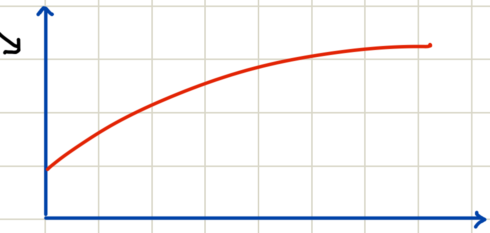
U-1 Basics of Signals & Systems

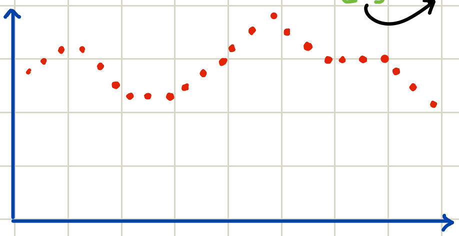
→ Signals are represented mathematically as functions of one or more independant variables

→ Signals are of 2 types

i) Continuous Time Signals – Independant variable is continuous → $x(t)$

ii) Discrete Time Signals – Independant variable takes on only discrete values → $x[n]$ → n : no of samples





Note : Discrete-time signal is defined only for integer values

Value (Amplitude) → Continuous, Discrete

Time → Continuous, Discrete

Continuous Value Continuous Time

Continuous Value Discrete Time

Discrete Value Continuous Time

Discrete Value Discrete Time

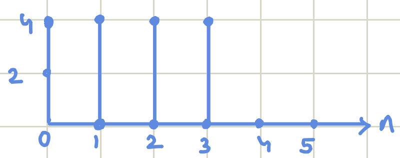
→ If signal is predictable, it is called deterministic signal else random signal

Q. Sketch the following continuous & discrete signals

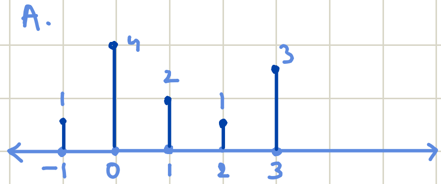
i) $x[n] = \begin{cases} 4 & , 0 < n \leq 3 \\ 0 & , \text{otherwise} \end{cases}$

ii) $x[n] = \{1, \underset{\uparrow n=0}{4}, 2, 1, 3\}$

A.

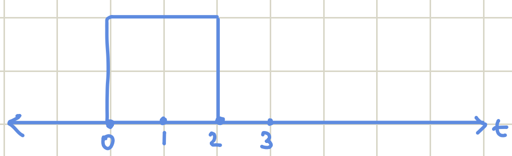


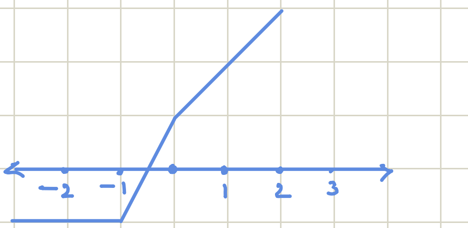
A.



iii) $x(t) = \begin{cases} 1 & , 0 \leq t \leq 2 \\ 0 & , \text{otherwise} \end{cases}$

iv) $x(t) = \begin{cases} t+1 & , 0 \leq t \leq 2 \\ -1 & , t < 0 \end{cases}$





→ Signal Energy & Power

	CT Signal	DT Signal
Total Energy	$\int_{t_1}^{t_2} \lvert x(t)\rvert^2 dt$	$\sum_{n=n_1}^{n_2} \lvert x[n]\rvert^2$
Average Power	$\frac{\int_{t_1}^{t_2} \lvert x(t)\rvert^2 dt}{t_2 - t_1}$	$\frac{\sum_{n=n_1}^{n_2} \lvert x[n]\rvert^2}{n_2 - n_1 + 1}$
E_∞	$\lim_{T\to\infty} \int_{-T}^{T} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\lim_{N\to\infty} \sum_{n=-N}^{N} \lvert x[n]\rvert^2 = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
P_∞	$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$	$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x[n]\rvert^2$

→ If E_∞ = Finite then $P_\infty = 0$ (Energy signal)
If $P_\infty > 0$ then $E_\infty = \infty$ (Power signal)

→ In general most periodic signals are power signals

Q. Given $x(t) = e^{-2t}u(t)$. Compute energy & power and state whether it is power or energy signal. Assume $u(t)$ is Unit Step function.

A. $x(t) = e^{-2t}$, $t \geq 0$

Now,

$$\int_{-T}^{T} |x(t)|^2 dt = \int_0^T |e^{-2t}|^2 dt = \int_0^T e^{-4t} dt = \left[\frac{e^{-4t}}{-4}\right]_0^T = \frac{1}{4} - \frac{e^{-4T}}{4}$$

Energy,

$$\underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt = \underset{T\to\infty}{Lt} \left(\frac{1}{4} - \frac{e^{-4T}}{4}\right) = \frac{1}{4} - 0 = \frac{1}{4} J$$

Power,

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \frac{1}{\infty}\left(\frac{1}{4}\right) = 0 \text{ Watts}$$

Since, $x(t)$ has finite energy & zero power, it is an energy signal

Q. $x(t) = \cos t$. Is it energy / power signal?

A.

$$E_\infty = \int_{-\infty}^{\infty} [\cos t]^2 dt = \int_{-\infty}^{\infty} \left(\frac{\cos 2t + 1}{2}\right) dt = \frac{1}{2}\left[\frac{\sin 2t}{2} + t\right]_{-\infty}^{\infty} = \left(\frac{\sin 2t}{4}\right)_{-\infty}^{\infty} + \frac{\infty - (-\infty)}{2} = \infty$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\cos t|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\left(\frac{\sin 2t}{2} + t\right)_{-T}^{T} = \lim_{T\to\infty} \frac{\left(\frac{\sin 2T - (\sin 2(-T))}{2} + T - (-T)\right)}{T} = \frac{1}{2}$$

Power Signal

Q. $\cos\frac{\pi}{4}n$. Find if power (or) energy signal?

A. $E_\infty = \lim_{N\to\infty}\sum_{-N}^{N}|x[n]^2| = \lim_{N\to\infty}\sum_{-N}^{N}\left(\cos\frac{\pi}{4}n\right)^2 = \lim_{N\to\infty}\sum_{-N}^{N}\left[\frac{\cos\frac{\pi n}{2}+1}{2}\right] = \sum_{n=-\infty}^{\infty}\frac{1}{2} + \cancel{\sum_{n=-\infty}^{\infty}\cos\frac{\pi n}{2}}^{\infty}$

$= \infty$

$P_\infty = \lim_{N\to\infty}\sum_{-N}^{N}\frac{|x[n]^2|}{2N+1} = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N}\left|\cos\frac{\pi n}{4}\right|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\left[\underbrace{\sum_{n=-N}^{N}\frac{1}{2}}_{\frac{2N+1}{2}} + \underbrace{\sum_{n=-N}^{N}\frac{\cos\frac{\pi n}{2}}{2}}_{0}\right]$

$= \lim_{N\to\infty}\frac{1}{2N+1}\left[\frac{2N+1}{2}+0\right] = \frac{1}{2}$

Q. $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Power or Energy Signal?

A. $E_\infty = \lim_{N\to\infty}\sum_{N=0}^{N}\left|\left(\frac{1}{2}\right)^n\right|^2 = \lim_{N\to\infty}\sum_{N=0}^{N}\left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

$P_\infty = \lim_{N\to\infty}\frac{\sum_{N=0}^{N}\left|\left(\frac{1}{2}\right)^n\right|^2}{2N+1} = \lim_{N\to\infty}\sum_{N=0}^{N}\frac{\left(\frac{1}{4}\right)^n}{2N+1} = \lim_{N\to\infty}\left|\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right| \times \frac{1}{2N+1}$

$= \lim_{N\to\infty}\left|1-\frac{1}{4^{N+1}}\right| \times \frac{1}{2N+1} \times \frac{4}{3}$

$= \frac{1}{\infty} = 0$

Q.  Energy Signal or Power Signal?

A. $x(t) = 1$

$E_\infty = \lim_{T\to\infty}\int_0^T |x(t)|^2\,dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty}(t)_0^T = \lim_{T\to\infty} t = \infty$

$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\times\int_0^T |x(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{2T}\int_0^T 1\,dt = \lim_{T\to\infty}\frac{\cancel{T}}{2\cancel{T}} = \frac{1}{2}$

→ Transformation of the Independant Variable

→ Signals & Systems analysis focuses on transformation of a signal to desired form for processing & subsequent use

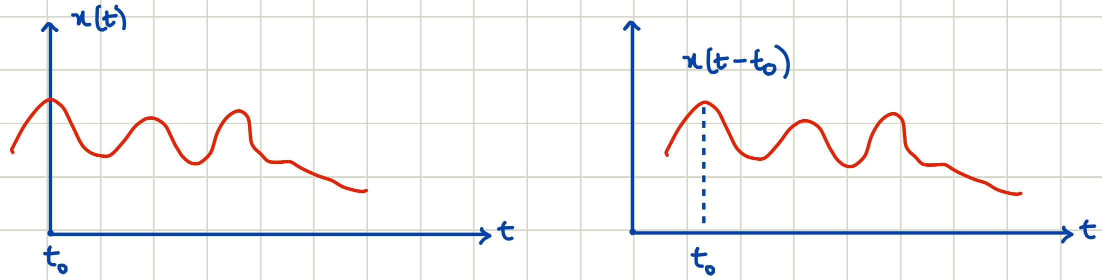
→ Time Shift

→ A signal (CT/DT) can be shifted on time axis (left/right)

→ When shifted to right by t_0 / n_0 ⇒ delayed

t_0 / n_0 is positive

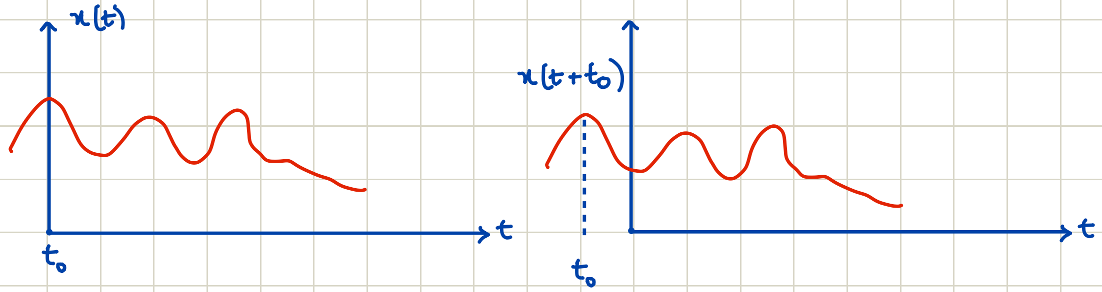
$x(t-t_0)$ or $x[n-n_0]$ is delayed version of $x(t)$ or $x[n]$



→ When shifted to left by t_0 / n_0 ⇒ advanced

t_0 / n_0 is negative

$x(t+t_0)$ or $x[n+n_0]$ is advanced version of $x(t)$ or $x[n]$



→ Time Reversal

→ A signal is 'time reversed' when it is 'reflected' about vertical axis

→ Denoted as $x(-t)$ or $x[-n]$

→ Time Scaling

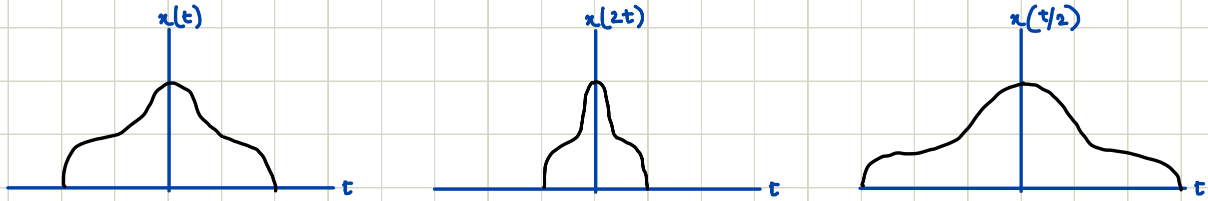
→ Signal is 'time scaled' when stretched / compressed linearly on time axis

→ Denoted by $x(\alpha t)$ or $x[\alpha n]$

→ When,

$$|\alpha| > 1 \Rightarrow \text{Signal is compressed}$$
$$|\alpha| < 1 \Rightarrow \text{Signal is stretched}$$
$$\alpha < 0 \Rightarrow \text{Signal is time reversed}$$

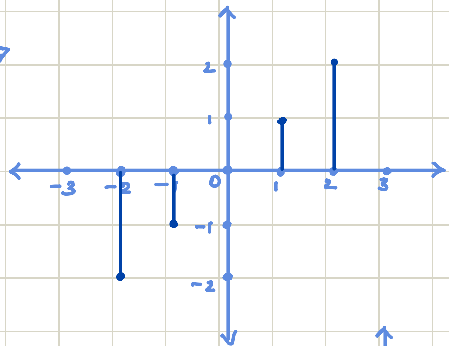
→ When signal is time scaled, amplitude / magnitude is preserved



