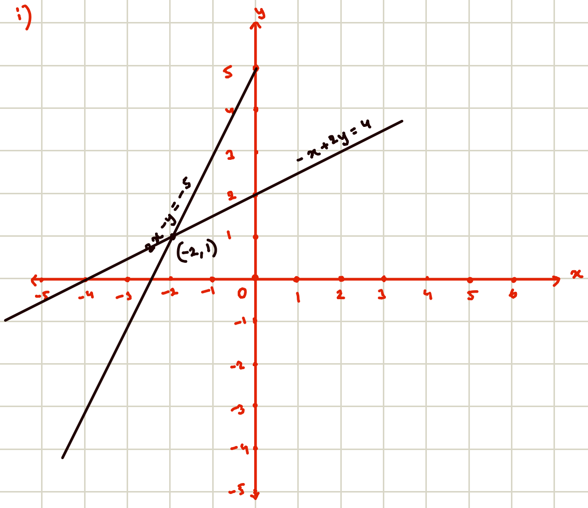
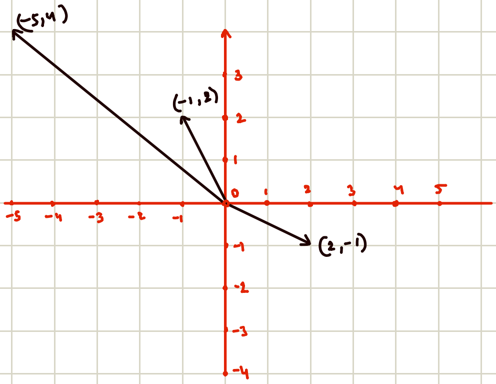
# Unit - 1

Q. Draw the row and column picture of these lines (i) $2x-y=-5$ ; $-x+2y=4$ (ii) $x+2y=7$; $x-y=1$ (iii) $x+2y=3$; $2x+4y=6$. Write down the solution if it exists.

A. i)



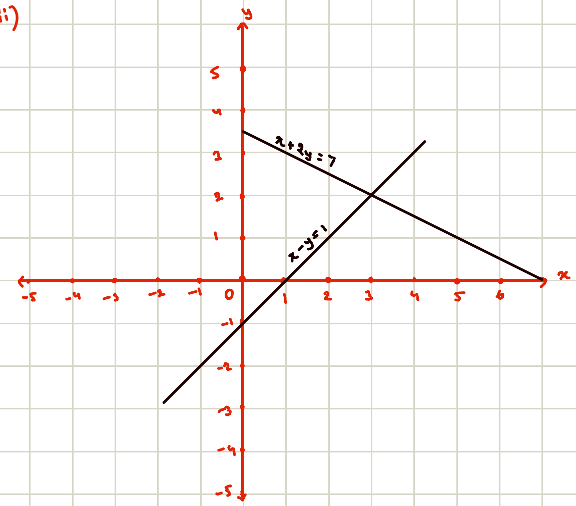

Row Picture



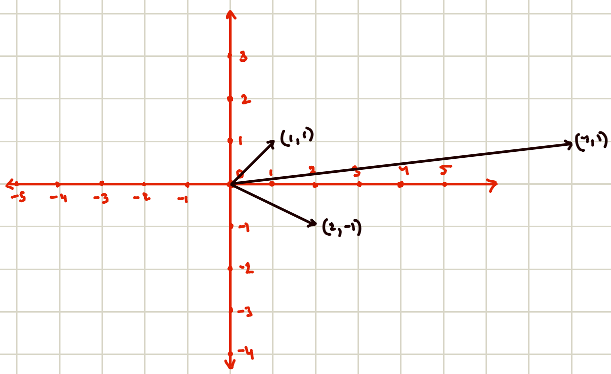

Column Picture

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

ii)



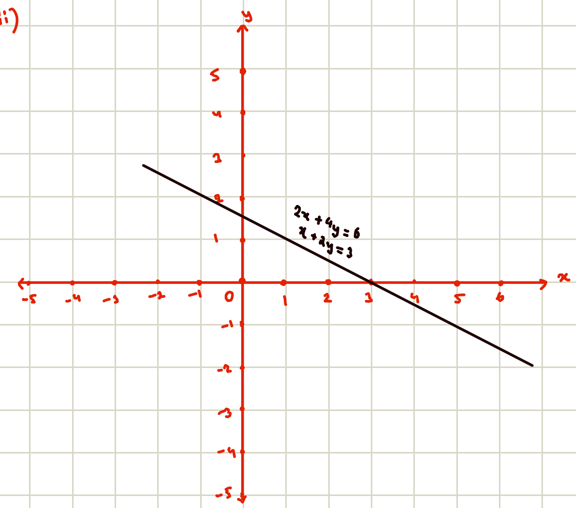

Row Picture



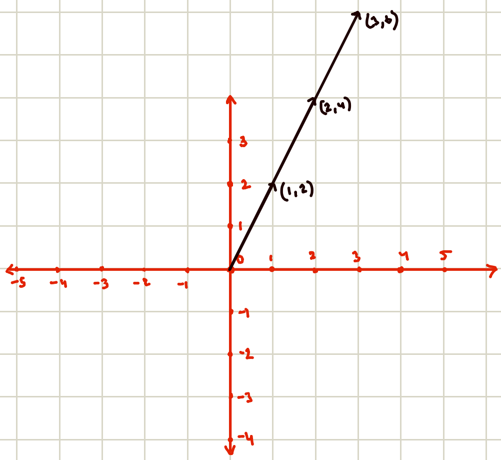

Column Picture

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} y = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

iii)




