- ## Gamma Functions :

① $\Gamma(1) = 1$

Proof: W.K.T $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\Rightarrow \Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx$$

$$\Rightarrow \Gamma(1) = \int_0^\infty e^{-x} dx$$

$$\Rightarrow \Gamma(1) = \left[-e^{-x}\right]_0^\infty$$

$$\Rightarrow \Gamma(1) = -(0-1) = 1$$

$$\left[\lim_{x\to\infty} e^{-x} = 0\right]$$

② $\Gamma(n+1) = n\Gamma(n)$ → Reduction formula

Proof: W.K.T $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$ and $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

Let $\Gamma(n+1) = I_1$ and $\Gamma(n) = I_2$

$$I_1 = \left[-x^n e^{-x} - \int n x^{n-1} (-e^{-x}) dx\right]_0^\infty$$

(By parts method)

$$I_1 = \left[-x^n e^{-x} + n \int x^{n-1} e^{-x} dx \right]_0^\infty$$

$$I_1 = (0-0) + n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Rightarrow I_1 = n \int_0^\infty e^{-x} x^{n-1} dx = n\Gamma(n)$$

$$n \quad I_1 = n\Gamma(n)$$

$$\Rightarrow \Gamma(n+1) = n\Gamma(n)$$

(iii) $\Gamma(n+1) = n!$

<u>Proof</u>: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

$$\vdots$$

$$\Gamma(2) = 1$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\because \Gamma(n) = (n-1)\Gamma(n-1)$$

$\Rightarrow \Gamma(n+1) = n(n-1)\Gamma(n-1)$

$\Rightarrow \Gamma(n+1) = n(n-1)(n-2)\Gamma(n-2)$

$\vdots$

$\Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots 2.1 = n!$

(4) $\Gamma(1/2) = \sqrt{\pi}$

Proof: W.K.T. $\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2} dx$

$\Rightarrow \Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx = I_1$

Similarly, $\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$

$\Rightarrow \Gamma(1/2) = 2\int_0^\infty e^{-y^2} dy = I_2$

Now, $I_1 \times I_2$

$\Rightarrow \Gamma(1/2) \cdot \Gamma(1/2) = 4\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$

$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$

Let $x = r\cos\theta$, $y = r\sin\theta$ and replace $dx\,dy \longrightarrow r\,dr\,d\theta$

$$\Rightarrow \left[\Gamma(1/2)\right]^2 = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2(\sin^2\theta+\cos^2\theta)} \cdot r\,dr\,d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} \cdot r\,dr\,d\theta$$

$$\Rightarrow 4\int_0^{\pi/2} d\theta \cdot \int_0^{\infty} r e^{-r^2} dr$$

Let $\boxed{r^2 = t}$ $\Rightarrow r = \sqrt{t}$

$\Rightarrow 2r\,dr = dt$

$\Rightarrow dr = \dfrac{dt}{2r} \Rightarrow dr = \dfrac{dt}{2\sqrt{t}}$

$$\Rightarrow \left[\Gamma(1/2)\right]^2 = 4\int_0^{\pi/2} d\theta \cdot \int_0^{\infty} \frac{\cancel{\sqrt{t}}\, e^{-t}}{2\cancel{\sqrt{t}}}\, dt$$

$$= \frac{4}{2}\left[\theta\right]_0^{\pi/2} \cdot \left[-e^{-t}\right]_0^{\infty}$$

$$\Rightarrow -2 \cdot \frac{\pi}{2} \cdot (-1) = \pi$$

$$\Rightarrow \left[\Gamma(1/2)\right]^2 = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

(6) $\int_0^\infty e^{-x^n} dx = \frac{1}{n} \Gamma\left(\frac{1}{n}\right)$; $n > 0$