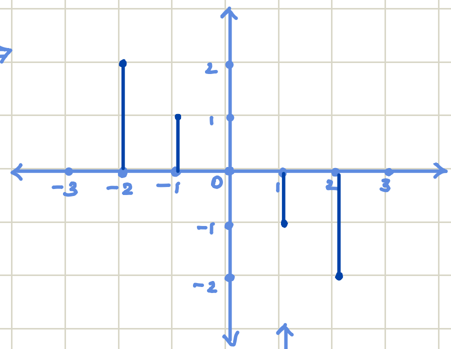
→ According to precedence rule, for $x(\alpha t + \beta)$ or $x[\alpha n + \beta]$, signal is time shifted by β & then time scaled / reversed by α

Q. Consider a DT sequence $x[n] = \{-2, -1, \mathbf{0}, 1, 2\}$. Sketch $x[n]$ and plot $x[-n]$ and $x[2-n]$. The sample at origin is indicated in bold

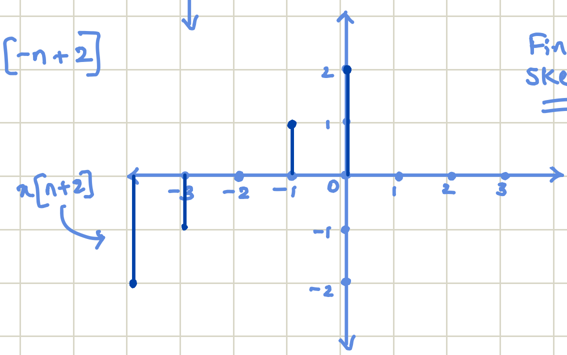
A. $x[n] \Rightarrow$



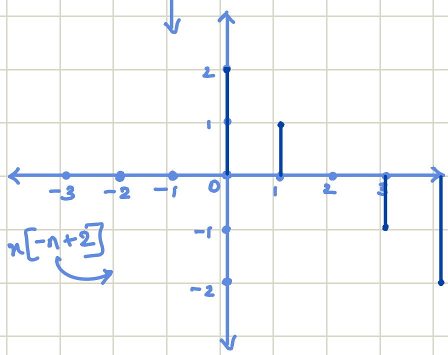
$x[-n] \Rightarrow$



$x[2-n] = x[-n+2]$

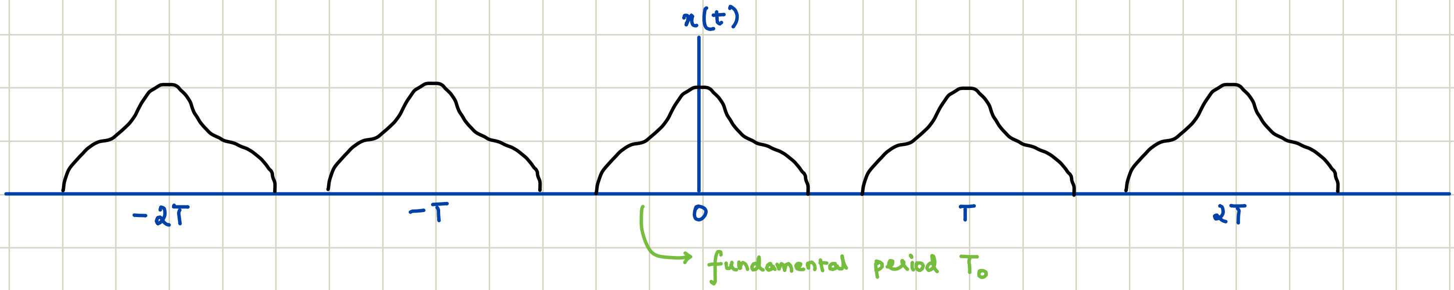


Final sketch ⇒



→ Periodic Signals

→ Signal is periodic if it satisfies $x(t) = x(t + mT)$ where T is positive & m is an integer for all t



→ Any signal that doesn't satisfy $x(t) = x(t+T)$ is Aperiodic

→ Constant signals has period undefined as there is no smallest value of T

→ A DT Signal that satisfies $x[n] = x[n+N]$ is periodic with period N, where N is some positive integer

→ Fundamental period N_0 is smallest value of N holding good for $x[n] = x[n+N]$

→ If DT signal is expressed in the form

$$x[n] = A\cos(\omega_0 n + \theta)$$

then it is periodic if fundamental frequency f_0 (or) $\frac{\omega_0}{2\pi}$ is a rational no. $\omega_0 = 2\pi f_0$

→ For a continuous time domain, assume 2 periodic signals with fundamental periods T_1 & T_2, then sum of these 2 signals is periodic if $\frac{T_1}{T_2}$ is a rational number.

Fundamental period of resultant periodic signal is $LCM(T_1, T_2)$

Q. Find the fundamental period of

i) $x(t) = \cos\left\{\frac{8\pi}{31}t\right\}$ ii) $x[n] = \cos\left\{\frac{8\pi}{31}n\right\}$

A. i) $x(t) = \cos\left\{\frac{8\pi}{31}t\right\} = \cos\omega_0 t$

$\omega_0 = \frac{8\pi}{31} = \frac{2\pi}{T} \Rightarrow T = \frac{31}{4} = 7.75$ secs

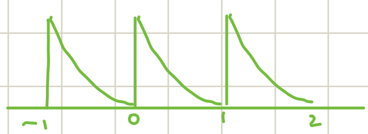
ii) $x[n] = \cos\left\{\frac{8\pi}{31}n\right\} = \cos\omega_0 n$

$\omega_0 = \frac{8\pi}{31}$, Then $\frac{\omega_0}{2\pi} = \frac{8\pi}{31 \times 2\pi} = \frac{4}{31} = \frac{m}{N}$

Therefore, fundamental period of $x[n]$ is $N = 31$ samples

Q. Check if given function is periodic or not. If periodic find T & f

i)

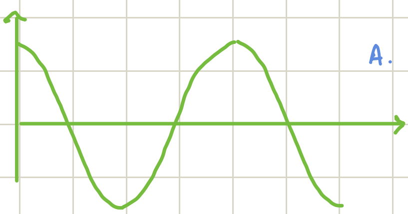


A. Periodic

$T = 1$ sec

$f = 1$ Hz

ii)



A. aperiodic (not present before origin)

iii) $x_3(t) = \cos(\pi t) + \sin(3\pi t)$

A. $x_1 = \cos\pi t$; $x_2 = \sin 3\pi t$

$\omega_1 = \frac{2\pi}{T_1} = \pi \Rightarrow T_1 = 2$ sec

$\omega_2 = \frac{2\pi}{T_2} = 3\pi \Rightarrow T_2 = \frac{2}{3}$ sec

$T = LCM(T_1, T_2) = 2$ sec

$\frac{T_1}{T_2} = \frac{3}{1} \Rightarrow$ rational $\downarrow$ periodic

Q. $x_4(t) = \cos(2t) + \sin 5\pi t$

A. $\omega_1 = \frac{2\pi}{T_1} = 2 \Rightarrow T_1 = \pi$ sec

$\omega_2 = \frac{2\pi}{T_2} = 5\pi \Rightarrow T_2 = \frac{2}{5}$ sec

$\frac{T_1}{T_2} = \frac{5\pi}{2} \Rightarrow$ aperiodic

→ If the discrete time signal $x[n]$ satisfies below cond^n, then the signal $x[n]$ is reffered periodic, otherwise a·periodic

$$x[n] = x[n+N] \quad \forall n$$
$$N \in Z$$

Q. $x_1[n] = \sin\left[\frac{7\pi n}{8}\right] + \cos\left[\frac{\pi n}{2}\right]$

A. $N_1 = \frac{2\pi}{\frac{7\pi}{8}} = \frac{16}{7}$ samples ⇒ Not integer

$N_2 = \frac{2\pi}{\frac{\pi}{2}} = 4$ samples

x_1 is aperiodic

Q. $x_2[n] = \sin\left[\frac{\pi n}{8}\right] + \cos\left[\frac{\pi n}{2}\right]$

A. $N_1 = \frac{2\pi}{\frac{\pi}{8}} = 16$ samples

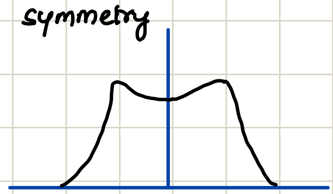
$N_2 = \frac{2\pi}{\frac{\pi}{2}} = 4$ samples

$\frac{N_1}{N_2} = \frac{16}{4} = 4$ ⇒ periodic

$LCM(N_1, N_2) = 16$

→ Even & Odd Signals

→ Signals exhibit some sort of symmetry when time reversed can be classified as 'even' & 'odd' based on symmetry



→ CT signal is even if $x(-t) = x(t)$
DT signal is even if $x[-n] = x[n]$
Also known as symmetric signal

→ CT signal is odd if $x(-t) = -x(t)$
DT signal is odd if $x[-n] = -x[n]$
Also known as anti-symmetric signal
(Odd signal has to pass through origin or should be 0 at $t=0$ or $n=0$ to satisfy $x(0) = -x(0)$ or $x[0] = -x[0]$

→ Even part of signal is given by
$$Ev\{x(t)\} = \frac{x(t) + x(-t)}{2} \quad ; \quad Ev\{x[n]\} = \frac{x[n] + x[-n]}{2}$$

→ Odd part of signal is given by
$$Od\{x(t)\} = \frac{x(t) - x(-t)}{2} \quad ; \quad Od\{x[n]\} = \frac{x[n] - x[-n]}{2}$$

Q. Evaluate even & odd part of given signal $x(t) = e^t$

A. $x(t) = e^t$
$x(-t) = e^{-t}$

$$\text{Even of } x(t) = \frac{x(t) + x(-t)}{2} = \frac{e^t + e^{-t}}{2} = \cosh t$$

$$\text{Odd of } x(t) = \frac{x(t) - x(-t)}{2} = \frac{e^t - e^{-t}}{2} = \sinh t$$

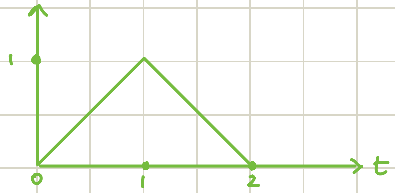
Q. Evaluate even & odd part of given signal $x(t) = (1+t^3)\cos^3(10t)$

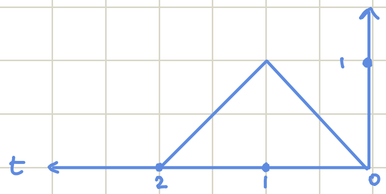
A $x(t) = (1+t^3)\cos^3(10t)$
$x(-t) = (1-t^3)\cos^3(10t)$

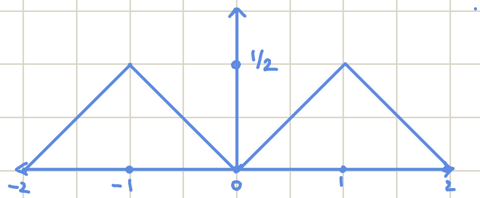
$$\text{Even of } x(t) = \frac{x(t) + x(-t)}{2} = \cos^3(10t)$$

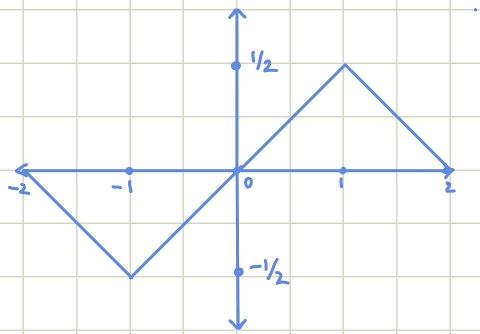
$$\text{Odd of } x(t) = \frac{x(t) - x(-t)}{2} = t^3\cos^3(10t)$$

Q. For the following signal, find the even & odd part

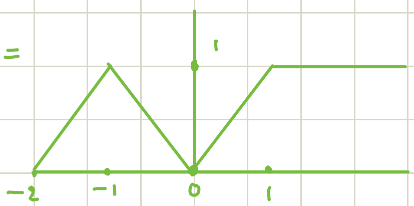
$x(t) =$ 

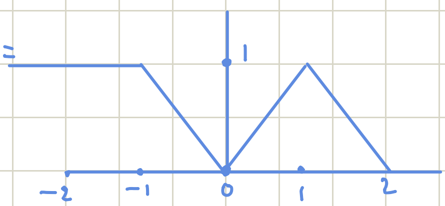
A. 

