UNIT-3 BAND LIMITED CHANNELS AND BASEBAND SIGNAL TRANSMISSION AND RECEPTION

Inter Symbol Interface

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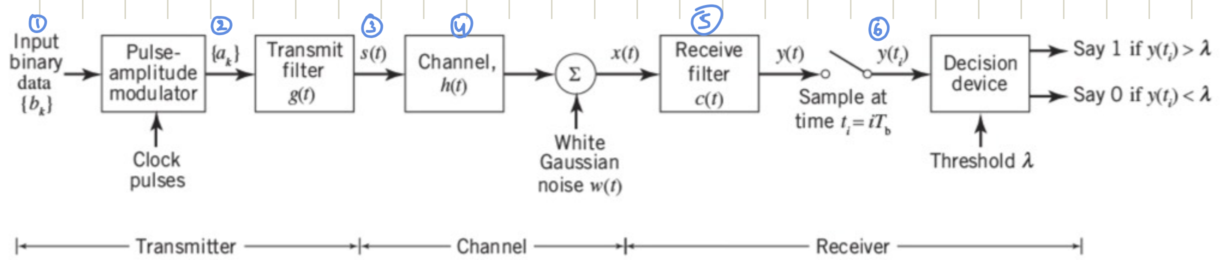


In transmitter side,

1. Let $\{b_k\}$ be a binary sequence emitted by source at time instants kT_b where $k = 0, \pm 1, \pm 2 \ldots$ and T_b is bit duration

2. Let $\{a_k\}$ be line code obtained based on $\{b_k\}$ $\left[a_k = \begin{cases} 1, & b_k = 1 \\ -1, & b_k = 0 \end{cases} \right]$

Now the sequence of short pulses is applied to a transmit filter whose impulse response is given by $g(t)$

3. and transmitted signal, $s(t) = \sum_k a_k\, g(t - kT_b)$ (or) $\{a_k\} * g(t)$

In channel side,

4. The PAM signal $s(t)$ is passed over a linear communication channel whose impulse response is $h(t)$

5. and channel output $x(t)$ is processed by receive-filter whose impulse response is $c(t)$

$\hookrightarrow x(t) = h(t) * s(t) + w(t)$

In receiver side,

6. The output of receive filter $y(t)$ is sampled synchronously with generator of clock pulses in transmitter

$$y(t) = x(t) * c(t)$$
$$= h(t) * g(t) * c(t) * \{a_k\}$$
$$y(t) = \sum_{k=-\infty}^{\infty} a_k\, p(t - kT_b)$$

where $p(t) = g(t) * h(t) * c(t)$ & $p(0) = 1$

The synchronized output of $y(t)$ at the time $t = iT_b$ is given by

$$y(t_i) = \sum_{k=-\infty}^{\infty} a_k\, p[(i-k)T_b] \quad , i = 0, \pm 1, \pm 2$$

$$= \underbrace{a_i}_{\substack{\text{Contribution} \\ \text{of } i^{th} \\ \text{transmitted bit}}} + \underbrace{\sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k\, p[(i-k)T_b]}_{\substack{\text{Residual effect of} \\ \text{all other transmitted} \\ \text{bits on the decoding} \\ \text{of the } i^{th} \text{ bit}}}$$

→ ISI (or) Inter Symbol Interface

If no ISI, $y(t_i) = a_i$

Signal Design for zero ISI

→ We idealize the design which has the criterion for distortionless transmission

→ The pulse shaping requirement is referred as signal design problem

Now for the transmission to be free of ISI,

$$a_k\, p(iT_b - kT_b) = 0 \quad \in k \neq i$$

and overall pulse-shape $p(t)$ must be,

$$p(iT_b - kT_b) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \longrightarrow \text{Nyquist Criterion for Distortionless binary baseband data transmission}$$

the $p(t)$ satisfying this condition is known as Nyquist Pulse

Ideal Nyquist Pulse for Distortionless binary baseband data transmission

→ For the sequence of samples $\{p(nT_b)\}$,

$$P_\delta(f) = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b)$$

where $R_b = \frac{1}{T_b}$ (bit rate per second)

$P_\delta(f)$: FT of infinite periodic sequence of delta functions of period T_b

So,
$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[p(mT_b)\, \delta(t - mT_b) \right] e^{-j2\pi ft}\, dt \qquad (m = i-k)$$

$$= p(0) \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt$$

$$= p(0) = 1$$

Hence, frequency-domain condition for zero ISI is satisfied, provided $\sum_{n=-\infty}^{\infty} P(f - nR_b) = T_b \longrightarrow ①$

→ The simplest way ① (zero ISI condition) to satisfy it is to specify the frequency function $p(f)$ to be in the form of rectangular function

$$P(f) = \begin{cases} \frac{1}{2W}, & -W < f < W \\ 0, & |f| > W \end{cases}$$

$$= \frac{1}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

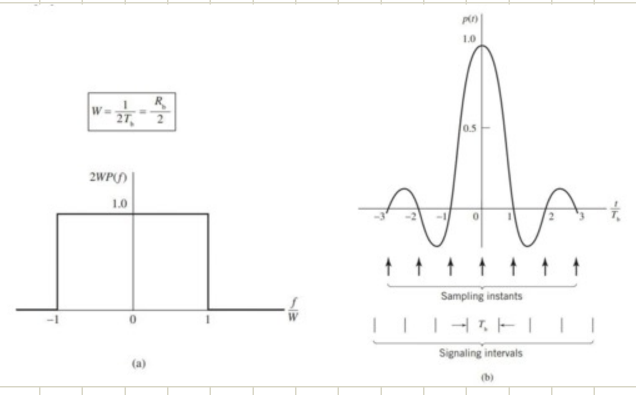
The overall baseband system bandwidth W is defined by $W = \frac{R_b}{2} = \frac{1}{2T_b}$

The signal waveform that produces zero ISI is defined by the sinc function $p(t) = \operatorname{sinc}(2Wt)$

$$= \frac{\sin(2\pi Wt)}{2\pi Wt}$$

Bit rate $R_b = 2W$ → Nyquist Rate

W : Nyquist Bandwidth



Represents normalized form of frequency function $P(f)$ plotted for +ve & −ve freqs. (Ideal Magnitude Response)

→ Includes signaling intervals & the corresponding centred sampling instants (Ideal basic pulse shape)

$p(t)$ has peak value at 0 & goes through zero at integer multiples of T_b

→ Practical difficulties for desiring zero ISI

→ It requires magnitude characteristic of $P(f)$ to be flat from $-W$ to W & zero elsewhere
This is physically unrealizable because of abrupt transitions at band edges $\pm W$

→ The pulse function $p(t)$ decreases as $\frac{1}{|t|}$ for large values of $|t|$ resulting in slow decay rate
This is also caused by discontinuity of $P(f)$ at $\pm W$
So, there is no margin of error in sampling times in the receiver

Raised Cosine Spectrum

→ To overcome the above mentioned difficulties encountered with ideal Nyquist pulse, we extend the bandwidth from $W = \frac{R_b}{2}$ to an adjustable value between W and $2W$

Basically we are trading off increased channel bandwidth for more robust signal tolerant of timing errors

→ Overall frequency response is designed as $P(f) + P(f-2W) + P(f+2W) = \frac{1}{2W}$, $-W \leq f \leq W$

→ A particular form of $P(f)$ that embodies many desirable features (flat portion & roll off portion that has sinusoidal form) is provided by Raised Cosine (RC) Spectrum

$$P(f) = \begin{cases} \frac{1}{2W} & , 0 \leq |f| < f_1 \\ \frac{1}{4W}\left\{1 + \cos\left[\frac{\pi}{2W\alpha}(|f| - f_1)\right]\right\} & , f_1 \leq |f| < 2W - f_1 \\ 0 & , |f_1| \geq 2W - f_1 \end{cases}$$

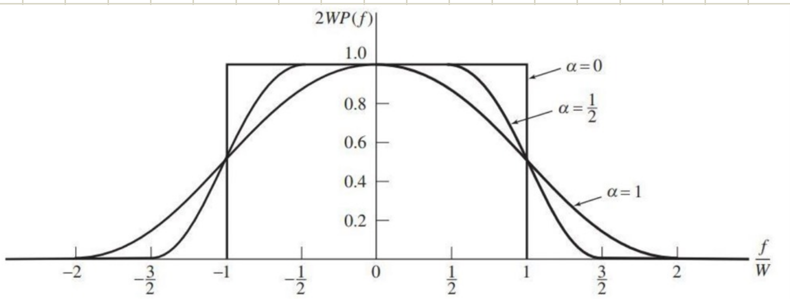
The new parameter α is known as roll off factor which indicates excess bandwidth over ideal sol^n W

$$\alpha = 1 - \frac{f_1}{W} \Rightarrow f_1 = W(1-\alpha)$$

Transmission Bandwidth
$$B_T = 2W - f_1$$
$$= 2W - W(1-\alpha)$$
$$= W + W\alpha$$
$$= W(1+\alpha)$$

Now lets observe $P(f)$ for different values of $\alpha = 0, 0.5, 1$

★



For $\alpha = 0.5, 1$ it rolls off gradually compared to ideal nyquist rate ($\alpha = 0$)

special case!

$\alpha = 1$ ($f_1 = 0$) → full-cosine roll-off

$$P(f) = \begin{cases} \frac{1}{4W}\left[1 + \cos\left(\frac{\pi f}{2W}\right)\right] & , 0 < |f| < 2W \\ 0 & , |f| > 2W \end{cases}$$

Note : $P(f)$ is normalized by multiplying it by a factor of $2W$

IFT

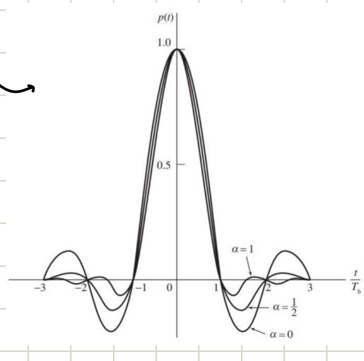
$$p(t) = \frac{\text{sinc}(4Wt)}{1 - 16W^2t^2}$$

The time response $p(t)$ is given by $p(t) = \text{sinc}(2Wt)\,\frac{\cos(2\pi\alpha Wt)}{1 - 16\alpha^2W^2t^2}$
↳ IFT of $P(f)$

→ $p(t)$ exhibits 2 interesting properties :

i) At $t = \pm T_b/2 = \pm 1/4W$, $p(t) = 0.5$
that is, pulse width measured at half amplitude = bit duration T_b

ii) There are zero crossings at $t = \pm\frac{3T_b}{2}, \pm\frac{5T_b}{2} \ldots$
in addition to regular $t = \pm T_b, \pm 2T_b \ldots$



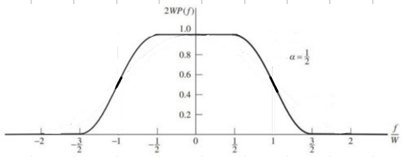
Q. A binary PAM signal is to be transmitted over a baseband channel with an absolute maximum bandwidth of 75 kHz. The bit duration is 10 μs. Find an RC spectrum that satisfies these requirements.

A. Given, $B_T = 75\,KHz$

$T_b = 10\,\mu s$

$B_T = 2W - f_1$

$W = \frac{1}{2T_b} = \frac{1}{20 \times 10^{-6}} = 50\,KHz$



$f_1 = 2 \times 50 - 75 = 25\,KHz$

$\alpha = 1 - \frac{f_1}{W} = 1 - \frac{25}{50} = 0.5$

Q. Consider a 4-ary PAM signal transmitted over a baseband channel. Suppose the channel bandwidth is 75 kHz. What is the bit rate that can be achieved for the cases where the signal is transmitted as RC pulses with roll-off factor α of 0.5 and 1 respectively?

A. $B = \frac{R_s}{2}(1+\alpha) \Rightarrow R_s = \frac{2B}{1+\alpha}$

$R_b = R_s \times \log_2 M$

$= \frac{2B}{1+\alpha} \times \log_2 M$

$M=4,\ B = 75\,KHz,\ \alpha = 0.5$

$R_B = \frac{150 \times 10^3}{1.5} \times \log_2 4 = 200\,Kbps$

$M=4,\ B = 75\,KHz,\ \alpha = 2$

$R_B = \frac{150 \times 10^3}{2} \times \log_2 4 = 150\,Kbps$

Q. An analog signal is sampled, quantized, and encoded into a binary PCM. Specifications of the PCM signal include the following:

Sampling rate, 8 kHz

Number of representation levels, 64

The PCM signal is transmitted over a baseband channel using discrete PAM. Determine the minimum bandwidth required for transmitting the PCM signal if each pulse is allowed to take on the following number of amplitude levels: 2, 4, or 8.