Proof : Let $x^n = t \Rightarrow x = t^{1/n}$

at $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$\Rightarrow n x^{n-1} dx = dt$

$$\Rightarrow dx = \frac{dt}{n x^{n-1}}$$

$$\Rightarrow dx = \frac{dt}{n t^{\frac{n-1}{n}}} \Rightarrow dx = \frac{dt}{n t^{1-1/n}}$$

$$\Rightarrow \int_0^\infty e^{-t} \cdot \frac{1}{n t^{1-1/n}} dt$$

$$\Rightarrow \frac{1}{n} \int_0^\infty e^{-t} \cdot t^{1/n - 1} dt = \frac{1}{n} \Gamma\left(\frac{1}{n}\right)$$

(7) $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n . n!}{(m+1)^{n+1}}$

Proof: Let $x = e^{-t}$ At $x = 0$, $t = \infty$

$dx = -e^{-t} dt$ $x = 1$, $t = 0$

$$\int_\infty^0 e^{-mt} (\log e^{-t})^n (-e^{-t}) \, dt$$

$$\Rightarrow -\int_\infty^0 e^{-mt} \cdot (-t)^n \cdot e^{-t} \, dt$$

$$\Rightarrow \int_0^\infty e^{-t(m+1)} \cdot (-t)^n \, dt$$

$$\left[\int_a^b x \, dx = -\int_b^a x \, dx \right]$$

$$\Rightarrow (-1)^n \int_0^\infty e^{-t(m+1)} \cdot t^n \, dt$$

Let $(m+1)t = u$

$\Rightarrow (m+1) dt = du \Rightarrow dt = \frac{du}{m+1}$ at $t = 0$, $u = 0$

$t = \infty$, $u = \infty$

$$\Rightarrow (-1)^n \int_0^\infty e^{-u} \cdot \left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)}$$

$$\Rightarrow \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^n \, du = \frac{(-1)^n}{(m+1)^{n+1}} \cdot \Gamma(n+1) \rightarrow (1)$$

$\because \Gamma(n+1) = n!$

$(1) = (-1)^n (m+1)^{-(n+1)} . n!$

⑧ $\int_0^1 x^{m-1} \cdot \left(\log \frac{1}{x}\right)^{n-1} dx = \frac{\Gamma(n)}{m^n}$

Proof: Let $x = e^{-t}$ at $x = 0,\ t = \infty$

$dx = -e^{-t}\, dt$ $x = 1,\ t = 0$

$$\int_\infty^0 e^{-t(m-1)} \cdot (\log e^t)^{n-1} \cdot (-e^{-t})\, dt$$

$$\Rightarrow -\int_\infty^0 e^{-t(m-1)-t} \cdot t^{n-1}\, dt$$

$$\Rightarrow \int_0^\infty e^{-t(m-1+1)} \cdot t^{n-1}\, dt = \int_0^\infty e^{-mt} \cdot t^{n-1}\, dt$$

Let $mt = u$ at $t = 0,\ u = 0$

$m\,dt = du$ $t = \infty,\ u = \infty$

$$\Rightarrow \int_0^\infty e^{-u} \cdot \frac{u^{n-1}}{m^{n-1}} \cdot \frac{du}{m}$$

$$\Rightarrow \frac{1}{m^n} \int_0^\infty e^{-u} \cdot u^{n-1}\, du = \frac{\Gamma(n)}{m^n}$$

• Beta Functions:

① $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{2m-1}\theta \, d\theta$

Proof: W.K.T. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ — ①

Let $x = \sin^2\theta$ at $x=0$, $\theta=0$

$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$ $x=1$, $\theta = \pi/2$

$$\Rightarrow \beta(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(m,n) = 2\cdot\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2(n-1)}\theta \cdot \sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(m,n) = 2\cdot\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

② $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Proof: $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$ — ①

Let $x = \tan^2\theta$ at $x=0$, $\theta=0$

$dx = 2\tan\theta\sec^2\theta\,d\theta$ $x=\infty$, $\theta = \pi/2$

$$\beta(m,n) = \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1} \cdot 2\tan\theta\sec^2\theta}{(1+\tan^2\theta)^{m+n}}\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \tan^{2m-2}\theta \tan\theta \sec^2\theta \sec^{-2(m+n)}\theta\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \tan^{2m-1}\theta \sec^{2-2m-2n}\theta\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta} \cdot \frac{1}{\cos^{2-2m-2n}\theta}\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{-(1-2n)}\theta\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