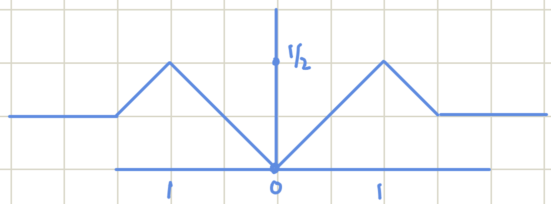
Even part of $x(t) = \dfrac{x(t) + x(-t)}{2} \Rightarrow$ 

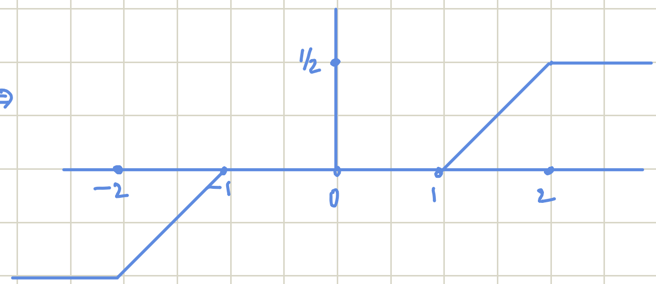
Odd part of $x(t) = \dfrac{x(t) - x(-t)}{2} \Rightarrow$ 

Q. For the following signal, find the even & odd part

$x(t) =$ 

$x(-t) =$ 

Even part of $x(t) = \dfrac{x(t) + x(-t)}{2} \Rightarrow$ 

Odd part of $x(t) = \dfrac{x(t) - x(-t)}{2} \Rightarrow$ 

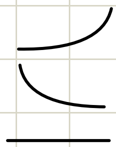
→ Exponential & Sinusoidal Signals

→ These signals exist in continuous as well as discrete domains

→ CT complex exponential is of the form :

$$x(t) = C e^{at}$$

real exponential ($x(t)$) real (C, a)



When a is +ve , $x(t)$ is growing exponential as t increases

When a is −ve , $x(t)$ is decaying exponential as t increases

When a is 0 , $x(t)$ is a constant signal

→ Important class of CT complex exponential is obtained by constraining a to be purely imaginary

$$x(t) = e^{j\omega_0 t}$$

$x(t)$ is periodic

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Thus, the signals $e^{j\omega_0 t}$ & $e^{-j\omega_0 t}$ have same fundamental period

→ Another important class of signals closely related to periodic complex exponential is sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \phi)$$

ω_0 – rad/s

t – s

ϕ – rad

$\omega_0 = 2\pi f_0$

f_0 – Hz

→ Euler's Identity ⇒ $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$

$$A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

So,

$$A\cos(\omega_0 t + \phi) = A \cdot \mathrm{Re}\{e^{j(\omega_0 t + \phi)}\}$$

$$A\sin(\omega_0 t + \phi) = A \cdot \mathrm{Im}\{e^{j(\omega_0 t + \phi)}\}$$

→ Harmonically Related CT Complex Exponentials

→ Defined as set of complex exponentials which are periodic with common time period T_0

→ Condition being

$$e^{j\omega T_0} = 1$$

which implies that ωT_0 is a multiple of 2π

$$\omega T_0 = 2\pi K$$

where $K = 0, \pm 1, \pm 2, \pm 3 \ldots$

So, $\phi_K(t) = e^{jK\omega_0 t}$

and, $\dfrac{2\pi}{|K|\omega_0} = \dfrac{T_0}{|K|}$

→ General CT Complex exponential signals

→ $x(t) = C e^{at}$

$C = |C| e^{j\theta}$, $a = r + j\omega_0$

→ $x(t) = C e^{at} = |C| e^{j\theta} e^{(r + j\omega_0)t} = |C| e^{rt} . e^{j(\omega_0 t + \theta)}$

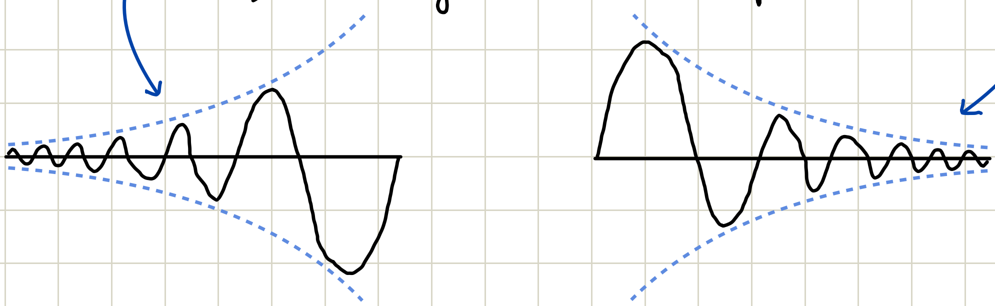
→ By euler's identity,

$$x(t) = C e^{at} = |C| e^{rt} \cos(\omega_0 t + \theta) + j |C| e^{rt} \sin(\omega_0 t + \theta)$$

For $r = 0$, the real & imaginary parts of complex exponential are sinusoidal

& when $r > 0$, the signal is a growing sinusoid,

$r < 0$, the signal is a damped sinusoid

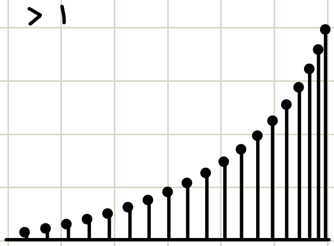


→ $x[n] = C\alpha^n = Ce^{\beta n}$
$(\alpha = e^{\beta})$

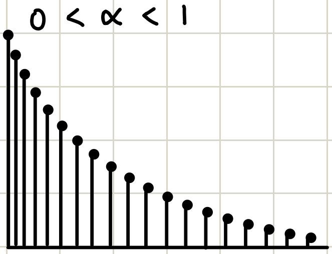
General complex numbers

When $\alpha = 1$, $x[n]$ is a constant signal of amplitude C
& if $\alpha = -1$, $x[n]$ alternates between $+C$ and $-C$

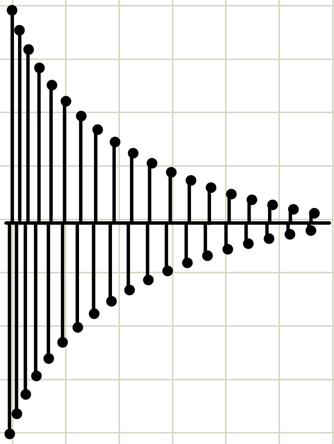
→ $\alpha > 1$



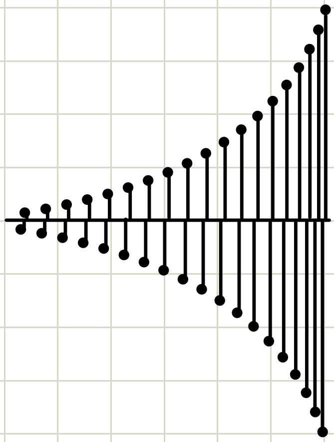
→ $0 < \alpha < 1$



→ $-1 < \alpha < 0$



→ $\alpha < -1$

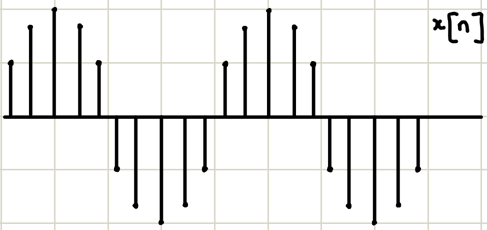


→ If we restrain β to be purely imaginary, such that $|\alpha| = 1$ &
$x[n] = e^{j\omega_0 n}$

and we define DT sinusoidal signal as $x[n] = A\cos(\omega_0 n + \phi)$

$$= \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

→ As in the case of CT, DT complex exponentials are also periodic with ∞ energy & avg. energy to be unity



$x[n] = \cos\left(\frac{2\pi n}{12}\right)$

→ General DT Complex exponential signals

→ Consider $x[n] = C\alpha^n$

→ expressed in the polar form of $C = |C|e^{j\theta}$, $\alpha = |\alpha|e^{j\omega_0}$

→ Then,

$$C\alpha^n = |C|\,|\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

$|\alpha| = 1$, real & imaginary parts of the complex exponential are sinusoidal

$|\alpha| < 1$, $x[n]$ is sinusoidal sequence multiplied by decaying exponential

$|\alpha| > 1$, $x[n]$ is sinusoidal sequence multiplied by growing exponential

→ Periodicity properties of DT Complex exponentials

→ In case of CT complex exponential, $x(t) = e^{j\omega_0 t}$

(i) higher value of ω_0, higher the $x(t)$

(ii) signal is periodic for any value of ω_0 except 0

→ Consider DT Complex exponential with frequency $\omega_0 + 2\pi$ then

$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}\, e^{j\omega_0 n} = e^{j\omega_0 n}$

We can only consider frequency interval of length 2π in which to choose ω_0

and because of implied periodicity, DT complex exponential doesn't have a continually increasing rate of oscillation as ω_0 is increased in magnitude

So, as ω_0 increases from 0 to π, oscillates rapidly

and ω_0 increases from π to 2π, oscillation decreases at same rate

→ In particular, for $\omega_0 = \pi$ or any multiple of π, this signal oscillates rapidly changing sign at each point in time

$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$

→ for a DT complex exponential to be periodic with a period $N > 0$, we need to have

$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$, So, $e^{j\omega_0 N} = 1$

So $\omega_0 n$ must be multiple of 2π

→ Hence, $\omega_0 N = 2\pi m \Rightarrow \dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$

→ If $x[n]$ is periodic with a fundamental period N, then fundamental frequency $= \frac{2\pi}{N}$

then, $\frac{\omega_0}{2\pi} = \frac{m}{N}$

→ If DT signal has $\omega_0 = 0$, fundamental period is undefined, which is same as CT counterpart

	$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
1)	Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 seperated by multiples of 2π
2)	Periodic for any choice of ω_0	Periodic only if $\omega_0 = \frac{2\pi m}{N}$ for some integers $N>0$ & m
3)	Fundamental frequency ω_0	Fundamental frequency $\frac{\omega_0}{m}$
4)	Fundamental period $\omega_0 = 0$; undefined $\omega_0 \neq 0$; $\frac{2\pi}{\omega_0}$	Fundamental period $\omega_0 = 0$; undefined $\omega_0 \neq 0$; $\frac{2\pi}{\omega_0} \times m$

Q. Find the period for the following

i) $\cos \frac{\pi n}{2}$ ii) $\cos \frac{\pi n}{4}$ iii) $\cos\left(\frac{\pi n}{2}\right) \cdot \cos\left(\frac{\pi n}{4}\right)$

A. i) $\omega N = 2\pi m$

$\frac{N}{m} = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$

$\therefore$ Periodic (Period = 4)

ii) $\omega N = 2\pi m$

$\frac{N}{m} = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{4}} = 8$

$\therefore$ Periodic (Period = 8)

iii) $\cos\left(\frac{\pi n}{2}\right) \cdot \cos\left(\frac{\pi n}{4}\right) \Rightarrow \cos a \cdot \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}$

$= \frac{\cos\left(\frac{3\pi n}{4}\right)}{2} + \frac{\cos\left(\frac{\pi n}{4}\right)}{2}$

$\frac{N_1}{m_1} = \frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3}$ $\frac{N_2}{m_2} = \frac{2\pi}{\frac{\pi}{4}} = 8$

$\therefore$ Periodic (Period = LCM(N_1, N_2) = LCM$(8,8)$ = 8)

Q. Check if Periodic

$2\cos \frac{\pi n}{4} + \sin \frac{\pi n}{8} - 2\cos\left(\frac{\pi n}{2} + \frac{\pi n}{6}\right)$

A. $\frac{N_1}{m_1} = \frac{2\pi}{\frac{\pi}{4}} = 8$

$\frac{N_2}{m_2} = \frac{2\pi}{\frac{\pi}{8}} = 16$

$\frac{N_3}{m_3} = \frac{2\pi}{\frac{\pi}{2}} = 4$

LCM(N_1, N_2, N_3) = LCM$(8,16,4)$ = 16

$\therefore$ Periodic (Period = 16)

→ Harmonically related DT complex exponentials

→ Defined as set of complex exponentials, all of which are periodic with common period N

→ The signals are multiples of $\frac{2\pi}{N}$

→ $$\phi_K[n] = e^{jK\left(\frac{2\pi}{T}\right)n}, \quad K = 0, \pm 1, \pm 2, \pm 3 \ldots$$

$$\phi_K(t) = e^{jK\omega_0 t} = e^{jK\left(\frac{2\pi}{T}\right)t}, \quad K = 0, \pm 1, \pm 2, \ldots$$

the exponentials are distinct

→ In case of DT,

$$e^{j(\omega_0 + 2\pi)n} = e^{j2\pi n} \cdot e^{j\omega_0 n} = e^{j\omega_0 n}$$

Specifically,

$$\phi_{K+N}[n] = e^{j(K+N)\left(\frac{2\pi}{N}\right)n} = e^{jK\left(\frac{2\pi}{N}\right)n} \cdot e^{j2\pi n} = \phi_K[n]$$

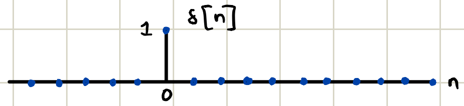
→ This implies there are only N distinct periodic exponentials in the set given by

$$\phi_K[n] = e^{jK\left(\frac{2\pi}{N}\right)n}, \quad K = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\left(\phi_K[n] = \phi_0[n], \quad \phi_{-1}[n] = \phi_{N-1}[n]\right)$$

→ DT Unit Impulse and Unit Step Sequences

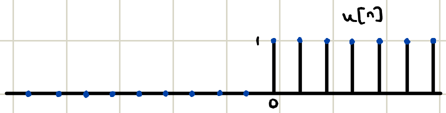
→ Unit impulse (or) Unit Sample



$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

→ Unit Step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



→ Unit impulse is the first difference of the DT unit step

$$\delta[n] = u[n] - u[n-1]$$

Similarly, DT Unit Step is the running sum of DT unit impulse

$$u[n] = \sum_{K=0}^{\infty} \delta[n-K]$$

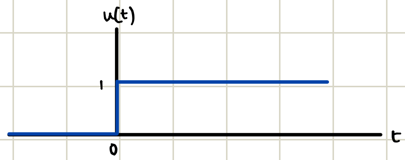
→ Important identities for sampling of a signal using unit impulse

