ELECTRO - MAGNETIC FIELD THEORY

(UE23EE241B)

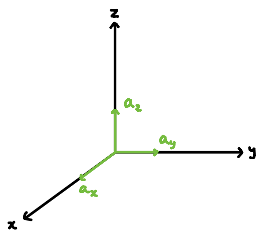
-Hitesh

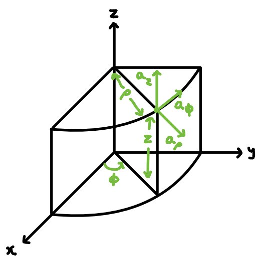
Unit - 0 Vector Analysis

Maxwell Equations

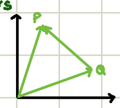
- $\nabla . D = \rho$
- $\nabla \times E = 0$
- $\nabla . B = 0$
- $\nabla \times H = J$

D : Electric Flux Density
ρ : Change Density
E : Electric Field Intensity
H : Magnetic Field Intensity
J : Convention Current Density





Vectors

→ 

$\vec{V}_{PQ} = \vec{V}_Q - \vec{V}_P$

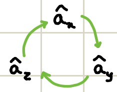
Dot & Cross Products

→ $\hat{a}_x . \hat{a}_x = 1$

$\hat{a}_x . \hat{a}_y = 0$

→ $\hat{a}_x \times \hat{a}_x = 0$

$\hat{a}_x \times \hat{a}_y = 1$



Orthogonal System : Co-ordinate surfaces are mutually perpendicular is usually choosen
↳ Popular co-ordinate include cartesian, cylindrical & spherical

Cartesian Co-ordinate System

→ Any point P can be represented as (x, y, z)

→ Range : $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$

→ $A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

Circular Cylindrical Coordinates

→ Any point P can be represented as (ρ, ϕ, z)

→ Range : $0 \le \rho < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$

→ $A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

→ Relation b/w the variables of cartesian & cylindrical systems :

$\rho = \sqrt{x^2 + y^2}$ $\qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\qquad z = z$

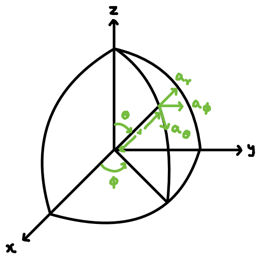
$x = \rho \cos\phi$ $\qquad y = \rho \sin\phi$

→ Relation b/w the unit vectors of cartesian & cylindrical systems are represented as :

$\hat{a}_x = \cos\phi\, \hat{a}_\rho - \sin\phi\, \hat{a}_\phi$	$\hat{a}_\rho = \cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y$
$\hat{a}_y = \sin\phi\, \hat{a}_\rho + \cos\phi\, \hat{a}_\phi$	$\hat{a}_\phi = -\sin\phi\, \hat{a}_x + \cos\phi\, \hat{a}_y$
$\hat{a}_z = \hat{a}_z$	$\hat{a}_z = \hat{a}_z$

Matrices :

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$



Spherical Coordinates

→ Any point P can be represented as (r, θ, ϕ)

→ Range : $0 \le \;\; < \infty$, $0 \le \theta < \pi$, $0 \le \phi < 2\pi$

→ $A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$

→ Relation b/w the variables of cartesian & spherical systems :

$$\rho = \sqrt{x^2+y^2} \qquad r = \sqrt{x^2+y^2+z^2} \qquad \theta = \tan^{-1}\left(\frac{\sqrt{y^2+x^2}}{z}\right) \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

→ Relation b/w the unit vectors of cartesian & spherical systems are represented as :

$$\hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\, a_\phi$$
$$\hat{a}_y = \sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\, a_\phi$$
$$\hat{a}_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$$

$$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$
$$\hat{a}_\theta = \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y - \sin\theta\,\hat{a}_z$$
$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

→ Relation b/w the variables of Cylindrical and Spherical systems are represented as :

$$r = \sqrt{\rho^2 + z^2} \qquad \theta = \tan^{-1}\left(\frac{\phi}{z}\right) \qquad \phi = \phi$$

$$\rho = r\sin\theta \qquad z = r\cos\theta \qquad \phi = \phi$$

→ Transformation from (A_ρ, A_ϕ, A_z) to (A_r, A_θ, A_ϕ)

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

→ Transformation from (A_r, A_θ, A_ϕ) to (A_ρ, A_ϕ, A_z)

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Differential Length, Area & Volume

→ **Cartesian Systems :**

- Differential Displacement : $d\mathbf{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$
- Differential Normal Surface Area : $d\mathbf{S} = dy\,dz\,\hat{a}_x$
  $= dx\,dz\,\hat{a}_y$
  $= dx\,dy\,\hat{a}_z$
- Differential Volume : $dv = dx\,dy\,dz$

→ **Cylindrical Systems :**

- Differential Displacement : $d\mathbf{l} = d\rho\,\hat{a}_\rho + \rho\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$
- Differential Normal Surface Area : $d\mathbf{S} = \rho\,d\phi\,dz\,\hat{a}_\rho$
  $= d\rho\,dz\,\hat{a}_\phi$
  $= \rho\,d\rho\,d\phi\,\hat{a}_z$
- Differential Volume : $dv = \rho\,d\rho\,d\phi\,dz$

→ **Spherical Systems :**

- Differential Displacement : $d\mathbf{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$
- Differential Normal Surface Area : $d\mathbf{S} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$
  $= r\sin\theta\,dr\,d\phi\,\hat{a}_\theta$
  $= r\,dr\,d\theta\,a_\phi$
- Differential Volume : $dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

Total Charge

$$Q = \int \rho_L\,dl = \int \rho_S\,dS = \int \rho_v\,dv$$

Q. Find the total charge on a circular disc defined by $\rho \le a$, $\rho_s = K\rho$ (C/m^2)

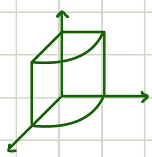
A. $dS = \rho\,d\rho\,d\phi\,\hat{a}_z$

$$Q = \int \rho_s\,dS = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} K\rho^2\,d\rho\,d\phi = 2\pi K\left(\frac{\rho^3}{3}\right)_0^a = \frac{2\pi K a^3}{3}$$

Q. A wire defined by $2 \le y \le 5$, $x = z = 0$ with charge density $\rho_L = ae^y$ (C/m). Find Q on the wire

A. $$Q = \int \rho_L\,dl = \int_2^5 \rho_L\,dy = \int_2^5 ae^y\,dy = a(e^5 - e^2) = 141.02\,a$$

Q. $0 \le \rho \le 3$, $0 \le z \le 2$, $Q = ?$



A. From the diagram, $0 \le \phi \le \frac{\pi}{2}$

$$Q = \int \rho_v\,dv = \rho_v \int_0^3\int_0^2\int_0^{\pi/2} \rho\,d\rho\,d\phi\,dz$$

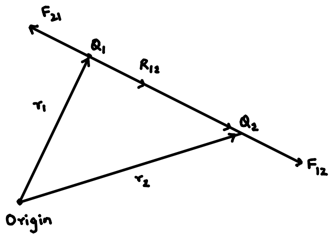
$$= \rho_v \; 2 \times \frac{3^2}{2} \times \frac{\pi}{2} = \frac{9\pi\rho_v}{2}$$

Unit - 1 Electrostatics

$K = \frac{1}{4\pi\epsilon_0} = 9\times10^9$ m/F

$\epsilon_0 = 8.854\times10^{-12} \simeq \frac{10^{-9}}{36\pi}$ F/m

↳ permittivity of free space



Electrostatics

→ Concepts applicable to static electric fields in free space

→ Electrostatic field is produced by 'static' charge distribution

→ Unit of electric charge : Coulomb (C)

Coulomb's Law

→ The law states that Force b/w 2 point charges Q_1 & Q_2 is :

- Along the line joining them
- Directly proportional to product Q_1Q_2 of the charges
- Inversely proportional to square of the distance b/w them