- Relation b/w Γ & β :

(1) $\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

Proof: W.K.T. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta$ — (1)

$$\Gamma(m) = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\, dx \quad — (2)$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-y^2} y^{2m-1}\, dy \quad — (3)$$

$$\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr \quad — (4)$$

$$\Gamma(m)\cdot\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx \cdot 2\int_0^{\infty} e^{-y^2} y^{2m-1} dy$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx dy \to r dr d\theta$

$$\Rightarrow 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)} \cdot (r\cos\theta)^{2n-1} \cdot (r\sin\theta)^{2m-1} r\, dr\, d\theta = \Gamma(m)\cdot\Gamma(n)$$

$$\Rightarrow 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} \cdot r^{2n-1+2m-1+1} \cdot \cos^{2n-1}\theta \cdot \sin^{2m-1}\theta\, dr\, d\theta = \Gamma(m)\cdot\Gamma(n)$$

$$\Rightarrow \Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\cdot r^{2(n+m)-1}\cdot \cos^{2n-1}\theta\cdot \sin^{2m-1}\theta\, dr\, d\theta$$

$$\Rightarrow \Gamma(m)\cdot\Gamma(n) = 2\int_{0}^{\infty} e^{-r^2}\cdot r^{2(n+m)-1}\, dr \cdot 2\int_{0}^{\pi/2} \cos^{2n-1}\theta\, \sin^{2m-1}\, d\theta$$

Using ① & ④:

$$\Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$$

$$\therefore \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$$

• Bessel's Func^n:

① $J_{-n}(x) = (-1)^n J_n(x)$

Proof: W.K.T. $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$ — ①

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r-n} \frac{1}{\Gamma(1+r-n)\cdot r!} \quad \text{— ②}$$

Consider $\Gamma(1+r-n) = \Gamma(r-(n-1))$

$= \Gamma(-r)$ for $r = 0, 1, 2, \ldots, (n-2)$

$$J_{-n}(x) = \sum_{r}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r-n} \cdot \frac{1}{\Gamma(-n+r+1)\, r!}$$

Let $r-n = s \Rightarrow r = s+n$

$$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{-n+2s+2n} \frac{1}{\Gamma(s+1)\cdot(s+n)!}$$

$$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n} \cdot \frac{1}{s!\cdot\Gamma(s+n+1)} \quad \text{--- (3)}$$

$[\Gamma(s+1) = s!]$

$[(s+n)! = \Gamma(s+n+1)]$

Using (1) & (3): $J_{-n}(x) = (-1)^n J_n(x)$

(2) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}$

Proof: W.K.T. $J_n(x) = \displaystyle\sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\cdot r!}$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{1/2+2r} \frac{1}{\Gamma(3/2+r)\cdot r!}$$

Taking $r = 0, 1, 2, \ldots$

at $r=0$, $J_{1/2}(x) = (-1)^0 \left(\frac{x}{2}\right)^{1/2+2\cdot 0} \dfrac{1}{\Gamma(3/2)\cdot 0!}$

$$\Rightarrow J_{1/2}(x) = \frac{(x/2)^{1/2}}{\Gamma(3/2)}$$

at $r=1$, $J_{1/2} = (-1)^1 (x/2)^{1/2+2.1} \dfrac{1}{\Gamma(3/2+1)\,1!}$

$$\Rightarrow J_{1/2}(x) = -(x/2)^{1/2} \cdot \frac{x^2}{4} \; \frac{1}{\Gamma(5/2)}$$

at $r=2$, $J_{1/2}(x) = (-1)^2 (x/2)^{1/2+2.2} \dfrac{1}{\Gamma(3/2+2)\,2!}$

$$\Rightarrow J_{1/2}(x) = (x/2)^{1/2} \frac{x^4}{16} \cdot \frac{1}{2\Gamma(7/2)}$$