A. Given, $f_s = 8\,KHz$

No. of quantization levels $= 64$

So for 64 levels, $\log_2 64 = 6$ bits required

Minimum Bandwidth $B_T = \frac{1}{2T} = \frac{R_s}{2}$

First we find bit rate R_b

$R_b = f_s \cdot n$

$= (8 \times 10^3) \times 6 = 48\,Kbps$

i) $M = 2$ Amplitude levels

$R_s = \frac{48000}{\log_2 2} = 48000$ symbols/sec

$B_{min} = \frac{48000}{2} = 24\,KHz$

ii) $M = 4$ Amplitude levels

$R_s = \frac{48000}{\log_2 4} = 24000$ symbols/sec

$B_{min} = \frac{24000}{2} = 12\,KHz = 12\,Kbaud$

iii) $M = 8$ Amplitude levels

$R_s = \frac{48000}{\log_2 8} = 16000$ symbols/sec

$B_{min} = \frac{16000}{2} = 8\,KHz$

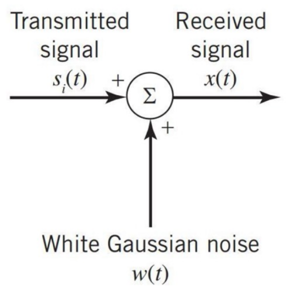
AWGN Model

→ Assume a transmitter part of a digital communication takes 1s & 0s emitted by source & encodes them into distinct signals $s_1(t)$ & $s_2(t)$ suitable for transmission over analog channel

→ The analog channel is represented by an AWGN Model and defined as

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T_b \\ i = 1, 2 \end{cases}$$

$w(t)$: channel noise



→ The receiver has a task to observe these signals over a duration of T_b and make a rough estimate of $s_i(t)$, however, due to channel noise, receiver will inevitably make errors.

The requirement is to design receiver such that it minimizes average probability of symbol error, P_e

→ The main motivation to minimize P_e is to make digital communication more reliable

→ To achieve this important design objective in a generic setting that involves M-ary alphabet whose symbols are denoted by $m_1, m_2, \ldots, m_M$ we need to understand 2 basic issues –

i) How to optimize design of receiver so as to minimize average probability of symbol error

ii) How to choose set of signals $s_1(t), s_2(t) \ldots s_M(t)$ for representing symbols $m_1, m_2, \ldots, m_M$

This is done by Geometric Representation of Signal

Geometric Representation of Signals

→ The essence of geometric representation of signals is to represent any set of M energy signals $\{s_i(t)\}$ as a linear combination of N orthonormal basis functions where $N \leq M$

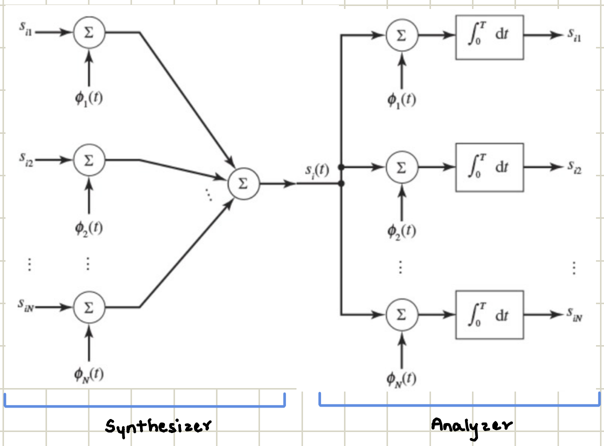
→ Given, a set of real-valued energy signals $s_1(t), s_2(t), \ldots, s_M(t)$ each of duration T seconds,

then, $$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{cases}$$

co-efficients, $$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt, \quad \begin{cases} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{cases}$$

The set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as N-dimensional signal vector denoted by s_i

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots M$$

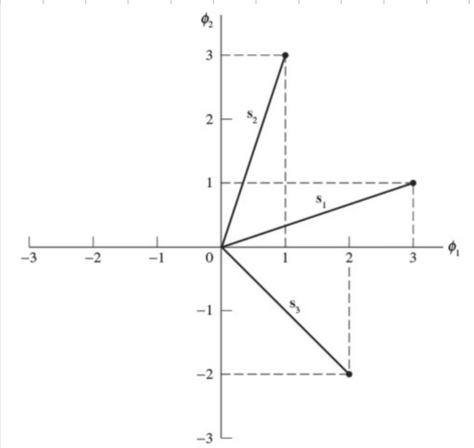


→ Consider the set of signal vectors $\{s_i \mid i = 1, 2, \ldots, M\}$ as defining a corresponding set of M points in an N-dimensional Euclidian Space

It has N mutually perpendicular axes labeled $\phi_1, \phi_2, \ldots, \phi_N$

This N-dimensional Euclidian Space is called Signal Space

→ This is a 2D signal space with 3 signals (N = 2, M = 3)



→ The length / norm of signal vector s_i is given by symbol $||s_i||$

→ The squared length of any signal vector s_i is defined to be inner product or dot product of s_i

$$||s_i||^2 = s_i^T s_i$$

$$= \sum_{j=1}^{N} s_{ij}^2 \quad , i = 1, 2, \ldots M$$

$$E_i = \int_0^T s_i^2(t)\, dt \quad , i = 1, 2, \ldots, M$$

$$E_i = \int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right]\left[\sum_{k=1}^{N} s_{ik}\phi_k(t)\right] dt$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N} s_{ij}\, s_{ik} \int_0^T \phi_j(t)\,\phi_k(t)\, dt$$

$$= \sum_{j=1}^{N} s_{ij}^2$$

$$= ||s_i^2||$$

→ Energy of signal $s_i(t)$ of duration T seconds is given by

$$E_i = \int_0^T s_i^2(t)\, dt \quad , i = 1, 2, \ldots, M$$

can be simplified as $E_i = \sum_{j=1}^{N} s_{ij}^2 = ||s_i^2||$

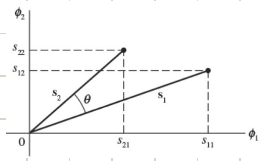
→ For a pair of signals $s_i(t)$ & $s_k(t)$ represented by signal vectors s_i & s_k, we have

$$\int_0^T s_i(t)\, s_k(t)\, dt = s_i^T s_k$$

Also, $$||s_i - s_k||^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2 = \int_0^T (s_i(t) - s_k(t))^2\, dt$$

$||s_i - s_k||$: Euclidian distance d_{ik} b/w points represented by signal vectors s_i & s_k

→ Finally, angle θ_{ik} subtended b/w 2 signal vectors s_i and s_k provides a complete geometric representation



$$\cos\theta_{ik} = \frac{s_i^T s_k}{||s_i||\, ||s_k||}$$

(if $s_i^T s_k = 0 \Rightarrow \theta_{ik} = 90^\circ \Rightarrow s_i$ & s_k are orthogonal to each other)

Q. An 8-level PAM signal is defined by

$$s_i(t) = A_i \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

where A_i = ±1, ±3, ±5, ±7. Formulate the signal constellation of $s_i\{(t)\}_{i=1}^{8}$.

A. Consider $s_1(t)$ as linearly independant ⟶ This means only 1 orthonormal basis function exists
& all others as linearly dependant

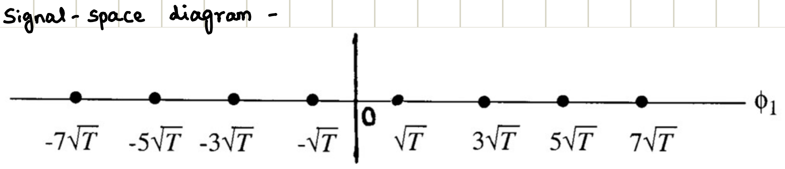
$$E_i = \int_0^T (A_i)^2\, dt$$

$$= A_i^2 T \quad , \quad A_i = \pm 1, \pm 3, \pm 5, \pm 7$$

So basis function $\phi_1(t) = \frac{s_i(t)}{\sqrt{E_i}} = \frac{s_i(t)}{A_i\sqrt{T}}$

$\Rightarrow$ $s_1(t) = \phi_1(t)\sqrt{T}$
$s_2(t) = -\phi_1(t)\sqrt{T}$
$s_3(t) = 3\phi_1(t)\sqrt{T}$
$s_4(t) = -3\phi_1(t)\sqrt{T}$
$s_5(t) = 5\phi_1(t)\sqrt{T}$
$s_6(t) = -5\phi_1(t)\sqrt{T}$
$s_7(t) = 7\phi_1(t)\sqrt{T}$
$s_8(t) = -7\phi_1(t)\sqrt{T}$

Signal-space diagram -



Gram-Schmidt Orthogonalization Procedure

→ Consider a set of M energy signals denoted by $s_1(t), s_2(t), \dots, s_M(t)$

First basis function is defined by $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$ (E_1 is energy of signal $s_1(t)$)

So, $s_1(t) = \sqrt{E_1}\,\phi_1(t)$

$= s_{11}\,\phi_1(t)\,dt$

The coefficient s_{21} can be defined as $s_{21} = \int_0^T s_2(t)\,\phi_1 dt$

Now we introduce an intermediate function $g_2(t) = s_2(t) - s_{21}\phi_1(t)$

Second basis function is defined by $\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$ (E_2 is energy of signal $s_2(t)$)

Also $\int_0^T \phi_1(t)\,\phi_2(t)\,dt = 0$ (orthonormal pair)

Hence we define a generalised intermediate function $g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\,\phi_j(t)$

and $s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt$, $j = 1, 2, \dots, i-1$

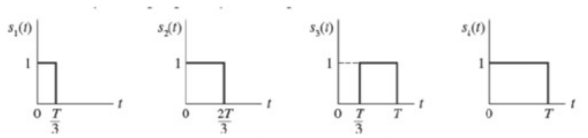
also $\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}}$

The dimension $N \leq$ No. of signals M for 2 possibilities

i) all the signals are linearly independant, Then $N = M$

ii) all the signals aren't linearly independant, Then $N < M$

Q. For the set of signals shown in the figure a) Using Gram-Schmidt procedure find orthonormal basis
b) Construct signal-space diagram



A. From the given signals $s_1(t)$, $s_2(t)$, $s_3(t)$ are linearly independant & $s_4(t)$ isn't. So 3 orthonormal basis functions

↳ $s_4(t) = s_1(t) + s_3(t)$

Energy of $s_1(t)$, $E_1 = \int_0^T s_1^2(t)\,dt = \int_0^{T/3} (1)^2\,dt = \frac{T}{3}$

First basis func^n, $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \frac{1}{\sqrt{T/3}}, & 0 \leq t \leq \frac{T}{3} \\ 0, & \text{otherwise} \end{cases}$

Projection of $s_2(t)$ on $\phi_1(t)$, $s_{21} = \int_0^T s_2(t)\,\phi_1(t)\,dt = \int_0^{2T/3} (1) \times \phi_1(t)\,dt = \int_0^{T/3} \sqrt{\frac{3}{T}}\,dt = \sqrt{\frac{3}{T}} \times \frac{T}{3} = \sqrt{\frac{T}{3}}$

Energy of $s_2(t)$, $E_2 = \int_0^T s_2^2(t)\,dt = \int_0^{2T/3} (1)^2\,dt = \frac{2T}{3}$

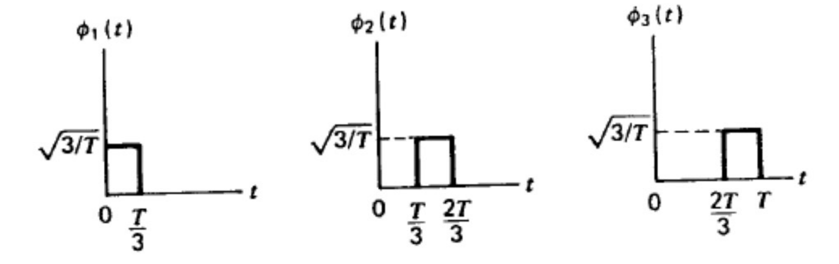
Second basis func^n, $\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} = \begin{cases} \frac{1 - \sqrt{\frac{T}{3}} \times \sqrt{\frac{3}{T}}}{\sqrt{\frac{2T}{3} - \frac{T}{3}}}, & 0 \leq t \leq \frac{T}{3} \\ \frac{1 - 0}{\sqrt{\frac{2T}{3} - \frac{T}{3}}}, & \frac{T}{3} \leq t \leq \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{T}{3} \leq t \leq \frac{2T}{3} \\ 0, & \text{otherwise} \end{cases}$