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$$

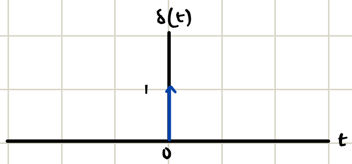
→ CT Unit Impulse and Unit Step Sequences

→ Unit Step function



$$u(t) = \begin{cases} 0 , & t < 0 \\ 1 , & t \geq 0 \end{cases}$$

→ Unit Impulse function



$$\delta(t) = \frac{du(t)}{dt}$$

→ Unit impulse is the first derivative of the CT Unit step

→ CT Unit step is the running integral of CT Unit Impulse

$$u(t) = \int_0^\infty \delta(t-\sigma)\, d\sigma$$

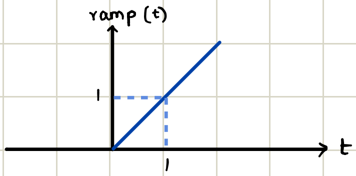
→ Important identities for sampling of a signal using unit impulse

$$x(t)\ \delta(t) = x(0)\ \delta(t)$$

$$x(t)\ \delta(t-t_0) = x(t_0)\ \delta(t-t_0)$$

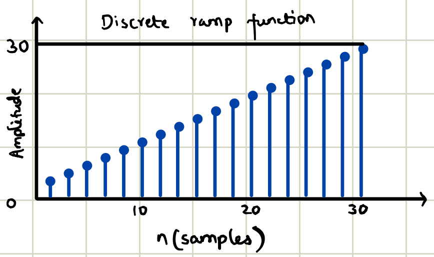
→ CT & DT Ramp Functions

→ CT Ramp



$$x(t) = t \quad \text{for } t > 0$$

→ DT ramp



$$x[n] = n \quad \text{for } n > 0$$

→ $\delta[n] = u[n] - u[n-1]$, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

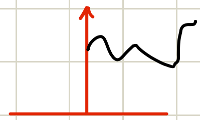
$u[n] = r[n+1] - r[n]$, $r[n] = \sum_{k=1}^{\infty} u[n-k]$

So,

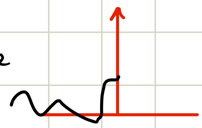
$$\delta(t) \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftarrows}} u(t) \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftarrows}} r(t)$$

→ Causal & Anti-Causal Signals

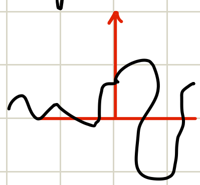
1) Causal Signals – The signals whose amplitude is zero for all negative time



2) Anti-Causal Signals – The signals whose amplitude is zero for all positive time



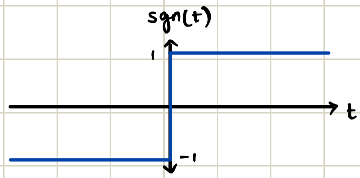
3) Non-Causal – The signals whose amplitude is non-zero in both positive & negative time axes



→ CT Signum & Sinc Functions

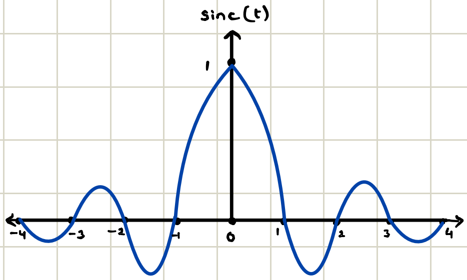
→ CT Signum Function

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} = 2u(t) - 1$$



→ CT Sinc Function

$$sinc(t) : \frac{\sin(\pi t)}{\pi t}$$

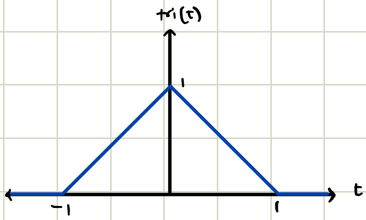


sinc (t) is the compressed version of sa (t)

$$\left(sa(t) = \frac{\sin t}{t} \right)$$

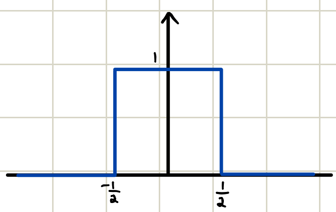
→ CT Triangular function

$$tri(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$



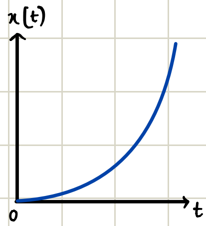
→ CT Rectangular function

$$rec(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 1/2, & |t| = \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$

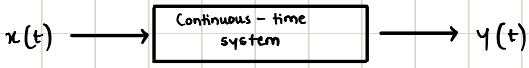


→ CT Parabolic function

$$x(t) = \begin{cases} t^2/2, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

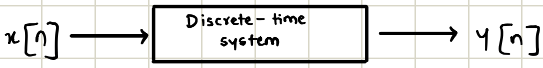


→ Continuous – time systems – Defined as the ones in which the input is a CT signal and the resultant output is also continuous



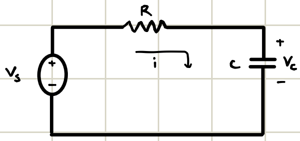
$x(t) \rightarrow y(t)$

→ Discrete - time systems – Defined as the ones in which the input is a DT signal and the resultant output is also discrete



$x[n] \rightarrow y[n]$

→ LTI Systems (Linear Time Invariant)



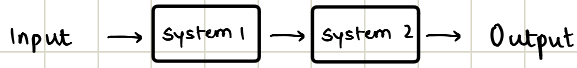
$$\frac{dV_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} V_s(t)$$

Linear constant co-efficient differential equation $\Rightarrow \dfrac{dy(t)}{dt} + ay(t) = bx(t)$

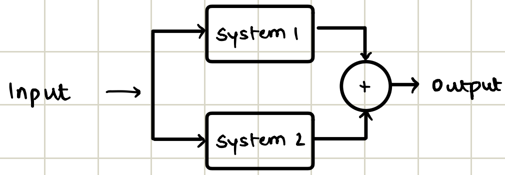
Linear constant co-efficient difference equation $\Rightarrow y[n] - 1.01\,y[n-1] = x[n]$

→ Interconnection of Systems

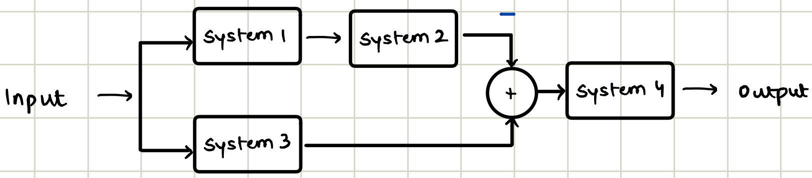
1) Series Interconnection ⇒



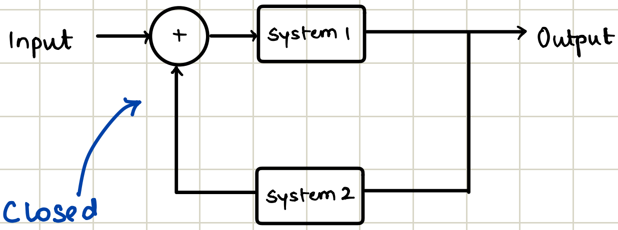
2) Parallel Interconnection ⇒



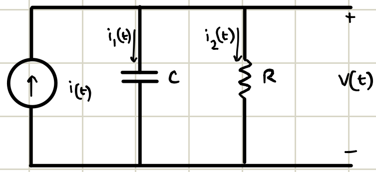
3) Series - Parallel Interconnection ⇒



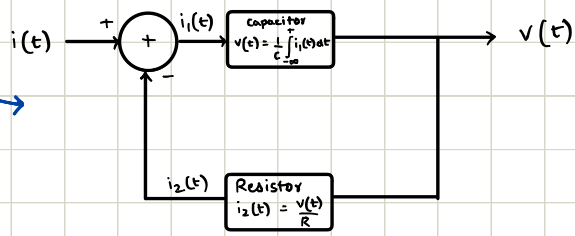
→ Interconnection of systems – Feedback



Closed Loop



Open loop



→ Basic System Properties

1) Memory

→ Systems with and without memory (Static Systems)

→ System is memoryless if output for each value of independant variable at a given time is dependant only on input at the same instant of time

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t)$$

→ Systems with memory (Dynamic Systems)

→ if output for each value of independant variable at a given time is dependant on input of same time and/or previous input/outputs

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n-1]$$

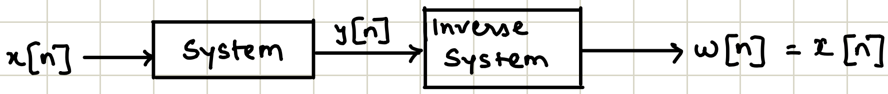
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(t)\,dt$$

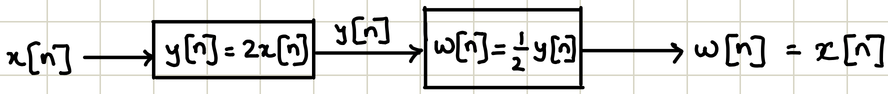
→ Identity System

→ $y(t) = x(t)$; $y[n] = x[n]$

2) Invertibility

→ Distinct input leads to distinct outputs





Steps to Find Inverse of a System

1) Write system equation
2) Analyze if system has 1-to-1 mapping. If no, not invertible
3) Try to find any loss of information (constants, squaring, differentiation etc.) if losses are there, not invertible
4) Try to express input in terms of output
5) verify that inverse recovers original Signal

→ Concept of inverse systems is important in encoding & decoding

→ Encoding is essential to ensure lossless & secured communication
Decoding is reverse process

3) Causality

→ If output at any time depends only on present & past input

→ Also referred as non-anticipative (Don't anticipate future values)

→ All memoryless systems are causal but not vice-versa

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Causal

$$y(t) = x(t+1)$$

Non-causal

→ Causality is not often an essential constraint in applications in which independant variable is not time, like image processing

→ $y(t) = x(-t)$

↳ Causal

But for negative values of t, depends on future, Hence Non-Causal

→ $y(t) = x(t)\cos(t+1)$

↳ Write as $g(t)$ ⇒ $y(t) = x(t)\,g(t)$

We deduce that only current value of input $x(t)$ influences output $y(t)$

Hence Causal

4) Stability

→ Stable system is one in which small inputs lead to outputs/responses which don't diverge

→ System is stable when Bounded Inputs lead to Bounded Outputs - BIBO Stability

→ $\ddot{y}(t) + a\dot{y}(t) + by(t) = x(t)$ & $a, b > 0$ is BIBO stable if $x(t)$ is bounded

→ $y[n] = \sum_{k=-\infty}^{n} x[k]$ is unstable, $x(t) = \sin t$ is stable, $x(t) = e^{t}$ is unstable

5) Time Invariance

→ System is said to be time invariant if behaviour & characteristics don't change with time!

→ It is time invariant if input-output satisfies :

$$y(t-t_0) = x(t-t_0)$$

[or]

$$y[n-n_0] = x[n-n_0]$$

Steps to check invariance

1. Find $y_1(t)$ for $x_1(t)$
2. Shift $x_1(t)$ by t_0 & consider as $x_2(t)$
3. Compute $y_2(t)$ using $x_2(t)$ and denote as eq A
4. Compute time shifted $y_1(t)$ by t_0 and denote as eq B
5. If eq A = eq B, system is time invariant

6) Linearity

→ Linear system is a system that possesses the property of superposition

→ Superposition consists of 2 parts : additivity and homogeneity

↳ If x_1 is an input that results y_1 as output & similarly x_2 results y_2. Then, $x_1 + x_2$ must result to $y_1 + y_2$

↳ If x results to y then upon scaling ax must result ay

Simultaneous!