→ $F = \frac{KQ_1Q_2}{R^2} = \frac{Q_1Q_2}{4\pi\epsilon_0 R^2}$

$K = \frac{1}{4\pi\epsilon_0} \simeq 9\times10^9$ m/F

→ If Q_1 & Q_2 are located at points having position vectors $\vec{r_1}$ & $\vec{r_2}$ then force $\vec{F_{12}}$ on Q_2 due to Q_1 is

$\vec{F_{12}} = \frac{Q_1Q_2}{4\pi\epsilon_0 R^2}\hat{a}_{R_{12}}$

where $\vec{R_{12}} = \vec{r_2} - \vec{r_1}$

$R = |R_{12}|$

$\hat{a}_{R_{12}} = \frac{R_{12}}{R} = \frac{R_{12}}{|R_{12}|}$

→ $\vec{F_{21}} = |\vec{F_{12}}|\hat{a}_{R_{21}} = |\vec{F_{12}}|(-\hat{a}_{R_{12}}) = -\vec{F_{12}}$

Principle of Superposition

→ It states that if there are N charges $Q_1, Q_2, \dots, Q_N$ located at respective position vectors $\vec{r_1}, \vec{r_2}, \vec{r_3}, \dots, \vec{r_N}$, the resultant force $\vec{F}$ on charge Q located at a point with its position vector r, is the vector sum of forces exerted on Q by $Q_1, Q_2, \dots, Q_N$

→ $F = \frac{QQ_1(\vec{r}-\vec{r_1})}{4\pi\epsilon_0|\vec{r}-\vec{r_1}|^3} + \frac{QQ_2(\vec{r}-\vec{r_2})}{4\pi\epsilon_0|\vec{r}-\vec{r_2}|^3} + \dots + \frac{QQ_N(\vec{r}-\vec{r_N})}{4\pi\epsilon_0|\vec{r}-\vec{r_N}|^3}$

$= \frac{Q}{4\pi\epsilon_0}\sum_{K=1}^{N}\frac{Q_K(\vec{r}-\vec{r_K})}{|\vec{r}-\vec{r_K}|^3}$

Electric Field Intensity / Electric Field Strength

→ $E = \frac{F}{Q} = \frac{Q_1(\vec{r}-\vec{r_1})}{4\pi\epsilon_0|\vec{r}-\vec{r_1}|^3} + \frac{Q_2(\vec{r}-\vec{r_2})}{4\pi\epsilon_0|\vec{r}-\vec{r_2}|^3} + \dots + \frac{Q_N(\vec{r}-\vec{r_N})}{4\pi\epsilon_0|\vec{r}-\vec{r_N}|^3} = \frac{1}{4\pi\epsilon_0}\sum_{K=1}^{N}\frac{Q_K(\vec{r}-\vec{r_K})}{|\vec{r}-\vec{r_K}|^3}$

Electric Field due to continuous charge distributions

→ Charge densities are represented as :

- Line Charge Density - ρ_L (C/m)
- Surface Charge Density - ρ_s (C/m^2)
- Volume Charge Density - ρ_v (C/m^3)

→ The charge element dQ & total charge Q can be obtained as :

$$dQ = \rho_L dl \longrightarrow \int_L \rho_L dl$$

$$dQ = \rho_s dS \longrightarrow \int_S \rho_s dS$$

$$dQ = \rho_v dv \longrightarrow \int_v \rho_v dv$$

→ $$E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_R = \frac{Q(r-r')}{4\pi\varepsilon_0 |r-r'|^3} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r$$

So, $$E = \int_L \frac{\rho_L dl}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$E = \int_S \frac{\rho_s dS}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$E = \int_v \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Q. Point charges 1 mC and −2 mC at (3,2,−1) and (−1,−1,4) respectively. Calculate the electric force on a 10nC charge located at (0,3,1) and the electric field intensity at that point

A. $$F = \frac{Q}{4\pi\varepsilon_0}\sum Q_k \frac{(r-r_k)}{|r-r_k|^3}$$

$$= 10^{-8}\times 9\times 10^{9}\left(\frac{10^{-3}\times(0\hat{i}-3\hat{i}+3\hat{j}-2\hat{j}+1\hat{k}+1\hat{k})}{|0\hat{i}-3\hat{i}+3\hat{j}-2\hat{j}+1\hat{k}+1\hat{k}|^3} - \frac{(2\times 10^{-3}\times(0\hat{i}+\hat{i}+3\hat{j}+\hat{j}+\hat{k}-4\hat{k}))}{(0\hat{i}+\hat{i}+3\hat{j}+\hat{j}+\hat{k}-4\hat{k})^3}\right)$$

$$= 90\left(10^{-3}\frac{(-3\hat{i}+\hat{j}+2\hat{k})}{(\sqrt{9+1+4})^3} - \frac{2\times 10^{-3}(\hat{i}+4\hat{j}-3\hat{k})}{(\sqrt{26})^3}\right)$$

$$= 90\times 10^{-3}\left(0.02(-3\hat{i}+\hat{j}+2\hat{k}) - 0.015(\hat{i}+4\hat{j}-3\hat{k})\right)$$

$$= 90\times 10^{-3}(-0.075\hat{i} - 0.04\hat{j} + 0.085\hat{k})$$

$$= (-6.75\hat{i} - 3.6\hat{j} + 7.6\hat{k})\ mN$$

Summary,

$$F = KQ_1 \sum \frac{Q_n}{|r_1-r_n|^2}\cdot\hat{a}_{1n}$$

Q. A charge $Q_1 = 3\times10^{-4}$ C is placed at $M(1,2,3)$ and $Q_2 = -10^{-4}$ C at $N(2,0,5)$ in vacuum. Find force exerted by Q_1 on Q_2

A. $F_2 = \frac{KQ_1Q_2}{R_{12}^2}\hat{a}_{12}$

$$= \frac{9\times10^9 \times 3\times10^{-4}\times(-10^{-4})}{\left((1-2)^2+(2-0)^2+(3-5)^2\right)} \left(\frac{(2-1)\hat{a}_x + (0-2)\hat{a}_y + (5-3)\hat{a}_z}{3}\right)$$

$$= \frac{-270}{1+4+4}\left(\frac{\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z}{3}\right)$$

$$= -10\left(\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z\right)$$

Q. A charge $Q_1 = -20\mu C$ is located at $P(-6,4,6)$ and a charge $Q_2 = 50\mu C$ is located at $R(5,8,-2)$ in a free space. Find i) r ii) $|r|$ iii) F_{Q_2}

A. i) $r = (5-(-6))\hat{a}_x + (8-4)\hat{a}_y + (-2-6)\hat{a}_z$

$$= 11\hat{a}_x + 4\hat{a}_y - 8\hat{a}_z$$

ii) $|r| = \sqrt{11^2+4^2+8^2} = \sqrt{201} = 14.17$

iii) $F_{Q_2} = \frac{KQ_1Q_2}{r_{12}^2}\hat{a}_{12} = \frac{(-20\times10^{-6})(50\times10^{-6})}{201}\cdot 9\times10^9 \cdot \left(\frac{11\hat{a}_x + 4\hat{a}_y - 8\hat{a}_z}{\sqrt{201}}\right)$

$$= -3.16\times10^{-3}\left(11\hat{a}_x + 4\hat{a}_y - 8\hat{a}_z\right)$$

$$= -0.034\hat{a}_x - 0.0126\hat{a}_y + 0.025\hat{a}_z$$

Q. Two point charges of magnitude 2mC and −7mC are located at places $P_1(4,7,-5)$ and $P_2(-3,2,-9)$ respectively in free space. Evaluate the vector force on charge at P_2

