2}-k\right)\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$= \sum_{k=1}^{M/2} b_k \cos\left(\omega\left(k-\frac{1}{2}\right)\right) \qquad \left(b_k = 2h\left(\frac{M}{2}-k\right)\right)$$

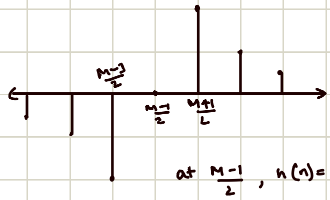
at $\omega=0$, $H_r(e^{j\omega})$ = non-zero

$\omega=\pi$, $H_r(e^{j\omega})$ = zero



LP   ~~HP~~   BP   ~~BS~~   ⇒ LP & BP can be designed

Type 3: $h(n) = -h(M-1-n)$ & M is odd



at $\frac{M-1}{2}$, $h(n)=0$

Then, $H(e^{j\omega}) = \sum_{k=0}^{M-1} h(n)e^{-j\omega n}$

$$= \underbrace{\sum_{k=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n}}_{(i)} + \underbrace{\cancel{h\left(\tfrac{M-1}{2}\right)e^{-j\omega\left(\frac{M-1}{2}\right)}}}_{(ii)\;\to\;0} + \underbrace{\sum_{k=\frac{M+1}{2}}^{M-1} h(n)e^{-j\omega n}}_{(iii)}$$

Consider (iii), $\displaystyle\sum_{k=\frac{M+1}{2}}^{M-1} h(n)e^{-j\omega n}$

$$= \sum_{n=\frac{M+1}{2}}^{M-1} -h(M-1-n)e^{-j\omega n}$$

Let $k = M-1-n \Rightarrow n = M-1-k$

$$\Rightarrow -\sum_{k=\frac{M-3}{2}}^{0} h(k)e^{-j\omega(M-1-k)} = \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega(M-1-n)}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}}\left(h(n)e^{-j\omega n} - h(n)e^{-j\omega(M-1-n)}\right)$$

Now, $e^{-j\omega n} - e^{-j\omega(M-1-n)}$

$$= e^{-j\omega n}e^{-j\omega\left(\frac{M-1}{2}\right)}e^{j\omega\left(\frac{M-1}{2}\right)} - e^{-j\omega\left(\frac{M-1}{2}\right)}e^{-j\omega\left(\frac{M-1}{2}\right)}e^{j\omega n}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)}\left(e^{-j\omega n}e^{j\omega\left(\frac{M-1}{2}\right)} - e^{-j\omega\left(\frac{M-1}{2}\right)}e^{j\omega n}\right)$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)}\left(e^{j\omega\left(\frac{M-1}{2}-n\right)} - e^{j\omega\left(n-\frac{M-1}{2}\right)}\right)$$

$$= 2je^{-j\omega\left(\frac{M-1}{2}\right)}\sin\left(\omega\left(\frac{M-1}{2}-n\right)\right)$$

$$H(e^{j\omega}) = 2je^{-j\omega\left(\frac{M-1}{2}\right)}\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right) = 2e^{j\frac{\pi}{2}}e^{-j\omega\left(\frac{M-1}{2}\right)}\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)$$

$$= 2e^{j\left(\frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)\right)}\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right), & H_r(e^{j\omega}) \geq 0 \\ \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)+\pi, & H_r(e^{j\omega}) < 0 \\ \frac{3\pi}{2}-\omega\left(\frac{M-1}{2}\right) & \end{cases}$$

Piecewise Linear

$$H_r(e^{j\omega}) = 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\left(\frac{M-1}{2}\right)-n\right)$$

Let $k = \frac{M-1}{2}-n \Rightarrow n = \frac{M-1}{2}-k$

when $n=0 \Rightarrow k = \frac{M-1}{2}$

$n = \frac{M-3}{2} \Rightarrow k=1$

$$H_r(e^{j\omega}) = 2\sum_{k=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2}-k\right)\sin\omega k$$

$$= 2\sum_{k=1}^{\frac{M-1}{2}} c(k)\sin\omega k \qquad \text{where } c(k) = h\left(\frac{M-1}{2}-k\right)$$

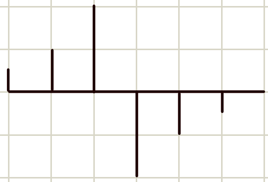
⇒ $\omega=0$, $H_r(e^{j\omega})=0$

$\omega=\pi$, $H_r(e^{j\omega})=0$

So only Bandpass filter can be designed

FIR differentiator & Hilbert Transform can also be design because The FT is imaginary

Type 4 : $h(n) = -h(M-1-n)$ & M is even



$$H(e^{jw}) = \sum_{n=0}^{M-1} h(n) e^{-jwn}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) e^{-jwn} + \sum_{n=\frac{M}{2}}^{M-1} h(n) e^{-jwn}$$

$$\sum_{n=\frac{M}{2}}^{M-1} h(n)\, e^{-jwn} = \sum_{n=\frac{M}{2}}^{M-1} -h(M-1-n) e^{-jwn}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} -h(n) e^{-jw(M-1-n)}$$

Then $$H(e^{jw}) = \sum_{n=0}^{\frac{M}{2}-1} \left(h(n) e^{-jwn} - h(n) e^{-jw(M-1-n)} \right)$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) \left(e^{-jwn} - e^{-jw(M-1-n)} \right)$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n)\, e^{-jw\left(\frac{M}{2}-1\right)}\, 2j \sin w\left(\frac{M-1}{2} - n\right)$$

$$H_r(e^{jw}) = 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin w\left(\frac{M-1}{2} - n\right)$$

$$\theta(w) = \begin{cases} \frac{\pi}{2} - w\left(\frac{M-1}{2}\right) & , H_r(e^{jw}) \geq 0 \\ \frac{\pi}{2} - w\left(\frac{M-1}{2}\right) + \pi & , H_r(e^{jw}) < 0 \\ \frac{3\pi}{2} - w\left(\frac{M-1}{2}\right) & \end{cases}$$

$$= 2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \sin w\left(\frac{M}{2} - n - \frac{1}{2}\right)$$

$k = \frac{M}{2} - n \Rightarrow n = \frac{M}{2} - k$

$n = 0$, $k = \frac{M}{2}$ & $n = \frac{M}{2} - 1$, $k = 1$

$$H_r(e^{jw}) = 2 \sum_{k=1}^{\frac{M}{2}} h\left(\frac{M}{2} - k\right) \sin\left(w\left(k - \frac{1}{2}\right)\right)$$

$$= \sum_{k=1}^{M/2} d(k) \sin\left(w\left(k - \frac{1}{2}\right)\right)$$ where $d(k) = 2h\left(\frac{M}{2} - k\right)$ $k=1$ to $\frac{M}{2}$

$$\left.\begin{aligned} w = 0 &\Rightarrow H_r(e^{jw}) = 0 \\ w = \pi &\Rightarrow H_r(e^{jw}) = \text{non-zero} \end{aligned}\right\} \rightarrow$$

only HP & BP

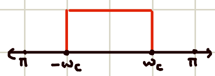
FIR differentiator & Hilbert Transform can also be design because The FT is imaginary

	$\theta(w)$	$H_r(e^{jw})$	Constraints	Types of Filters Designable	
Type - I $h(n) = h(M-1-N)$ M = odd	$\begin{cases} -w\left(\frac{M-1}{2}\right) \\ -w\left(\frac{M-1}{2}\right) + \pi \end{cases}$	$h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) \cos w\left(\frac{M-1}{2} - n\right)$	–	LP, HP, BP, BS	→ Symmetric
Type - II $h(n) = h(M-1-N)$ M = Even	$\begin{cases} -w\left(\frac{M-1}{2}\right) \\ -w\left(\frac{M-1}{2}\right) + \pi \end{cases}$	$2 \sum_{n=0}^{\frac{M}{2}-1} h(n) \cos\left(w\left(\frac{M-1}{2} - n\right)\right)$	$w=0 \Rightarrow H_r(e^{jw}) \neq 0$ $w=\pi \Rightarrow H_r(e^{jw}) = 0$	LP, BP	
Type - III $h(n) = -h(M-1-N)$ M = odd	$\begin{cases} \frac{\pi}{2} - w\left(\frac{M-1}{2}\right) \\ \frac{3\pi}{2} - w\left(\frac{M-1}{2}\right) \end{cases}$	$2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin w\left(\left(\frac{M-1}{2}\right) - n\right)$	$w=0 \Rightarrow H_r(e^{jw}) = 0$ $w=\pi \Rightarrow H_r(e^{jw}) = 0$	BP, FIR Diff, HT	→ Anti - symmetric
Type - IV $h(n) = -h(M-1-N)$ M = Even	$\begin{cases} \frac{\pi}{2} - w\left(\frac{M-1}{2}\right) \\ \frac{3\pi}{2} - w\left(\frac{M-1}{2}\right) \end{cases}$	$2 \sum_{n=1}^{\frac{M}{2}} h\left(\frac{M}{2} - n\right) \sin\left(w\left(n - \frac{1}{2}\right)\right)$	$w=0 \Rightarrow H_r(e^{jw}) = 0$ $w=\pi \Rightarrow H_r(e^{jw}) \neq 0$	HP, BP, FIR Diff, HT	