Projection of $s_3(t)$ on $\phi_1(t)$, $s_{31} = \int_0^T s_3(t)\,\phi_1(t)\,dt = \int_0^{T/3} 0 \times \sqrt{\frac{3}{T}}\,dt + \int_{T/3}^{T} 1 \times 0\,dt = 0$

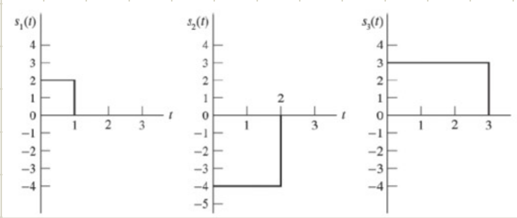
Projection of $s_3(t)$ on $\phi_2(t)$, $s_{32} = \int_0^T s_3(t)\,\phi_2(t)\,dt = \int_{T/3}^{2T/3} 1 \times \sqrt{\frac{3}{T}}\,dt = \sqrt{\frac{3}{T}} \times \left(\frac{2T-T}{3}\right) = \sqrt{\frac{T}{3}}$

Energy of $s_3(t)$, $E_3 = \int_0^T s_3^2(t)\,dt = \int_{T/3}^{T} (1)^2\,dt = \frac{2T}{3}$

Third basis func^n, $\phi_3(t) = \dfrac{s_3(t) - s_{31}\,\phi_1(t) - s_{32}\,\phi_2(t)}{\sqrt{E_3 - s_{31}^2 - s_{32}^2}} = \begin{cases} \dfrac{1 - 0 - \sqrt{\frac{T}{3}} \times \sqrt{\frac{3}{T}}}{\sqrt{\frac{2T}{3} - 0 - \frac{T}{3}}}, & \frac{T}{3} \le t \le \frac{2T}{3} \\ \dfrac{1 - 0 - 0}{\sqrt{\frac{2T}{3} - 0 - \frac{T}{3}}}, & \frac{2T}{3} \le t \le T \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \sqrt{\frac{3}{T}}, & \frac{2T}{3} \le t \le T \\ 0, & \text{otherwise} \end{cases}$



Q. Using Gram–Schmidt orthogonalization procedure, find a set of orthonormal basis functions to represent the three signals $s1(t)$, $s2(t)$, and $s3(t)$ shown in figure below. Express each of these signals in terms of the set of basis functions found



A. All 3 linearly independant. So 3 orthonormal basis functions

$E_1 = \int_0^1 (2)^2\,dt = 4$

$\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

$s_{21} = \int_0^2 s_2(t)\,\phi_1(t)\,dt = \int_0^1 -4 \times 1\,dt = -4$

$E_2 = \int_0^2 (-4)^2\,dt = 16 \times 2 = 32$

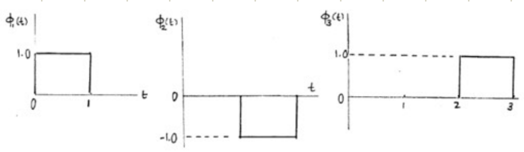
$\phi_2(t) = \dfrac{s_2(t) - s_{21}\,\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} = \begin{cases} \dfrac{-4 - (-4)1}{\sqrt{32-16}}, & 0 \le t \le 1 \\ \dfrac{-4 - 0}{\sqrt{32-16}}, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

$s_{31} = \int_0^3 s_3(t)\,\phi_1(t)\,dt = \int_0^1 3 \times 1\,dt = 3$

$s_{32} = \int_0^3 s_3(t)\,\phi_2(t)\,dt = \int_1^2 3 \times (-1)\,dt = -3$

$E_3 = \int_0^3 s_3^2(t)\,dt = \int_0^3 9\,dt = 27$

$$\phi_3(t) = \frac{s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)}{\sqrt{E_3 - s_{31}^2 - s_{32}^2}} = \begin{cases} \dfrac{3 - 3(1) - 0}{\sqrt{27-9-9}}, & 0 \le t \le 1 \\ \dfrac{3 - 0 - (-3)(-1)}{\sqrt{27-9-9}}, & 1 \le t \le 2 \\ \dfrac{3 - 0 - 0}{\sqrt{27-9-9}}, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$



$s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

Q. An orthogonal set of signals is characterized by the property that the inner product of any pair of signals in the set is zero. Figure P7.4 shows a pair of signals $s_1(t)$ and $s_2(t)$ that satisfy this definition. Construct the signal constellation for this pair of signals.

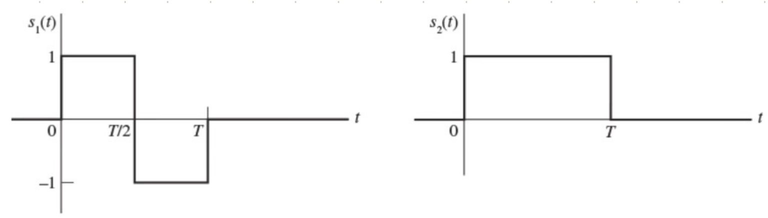


Figure P7.4

A. $s_1(t)$ & $s_2(t)$ are orthogonal to each other

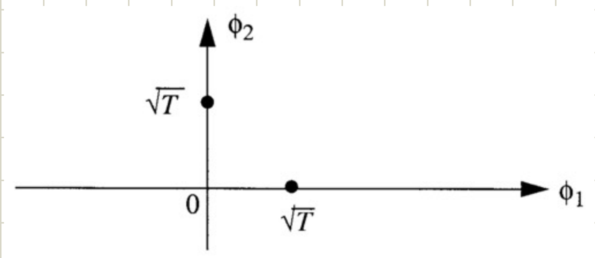
$$E_1 = \int_0^{T/2} (1)^2\,dt + \int_{T/2}^{T} (-1)^2\,dt = \frac{T}{2} + \left(T - \frac{T}{2}\right) = T$$

$$\phi_1 = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

$$E_2 = \int_0^{T} (1)^2\,dt = T$$

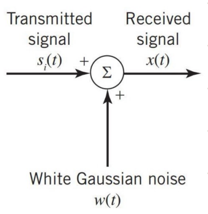
$$\phi_2 = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

Signal space diagram for $s_1(t)$ & $s_2(t)$



Conversion of the continuous AWGN Channel into Vector Channel

→ Assume the input to analyzer wasn't $s_i(t)$ but was $x(t)$ defined in accordance with AWGN channel as in the figure



$$\Longrightarrow \quad x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

where $w(t)$ is a sample function of white Gaussian Noise process $W(t)$ of zero mean & Power Spectral Density $\frac{N_0}{2}$

The output of correlator j is given by
$$x_j = \int_0^T x(t) \cdot \phi_j(t)\, dt$$
$$= s_{ij} + w_j \quad , \quad j = 1, 2, \ldots, N$$

s_{ij} is the deterministic component of x_j due to transmitted signal $s_i(t)$
$$s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt$$

w_j is the sample value of a random variable W_j due to channel noise $w(t)$
$$w_j = \int_0^T w(t)\, \phi_j(t)\, dt$$

→ The j^{th} correlator output x_j is a Gaussian random variable & $X(t)$ is a Gaussian Random Process
The mean & variance of x_j are found as follows

$$\mu_{x_j} = E[X_j]$$
$$= E[s_{ij} + w_j]$$
$$= s_{ij} + E[w_j]$$
$$= s_{ij}$$

★
$$\sigma^2_{x_j} = var[x_j]$$
$$= E[(x_j - \mu_{x_j})^2] = E[w_j^2]$$
$$= E\left[\int_0^T w(t)\, \phi_j(t)\, dt \int_0^T w(u)\, \phi_j(u)\, du\right]$$
$$= E\left[\int_0^T \int_0^T \phi_j(t)\phi_j(u)\, w(t)\, w(u)\, dt\, du\right]$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)\, E[w(t)\, w(u)]\, dt\, du$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)\, R_w(t,u)\, dt\, du$$
$$= \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t)\phi_j(u)\, \delta(t-u)\, dt\, du \qquad \left(R_w(t,u) = \frac{N_0}{2}\delta(t-u)\right)$$
$$= \frac{N_0}{2} \int_0^T \phi_j^2(t)\, dt$$
$$= \frac{N_0}{2} \quad \epsilon_j \qquad \left(\|\phi_j(t)\|^2 = 1\right)$$

Covariance derivation

$$cov[X_j X_k] = E\left[(X_j - \mu_{X_j})(X_k - \mu_{X_k})\right] = E[W_j W_k]$$
$$= E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_k(u)du\right]$$
$$= E\left[\int_0^T \int_0^T \phi_j(t)\phi_k(u)W(t)W(u)dtdu\right]$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_j(u)E[W(t)W(u)]dtdu$$
$$= \int_0^T \int_0^T \phi_j(t)\phi_k(u)R_W(t,u)dtdu$$
$$= \frac{N_0}{2}\int_0^T \int_0^T \phi_j(t)\phi_k(u)\delta(t-u)dtdu \quad \text{as } R_W(t,u) = \frac{N_0}{2}\delta(t-u)$$
$$= \frac{N_0}{2}\int_0^T \phi_j(t)\phi_k(t)dt = 0 \quad \text{as } \langle\phi_j(t), \phi_k(t)\rangle = 0$$

Yes, derivations are there 😔🖐

→ Define the vector $\vec{X}$ of N statistically independant Gaussian R.V

$X = [X_1, \dots, X_N]^T$

Each Gaussian R.V has mean S_{ij} & variance $\frac{N_0}{2}$

Then the conditional probability density function of X given the emitted symbol m_i / transmitted signal $s_i(t)$ as

$$f_X(x|m_i) = \prod_{j=1}^{N} f_{X_j}(x_j|m_i) \qquad (i = 1,2,\dots,M)$$

Channel satisfying this is said to be memoryless

$\vec{x}$ & x_j are sample values of X and X_j respectively

observation vector

element of observation vector

→ The conditional probability density function for observing an element x_j given symbol m_i emitted by source is

$$f_X(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{1}{N_0}(x_j - s_{ij})^2}, \quad \begin{cases} j = 1,2,\dots,N \\ i = 1,2,\dots,M \end{cases}$$

and for vector $\vec{x}$

$$f_X(\vec{x}|m_i) = (\pi N_0)^{-N/2} e^{-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2}, \quad i = 1,2,\dots,M$$

→ Likelihood function

→ The conditional probability density functions $f_X(\vec{x}|m_i)$ characterize the AWGN channel
Their derivation leads to a functional dependance on the observation vector $\vec{x}$ given the transmitted message symbol m_i, but at receiver we have completely opposite situation
Because we are given $\vec{x}$ & expected to estimate message symbol m_i responsible for generating $\vec{x}$

To emphasize on this, we introduce the idea of likelihood function

$$L(m_i) = f_X(\vec{x}|m_i), \quad i = 1,2,\dots,M$$

In practise, we use log-likelihood function

$$l(m_i) = \log L(m_i)$$

$$= \frac{-1}{N}\sum_{j=1}^{N}(x_j - s_{ij})^2, \quad i = 1,2,\dots,N$$

The constant term $\frac{-N}{2}\ln\left(\frac{\pi}{N_0}\right)$ is neglected

Optimum Receivers using Coherent Detection

→ **Maximum Likelihood Decoding**

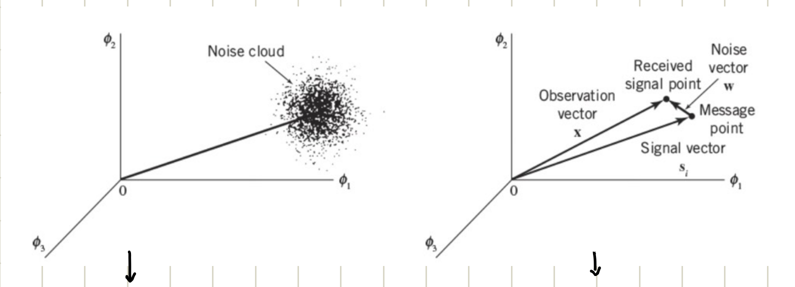
→ Given the observation vector x, perform mapping of $\vec{x}$ to an estimate of the transmitted symbol m_i in a way that would minimize the probability of error in decision making process

$$P_e(m_i \mid \vec{x}) = p(m_i \text{ not sent} \mid \vec{x}) = 1 - p(m_i \text{ sent} \mid \vec{x})$$

→ The optimum decision rule to minimize probability of error is

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad P(m_i \text{ sent} \mid \vec{x}) \geq P(m_k \text{ sent} \mid \vec{x}) \quad \forall\, k \neq i$$

→ This decision rule is known as Maximum A Posteriori (MAP) rule



↓ Noise Perturbation ↓ Location of received signal point

→ The MAP rule can be expressed in terms of likelihood functions & prior possibilities as

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad \frac{P_k f_x(\vec{x} \mid m_i)}{f_x(\vec{x})} \text{ is maximum for } k = i$$