A. $F = \frac{KQ_1Q_2}{R_{12}^2}\hat{a}_{12}$

$$= \frac{9\times10^9 \times 2\times10^{-3}\times(-7\times10^{-3})}{(-7)^2+(-5)^2+(-4)^2} \cdot \frac{(-7\hat{a}_x - 5\hat{a}_y - 4\hat{a}_z)}{\sqrt{7^2+5^2+4^2}}$$

$$= -147.57\left(-7\hat{a}_x - 5\hat{a}_y - 4\hat{a}_z\right)$$

$$= 1033\hat{a}_x + 737.85\hat{a}_y + 590.28\hat{a}_z$$

Electric Field due to line charge

→ Consider a line charge with uniform charge density ρ_L extending from A to B along z-axis

→ Charge element dQ associated with dl is written as

$$dQ = \rho_L dl = \rho_L dz$$

So,

$$Q = \int_{z_A}^{z_B} \rho_L dz$$

→ Electric field intensity $\vec{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_0 R^2} \hat{a}_R$

$$= \int_L \frac{\rho_L dz}{4\pi\varepsilon_0 R^2} \hat{a}_R$$

→ The distance vector $\vec{R} = (x, y, z) - (0, 0, z')$

$$= x\hat{a}_x + y\hat{a}_y + (z-z')\hat{a}_z$$

$$= \rho\hat{a}_\rho + (z-z')\hat{a}_z$$

$$R^2 = \rho^2 + (z-z')^2$$

So, $$\frac{\hat{a}_R}{R^2} = \frac{\vec{R}}{|\vec{R}|^3} = \frac{\rho\hat{a}_\rho + (z-z')\hat{a}_z}{(\rho^2 + (z-z')^2)^{3/2}}$$

Then, $$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho\hat{a}_\rho + (z-z')\hat{a}_z}{(\rho^2 + (z-z')^2)^{3/2}} dz'$$

→ $$\vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\,[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]\,d\alpha}{\rho^2\sec^2\alpha}$$

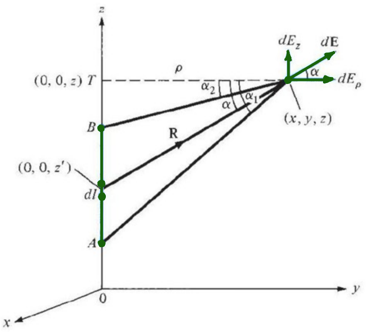
$$\vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0\rho} \int_{\alpha_1}^{\alpha_2} [\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z]\,d\alpha$$

For finite line charges,

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\hat{a}_z\right]$$

For infinite line charges, $\Rightarrow$ $A(0,0,-\infty)$ and $B(0,0,\infty)$, $\alpha_1 = \frac{\pi}{2}$ $\alpha_2 = \frac{-\pi}{2}$

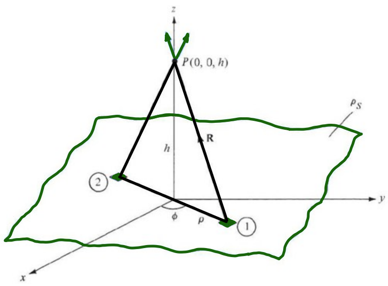
$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(-1-1)\hat{a}_\rho\right] = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$



$$(\rho^2 + (z-z')^2)^{1/2}$$
$$= (\rho^2 + \rho^2\tan^2\alpha)^{1/2}$$
$$= \rho(1+\tan^2\alpha)^{1/2}$$
$$= \rho(\sec^2\alpha)^{1/2}$$
$$= \rho\sec\alpha$$

$$\tan\alpha = \frac{z-z'}{\rho}$$
$$dl = OT - (z-z')$$
$$z' = OT - (z-z')$$
$$z' = OT - \rho\tan\alpha$$
$$dz' = -\rho\sec^2\alpha\,d\alpha$$

$$E = K \sum \frac{Q_n}{r_{1n}^2}\hat{a}_{1n}$$



Electric Field due to surface charge

→ Consider an infinite sheet of charge in xy-plane with uniform charge density ρ_s

→ Charge element dQ associated with dS is written as

$$dQ = \rho_s \, dS$$

→ Electric field intensity $\vec{E} = \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0 R^2} \hat{a}_R$

The contribution to the Electric field ($\vec{E}$) at a point $(0,0,h)$ by the charge dQ on 1 is

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 R^2} \hat{a}_R$$

→ From the diagram,

$$\vec{R} = -\rho\hat{a}_\rho + h\hat{a}_z \quad \Rightarrow \quad h\hat{a}_z = \rho\hat{a}_\rho + \vec{R}$$

$$|\vec{R}| = \sqrt{\rho^2 + h^2} = R \qquad \& \quad \hat{a}_R = \frac{\vec{R}}{R}$$

$$dQ = \rho_s \, dS = \rho_s \, \rho \, d\phi \, d\rho$$

Now, $$d\vec{E} = \frac{\rho_s dS}{4\pi\varepsilon_0 R^2} \cdot \frac{\vec{R}}{R} = \frac{\rho_s \rho \, d\phi \, d\rho}{4\pi\varepsilon_0 R^3} \cdot \vec{R} = \frac{\rho_s \rho \, d\phi \, d\rho \left(-\rho\hat{a}_\rho + h\hat{a}_z\right)}{4\pi\varepsilon_0 (\rho^2+h^2)^{3/2}}$$

$\rho^2 + h^2 = v^2$

$\cancel{2}\rho d\rho = \cancel{2} v dv$

$\rho \Rightarrow 0$ to ∞

$v \Rightarrow h$ to ∞

$$\vec{E} = \int_S d\vec{E}_z = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h \rho \, d\phi \, d\rho}{(\rho^2+h^2)^{3/2}} \hat{a}_z = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{v=h}^{\infty} \frac{h \, d\phi \, v dv}{v^3} \hat{a}_z$$

$$= \frac{\rho_s h}{4\pi\varepsilon_0} (\phi)_0^{2\pi} \cdot \left(\frac{-1}{v}\right)_h^{\infty} \hat{a}_z = \frac{\rho_s h}{\cancel{4\pi}\varepsilon_0 \, 2} \times \cancel{2\pi} \left(\frac{-1}{\infty} - \left(\frac{-1}{h}\right) \right) \hat{a}_z = \frac{\rho_s h}{2\varepsilon_0 h} \hat{a}_z$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_z}$$

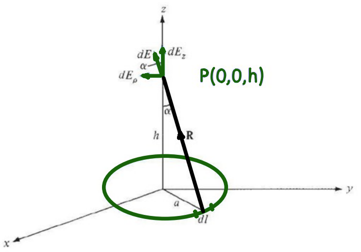
→ $\vec{E}$ expression is valid only $h > 0$

For $h < 0 \Rightarrow \hat{a}_z$ is replaced $\hat{a}_z$

For infinite sheet of charge $\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_n$

Note: $\vec{E}$ is normal to the sheet & independant of distance b/w sheet & P (P : Point of observation)

→ For parallel plate capacitor, $\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_n + \frac{-\rho_s}{2\varepsilon_0} (-\hat{a}_n) = \frac{\rho_s}{\varepsilon_0} \hat{a}_n$



Q. A Circular ring of radius 'a' carries a uniform charge ρ_L is placed on the xy-plane with axis same as z-axis.

a) Show that $E(0,0,h) = \dfrac{\rho_L a h}{2\varepsilon_0 (h^2+a^2)^{3/2}} \hat{a}_z$

b) What values of h gives the maximum value of E = ?

c) If the total charge on the ring is Q, Find E as 'a' tends to 0

$$\left(E = \frac{1}{4\pi\varepsilon_0} \int_L \rho_L \frac{dL}{R^2} \hat{a}_R \right)$$