Desired Unit Sample Response

→ Ideal response is desired response

1) Desired LPF * $H_d(e^{j\omega}) = \begin{cases} 1 & , |\omega| \le \omega_c \\ 0 & , \omega_c < |\omega| \le \pi \end{cases}$



$h_d(n)$ = unit sample response of $H_d(e^{j\omega})$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{\sin \omega_c n}{\pi n}$$

★ $$h_d(n) = \begin{cases} \dfrac{\sin \omega_c n}{\pi n} & , n \ne 0 \\ \dfrac{\omega_c}{\pi} & , n = 0 \end{cases}$$

→ Apply L'Hospital Rule

→ Not causal, Symmetric about $n=0$, ∞ on both sides

In order to make it causal, we shift by $\frac{M-1}{2}$

Then, * $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & , |\omega| \le \omega_c \\ 0 & , \omega_c < |\omega| \le \pi \end{cases}$ where $\alpha = \frac{M-1}{2}$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left(\frac{e^{j(n-\alpha)\omega}}{j(n-\alpha)}\right)_{-\omega_c}^{\omega_c} = \frac{e^{j(n-\alpha)\omega_c} - e^{-j(n-\alpha)\omega_c}}{2\pi j(n-\alpha)} = \frac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)}$$

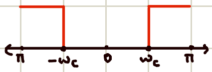
when $n = \alpha$, apply LH rule, $\frac{\omega_c}{\pi}$

* $$h_d(n) = \begin{cases} \dfrac{\sin(n-\alpha)\omega_c}{\pi(n-\alpha)} & , n \ne \alpha \\ \dfrac{\omega_c}{\pi} & , n = \alpha \end{cases}$$

⇒ $h_d(n)$ symmetric wrt $n = \alpha = \frac{M-1}{2}$

∞ on B.S

2) Desired HPF $H_d(e^{j\omega}) = \begin{cases} 1 & , \ \omega_c \le |\omega| \le \pi \\ 0 & , \ |\omega| < \omega_c \end{cases}$



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \int_{\omega_c}^{\pi} e^{j\omega n} d\omega\right] = \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi}\right]$$

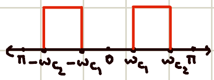
$$= \frac{1}{2j\pi n}\left[e^{-j\omega_c n} - e^{-j\pi n} + e^{j\pi n} - e^{j\omega_c n}\right] = \frac{1}{\pi n} \times \left(\sin \pi n - \sin \omega_c n\right)$$

Applying LH rule, $h_d(n) = \begin{cases} \dfrac{(\sin \pi n - \sin \omega_c n)}{\pi n} \\ \\ \dfrac{\pi - \omega_c}{\pi} \end{cases}$

After shifting, $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & , \ \omega_c \le |\omega| \le \pi \\ 0 & , \ |\omega| < \omega_c \end{cases}$

Then $h_d(n) = \begin{cases} \dfrac{\sin \pi(n-\alpha) - \sin \omega_c (n-\alpha)}{\pi(n-\alpha)} & , \ n \ne \alpha \\ \\ 1 - \dfrac{\omega_c}{\pi} & , \ n = \alpha \end{cases}$

3) Desired BPF $H_d(e^{j\omega}) = \begin{cases} 1 & , \ \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & , \ |\omega| < \omega_{c_1} \ \& \ |\omega| > \omega_{c_2} \end{cases}$



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega n} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega n} d\omega\right] = \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c_2}}^{-\omega_{c_1}} + \left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c_1}}^{\omega_{c_2}}\right]$$

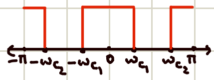
$$= \frac{1}{2j\pi n}\left[e^{-j\omega_{c_1} n} - e^{-j\omega_{c_2} n} + e^{j\omega_{c_2} n} - e^{j\omega_{c_1} n}\right] = \frac{1}{\pi n} \times \left(\sin \omega_{c_2} n - \sin \omega_{c_1} n\right)$$

Applying LH rule, $h_d(n) = \begin{cases} \dfrac{(\sin \omega_{c_2} - \sin \omega_{c_1} n)}{\pi n} \\ \\ \dfrac{\omega_{c_2} - \omega_{c_1}}{\pi} \end{cases}$

After shifting, $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & , \ \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & , \ |\omega| < \omega_{c_1} \ \& \ |\omega| > \omega_{c_2} \end{cases}$

Then $h_d(n) = \begin{cases} \dfrac{\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)}{\pi(n-\alpha)} & , \ n \ne \alpha \\ \\ \dfrac{\omega_{c_2} - \omega_{c_1}}{\pi} & , \ n = \alpha \end{cases}$

4) Desired BSF $H_d(e^{j\omega}) = \begin{cases} 1 , & |\omega| < \omega_{c_1} \ \& \ \pi \geq |\omega| > \omega_{c_2} \\ 0 , & \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \end{cases}$



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\omega_{c_2}}^{-\pi} e^{j\omega n} d\omega + \int_{-\omega_{c_1}}^{\omega_{c_1}} e^{j\omega n} d\omega + \int_{\omega_{c_2}}^{\pi} e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c_2}} + \left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c_1}}^{\omega_{c_1}} + \left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c_2}}^{\pi}\right]$$

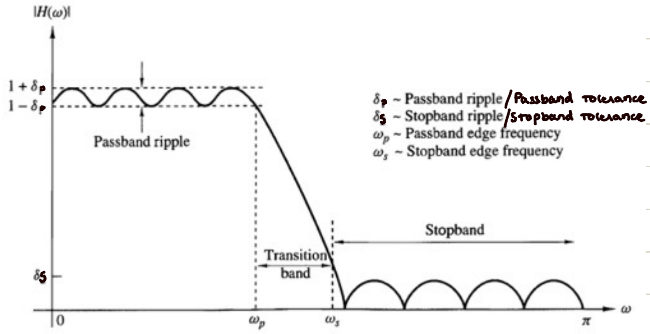
$$= \frac{1}{2j\pi n}\left[e^{-j\omega_{c_2}n} - e^{-j\pi n} + e^{j\omega_{c_1}n} - e^{-j\omega_{c_1}n} + e^{j\pi n} - e^{j\omega_{c_2}n}\right] = \frac{1}{\pi n} \times \left(\sin \pi n + \sin \omega_{c_1} n - \sin \omega_{c_2} n\right)$$

Applying LH rule, $h_d(n) = \begin{cases} \dfrac{(\sin \pi n - \sin \omega_{c_2} + \sin \omega_{c_1} n)}{\pi n} \\ \\ 1 + \dfrac{\omega_{c_2} + \omega_{c_1}}{\pi} \end{cases}$

After shifting, $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} , & |\omega| < \omega_{c_1} \ \& \ \pi \geq |\omega| > \omega_{c_2} \\ 0 , & \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \end{cases}$

Then $h_d(n) = \begin{cases} \dfrac{\sin \pi(n-\alpha) + \sin \omega_{c_1}(n-\alpha) - \sin \omega_{c_2}(n-\alpha)}{\pi(n-\alpha)} , & n \neq \alpha \\ \\ \dfrac{\pi + \omega_{c_1} - \omega_{c_2}}{\pi} , & n = \alpha \end{cases}$

Frequency Response of FIR Filter



at $0 \le \omega \le \omega_p$, $1 - \delta_p \le |H(\omega)| \le 1 + \delta_p$

$\omega_s \le \omega \le \pi$, $|H(\omega)| \le \delta_s$

$\omega_p = \dfrac{2\pi f_p}{F_T}$ $\omega_s = \dfrac{2\pi f_s}{F_T}$ F_T : Sampling Frequency

On a linear scale, Gain $G = |H(e^{j\omega})|$ $\Rightarrow$ in dB , $G = 20\log_{10}|H(e^{j\omega})|$

Attenuation $A = \dfrac{1}{G} = \dfrac{1}{|H(e^{j\omega})|}$ $\Rightarrow$ in dB , $A = 20\log_{10}\left|\dfrac{1}{G}\right| = 20\log_{10}\left|\dfrac{1}{H(e^{j\omega})}\right|$

$= -20\log_{10}|H(e^{j\omega})|$

Absolute Specification

→ $1 - \delta_p \le |H(e^{j\omega})| \le 1 + \delta_p$, $0 \le \omega \le \omega_p$

$|H(e^{j\omega})| \le \delta_s$, $\omega_s \le \omega_s \le \pi$

Relative Specification

→ Absolute Specification is normalized

$\dfrac{1-\delta_p}{1+\delta_p} \le |H(e^{j\omega})| \le 1$ $\Rightarrow$ $0 \le \omega \le \omega_p$

$|H(e^{j\omega})| \le \dfrac{\delta_s}{1+\delta_p}$ $\Rightarrow$ $\omega_s \le \omega \le \pi$

→ Gain , $-A_p \le |H(e^{j\omega})|_{dB} \le 0$, $0 \le \omega \le \omega_p$

$|H(e^{j\omega})|_{dB} \le -A_s$, $\omega_s \le \omega \le \pi$

Attenuation, $|H(e^{j\omega})|_{dB} \le A_p$, $0 \le \omega \le \omega_p$

$|H(e^{j\omega})|_{dB} \ge A_s$, $\omega_s \le \omega \le \pi$

$A_p = 20\log_{10}\left(\dfrac{1+\delta_p}{1-\delta_p}\right) = -20\log_{10}\left(\dfrac{1-\delta_p}{1+\delta_p}\right)$