(Denominator is independant of transmitted symbol
The prior possibilities p_k could be equiprobable)

→ The optimal decision rule can also be stated in log-likelihood function

set $\hat{m} = m_i$ if $l(m_i)$ is maximum for $k = i$

→ This decision rule is known as Maximum Likelihood (ML) rule

→ The ML decoder computes log-likelihood functions for all M symbols & decides in the favour of maximum
Unlike MAP, ML decoder assumes equiprobable message symbols

→ ML rule is also given as

$$l(m_i) = \log L(m_i) = \frac{-1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2$$

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad \sum_{j=1}^{N} (x_j - s_{ij})^2 \text{ is minimum for } k = i$$

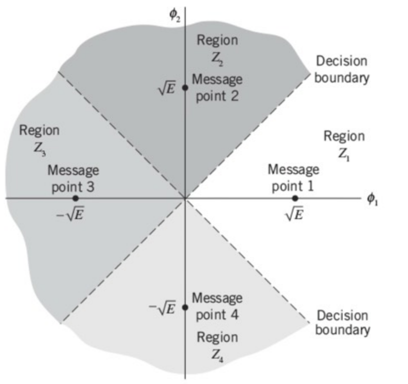
other versions include, $$\sum_{j=1}^{N} (x_j - s_{ij})^2 = \| x - s_k \|^2$$

$$= \sum_{j=1}^{N} x_j^2 + E_k - 2 \sum_{j=1}^{N} x_j s_{kj}$$

ML rule further modified as

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad \| x - s_k \| \text{ is minimum for } k = i$$

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad \| x - s_k \| \text{ is minimum for } k = i$$



↳ Graphical illustration of ML decoding $(N = 2, M = 4)$

There are M decision regions (z_1, z_2, z_3, z_4)

ML rule can be stated as $\vec{x}$ lies in z_i if $L(m_k)$ is max for $k = i$ $(k = 1, 2, \ldots M)$

If tie occurs ($\vec{x}$ on boundary), it is decided by coin flip

→ Different ways of constructing a detector (convert noisy signal $x(t)$ into N dimensional signal vector $\vec{x}$)

i) Correlation Receiver

ii) Matched Filter Receiver

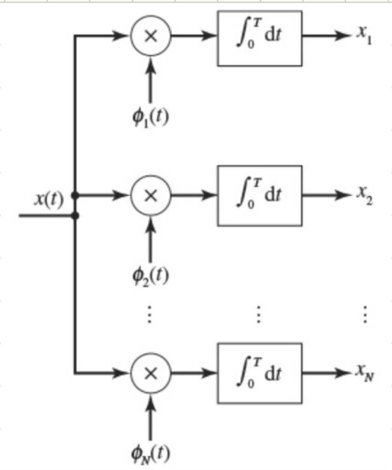
Correlation Receiver

→ The optimum receiver used for recovery of equiprobable M-ary symbols transmitted over an AWGN channel has 2 sub systems :

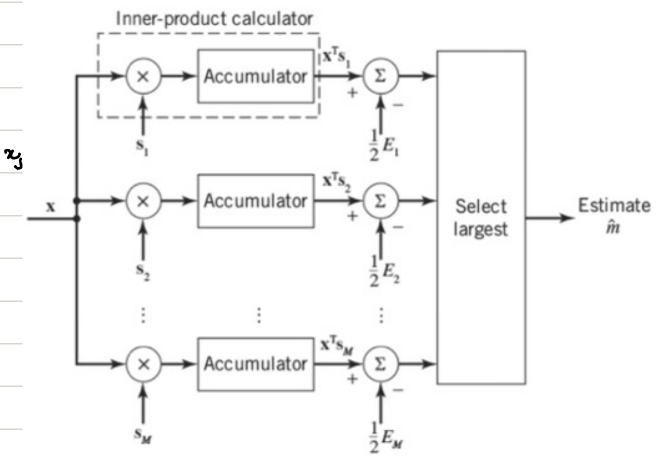
i) Detector / Demodulator - A bank of N correlators (product integrators) which produce an observation vector $\vec{x}$ corresponding to received signal $x(t)$ for $0 \le t \le T$

ii) Signal Transmission Decoder - An ML decoder which produces an estimate of the message based on observation vector $\vec{x}$ with minimum probability of error

$$\text{Set } \hat{m} = m_i \quad \text{if} \quad \sum_{j=1}^{N} s_{kj} - \frac{1}{2}E_k \quad \text{is maximum for } k = i$$



Detector / Demodulator



Signal Transmission Decoder

Matched Filter Receiver

→ The previous detector uses correlators (multipliers) which are difficult to design

→ Assume that received signal $x(t)$ is passed through a filter having impulse response $h_j(t)$ such that,

$$h_j(t) = \phi_j(T-t) \quad \in j = 1, 2, \dots, N$$

Then output $y_j(T) = \int_0^T x(\tau)\,\phi_j(\tau)\,d\tau$

$$\phi_j(t) \begin{cases} \neq 0, & 0 \le t \le T \\ = 0, & \text{otherwise} \end{cases}$$

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\, h_j(t-\tau)\, d\tau$$

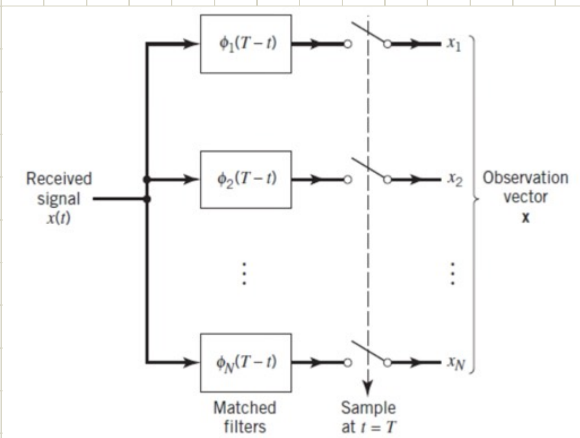
$$= \int_{-\infty}^{\infty} x(\tau)\, \phi_j(T-t+\tau)\, d\tau$$

(Note : $y_j(T) = x_j$ i.e, j^{th} correlator o/p)

Output of j^{th} correlator is $x_j = \int_0^T x(t)\,\phi_j(t)\,dt$

Condition imposed on desired impulse response of filter is given as

$h_j(t) = \phi_j(T-t)$ for $0 \le t \le T$ & $j = 1, 2, \dots, M$



So a time-invariant filter defined in this way is called matched filter, Correspondingly, optimum receiver using matched filters in place of correlators is called matched filter receiver

Matched Filter Properties

→ Optimization of impulse response of a filter is to maximize peak SNR at o/p of filter

→ Using FT, filter o/p $g_o(t)$ can be expressed in terms of $G(f)\,H(f)$

$$g_o(t) = \int_{-\infty}^{\infty} H(f)\,G(f)\, e^{j2\pi f t}\, df$$

At sample time $t = T$, signal power is given by

$$|g_o(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)\,G(f)\, e^{j2\pi f t}\, df \right|^2$$

i) Power Spectral Density, $S_{NN}(f) = \frac{N_0}{2} |H(f)|^2$

ii) Avg. o/p Power, $E[n^2(t)] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df$

iii) Peak SNR, $\eta = \dfrac{\left| \int_{-\infty}^{\infty} H(f)G(f)\, e^{j2\pi f t}\, df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df}$

iv) Schwartz's Inequality, $\left| \int_{-\infty}^{\infty} H(f)G(f) e^{j2\pi f t}\, df \right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\, df \int_{-\infty}^{\infty} |G(f)|^2\, df$

v) Peak SNR, $\eta \le \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2\, df$

vi) Optimum value of $H(f)$, $H_{opt}(f) = k\, G^*(f)\, e^{-j2\pi f T}$

(k : scaling factor
$G^*(f)$: Complex Conjugate of FT of $g(t)$)

Applying IFT, $h_{opt}(t) = k \int_{-\infty}^{\infty} G^*(f)\, e^{-j2\pi f(T-t)}\, df$

For real signal $G^*(f) = G(-f)$, $h_{opt}(t) = k \int_{-\infty}^{\infty} G(-f)\, e^{-j2\pi f(T-t)}\, df = k\,g(T-t)$

↳ This shows h_{opt} is delayed version of input signal $g(t)$

↳ $h_{opt}(t)$ is matched to i/p signal

a linear time-invariant filter defined in this way is called matched filter

→ The peak pulse SNR of a matched filter depends only on the ratio of signal energy to the power spectral density of white noise at the filter i/p

$$\begin{aligned} G_o(f) &= H_{opt}(f)\, G(f) \\ &= k\, G^*(f)\, e^{-j2\pi f T} \\ &= k\,|G(f)|^2 e^{-j2\pi f T} \end{aligned} \qquad \begin{aligned} g_o(T) &= k \int_{-\infty}^{\infty} G_o(f)\, e^{j2\pi f T}\, df \\ &= k \int_{-\infty}^{\infty} |G(f)|^2\, df \\ &= kE \end{aligned}$$

→ Average Output Noise Power, $E[n^2(t)] = \frac{k^2 N_0}{2} \int_{-\infty}^{\infty} |G(f)|^2\, df$

$$= \frac{k^2 N_0 E}{2}$$

→ Peak pulse SNR has the maximum value, $\eta_{max} = \dfrac{(kE)^2}{k^2 N_0 E/2} = \dfrac{2E}{N_0}$

Q. One of four equally likely symbols is transmitted over an AWGN channel. The signal space representation of the symbols is $s_1 = [1,0]$, $s_2 = [-1,0]$, $s_3 = [0,1]$ and $s_4 = [0,-1]$. The output of the bank of correlators is $x = [0.4, 0.5]$. Find the decision of the MAP decoder.

A. Set $\hat{M} = M_i$ if $\|x - s_k\|$ is min for some $k = i$ where $k = 1, 2, \ldots M$

MAP Scale

$$\left.\pi_k = \frac{1}{M} \text{ for all } k \right\} \longrightarrow \text{ML rule}$$

$s_1 = [1,0]$ $s_2 = [-1,0]$ $s_3 = [0,1]$ $s_4 = [0,-1]$

$x = [0.4, 0.5]$

$$\left.\begin{aligned}
\|x - s_1\|^2 &= (0.4-1)^2 + (0.5-0)^2 = 0.61 \\
\|x - s_2\|^2 &= (0.4+1)^2 + (0.5-0)^2 = 2.21 \\
\|x - s_3\|^2 &= (0.4-0)^2 + (0.5-1)^2 = 0.41 \\
\|x - s_4\|^2 &= (0.4-0)^2 + (0.5+1)^2 = 2.41
\end{aligned}\right] \longrightarrow \text{From this, output of ML decoder is message point } s_3$$

Q. Suppose the signal space is represented by $s_1 = [1,0,0]$, $s_2 = [0,1,0]$, $s_3 = [0,0,1]$ and $s_4 = [1,1,1]$. Find the output of the ML decoder given the noisy received signal vector $x = [0.4, 0.5, 0.6]$.