A. a) $E = K \int_L \dfrac{\rho_L \, dL}{R^2} \hat{a}_R$

$$a\hat{a}_\rho + R = h\hat{a}_z$$

$$R = h\hat{a}_z - a\hat{a}_\rho$$

$$|R| = \sqrt{h^2+a^2}$$

$$E = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \frac{\rho_L a \, d\phi}{(h^2+a^2)^{3/2}} (h\hat{a}_z - a\hat{a}_\rho)$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \frac{(h\hat{a}_h - \cancel{a\hat{a}_\rho}^{\,0})\, a\, 2\pi}{(h^2+a^2)^{3/2}}$$

$$= \frac{\rho_L a h \hat{a}_z}{2\varepsilon_0 (h^2+a^2)^{3/2}}$$

b) $\dfrac{dE}{dh} = 0$

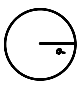
$$E = \frac{\rho_L a h}{2\varepsilon_0 (h^2+a^2)^{3/2}}$$

$$\frac{dE}{dh} = \frac{\rho_L a}{2\varepsilon_0} \frac{d}{dh}\left(\frac{h}{(h^2+a^2)^{3/2}} \right)$$

$$= \frac{\rho_L a}{2\varepsilon_0} \left(\frac{(1)(h^2+a^2)^{3/2} - h(3/2)(h^2+a^2)^{1/2}(2h)}{(h^2+a^2)^3} \right)$$

$$= \frac{\rho_L a}{2\varepsilon_0} (h^2+a^2)^{-5/2} (h^2+a^2-3h^2) = 0$$

$$0 = a^2 - 2h^2 \Rightarrow h = \pm \frac{a}{\sqrt{2}}$$



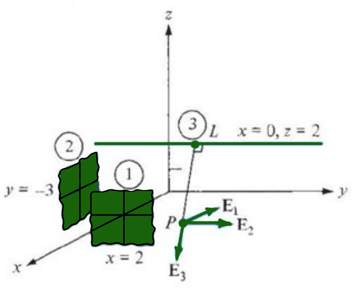
c) $L = 2\pi a$

$$\rho_L = \frac{Q}{L} = \frac{Q}{2\pi a}$$

$$E = \lim_{a \to 0} \frac{Q}{2\pi \cancel{a}} \cdot \frac{\cancel{a} h}{2\varepsilon_0 (h^2+a^2)^{3/2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{Qh}{h^3} \hat{a}_z$$

Q. Planes $x=2$ & $y=3$ respectively carry charges $10\ nC/m^2$. If the line $x=0$ & $z=2$ carries charge $10\pi\ nC/m$, Calculate E at $(1,1,-1)$ due to the 3 charge distribution

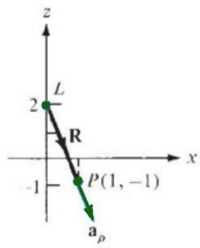


A. $\vec{E}$ for surface $= \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_z$

$\vec{E}$ for line $= \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$

$$\vec{E}_1 = \frac{\rho_{s_1}}{2\varepsilon_0}(-\hat{a}_z) = \frac{-10\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\hat{a}_x = -180\pi\,\hat{a}_x$$

$$\vec{E}_2 = \frac{\rho_{s_2}}{2\varepsilon_0}(\hat{a}_y) = \frac{15\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\hat{a}_y = 270\pi\,\hat{a}_y$$



$$\vec{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho = \frac{10\pi\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{10}}\hat{a}_\rho = \frac{180\pi}{\sqrt{10}}\hat{a}_\rho$$

$\hat{a}_\rho = \hat{a}_R \Rightarrow R = \hat{a}_x - 3\hat{a}_z$

$|R| = \sqrt{1^2+9^2} = \sqrt{10}$

$\rho = |R| = \sqrt{10}$

$$a_R = \frac{\vec{a}_R}{|\vec{a}_R|} = \frac{\hat{a}_x - 3\hat{a}_z}{\sqrt{10}}$$

$$\vec{E}_3 = \frac{180\pi}{\sqrt{10}}\cdot\frac{(\hat{a}_x - 3\hat{a}_z)}{\sqrt{10}}$$

$$= 18\pi\,(\hat{a}_x - 3\hat{a}_z)$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$= \left(-162\pi\,\hat{a}_x + 270\pi\,\hat{a}_y - 54\pi\,\hat{a}_z\right)\ V/m$$

Electric Flux Density

→ The flux due to any vector field $\vec{A}$ through any surface is given

$$\Psi = \int_s \vec{A} \cdot d\vec{s}$$

and flux due to an electric field $\vec{E}$ is

$$\Psi = \int_s \vec{E}\, d\vec{s}$$

But E is dependant of the medium in which change is placed

So we use $\vec{D}$ (Electric Flux Density / Electric Displacement)

$$\Psi = \int_s \vec{D}\, d\vec{s}$$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$= \frac{Q}{4\pi R^2}\, \hat{a}_R \quad \text{- due to point charge}$$

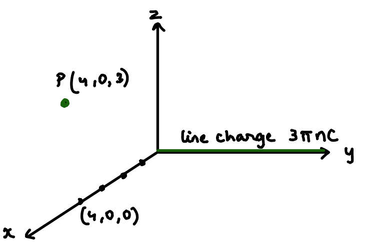
→ $\vec{E} = \dfrac{\rho_s}{2\varepsilon_0}\hat{a}_n \Rightarrow \vec{D} = \varepsilon_0\vec{E} = \dfrac{\rho_s}{2}\hat{a}_n$ - due to infinite surface charge

$\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_0}\hat{a}_\rho \Rightarrow \vec{D} = \varepsilon_0\vec{E} = \dfrac{\rho_L}{2\pi\rho}\hat{a}_\rho$ - due to infinite line charge

$\vec{E} = \displaystyle\int_v \frac{\rho_v\, dv}{4\pi\varepsilon_0 R^2}\hat{a}_R \Rightarrow \vec{D} = \varepsilon_0\vec{E} = \int_v \frac{\rho_v\, dv}{4\pi R^2}\hat{a}_R$ - due to volume charge distribution

Q. Determine D at (4, 0, 3) if there is a point charge -5π mC at (4,0,0) and line charge 3πmC along y-axis

Note : Since line-change is ∞ along y-direction, any displacement along y-direction is negligible compared to length of line, hence, the effect of displacement in y-direction can be ignored



A. Point charge :

$$D = \frac{Q}{4\pi R^2}\hat{a}_R$$

$$\hat{a}_R = \frac{\vec{R}}{|R|} = \frac{3\hat{a}_z}{3} = \hat{a}_z$$

$$D = \frac{-5\cancel{\pi}\times 10^{-3}}{4\cancel{\pi}(3)^2}\times \hat{a}_z = -0.139\,m\,\hat{a}_z\ C/m^2$$

Line charge :

$$R = 4\hat{a}_x + 3\hat{a}_z \qquad \hat{a}_\rho = \frac{R}{|\vec{R}|} = \frac{4\hat{a}_x + 3\hat{a}_z}{\sqrt{4^2+3^2}}$$

$$D = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$= \frac{3\pi\times 10^{-3}}{2\pi(5)} \cdot \frac{4\hat{a}_x + 3\hat{a}_z}{\sqrt{4^2+3^2}}$$

$$= (0.24\,\hat{a}_x + 0.18\,\hat{a}_z)\ mC/m^2$$

positive divergence   negative divergence   nil divergence

Divergence of vector

→ The net outward flow of flux per unit volume over closed incremental surface

(or)

Divergence at point P is the outward flux per unit volume as the volume shrinks at P

$$\text{div}\,\vec{A} = \nabla\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A}.d\vec{s}}{\Delta v}$$

Cartesian – $\nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Cylindrical – $\nabla.\vec{A} = \frac{1}{\rho}.\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}.\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

Spherical – $\nabla.\vec{A} = \frac{1}{r^2}.\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}.\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}.\frac{\partial A_\phi}{\partial \phi}$

→ Properties of Divergence of Vector field

1) It produces scalar field (because of dot product)
2) $\nabla(\vec{A} + \vec{B}) = \nabla\vec{A} + \nabla\vec{B}$
3) $\nabla(v\vec{A}) = v\nabla\vec{A} + A\nabla\vec{v}$