$A_s = 20\log_{10}\left(\dfrac{1+\delta_p}{\delta_s}\right) = -20\log_{10}\left(\dfrac{\delta_s}{1+\delta_p}\right)$

From this, $\delta_p = \dfrac{10^{A_p/20} - 1}{10^{A_p/20} + 1}$ & $\delta_s = 10^{-A_s/20}(1 + \delta_p)$

Q. A low pass digital filter is specified by the following relative specifications

$\omega_p = 0.3\pi$ $A_p = 0.1\text{dB}$

$\omega_s = 0.5\pi$ $A_s = 35\text{dB}$

Find δ_p & δ_s

A.

$$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} + 1} \quad \& \quad \delta_s = 10^{-A_s/20}(1+\delta_p)$$

$$= \frac{10^{0.1/20} - 1}{10^{0.1/20} + 1} \qquad = 10^{-35/20}(1 + 5.756 \times 10^{-3})$$

$$= 0.0179$$

$$= 5.756 \times 10^{-3}$$

Q. Estimate a filter of a linear phase low pass FIR filter using Kaiser formula with the following specifications

$f_p = 1.8\text{KHz} \rightarrow$ PB edge freq ; Sampling freq, $F_T = 12$ KHz

$f_s = 3$ KHz $\rightarrow$ SB edge freq ;

Max P.B Attenuation, $A_p = 0.1\text{dB}$; Min SB attenuation, $A_s = 35\text{dB}$

A. $$M = \frac{-20\log_{10}(\sqrt{\delta_p \delta_s}) - 13}{14.6 \frac{\Delta\omega}{2\pi}} + 1 \Rightarrow \text{Kaiser Formula} \qquad (\Delta\omega = \omega_s - \omega_p)$$

$$\omega_p = \frac{2\pi f_p}{F_T} = \frac{2\pi(1.8 \times 10^3)}{12 \times 10^3} = 0.3\pi$$

$$\Rightarrow \Delta\omega = 0.2\pi$$

$$\omega_s = \frac{2\pi f_s}{F_T} = \frac{2\pi(3 \times 10^3)}{12 \times 10^3} = 0.5\pi$$

$$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} + 1} = \frac{10^{\frac{0.1}{20}} - 1}{10^{\frac{0.1}{20}} + 1} = 0.0058 \quad ; \quad \delta_s = (\delta_p + 1)10^{-A_s/20} = (1 + 0.0058)10^{-3/20} = 0.0179$$

$$M = \frac{-20\log_{10}(\sqrt{0.0058 \times 0.0179}) - 13}{14.6 \times \frac{0.2\pi}{2\pi}} + 1 = 19.38 = 21$$

(Next odd no.)

↳ For Symmetry, we don't use even no.

Design of FIR filter (windowing method)

→ $x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$

FIR Process

$h(n)$ must be finite length

$H_d(e^{j\omega}) \rightarrow H_d(\omega) \Rightarrow$ Desired frequency response

→ Procedure

$\boxed{H_d(\omega)} \rightarrow \boxed{h_d(n)} \xrightarrow{\text{make finite}} \boxed{h(n)}$

Given — Infinite — FIR

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n} \longrightarrow \text{Desired Response}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \longrightarrow \text{Desired Unit Sample Response}$$

Lets introduce window function,

⇒ Rectangular window function, $w_R(n) = \begin{cases} 1, & n = 0 \text{ to } M-1 \\ 0, & \text{elsewhere} \end{cases}$

$h(n) = h_d(n) \times w_R(n)$ ⟶ Windowing method

$h_d(n) \Rightarrow$ Symmetric about $\frac{M-1}{2}$

$w_R(n) \Rightarrow$ Symmetric about $\frac{M-1}{2}$

$h(n) \Rightarrow$ Symmetric and finite

⇒ $H_d(\omega) \longrightarrow h_d(n) \longrightarrow h_d(n) \times w_R(n) \longrightarrow h(n) \longrightarrow$ FIR filter $n=0$ to $M-1$

Window Function

→ $w_R(n) = \begin{cases} 1, & n = 0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$

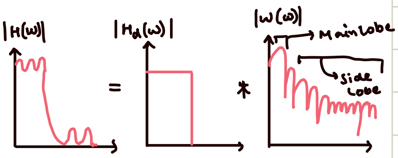
$$W(\omega) = \sum_{n=0}^{M-1} w_R(n) e^{-j\omega n} = \sum_{n=0}^{M-1} (e^{-j\omega})^n = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)} = \frac{e^{-j\omega\frac{M}{2}} \sin\left(\frac{\omega M}{2}\right)}{e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right)}$$

$$= e^{j\theta(\omega)} |W(\omega)|$$

→ $\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right), & |W(\omega)| \geq 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi, & |W(\omega)| < 0 \end{cases}$ ⇒ Piecewise Linear

Linear Phase Guaranteed

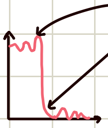
→ $|W(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|}$



$h(n) = h_d(n) \cdot w(n)$

→ In FD, $H(\omega) = H_d(\omega) * W(\omega)$

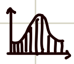
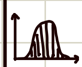
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(v) \cdot W(\omega - v)\, dv$$



→ Large Oscillations are due to side lobe which is called Ringing Effect or Gibbs Phenomenon

→ Abrupt discontinuity of $w_R(n)$ causes side lobes

So to reduce ringing effect, we use window function that gradually decrease to 0

Sl No	Function	Transition Region $\Delta\omega$	Window func^n attenuation	Filter Stopband Attenuation
1	Rectangular $w(n) = \begin{cases} 1, & n=0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{4\pi}{M}$	−13 dB	−21 dB
2 	Bartlett → Triangular $w(n) = \begin{cases} 1 - \frac{2\left\|n-\frac{M-1}{2}\right\|}{M-1}, & n=0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{8\pi}{M}$	−25 dB	−25 dB
3 	Hanning Window $w(n) = \begin{cases} 0.5 - 0.5\cos\left[\frac{2\pi n}{M-1}\right], & n=0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{8\pi}{M}$	−31 dB	−44 dB
4 	Hamming Window $w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right), & n=0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{8\pi}{M}$	−41 dB	−53 dB
5 	Blackman Window $w(n) = \begin{cases} 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}, & n=0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{12\pi}{M}$	−57 dB	−74 dB
6	Kaiser Window $w(n) = \frac{I_0\left[\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0\beta}$ β : wrt side lobe attenu^n N : order	Gives flexibility for $\Delta\omega$, attenuation (No fixed values) β : Helps control side lobe attenuation to reduce ringing effect		

Q. Design FIR filter LPF where length of filter is 7 & cutoff frequency is 1 rad/sec using rectangular window

A. $$H_d(e^{jw}) = \begin{cases} e^{-jw\alpha} & , |w| \leq w_c \\ 0 & , \text{otherwise} \end{cases}$$

$M = 7$

$w_c = 1$ rad/sec

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{jw})\, e^{jwn}\, dw$$

$$= \begin{cases} \dfrac{\sin w_c(n-\alpha)}{\pi(n-\alpha)} & , n \neq \alpha \\ \dfrac{w_c}{\pi} & , n = \alpha \end{cases}$$

$$h(n) = h_d(n) \times w(n) \qquad n: 0 \to M-1$$

$\hookrightarrow$ window function $\quad w(n) = \begin{cases} 1 & , n = 0 \text{ to } M-1 \\ 0 & , \text{otherwise} \end{cases}$

So, $$h_d(n) = \begin{cases} \dfrac{\sin\left(n-\left(\frac{7-1}{2}\right)\right)}{\pi\left(n-\left(\frac{7-1}{2}\right)\right)} = \dfrac{\sin(n-3)}{\pi(n-3)} & , n \neq 3 \\ \dfrac{1}{\pi} & , n = 3 \end{cases}$$

$$\left(\alpha = \frac{M-1}{2} = \frac{7-1}{2}\right)$$

n varies from 0 to $M-1$ $\Rightarrow$ 0 to 6

$$h_d(n) = \{\underset{0}{\overset{\uparrow}{0.0149}},\ \underset{1}{\overset{\uparrow}{0.1447}},\ \underset{2}{\overset{\uparrow}{0.2678}},\ \underset{3}{\overset{\uparrow}{0.3183}},\ \underset{4}{\overset{\uparrow}{0.2678}},\ \underset{5}{\overset{\uparrow}{0.1447}},\ \underset{6}{\overset{\uparrow}{0.0149}}\}$$

$$h(n) = \underbrace{h_d(n)}_{\substack{\text{symmetric} \\ \text{infinite}}} \times w(n)$$

$$h(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} \times w_R(n) & , n \neq 3 \\ \dfrac{w_R(n)}{\pi} & , n = 3 \end{cases}$$

$$= \{0.0149,\ 0.1447,\ 0.2678,\ 0.3183,\ 0.2678,\ 0.1447,\ 0.0149\}$$

Now to prove $h(n)$ is symmetric,

$h(n) = h(M-1-n) \Rightarrow h(n) = h(6-n)$

$h(0) = h(6)$ ✓

$h(1) = h(5)$ ✓ $\quad \Rightarrow$ Hence symmetric

$h(2) = h(4)$ ✓

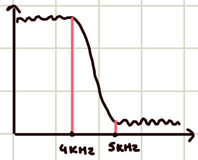
$h(3) = h(3)$ ✓

Q. Design a filter that guarantees linear phase where passband edge frequency is 4 KHz and stopband edge frequency is 5 KHz with atleast 50 dB stopband attenuation. Let sampling frequency be 20 KHz.