CT ⇒ $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

DT ⇒ $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$

→ If input is zero, output must also be zero

→ Both conditions must be met simultaneously

If either of them only, linearity isn't guaranteed

Special Case ⇒ System is non-linear but has linear element (ex: $y[n] = 2x[n] + 3$)

→ when input is zero, output is given ⇒ zero input impedance

→ So, overall output is in superposition of response of linear system & zero-input response

→ They are called Incrementally Linear Systems

→ Difference in responses for any 2 inputs is always linear $(y_1[n] - y_2[n])$

Q. Check if the signal has the following properties

i) Memory ii) Linearity iii) Causality iv) Stability v) Time Invariance

a) $y(t) = \cos 3t \; x(t)$

b) $y(t) = x(t/3)$

c) $Ev\left[x[n-1]\right]$

d) $y[n] = \begin{cases} x[n], & n > 0 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$

e) $y[n] = \log_{10} x[n]$

f) $y(t) = e^{x(t)}$

A. a) i) No memory / Static

ii) linear

$a x_1(t) + b x_2(t) = a y_1(t) + b y_2(t)$

$a \cos 3t \; x_1(t) + b \cos 3t \; x_2(t) = a y_1(t) + b y_2(t)$

iii) Causal - Depends on previous/present values

iv) Stable - Output is bounded b/w 1 & -1

v) Time variant

$y_1(t) = \cos 3t \; x_1(t)$

$x_2(t) = x_1(t - t_0) \Rightarrow y_2(t) = \cos 3t \; x(t - t_0) \rightarrow A$

$y_1(t - t_0) = \cos 3(t - t_0) \, x(t - t_0) \rightarrow B$

$A \neq B$

b) i) Has memory - $y(3) = x(1) \Rightarrow$ Depends on previous values

ii) Linear - $a x_1(t/3) + b x_2(t/3) = a y_1(t) + b y_2(t)$

iii) Not Causal - For negative values, depends on future

iv) Not Stable - Depends on input signal.

v) Time variant - $y_1(t) = x_1(t/3)$

$x_2(t) = x_1(t - t_0) \Rightarrow y_2(t) = x_1\left(\frac{t_2 - t_0}{3}\right) \rightarrow A$

$y_1(t - t_0) = x_1\left(\frac{t}{3} - t_0\right) \longrightarrow B$

$A \neq B$

c) $Ev\left[x[n-1]\right] = \dfrac{x[n-1] + x[-n+1]}{2}$

i) Dynamic - Depends on past values

ii) Linear - $a\left(Ev\left[x_1[n-1]\right]\right) + b\left(Ev\left[x_2[n-1]\right]\right) = a y_1[n] + b y_2[n]$

iii) Not Causal - Anticipates future values

iv) Stable

v) Time variant

d) i) Dynamic

ii) Not causal

iii) Stable

iv) Linear

v) Time Variant - $y[n - n_0] = \begin{cases} x[n - n_0], & n - n_0 > 0 \text{ (or) } n > n_0 \text{ (or) } n - n_0 \geq 1 \\ 0, & n = n_0 \\ x[n + 1 - n_0], & n + 1 \leq n_0 \end{cases}$

e) $y[n] = \log_{10} x[n]$

i) Static - Doesn't depend on previous values

ii) Stable

iii) Causal

iv) Non-linear - $a y_1[n] = \log_{10} a x[n] = \log_{10} a + \log_{10} x[n]$

v) Time Invariant

f) $y(t) = e^{x(t)}$

i) Static - Doesn't depend on previous values

ii) Stable

iii) Causal

iv) Non - Linear - $a y(t) = e^{a x(t)} = e^{(x(t))^a} = a e^{x(t)}$

v) Time Invariant -

$x_2(t) = x_1(t - t_0) \Rightarrow y_2(t) = e^{x_1(t-t_0)} \rightarrow A$

$y_1(t - t_0) = e^{x_1(t-t_0)} \rightarrow B$

$A = B$

Q. Determine if the following signals are invertible or not

i) $y(t) = x(t-4)$

ii) $y(t) = \cos x(t)$

iii) $y[n] = n\, x[n]$

iv) $y[n] = x[-n]$

v) $y(t) = \dfrac{d x(t)}{dt}$

vi) $y(t) = \displaystyle\int_{-\infty}^{2t} x(z)\, dz$

A. i) $y(t) = x(t-4) \Rightarrow x(t) = y(t+4)$

ii) $y(t) = \cos x(t) \Rightarrow$ Not invertible, since multiple values can give Same output

iii) $y[n] = n x[n] \Rightarrow$ Not invertible, $x[n] = \dfrac{y[n]}{n}$ not valid for $n = 0$

iv) $y[n] = x[-n] \Rightarrow x[n] = y[-n]$

v) $y(t) = \dfrac{dx(t)}{dt} \Rightarrow$ Not invertible, integration of $x(t)$ without limits can't determine exact inverted state

vi) $y(t) = \displaystyle\int_{-\infty}^{2t} x(z)\, dz \Rightarrow \dfrac{dy(t)}{dt} = 2\,x(2t)$

$x(t) = \dfrac{1}{2} y(t/2)$

Q. Consider a system defined by $y(t) = \sin\{x(t)\}$. Check the time invariance.

A. Let $x_1(t)$ be input,

then $y_1(t) = \sin\{x_1(t)\}$

Let $x_2(t) = x_1(t-t_0)$

Then $y_2(t) = \sin\{x_2(t)\} = \sin\{x_1(t-t_0)\} \longrightarrow A$

Now $y_1(t-t_0) = \sin\{x_1(t-t_0)\} \longrightarrow B$

$A = B$, hence time invariant

Q. Consider a system defined by $y[n] = n\,x[n]$. Check the time invariance.

A. Let $x_1[n]$ be input,

then $y_1[n] = n\,x_1[n]$

Let $x_2[n] = x_1[n-n_0]$

Then $y_2[n] = n\,x_1[n-n_0] \rightarrow A$

Now $y_1[n-n_0] = (n-n_0)\,x_1[n-n_0] \rightarrow B$

$A \neq B$, hence time variant

Q. Consider a system defined by $y(t) = x(2t)$. Check the time invariance.

A. Let $x_1(t)$ be input,

then $y_1(t) = x_1(2t)$

Let $x_2(t) = x_1(t-t_0)$

Then $y_2(t) = x_2(2t) = x_1(2t - t_0) \longrightarrow A$

Now $y_1(t-t_0) = x_1(2(t-t_0)) \longrightarrow B$

$A \neq B$, hence time variant

Q. Consider a system defined by $y(t) = t\,x(t)$. Check the linearity

A. Consider 2 arbitrary inputs

$x_1(t) \longrightarrow y_1(t) = t\,x_1(t)$

$x_2(t) \longrightarrow y_2(t) = t\,x_2(t)$

Let some $x_3(t) = a x_1(t) + b x_2(t)$

then $y_3(t) = t\left(a x_1(t) + b x_2(t)\right)$

$= a t x_1(t) + b t x_2(t)$

$= a y_1(t) + b y_2(t)$

Since properties of additivity & homogenity are satisfied

Hence, system is linear

Q. Consider a system defined by $y(t) = x^2(t)$. Check the linearity

A. Consider 2 arbitrary inputs

$x_1(t) \rightarrow y_1(t) = x_1^2(t)$

$x_2(t) \rightarrow y_2(t) = x_2^2(t)$

Let some $x_3(t) = a x_1(t) + b x_2(t)$

then $y_3(t) = x_3^2(t)$

$$= (a x_1(t) + b x_2(t))^2$$

$$= a^2 x_1^2(t) + b^2 x_2^2(t) + 2ab\, x_1(t) x_2(t)$$

$$= a^2 y_1(t) + b^2 y_2(t) + 2ab\, x_1(t) x_2(t)$$

$$\neq a y_1(t) + b y_2(t)$$

Hence system is non-linear

Q. Consider a system defined by $y[n] = 2x[n] + 3$. Check the linearity

A. Consider 2 arbitrary inputs

$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$

$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3$

Let some $x_3[n] = a x_1[n] + b x_2[n]$

then $y_3[n] = 2x_3[n] + 3$

$$= 2(a x_1[n] + b x_2[n]) + 3$$

$$= 2a x_1[n] + 3 + 2b x_2[n]$$

$$\neq a y_1[n] + b y_2[n]$$

Hence system is non-linear

But for zero input / zero output condition is not obeyed,

$y[n] = 2x[n]$ is linear

$y_0[n] = 3 \rightarrow$ zero input response

Hence system is incrementally linear system

Q. Consider a system defined by $y(t) = x(t) + \frac{d\,x(t)}{dt}$. Check the time invariance

A. Let $x_1(t)$ be input,

then $y_1(t) = x_1(t) + \frac{dx_1(t)}{dt}$

Let $x_2(t) = x_1(t - t_0)$

Then $y_2(t) = x_2(t) + \frac{dx_2(t)}{dt} = x_1(t - t_0) + \frac{d}{dt} x_1(t - t_0) \longrightarrow A$

Now $y_1(t - t_0) = x_1(t - t_0) + \frac{d}{dt} x_1(t - t_0) \longrightarrow B$

$A = B$, Hence system is time invariant

Q. Consider a system defined by $y(t) = x(t) + \int x(t)\,dt$. Check the time invariance

A. Let $x_1(t)$ be input,

then $y_1(t) = x_1(t) + \int x_1(t)\,dt$

Let $x_2(t) = x_1(t-t_0)$

Then $y_2(t) = x_2(t) + \int x_2(t)\,dt = x_1(t-t_0) + \int x_1(t-t_0)\,dt \longrightarrow A$

Now $y_1(t-t_0) = x_1(t-t_0) + x_1(t-t_0)\,dt \longrightarrow B$

$A = B$, Hence system is time invariant

Q. Consider a system defined by $y(t) = 4x(t) + 2\frac{dx(t)}{dt}$. Check the linearity

A. Consider 2 arbitrary inputs

$x_1(t) \rightarrow y_1(t) = 4x_1(t) + 2\frac{dx_1(t)}{dt}$

$x_2(t) \rightarrow y_2(t) = 4x_2(t) + 2\frac{dx_2(t)}{dt}$

Let some $x_3(t) = a x_1(t) + b x_2(t)$

then $y_3(t) = 4x_3(t) + 2\frac{d x_3(t)}{dt}$

$$= 4\left(ax_1(t) + bx_2(t)\right) + 2\frac{d}{dt}\left(ax_1(t) + bx_2(t)\right)$$

$$= 4ax_1(t) + 2a\frac{dx_1(t)}{dt} + 4b\,x_2(t) + 2b\frac{dx_2(t)}{dt}$$

$$= ay_1(t) + by_2(t)$$

Since properties of additivity & homogenity are satisfied

Hence, system is linear

Q. Consider a system defined by $y[n] = a^{x[n]}$. Check the time invariance

A. Let $x_1[n]$ be input,

then $y_1[n] = a^{x_1[n]}$

Let $x_2[n] = x_1[n-n_0]$

then $y_2[n] = a^{x_2[n]} = a^{x_1[n-n_0]} \longrightarrow A$

Now $y_1[n-n_0] = a^{x_1[n-n_0]} \longrightarrow B$

$A = B$, Hence system is time invariant

Q Consider a system defined by $y[n] = x[n^2]$. Check the linearity

A. Consider 2 arbitrary inputs

$x_1[n] \rightarrow y_1[n] = x_1[n^2]$

$x_2[n] \rightarrow y_2[n] = x_2[n^2]$

Let some $x_3[n] = a x_1[n] + b x_2[n]$

then $y_3[n] = x_3[n^2] = a x_1[n^2] + b x_2[n^2]$

$$= ay_1[n] + by_2[n]$$

Since properties of additivity & homogenity are satisfied

Hence, system is linear

Linear Time Invariant Systems

→ Mathematical Modelling

- Impulse Response
- Linear constant
- State-space representation
- Transfer function

→ Time domain modelling (Impulse Response, Linear constant, State-space representation)

→ Frequency domain modelling (Transfer function)

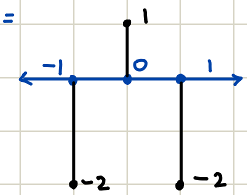
→ Impulse Response

→ The impulse response $h[n]$ of LTI system is the output of the system when unit impulse $\delta[n]$ is applied as input to LTI Systems

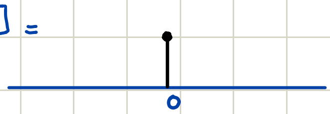
$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n] = y[n]$$

All initial conditions are zero

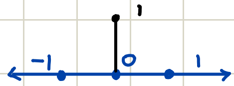
ex - $x[n] =$



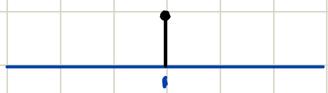
$\delta[n-0] =$



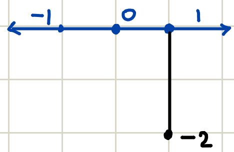
$\Rightarrow \delta[n-0].\, x[n] = x[0] = 1$



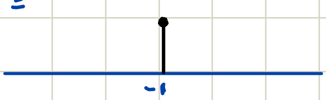
$\delta[n-1] =$



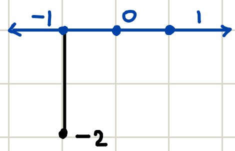
$\Rightarrow \delta[n-1].\, x[n] = x[1] = -2$



$\delta[n+1] =$



$\Rightarrow \delta[n+1].\, x[n] = x[-1] = -2$



Convolution Sum

→ An arbitrary input sequence $x[n]$ can be expressed as a weighted linear combination of unit impulse function given by

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

→ Now, the discrete time system response $y[n]$ is given by

$$y[n] = H[x[n]]$$

$$= H\left[x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,H[\delta[n-k]]$$ (By principle of superposition)

$$h_k[n] = H[\delta[n-k]]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]} \longrightarrow \text{Convolution Sum}$$

→ Computing convolution b/w $x(k)$ and $h(k)$

1) Plot the given func^n in terms of k
2) Obtain $h(-k)$ by folding $h(k)$ about $k=0$
3) Shift $h(-k)$ by n to right if n is +ve & left if n is -ve
4) Multiply $x(k)$ by $h(n_0-k)$ to obtain product sequence

(or)

Sum of the values of the product sequence to obtain the value of output at time $n=n_0$