Divergence Theorem (Gauss- Otrogradsky Theorem)

→ It states that total outward flux of vector field $\vec{A}$ through the closed surface S is same as volume integral of divergence of A

→ $\oint_s \vec{A}.d\vec{s} = \oint_v \nabla\vec{A}\,dv$

Gauss' Law

→ The total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface

→ $\Psi = Q_{enclosed}$

$\Psi = \oint_s d\Psi = \oint_s \vec{D}.d\vec{s}$ & $Q = \int \rho_v\,dv$

so,

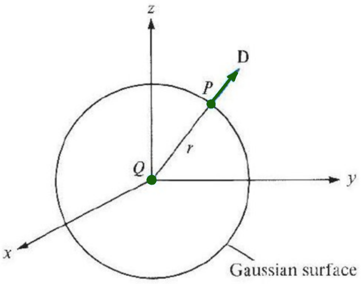
$Q = \oint_s \vec{D}.d\vec{s} = \int_v \rho_v\,dv = \Psi \longrightarrow ①$

and

$\oint_s \vec{D}.d\vec{s} = \int_v \nabla.\vec{D}\,dv \longrightarrow ②$

Then from ① & ②, $\rho_v = \nabla.\vec{D}$

Application's of Gauss' Law — Point Charge



→ Consider a charge is located at the origin, so we construct a gaussian surface (sphere) passing through point P & centred at origin

→ $\vec{D}$ is normal everywhere to the gaussian surface,

$\vec{D} = D_r \hat{a}_r$ (D_r is scalar)

→ $Q = \oint_s \vec{D} \cdot d\vec{S} = D_r \oint_s d\vec{S}$

$dS = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$

$Q = D_r \int_0^{\pi} r^2 \sin\theta \, d\theta \int_0^{2\pi} d\phi$

$= D_r r^2 (-\cos\theta)_0^{\pi} \times (2\pi - 0)$

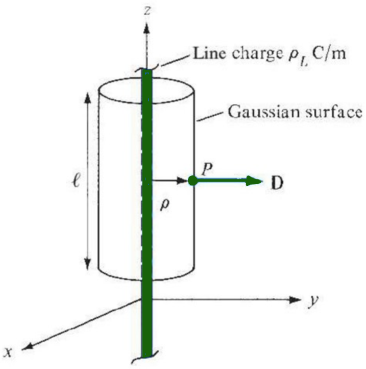
$= D_r r^2 (-(-1)-(-1))(2\pi)$

$Q = 4\pi r^2 D_r$

$D_r = \frac{Q}{4\pi r^2}$ with $\vec{D} = D_r \hat{a}_r$

$\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r \Longrightarrow \vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{a}_r$

Application's of Gauss' Law — Infinite Line Charge



→ Consider an infinite line of uniform charge ρ_L lies on z-axis, so we construct a cylindrical surface containing P to satisfy symmetry

→ $\vec{D}$ is constant on & normal to cylindrical Gaussian surface

$\vec{D} = D_\rho \hat{a}_\rho$

→ $\rho_L l = Q = \oint_s \vec{D} \cdot d\vec{S} = D_\rho \oint_s dS$

$dS = (\rho \, d\phi \, dz) \hat{a}_\rho$

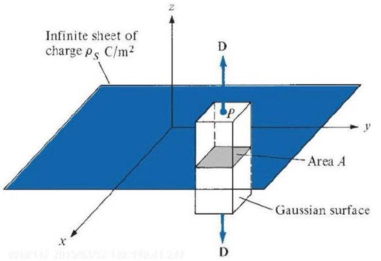
$Q = D_\rho \int_0^{l} \rho \, dz \int_0^{2\pi} d\phi = D_\rho \, 2\pi\rho l$

$D_\rho = \frac{Q}{2\pi\rho l}$

$\oint_s dS = 2\pi\rho l$

$\rho_L l = D_\rho \, 2\pi\rho l$

$\vec{D} = \frac{\rho_L}{2\pi\rho} \hat{a}_\rho$



Application's of Gauss' Law — Infinite Sheet Charge

→ Consider an infinite sheet of uniform charge ρ_s C/sq.m lying on xy-plane / z=0

→ Now take out a rectangular box which is symmetrical by sheet charge

→ $\vec{D}$ is normal to the sheet

$\vec{D} = D_z \hat{a}_z$ in z-direction

→ By Gauss' Law

$$\rho_s \int_s dS = Q = \oint_s \vec{D} \cdot \vec{dS}$$

$$\rho_s A = D_z \left\{ \int_{top} dS + \int_{bottom} dS \right\}$$

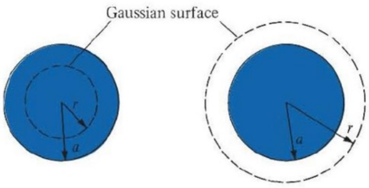
$$= D_z (A + A)$$

$$\rho_s A = 2 D_z A$$

$$D_z = \frac{\rho_s}{2}$$

$$\vec{D} = D_z \hat{a}_z = \frac{\rho_s \hat{a}_z}{2}$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_z$$



Application's of Gauss' Law — Uniformly Charged Sphere

→ Consider sphere of radius 'a' with uniform charge ρ_0 C/m³

→ To determine $\vec{D}$, 2 cases:

$r \leq a$ & $r \geq a$

→ $$Q_{enc} = \int_v \rho_0 \, dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \left(\frac{r^3}{3}\right)_0^r \left(-\cos\theta\right)_0^{\pi} (2\pi)$$

$$= \rho_0 \frac{4\pi r^3}{3}$$

$$\Psi = \oint_s \vec{D} \cdot \vec{dS} = D_r \oint_s dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r 4\pi r^2$$

$$\Psi = Q_{enc}$$

$$D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi r^3$$

$$D_r = \frac{r}{3} \rho_0$$

$$\vec{D} = D_r \hat{a}_r = \frac{r}{3} \rho_0 \hat{a}_r \quad , \quad 0 < r \leq a$$

→ $$Q_{enc} = \int_v \rho_0 \, dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \frac{4}{3} \pi a^3$$

$$\Psi = D_r 4\pi r^2$$

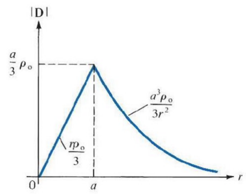
$$\Psi = Q_{enc}$$

$$D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi a^3$$

$$D_r = \frac{a^3}{3r^2} \rho_0$$

$$\vec{D} = \frac{a^3}{3r^2} \rho_0 \hat{a}_r \quad , \quad r > a$$

$$\Rightarrow \vec{D} = \begin{cases} \frac{r}{3} \rho_0 \hat{a}_r & , \ 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_0 \hat{a}_r & , \ r > a \end{cases}$$



Q. Given, $\vec{D} = z\rho\cos^2\phi\,\hat{a}_z$ C/m². Calculate the charge density at $(1, \frac{\pi}{4}, 3)$ & the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2$

A. **Method 1:**

$$\vec{D} = z\rho\cos^2\phi\,\hat{a}_z$$

$$\rho_v = \nabla \cdot D$$

$$= \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right)(D_z \hat{a}_z)$$

$$= \frac{\partial}{\partial z} D_z = \rho\cos^2\phi \Big|_{1,\frac{\pi}{4},3}$$

$$= 1 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 0.5$$

$$Q = \int \phi_v\, dv$$

$$= \int_0^1 \int_0^{2\pi} \int_{-2}^{2} \rho\cos^2\phi\; \rho\, d\rho\, d\phi\, dz \qquad \left(\frac{1+\cos 2\theta}{2} = \cos^2\theta\right)$$

$$= \left(z\right)_{-2}^{2} \left(\frac{\rho^3}{3}\right)_0^1 \left(\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right)_0^{2\pi}$$

$$= (2-(-2))\left(\frac{1}{3}\right)\left(\pi + 0 - 0 - 0\right) = \frac{4\pi}{3}$$