$$\Rightarrow J_{1/2}(x) = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{1/2} \cdot x^2}{4\Gamma(5/2)} + \frac{(x/2)^{1/2} x^4}{32\Gamma(7/2)} - \cdots$$

$$\Rightarrow J_{1/2}(x) = (x/2)^{1/2} \left(\frac{1}{\Gamma(3/2)} - \frac{x^2}{4\Gamma(5/2)} + \frac{x^4}{32\Gamma(7/2)} - \cdots \right)$$

$$\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \Gamma(1+3/2) = \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma(7/2) = \Gamma(1+5/2) = \frac{5}{2}\Gamma(5/2) = \frac{15\sqrt{\pi}}{8}$$

$$\Rightarrow J_{1/2}(x) = (x/2)^{1/2} \left(\frac{2}{\sqrt{\pi}} - \frac{x^2}{\cancel{4}} \cdot \frac{\cancel{4}}{3\sqrt{\pi}} + \frac{x^4}{\cancel{32}_{4}} \cdot \frac{\cancel{8}}{15\sqrt{\pi}} - \cdots \right)$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{x}{2}} \cdot \frac{1}{\sqrt{\pi}} \left(2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots \right) \quad (1)$$

W.k.T. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Multiply and divide ① by $\frac{x}{2}$

$$\Rightarrow \sqrt{\frac{x}{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{2}{x} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots \right) = J_{1/2}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) = J_{1/2}(x)$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

③ $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Proof: W.k.T. $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{2r-n} \frac{1}{\Gamma(r+1-n) \cdot r!}$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{2r-1/2} \frac{1}{\Gamma(1/2+r) \cdot r!}$$

Taking $r = 0, 1, 2, \dots$

at $r=0$, $J_{-1/2}(x) = (-1)^0 (x/2)^{2.0-1/2} \frac{1}{\Gamma(1/2) \cdot 0!}$

$$\Rightarrow J_{-1/2}(x) = \frac{(x/2)^{-1/2}}{\Gamma(1/2)}$$

at $r=1$, $J_{-1/2} = (-1)^1 (x/2)^{2\cdot 1 - 1/2} \frac{1}{\Gamma(1/2+1)\,1!}$

$$\Rightarrow J_{-1/2}(x) = -(x/2)^{-1/2} \cdot \frac{x^2}{4} \frac{1}{\Gamma(3/2)}$$

at $r=2$, $J_{-1/2}(x) = (-1)^2 (x/2)^{2\cdot 2 - 1/2} \frac{1}{\Gamma(1/2+2)\,2!}$

$$\Rightarrow J_{-1/2}(x) = (x/2)^{-1/2} \frac{x^4}{16} \cdot \frac{1}{2\Gamma(5/2)}$$

$$\Rightarrow J_{-1/2}(x) = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{-1/2} \cdot x^2}{4\Gamma(5/2)} + \frac{(x/2)^{-1/2} x^4}{32\Gamma(7/2)} - \cdots$$

$$\Rightarrow J_{-1/2}(x) = (x/2)^{-1/2} \left(\frac{1}{\Gamma(1/2)} - \frac{x^2}{4\Gamma(3/2)} + \frac{x^4}{32\Gamma(5/2)} - \cdots \right)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \Gamma(1+3/2) = \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}$$

$$\Rightarrow J_{-1/2}(x) = (x/2)^{-1/2}\left(\frac{1}{\sqrt{\pi}} - \frac{x^2}{\cancel{4}_2}\cdot\frac{\cancel{k}}{\sqrt{\pi}} + \frac{x^4}{\cancel{32}_8}\cdot\frac{\cancel{4}}{3\sqrt{\pi}} - \cdots\right)$$

$$\Rightarrow J_{-1/2}(x) = \sqrt{\frac{2}{x}}\cdot\frac{1}{\sqrt{\pi}}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} \cdots\right) = \sqrt{\frac{2}{\pi x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

W.k.T. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\therefore J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

(4) $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$ and

$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + \cdots$

Proof: W.k.T. $e^{\frac{x}{2}(t - 1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