A. Stopband attenuation = 50 dB $\Rightarrow$ Hamming window function

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right), & n = 0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta w \geq \frac{8\pi}{M}$$



$$h(n) = h_d(n) \times w_{HM}(n)$$

$$h_d(n) = \begin{cases} \frac{\sin w_c(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ \frac{w_c}{\pi}, & n = \alpha \end{cases}$$

$$w_c = \frac{w_s + w_p}{2}$$

$$w_p = \frac{2\pi f_p}{F_T} = \frac{2\pi \times 4000}{20000} = 1.257 \text{ rad/s}$$

$$w_s = \frac{2\pi f_s}{F_T} = \frac{2\pi \times 5000}{20000} = 1.571 \text{ rad/s}$$

$$w_c = 1.414 \text{ rad/s}$$

$$\Delta w = w_s - w_p = 0.314 = \pi/10$$

$$\Delta w \geq \frac{8\pi}{M} \Rightarrow M \geq \frac{8\pi}{\Delta w} = \frac{8\pi}{\frac{\pi}{10}} = 80 \Rightarrow M \geq 80 \Rightarrow M = 81$$

$$h(n) = \begin{cases} \frac{\sin \frac{9\pi}{20}\left(\frac{81-1}{2} - n\right)}{\pi(40-n)} \times \left(0.54 - 0.46\cos\left(\frac{2\pi n}{80}\right)\right), & n \neq 40 \\ \frac{9}{20}\left(0.54 - 0.46\cos\left(\frac{2\pi n}{80}\right)\right), & n = 40 \end{cases}$$

where n varies from 0 to 80

Q. Design an FIR filter to meet the following Specifications

$f_p = 4\,KHz$, $f_s = 2\,KHz$

$K_p = -2dB$, $K_s = -40dB$, $F_s = 20\,KHz$

A. It is high pass filter because $f_s < f_p$

$$w_{HN}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & , n = 0 \text{ to } M-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) - \sin w_c(n-\alpha)}{\pi(n-\alpha)} \times \left(0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right)\right) & , n \neq \alpha \\ \left(1 - \frac{w_c}{\pi}\right)\left(0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right)\right) & , n = \alpha \end{cases}$$

where n varies from 0 to $M-1$

$$w_p = \frac{2\pi f_p}{F_T} = \frac{2\pi}{5} \text{ rad/s}$$

$$w_s = \frac{2\pi f_s}{F} = \frac{\pi}{5} \text{ rad/s}$$

$$\Delta w = \frac{\pi}{5} \quad \& \quad w_c = \frac{2\pi + \pi}{5 \times 2} = \frac{3\pi}{10}$$

$$\Delta w \geq \frac{8\pi}{M} \Rightarrow M \geq \frac{8\pi}{\Delta w} = \frac{8\pi}{\frac{\pi}{5}} = 40$$

$M = 41$

$$\alpha = \frac{M-1}{2} = 20$$

$$h(n) = \begin{cases} \dfrac{\sin\pi(n-20) - \sin\left(\frac{3\pi}{10}\right)(n-20)}{\pi(n-20)} \times \left(0.5 - 0.5\cos\left(\frac{2\pi n}{40}\right)\right) & , n \neq 20 \\ (0.7)\left(0.5 - 0.5\cos\left(\frac{2\pi n}{40}\right)\right) & , n = 20 \end{cases}$$

$$1 - \frac{3\cancel{\pi}}{10\cancel{\pi}} = 1 - 0.3 = 0.7$$

Q. Design a HP FIR filter, $M = 5$, $w_c = 1.5$ rad/s using hanning window

A. $$w_{HM}(n) = 0.50 - 0.50\cos\left(\frac{2\pi n}{M-1}\right)$$

$$h_D(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) - \sin w_c(n-\alpha)}{\pi(n-\alpha)} & , n \neq \alpha \\ 1 - \frac{w_c}{\pi} & , n = \alpha \end{cases} \qquad \left(\alpha = \frac{5-1}{2} = 2\right)$$

$$h(n) = \begin{cases} \left(\dfrac{\sin\pi(n-2) - \sin 1.5(n-2)}{\pi(n-2)}\right)\left(0.5 - 0.5\cos\left(\frac{2\pi n}{4}\right)\right) & , n \neq 2 \\ 1 - \frac{1.5}{\pi} & , n = 2 \end{cases}$$

$$h(n) = \{0, -0.1587, 0.5225, -0.1587, 0\}$$

Q. Design Band Pass filter, $M=7$, $\omega_{c_1} = 0.3$ rad $\omega_{c_2} = 0.65$ rad use a hamming window in designing.

A. $$h_d(n) = \begin{cases} \dfrac{\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\[2ex] \dfrac{\omega_{c_2}-\omega_{c_1}}{\pi}, & n = \alpha \end{cases}$$

$$\alpha = \frac{7-1}{2} = 3$$

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M-1}\right), & n = 0 \text{ to } M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} \left(\dfrac{\sin 0.65(n-3) - \sin 0.3(n-3)}{\pi(n-3)}\right) \times \left(0.54 - 0.46\cos\left(\dfrac{2\pi n}{6}\right)\right), & n \neq \alpha \\[2ex] \left(\dfrac{0.35}{\pi}\right) \times \left(0.54 - 0.46\cos\left(\dfrac{2\pi n}{6}\right)\right), & n = \alpha \end{cases}$$

$$h(n) = \{0.00123,\ 0.01965,\ 0.0758,\ 0.1114,\ 0.0758,\ 0.01965,\ 0.00123\}$$

Q. Design Band stop filter $M=5$, $\omega_{c_1} = 0.5$ rad, $\omega_{c_2} = 0.75$, use Hanning window.

A. $$h(n) = \begin{cases} \dfrac{\sin\pi(n-\alpha) + \sin\omega_{c_1}(n-\alpha) - \sin\omega_{c_2}(n-\alpha)}{\pi(n-\alpha)} \times \left(0.5 - 0.5\cos\left(\dfrac{2\pi n}{M-1}\right)\right), & n \neq \alpha \\[2ex] \dfrac{\pi + \omega_{c_1} - \omega_{c_2}}{\pi} \times \left(0.5 - 0.5\cos\left(\dfrac{2\pi n}{M-1}\right)\right), & n = \alpha \end{cases}$$

$\alpha = 2$

$$h(n) = \begin{cases} \dfrac{\sin\pi(n-2) + \sin 0.5(n-2) - \sin 0.75(n-2)}{\pi(n-2)} \times \left(0.5 - 0.5\cos\left(\dfrac{2\pi n}{4}\right)\right) \\[2ex] \dfrac{\pi - 0.25}{\pi} \times \left(0.5 - 0.5\cos\left(\dfrac{2\pi n}{4}\right)\right) \end{cases}$$

$$h(n) = \{0,\ -0.0321,\ 0.9204,\ -0.0321,\ 0\}$$

Q. Design FIR bandpass filter where lower stop band : 0-500 Hz

pass band : 1600-2300 Hz

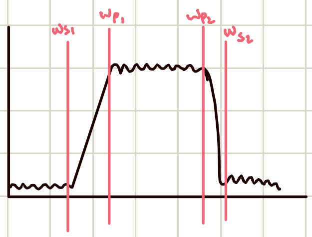
upper stop band : 3500-4000 Hz

Passband attenuation (ripple): 0.05 dB

Stopband attenuation : 50 dB

Sampling rate : 8 KHz

A.



$$\omega_{p_1} = \frac{2\pi \times 1600}{8000} = \frac{2\pi}{5} \qquad \omega_{p_2} = \frac{2\pi \times 2300}{8000} = \frac{23\pi}{40}$$

$$\omega_{s_1} = \frac{2\pi \times 500}{8000} = \frac{\pi}{8} \qquad \omega_{s_2} = \frac{2\pi \times 3500}{8000} = \frac{7\pi}{8}$$

$$\Delta\omega = \min(\Delta\omega_1, \Delta\omega_2)$$

$$\Delta\omega_1 = \omega_{p_1} - \omega_{s_1} = \frac{2\pi}{5} - \frac{\pi}{8} = \frac{11\pi}{40}$$

$$\Delta\omega_2 = \omega_{s_2} - \omega_{p_2} = \frac{7\pi}{8} - \frac{23\pi}{40} = \frac{12\pi}{40}$$

$$\omega_{c_1} = \omega_{p_1} - \frac{\Delta\omega}{2} = \frac{2\pi}{5} - \frac{11\pi}{80} = \frac{21}{80}\pi$$

$$\omega_{c_2} = \omega_{p_2} + \frac{\Delta\omega}{2} = \frac{23\pi}{40} + \frac{11\pi}{80} = \frac{57}{80}\pi$$

$$\Delta\omega = \frac{11\pi}{40}$$

Hamming window : $\Delta\omega \geq \frac{8\pi}{M} \Rightarrow M \geq \frac{8\pi}{\frac{11\pi}{40}} = 29.09 \Rightarrow M = 31 \Rightarrow \alpha = \frac{31-1}{2} = 15$

$$h_d(n) = \begin{cases} \dfrac{\sin\frac{57\pi}{80}(n-15) - \sin\frac{21\pi}{80}(n-15)}{\pi(n-15)}, & n \neq 15 \\[2ex] \dfrac{36}{80}, & n = 15 \end{cases}$$