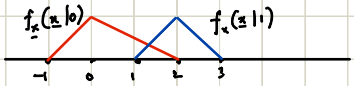
A.

$$\begin{aligned}
\|x - s_1\|^2 &= (0.4-1)^2 + (0.5-0)^2 + (0.6-0)^2 = 0.97 \\
\|x - s\|^2 &= (0.4-0)^2 + (0.5-1)^2 + (0.6-0)^2 = 0.77 \\
\|x - s\|^2 &= (0.4-0)^2 + (0.5-0)^2 + (0.6-1)^2 = 0.57 \longrightarrow \|x - s_3\| \text{ is the} \\
\|x - s\|^2 &= (0.4-1)^2 + (0.5-1)^2 + (0.6-1)^2 = 0.77
\end{aligned}$$

ex: For the 2 likelihood functions below,

a) ML Rule

b) MAP Rule when $p(0) = 1/4$ & $P(1) = 3/4$



a) Area under $f_x(x|0) = 1 = \frac{1}{2} \times (2-(-1)) \times h_1 \Rightarrow h_1 = \frac{2}{3}$

Line segment connecting $(0, \frac{2}{3})$ & $(2,0)$ is $y = m_1 x + c_1$

$$M_1 = \frac{\frac{2}{3} - 0}{0 - 2} = \frac{-1}{3}$$

$$0 = \frac{-1}{3} \cdot 2 + c_1 \Rightarrow c_1 = \frac{2}{3}$$

Area under $f_x(x|1) = 1 = \frac{1}{2} \times (3-1) \times h_2 \Rightarrow h_2 = 1$. Line segment connecting $(1,0)$ and $(2,1)$ is given by $y = m_2 x + c_2$ $\therefore f_x(x|1) = x - 1$

set $\hat{M} = 1$ if $x - 1 \geq \frac{-1}{3}x + \frac{2}{3}$

Set $\hat{M} = 0$ otherwise

set $\hat{M} = 1$ if $x \geq 1.25$

set $\hat{M} = 0$ otherwise

ML Rule : set $\hat{m} = 1$ if $f_x(x|m_i = 1) \geq f_x(x|m_i = 0)$

set $\hat{m} = 0$ if $f_x(x|m_i = 0) \geq f_x(x|m_i = 1)$

Probability of error

- To evaluate performance of a receiver in the presence of AWGN the observation space Z is partitioned into a set of regions, $\{Z_i\}$ i= 1 to M, *b*ased on the maximum-likelihood decision rule.
- Suppose that symbol *mi* (or, equivalently, signal vector **s***i*) is transmitted and an observation vector **x** is received then, an error occurs whenever
  - The received signal point represented by **x** does not fall inside region Z_i associated with the message point $\mathbf{s}_i$.
  - Averaging over all possible transmitted symbols (assumed to be equiprobable), the *average probability of symbol error* is calculated

- The *average probability of symbol error* is given by:

$$P = \sum_{i=1}^{M} \pi_i P(\mathbf{x}\ does\ not\ lie\ in\ Z_i|m_i\ sent)$$

$$= \frac{1}{M}\sum_{i=1}^{M} P(\mathbf{x}\ does\ not\ lie\ in\ Z_i|m_i\ sent)\,, \pi_i = \frac{1}{M}$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M} P(\mathbf{x}\ lies\ in\ Z_i|m_i\ sent)$$

In terms of Likelihood function, given that the message symbol m_i is sent:

$$P_e = 1 - \frac{1}{M}\sum_{i=1}^{M}\int f_X\,(\mathbf{x}|m_i)d\mathbf{x}$$

6.11 A rectangular pulse existing between $t = 0$ and $t = T$ and having an amplitude A is to be received by a matched filter.

a. Draw the impulse response of the matched filter labeling all important points and the axes.
b. Find the signal shape at the filter output. Label all important points and axes.

6.12 For the input signal *s*(*t*), as shown in Figure P6.12,

a. Sketch the impulse response *h*(*t*) of the matched filter.
b. Sketch the output *y*(*t*) of the matched filter.
c. If the input to the matched filter consists of the signal *s*(*t*) plus additive white noise of two-sided spectral density of 10^{-6} W/Hz, find the maximum output SNR of the matched filter.

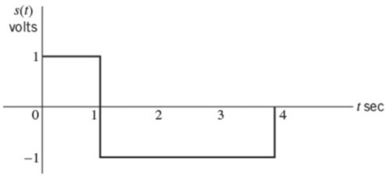


Figure P6.12

6.13 For the signal *s*(*t*), as shown in Figure P6.13,

a. Find the impulse response *h*(*t*) of a filter matched to *s*(*t*).
b. Sketch the matched filter output *y*(*t*).
c. Determine the peak value of the output *y*(*t*).

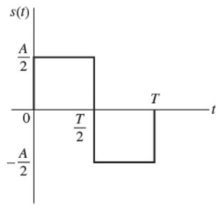


Figure P6.13

6.14 If the signal *s*(*t*) at the input of the matched filter is

$$s(t) = -6u(t) + 12u\left(t - \frac{T}{4}\right) - 12u\left(t - \frac{3T}{4}\right) + 6u(t - T)$$

where $u(\cdot)$ is a unit-step function. Find the impulse response *h*(*t*) of the matched filter.

6.15 A signal *s*(*t*) as shown in Figure P6.15, is applied to the input of the matched filter. Find the maximum SNR at the output *y*(*t*) of the filter.

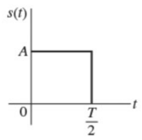
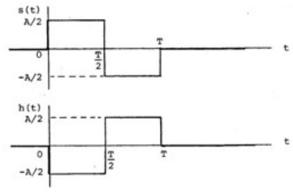


Figure P6.15

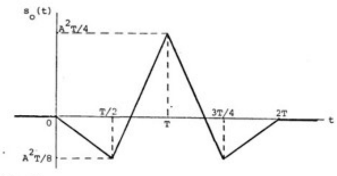
(a) The impulse response of the matched filter is

$h(t) = s(T-t)$

The s(t) and h(t) are shown below:



(b) The corresponding output of the matched filter is obtained by convolving h(t) with s(t). The result is shown below:



(c) The peak value of the filter output is equal to $A^2T/4$, occuring at t=T.

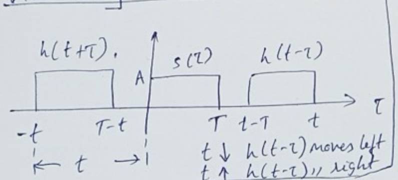
6.11 Find $h(t)$: impulse response of matched filter given input $s(t) = \begin{cases} A & \text{if } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$. Also find the output of the matched filter, denoted by $y(t)$.

a) $s(t) \longrightarrow$ [matched filter $h(t)$] $\longrightarrow y(t) = s(t) * h(t)$

As $h(t) = s(T-t) = \begin{cases} A & \text{if } 0 \le t \le T \\ 0 & \text{otherwise.} \end{cases}$

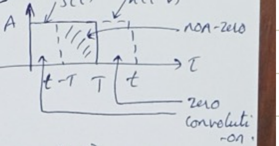
We can write $y(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau$. w.r.t. τ.

<u>Visualizing $s(\tau)$ and $h(t-\tau)$.</u>



To find $y(t)$, find cases where $s(\tau)$ and $h(t-\tau)$ overlap for some t.

<u>Case 1:</u> When $t-T \le \tau \le T$ we have

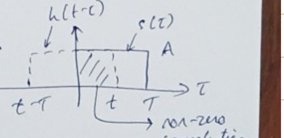


$y(t) = \int_{t-T}^{T} A \cdot A \cdot d\tau = A^2(2T-t)$

from the figure we see that $t-T \le \tau$ and $0 \le \tau \le T$

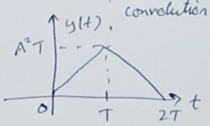
$\Rightarrow t \ge T$ and $t \le 2T$.

<u>Case 2</u>: When $0 \le \tau \le t$ we get



$y(t) = \int_0^t A \cdot A \cdot d\tau = A^2 t$. From the figure we see that $0 \le t \le T$.

$\therefore\ y(t) = \begin{cases} A^2 t & \text{if } 0 \le t \le T \\ A^2(2T-t) & \text{if } T \le t \le 2T \end{cases}$



UNIT-4 PASSBAND SIGNAL TRANSMISSION, RECEPTION AND ANALYSIS

Passband Modulation

→ In unit 3, we worked on baseband signals (pulses around DC (0 Hz))
But real world channels usually operate around high carrier frequencies (MHz or GHz)

→ So we take a digital sequence, shape it as baseband symbols & Modulate it onto a carrier with frequency f_c, which makes it a passband signal

→ Passband lets us to:
i) Fit into allocated Radio Frequency Bands
ii) Use smaller antennas (antenna size ≃ $\lambda/2$)
iii) Frequency-division multiplexing

→ So the unit's goal is to take a digital bit and a carrier, then we must build & analyze bandpass waveforms optimally.

Hilbert Transform

→ Every real signal $x(t)$ will have symmetrical spectra in +ve & -ve frequencies : $X(-f) = X^*(f)$
But for mathematical convenience, it is useful to have complex signal containing only +ve frequencies
To construct this, we need a tool which gives a version of $x(t)$ that differs by 90° phase shift at all freq.

→ Consider a real signal $g(t)$ which is made of many sinusoids of different frequencies

→ The Hilbert Transform provides the quadrature version of a real signal. (quadrature filter)

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau}\, d\tau = H\{g(t)\} \quad \left(\text{Convolution of } g(t) \text{ with } \frac{1}{\pi t}\right)$$

→ It shifts every spectral component by -90° for positive frequencies but magnitude remains same
$+90^\circ$ for negative frequencies

ex: If $g(t) = \cos(2\pi f_0 t)$ then $\hat{g}(t) = \sin(2\pi f_0 t)$ or $-\sin(2\pi f_0 t)$ based on conventions

→ Fourier Transform of $\hat{g}(t)$ is $\hat{G}(f) = -j\,\mathrm{sgn}(f)\,G(f) = \begin{cases} -jG(f), & f>0 \\ +jG(f), & f<0 \\ 0, & f=0 \end{cases}$ $\left(\mathrm{sgn}(f) = \begin{cases} 1, & f>0 \\ -1, & f<0 \\ 0, & f=0 \end{cases}\right)$

[Hilbert Transform = Multiply spectrum by $-j\,\mathrm{sgn}(f)$]

→ **Properties of Hilbert Transform**

Unlike FT, Hilbert Transform exclusively operates in time domain

i) Magnitude Preservation
→ $|\hat{G}(f)| = |G(f)|$
Hilbert Transform doesn't change energy or power of the signal ★

ii) Double Hilbert = Negative Original
→ $H\{H\{g(t)\}\} = -g(t)$

$(-j\,\mathrm{sgn}(f))^2 = (-j)^2\,\mathrm{sgn}^2(f) = -1$

iii) Orthogonality
→ For finite energy $g(t)$, $\int_{-\infty}^{\infty} g(t)\,\hat{g}(t)\,dt = 0$

So, $g(t)$ and $\hat{g}(t)$

iv) Analytic / Complex Signal construction

→ We can construct

$$g_+(t) = g(t) + j\hat{g}(t)$$

called the analytic signal which has only positive-frequency content & it is central to complex envelope representation

v) Hilbert Pairs

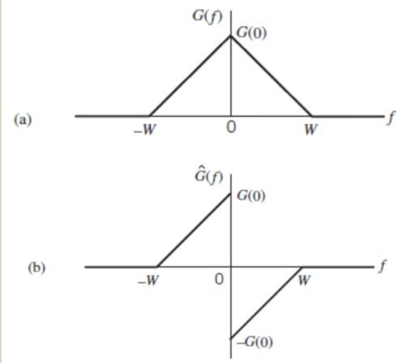
→ Cosine ⟷ Sine pairs like :

$H\{\cos(2\pi f_0 t)\} = \sin(2\pi f_0 t)$

$H\{\sin(2\pi f_0 t)\} = -\cos(2\pi f_0 t)$

Time function	Hilbert transform
1. $m(t)\cos(2\pi f_c t)$	$m(t)\sin(2\pi f_c t)$
2. $m(t)\sin(2\pi f_c t)$	$-m(t)\cos(2\pi f_c t)$
3. $\cos(2\pi f_c t)$	$\sin(2\pi f_c t)$
4. $\sin(2\pi f_c t)$	$-\cos(2\pi f_c t)$
5. $\frac{\sin t}{t}$	$\frac{1-\cos t}{t}$
6. $\mathrm{rect}(t)$	$-\frac{1}{\pi}\ln\left\|\frac{t-1/2}{t+1/2}\right\|$
7. $\delta(t)$	$\frac{1}{\pi t}$
8. $\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$
9. $\frac{1}{t}$	$-\pi\delta(t)$

ex :



→ Spectrum of the signal $G(f)$

→ Spectrum of the Hilbert Transform $\hat{G}(f)$

Pre-envelope / Analytic Signal

→ From a real signal $g(t)$, we can build a complex signal called analytic signal / pre-envelope and it can be written as $g_+(t) = g(t) + j\hat{g}(t)$

Now apply fourier transform, $G_+(f) = G(f) + \mathrm{sgn}(f)\,G(f)$

$$= \begin{cases} 2G(f) & , f>0 \\ G(f) & , f=0 \\ 0 & , f<0 \end{cases}$$

So we can observe that $G_+(f)$ has only positive frequencies (negative frequency part is suppressed)

Thus we can say it is analytic.