Row Picture




Column Picture

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 4 \end{bmatrix} y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Q. Check the following system of equations for consistency and solve if consistent using Gaussian Elimination:

(i) x+ 2y+ 3z = 6, 2x -3y+2z =14, 3x +y -z =-2

**Answer:** x=1, y=-2, z=3

(i) x+2y +z =4, y-z =1, x+3y=5

**Answer:** x=2-3k, y=1+k, z=k

(ii) x+2y +3z+4w=5 , x+3y+5z+7w =11, x-z-2w=-6

**Answer:** No solution

A. i)

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{5}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & \frac{-50}{7} & \frac{-150}{7} \end{array}\right]$$

$$\left.\begin{array}{r} x + 2y + 3z = 6 \\ -7y - 4z = 2 \\ -\frac{50z}{7} = \frac{-150}{7} \end{array}\right\} \to \begin{array}{l} x = 1 \\ y = -2 \\ z = 3 \end{array}$$

ii)

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{r} x + 2y + z = 4 \\ y - z = 1 \end{array}\right\} \to \begin{array}{l} x = 2 - 3K \\ y = 1 + K \\ z = K \end{array}$$

iii)

$$[A:b] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & -1 & -4 & -6 & -11 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & -2 & -3 & -7 \end{array}\right]$$

$$\left.\begin{array}{r} x + 2y + 3z + 4w = 5 \\ y + 2z + 3w = 6 \\ -2z - 3w = -7 \end{array}\right\} \to \text{No Solution}$$

Q. Find all values of a for which the linear system x + y - z = 3, x - y + 3z = 4, x + y + ($a^2$-10)z = a has (i)no solution, (ii) a unique solution and (iii)infinitely many solutions.

A.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & -1 & 3 & 4 \\ 1 & 1 & a^2-10 & a \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right]$$

i) For no sol$^n$, rank(A) $\neq$ rank(A : b) $\Rightarrow$ $a^2 - 9 = 0$ $\quad a - 3 \neq 0$ $\Rightarrow$ $a = -3$

ii) For unique sol$^n$, rank(A) = rank(A : b) = r = n $\Rightarrow$ $a \neq \pm 3$

iii) For $\infty$ sol$^n$, rank(A) = rank(A:b) = r < n $\Rightarrow$ $a = 3$

Q. Which elementary matrices put A into upper triangular form U? Hence find L and factor A into LU given $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

A.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{R_1}{2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 7/4 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{4}{7}R_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 7/4 & -1 & 0 \\ 0 & 0 & 10/7 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 + \frac{7}{10}R_3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 7/4 & -1 & 0 \\ 0 & 0 & 10/7 & -1 \\ 0 & 0 & 0 & \frac{13}{10} \end{bmatrix} = U$$

$$L = E_{21}^{-1} E_{32}^{-1} E_{43}^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -4/7 & 1 & 0 \\ 0 & 0 & -7/10 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 7/4 & -1 & 0 \\ 0 & 0 & 10/7 & -1 \\ 0 & 0 & 0 & \frac{13}{10} \end{bmatrix}$$

**Q.** Compute LU and LDU factorization for the matrix $\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}$

**A.** $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & 6 & 8 \end{bmatrix}$

$\xrightarrow{R_4 \to R_4 - 4R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**Q.** Apply elimination to produce the factors L and U for A= $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

Is A=LU possible? Explain. Write down the permutation matrices if any. Also find PA=LDU factorization.

**A.** $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$LDU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Q.** For the given matrix $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{pmatrix}$. Find A=LDU and discuss how is L and U related and why? Also find Cholesky factorization for (i) $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{pmatrix}$ and

(ii) $B = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$

**A.** $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

From this $L = U^T$

i) $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

$l_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$

$l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{3}{1} = 3$

$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{12-9} = \sqrt{3}$

$l_{31} = \dfrac{a_{31}}{l_{11}} = \dfrac{5}{1} = 5$

$l_{32} = \dfrac{a_{32} - l_{31}l_{21}}{l_{22}} = \dfrac{18-(5\times 3)}{\sqrt{3}} = \sqrt{3}$

$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{30-25-3} = \sqrt{2}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & \sqrt{3} & 0 \\ 5 & \sqrt{3} & \sqrt{2} \end{bmatrix}$ $\quad L^T = \begin{bmatrix} 1 & 3 & 5 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

ii) $B = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$

$l_{11} = \sqrt{a_{11}} = 2$

$l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{12}{2} = 6$

$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{37-36} = 1$

$l_{31} = \dfrac{a_{31}}{l_{11}} = \dfrac{-16}{2} = -8$

$l_{32} = \dfrac{a_{32} - l_{31}l_{21}}{l_{22}} = \dfrac{-43-(-8\times 6)}{1} = 5$

$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{98-64-25} = 3$

$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$ $\quad L^T = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Q. Compute inverse of the following matrices by Gauss Jordan method.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

A. a) $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$

$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 5/2 & -1 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$ $[I \mid A^{-1}]$

b) $[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + \frac{R_1}{2}} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + \frac{2}{3}R_2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{3}{4}R_3} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \xrightarrow{R_2 \to \frac{2}{3}R_2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \xrightarrow{R_1 \to \frac{R_1}{2}} [I \mid A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$

Q. Find the quadratic polynomial that interpolates the given points (1,2), (3,3), (5,8)

A. In general,

$p(x) = ax^2 + bx + c$

1) $a(1)^2 + b(1) + c = 2 \Rightarrow a + b + c = 2$

2) $a(3)^2 + b(3) + c = 3 \Rightarrow 9a + 3b + c = 3$

3) $a(5)^2 + b(5) + c = 8 \Rightarrow 25a + 5b + c = 8$

$\mapsto$ $a = 1/2$, $b = -3/2$, $c = 3$

$p(x) = \dfrac{x^2 - 3x + 6}{2}$

Q. Producing x trucks and y planes requires x+50y tons of steel, 40x+1000y pounds of rubber, and 2x + 50y months of labour. If the unit costs $z_1, z_2, z_3$ are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane?