Q. $h[n] = \{+2, 0, 2\}$
$x[n] = \{+1, 1\}$

A. $h[n] = \{\underset{-1}{\overset{\uparrow}{+2}}, \underset{0}{\overset{\uparrow}{0}}, \underset{1}{\overset{\uparrow}{2}}\}$
$x[n] = \{\underset{0}{\overset{\uparrow}{+1}}, \underset{1}{\overset{\uparrow}{1}}\}$

$$y[n] = \sum_{k=0}^{1} x[k]\,h[n-k]$$

$$= x[0]\,h[n] + x[1]\,h[n-1]$$

$$= h[n] + h[n-1]$$

$$y[n=-1] = x[0]\,h[-1] + x[1]\,h[-2]$$
$$= (1\times 2) + (1\times 0)$$
$$= 2$$

$$y[n=1] = x[0]\,h[1] + x[1]\,h[0]$$
$$= (1\times 2) + (1\times 0)$$
$$= 2$$

$$y[n=0] = x[0]\,h[0] + x[1]\,h[-1]$$
$$= (1\times 0) + (1\times 2)$$
$$= 2$$

$$y[n=2] = x[0]\,h[2] + x[1]\,h[1]$$
$$= (1\times 0) + (1\times 2) = 2$$

$$h[n] = \begin{cases} 1, & n = \pm 1 \\ 2, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} 2, & n=0 \\ 3, & n=1 \\ -2, & n=2 \end{cases}$$

$$y[n] = \sum_{K=0}^{2} x[K] * h[K] = \sum_{n=0}^{2} x[K] \cdot h[n-K]$$

$$= x[0] \cdot h[n] + x[1] h[n-1] + x[2] h[n-2]$$

$$= 2h[n] + 3h[n-1] - 2h[n-2]$$

$$y[-1] = 2h[-1] + 3h[-2] - 2h[-3]$$
$$= 2$$

$$y[0] = 2h[0] + 3h[-1] - 2h[-2]$$
$$= 4 + 3$$
$$= 7$$

$$y[1] = 2h[1] + 3h[0] - 2h[-1]$$
$$= 2 + 6 - 2$$
$$= 6$$

$$y[2] = 2h[2] + 3h[1] - 2h[0]$$
$$= 0 + 3 - 4$$
$$= -1$$

$$y[3] = 2h[3] + 3h[0] - 2h[1]$$
$$= -2$$

$$y[n] = \{2, 7, \underset{\downarrow}{6}, -1, -2\}$$

Q. Convolve $x[n] = \{\mathbf{1}, 2, 2, 3\}$ and $h[n] = \{\mathbf{2}, -1, 3\}$. *The sample at origin is indicated in bold.*

A.
$$y[n] = \sum_{K=0}^{3} x[K]\, h[n-K]$$
$$= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3]$$
$$= h[n] + 2h[n-1] + 2h[n-2] + 3h[n-3]$$
$$y[0] = h[0] + 2h[-1] + 2h[-2] + 3h[-3]$$
$$= 2$$
$$y[1] = h[1] + 2h[0] + 2h[-1] + 3h[-2]$$
$$= -1 + 4 = 3$$
$$y[2] = h[2] + 2h[1] + 2h[0] + 3h[-1]$$
$$= 3 - 2 + 4 = 5$$
$$y[3] = h[3] + 2h[2] + 2h[1] + 3h[0]$$
$$= 0 + 6 - 2 + 6 = 10$$
$$y[4] = h[4] + 2h[3] + 2h[2] + 3h[1]$$
$$= 0 + 0 + 6 - 3 = 3$$
$$y[5] = h[5] + 2h[4] + 2h[3] + 3h[2]$$
$$= 0 + 0 + 0 + 9 = 9$$

$$y[n] = \{\mathbf{2}, 3, 5, 10, 3, 9\}$$

- **Important inferences from DT Convolution Sum:**
  - **Convolution Sum expresses the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.**
  - **From this, we can infer that an LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.**
  - **The response due to the input *x[k]* applied at time *k* is *x[k]h[n - k]*; i.e., it is a shifted and scaled version (an "echo") of *h[n]*.**
  - **As before, the actual output is the superposition of all these responses.**

Important Summation Formulaes!

1)
$$\sum_{K=1}^{N} K = \frac{N(N+1)}{2}$$

2)
$$\sum_{K=1}^{N} K^2 = \frac{N(N+1)(2N+1)}{6}$$

3)
$$\sum_{K=0}^{N} \alpha^K = \begin{cases} \dfrac{1-\alpha^{N+1}}{1-\alpha} = \dfrac{\alpha^{N+1}-1}{\alpha-1}, & |\alpha| < 1 \\ N+1, & \alpha = 1 \end{cases}$$

Convolution Integral

→ Continuous-time LTI system response $y(t)$ is given by

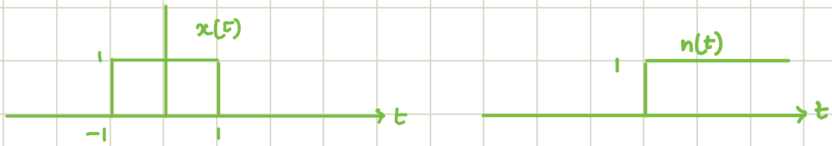
$$y(t) = H\left[x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)\,H\left[\delta(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

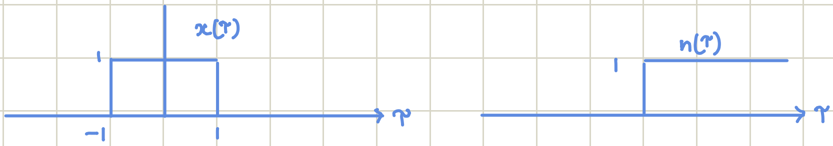
$$= x(t) \cdot h(t)$$

Q.

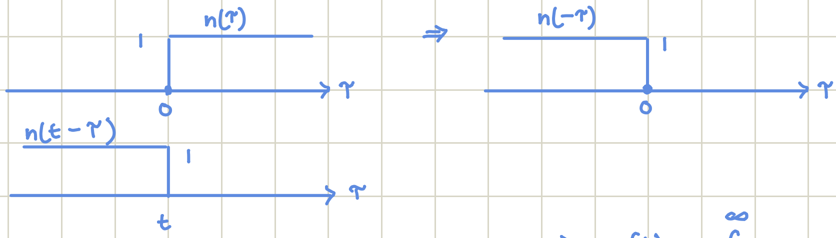


Convolve these 2

A. 1) Change independant variable t to τ



2) Plot $h(t-\tau)$



3)

$$y(t) = \begin{cases} 0, & t < -1 \\ t+1, & -1 \leq t \leq 1 \\ 2, & t > 1 \end{cases}$$

$y(t) = \int_{-\infty}^{\infty} 0 \cdot d\tau = 0$

$y(t) = \int_{-1}^{t} 1 \cdot 1\, d\tau = [\tau]_{-1}^{t} = t+1$

- **Important inferences from CT Convolution Integral:**
  - Convolution Integral expresses the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.
  - From this, we can infer that an LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.
  - The response due to the input $x(\tau)$ applied at time τ is $x(\tau)h(t-\tau)$; i.e., it is a shifted and scaled version (an "echo") of $h(t)$.
  - As before, the actual output is the superposition of all these responses.

$$\left(\begin{aligned} x(t) * \delta(t) &= x(t) \\ x(t) * \delta(t-\tau) &= x(t-\tau) \end{aligned}\right)$$

$$\left(\begin{aligned} x[n] * \delta[n] &= x[n] \\ x[n] * \delta[n-k] &= x[n-k] \end{aligned}\right)$$

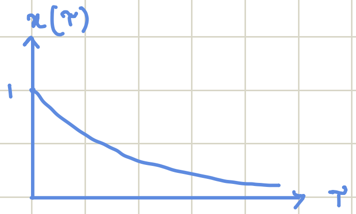
(Convolution of odd & odd = even
odd & even = odd
even & even = even)

Q. Convolve $x(t) = e^{-bt}u(t); b > 0$ and $h(t) = u(t)$.

A. Let $y(t) = x(t) * h(t)$

We rewrite $x(t)$ & $h(t)$ as $x(\tau)$ & $h(\tau)$

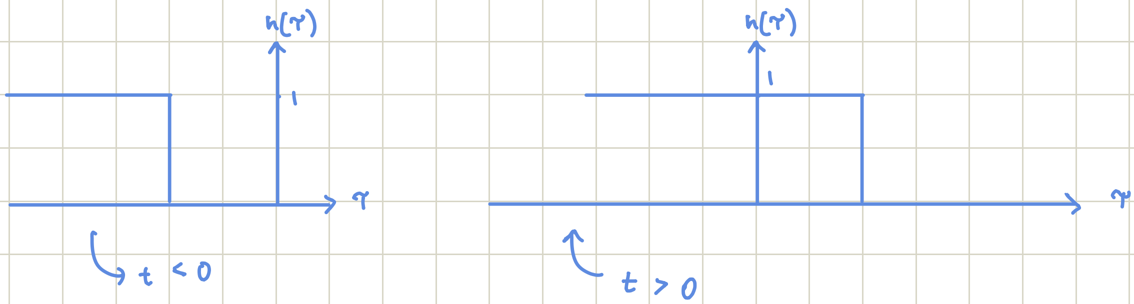
So,



and

$h(-\tau)$

We now plot $h(t-\tau)$ for $t<0$ and $t>0$



By observation,

there is no overlap when $t<0$

but there is overlap when $t>0$

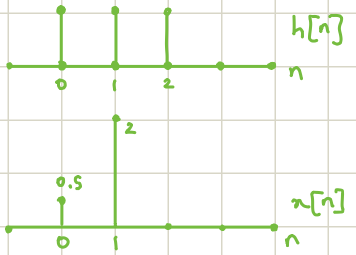
Hence,

$$x(\tau)\,h(t-\tau) = \begin{cases} e^{-b\tau}, & 0 \le \tau < t \\ 0, & \text{otherwise} \end{cases}$$

So, $$y(t) = \int_0^t e^{-b\tau}\,d\tau = \left(\frac{e^{-b\tau}}{-b}\right)_0^t = \frac{1-e^{-bt}}{b}, \quad t>0$$

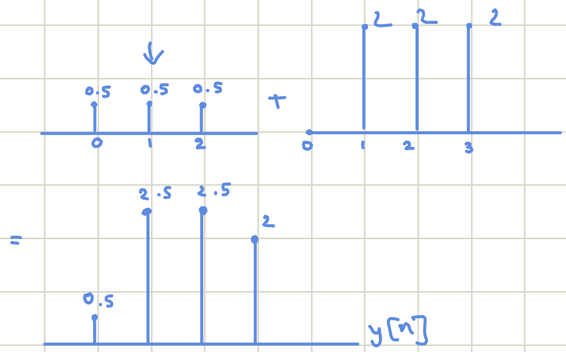
(or) $$\left(\frac{1-e^{-bt}}{b}\right)u(t)$$

Q. Consider $x[n]$ & $h[n]$ as shown



A. $y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$

$$y[n] = \sum_{K=0}^{1} x[K]\,h[n-K] = x[0]\,h[n-0] + x[1]\,h[n-1] = 0.5\,h[n] + 2\,h[n-1]$$



DT Unit Step Response from Impulse Response

→ With $u[n]$ as input & $h[n]$ as impulse response of a DT system,

$$y[n] = s[n] = u[n] * h[n] = h[n] * u[n]$$

$$= \sum_{K=-\infty}^{\infty} h[K]\,u[n-K]$$

Since input is $u[n]$, $s[n] = \sum_{K=-\infty}^{n} h[K]$

$s[n]$ → Step response

$h[K]$ → impulse response

also, $h[n] = s[n] - s[n-1]$

CT Unit Step Response from Impulse Response

→ With $u(t)$ as input & $h(t)$ as impulse response of a CT system,

$$y(t) = s(t) = u(t) * h(t) = h(t) * u(t)$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$s(t)$ → Step response

$h(\tau)$ → impulse response

also, $h(t) = \frac{d\,s(t)}{dt}$

Properties of LTI Systems

1) Commutative Property $\rightarrow x[n] * h[n] = h[n] * x[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K] = \sum_{K=-\infty}^{\infty} h[K]\, x[n-K]$

$\rightarrow x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

2) Distributive Property $\rightarrow x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$

$\rightarrow x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$

3) Associative Property $\rightarrow x[n] * \{h_1[n] * h_2[n]\} = \{x[n] * h_1[n]\} * h_2[n]$

$\rightarrow x(t) * \{h_1(t) * h_2(t)\} = \{x(t) * h_1(t)\} * h_2(t)$

LTI Systems with and without memory

→ System is memoryless if output at any time depends only on input at the same time

→ DT system is memoryless if $h[n] = 0$ for $n \neq 0$

CT system is memoryless if $h(t) = 0$ for $t \neq 0$

→ Impulse response $\Rightarrow$ $h[n] = K\,\delta[n]$ (or) $h(t) = K\,\delta(t)$

$\hookrightarrow \delta[0] \Rightarrow$ Constant $\Leftarrow \delta(0)$

→ Output $\Rightarrow$ $y[n] = K\, x[n]$ (or) $y(t) = K\, x(t)$

Invertibility of LTI Systems

→ Assume $h(t)$ or $h[n]$ is impulse response of first system

$h_1(t)$ or $h_1[n]$ is impulse response of second system in cascade,

Then first system is invertible if

$$h(t) * h_1(t) = \delta(t)$$

(or)

$$h[n] * h_1[n] = \delta(t)$$

ex: $h(t) = \delta(t+2) \Longrightarrow h_1(t) = \delta(t-2)$

$h(t) = \delta[n-3] \Longrightarrow h_1[n] = \delta[n+3]$

Causality of LTI Systems

→ If output at any time depends only on present & past input values, it's causal

→ Condition for LTI System to be causal is :