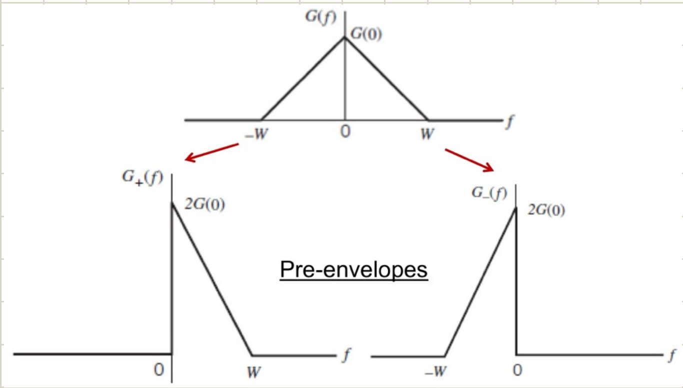
We can also write as $g_+(t) = A(t)e^{j\phi(t)}$

where, $A(t)$: Instantaneous envelope

$\phi(t)$: Instantaneous phase

So, analytical signal convenienty expresses amplitude + phase evolution of a real bandpass waveform

ex :



Similarly, pre-envelope for $g_-(t)$ from $g(t)$ can be obtained eliminating +ve frequencies

$g_-(t) = g(t) - j\hat{g}(t)$ $\quad (g_-(t) = g_+^*(t))$

$$G_-(f) = \begin{cases} 0 & , f>0 \\ G(0) & , f=0 \\ 2G(f) & , f<0 \end{cases}$$

The pre-envelope graphs are given in the figure to the left

→ Instantaneous Envelope and Phase

→ Analytic signal allows writing : $x_+(t) = A(t)\,e^{j\phi(t)}$

where,

instantaneous envelope $A(t) = |x_+(t)| = \sqrt{x^2(t) + \hat{x}^2(t)}$

instantaneous phase $\phi(t) = \arg(x_+(t)) = \tan^{-1}\left(\frac{\hat{x}(t)}{x(t)}\right)$

instantaneous frequency $f_i(t) = \frac{1}{2\pi} \cdot \frac{d\phi(t)}{dt}$

→ Energy of Pre-envelope

→ $\int |x_+(t)|^2\,dt = 2\int |x(t)|^2\,dt$

Pre-envelope contains +ve frequencies

They are doubled in magnitude, giving 4x amplitude → 2x energy

ex: Let $x(t) = A\cos(2\pi f_0 t)$

Then $\hat{x}(t) = A\sin(2\pi f_0 t)$

Hence $x_+(t) = A\cos(2\pi f_0 t) + jA\sin(2\pi f_0 t)$

This simplies to $x_+(t) = A\,e^{j2\pi f_0 t}$

Complex Envelopes

→ Usually a passband signal is a real-valued signal whose frequency content is concentrated around a non-zero carrier frequency f_c

But for mathematical ease, we want to remove the carrier $e^{j2\pi f_c t}$, convert passband to baseband signal, and represent low frequency components which are easy to manipulate

→ This is done using Complex Envelope

→ For a real passband signal $s(t)$, it's complex envelope $\tilde{s}(t) = s_+(t)\,e^{-j2\pi f_c t}$

$s_+(t) = s(t) + j\hat{s}(t)$

Thus $\tilde{s}(t) = I(t) + jQ(t)$

where $\tilde{s}(t)$ is complex, lowpass (lowpass BW ≃ baseband BW)

it is also contains modulation info & easy to represent in vector space

→ We can recover bandpass signal from complex envelope,

$$s(t) = \mathrm{Re}\left\{\tilde{s}(t)\,e^{j2\pi f_c t}\right\}$$

Derivation,

$$\tilde{s}(t) = s_I(t) + j s_Q(t)$$

$$\begin{aligned} s(t) &= \mathrm{Re}\left\{\tilde{s}(t)\,e^{j2\pi f_c t}\right\} \\ &= \mathrm{Re}\left\{(s_I(t) + j s_Q(t))\,e^{j2\pi f_c t}\right\} \\ &= \mathrm{Re}\left\{(s_I(t) + j s_Q(t))(\cos 2\pi f_c t + j\sin 2\pi f_c t)\right\} \\ &= \mathrm{Re}\left\{s_I(t)\cos 2\pi f_c t - s_Q \sin 2\pi f_c t + j\,(s_Q(t)\cos 2\pi f_c t + s_I(t)\sin 2\pi f_c t)\right\} \end{aligned}$$

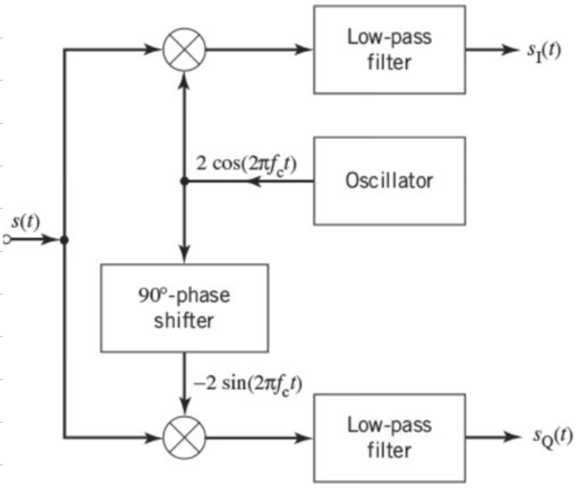
$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$ ⟵ Canonical representation of bandpass signal

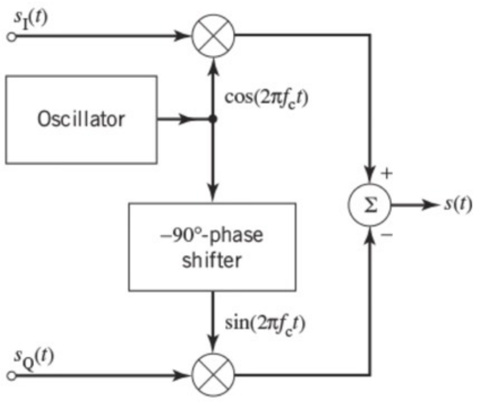
$= s_I(t)\,c(t) - s_Q(t)\,\hat{c}(t)$ $\quad (c(t) = \cos 2\pi f_c t)$

Gives final Radio Frequency signal

$s(t) = A(t)\cos(2\pi f_c t + \phi(t))$ ⟵ Polar Form ✗ NOT IN SYLLABUS

58	Sections 2.13-2.14, pages 45-48	Bandpass signal analysis: Complex envelopes and canonical representation of bandpass signals (no polar form)	1	79





Deriving in-phase & quadrature components of band-pass signal g(t) (Analyzer)

Reconstructing band pass signal from its in-phase & quadrature components (Synthesizer)

Q) Verify that the $\mathcal{H}\{sinc(t)\} = sinc(t/2)\sin(\pi t/2)$

Answer

Let $g(t) = sinc(t)$ and $\hat{g}(t) = \mathcal{H}\{sinc(t)\} = \frac{1}{\pi t} * sinc(t)$

The Fourier transfom of $\hat{g}(t)$ is given by

$$\hat{G}(f) = -j\, sgn(f)G(f) = -jsgn(f)rect(f) = \begin{cases} j & -\frac{1}{2} < f < 0 \\ -j & 0 < f < \frac{1}{2} \end{cases}$$

(sinc $\xrightleftharpoons{FT}$ rectangle)

Therefore, taking the inverse Fourier transform we get

$$\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)e^{j2\pi ft}df = \int_{-1/2}^{0} je^{j2\pi ft}df + \int_{0}^{1/2} -je^{j2\pi ft}df$$

$$\Rightarrow \hat{g}(t) = \frac{1}{\pi t}[1-\cos\pi t] = \frac{1}{\pi t}2\sin^2(\pi t/2) = \frac{\sin\left(\frac{\pi t}{2}\right)}{\pi t/2}\sin\frac{\pi t}{2} = sinc\left(\frac{t}{2}\right)\sin\frac{\pi t}{2}$$

Find the pre-envelope, complex envelope and canonical representations of the bandpass signal $g(t) = \cos 2\pi f_c t$

Answer

$\hat{g}(t) = \sin 2\pi f_c t$

Pre $-$ envelope: $g_+(t) = g(t) + j\hat{g}(t) = \cos 2\pi f_c t + j\sin 2\pi f_c t$

Alternatively, $g_+(t) = e^{j2\pi f_c t}$ this implies the complex envelope is

$\tilde{g}(t) = 1 + j0 = 1 \Rightarrow g_I(t) = 1$ and $g_Q(t) = 0$; $a(t) = 1$ and $\phi(t) = 0$

Canonical representations of $g(t)$ are given below

$g(t) = g_I(t)\cos 2\pi f_c t - g_Q(t)\sin 2\pi f_c t = \cos 2\pi f_c t$

$g(t) = a(t)\cos(2\pi f_c t + \phi(t)) = \cos 2\pi f_c t$

Binary Phase Shift Keying (BPSK)

→ It is the simplest & most fundamental passband digital modulation scheme

→ Each bit is mapped to a phase shift of the carrier
Phase changes b/w 0° & 180° depending on the bit

→ Mathematically,

Bit "1" ⟶ Carrier with 0° Phase

Bit "0" ⟶ Carrier with 180° Phase

This makes the signals antipodal (Exact negatives of each other)

Antipodal waveforms maximize distance in signal space, giving best performance among binary modulations

→ Mathematical Model of BPSK

→ Let the bit duration be T_b, Then 2 possible waveforms are,

Bit 1 : $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$, $0 \le t \le T_b$

Bit 0 : $s_2(t) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$

Where, E_b : Bit Energy

f_c : Carrier Frequency

→ These 2 signals only differ by phase shift of π, hence, "phase shift Keying"

→ Signal space methods (orthonormal basis representation) allow us to describe signals using geometric vectors
Which gives the foundation for :

i) Optimal receivers (Maximum Likelihood Detection)

ii) Constellation Diagrams

iii) Performance Comparisions

iv) Vector - Channel Models

→ Signal Space Representation

Step i → Define an Orthonormal Basis function

We choose, $\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$, $0 \le t \le T_b$

The function is normalized $\int_0^{T_b} \phi_1^2(t)\, dt = 1$ & Orthogonal to any function outside the dimension

Since BPSK signals differ only by sign, Single basis function is sufficient
⟶ Usually it is one-dimensional modulation scheme

Step ii → Represent the Waveforms using the Basis

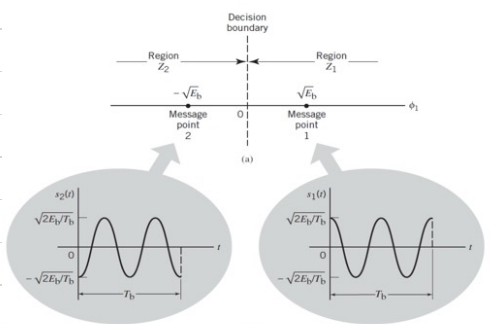
bit 1 : $s_1(t) = +\sqrt{E_b}\, \phi_1(t)$

bit 0 : $s_2(t) = -\sqrt{E_b}\, \phi_1(t)$

So signal vectors are $\vec{s_1} = [+\sqrt{E_b}]$ & $\vec{s_2} = [-\sqrt{E_b}]$

Step iii → Signal Constellation for BPSK

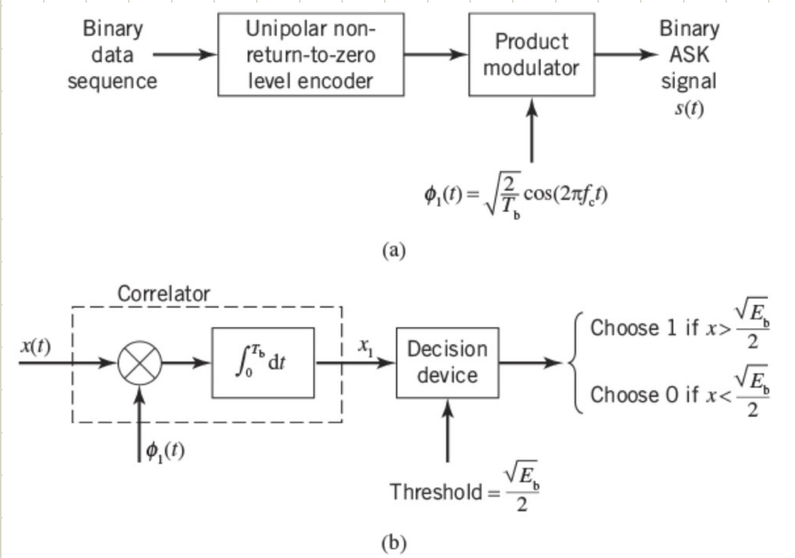
Constellation Diagram :



These points are located symmetrically about origin

distance b/w points : $d = |\sqrt{E_b} - (-\sqrt{E_b})|$
$= 2\sqrt{E_b}$

→ Block diagram for binary ASK transmitter and Coherent binary ASK receiver



Transmitter consists of Unipolar NRZ & Product Modulator

Encoder which represents 1 & 0 of incoming signals by amplitude levels $+\sqrt{E_b}$ & 0

Multiplies o/p of Unipolar NRZ encoder by basis function $\phi_1(t)$

→ BPSK Error Probability

→ In DC, receiver must decide which symbol was transmitted based on noisy observation of received signal and for BPSK, sends one of the 2 antipodal signals