A. Resources $A = \begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}$

Unit cost vector $z = \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix}$

Cost $= z^T A \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 700 & 3 & 3000 \end{bmatrix} \begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} = \begin{bmatrix} 6820 & 188000 \end{bmatrix}$

6820 ↓ Cost of 1 Truck

188000 ↓ Cost of 1 Plane

# Unit - 2

**Q.** Examine if the following sets of vectors are linearly independent. If not, find a relation between the vectors:

(a) $\{(4,2,-1,3), (6,5,-5,1), (2,-1,3,5)\}$

(b) $\{sin t, e^t, t^2\}$

(c) $\{t^2+t+2, 2t^2, 3t^2+2t+2\}$

(d) $\left\{\begin{pmatrix}1 & 2\\1 & 0\end{pmatrix}, \begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}, \begin{pmatrix}3 & 1\\2 & 2\end{pmatrix}\right\}$ in $M_{2x2}(R)$

**A.** a)

$$\begin{bmatrix}4 & 6 & 2\\2 & 5 & -1\\-1 & -5 & 3\\3 & 1 & 5\end{bmatrix} \xrightarrow[R_2 \to R_2 - \frac{2}{4}R_1]{R_4 \to R_4 - \frac{3}{4}R_1,\ R_3 \to R_3 + \frac{1}{4}R_1} \begin{bmatrix}4 & 6 & 2\\0 & 2 & -2\\0 & -7/2 & 7/2\\0 & -7/2 & 7/2\end{bmatrix} \xrightarrow[R_4 \to R_4 + \frac{7}{4}R_2]{R_3 \to R_3 + \frac{7}{4}R_2} \begin{bmatrix}4 & 6 & 2\\0 & 2 & -2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$4a + 6b + 2c = 0$

$2b = 2c \Rightarrow b = c \quad ; \quad a = -2c$

$$\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

$-2v_1 + v_2 + v_3 = 0$

b)

$$\begin{bmatrix}\sin t\\e^t\\t^2\end{bmatrix} \longrightarrow \text{linearly independant}$$

c)

$$\begin{bmatrix}1 & 2 & 3\\1 & 0 & 2\\2 & 0 & 2\end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix}1 & 2 & 3\\0 & -2 & -1\\0 & -4 & -4\end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix}1 & 2 & 3\\0 & -2 & -1\\0 & 0 & -2\end{bmatrix}$$

It has pivots, linearly independant

d)

$$\begin{bmatrix}1 & 2 & 3\\2 & -1 & 1\\1 & 1 & 2\\0 & 2 & 2\end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 2 & 3\\0 & -5 & -5\\0 & -1 & -1\\0 & 2 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 3\\0 & -5 & -5\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$a + 2b + 3c$

$-b = c$

$$\begin{bmatrix}-1\\-1\\1\end{bmatrix} \Rightarrow v_1 + v_2 = v_3$$

**Q.** (a)Determine whether these vectors $\{(1,2,-2,1), (-3,0,-4,3), (2,1,1,-1), (-3,3,-9,6)\}$ form a basis of $\mathbb{R}^4$. If not, find the dimension of the subspace S they span. If S is a subspace of $\mathbb{R}^4$, extend the basis of S to a basis of $\mathbb{R}^4$.

(b) Do these vectors $\{(2,2,3), (-1,-2,1), (0,1,0)\}$ span $\mathbb{R}^3$.

**A.** a)

$$\begin{bmatrix}1 & -3 & 2 & -3\\2 & 0 & 1 & 3\\-2 & -4 & 1 & -9\\1 & 3 & -1 & 6\end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + 2R_1} \begin{bmatrix}1 & -3 & 2 & -3\\0 & 6 & -3 & 9\\0 & -10 & 5 & -15\\0 & 6 & -3 & 9\end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + \frac{10}{6}R_2} \begin{bmatrix}1 & -3 & 2 & -3\\0 & 6 & -3 & 9\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$a - 3b + 2c - 3d = 0$

$6b - 3c + 9d = 0$

They don't form a basis of $\mathbb{R}^4$

They span subspace $S = \{(1,2,-2,1), (-3,0,-4,3)\}$ of dimension 2

b)

$$\begin{bmatrix}2 & -1 & 0\\2 & -2 & 1\\3 & 1 & 0\end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{3}{2}R_1]{R_2 \to R_2 - R_1} \begin{bmatrix}2 & -1 & 0\\0 & -1 & 1\\0 & 2 & 0\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}2 & -1 & 0\\0 & 2 & 0\\0 & -1 & 1\end{bmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \begin{bmatrix}2 & -1 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$$

Yes. They span $\mathbb{R}^3$

**Q.** Reduce the following matrices to Row Reduced Echelon form and determine their ranks $\begin{pmatrix}1 & 3 & -2 & 5\\2 & 1 & 3 & 2\\4 & 7 & -1 & 12\end{pmatrix}, \begin{pmatrix}0 & 1 & -2 & -1\\1 & 2 & -1 & 3\\3 & 7 & -8 & 3\\4 & 5 & -7 & 0\end{pmatrix}$. Identify the pivot variables and free variables. Find the special solutions to Ax=0.