Method 2:

$$Q_{enc} = \Psi = \oint_S D \cdot dS = \Psi_t + \Psi_b + \Psi_s$$

Ψ_t: Top surface
Ψ_b: Bottom Surface
Ψ_s: Sides

$$\Psi_t = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} (z\rho\cos^2\phi)\,\hat{a}_z\, \rho\, d\rho\, d\phi \qquad \text{at } z=2$$

$$= z\left(\frac{\rho^3}{3}\right)_0^1 \left(\frac{\phi}{2} + \frac{\sin 2\phi}{2\times 2}\right)_0^{2\pi} \qquad \text{at } z=2$$

$$= \frac{2\pi}{3}$$

$$\Psi_b = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} (z\rho\cos^2\phi)(-\hat{a}_z)\, \rho\, d\rho\, d\phi \qquad \text{at } z=-2$$

$$= -z\left(\frac{\rho^3}{3}\right)\left(\frac{\phi}{2} + \frac{\sin 2\phi}{2\times 2}\right)_0^{2\pi} \qquad \text{at } z=-2$$

$$= \frac{-(-2)\pi}{3} = \frac{2\pi}{3}$$

$$\Psi_s = 0$$

$$\Psi = \frac{2\pi}{3} + \frac{2\pi}{3} + 0 = \frac{4\pi}{3}$$

Q. Charge with spherical symmetry $\rho = \begin{cases} \rho_0 \frac{r}{R} & , 0 \le r \le R \\ 0 & , r > R \end{cases}$

Determine $E = ?$

A. $Q_{enc} = \oint_s \vec{D}.d\vec{S} = \varepsilon_0 \oint_s \vec{E}\, d\vec{S} = \int_v \rho_v\, dv$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \rho_v r^2 \sin\theta\, dr\, d\theta\, d\phi$$

i) $r < R$,

$$Q_{enc} = \int_0^r \frac{\rho_0 r}{R} \cdot r^2 (-\cos\theta)_0^{\pi} \times (\phi)_0^{2\pi}$$

$$= \frac{4\pi\rho_0}{R} \int_0^r r^3\, dr = \frac{4\pi\rho_0}{R} \frac{r^4}{4} = \frac{\pi\rho_0 r^4}{R}$$

$$E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \hat{a}_r = \frac{\rho_0 r^2}{4\varepsilon_0 R} \hat{a}_r$$

ii) $r > R$,

$$Q_{enc} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\int_{r=0}^{R} \frac{\rho_0 r}{R} + \int_R^r 0 \right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} \frac{\rho_0 r^3}{R} \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\rho_0}{R} (-\cos\theta)_0^{\pi} \left(\frac{r^4}{4} \right)_0^R (\phi)_0^{2\pi}$$

$$= \frac{\rho_0}{R} \left(-(-1) - (-1) \right) \left(\frac{R^4}{4} \right) (2\pi)$$

$$= \frac{\rho_0}{R} (2) \times \frac{R^4}{4} \times 2\pi = \rho_0 \pi R^3$$

$$E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} \hat{a}_r = \frac{\rho_0 R^3}{4\varepsilon_0 r^2} \hat{a}_r$$

Electric Potential

→ There are 3 ways to calculate $\vec{E}$

i) Using Coulomb's Law if charge distribution is kno

ii) Using Gauss' Law if charge distribution is symmetric

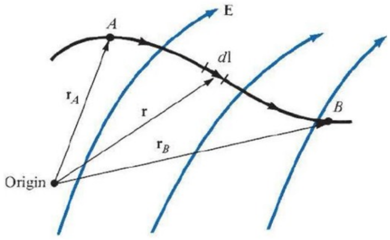
iii) Using Scalar Electric Potential V

→ From Coulomb's law, Force on $Q \Rightarrow \vec{F} = Q\vec{E}$

so work done to displace charge by $\vec{dl}$ is $dW = -\vec{F}.\vec{dl} = -Q\vec{E}.\vec{dl}$

−ve indicates work done by external agent

→ $W = -Q\int_A^B \vec{E}.\vec{dl}$



$$V_{BA} = \frac{W}{Q} = -\int_A^B \vec{E}.\vec{dl}$$

→ If −ve, there is loss in potential energy when moving Q from A to B

↳ Work done by field

→ If +ve, there is gain in potential energy when moving Q from A to B

↳ Work done by external agent

V_{BA} : independant of path

↳ J/C or V

→ $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r}\hat{a}_r$

$$V_{BA} = -\int_A^B \vec{E}.\vec{dl} = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r\, dr\, \hat{a}_r = \left[\frac{Q}{4\pi\varepsilon_0 r}\right]_A^B = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = V_B - V_A$$

V_B & V_A : Potentials at B & A respectively

V_{BA} = Potential at B wrt A

→ $V_A = 0$ at $r_A \to \infty$

So the potential at any point $r_B \to r$ is $V = \frac{Q}{4\pi\varepsilon_0 r}$

Conservative : vector whose line integral doesn't depend on the path of integration

→ $\vec{E}$ is conservative

→ The potential at any point is the potential difference between that point & a chosen point at which potential is 0

$$V = -\int_\infty^r \vec{E}.\vec{dl} \Rightarrow V(r) = \frac{Q}{4\pi\varepsilon_0|r-r'|}$$

′ indicate source point.

No ′ refers to the field point at which V is to be determined

By superposition principle,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k}{|\vec{r}-\vec{r}_k|}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(r')dl'}{|\vec{r}-\vec{r}'|} \rightarrow \text{line charge}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_L(r')ds'}{|\vec{r}-\vec{r}'|} \rightarrow \text{surface charge}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_L(r')dv'}{|\vec{r}-\vec{r}'|} \rightarrow \text{volume charge}$$

If reference has 0 potential, $V = \frac{Q}{4\pi\varepsilon_0 r}$ but if any other point, $V = \frac{Q}{4\pi\varepsilon_0 r} + C$, C must be determined

→ Potential at any point can be evaluated in 2 ways

1) If charge distribution is known, we can use previous expressions
2) If $\vec{E}$ is known, $V = -\int \vec{E}\, d\vec{l} + C$

$$V_{BA} = V_B - V_A = -\int_A^B \vec{E}.d\vec{l} = \frac{W}{Q}$$

Q. 2 point sources $-4\mu C$ & $5\mu C$ are located at $(2,-1,3)$ $(0,4,-2)$ respectively. Find the potential at $(1,0,1)$ assuming zero potential at ∞.

A. $$V(r) = \frac{Q_1}{4\pi\varepsilon_0|r-r_1|} + \frac{Q_2}{4\pi\varepsilon_0|r-r_2|} + \cancel{C_0}^{0}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{-4\mu}{\sqrt{(1-2)^2+(0-(-1))^2+(1-3)^2}} + \frac{5\mu}{\sqrt{(1-0)^2+(0-4)^2+(1-(-2))^2}}\right) = \frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right)$$

$$= 9\times10^{9} \times 10^{-6}\left(-1.633 + 0.98\right)$$

$$= 9000(-0.6524)$$

$$= -5871.774 \text{ V}$$

Q. If point charge $3\mu C$ is located at origin in addition to the 2 charges of example of previous example. Find potential at $(-1,5,2)$ assuming $V(\infty) = 0$

A. $$V(r) = \frac{Q_1}{4\pi\varepsilon_0|r-r_1|} + \frac{Q_2}{4\pi\varepsilon_0|r-r_2|} + \frac{Q_3}{4\pi\varepsilon_0|r-r_3|} + \cancel{C_0}^{0}$$

$$= \frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{-4}{\sqrt{(-1-2)^2+(5+1)^2+(2-3)^2}} + \frac{5}{\sqrt{(-1)^2+(5-4)^2+(2+2)^2}} + \frac{3}{\sqrt{1^2+5^2+2^2}}\right)$$

$$= 9\times10^{9}\times10^{-6}\left(\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}\right)$$

$$= 9000\,(1.1364) = 10228.2\text{V}$$

Q. A point charge 5nC is located at $(-3,4,0)$ while line $y=1$, $z=1$ carries uniform charge $2nC/m$

a) If $V = 0$ V at $O(0,0,0)$. Find V at $A(5,0,1)$
b) If $V = 100$V at $B(1,2,1)$. Find V at $C(-2,5,3)$
c) If $V = -5$V at $O(0,0,0)$. Find V_{BC}

