Formulas

$+ \quad \times$

$- \quad \div$

Partial Differentiation

• Partial Differentiation

If $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

z_x

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

z_y

$$z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

• Mixed Partial Derivatives

$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \, \partial x}$$

$$z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \, \partial y}$$

Clairut's Theorem ⇒

$$z_{xy} = z_{yx}$$

Mostly dealing with these type of questions

• Composite Functions

If $z = f(g(\phi(x)))$

Then, $\dfrac{dz}{dx} = f'(g(\phi(x))) \cdot g'(\phi(x)) \cdot \phi'(x)$

• Total Derivative Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

- **Chain Rule**

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

- **Implicit Functions**

If $z = f(x,y)$ and $\frac{dz}{dx} = 0$

Then $z_x \cdot \cancel{\frac{dx}{dx}}^{\,1} + z_y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-z_x}{z_y}$

- **Homogeneous Functions**

It is homogeneous if it can be expressed as

$x^n g\left(\frac{y}{x}\right)$ (or) $y^n g\left(\frac{x}{y}\right)$ [For $u = f(x,y)$

ex: $u = 3x + 4y \Rightarrow u = x\left(3 + \frac{4y}{x}\right)$

Degree = 1

$x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ (or) $y^n g\left(\frac{x}{y}, \frac{z}{y}\right)$ (or) $z^n g\left(\frac{x}{z}, \frac{y}{z}\right)$

[For $u = f(x,y,z)$]

• Euler's Theorem on homogeneous functions

1) Statement 1:

If $u = f(x,y)$ is homogeneous functions of degree n

$$\rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

2) Corollary 1:

$$\rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

3) Statement 2:

$$\rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Deductions from Euler's Theorem

1) If $z = f(u)$, is a homogeneous eq, of degree n, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$$

2) $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\left(g'(u) - 1\right)$$

where $$g(u) = n \cdot \frac{f(u)}{f'(u)}$$

• Taylor's Series

$$\underset{\text{at } (a,b)}{f(x,y)} = f(a,b) + \frac{1}{1!}\Big((x-a) f_x(a,b) + (y-b) f_y(a,b)\Big)$$
$$+ \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-a) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big)$$
$$+ \cdots$$

• Maclaurin's Series

$$f(x,y) = f(0,0) + \frac{1}{1!}\left\{x f_x(0,0) + y f_y(0,0)\right\} + \frac{1}{2!}\left\{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right\}$$
$$+ \cdots$$

• Maxima & Minima for a function of 2 variables

Necessary Condition :

$rt - s^2 > 0$ where $r = f_{xx}$; $t = f_{yy}$; $s = f_{xy}$

If $rt - s^2 < 0 \Rightarrow$ Saddle point

$rt - s^2 = 0 \Rightarrow$ Can't be determined

Working Rule :

i) Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ and equate to 0. Solve simultaneously

ii) Calculate r, s, t for each pair of values

iii) $rt - s^2 > 0$ & $r < 0$ at (a, b), $f(a, b) = \max$.

$rt - s^2 > 0$ & $r > 0$ at (a, b), $f(a, b) = \min$

$rt - s^2 < 0$, $f(a, b) =$ saddle point

$rt - s^2 = 0$ [Further Investigation required]

• Lagrange's Method of Undetermined Multipliers

→ To maximize (or) minimize $f(x, y, z)$ & subject to constraint $\phi(x, y, z)$

Working :

i) Form $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

ii) Partially differentiate F wrt x, y, z

iii) Solve $F_x = 0$, $F_y = 0$, $F_z = 0$ & constraint for λ & stationary values x, y, z

Differential Equations

• General form of DE

$$\frac{dy}{dx} + Py = Q$$

$$IF = e^{\int P dx}$$

$$y . IF = \int Q . IF . dx + C$$

$$\frac{dx}{dy} + Px = Q$$

$$IF = e^{\int P dy}$$

$$x . IF = \int Q . IF . dy + C$$

• Differential Equations

- Variable Seperable Method
- Homogeneous Equations
- Linear Equations
- Exact Equations

• Bernoulli's D.E

If in the form of

$$\frac{dy}{dx} + Py = Qy^n$$

i) $\frac{1}{y^n} \cdot \frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$

ii) $t = y^{n-1}$; $\frac{dt}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

and $\frac{dt}{dx} + P't = Q'$

iii) $IF = e^{\int P' dx}$

$$t . IF = \int Q' . (IF) dx + C$$

$$\frac{dx}{dy} + Px = Qx^n$$

i) $x^{-n} \cdot \frac{dx}{dy} + Px^{1-n} = Q$

ii) $t = x^{1-n}$; $\frac{dt}{dy} = (1-n)x^{-n} \cdot \frac{dx}{dy}$

and $\frac{dt}{dy} + P't = Q'$

iii) $IF = e^{\int P' dy}$

$$t . IF = \int Q' . (IF) dy + C$$

- **Exact Differential Equations**

DE in the form $M(x,y).dx + N(x,y)dy = 0$

Then solution is $u(x,y) = c$

Working :

i) Check for exactness,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

ii) Solution is, $N(y)$ = terms of N independant of x.