**A.** i)

$$\begin{bmatrix}1 & 3 & -2 & 5\\2 & 1 & 3 & 2\\4 & 7 & -1 & 12\end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 3 & -2 & 5\\0 & -5 & 7 & -8\\0 & -5 & 7 & -8\end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix}1 & 3 & -2 & 5\\0 & -5 & 7 & -8\\0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{-5}} \begin{bmatrix}1 & 3 & -2 & 5\\0 & 1 & -7/5 & 8/5\\0 & 0 & 0 & 0\end{bmatrix}$$

Rank = 2

Pivot variables : u, v

Free variables : w, x

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -7/5 & 8/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} u + \frac{w}{5} + \frac{x}{5} = 0 \\ v - \frac{7w}{5} + \frac{8x}{5} = 0 \end{matrix}$$

Special sol$^n$ $\Rightarrow \begin{pmatrix} -\frac{w}{5} - \frac{x}{5} \\ \frac{7w}{5} - \frac{8x}{5} \\ w \\ x \end{pmatrix} = w\begin{pmatrix} \frac{-1}{5} \\ \frac{7}{5} \\ 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} \frac{-1}{5} \\ \frac{-8}{5} \\ 0 \\ 1 \end{pmatrix}$

ii) $$\begin{bmatrix} 0 & 1 & -2 & -1 \\ 1 & 2 & -1 & 3 \\ 3 & 7 & -8 & 3 \\ 4 & 5 & -7 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 3 & 7 & -8 & 3 \\ 4 & 5 & -7 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 4R_1]{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -5 & -6 \\ 0 & -3 & -3 & -12 \end{bmatrix} \xrightarrow[R_4 \to R_4 + 3R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & -9 & -15 \end{bmatrix} \xrightarrow[R_2 \to R_2 + 2R_1]{R_4 \to R_4 - 3R_3,\ R_3 \to \frac{R_3}{-3}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

Pivot variables : u, v

Free variables : w, x

$$\xrightarrow{R_1 \to R_1 + R_3} \begin{bmatrix} 1 & 2 & 0 & 14/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Bigg\} \rightarrow \begin{matrix} u = 0 \\ v + \frac{7x}{3} = 0 \\ w + \frac{5}{3}x = 0 \end{matrix}$$

Special sol$^n$ $\Rightarrow t\begin{bmatrix} 0 \\ -7/3 \\ -5/3 \\ 1 \end{bmatrix}$

Q For the following vectors

(i) $S = \{ (0,0,1), (1,0,1), (0,1,1)\}$. Is u=(1, 1,1) in span of S.

(ii) $S = \{v_1 = 2t^2 + t + 2, v_2 = t^2 - 2t, v_3 = 5t^2 - 5t + 2,\ v_4 = -t^2 - 3t - 2\}$. Does $u = t^2 + t + 2 \in \text{span}\{ v_1, v_2, v_3, v_4 \}$. Find a basis for the subspace W which spans S and what is its dimension.

(iii) $\{ (1,2,-1), (2,3,4), (0,0,1)\}$. Do these vectors span $\mathbb{R}^3$.(Home-work)

A. i) $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$a\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} b = 1 \\ c = 1 \\ a + b + c = 1 \end{matrix} \Bigg\} \rightarrow \begin{matrix} a = -1 \\ b = 1 \\ c = 1 \end{matrix}$$

Yes, u is in span of S

ii) $$\begin{bmatrix} 2 & 1 & 5 & -1 \\ 1 & -2 & -5 & -3 \\ 2 & 0 & 2 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & -5/2 & -15/2 & -5/2 \\ 0 & -1 & -3 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{2}{5}R_2} \begin{bmatrix} 2 & 1 & 5 & -1 \\ 0 & -5/2 & -15/2 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C(A) = \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\}$

$$a\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} 2a + b = 1 \\ a - 2b = 1 \\ 2a = 2 \end{matrix} \Bigg\} \rightarrow \text{Not possible}$$

No, $u = t^2 + t + 2$ doesn't span $\{v_1, v_2, v_3, v_4\}$

$W = \{v_1, v_2\}$ spans S which is dimension 2

iii) $$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ -1 & 4 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 6 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 6R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow$ Yes, it spans $\mathbb{R}^3$

**Q** Let $U=\{(a,b,c)/a+b+c=0\}, V=\{(0,0,c)\}$. Show that $\mathbb{R}^3=U+V$. Is U+V a direct sum.

**A.** $a+b+c=0$

$c=-a-b$

$$U=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\b\\-a-b\end{pmatrix}=a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix}\Rightarrow \text{Dimension}=2$$

$$V=\begin{pmatrix}0\\0\\c\end{pmatrix}=c\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow \text{Dimension}=1$$

$U+V \Rightarrow$ Dimension $=2+1=3$

$$U+V=a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

**Q.** Find a basis and dimension of U+V given;

(i) $U=span\{(1,-1,-1.-2),(1,-2,-2,0),(1,-1,-2,-2)\}$.

(ii) $V=span\{(1,-2,-3.0),(1,-1,-3,2),(1,-1,-2,2)\}$.

What is the dimension of $U\cap V$.