A. $V = V_Q + V_L$

point charge, $V_Q = -\int \vec{E}.d\vec{l} = \int \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r . dr\,\hat{a}_r = \frac{Q}{4\pi\varepsilon_0 r} + C_1$

∞ line charge, $V_L = -\int \vec{E}.d\vec{l} = \int \frac{\rho_L}{2\pi\varepsilon_0 \rho}\hat{a}_\rho\, d\rho\,\hat{a}_\rho = \frac{-\rho_L}{2\pi\varepsilon_0}\ln\rho + C_2$

$$V = \frac{Q}{4\pi\varepsilon_0 r} - \frac{\rho_L}{2\pi\varepsilon_0}\ln\rho + C$$

a) $\rho = |(x,y,z) - (x,1,1)|$

$= \sqrt{(x-x)^2 + (y-1)^2 + (z-1)^2}$

$= \sqrt{(y-1)^2 + (z-1)^2}$

$\rho_0 = |(0,0,0) - (0,1,1)| = \sqrt{2}$

$r_0 = |(0,0,0) - (-3,4,0)| = 5$

$\rho_A = |(5,0,1) - (5,1,1)| = 1$

$r_A = |(5,0,1) - (-3,4,0)| = 9$

$$V_0 - V_A = \frac{-\rho_L}{2\pi\varepsilon_0} \ln \frac{\rho_0}{\rho_A} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r-r_1} - \frac{1}{r-r_2}\right)$$

$$0 - V_A = -2 \times 9\times10^9 \times (2\times10^{-9}) \ln \frac{\sqrt{2}}{1} + 5\times10^{-9} \times 9\times10^9 \left(\frac{1}{5} - \frac{1}{9}\right)$$

$-V_A = -36 \ln\sqrt{2} + 4$

$V_A = 36 \ln\sqrt{2} - 4$

$= 8.477\text{V}$

b) $\rho_B = |(1,2,1) - (1,1,1)| = 1$

$r_B = |(1,2,1) - (-3,4,0)| = \sqrt{21}$

$\rho_C = |(-2,5,3) - (-2,1,1)| = \sqrt{20}$

$r_C = |(-2,5,3) - (-3,4,0)| = \sqrt{11}$

$$V_B - V_C = \frac{-\rho_L}{2\pi\varepsilon_0} \ln \frac{\rho_B}{\rho_C} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_C}\right)$$

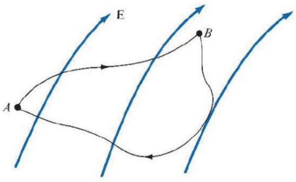
$$= -2\times10^{-9} \times 2\times9\times10^9 \ln\left(\frac{1}{\sqrt{20}}\right) + 5\times10^{-9}\times9\times10^9\left(\frac{1}{\sqrt{21}} - \frac{1}{\sqrt{11}}\right)$$

$100 - V_C = 50.175\text{ V}$

$V_C = 49.825\text{V}$

c) $V_{BC} = V_C - V_B = 49.825 - 100$ (We don't need potential reference if common reference is used)

$= -50.175\text{V}$



Relation b/w $\vec{E}$ and $\vec{V}$ - Maxwell's Equation

→ Potential difference b/w A & B is independant of path

So, $V_{BA} = -V_{AB} \Rightarrow V_{AB} + V_{BA} = 0$

$$\oint_L \vec{E}\cdot d\vec{l} = 0$$

→ This also means, no net work is done in moving a charge along a closed path in an electrostatic field

$$\oint_L \vec{E}\cdot d\vec{l} = 0 \quad \text{or} \quad \nabla \times \vec{E} = 0$$

↳ Maxwell's Second Equation

→ $V = \int \vec{E}\cdot d\vec{l}$

$dV = -\vec{E}\cdot d\vec{l}$

$= -E_x\,dx - E_y\,dy - E_z\,dz$

$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

So, $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, $E_z = -\frac{\partial V}{\partial z}$

Thus $E = -\nabla V$

Q. Given $V = \frac{10}{r^2}\sin\theta\cos\phi$

a) Find electric flux density $\vec{D}$ at $(2, \pi/2, 0)$

b) Calculate work done in moving a $-10\,\mu C$ charge from $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$

A. a) $\vec{D} = \varepsilon_0\vec{E}$

$\vec{E} = -\nabla V$

$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$

$= \frac{-20}{r^3}\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\cdot\frac{10}{r^2}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{1}{r\sin\theta}\cdot\frac{10}{r^2}\sin\theta(-\sin\phi)\,\hat{a}_\phi$

$= \frac{10}{r^3}\left(-2\sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi\right) = \frac{10}{8}\left(-2\hat{a}_r + 0 - 0\right) = -2.5\,\hat{a}_r$

$-\nabla V = 2.5\,\hat{a}_r = \vec{E}$

$D = 2.5\,\varepsilon_0\,\hat{a}_r \; C/m^2 = 22.1\,\hat{a}_r\; pC/m^2$

b) $W = -Q\int \vec{E}\cdot d\vec{l} = QV_{AB} = Q(V_B - V_A)$

$= 10^{-5}\left(\frac{10}{16}\sin 90\cos 60 - \frac{10}{1}\sin 30\cos 120\right)$

$= 10^{-5}\left(\frac{10}{32} - \left(\frac{-5}{2}\right)\right)$

$= 28.125\,\mu J$

Curl of a Vector

→ Circulation of a vector field $\vec{A}$ around closed path L is $\oint_L \vec{A}.dL$

→ The curl of A is an axial vector whose magnitude is maximum circulation of A per unit area as area tends to zero & direction normal to area

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \left\{ \lim_{\Delta s \to 0} \frac{\oint_L \vec{A}.d\vec{l}}{\Delta s} \right\}_{max} \hat{a}_n$$

→ Cartesian :

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{a}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{a}_z$$

Cylindrical :

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{a}_\rho - \left(\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right)\hat{a}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)\hat{a}_z$$

Spherical :

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{a}_r - \frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi}\right)\hat{a}_\theta + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{a}_\phi$$

→ Properties

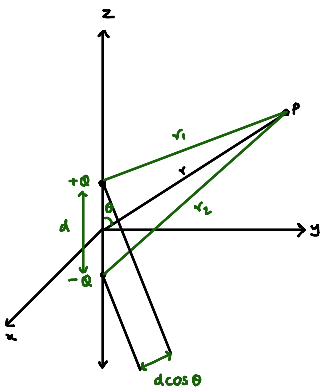
- $\nabla \times (A + B) = \nabla \times A + \nabla \times B$
- $\nabla \times (A \times B) = A(\nabla . B) - B(\nabla . A) + (B.\nabla)A - (A.\nabla)B$
- $\nabla \times (VA) = V\nabla \times A + \nabla V \times A$ (V - Scalar field)
- $\nabla . (\nabla \times A) = 0$ (Divergence of curl = 0)
- $\nabla \times (\nabla V) = 0$ (Curl of Gradient = 0)

→ Curl indicates direction along which maximum value occurs

Stoke's Theorem (Curl of a Vector)

→ Circulation of vector field $\vec{A}$ (around a closed path L) = surface integral of curl of A (over open surface S bounded by L)

$$\oint_L \vec{A}.d\vec{l} = \int_S (\vec{\nabla} \times \vec{A}).d\vec{S}$$



Electric Dipole

→ An electric dipole is formed when 2 point charges of equal magnitude but opposite signs are seperated by small distance