R.H.S.: $\sum_{n=-\infty}^{\infty} t^n J_n(x) = J_0(x) + tJ_1(x) + t^2 J_2(x) + \cdots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \cdots$

When $n \in I^+$, we have $J_{-n}(x) = (-1)^n J_n(x)$

$$\Rightarrow e^{\frac{x}{2}(t-1/t)} = J_0 + tJ_1 + t^2 J_2 + \cdots + t^{-1}(-J_1) + t^{-2}J_2 + \cdots$$

$$\Rightarrow e^{\frac{x}{2}(t-1/t)} = J_0 + J_1(t - t^{-1}) + J_2(t^2 + t^{-2}) + \dots$$

Put $t = e^{i\theta} \Rightarrow t^{-1} = e^{-i\theta}$

$$\Rightarrow e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = J_0 + (e^{i\theta} - e^{-i\theta})J_1 + (e^{2i\theta} + e^{-2i\theta})J_2 + \dots$$

$e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$e^{i\theta} - e^{-i\theta} = -2i\sin\theta$

$$\Rightarrow e^{\frac{x}{\cancel{2}}\cdot\cancel{2}i\sin\theta} = J_0 + 2J_1 i\sin\theta + 2J_2\cos2\theta + \dots$$

$$\Rightarrow e^{ix\sin\theta} = J_0 + 2J_1 i\sin\theta + 2J_2\cos2\theta + \dots$$

$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + 2J_1 i\sin\theta + 2J_2\cos2\theta + \dots$$

Equating real part and imaginary part:

$$\cos(x\sin\theta) = J_0 + 2J_2\cos2\theta + 2J_4\cos4\theta + \dots$$

$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin3\theta + \dots$$

⑤ $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta \;;\; n \in I^+$

<u>Proof</u>: W.K.T.: $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots$ — ①

$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + \dots$ — ②

Multiply eqn ① by $\cos n\theta$ and then integrate w.r.t. θ b/w limits 0 & π

$$\int_0^{\pi}\cos(x\sin\theta)\cos(n\theta)\,d\theta = \int_0^{\pi} J_0\cos(n\theta)\,d\theta + 2\int_0^{\pi} J_2\cos(2\theta)\cos(n\theta)\,d\theta + 2\int_0^{\pi} J_4\cos(4\theta)\cos(n\theta)\,d\theta + \dots$$

Now, $$\int_0^{\pi} 2J_m\cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} \pi J_m \;;\; m=n \text{ and } n \text{ is even} \\ 0 \;;\; m \neq n \end{cases}$$

Consider $$\int_0^{\pi} J_0\cos(n\theta)\,d\theta = J_0\left[\frac{\sin(n\theta)}{n}\right]_0^{\pi} = 0$$

At $n=0$, $$\int_0^{\pi} J_0\cos(0^\circ)\,d\theta = J_0[\theta]_0^{\pi} = \pi J_0$$

$$\therefore \int_0^{\pi}\cos(x\sin\theta)\cdot\cos(n\theta)\,d\theta = \pi J_n \;;\; n = 0,2,4,6,8,\dots$$ — ③

Multiply eqn ② by $\sin(n\theta)$ and integrate w.r.t. θ b/w 0 & π

$$\int_0^{\pi} \sin(x\sin\theta)\sin(n\theta)\,d\theta = 2J_1\int_0^{\pi}\sin\theta\sin(n\theta)\,d\theta + 2J_3\int_0^{\pi}\sin(3\theta)\sin(n\theta)\,d\theta + \dots \quad (4)$$

Now, $\displaystyle\int_0^{\pi}\sin(m\theta)\sin(n\theta)\,d\theta = \begin{cases} \frac{\pi}{2}\ ; & m=n \text{ and } n \text{ is even} \\ 0\ ; & m\neq n \end{cases}$

From (4): $\displaystyle\int_0^{\pi}\sin(x\sin\theta)\,d\theta = \pi J_n\ ;\ n=1,3,5,7,9\dots \quad (5)$