**A.**

$$U=\begin{bmatrix}1&1&1\\-1&-2&-1\\-1&-2&-2\\-2&0&-2\end{bmatrix}\xrightarrow[R_4\to R_4+2R_1]{R_2\to R_2+R_1,\ R_3\to R_3+R_1}\begin{bmatrix}1&1&1\\0&-1&0\\0&-1&-1\\0&2&0\end{bmatrix}\xrightarrow[R_4\to R_4+2R_2]{R_3\to R_3-R_2}\begin{bmatrix}1&1&1\\0&-1&0\\0&0&-1\\0&0&0\end{bmatrix}\quad \text{Dim}=3$$

$$V=\begin{bmatrix}1&1&1\\-2&-1&-1\\-3&-3&-2\\0&2&2\end{bmatrix}\xrightarrow[R_3\to R_3+3R_1]{R_2\to R_2+2R_1}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\\0&2&2\end{bmatrix}\xrightarrow{R_4\to R_4-2R_2}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\\0&0&0\end{bmatrix}\quad \text{Dim}=3$$

$$U+V=\begin{bmatrix}1&1&1&1&1&1\\-1&-2&-1&-2&-1&-1\\-1&-2&-2&-3&-3&-2\\-2&0&-2&0&2&2\end{bmatrix}\xrightarrow[R_4\to R_4+2R_1]{R_2\to R_2+R_1,\ R_3\to R_3+R_1}\begin{bmatrix}1&1&1&1&1&1\\0&-1&0&-1&0&0\\0&-1&-1&-2&-2&-1\\0&2&0&2&4&4\end{bmatrix}\xrightarrow[R_4\to R_4+2R_2]{R_3\to R_3-R_2}\begin{bmatrix}1&1&1&1&1&1\\0&-1&0&-1&0&0\\0&0&-1&-1&-2&-1\\0&0&0&0&4&4\end{bmatrix}\quad \text{Dim}=4$$

$$\text{Dim}(U\cap V)=\text{Dim}(U)+\text{Dim}(V)-\text{Dim}(U+V)=3+3-4=2$$

**Q.** Find the Column space and Null space for the following matrices:

$$\begin{pmatrix}1&1&-3&-2\\2&9&8&3\\1&-1&3&2\end{pmatrix}\qquad\begin{pmatrix}1&2&-1&3\\3&5&2&0\\0&1&2&1\\-1&0&-2&7\end{pmatrix}$$

**A.** i)

$$\begin{bmatrix}1&1&-3&-2\\2&9&8&3\\1&-1&3&2\end{bmatrix}\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1}\begin{bmatrix}1&1&-3&-2\\0&7&14&7\\0&-2&-6&4\end{bmatrix}\xrightarrow{R_3\to R_3+\frac{2}{7}R_2}\begin{bmatrix}1&1&-3&-2\\0&7&14&7\\0&0&-2&6\end{bmatrix}$$

$$C(A)=\left\{\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}1\\9\\-1\end{pmatrix},\begin{pmatrix}-3\\8\\3\end{pmatrix}\right\}\quad C(A) \text{ is 3d plane in } \mathbb{R}^3$$

$$\begin{bmatrix}1&1&-3&-2\\0&7&14&7\\0&0&-2&6\end{bmatrix}\begin{bmatrix}x\\y\\z\\u\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{aligned}x+y-3z-2u&=0\\7y+14z+7u&=0\\-2z+6u&=0\end{aligned}\Rightarrow\begin{aligned}x&=18u\\y&=-7u\\z&=3u\end{aligned}\Rightarrow u\begin{Bmatrix}18\\-7\\3\\1\end{Bmatrix}\longrightarrow N(A) \text{ is a line in } \mathbb{R}^4$$

ii)

$$\begin{bmatrix}1&2&-1&3\\3&5&2&0\\0&1&2&1\\-1&0&-2&7\end{bmatrix}\xrightarrow[R_4\to R_4+R_1]{R_2\to R_2-3R_1}\begin{bmatrix}1&2&-1&3\\0&-1&5&-9\\0&1&2&1\\0&2&-3&10\end{bmatrix}\xrightarrow[R_4\to R_4+2R_2]{R_3\to R_3+R_2}\begin{bmatrix}1&2&-1&3\\0&-1&5&-9\\0&0&7&-8\\0&0&7&-8\end{bmatrix}\xrightarrow{R_4\to R_4-R_3}\begin{bmatrix}1&2&-1&3\\0&-1&5&-9\\0&0&7&-8\\0&0&0&0\end{bmatrix}$$

$$C(A)=\left\{\begin{pmatrix}1\\3\\0\\-1\end{pmatrix},\begin{pmatrix}2\\5\\1\\0\end{pmatrix},\begin{pmatrix}-1\\2\\2\\-2\end{pmatrix}\right\}$$

3D space in $\mathbb{R}^4$

$$\begin{bmatrix}1&2&-1&3\\0&-1&5&-9\\0&0&7&-8\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\u\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{lll} x+2y-z+3u=0 & \Rightarrow & x=\frac{33u}{7}\\ -y+5z-9u=0 & \Rightarrow & y=\frac{-23u}{7}\\ 7z-8u=0 & \Rightarrow & z=\frac{8u}{7}\end{array} \Rightarrow u\begin{Bmatrix}33/7\\-23/7\\8/7\\1\end{Bmatrix} \Rightarrow$$ N(A) is origin in $\mathbb{R}^4$

Q. For which vector (a, b, c) does the following system Ax=b have a solution?
$x + 2y - 3z = a$; $2x + 3y + 3z = b$; $5x + 9y - 6z = c$

A.
$$\left[\begin{array}{ccc|c}1&2&-3&a\\2&3&3&b\\5&9&-6&c\end{array}\right]\xrightarrow[R_3\to R_3-5R_1]{R_2\to R_2-2R_1}\left[\begin{array}{ccc|c}1&2&-3&a\\0&-1&9&a-2b\\0&-1&9&c-5b\end{array}\right]\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{ccc|c}1&2&-3&a\\0&-1&9&a-2b\\0&0&0&c-3b-a\end{array}\right]$$

$x+2y-3z=a$

$-y+9z=a-2b$

$0=c-3b-a \Rightarrow a-3b-c=0$ is the vector for $Ax=b$ to have a soln

Q. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane x + y - 3z = 0.