→ $V = \frac{Q}{4\pi\varepsilon_0 r_1} + \frac{-Q}{4\pi\varepsilon_0 r_2}$

$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$

→ If $r >> d$, $r_2 - r_1 \simeq d\cos\theta$ and $r_1 r_2 \simeq r^2$

Then $V = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{d\cos\theta}{r^2}$

Let $\vec{d} = d\hat{a}_z$, $\vec{d}.\hat{a}_r = d|\hat{a}_r|\cos\theta = d\cos\theta$

→ Dipole Moment ⇒ $\vec{p} = Q\vec{d}$

$\vec{p}.\hat{a}_r = |\vec{p}||\hat{a}_r|\cos\theta = Q|\vec{d}|\cos\theta = Qd\cos\theta$

So, $V = \frac{\vec{p}.\hat{a}_r}{4\pi\varepsilon_0 r^2}$ (Dipole moment directed from +Q to -Q)

→ If dipole centre is at r' ⇒ $V(r) = \frac{\vec{p}\,(r - r')}{4\pi\varepsilon_0 |r - r'|^3}$

Q. A Dipole moment $\vec{p} = 3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z$ nCm is located at Q(1,2,-4) in free space. Find V at P(2,3,4)

A. $V(r) = \frac{\vec{p}\,(r - r')}{4\pi\varepsilon_0 |r - r'|^3}$

$= \frac{(3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z)\times 10^{-9}\left((2-1)\hat{a}_x + (3-2)\hat{a}_y + (4+4)\hat{a}_z\right)}{4\pi\varepsilon_0\left(\sqrt{1^2 + 1^2 + 8^2}\right)^3}$

$= \frac{(3 - 5 + 80)\times 10^{-9} \times 9\times 10^{9}}{(66)^{3/2}} = 1.307$ J/c

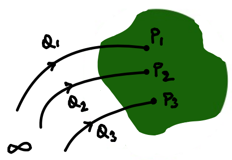
Q. 2 dipoles with dipole moments $-5\hat{a}_z$ nC/m and $9\hat{a}_z$ nC/m are located at points (0,0,-2) and (0,0,3). Find potential at origin

A. $V = \sum_{k=1}^{2} \frac{\vec{P}_k . \vec{r}_k}{4\pi\varepsilon_0 r_k^3} = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{P}_1 . \vec{r}_1}{r_1^3} + \frac{\vec{P}_2 . \vec{r}_2}{r_2^3}\right]$

$V = 9\times 10^9\left[\frac{-5\hat{a}_z\left((0,0,0)-(0,0,-2)\right)}{2^3} + \frac{9\hat{a}_z\left((0,0,0)-(0,0,3)\right)}{3^3}\right]\times 10^{-9}$

$= 9\left[\frac{-10}{8} - \frac{27}{3^3}\right] = 9\times(-2.25) = -20.25$ V

Energy Density in Electrostatic Fields



→ To find energy, we calculate work necessary to assemble them

Take the diagram, W_1, W_2, W_3 be work required to bring from ∞ to P_1, P_2, P_3

Initially, work done to bring Q_1 to P_1 is zero because initially change free, no electric field. Work done on Q_2 = $Q_2 \times V_{21}$ at P_2 due to Q_1, similar for Q_3

$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$

also, $$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13})$$

Then, $$2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$
$$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

$$\boxed{W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k} \quad (J)$$

→ For continuous charge distribution,

- line charge – $\frac{1}{2}\int_L \rho_L V \, dl$
- surface charge – $\frac{1}{2}\int_S \rho_S V \, dS$
- volume charge – $\frac{1}{2}\int_v \rho_v V \, dv$

→ We know $\rho_v = \nabla \cdot \vec D$,

$$W_E = \frac{1}{2}\int_v (\vec\nabla \cdot \vec D) V \, dv$$
$$= \frac{1}{2}\int_v (\vec\nabla \cdot V\vec D) dv - \frac{1}{2}\int_v (\vec D \cdot \vec\nabla V) dv$$
$$= \frac{1}{2}\oint_S (V\vec D) \cdot d\vec S - \frac{1}{2}\int_v (\vec D \cdot \vec\nabla V) dv$$

↳ tends to zero ($V\vec D$ varies atleast as $\frac{1}{r^3}$, dS varies as r^2)

$$= \frac{-1}{2}\int_v (\vec D \cdot \vec\nabla V) dv = \frac{1}{2}\int_v (\vec D \cdot \vec E) dv$$

($\vec E = -\nabla V$ & $\vec D = \varepsilon_0 \vec E$)

$$W_E = \frac{1}{2}\int_v \vec D \cdot \vec E \, dv = \frac{1}{2}\int_v (\vec D \cdot \vec E) dv = \frac{1}{2}\int_v \varepsilon_0 E^2 dv$$

So, $$\boxed{W_E = \int_v w_E \, dv}$$

Q. The point charges $-1\,nC$, $4nC$, $3nC$ are located at $(0,0,0)$, $(0,0,1)$ and $(1,0,0)$ respectively. Find energy of the system

A. $W = W_1 + W_2 + W_3$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$

$$= Q_2 \cdot \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{|(0,0,1)-(0,0,0)|}\right) + Q_3 \cdot \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{|(1,0,0)-(0,0,0)|} + \frac{Q_2}{|(1,0,0)-(0,0,1)|}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(Q_2Q_1 + Q_3Q_1 + \frac{Q_3Q_2}{\sqrt{2}}\right)$$

$$= 9\times10^9 \left(-4 - 3 + \frac{12}{\sqrt{2}}\right) \times 10^{-18}$$

$$= 13.37\ nJ$$

Q. If $V = (x - y + xy + 2z)\,V$, find $\vec{E}$ at $(1,2,3)$ & electrostatic energy stored in cube of side $= 2m$, centred at origin

A. $E = -\nabla V$

$$= -\frac{\partial V}{\partial x}\hat{a}_x - \frac{\partial V}{\partial y}\hat{a}_y - \frac{\partial V}{\partial z}\hat{a}_z$$

$$= -(1+y)\hat{a}_x - (-1+x)\hat{a}_y - (2)\hat{a}_z$$

$$E_{(1,2,3)} = -(1+2)\hat{a}_x - (-1+1)\hat{a}_y - 2\hat{a}_z$$

$$= -3\hat{a}_x - 2\hat{a}_z$$

$$W = \frac{\varepsilon_0}{2}\int_v |E|^2\, dV$$

$$= \frac{\varepsilon_0}{2}\int_v \left((1+y)^2 + (-1+x)^2 + 4\right) dV$$

$$= \frac{\varepsilon_0}{2}\int_v \left(1 + y^2 + 2y + 1 + x^2 - 2x + 4\right) dV$$

$$= \frac{\varepsilon_0}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x^2 + y^2 - 2x + 2y + 6)\, dx\, dy\, dz$$

$$= \frac{\varepsilon_0}{2}\int_{-1}^{1}\int_{-1}^{1} \left(\frac{x^3}{3} + y^2x - x^2 + 2xy + 6x\right)_{-1}^{1} dy\, dz$$

$$= \frac{\varepsilon_0}{2}\int_{-1}^{1}\int_{-1}^{1} \left(\frac{2}{3} + 2y^2 - 0 + 4y + 12\right) dy\, dz$$

$$= \frac{\varepsilon_0}{2}\int_{-1}^{1}\left(\frac{2y^3}{3} + 2y^2 + \frac{38y}{3}\right)_{-1}^{1} dz = \frac{\varepsilon_0}{2}\int_{-1}^{1}\left(\frac{4y}{3} + \frac{76}{3}\right) dz = \frac{\varepsilon_0}{2}\left(\frac{2y^2}{3} + \frac{76z}{3}\right)_{-1}^{1} = \frac{76\varepsilon_0}{3} \approx 0.2353\ nJ$$