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

$$s_2(t) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

→ In absence of noise, receiver can determine transmitted symbol perfectly because $s_1(t)$ and $s_2(t)$ are orthogonally placed in signal space & matched filter / correlator outputs would be exactly $\pm\sqrt{E_b}$

BUT for practical signals, AWGN corrupts signal,

$$x(t) = s_i(t) + n(t)$$

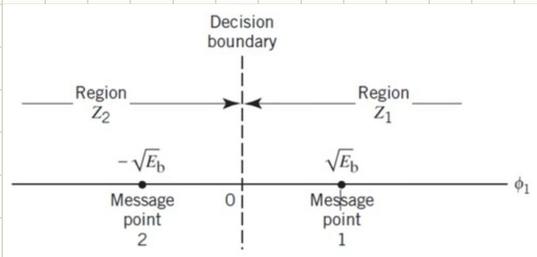
and $$y = \int_0^{T_b} x(t)\phi(t)\,dt$$

$$= \underbrace{\int_0^{T_b} s_i(t)\phi(t)\,dt}_{\text{signal part}} + \underbrace{\int_0^{T_b} n(t)\phi(t)\,dt}_{\text{noise part}}$$

This integral gives $n' \sim N(0, N_0/2)$

This gives $+\sqrt{E_b}$ if bit 1 was transmitted
$-\sqrt{E_b}$ if bit 0 was transmitted

So, if bit 1 was sent, $y = \sqrt{E_b} + n'$
if bit 0 was sent, $y = -\sqrt{E_b} + n'$



Noise can push the bit 1 & 0 points closer to zero & sometimes across zero

→ Basically errors occur when x_1 falls in Z_2 but $s_1(t)$ was transmitted &
when x_1 falls in Z_1 but $s_2(t)$ was transmitted

So, conditional PDF of RV X given symbol 0 was transmitted is given by

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}}\exp\left(\frac{-1}{N_0}(x_1 - s_{21})^2\right)$$

$$= \frac{1}{\sqrt{\pi N_0}}\exp\left(\frac{-1}{N_0}(x_1 + \sqrt{E_b})^2\right)$$

→ Probability of error is given by

P_{10} = P (receiver decoded x_1 $s_1(t) | s_2(t)$ was sent)

= P (receiver decoded x_1 symbol 1 | symbol 0 was sent)

The AWGN causes the received signal point x_1 to fall in region Z_1 (decoded as symbol 0)

$$P_{10} = \int_0^{\infty} f_{x_1}(x_1 | 0)\, dx_1$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left(\frac{-1}{N_0} (x_1 + \sqrt{E_b})^2 \right) dx_1$$

Let $z = \sqrt{\frac{2}{N_0}} (x_1 + E_b)$ & change integration variable from x_1 to z

then $$P_{10} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E_b}{N_0}}}^{\infty} \exp\left(\frac{-z^2}{2} \right) dz$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}} \right)$$

(Using Q-function, we rewrite conditional probability that receiver decoded in favour of symbol 1 as $P_{10} = Q\left(\sqrt{2E_b/N_0}\right)$)

The standard normal distribution ($\mu = 0$ & $\sigma^2 = 1$) is expressed as

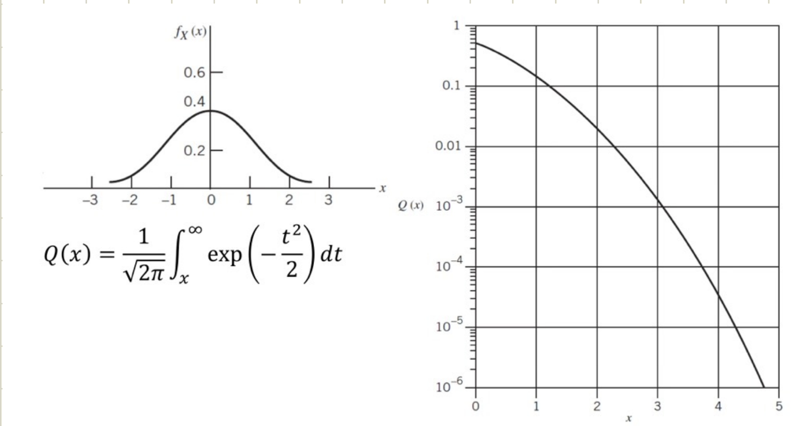
$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} \right)$$

The distribution function $F_X(x) = P(-\infty \le x)$ is given as

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-t^2}{2} \right) dt$$

The Q function is given as $Q(x) = 1 - F(x)$

↳ Gives the area under $f_X(x)$ in the range $[x, \infty]$



→ P_{10} : Conditional Probability of receiver decoding in favour of symbol 1 given symbol 0 was transmitted

P_{01} : Conditional Probability of receiver decoding in favour of symbol 0 given symbol 1 was transmitted

$P_{10} = P_{01}$

Therefore, average probability of error, $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}} \right)$

↳ Bit Error Rate of coherent BPSK

→ BPSK Power Spectra

→ Based on canonical representation of bandpass signals, BPSK signal has only in-phase component & baseband signal is given by $\pm g(t)$

$g(t)$ is the pulse shaping function is given by $g(t) = \sqrt{\frac{2E_b}{T_b}}$ where $t \in [0, T_b]$

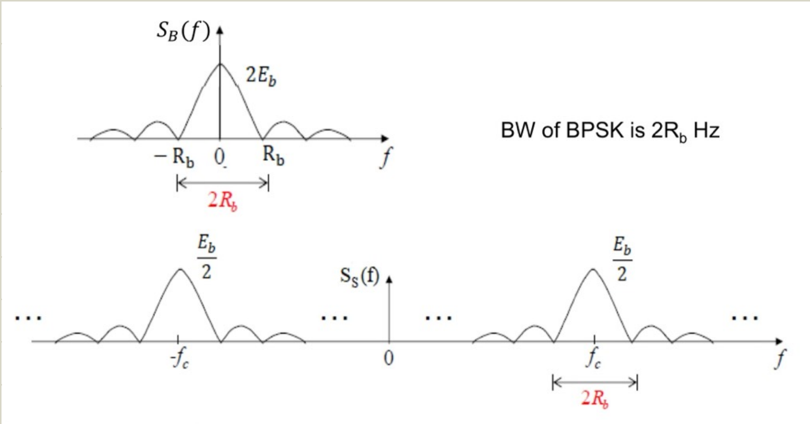
→ As baseband component resembles polar NRZ signal, we can express baseband power spectral density as

$$S_B(f) = a^2 T_b \operatorname{sinc}^2(f T_b)$$
$$= 2E_b \operatorname{sinc}^2(f T_b) \qquad \left(a = \sqrt{\frac{2E_b}{T_b}}\right)$$

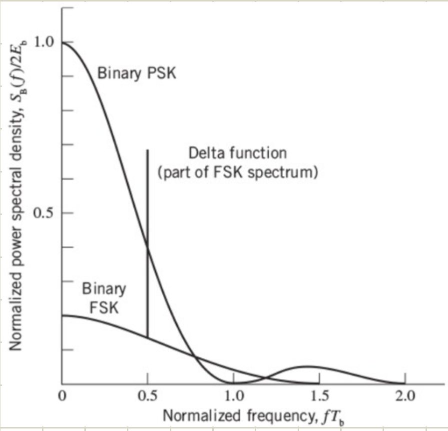
Baseband power spectral density is symmetric about vertical axis & zero crossings at integer multiples of $\pm \frac{1}{T_b}$

→ Bandpass power spectrum is given by frequency shift property of FT,

$$S_S(f) = \frac{E_b}{2}\left[\operatorname{sinc}^2((f-f_c)T_b) + \operatorname{sinc}^2((f+f_c)T_b)\right]$$



Baseband & Passband Power Spectra



Power Spectra of binary PSK & FSK signals

M-Ary QPSK

→ BPSK sends only 1 bit per symbol,

But if we wanted to send more bits w/o increasing Bandwidth, we need a better solution.

→ An alternative is to increase no. of distinct phases assigned to symbols

→ M-PSK uses M different phases,

$\log_2 M$ bits per symbol

ex: $M = 2 \Rightarrow$ BPSK (1 bit/symbol)

$M = 4 \Rightarrow$ QPSK (2 bit/symbol)

$M = 8 \Rightarrow$ 8-PSK (3 bit/symbol)

$M = 16 \Rightarrow$ 16-PSK (4 bit/symbol)

→ Spectral efficiency = $\log_2 M$ bits/Hz

→ Trade off:

i) Higher M → Higher Bandwidth Efficiency

ii) Constellation points get closer → BER increases

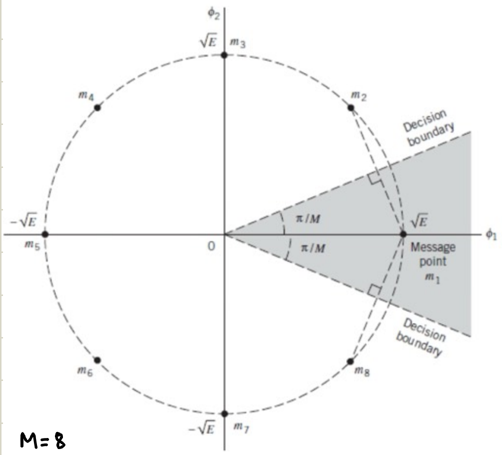
→ From this, during each signaling interview of duration T, one of the M possible transmitted waveforms are:

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t + \frac{2\pi(i-1)}{M}\right), \quad i = 1, 2, \ldots M$$

(Condition: $f_c = \frac{n_c}{T}$)

→ So symbols lie uniformly spaced around a circle

ex:



In General,

Radius of the constellation is $\sqrt{E}$

Distance b/w adjacent points is $2\sqrt{E}\sin\left(\frac{\pi}{M}\right)$

$$P_e \simeq 2Q\left(\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

M=8

→ signal vectors are $s_i = \left[\sqrt{E}\cos\left(\frac{\pi}{4}(i-1)\right), \sqrt{E}\sin\left(\frac{\pi}{4}(i-1)\right)\right]^T$

$s_1 = [\sqrt{E}, 0]^T$ $\qquad$ $s_5 = [-\sqrt{E}, 0]^T$

$s_2 = [\sqrt{E}/2, \sqrt{E}/2]^T$ $\qquad$ $s_6 = [-\sqrt{E}/2, -\sqrt{E}/2]^T$

$s_3 = [0, \sqrt{E}]^T$ $\qquad$ $s_7 = [0, -\sqrt{E}]^T$

$s_4 = [-\sqrt{E}/2, \sqrt{E}/2]^T$ $\qquad$ $s_8 = [\sqrt{E}/2, -\sqrt{E}/2]^T$

→ Power Spectra

→ Baseband power spectra can be generated NRZ pulses with different power levels as below,

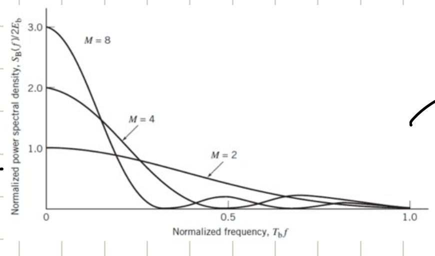
$$S_B(f) = 2E\,\text{sinc}^2(fT)$$
$$= 2E_b \log_2 M\ \text{sinc}^2(f T_b \log_2 M)$$

→ The zero crossings for the baseband power spectrum occur at integer multiples of $t = \pm\frac{1}{T}$

→ Bandwidth of M-ary PSK signal is $B = \frac{2}{T} = \frac{2R_b}{\log_2 M}$

Bandwidth efficiency is $\rho = \frac{R_b}{B} = \frac{\log_2 M}{2}$

M	2	4	8	16	32	64
ρ (bit/(s/Hz))	0.5	1	1.5	2	2.5	3



→ Power Spectra for M = 2, 4, 8

Q. **Computation of Bits Per Symbol**

An *M*-ary PSK modulation scheme is used to transmit independent binary digits over a band-pass channel with bandwidth 100 kHz. The bit rate is 200 kbps and the system characteristic is a raised-cosine spectrum with 100% excess bandwidth. Bandwidth for *M*-ary PSK system with raised-cosine characteristic is

$$B_T = \frac{1}{T}(1+\alpha)$$

where T is symbol duration and α is roll-off factor. It is given that B_T = 100 kHz and α = 1, so T = 20 μs. As the bit duration $T_b = \frac{1}{200} \times 10^3 \text{s} = 5\,\mu\text{s}$,

$$\text{Bit duration} = \frac{\text{Symbol duration}}{\log_2 M} \Rightarrow \log_2 M = \frac{20\ \mu\text{s}}{5\ \mu\text{s}} = 4$$

where $\log_2 M$ is the number of bits per symbol. After simplification, it gives M = 16.