$$\int M dx + \int N(y) dy = c$$

(or)

$$\int M(x) dx + \int N dy = c$$

$M(x)$ = terms of M independant of y

- **Equations Reducible to Exact Form**

→ Some DE's aren't exact but can be, by multiplying IF

Case 1 : $IF = e^{\int f(x)dx}$ (or) $e^{-\int f(y)dy}$

Procedure : i) Find $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$

ii) If difference is close to M, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y)$ & $IF = e^{-\int f(y)dy}$

or If difference is close to N, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ & $IF = e^{\int f(x)dx}$

iii) Multiply IF with $Mdx + Ndy = 0$ & Find Sol^n.

Case 2 : If $M(x,y)$ & $N(x,y)$ are homogeneous of same degree

$$IF = \frac{1}{Mx+Ny} \quad \text{Provided } Mx+Ny \neq 0$$

But, if $Mx+Ny=0$, $IF = \frac{1}{x^2}$ (or) $\frac{1}{xy}$ (or) $\frac{1}{y^2}$

Case 3 : If DE in form, $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

then, $IF = \frac{1}{Mx-Ny}$

where $M = y \cdot f(x,y)$ & $N = x \cdot g(x,y)$

& $Mx - Ny \neq 0$

• Orthogonal Trajectories

→ If every member of one family intersects with the other family orthogonally.

Procedure : i) Differentiate $f(x,y,c)=0$ wrt x & eliminate c between y & y'

ii) Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to obtain DE

iii) Solve DE to get orthogonal family of curves

(if you get same eq after replacing $\frac{dr}{d\theta}$ with $-\frac{d\theta}{dr}$, then it is self-orthogonal)

- Orthogonal Trajectories – Polar Form

Procedure :

i) Form DE of given family curves $F(r,\theta,c)=0$ in the form $f\left(r,\theta,\frac{dr}{d\theta}\right)=0$

ii) Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ to obtain DE of required orthogonal family of curves

iii) Solve DE

(if you get same eq after replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$, then it is self-orthogonal)

- Non-linear DE

→ DE of 1st order & higher degree of the form
$f(x, y, p) = 0$
where $p = \frac{dy}{dx}$

Solution obtained by reducing to 1st order & degree

Case 1 : Solving p

Procedure : i) nth degree non-linear DE expressed as nth degree polynomial in p.

ii) Resolve to n linear real factors
$(p-b_1)(p-b_2)(p-b_3)\dots(p-b_n) = 0$

iii) Equate each to zero &
$(y' = b_1), (y' = b_2), \dots (y' = b_n)$

iv) Obtain following solutions
$f_1(x,y,c) = 0, f_2(x,y,c) = 0, \dots, f_n(x,y,c) = 0$

v) general solution is
$f_1(x,y,c) \cdot f_2(x,y,c) \dots f_n(x,y,c) = 0$

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Case 2 : Solving for x

Procedure : i) Rewrite $f(x,y,p)=0$ in the form of $x = F(y,p) \longrightarrow (1)$

ii) Differentiate wrt y & form $\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \longrightarrow (2)$

iii) Solve (2) & get solution of the form $G(y,p,c) = 0 \longrightarrow (3)$

iv) Eliminate p from (1) & (3)

(If its not possible to eliminate p, solution of DE (1) is given by parametric eq. $x = x(p, c)$ & $y = y(p, c)$

Refer problems for further understanding

Case 3 : Solvable for y

Procedure : i) Rewrite $f(x,y,p)=0$ to $y = F(x,p) \rightarrow ①$

ii) Differentiate wrt x and get

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \longrightarrow ②$$

iii) Solve ②, and solution of form $G(x,p,c)=0 \rightarrow ③$

iv) Eliminate p from ① & ③, req. solⁿ of ①

(If not possible to eliminate p, solⁿ of ① is given by parametric eq $x = x(p,c)$ & $y = y(p,c)$)