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{n\pi}{15}\right), & n = 0 \text{ to } 40 \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} \left(\dfrac{\sin\frac{57\pi}{80}(n-15) - \sin\frac{21\pi}{80}(n-15)}{\pi(n-15)}\right)\left(0.54 - 0.46\cos\left(\dfrac{n\pi}{15}\right)\right) \\[2ex] \dfrac{36}{80}\left(0.54 - 0.46\cos\left(\dfrac{n\pi}{15}\right)\right) \end{cases}$$

Designing an FIR Filter using Kaiser Window

→ $w_K(n) = K(\beta, N) = \dfrac{I_0\left[\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0[\beta]}$ $\left(\alpha = \dfrac{M-1}{2} \quad n : 0 \to M-1\right)$

β: Side lobe attenuation

N: width of main lobe

$I_0\{x\} \longrightarrow 0^{th}$ order bessel function

$I_0\{x\} \longrightarrow 1 + \sum_{n=1}^{\infty}\left[\left(\frac{x}{2}\right)^n \frac{1}{n!}\right]^2$

Design Procedure

→ $\delta_p \to$ Passband ripple

$A_p \to$ Passband ripple in dB

$A_s \to$ Stopband attenuation in dB

$$\delta_p = \frac{10^{A_p/20} - 1}{10^{A_p/20} + 1}$$

$$\delta_s = (1+\delta_p)\, 10^{-A_s/20}$$

$$\delta = \min(\delta_p, \delta_s)$$

Attenuation $A_{dB} = -20 \log_{10} \delta$

$M = \dfrac{A - 8}{2.285\, \Delta\omega}$ where $\Delta\omega = \omega_s - \omega_p$

$$\beta = \begin{cases} 0.1102\,(A - 8.7) & , A > 50 \\ 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21) & , 21 \le A \le 50 \\ 0 & , A < 21 \end{cases}$$

β: Degree of taper

→ $h(n) = h_d(n)\, w_K(n)$

$$h(n) = \begin{cases} h_d(n) \times \dfrac{I_0\left[\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right]}{I_0[\beta]} & , n \neq \alpha \\ h_d(n) \times w_K(n) & , n = \alpha \end{cases}$$

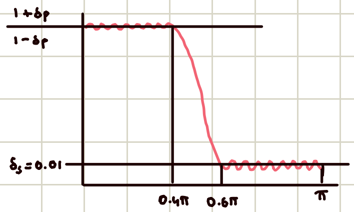
$h(n)$: Unit sample response of filter

Q. Design an FIR Low pass filter to meet the following specifications using kaiser window

$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad |\omega| \le 0.4\pi$

$|H(e^{j\omega})| \le 0.01 \qquad 0.6\pi \le |\omega| \le \pi$

A.



$1 + \delta_p = 1.01$

$\delta_p = 0.01$

$\delta = \min(\delta_p, \delta_s) = 0.01$

$A = -20 \log_{10} \delta = -20 \log_{10}(0.01) = 40\,dB$

$\Delta\omega = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$

$\omega_c = \dfrac{\omega_s + \omega_p}{2} = \dfrac{0.6\pi + 0.4\pi}{2} = \dfrac{\pi}{2}$

$M \geq \dfrac{A - 8}{2.285\,\Delta\omega} \geq \dfrac{40 - 8}{2.285 \times 0.2\pi} \geq 22.28 \qquad M = 23$

$\beta = 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21)$

$= 3.395$

$h(n) = h_d(n) \times \omega_k(n)$

$\omega_k(n) = \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right)}{I_0(\beta)}$

$\alpha = \dfrac{23-1}{2} = 11$

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\[2ex] \dfrac{\omega_c}{\pi}, & n = \alpha \end{cases}$$

$$h(n) = \begin{cases} \dfrac{\sin \frac{\pi}{2}(n-11)}{\pi(n-11)} \times \dfrac{I_0\left(3.4\sqrt{1-\left(\frac{n-11}{11}\right)^2}\right)}{I_0(3.4)}, & n \neq 11 \\[2ex] \dfrac{1}{2} \times \dfrac{I_0\left(3.4\sqrt{1-\left(\frac{n-11}{11}\right)^2}\right)}{I_0(3.4)}, & n = 11 \end{cases}$$

Q. Design L.P FIR Filter with cutoff freq $\omega_c = \frac{\pi}{4}$ rad & $\Delta\omega = 0.02\pi$ & stopband ripple/tolerance $\delta_s = 0.01$. Use kaiser window in the design

A.

$$h(n) = \begin{cases} \dfrac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \times \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right)}{I_0(\beta)}, & n \neq \alpha \\[2ex] \dfrac{\omega_c}{\pi} \times \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n-\alpha}{\alpha}\right)^2}\right)}{I_0(\beta)}, & n = \alpha \end{cases}$$

$A = -20\log_{10}(0.01) = 40$

$M \geq \dfrac{A-8}{2.285\,\Delta\omega} \geq \dfrac{40-8}{2.285 \times 0.02\pi} \geq 222.88$

$M = 223$

$\alpha = \dfrac{223-1}{2} = 111$

$\beta = 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21)$

$= 3.395 \approx 3.4$

$$h(n) = \begin{cases} \dfrac{\sin \frac{\pi}{4}(n-111)}{\pi(n-111)} \times \dfrac{I_0\left(3.4\sqrt{1-\left(\frac{n-111}{111}\right)^2}\right)}{I_0(3.4)}, & n \neq 111 \\[2ex] \dfrac{1}{4} \times \dfrac{I_0\left(3.4\sqrt{1-\left(\frac{n-111}{111}\right)^2}\right)}{I_0(3.4)}, & n = 111 \end{cases}$$

Designing an FIR filter using frequency sampling method

$$\rightarrow \quad H(e^{j\omega})\Big|_{\omega=\frac{2\pi K}{M}} = H(K)$$

$$H(K) = H(\omega)\Big|_{\omega=\frac{2\pi K}{M}}$$

$$h(n) = \text{IDFT}\{H(K)\}$$

$$h(n) = \frac{1}{M}\sum_{K=0}^{M-1} H(K)\, e^{\frac{j2\pi Kn}{M}}$$

$$= \frac{1}{M}\left[H(0) + H(1)e^{\frac{j2\pi n}{M}} + H(2)e^{\frac{j2\pi 2n}{M}} + H(3)e^{\frac{j2\pi 3n}{M}} + \dots + H(M-2)e^{\frac{j2\pi(M-2)n}{M}} + H(M-1)e^{\frac{j2\pi(M-1)n}{M}}\right]$$

We know, $H(K) = H^*(M-K)$

$$h(n) = \frac{1}{M}\left[H(0) + \left(H(1)e^{\frac{j2\pi n}{M}} + H^*(1)e^{\frac{-j2\pi n}{M}}\right) + \left(H(2)e^{\frac{j2\pi(2n)}{M}} + H^*(2)e^{\frac{-j2\pi(2n)}{M}}\right) + \dots\right]$$

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} \text{Re}\left\{H(K)e^{\frac{j2\pi Kn}{M}}\right\}\right] \quad \text{when } M \text{ is odd}$$

and,

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M}{2}-1} \text{Re}\left\{H(K)e^{\frac{j2\pi Kn}{M}}\right\} + \cancel{H\left(\frac{M}{2}\right)}^{\,0}\right] \quad \text{when } M \text{ is even}$$

Q. Design a L.P FIR filter using frequency sampling method for the desired frequency responsed given $H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 \le |\omega| \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$

A. $\alpha = 3 \Rightarrow M = 2\alpha + 1 = 7$

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 \le |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

$$H(K) = H_d(\omega)\Big|_{\omega=\frac{2\pi K}{7}}$$

$$= \begin{cases} e^{-j3\left(\frac{2\pi K}{7}\right)}, & 0 \le \frac{2\pi K}{7} \le \pi/2 \\ 0, & \pi/2 \le \frac{2\pi K}{7} \le \pi \end{cases}$$

$$= \begin{cases} e^{-j\frac{6\pi K}{7}}, & 0 \le K \le 1.75 \Rightarrow K = 0,1 \\ 0, & 1.75 < K \le 3.5 \Rightarrow K = 2,3 \end{cases}$$