$h[n] = 0$ for $n < 0$ (or) $h(t) = 0$ for $t < 0$

Stability of LTI Systems

→ Condition : input $x[n]$ or $x(t)$ is bounded such that $|x[n]| < B < \infty$ or $|x(t)| < B < \infty$

→ If impulse response is $h[n]$ or $h(t)$, the convolution sum is : $|y[n]| = \left|\sum_{K=-\infty}^{\infty} x[K]h[n-K]\right| \leq \sum_{K=-\infty}^{\infty} |h[K]|\, |x[n-K]|$

$|y[n]| \leq B \sum_{K=-\infty}^{\infty} |h[K]| \Rightarrow \sum_{K=-\infty}^{\infty} |h[K]| < \infty$

$|y(t)| = \left|\int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, d\tau\right| \leq \int_{-\infty}^{\infty} |x(t)|\, |x(t-\tau)|\, d\tau$

$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

Impulse response of DT/CT should be absolutely summable /integrable

Q. For the following impulse responses, determine if corresponding system is

i) memoryless ii) causal iii) stable

a) $h(t) = \cos \pi t$ b) $h(t) = u(t+1)$ b) $h(t) = 3\delta(t)$ d) $h[n] = 2u[n] - 2u[n-5]$ e) $h[n] = \sin\left(\frac{\pi n}{2}\right)$

A. a) $h(t) = \cos \pi t$

→ $h(t) = 0$ for $t \neq 0$

So given system is memory system

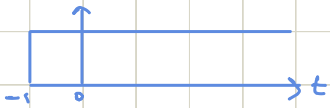
→ $h(t) = 0$ for $t < 0$

Non-causal

→ $\int_{-\infty}^{\infty} |h(\tau)| d\tau \not< \infty$

Not-stable

b) $h(t) = u(t+1)$



→ $h(t) = 0$ for $t \neq 0$

Memory System

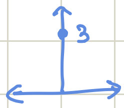
→ $h(t) = 0$ for $t < 0$

Not causal

→ $\int_{-\infty}^{\infty} |h(\tau)| d\tau \not< \infty$

Not-stable

c) $h(t) = 3\delta(t)$



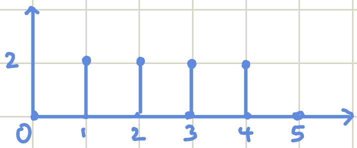
→ $h(t) \neq 0$ for $t \neq 0$

Non-memory

→ Causal

→ Stable

d)



→ Memory

→ Causal

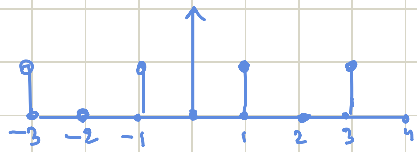
→ Stable

e) $h[n] = \sin\left(\frac{\pi n}{2}\right)$

→ Memory

→ Non-causal

→ Non-stable



Q. Evaluate step response for LTI System

a) $h[n] = \left(\frac{1}{2}\right)^n u^n$ b) $h(t) = u(t)$

A. a) $s[n] = \sum_{k=-\infty}^{n} h[k]$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n = \frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n+1}\right)$$

→ Memory

→ Causal

→ Non-stable

b) $h(t) = u(t)$

$$s(t) = \int_{-\infty}^{t} u(t)\, dt = \int_{0}^{t} 1 \cdot dt$$

$$= [t]_0^t = t$$

Causal LTI System defined by Linear Constant Coefficient Differential Eq

→ Differential equation conventions :

$$\dot{y} = \frac{dy}{dt} \quad \& \quad \ddot{y} = \frac{d^2y}{dt^2}$$

→ Usually we are given an implicit relation b/w $x(t)$ & $y(t)$
But we need an explicit expression
↳ express $y(t)$ as a function of $x(t)$

So, we have to solve the differential equation

→ The response to an input $x(t)$ consists the sum of homogeneous & particular solution

$y_p(t)$: input response

$y_h(t)$: natural response

→ An n^{th} order differential equation is expressed as :

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y(t) = b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^m} + \dots + b_1 \frac{dx}{dt} + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt}$$

(usually $n \geq m$)

❖ Method 1: To be used only when the complete solution (or) total response is required – a quick method.

- ❖ Compute the 'homogeneous' solution; do not evaluate the unknowns, now !
- ❖ Compute the 'particular solution', evaluate the unknowns.
- ❖ The 'complete' solution would be the sum of the above solutions.
- ❖ Now compute the unknowns of the homogeneous solution using the given initial conditions onto the complete solution to get final form of complete solution (or) total response.

Nature of roots	Form of Homogeneous Solution
Real and distinct	$\sum_{i=1}^{n} c_i e^{r_i t}$
Real and repeated 'p' times	$e^{rt}\left[c_0 + c_1 t + \dots\dots\dots + c_{p-1}t^{p-1}\right]$
Complex Conjugate r = α ± j ω	$e^{\alpha t}\left[c_1 \cos\omega t + c_2 \sin\omega t\right]$
Imaginary r = ± j ω	$c_1 \cos\omega t + c_2 \sin\omega t$

❖ Method 2: To be used when the individual responses are required and then to arrive at the final or total response.

- ❖ Compute the 'homogeneous' solution; do evaluate the unknowns, now !. Let this be the 'natural' response $y_n(t)$.
- ❖ Compute the 'particular solution' $y_p(t)$, evaluate the unknowns.
- ❖ The 'forced' response $y_f(t)$ would be the sum of the above solutions but with unknown homogeneous form.
- ❖ Now apply zero initial conditions to the above forced response to compute the unknowns in the homogeneous solution.
- ❖ The complete response will now be the sum of the natural response and forced response.

Input	Form of Particular Solution
Constant C_0	Another constant C_1
$e^{\alpha t}$	$Ce^{\alpha t}$
$\cos(\omega t + \beta)$	$C_1 \cos \omega t + C_2 \sin \omega t$
$e^{\alpha t}\cos(\omega t + \beta)$	$e^{\alpha t}\{C_1 \cos \omega t + C_2 \sin \omega t\}$
t	$C_0 + C_1 t$
$te^{\alpha t}$	$e^{\alpha t}\{C_0 + C_1 t\}$
t^p	$C_0 + C_1 t + C_2 t^2 + \dots + C_p t^p$

Q. Determine homo sol^n for $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 2x(t) + \dot{x}(t)$

$y(0)=3$, $\left.\frac{dy(t)}{dt}\right|_{y=0} = -7$, $x(t) = e^{-t}u(t)$

A. homo sol^n of $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 0$

$$y^2 + 5y + 6 \Rightarrow y = -2, -3$$

$$y_N(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$y_N(0) = c_1 + c_2 = 3$$

$$\frac{dy_N(t)}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$\frac{dy_N(0)}{dt} = -2c_1 - 3c_2 = -7$$

$$c_2 = 1, \quad c_1 = 2$$

$$y_N(t) = 2e^{-2t} + e^{-3t}$$

particular sol^n $\frac{d^2}{dt^2} y_p(t) + 5\frac{d}{dt} y_p(t) + 6y_p(t) = 2x(t) + \frac{d}{dt}x(t)$

$$y_p(t) = Ke^{-t}, \quad t \geq 0$$

$$\frac{d^2}{dt^2}(Ke^{-t}) + 5\frac{d}{dt}(Ke^{-t}) + 6Ke^{-t} = 2e^{-t} + \frac{d}{dt}(e^{-t})$$

$$Ke^{-t} - 5Ke^{-t} + 6Ke^{-t} = 2e^{-t} - e^{-t}$$

$$2K\cancel{e^{-t}} = \cancel{e^{-t}}$$

$$K = 1/2$$

Q. Evaluate the **natural**, **forced,** and **total response** of a CT system described by $\dot{y}(t) + 5y(t) = x(t)$ with $x(t) = u(t)$ and $y(0) = 2$.

A. $\dot{y}(t) + 5y(t) = x(t)$

$y + 5 = 0 \Rightarrow y = -5$

$\Rightarrow \quad y_N(t) = c_1 e^{-5t}$

$\Rightarrow$ assume $\quad y_F(t) = C$

$$0 + 5C = 1$$

$$C = \frac{1}{5} = 0.2$$

$y(t) = c_1 e^{-5t} + 0.2$

Apply $\quad y(0) = c_1 + 0.2$

$2 = c_1 + 0.2 \quad \Longrightarrow \quad y(t) = 1.8e^{-5t} + 0.2$

$c_1 = 1.8$

Causal LTI System defined by Linear Constant Coefficient Difference Eq

→ Similar to differential eq,

For a n^{th} order Linear Constant Co-efficient difference eq,

$$\sum_{K=0}^{n} a_n y[n-K] = \sum_{K=0}^{m} b_m x[n-K]$$

→ We can solve the above equation recursively,

$$y[n] = \frac{1}{a_0}\sum_{K=0}^{m} b_K x[n-K] - \frac{1}{a_0}\sum_{K=1}^{n} a_K y[n-K]$$ (usually $n \geq m$)

Special case $(n=0)$: $y[n] = \frac{1}{a_0}\sum_{K=0}^{m} b_K x[n-K]$

→ Current Output depends on current & previous values of inputs only

Hence Non-recursive equation (or) Explicit Relation

→ They have Impulse response of finite duration and hence called Finite Impulse Response Systems

BUT! for $n \geq 1$, They have Impulse response of infinite duration and hence called Infinite Impulse Response Systems

❖ **Method 1: To be used only when the complete solution (or) total response is required – a quick method.**

- ❖ **Compute the 'homogeneous' solution; do not evaluate the unknowns, now !**
- ❖ **Compute the 'particular solution', evaluate the unknowns.**
- ❖ **The 'complete' solution would be the sum of the above solutions.**
- ❖ **Now compute the unknowns of the homogeneous solution using the given initial conditions onto the complete solution to get final form of complete solution (or) total response.**

Nature of roots	Form of Homogeneous Solution
Real and distinct	$\sum_{i=1}^{n} c_i r_i^n$
Real and repeated 'p' times	$r^n\left[c_0 + c_1 n + \ldots\ldots\ldots + c_{p-1} n^{p-1}\right]$
Complex $re^{j\omega}$	$r^n\left[c_1 \cos n\omega + c_2 \sin n\omega\right]$

❖ **Method 2: To be used when the individual responses are required and then to arrive at the final or total response.**

- ❖ **Compute the 'homogeneous' solution; do evaluate the unknowns, now !. Let this be the 'natural' response $y_n[n]$.**
- ❖ **Compute the 'particular solution' $y_p[n]$, evaluate the unknowns.**
- ❖ **The 'forced' response $y_f[n]$ would be the sum of the above solutions but with unknown homogeneous form.**
- ❖ **Now apply zero initial conditions to the above forced response to compute the unknowns in the homogeneous solution.**
- ❖ **The complete response will now be the sum of the natural response and forced response.**

Input	Form of Particular Solution
Constant C_0	Another constant C_1
α^n	$C\alpha^n$
$\cos(n\omega + \beta)$	$C_1 \cos n\omega + C_2 \sin n\omega$
$\alpha^n \cos(n\omega + \beta)$	$\alpha^n\{C_1 \cos n\omega + C_2 \sin n\omega\}$
n	$C_0 + C_1 n$
n^p	$C_0 + C_1 n + C_2 n^2 + \ldots + C_p n^p$
$n\alpha^n$	$\alpha^n \{C_0 + C_1 n\}$
$n^p \alpha^n$	$\alpha^n\{C_0 + C_1 n + C_2 n^2 + \ldots + C_p n^p\}$

Q. Determine homogeneous sol^n for system described by following difference eq

$$y[n] - \frac{9}{16} y[n-1] = x[n-1]$$

A. homogeneous eq is,

$$y[n] - \frac{9}{16} y[n-1] = 0$$

$$\lambda^n - \frac{9}{16} \lambda^{n-1} = 0$$

$\lambda = 9/16 \longrightarrow$ real & distinct

$$y[n] = c\left(\frac{9}{16}\right)^n$$

for $n=0$, $y[0] = c$

Q. Determine Particular sol^n for system described by following difference eq

$$y[n] - \frac{9}{16} y[n-1] = x[n-1]$$

input $x[n] = 2$

A. $y_p[n] - \frac{9}{16} y_p[n-1] = x[n-1]$ $\qquad y_p[n] = K$

$$K - \frac{9}{16} K = 2$$

$$\frac{7K}{16} = 2 \Rightarrow K = \frac{32}{7} = y_p[n]$$

$$y_F[n] = y_h[n] + y_p[n]$$

$$= c\left(\frac{9}{16}\right)^n + \frac{32}{7}$$

For $n=0$

$$y[0] - \left(\frac{9}{16}\right) y[-1] = x[-1]$$

$$y[0] = 0$$

$$y_r[0] = c + \frac{32}{7} = 0$$

$$c = \frac{-32}{7}$$

$$y_r[n] = -\frac{32}{7}\left(\frac{9}{16}\right)^n + \frac{32}{7}$$

$$y_c[n] = y_N[n] + y_F[n]$$

$$y_c[n] = -\frac{4}{3}\left(\frac{9}{16}\right)^n - \frac{32}{7}\left(\frac{9}{16}\right)^n + \frac{32}{7}$$

UNIT - 2

Explanation of Complex Exponentials

→ Any signal that could be represented as a linear combination of some basic signals possesses the following properties :

i) Set of basic signals can be used to construct a broad & useful class of signals

ii) Response of LTI system to such a signal should be simple to provide a convenient representation for response of system to any signal constructed as linear combination of the basic signals

→ Continuous Time : e^{st} (s : complex no.)