- **Newton's law of cooling**

$$\rightarrow \frac{dT}{dt} = -k(T - t_2)$$

$$\hookrightarrow T = t_2 + (t_1 - t_2)e^{-kt}$$

T : Temp of body at any time

t_1 : Initial temp of body

t_2 : constant temp of medium

t : time ; k : Proportionality constant

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = X$$

• Higher Order Linear D.E

Homogeneous LDE

$X = 0$

ex: $3\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 4 = 0$

Non-Homogeneous LDE

X is non-zero function of x

ex: $3\frac{d^2 y}{dx^2} - 9\frac{dy}{dx} + 4y = \sin^2 x$

Differential Operator

We shall write $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = X$

as $D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y = X$

$\Rightarrow f(D) y = X$

and auxillary of eq as $m^n y + a_1 m^{n-1} y + \dots$

• Solving Higher order homogeneous LDE w constant coeff

Case 1: Auxillary eq has real & distinct roots

If m_1 & m_2 are the roots,

$$G.S \Rightarrow y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2: Auxillary has only 1 real root

If root is m,

$$y = (c_1 + c_2 x) e^{mx}$$

Note: Suppose m_1, m_2, m_3, m_4 are real & equal and remaining $(n-4)$ are real & distinct, then,

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^{mx} + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

Case 3: Auxillary has complex roots

Let $m = \alpha \pm i\beta$

Then $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Note 1: If 2 pairs of imag. roots, $\alpha \pm \beta i$ & $\gamma \pm \delta i$, then,

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + e^{\gamma x}(c_3 \cos\delta x + c_4 \sin\delta x)$$

Note 2: If a pair of roots appear twice,

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right]$$

• Non-Homogeneous Linear DE

Type $f(D)y = X$ when $X = e^{ax}$

• Inverse differential operator $\Rightarrow \underbrace{\dfrac{1}{f(D)}}_{\text{Particular Integral}}(X)$ is particular solⁿ of $f(D)y = X$

y = Complimentary function + Particular Integral

Important Results

1) $\dfrac{1}{D}(X) = \int X \cdot dx$

2) $\dfrac{1}{D-a}(X) = e^{ax}\int e^{-ax} \cdot X\, dx$

3) If $f(D) = ((D-a)(D-b))$

$$\frac{1}{f(D)}(X) = \frac{1}{(D-a)(D-b)}(X) = \frac{1}{D-a}\left(\frac{(X)}{D-b}\right)$$

Rules to find P.I

• Type (i) : $X = e^{ax}$

$$\frac{1}{f(D)}(e^{ax}) = \frac{e^{ax}}{f(a)} \qquad (f(a) \neq 0)$$

If $f(a) = 0 \Rightarrow$ Differentiate denominator & multiply x

$$\Rightarrow \frac{1}{f'(a)} \cdot x e^{ax}$$

- **Type (ii)** : $X = \sin(ax)$ or $\sin(ax+b)$

  $\cos(ax)$ or $\cos(ax+b)$

$$\frac{1}{f(D)} \cdot \sin ax = \frac{1}{f(D)}\bigg|_{D^2 \to -a^2} \sin ax$$

Suppose we get something like $\frac{1}{5-D}$ then rationalize

$$\frac{1}{5-D} \times \frac{5+D}{5+D} = \frac{5+D}{25-D^2} \Rightarrow \text{Then } \frac{5\sin ax + a\cos ax}{25+a^2}$$

- **Type (iii)** : $X = \phi(x)$ where $\phi(x)$ is a polynomial of x

Method 1 : $(D^2-2D-3)y = 2x^2+6x$

PI $\Rightarrow$

$$\begin{array}{r|l} & -\frac{2x^2}{3} - \frac{10x}{9} + \frac{8}{27} \\ \hline -3-2D+D^2 & 2x^2+6x \\ & \underline{-\left(2x^2 + \frac{8x}{3} - \frac{4}{3}\right)} \\ & \frac{10x}{3} + \frac{4}{3} \\ & \underline{-\left(\frac{10x}{3} + \frac{20}{9} + 0\right)} \\ & -\frac{8}{9} \\ & \underline{-\frac{8}{9}} \\ & 0 \end{array}$$

$$PI = -\frac{2x^2}{3} - \frac{10x}{9} + \frac{8}{27}$$

Avoid Method 2 :)

Type (iv) : $X = e^{ax} V(x)$ where $V(x) = \sin bx, \cos bx$ (or) polynomial in x

$$PI = \frac{1}{f(D)} \cdot e^{ax} V(x) \quad D \to D+a$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} \cdot V(x)$$ and use type (ii) or (iii)

Type (v) : $X = x\,\phi(x)$ where $\phi(x) = \sin ax$ (or) $\cos ax$

then, $$PI = \left(x - \frac{f'(D)}{f(D)}\right) \frac{\phi(x)}{f(D)}$$

$\rightarrow$ $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\rightarrow$ $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\rightarrow$ $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\rightarrow$ $\sin^2 x = \frac{1 - \cos 2x}{2}$	$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$	$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$
$\rightarrow$ $\cos^2 x = \frac{1 + \cos 2x}{2}$	$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$	$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$