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} \text{Re}\left(H(K)e^{\frac{j2\pi Kn}{M}}\right)\right]$$

$$= \frac{1}{7}\left[1 + 2\sum_{K=1}^{3} \text{Re}\left\{H(K)e^{\frac{j2\pi Kn}{7}}\right\}\right]$$

$$= \frac{1}{7}\left[1 + 2\,\text{Re}\left\{e^{\frac{-j6\pi}{7}}e^{\frac{j2\pi n}{7}}\right\}\right] = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right]$$

$$h(n) = \{-0.114,\ 0.079,\ 0.321,\ 0.428,\ 0.321,\ 0.079,\ -0.114\}$$

Q. Design L.P FIR filter where $f_c = 5KHz$, $F_T = 18KHz$, $M = 9$. Find coefficients

A. $\omega_c = \frac{2\pi f_c}{F_T} = \frac{10\pi}{18} = \frac{5\pi}{9}$

$M = 9 \Rightarrow \alpha = 4$

$$H_d(\omega) = \begin{cases} e^{-j4\omega}, & 0 \leq |\omega| \leq \frac{5\pi}{9} \\ 0, & \frac{5\pi}{9} < |\omega| \leq \pi \end{cases}$$

$$H(K)\Big|_{\omega = \frac{2\pi K}{9}} = \begin{cases} e^{-j4\left(\frac{2\pi K}{9}\right)}, & 0 \leq \left|\frac{2\pi K}{9}\right| \leq \frac{5\pi}{9} \\ 0, & \frac{5\pi}{9} < \left|\frac{2\pi K}{9}\right| \leq \pi \end{cases}$$

$$= \begin{cases} e^{-j\frac{8\pi K}{9}}, & 0 \leq K \leq 2.5 \Rightarrow K = 0,1,2 \\ 0, & 2.5 < K \leq 4.5 \Rightarrow K = 3,4 \end{cases}$$

$$h(n) = \frac{1}{9}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} Re\left(H(K)e^{\frac{j2\pi Kn}{M}}\right)\right]$$

$$= \frac{1}{9}\left[1 + 2\sum_{K=1}^{4} Re\left\{H(K)e^{\frac{j2\pi Kn}{9}}\right\}\right]$$

$$= \frac{1}{9}\left[1 + 2\,Re\left(e^{-j\frac{8\pi}{9}}e^{j\frac{2\pi n}{9}} + e^{-j\frac{16\pi}{9}}e^{j\frac{4\pi n}{9}}\right)\right]$$

$$= \frac{1}{9}\left[1 + 2\,Re\left(e^{j\frac{2\pi}{9}(n-4)} + e^{j\frac{4\pi}{9}(n-4)}\right)\right]$$

$$= \frac{1}{9}\left[1 + 2\cos\frac{2\pi}{9}(n-4) + 2\cos\frac{4\pi}{9}(n-4)\right]$$

Q. Design Bandpass filter using frequency sampling method

$F_1 = 1KHz$ $\quad F_2 = 3KHz$

$F_T = 8KHz$ $\quad M = 7$

A. $$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & , \ \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \\ 0 & , \ \omega_{c_2} < |\omega| \leq \pi \end{cases}$$

$$H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{7}} = \begin{cases} e^{-j3\left(\frac{2\pi K}{7}\right)} & , \ \frac{\pi}{4} \leq \frac{2\pi K}{7} \leq \frac{3\pi}{4} \\ 0 & , \ \frac{3\pi}{4} \leq \frac{2\pi K}{7} \leq \pi \ \& \ \frac{2\pi K}{7} < \frac{\pi}{4} \end{cases}$$

$$= \begin{cases} e^{-j\frac{6\pi K}{7}} & , \ \frac{7}{8} \leq K \leq \frac{21}{8} \Rightarrow 1, 2 \\ 0 & , \ \frac{21}{8} \leq K \leq 3.5 \ \& \ K < \frac{7}{8} \Rightarrow 0, 3 \end{cases}$$

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} \text{Re}\left(H(K)e^{j\frac{2\pi K n}{M}}\right)\right]$$

$$= \frac{1}{7}\left[0 + 2\sum_{K=1}^{2} \text{Re}\left(e^{-j\frac{6\pi K}{7}} e^{j\frac{2\pi K n}{7}}\right) + 2(0)\right]$$

$$= \frac{1}{7}\left[2\sum_{K=1}^{2} \text{Re}\left(e^{j\frac{2\pi K}{7}(n-3)}\right)\right]$$

$$= \frac{2}{7}\left[\cos\left(\frac{2\pi}{7}(n-3)\right) + \cos\left(\frac{4\pi}{7}(n-3)\right)\right]$$

$$h(n) = \{-0.079, -0.321, 0.114, 0.571, 0.114, -0.321, -0.079\}$$

Location of zeros in FIR filter

→ All poles lie at origin. FIR filters have only zeros

→ $h(n) = \pm h(M-1-n)$

$h(n) \xleftrightarrow{ZT} H(z)$

$h(M-1+n) \longleftrightarrow \pm z^{(M-1)} H(z)$ (time-shift)

$h(M-1-n) \longleftrightarrow \pm z^{-(M-1)} H(z^{-1})$ (time-reversal property)

$H(z) = \pm z^{-(M-1)} H(1/z)$

→ Any zero will appear as a reciprocal pair

→ When there is zero is at z_1, then there is zero at $\frac{1}{z_1}$ where $|z_1| = 1$

→ If there is a real zero at z_2, then $\frac{1}{z_2}$ is also a zero where $|z_2| < 1$

→ If there is a complex zero at z_3, then zero at $\frac{1}{z_3} = z_3^*$

→ If there is a complex zero at z_4, where $|z_4| \neq 1$, then there will be 4 zeros at z_4, z_4^*, $\frac{1}{z_4}$, $\frac{1}{z_4^*}$

Group Delay & Phase Delay

→ $x(n)$ — LTI — $y(n)$
- Phase Delay (Single freq. component, $\tau_p(\omega) = \frac{-\theta(\omega)}{\omega}$)
- Group Delay (Multiple freq. component, $\tau_g(\omega) = \frac{-d\theta(\omega)}{d\omega}$, same delay)
  - ↳ Not harmonically related

→ In case of an LTI system, Phase & Group delay are constant

Q. $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

A. $H(\omega) = H(z)\Big|_{z=e^{j\omega}} = 1 + 2(e^{-j\omega}) + 2e^{-2j\omega} + e^{-3j\omega}$

$$= e^{-j\frac{3\omega}{2}}\left(e^{j\frac{3\omega}{2}} + 2e^{j\frac{\omega}{2}} + 2e^{-j\frac{\omega}{2}} + e^{-j\frac{3\omega}{2}}\right)$$

$$= e^{j\theta(\omega)}\,|H(e^{j\omega})|$$

$$\theta(\omega) = \frac{-3\omega}{2}$$

$$\left.\begin{aligned} \tau_p(\omega) &= \frac{-\theta(\omega)}{\omega} = \frac{+3\omega}{2\omega} = 1.5 \\ \tau_g(\omega) &= \frac{-d\theta(\omega)}{d\omega} = \frac{3}{2} = 1.5 \end{aligned}\right\} \rightarrow \tau_p(\omega) = \tau_g(\omega) \Rightarrow \text{LTI System}$$

Phase Delay

$$\tau_p(\omega) = \frac{-\angle H(e^{j\omega})}{\omega} = \frac{-\theta(\omega)}{\omega}$$

If −ve phase response for +ve freq.

$x(n)$ — LTI — $y(n)$ experiences a delay

If +ve phase response for −ve freq.

$x(n)$ — LTI — $y(n)$ signal is advanced

Group Delay

$x(n)$ — LTI — $\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = \frac{-dH(\omega)}{d\omega}$

different frequency components

FIR Differentiator

→ Ideal frequency response to find the derivative using window method

→ Steps : $H_d(e^{jw}) \longrightarrow h_d(n) \longrightarrow \overset{h_d'(n)}{h_d(n-\alpha)} \longrightarrow h(n) = h_d'(n) \times w(n)$

$h_d(n)$ ↓ $n=0$

$h_d(n-\alpha)$ ↓ $n \neq \alpha$ Infinite & anti-symmetric wrt α

$h(n)$ ↓ Finite length & anti-symmetric wrt α

$$H_d(e^{jw}) = jw \qquad |w| \leq \pi$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{jw})\, e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} jw\, e^{jwn}\, dw$$

$$= \begin{cases} \dfrac{\cos n\pi}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

$h_d(n) = -h_d(-n)$ because it is anti-symmetric and M must be odd length signal

$$h_d'(n) = h_d(n-\alpha) = \begin{cases} \dfrac{\cos(n-\alpha)\pi}{n-\alpha}, & n \neq \alpha \\ 0, & n = \alpha \end{cases}$$

$h(n) = h_d'(n) \times w(n)$

↳ Finite & anti-symmetric about α

$h(n) = -h(M-1-n)$ M is odd

$$|H(w)| = \left| 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin w\left(\frac{M-1}{2} - n\right) \right|$$

Q. Design an FIR Differentiator (7-tap) using Hamming window. The ideal frequency response of an ideal FIR differentior is given below:

$H_d(w) = jw \quad , \quad -\pi \leq w \leq \pi$

Find an equation for magnitude response of the system

A. $H_d(w) = jw \quad , \quad -\pi \leq w \leq \pi$

$$h_d'(n) = \begin{cases} \dfrac{\cos(n-\alpha)\pi}{n-\alpha} & , \; n \neq \alpha \\ 0 & , \; n = \alpha \end{cases}$$

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$$

$$h(n) = h_d'(n) \times w(n)$$

$$= \begin{cases} \dfrac{\cos(n-3)\pi}{n-3} \times \left(0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right)\right) & , \; n \neq 3 \\ 0 & , \; n = 3 \end{cases}$$

$$= \{0.026, -0.155, 0.77, 0, -0.77, 0.155, -0.026\} \quad)$$

$$|H(w)| = \left| 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin\left(w\left(\frac{M-1}{2} - n\right)\right) \right|$$

$$= \left| 2 \sum_{n=0}^{2} h(n) \sin\left(w(3-n)\right) \right|$$

$$= \left| 2\left(0.026\sin 3w - 0.155\sin 2w + 0.77\sin w\right) \right|$$

Hilbert Transform

→ All pass transform

→ 90° Phase shifter

→ Ideal frequency response $H_d(\omega) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases}$