Eqn (3) + Eqn (5)

$$\int_0^{\pi}\cos(x\sin\theta)\cos(n\theta)\,d\theta + \int_0^{\pi}\sin(x\sin\theta)\sin(n\theta)\,d\theta = \pi J_n\ ,\ n=1,2,3,\dots$$

$$\Rightarrow \int_0^{\pi}\left[\cos(x\sin\theta)\cdot\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta)\right]d\theta = \pi J_n$$

$$\Rightarrow \int_0^{\pi}\cos(x\sin\theta - n\theta)\,d\theta = \pi J_n \qquad [\cos A\cos B + \sin A\sin B = \cos(A-B)]$$

$$\therefore\ J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta\ ;\ n\in I^+$$

⑥ $J_0^2 + J_1^2 + 2J_2^2 + \dots = 1$

Proof: w.k.t. $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots$ — ①

$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + \dots$ — ②

Square eqn ① on both sides and integrate w.r.t. θ b/w 0 and π

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi (J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots)^2\,d\theta$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi (J_0^2 + 4J_2^2\cos^2\theta + 4J_4^2\cos^2 4\theta + \dots) + 2(2J_0J_2\cos 2\theta + 2J_0J_4\cos 4\theta + \dots)\,d\theta$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi J_0^2 + 4J_2^2\int_0^\pi \cos^2\theta\,d\theta + 4J_4^2\int_0^\pi \cos^2 4\theta\,d\theta + \dots + \int_0^\pi 4J_0J_2\cos 2\theta\,d\theta$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \pi J_0^2 + 4J_2^2\cdot\frac{\pi}{2} + \frac{4\pi}{2}J_4^2 + \dots + 0 + 0 + \dots$$

$$\Rightarrow \int_0^\pi \cos^2(x\sin\theta)\,d\theta = \pi(J_0^2 + 2J_2^2 + 2J_4^2 + \dots) \quad \text{— ③}$$

Similarly:

$$\int_0^\pi \sin^2(x\sin\theta)\,d\theta = 2\pi(J_1^2 + J_3^2 + J_5^2 + \dots) \quad \text{— ④}$$

Adding eqn ③ and eqn ④

$$\int_0^{\pi}\left[\sin^2(x\sin\theta)+\cos^2(x\sin\theta)\right]d\theta=\pi\left(J_0^2+2J_1^2+2J_2^2+2J_3^2+\dots\right)$$

$$\Rightarrow \int_0^{\pi}d\theta=\pi\left(J_0^2+2J_1^2+2J_2^2+2J_3^2+\dots\right)$$

$$\Rightarrow \cancel{\pi}=\cancel{\pi}\left(J_0^2+2J_1^2+2J_2^2+2J_3^2+\dots\right)$$

$$\therefore \left(J_0^2+2J_1^2+2J_2^2+2J_3^2+\dots\right)=1$$

Orthogonality of Bessel's Func^n:

Eqn: $$\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx=\begin{cases}\dfrac{a^2}{2}J_{n+1}^2(a\alpha) & ;\ \alpha=\beta\\[2ex] 0 & ;\ \alpha\neq\beta\end{cases}$$

Proof: Let $u=J_n(\alpha x)$ and $v=J_n(\beta x)$ respectively be the solutions of eqns reducible to Bessel's D.E.

$$x^2u''+xu'+(\alpha^2x^2-n^2)u=0 \quad \text{—}(1)$$

$$x^2v''+xv'+(\beta^2x^2-n^2)v=0 \quad \text{—}(2)$$

Multiply eq (1) by $\frac{v}{x}$ and eqn (2) by $\frac{u}{x}$

From eqn 1:

$$xvu'' + vu' + (\alpha^2 x^2 - n^2)\frac{uv}{x} = 0 \quad \text{③}$$

From eqn 2:

$$xuv'' + uv' + (\beta^2 x^2 - n^2)\frac{uv}{x} = 0 \quad \text{④}$$

Eqn 3 − Eqn 4: $x(vu'' - uv'') + (vu' - uv') + \frac{uv}{\cancel{x}}(\alpha^2 x^{\cancel{2}} - \beta^2 x^{\cancel{2}}) = 0$

$$\Rightarrow x(vu'' - uv'') + (vu' - uv') + xuv(\alpha^2 - \beta^2) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] + xuv(\alpha^2 - \beta^2) = 0$$