A. $A=\begin{bmatrix}1&1&-3\end{bmatrix}$

$x+y-3z=0$

$x=-y+3z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-y+3z\\y\\z\end{pmatrix}=y\begin{pmatrix}-1\\1\\0\end{pmatrix}+z\begin{pmatrix}3\\0\\1\end{pmatrix}$$

Q. For which vector ($b_1,b_2,b_3,b_4$) is this system solvable? $\begin{pmatrix}1&2&-1\\1&9&-1\\-3&8&3\\-2&3&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$.

A.
$$\left[\begin{array}{ccc|c}1&2&-1&b_1\\1&9&-1&b_2\\-3&8&3&b_3\\-2&3&2&b_4\end{array}\right]\xrightarrow[\substack{R_3\to R_3+3R_1\\R_4\to R_4+2R_1}]{R_2\to R_2-R_1}\left[\begin{array}{ccc|c}1&2&-1&b_1\\0&7&0&b_2-b_1\\0&14&0&b_3+3b_1\\0&7&0&b_4+2b_1\end{array}\right]\xrightarrow[R_4\to R_4-R_2]{R_3\to R_3-2R_2}\left[\begin{array}{ccc|c}1&2&-1&b_1\\0&7&0&b_2-b_1\\0&0&0&b_3-2b_2+5b_1\\0&0&0&b_4-b_2+3b_1\end{array}\right]$$

$\Rightarrow b_3-2b_2+5b_1=0$

and $b_4-b_2+3b_1=0$

Q. If the set of vectors {u,v,w} are linearly independent vectors, then show that the set $\{u+v-2w,\ u-v-w,\ u+w\}$ is linearly independent.

A. $c_1u+c_2v+c_3w=0$

$c_1(u+v-2w)+c_2(u-v-w)+c_3(u+w)=0$

$u(c_1+c_2+c_3)+v(c_1-c_2)+w(-2c_1-c_2+c_3)=0$

$c_1+c_2+c_3=0 \Rightarrow c_1=-c_3/2$

$c_1-c_2=0 \Rightarrow c_1=c_2$

$-2c_1-c_2+c_3=0 \Rightarrow c_1=c_3/3$

$-\frac{c_3}{2}=\frac{c_3}{3} \Rightarrow c_3=0$

$c_2=0$

$c_1=0$

$\Big\}\rightarrow$ hence, linearly independant

Q. Find a basis and dimension of the subspace W of V=$M_{2\times 2}$ spanned by
$\left\{\begin{pmatrix}1&2\\1&2\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}0&2\\0&1\end{pmatrix},\begin{pmatrix}2&4\\2&4\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$. (Home-work)

A.
$$\begin{bmatrix}1&1&0&2&1\\2&0&2&4&0\\1&1&0&2&0\\2&1&1&4&1\end{bmatrix}\xrightarrow[\substack{R_3\to R_3-R_1\\R_4\to R_4-2R_1}]{R_2\to R_2-2R_1}\begin{bmatrix}1&1&0&2&1\\0&-2&2&0&-2\\0&0&0&0&-1\\0&-1&1&0&-1\end{bmatrix}\xrightarrow{R_4\to R_4-\frac{1}{2}R_2}\begin{bmatrix}1&1&0&2&1\\0&-2&2&0&-2\\0&0&0&0&-1\\0&0&0&0&0\end{bmatrix}$$

Basis of W = $(M_1, M_2, M_5)$ Dimension = 3

Q. Find a basis and the dimension of the subspaces of $U=\{(a,b,c,d)/b-2c+d=0\}$ and $V=\{(a,b,c,d)/a=d, b=2c\}$ in $\mathbb{R}^4$. Find basis and dim of $U\cap V$.

A.
$$U=\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}, b-2c+d=0\right\}=\begin{bmatrix}a\\b\\c\\2c-b\end{bmatrix}=a\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}+c\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$$

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, a = d, b = 2c \right\} = \begin{bmatrix} a \\ 2c \\ c \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Basis of } U \cap V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{array}{c} b - 2c + d = 0,\ a = d,\ b = 2c \\ \downarrow \qquad\qquad \downarrow \\ 2c - 2c + d = 0 \quad a = d = 0 \\ \downarrow \\ d = 0 \end{array} \right\} = \begin{bmatrix} 0 \\ 2c \\ c \\ 0 \end{bmatrix} = c \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{Dimension} = 1$$

**Q.** If the column space of A is spanned by the vectors (1,2,7,5), (-2,-1,-8,-7), (-1,3,3,0) find all those vectors that span the null space of A. What are the bases and dimensions of $C(A^T)$ and $N(A^T)$.