→ Steps : $H_d(\omega) \longrightarrow h_d(n) \xrightarrow{n-\alpha} h_d'(n) \longrightarrow h(n) = h_d'(n) \times w(n)$

$h_d(n)$ ↓ anti-symm about $n=0$ & odd

$h_d'(n)$ ↓ anti-symm about $n=\alpha$ & odd

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j e^{j\omega n} d\omega + \int_{0}^{\pi} (-j) e^{j\omega n} d\omega\right]$$

$$= \begin{cases} \frac{2}{n\pi}\sin^2\left(\frac{n\pi}{2}\right), & n \neq 0 \\ 0, & n = 0 \end{cases} \qquad -\infty < n < \infty$$

$h_d(n)$ is antisymmetric wrt $n=0$ & $h_d(n) = -h_d(-n)$ & M is odd

$$h_d'(n) = h_d(n-\alpha)$$

$$= \begin{cases} \dfrac{2\sin^2\left(\frac{n-\alpha}{2}\pi\right)}{(n-\alpha)\pi}, & n \neq \alpha \\ 0, & n = \alpha \end{cases}$$

$$h(n) = h_d'(n) \times w(n)$$

$$|H(\omega)| = \left| 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin\left(\omega\left(\frac{M-1}{2} - n\right)\right) \right|$$

Q. Design 11 tap Hilbert Transformer using a rectangular window

$$H_d(\omega) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases}$$

a) Find unit sample response of HT

b) Find an expression for magnitude response of HT

c) Find transfer function of HT $\quad H(z) = \sum_{n=0}^{M-1} h(n)\, z^{-n}$

A. a) $$H_d(\omega) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases}$$

$$h_d'(n) = \begin{cases} \dfrac{2\sin^2\left(\frac{n-\alpha}{2}\pi\right)}{(n-\alpha)\pi}, & n \ne \alpha \\ 0, & n = \alpha \end{cases} \qquad \alpha = \frac{M-1}{2} = \frac{11-1}{2} = 5$$

$$= \begin{cases} \dfrac{2\sin^2\left(\frac{n-5}{2}\pi\right)}{(n-5)\pi}, & n \ne 5 \\ 0, & n = 5 \end{cases}$$

$$= \{\underset{\uparrow_0}{-0.127}, 0, -0.212, 0, -0.636, \underset{\uparrow_5}{0}, 0.636, 0, 0.212, 0, \underset{\uparrow_{10}}{0.127}\}$$

b) $$|H(\omega)| = \left| 2 \sum_{n=0}^{\frac{M-3}{2}} h(n) \sin\left(\omega\left(\tfrac{M-1}{2} - n\right)\right) \right|$$

$$= \left| 2\left(-0.127 \sin 5\omega - 0.212 \sin 3\omega - 0.636 \sin\omega\right) \right|$$

c) $$H(z) = \sum_{n=0}^{10} h(n) \cdot z^{-n}$$

$$= -0.127 - 0.212 z^{-2} - 0.636 z^{-4} + 0.636 z^{-6} + 0.212 z^{-8} + 0.127 z^{-10}$$

Realization of FIR filter

i) Direct form realization

ii) Cascaded realization

iii) Lattice realization

→ Direct Form

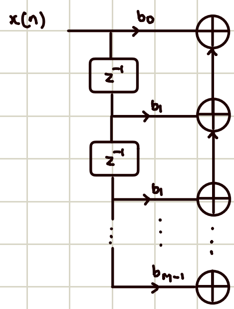
$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{M-1} z^{-M+1}$$

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{M-1} z^{-M+1}$$

$$\Rightarrow Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{M-1} z^{-M+1}) X(z)$$

Applying Inverse z-transform

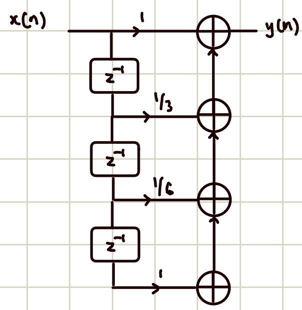
$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots + b_{M-1} x(n-M+1)$$



Q. $H(z) = 1 + \frac{1}{3} z^{-1} + \frac{1}{6} z^{-2} + z^{-3}$. Realize Direct form method & draw structure

A. (write diff eq),

$$H(z) = \frac{Y(z)}{X(z)} = x(n) + \frac{1}{3} x(n-1) + \frac{1}{6} x(n-2) + x(n-3)$$



→ **Cascaded Realization**

$$H(z) = H_1(z) \cdot H_2(z) \ldots$$

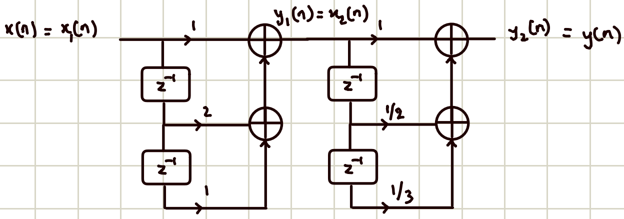
Q. $H(z) = (1 + 2z^{-1} + z^{-2})(1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})$

Realize using Cascaded form

A. $H(z) = H_1(z) \cdot H_2(z)$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = 1 + 2z^{-1} + z^{-2} \Rightarrow y_1(n) = x_1(n) + 2x_1(n-1) + x_1(n-2)$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} \Rightarrow y_2(n) = x_2(n) + \frac{1}{2}x_2(n-1) + \frac{1}{3}x_2(n-2)$$

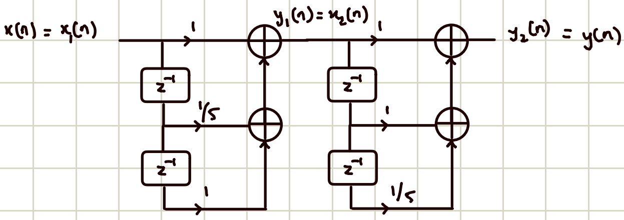


Q. $H(z) = (1 + \frac{1}{5}z^{-1} + z^{-2})(1 + z^{-1} + \frac{1}{5}z^{-2})$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = 1 + \frac{1}{5}z^{-1} + z^{-2} \Rightarrow y_1(n) = x_1(n) + \frac{1}{5}x_1(n-1) + x_1(n-2)$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = 1 + z^{-1} + \frac{1}{5}z^{-2} \Rightarrow y_2(n) = x_2(n) + x_2(n-1) + \frac{1}{5}x_2(n-2)$$



→ Lattic Realization

→ Based on recursion of polynomial

→ Define some $A_m(z) = H(z)$

$$A_m(z) = a_m(0) + a_m(1)z^{-1} + a_m(2)z^{-2} + \dots + a_m(m)z^{-m}$$

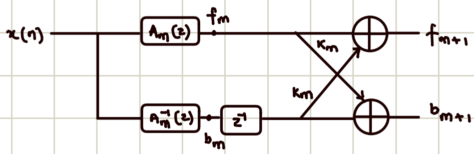
$$A_m^{-1}(z) = a_m(m) + a_m(m-1)z^{-1} + \dots + a_m(1)z^{-(m-1)} + a_m(0)z^{-m} \longrightarrow \text{Reverse Polynomial}$$

$$A_m(z^{-1}) = a_m(0) + a_m(1)z^{1} + a_m(2)z^{2} + \dots + a_m(m)z^{m}$$

$$A_m(z^{-1}) \times z^{-m} = a_m(m) + a_m(m-1)z^{-1} + \dots + a_m(1)z^{-(m-1)} + a_m(0)z^{-m} = A_m^{-1}(z)$$

So, $z^{-m} A_m(z^{-1}) = A_m^{-1}(z)$

Structure will be,



K_m : reflection co-efficient

So, $$\boxed{A_{m+1}(z) = A_m(z) + K_m z^{-1} A_m^{-1}(z)}$$

$$A_{m+1}^{-1}(z) = A_m(z)K_m + z^{-1}A_m^{-1}(z)$$

Now in matrix form, $\begin{bmatrix} A_{m+1} \\ A_{m+1}^{-1} \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ K_m & 1 \end{bmatrix} \begin{bmatrix} A_m \\ z^{-1}A_m^{-1} \end{bmatrix}$

Now if we invert it, $\begin{bmatrix} A_m \\ z^{-1}A_m^{-1} \end{bmatrix} = \frac{1}{1-K_m^2} \begin{bmatrix} 1 & -K_m \\ -K_m & 1 \end{bmatrix} \begin{bmatrix} A_{m+1} \\ A_{m+1}^{-1} \end{bmatrix}$

Then, $$\boxed{A_m = \frac{A_{m+1} - K_m A_{m+1}^{-1}}{1-K_m^2}}$$

$$z^{-1}A_m^{-1} = \frac{-K_m A_{m+1} + A_{m+1}^{-1}}{1-K_m^2}$$

$$\begin{bmatrix} a_m(0) + a_m(1)z^{-1} + \dots \\ z^{-1}(a_m(m) + a_m(m-1)z^{-1} + \dots \end{bmatrix} = \frac{1}{1-K_m^2} \begin{bmatrix} 1 & -K_m \\ -K_m & 1 \end{bmatrix} \begin{bmatrix} a_{m+1}(0) + a_{m+1}(1)z^{-1} + \dots \\ a_{m+1}(m+1) + a_{m+1}(m)z^{-1} + \dots \end{bmatrix}$$

$$\begin{bmatrix} a_m(0) + a_m(1)z^{-1} + \dots \\ 0 + a_m(m)z^{-1} + a_m(m-1)z^{-2} + \dots \end{bmatrix} = \frac{1}{1-K_m^2} \begin{bmatrix} 1 & -K_m \\ -K_m & 1 \end{bmatrix} \begin{bmatrix} a_{m+1}(0) + a_{m+1}(1)z^{-1} + \dots \\ a_{m+1}(m+1) + a_{m+1}(m)z^{-1} + \dots \end{bmatrix}$$