Discrete Time : z^n ; $z = |z|e^{j\theta}$ (z : complex no.)

So, response of LTI system to a complex exponential input is same complex exponential with only change in amplitude

Continuous Time : $e^{st} \longrightarrow y(t) = \underline{H(s)\, e^{st}}$ ($H(s)$ & $H(z)$: complex amplitudes)

Discrete Time : $z^n \qquad y[n] = \underline{H(z)\, z^n}$ ↳ Eigen values

Eigen Functions

→ All complex exponentials are Eigen functions of LTI Systems

→ In CT Domain, input ⇒ e^{st}

Then, $H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau = H(s)\, e^{st}$$

→ In DT Domain, input ⇒ z^n

Then, $H(z) = \sum_{-\infty}^{\infty} h[k]\, z^{-k}$

$$y(t) = \sum_{-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{-\infty}^{\infty} h[k]\, z^{n-k} = z^n \sum_{-\infty}^{\infty} h[k]\, z^{-k} = z^n H(z)$$

→ For CT, $x(t) = \sum_k a_k e^{s_k t} \longrightarrow \boxed{y(t) = \sum_k a_k H(s_k)\, e^{s_k t}}$

→ For DT, $x[n] = \sum_k a_k z_k^n \longrightarrow \boxed{y[n] = \sum_k a_k H(z_k)\, z_k^n}$

→ In Future, $s \to j\omega$ (or) $e^{st} = e^{j\omega t}$

$z \to e^{j\omega}$ (or) $z^n = e^{j\omega n}$

Q. Consider LTI System $y(t) = x(t-3)$. Verify Eigen values

A. Let input be $x(t) = e^{j2t} = e^{st}$

Then $y(t) = e^{j2(t-3)}$

$$= \underbrace{e^{-j6}}_{\text{Eigen Value}} \cdot \underbrace{e^{j2t}}_{\text{Eigen Function}}$$

For verification,

$$h(t) = \delta(t-3)$$

Therefore, $H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} \underbrace{\delta(t-3)}_{\text{This term is 1 only when } t=3} e^{-st}\, dt$$

$$= e^{-3s}$$

$H(j2) = e^{-j6}$ $\Rightarrow$ Same as earlier

Fourier Series for CT Periodic Signals

→ Harmonically related complex exponentials – set of complex exponentials which are periodic with common period T_0 or T
– set of periodic exponentials with fundamental frequencies

→ Condition for complex exponential to be periodic $\Rightarrow$ $e^{j\omega_0 t} = 1$

which means $\omega T_0 = 2\pi K$

$K = 0, \pm 1, \pm 2, \ldots$

So, $\phi_K(t) = e^{jK\omega_0 t}$, $K = 0, \pm 1, \pm 2, \ldots$

→ A linear combination of harmonically related complex exponentials is :

$$x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$$

$$= \sum_{K=-\infty}^{\infty} a_K e^{jK\left(\frac{2\pi}{T}\right)t}$$

→ Fourier Series representation of CT periodic signal

→ when $K = 0$ $\Rightarrow$ $x(t) = a_0$ $\Rightarrow$ constant

$K = -1$ or $+1$ $\Rightarrow$ fundamental (or) first harmonic components

$K = -2$ or $+2$ $\Rightarrow$ second harmonic components

$K = -N$ or $+N$ $\Rightarrow$ N^{th} harmonic components

Q. Consider periodic signal $x(t)$ with $\omega_0 = 2\pi$ expressed as

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

$a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$

Write complete expression of $x(t)$

A. $$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$= 1 + \frac{1}{4} \times (2\cos 2\pi t) + \frac{1}{2}(2\cos 4\pi t) + \frac{1}{3}(2\cos 6\pi t)$$

$$= 1 + \frac{\cos 2\pi t}{2} + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

→ For real periodic CT signals, Fourier coefficients are conjugate symmetric

$x(t) = x^*(t)$

→ If $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ → Synthesis equation

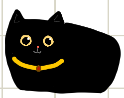
Then $a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$ → Analysis equation

$$= \frac{1}{T}\int_0^T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt$$

Proof ⇒ $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(t) \cdot a_k e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t}$$

$$\int_0^T x(t) \cdot a_k e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot e^{-jn\omega_0 t} dt$$



$$= \sum_{k=-\infty}^{\infty} a_k \left\{ \int_0^T e^{j(k-n)\omega_0 t} dt \right\}$$

Case (i) ⇒ $n = 1$ ⇒ $\int_0^T e^{j(k-n)\omega_0 t} dt = T$

Case (ii) ⇒ $n \neq k$, Let $k - n = p$

$$\int_0^T e^{jp\omega_0 t} dt = \left[\frac{e^{jp\omega_0 T}}{jp\omega_0}\right]_0^T = \frac{e^{jp\omega_0 T} - 1}{jp\omega_0} = \frac{e^{jp\omega_0 \cdot \frac{2\pi}{\omega_0}} - 1}{jp\omega_0}$$

$$= \frac{e^{j2\pi p} - 1}{jp\omega_0} = \frac{1-1}{jp\omega_0} = 0$$

So, $$\int_0^T e^{j(k-n)\omega_0 t} = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

Hence, $$\sum_{k=-\infty}^{\infty} a_k \left\{ \int_0^T e^{j(k-n)\omega_0 t} dt \right\} = a_k T \Rightarrow \boxed{a_k = \frac{1}{T}\int_0^T x(t) \cdot e^{-j\omega_0 n t} dt}$$

→ Frequency Spectrum (Line Spectrum) of periodic CT signals

a_K is complex,

So, $a_K = |a_K| \angle a_K$

⇒ Magnitude Spectrum ⇒ $|a_K|$ vs K

⇒ Phase Spectrum ⇒ $\angle a_K$ vs K

Q. Consider $x(t) = \sin \omega_0 t$, fundamental frequency $= \omega_0$

Find fourier coefficients

A. $\sin \omega_0 t = \frac{1}{2j}\left\{ e^{j\omega_0 t} - e^{-j\omega_0 t} \right\}$

$$= \frac{1}{2j} \cdot e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \longrightarrow (1)$$

Comparing (1) with $\sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$

$$a_1 = \frac{1}{2j}, \quad a_{-1} = \frac{-1}{2j}$$

$a_K = 0$ for $K \neq 1$ or -1

Q. Consider $x(t) = \cos \omega_0 t$, fundamental frequency $= \omega_0$

Find fourier coefficients

$\cos \omega_0 t = \frac{1}{2}\left\{ e^{j\omega_0 t} + e^{-j\omega_0 t} \right\}$

$$= \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} \longrightarrow (1)$$

Comparing (1) with $\sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$

$$a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}$$

$a_K = 0$ for $K \neq 1$ or -1

Q. Consider $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

Find fourier coefficients

A. $\sin\omega_0 t = \frac{1}{2j}\left\{e^{j\omega_0 t} - e^{-j\omega_0 t}\right\}$

$2\cos\omega_0 t = e^{j\omega_0 t} + e^{-j\omega_0 t}$

$\cos\left(2\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2}\left\{e^{j\left(2\omega_0 t + \frac{\pi}{4}\right)} + e^{-j\left(2\omega_0 t + \frac{\pi}{4}\right)}\right\}$

$$x(t) = 1 + \left\{1 + \frac{1}{2j}\right\}e^{j\omega_0 t} + \left\{1 - \frac{1}{2j}\right\}e^{-j\omega_0 t} + \left\{\frac{1}{2}e^{j(\pi/4)}\right\}e^{j2\omega_0 t} + \left\{\frac{1}{2}e^{-j\frac{\pi}{4}}\right\}e^{-j2\omega_0 t}$$

$a_0 = 1$

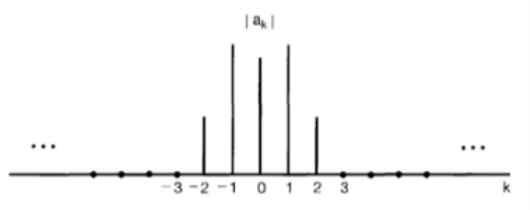
$a_1 = 1 + \frac{1}{2j}$

$a_{-1} = 1 - \frac{1}{2j}$

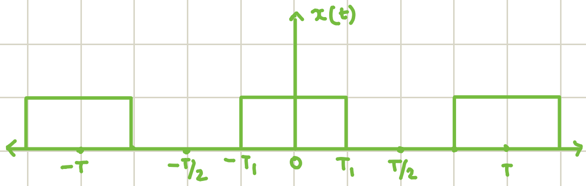
$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j)$

$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j)$

$a_k = 0$ for $|k| > 2$



Q. Consider a sequence wave



$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$ for one period

A. Fundamental frequency $\omega_0 = \frac{2\pi}{T}$

Because of symmetry about $t = 0$, The interval is $-\frac{T}{2} \leq t \leq \frac{T}{2}$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

From $x(t)$, limit changes to $-T_1 \leq t \leq T_1$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-jk\omega_0 t}\, dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} = \frac{2T_1}{T}$$

$$a_{k\neq 0} = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = \frac{1}{-jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} = \frac{2}{k\omega_0 T}\left\{\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right\}$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \left(\omega_0 = \frac{2\pi}{T}\right)$$

→ Consider $x(t)$ with fundamental period T, Fourier Series in trigonometric form ⇒

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt \;;\quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt \;;\quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt$$

where a_n & b_n are max amplitudes of n^{th} harmonic component

$$a_{K=0} = \frac{a_0}{2}$$

$$a_K = \frac{1}{2}\{a_n - jb_n\} \qquad K = 1,2,3,4\ldots$$

$$a_{-K} = \frac{1}{2}\{a_n + jb_n\} \qquad K = -1,-2,-3,-4\ldots$$

$$|a_K| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \qquad \text{for all } K \text{ except } 0$$

Convergence of CT Fourier Series

→ Must follows the 3 conditions of Dirichlet

1) over one period, $x(t)$ must be completely integrable

$$\int_T |x(t)|\,dt < \infty$$

2) There must be finite no. of maxima & minima during any single period

3) In any finite interval of time, there are only finite no. of discontinuities

→ For a periodic signal with Finite no. of discontinuities in each period, the FS representation equals to signal everywhere except at isolated points of discontinuity, at which series converges to avg. value of signal on either side of discontinuity

→ If energy of error b/w original signal & FS representation is zero, then both are same

Properties of CT Fourier series

→ Assume $x(t)$ is periodic with period T & fundamental frequency $\omega_0 = \frac{2\pi}{T}$ and $x(t) \overset{FS}{\Longleftrightarrow} a_k$

1) **Linearity** – Let $x(t)$ & $y(t)$ be 2 periodic signals with same fundamental period T. Any linear combination of these 2 signals will also be periodic with period T

- If $x(t) \overset{FS}{\Longleftrightarrow} a_k$ & $y(t) \overset{FS}{\Longleftrightarrow} b_k$
then $z(t) = Ax(t) + By(t) \overset{FS}{\Longleftrightarrow} c_k = Aa_k + Bb_k$

2) **Time Shift** – When time shift is applied, the period is preserved

- If $x(t) \overset{FS}{\Longleftrightarrow} a_k$
then $x(t-t_0) \overset{FS}{\Longleftrightarrow} b_k = e^{-jk\omega_0 t_0} a_k$
- Magnitudes are unaltered when time shift is applied

3) **Time Reversal** – When time reversal is applied, the period is preserved

- $x(t) \overset{FS}{\Longleftrightarrow} a_k$
$x(-t) \overset{FS}{\Longleftrightarrow} a_{-k}$
- If $x(t)$ is even, $a_k = a_{-k}$
$x(t)$ is odd, $a_k = -a_{-k}$

4) **Time Scaling** – Changes period of considered signal

- If $x(t)$ is periodic, period $= T$, fundamental freq $= \omega_0$
then $x(\alpha t)$ is periodic, period $= \frac{T}{\alpha}$, fundamental freq $= \alpha\omega_0$
- $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$

- Fourier co-efficients don't change with time signal

5) **Multiplication** – If $x(t)$ & $y(t)$ are periodic, then their product is also periodic

- If $x(t) \overset{FS}{\Longleftrightarrow} a_k$ & $y(t) \overset{FS}{\Longleftrightarrow} b_k$
Then $x(t)\,y(t) \overset{FS}{\Longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k$

6) **Conjugation & Conjugation Symmetry** – Complex conjugate of periodic signal has effect of complex conjugation & time reversal on corresponding Fourier Series coefficients

- If $x(t) \overset{FS}{\Longleftrightarrow} a_k$ then $x^*(t) \overset{FS}{\Longleftrightarrow} a^*_{-k}$
- For real $x(t)$, $|a_k| = |a_{-k}|$
For $x(t)$ is real & even, $a_k = a_{-k}$
For $x(t)$ is real & odd, a_k is purely imaginary & odd

7) **Parseval's Identity** – For CT, $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

↳ Total avg power ↳ Sum of avg powers in all of its harmonic components

Summary !

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$