**A.**

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 7R_1 \\ R_4 \to R_4 - 5R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \\ 0 & 3 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} x_1 - 2x_2 - x_3 = 0 \\ 3x_2 + 5x_3 = 0 \end{array} \right\} \rightarrow \begin{array}{l} x_1 = \dfrac{-7x_3}{3} \\ x_2 = \dfrac{-5x_3}{3} \end{array} \Rightarrow N(A) = x_3 \begin{bmatrix} -7/3 \\ -5/3 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 7 & 5 \\ -2 & -1 & -8 & -7 \\ -1 & 3 & 3 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 7 & 5 \\ 0 & 3 & 6 & 3 \\ 0 & 5 & 10 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{5}{3}R_2} \begin{bmatrix} 1 & 2 & 7 & 5 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A^T) = \left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\} \Rightarrow \text{Dimension} = 2$$

$$\begin{bmatrix} 1 & 2 & 7 & 5 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 2x_2 + 7x_3 + 5x_4 = 0 \Rightarrow x_1 = -3x_3 - 3x_4 \\ 3x_2 + 6x_3 + 3x_4 = 0 \quad \Rightarrow x_2 = -2x_3 - x_4 \end{array}$$

$$N(A^T) = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Dimension} = 2$$

**Q.** Obtain the four fundamental subspaces, their basis and dimension given $\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 1 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \\ 0 & 0 & 3 & 1 & 4 \end{pmatrix}$. Also describe the four fundamental subspaces.

**A.**

$$\begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 1 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 6 & 3 & 3 & 1 \\ 0 & 0 & 4 & 0 & 3 \\ 0 & 0 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{R_4 \to R_4 - \frac{3}{4}R_3} \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 6 & 3 & 3 & 1 \\ 0 & 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 7/4 \end{bmatrix}$$

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 12 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \\ 1 \end{bmatrix} \right\}$$

For $N(A)$,

$$\begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 6 & 3 & 3 & 1 \\ 0 & 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 7/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{r} x_1 + 3x_2 + x_3 + 2x_4 + x_5 = 0 \\ 6x_2 + 3x_3 + 3x_4 + x_5 = 0 \\ 4x_3 + \quad 3x_5 = 0 \\ x_4 + \frac{7}{4}x_5 = 0 \end{array} \Rightarrow N(A) = x_5 \begin{bmatrix} -7/2 \\ 9/4 \\ -3/4 \\ -7/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 3 & 9 & 12 & 0 \\ 1 & 4 & 8 & 3 \\ 2 & 5 & 8 & 1 \\ 1 & 2 & 7 & 4 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \\ R_5 \to R_5 - R_1}]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 3 & 4 & 3 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \xrightarrow[\substack{R_4 \to R_4 - \frac{R_2}{2} \\ R_5 \to R_5 - \frac{R_2}{6}}]{R_3 \to R_3 - \frac{R_2}{2}} \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \xrightarrow{R_5 \to R_5 - \frac{3}{4}R_3} \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7/4 \end{bmatrix}$$

$$C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \\ 8 \\ 8 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 4 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 + 4x_3 &= 0 \\ 6x_2 &= 0 \\ 4x_3 + 3x_4 &= 0 \\ x_4 &= 0 \\ 7/4\, x_4 &= 0 \end{aligned} \qquad N(A^T) = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Column Space - Set of all linear combinations of columns of A

Null Space - Set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = 0$

Row space - Set of all linear combinations of rows of A

Left Null Space - Set of all vectors $\vec{y} \in \mathbb{R}^n$ such that $A^T\vec{y} = 0$

Q. Find left / right inverse (whichever possible) for the following matrices:

(i) $\begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix}$ (Home-work).

A. i) $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow 2 \times 3 \quad (m \times n)$

$\rho(A) = 2 = m \Rightarrow$ Right Inverse $= A^T(AA^T)^{-1}$

$$\begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \\ 1/8 & -3/8 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix} \Rightarrow 3 \times 2$

$\rho(A) = n = 2 \Rightarrow$ Left Inverse $(A^TA)^{-1}A^T$

$$\left(\begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

Q. Which of these transformations are not linear? Give reasons.

$(i) T(x, y, z) = (x + y + z, 2x - 3y + 4z)$

$(ii) T(x, y) = (x + 3, 2y, x + y)$ $(iii)$ $(iii) T(x, y) = (xy, x)$

A. i) $T(x, y, z) = (x + y + z,\ 2x - 3y + 4z)$

Rule of linearity $T(cx + dy) = cT(x) + dT(y)$

$T(0,0,0) = (0,0)$

$T(1,0,0) = (1,2)$

To verify $T(cx) = cT(x)$

Let $c = 2$

$T(2,0,0) = (2,4) = 2(1,2) = cT(x)$ $\Rightarrow$ Hence, it is linear transform

ii) $T(x,y) = (x+3,\ 2y,\ x+y)$

$T(0,0) = (3,0,0) \neq (0,0,0)$ $\Rightarrow$ Not linear transform

iii) $T(x,y) = (xy, x)$

$T(0,0) = (0,0)$

$T(1,1) = (1,1)$

$T(2,2) = (4,2)$

$T(cx) \neq cT(x) \Rightarrow$ Not linear

Q. For each of the following linear transformations T, find a basis and the dimension of the range and kernel of T:

(i) $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by T(x, y, z)=(2x+z, x+y)

(ii) $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by T(x, y,z)=(x+2y+3z, -3x-2y-z, -2x+2z)

A. i) $T: \mathbb{R}^3 \to \mathbb{R}^2$ , $T(x,y,z) = (2x+z,\ x+y)$

(If basis isn't given, consider standard basis of domain)

Basis of $\mathbb{R}^3$ : $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T(e_1) = T(1,0,0) = (2,1)$

$T(e_2) = T(0,1,0) = (0,1)$

$T(e_3) = T(0,0,1) = (1,0)$

$\Rightarrow A = \begin{pmatrix}2 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{R_1}{2}} \begin{pmatrix}2 & 0 & 1\\ 0 & 1 & -\frac{1}{2}\end{pmatrix} \Rightarrow C(A) = \left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ Dimension = 2