Integrating both sides w.r.t. x b/w limits 0 & a

$$\int_0^a \frac{d}{dx}\left[x(vu' - uv')\right] dx + \int_0^a (\alpha^2 - \beta^2) xuv\, dx = 0$$

$$\int_0^a \left[x(vu' - uv')\right] dx = -\int_0^a (\alpha^2 - \beta^2)\, xuv\, dx$$

$$\Rightarrow \int_0^a \left[x(vu' - uv')\right] dx = \int_0^a (\beta^2 - \alpha^2) xuv\, dx$$

$$\Rightarrow \frac{\left[x(vu' - uv')\right]_0^a}{\beta^2 - \alpha^2} = \int_0^a xuv\, dx$$

W.K.T. $u = J_n(\alpha x)$ $\quad v = J_n(\beta x)$

$\Rightarrow u' = \alpha J_n'(\alpha x)$ $\quad v' = \beta J_n'(\beta x)$

$$\Rightarrow \frac{a\left[J_n(\beta a)\cdot \alpha J_n'(\alpha a) - J_n(\alpha a)\cdot \beta J_n'(\beta a)\right]}{\beta^2-\alpha^2} = \int_0^a x J_n(\alpha x) J_n(\beta x)\,dx \quad (5)$$

α & β are the roots of $J_n(ax) = 0$

$\Rightarrow J_n(a\alpha) = J_n(a\beta) = 0$

Case (1): $\alpha \neq \beta$

From (5): $\displaystyle\int_0^a x J_n(\alpha x) J_n(\beta x) = \frac{0}{\beta^2-\alpha^2} = 0$

Case (2): $\alpha = \beta$

$$\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \frac{a\left\{\left[J_n(\beta a)\cdot \alpha J_n'(\alpha a)\right] - \left[J_n(\alpha a)\cdot \beta J_n'(\beta a)\right]\right\}}{\beta^2-\alpha^2}$$

Take limit $\beta \to \alpha$ on both sides

$$\Rightarrow \lim_{\beta\to\alpha}\int_0^a x J_n(\alpha x) J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{a}{\beta^2-\alpha^2}\left\{\left[J_n(\beta a)\cdot \alpha J_n'(\alpha a)\right] - \left[J_n(\alpha a)\cdot \beta J_n'(\beta a)\right]\right\}$$

Applying L'Hospital rule

$$R.H.S. : \lim_{\beta \to \alpha} \left\{ \frac{a}{\beta^2 - \alpha^2} \left[\alpha J_n'(\alpha a) - J_n(\beta a) - 0 \right] \right\}$$

$$= \lim_{\beta \to \alpha} \frac{a \alpha J_n'(\alpha a) - a J_n(\beta a)}{2\beta}$$

Substituting limit

$$R.H.S. = \frac{a \cancel{\alpha} J_n'(\alpha a) \cdot a J_n'(\alpha a)}{2 \cancel{\alpha}}$$

$$\Rightarrow R.H.S. = \frac{a^2}{2} J_n'^2(\alpha a) \quad \text{—⑥}$$

$$W.k.T. : x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

$$\Rightarrow J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\Rightarrow J_n'(\alpha a) = \frac{n}{\alpha a} \cancelto{0}{J_n(\alpha a)} - J_{n+1}(\alpha a)$$

$$\Rightarrow J_n'(\alpha a) = - J_{n+1}(\alpha a)$$

Substituting value of $J_n'(\alpha a)$ in eqn ⑥

$$\Rightarrow \frac{a^2}{2} J_n'^2(\alpha a) = \frac{a^2}{2} J_{n+1}^2(\alpha a)$$

$$\therefore \int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} \frac{a^2}{2} J_{n+1}^2(\alpha a) & ; \alpha = \beta \\ 0 & ; \alpha \neq \beta \end{cases}$$