Comparing coefficients of a_{m+1},

$$0 = -K_m a_{m+1}(0) + a_{m+1}(m+1)$$

$$\Rightarrow \boxed{K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)}}$$

→ Reflection Coefficient

→ Equations are same for both IIR & FIR filter

Summary!

$$\boxed{A_{m+1}(z) = A_m(z) + K_m z^{-1} A_m^{-1}(z)}$$

$$\boxed{A_m = \frac{A_{m+1} - K_m A_{m+1}^{-1}}{1-K_m^2}}$$

$$\boxed{K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)}}$$

Q. Given the TF $H(z) = 1 + 3z^{-1} + 2z^{-2}$. Realize in lattice Form

A. $H(z) = A_2(z) = 1 + 3z^{-1} + 2z^{-2} = a_2(0) + a_2(1)\, z^{-1} + a_2(2)\, z^{-2}$

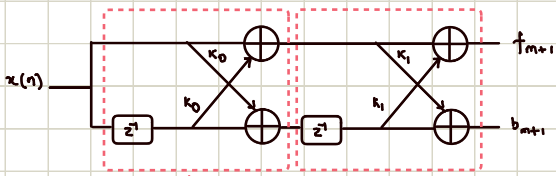
$A_2^{-1}(z) = 2 + 3z^{-1} + z^{-2}$

$$K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)}$$

$$K_1 = \frac{a_2(2)}{a_2(0)} = \frac{2}{1} = 2$$

$$A_1 = \frac{A_2 - K_1 A_2^{-1}}{1 - K_1^2} = \frac{1 + 3z^{-1} + 2z^{-2} - 2(2 + 3z^{-1} + z^{-2})}{1 - 4} = \frac{-3 - 3z^{-1}}{-3} = 1 + z^{-1}$$

$$K_0 = \frac{a_1(1)}{a_1(0)} = \frac{1}{1} = 1$$



Stage 1
$K_0 = 1$

Stage 2
$K_1 = 2$

Q. Given $K_0 = 0.1$, $K_1 = 0.2$, $K_2 = 0.3$ are reflection coefficients, Find $H(z)$ of filter

A. $K_0 = 0.1$; $K_1 = 0.2$; $K_2 = 0.3$

$A_0(z) = 1$

$A_{m+1}(z) = A_m(z) + K_m z^{-1} A_m^{-1}(z)$

$A_1(z) = A_0(z) + K_0 z^{-1} A_0^{-1}(z)$

$= 1 + 0.1 z^{-1}$

$A_2(z) = 1 + 0.1z^{-1} + 0.2z^{-1}(z^{-1} + 0.1) = 1 + 0.1z^{-1} + 0.2z^{-2} + 0.02z^{-1}$

$= 1 + 0.12z^{-1} + 0.2z^{-2}$

$A_3(z) = (1 + 0.12z^{-1} + 0.2z^{-2}) + 0.3z^{-1}(z^{-2} + 0.12z^{-1} + 0.2) = 1 + 0.12z^{-1} + 0.2z^{-2} + 0.3z^{-3} + 0.036z^{-2} + 0.06z^{-1}$

$= 1 + 0.18z^{-1} + 0.236z^{-2} + 0.3z^{-3}$

$= H(z)$ //

Unit-3 (Lattice Realization of IIR Filter)

Lattice Realization

→ For an all pole system (We don't have systems with both zeros & poles)

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{1}{A_m(z)}$$

→ IIR Filter is stable if $|K_m| < 1$ or else unstable

↳ Schur - Cohn Stability Test

Q. For $H(z) = \dfrac{1}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}}$. Realize the given all pole system using Lattice realization

A. $H(z) = \dfrac{1}{A_3(z)}$ & We need K_0, K_1, K_2

$$A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$

$$A_3(z) = a_3(0) + a_3(1)z^{-1} + a_3(2)z^{-2} + a_3(3)z^{-3}$$

$$K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)} \qquad A_m = \frac{A_{m+1} - K_m \tilde{A}_{m+1}}{1-K_m^2}$$

$$K_2 = \frac{a_3(3)}{a_3(0)} = \frac{1/3}{1} = 1/3$$

$$A_2 = \frac{A_3 - K_2\tilde{A}_3}{1-K_2^2} = \frac{\left(1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}\right) - \frac{1}{3}\left(z^{-3}+\frac{13}{24}z^{-2}+\frac{5}{8}z^{-1}+\frac{1}{3}\right)}{1-\frac{1}{9}}$$

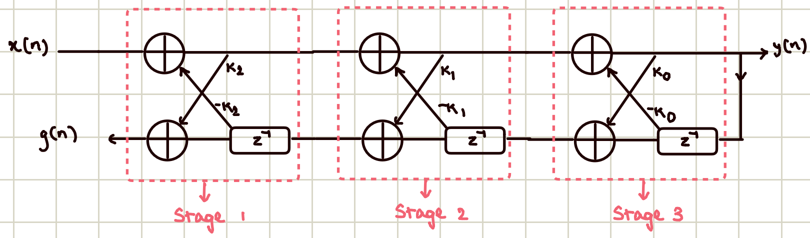
$$= \frac{\frac{8}{9}+\frac{1}{3}z^{-1}+\frac{4}{9}z^{-2}}{\frac{8}{9}} = 1+\frac{3}{8}z^{-1}+\frac{1}{2}z^{-2}$$

$$K_1 = \frac{a_2(2)}{a_2(0)} = \frac{1/2}{1} = 1/2$$

$$A_1 = \frac{A_2 - K_1\tilde{A}_2}{1-K_1^2} = \frac{\left(1+\frac{3}{8}z^{-1}+\frac{1}{2}z^{-2}\right)-\frac{1}{2}\left(z^{-2}+\frac{3}{8}z^{-1}+\frac{1}{2}\right)}{3/4} = \frac{\frac{3}{4}+\frac{3}{16}z^{-1}}{3/4} = 1+\frac{1}{4}z^{-1}$$

$$K_0 = \frac{a_1(1)}{a_1(0)} = \frac{1/4}{1} = 1/4$$

A_0 not required, it is $= 1$



↪ System has feedback

Q. If $H(z) = \dfrac{1}{A_3(z)} = \dfrac{1}{1 + 0.18z^{-1} + 0.236z^{-2} + 0.3z^{-3}}$. Realize $H(z)$ using lattice realization

A. $H(z) = \dfrac{1}{A_3(z)}$ & We need K_0, K_1, K_2

$$A_3(z) = 1 + 0.18z^{-1} + 0.236z^{-2} + 0.3z^{-3}$$

$$A_3(z) = a_3(0) + a_3(1)z^{-1} + a_3(2)z^{-2} + a_3(3)z^{-3}$$

$$K_m = \frac{a_{m+1}(m+1)}{a_{m+1}(0)} \qquad A_m = \frac{A_{m+1} - K_m A^{-1}_{m+1}}{1 - K_m^2}$$

$$K_2 = \frac{a_3(3)}{a_3(0)} = \frac{0.3}{1} = 0.3$$

$$A_2 = \frac{A_3 - K_2 A_3^{-1}}{1 - K_2^2} = \frac{(1 + 0.18z^{-1} + 0.236z^{-2} + 0.3z^{-3}) - 0.3(z^{-3} + 0.18z^{-2} + 0.236z^{-1} + 0.3)}{1 - 0.09}$$

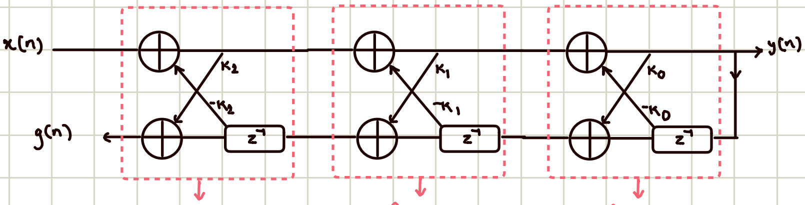
$$= \frac{0.1 + 0.109z^{-1} + 0.182z^{-2}}{0.91} = 1 + 0.12z^{-1} + 0.2z^{-2}$$

$$K_1 = \frac{a_2(2)}{a_2(0)} = \frac{0.2}{1} = 0.2$$

$$A_1 = \frac{A_2 - K_1 A_2^{-1}}{1 - K_1^2} = \frac{(1 + 0.12z^{-1} + 0.2z^{-2}) - 0.2(z^{-2} + 0.12z^{-1} + 0.2)}{0.96} = \frac{0.96 - 0.096z^{-1}}{0.96} = 1 - 0.12$$

$$K_0 = \frac{a_1(1)}{a_1(0)} = \frac{0.1}{1} = 0.1$$

A_0 not required, it is $= 1$



Stage 1 $K_2 = 0.3$ | Stage 2 $K_1 = 0.2$ | Stage 3 $K_0 = 0.1$

System has feedback

Syllabus Done :)