$N(A) \Rightarrow 2x_1 + x_3 = 0$

$x_2 - \frac{x_3}{2} = 0$

$\Rightarrow N(A) = x_1 \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\end{bmatrix}$ Dimension = 1

ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ , $T(x,y,z) = (x+2y+3z,\ -3x-2y-z,\ -2x+2z)$

basis of $\mathbb{R}^3$ : $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T(e_1) = T(1,0,0) = (1,-3,-2)$

$T(e_2) = T(0,1,0) = (2,-2,0)$

$T(e_3) = T(0,0,1) = (3,-1,2)$

$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ -3 & -2 & -1\\ -2 & 0 & 2\end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 8\\ 0 & 4 & 8\end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 8\\ 0 & 0 & 0\end{bmatrix}$

$C(A) = \left\{\begin{pmatrix}1\\-3\\-2\end{pmatrix}, \begin{pmatrix}2\\-2\\0\end{pmatrix}\right\}$ Dimension = 2

$N(A) \Rightarrow x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = x_3$

$4x_2 + 8x_3 = 0 \Rightarrow x_2 = -2x_3$

$N(A) = x_3 \begin{bmatrix}1\\-2\\1\end{bmatrix}$ Dimension = 1

Q. Find the matrix of the linear transformation T on $\mathbb{R}^3$ defined by T(x,y,z)=(2y+z, x-4y, 3x) with respect to

(i)the standard basis (1,0,0),(0,1,0), ((0,0,1)

(ii)the basis (1,1,1),(1,1,0), ((1,0,0)

A. $T(x,y,z) = (2y+z,\ x-4y,\ 3x)$

i) Basis of $\mathbb{R}^3$ : $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

$T(e_1) = T(1,0,0) = (0,1,3)$

$T(e_2) = T(0,1,0) = (2,-4,0)$

$T(e_3) = T(0,0,1) = (1,0,0)$

$\Rightarrow A = \begin{bmatrix}0 & 2 & 1\\ 1 & -4 & 0\\ 3 & 0 & 0\end{bmatrix}$

ii) Basis of $\mathbb{R}^3$ : $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$

$T(e_1) = T(1,1,1) = (3,-3,3)$

$T(e_2) = T(1,1,0) = (2,-3,3)$

$T(e_3) = T(1,0,0) = (0,1,3)$

For 1st column, $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix}3\\-3\\3\end{pmatrix} \Rightarrow c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3\\-3\\3\end{pmatrix} \Rightarrow$ $c_1 = 3$, $c_2 = -6$, $c_3 = 6$

For 2nd column, $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix}2\\-3\\3\end{pmatrix} \Rightarrow c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-3\\3\end{pmatrix} \Rightarrow$ $c_1 = 3$, $c_2 = -6$, $c_3 = 5$

For 3rd column, $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix}0\\1\\3\end{pmatrix} \Rightarrow c_1\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\3\end{pmatrix} \Rightarrow$ $c_1 = 3$, $c_2 = -2$, $c_3 = -1$

$\Rightarrow A = \begin{bmatrix}3 & 3 & 3\\ -6 & -6 & -2\\ 6 & 5 & -1\end{bmatrix}$

Q. Find a linear mapping $T: \mathbb{R}^4 \to \mathbb{R}^3$ whose Kernel is spanned by (-2,1,0,0) and (1,0,-1,1).

A. $\mathbb{R}^4 \to \mathbb{R}^3$

$Ax = 0$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Free variables

$-2a + b = 0 \Rightarrow b = 2a$ $\quad$ $a - c + d = 0 \Rightarrow a = c - d$

$-2e + f = 0 \Rightarrow f = 2e$ $\quad$ $e - g + h = 0 \Rightarrow e = g - h$

Let $a = 1$ (pivot element)

$e = 0 \Rightarrow f = 0$

Let $g = 1$ (pivot element) and $d = 0$

$a = c = 1$ $\Rightarrow$ T is one:one if all columns have pivot

$b = 2$ $\quad$ T is onto if all rows have pivot elements

$h = 1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \Rightarrow T(x,y,z,t) = (x+2y+z,\ z+t,\ 0)$$

Q. (i) If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation defined by T(x,y)=(3x+y,5x+7y, x+3y). Show that T is one-to-one. Is T onto?

(ii) Is T invertible? Find a rule for $T^{-1}$ like the one which defines T where T(x,y,z)=(3x,x-y, 2x+y+z) is a transformation in $\mathbb{R}^3$.

i) $T(x,y) = (3x+y,\ 5x+7y,\ x+3y)$

Basis for $\mathbb{R}^2$ : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T(1,0) = (3,5,1)$

$T(0,1) = (1,7,3)$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{3}R_1]{R_2 \to R_2 - \frac{5}{3}R_1} \begin{bmatrix} 3 & 1 \\ 0 & 16/3 \\ 0 & 8/3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 3 & 1 \\ 0 & 16/3 \\ 0 & 0 \end{bmatrix}$$

T is one to one, All columns have pivots

T is not onto, zero row exists

ii) $T(x,y,z) = (3x,\ x-y,\ 2x+y+z)$

Basis for $\mathbb{R}^3$ : $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(1,0,0) = (3,1,2)$

$T(0,1,0) = (0,-1,1)$

$T(0,0,1) = (0,0,1)$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{2}{3}R_1]{R_2 \to R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

T is one to one, All columns have pivots

T is onto, zero row doesn't exists

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \Rightarrow T^{-1}(x,y,z) = \left(\frac{x}{3},\ \frac{x}{3} - y,\ -x + y + z\right)$$