M-Ary Quadrature Amplitude Modulation

→ In M-Ary PSK, as M increases, points get closer in angle causing poorer performance

→ Instead of varying only phase, we can vary both phase & amplitude

We can use the I-Q plane to place more points while maintaining distance

→ QAM increases no. of distinguishable symbols w/o reducing distance as severely as M-Ary PSK

→ High Spectral Efficiency with better BER than M-ary PSK

→ The core idea of QAM is that it uses 2 carriers

i) In-phase carrier : $\cos(2\pi f_c t)$

ii) Quadrature carrier : $\sin(2\pi f_c t)$

Symbol is formed as $s(t) = I\cos(2\pi f_c t) - Q\sin(2\pi f_c t)$

I : In-phase Amplitude

Q : Quadrature Amplitude

} → Represent a point in 2D Plane

→ So for a square M-QAM : $L = \sqrt{M}$

ex : 4-QAM ⟶ 2×2 Grid

16-QAM ⟶ 4×4 Grid

64-QAM ⟶ 8×8 Grid

Each axis has L equally spaced levels : $I, Q \in \left\{ \pm(2k+1)d \mid k = 0,1,2,\dots,\frac{L}{2}-1 \right\}$

→ M-Ary QAM = M-Ary PSK w/o constant envelope constraint for In-Phase & Quadrature-Phase components

→ The modulated signal is given by

$$s_i(t) = \sqrt{\frac{2E_0}{T}}\, a_i \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_i \sin(2\pi f_c t) \quad , \quad i = 0,1,\dots,M-1 \ \ \& \ \ 0 \le t \le T$$

→ Mathematical Representation

→ Using the basis functions :

$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t)$$

→ A QAM signal is expressed as $s_k(t) = \sqrt{\frac{2E_0}{T}}\, a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}}\, b_k \sin(2\pi f_c t)$ $\begin{cases} 0 \le t \le T \\ k = 0, \pm1, \pm2, \dots \end{cases}$

where $d_{min} = 2\sqrt{E_0}$ & $i = 0,1,\dots M-1$

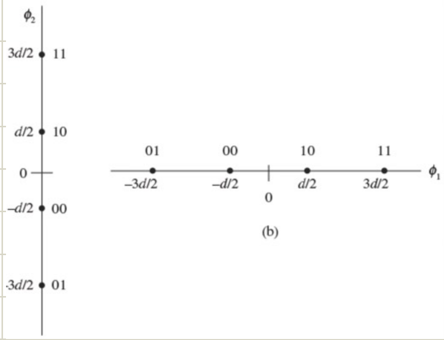
→ Signal space diagram :

$$\left\{ a_i,\ b_i \right\} = \begin{bmatrix} (-L+1,\ L-1) & (-L+3,\ L-1) & \cdots & (L-1,\ L-1) \\ (-L+1,\ L-3) & (-L+3,\ L-3) & \cdots & (L-1,\ L-3) \\ & & & \\ (-L+1,\ -L+1) & (-L+3,\ -L+1) & \cdots & (L-1,\ -L+1) \end{bmatrix} \quad (L = \sqrt{M})$$

QAM has 2 special cases :

i) When $b_i = 0 \ \forall i$, we get M-Ary ASK (Amplitude Shift Keying)

ii) When $E_0 = E$ & $\sqrt{E_{a_i}^2 + E_{b_i}^2} = \sqrt{E}$, we get M-Ary PSK

Q. 

Consider 2 orthogonal 4-ary PAM signals as below. Obtain 16-QAM.

A. Step 1: Use codewords from the part $\phi_1(t)$ & $\phi_2(t)$ to get first quadrant constellation

$$\underset{\text{Top to Bottom}}{\begin{bmatrix}11\\10\end{bmatrix}}\underset{\text{Left to Right}}{\begin{bmatrix}10 & 11\end{bmatrix}} \rightarrow \underset{Q1}{\begin{bmatrix}1110 & 1111\\1010 & 1011\end{bmatrix}}$$

Step 2: 2nd Quadrant

$$\underset{\text{Top to Bottom}}{\begin{bmatrix}11\\10\end{bmatrix}}\underset{\text{Left to Right}}{\begin{bmatrix}01 & 00\end{bmatrix}} \rightarrow \underset{Q2}{\begin{bmatrix}1101 & 1100\\1001 & 1000\end{bmatrix}}$$

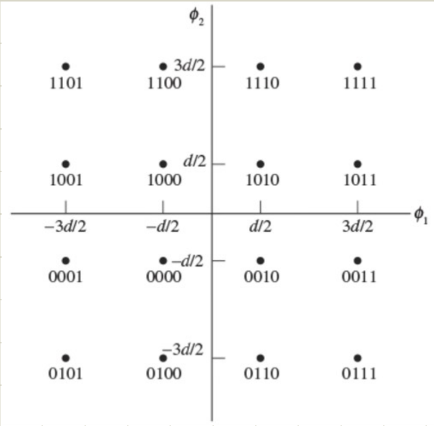
Step 3: 3rd Quadrant

$$\underset{\text{Top to Bottom}}{\begin{bmatrix}00\\01\end{bmatrix}}\underset{\text{Left to Right}}{\begin{bmatrix}10 & 11\end{bmatrix}} \rightarrow \underset{Q3}{\begin{bmatrix}0010 & 0011\\0110 & 0111\end{bmatrix}}$$

Step 4: 4th Quadrant

$$\underset{\text{Top to Bottom}}{\begin{bmatrix}00\\01\end{bmatrix}}\underset{\text{Left to Right}}{\begin{bmatrix}01 & 00\end{bmatrix}} \rightarrow \underset{Q4}{\begin{bmatrix}0001 & 0000\\0101 & 0100\end{bmatrix}}$$

→ Step 5



→ Average Probability of error

→ The probability of correct detection for M-ary QAM is

$$P_c = (1 - P_e')^2$$

P_e' : Probability of symbol error for the L-ary PAM

We know $L = \sqrt{M}$,

$$P_e' = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{2E_0}{N_0}}\right)$$

→ The probability of symbol error for M-ary QAM

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - (1 - P_e')^2 \\ &= \cancel{1} - (\cancel{1} + P_e'^2 - 2P_e') \\ &= 2P_e' + P_e'^2 \\ &\simeq 2P_e' \quad (P_e'^2 \text{ is too small}) \end{aligned}$$

→ Average Probability of error is

$$P_e \simeq 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{2E_0}{N_0}}\right)$$

→ As transmitted energy in M-ary QAM varies wrt symbol, the average energy E_{av} is used to express the average probability of error

$$E_{avg} = 2\left[\frac{2E_0}{L}\sum_{i=0}^{L/2}(2i-1)^2\right] = \frac{2(L^2-1)E_0}{3} = \frac{2(M-1)E_0}{3}$$

Q. **Computation of Roll-off Factor in an *M*-ary QAM**

An ideal band-pass channel 500 Hz to 2000 Hz is deployed for communication. A modem is designed to transmit bits at the rate of 4800 bits/s using 16-QAM. The Nyquist bandwidth $W = 4800/\log_2 16 = 1200$ Hz and the transmission bandwidth $B_T = 2000 - 500 = 1500$ Hz, used by the raised-cosine spectrum, can be expressed as

$$B_T = W(1+\alpha) \Rightarrow 1500 = 1200(1+\alpha)$$

The above relationship gives roll-off factor $\alpha = 0.25$.

Coherent BSFK

→ We have seen BPSK and M-PSK information is carried by phase and in QAM, by Amplitude + Phase

BUT the problem with both these methods is that in many channels, especially older Radio Freq./Analog Systems,

i) Phase can't be tracked accurately

ii) Amplitude varies due to fading/shadowing

→ To overcome this we need a modulation that doesn't rely on phase or amplitude, using Frequency Shift Keying

→ In FSK, information is delivered using frequency differences

→ Binary FSK uses 2 different frequencies :

i) One frequency for bit "1"

ii) Another frequency for bit "0"

Which makes it robust to amplitude variations

→ Basically in coherent BSFK, receiver knows the precise frequency + phase of both signals

→ BSFK is a non-linear modulation scheme and symbols 1 & 0 are represented by 2 sinusoidal waves having different frequencies

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) & , \; 0 \le t \le T_b \;\; \& \;\; i = 1,2 \\ 0 & , \; \text{elsewhere} \end{cases} \qquad \begin{pmatrix} i = 1 \Rightarrow \text{bit "1"} \\ i = 2 \Rightarrow \text{bit "0"} \end{pmatrix}$$

$$f_i = \frac{n_c + i}{T_b} \quad , \; n_c \text{ is integer}$$

This BSFK scheme is called Sunde's FSK

→ Unlike BSPK ($M=2$ & $N=1$), BSFK needs 2 orthonormal bases ($M=2$ & $N=2$)

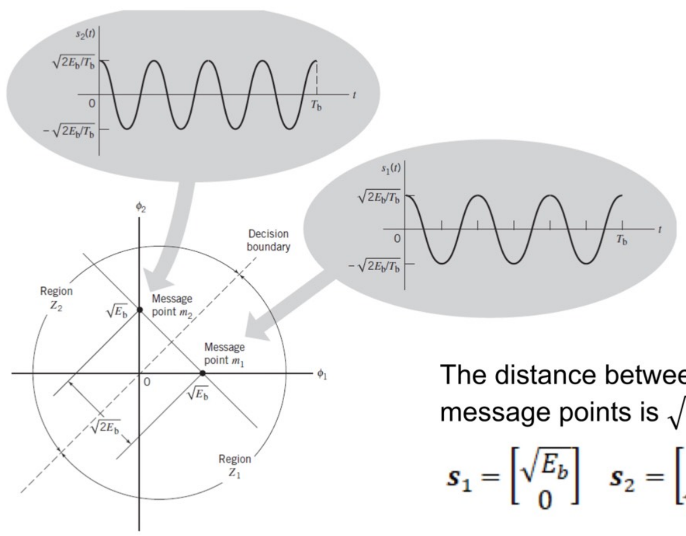
The orthonormal bases are given by $\phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) & , \; 0 \le t \le T_b \text{ and } i=1,2 \\ 0 & , \; \text{elsewhere} \end{cases}$

To make detection optimal, coordinates of the signals are orthogonal over one bit period :

$$s_{ij} = \int_0^{T_b} s_i(t)\, \phi_j(t)\, dt \quad , \; i=1,2 \;\; \& \;\; j=1,2$$

$$= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_j t)\, dt$$

$$= \begin{cases} \sqrt{E_b} & , \; i=j \\ 0 & , \; i \ne j \end{cases}$$

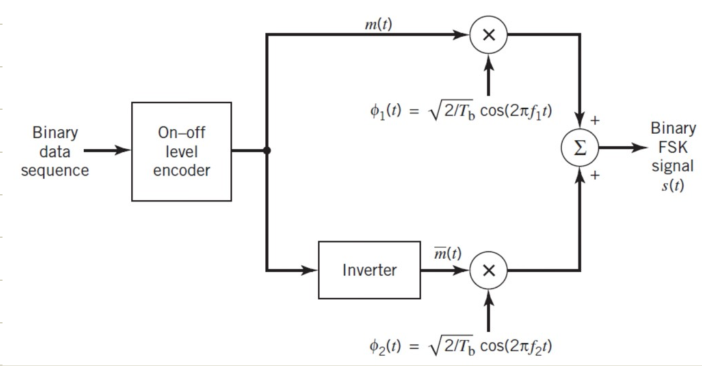


The distance between the message points is $\sqrt{2E_b}$

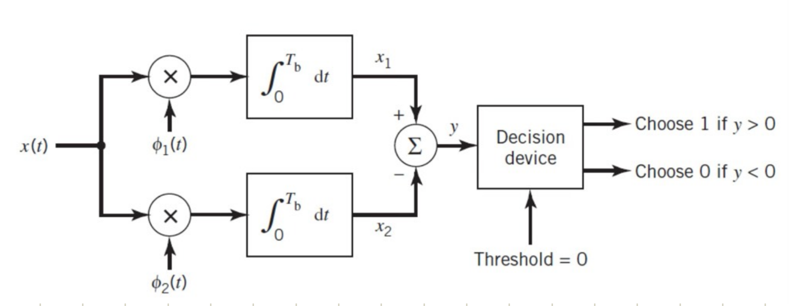
$$s_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

Coherent BFSK Generation Detection

Transmitter block diagram



Receiver block diagram



Coherent BFSK - Probability of Error

→ Elements of observation vector $\vec{x}$ are given by

$$x_j = \int_0^{T_b} x(t)\,\phi_j(t)\,dt \qquad j = 1, 2$$

$x(t)$: received signal

The receiver decides in favour of symbol 1 if $x_1 > x_2$

0 if $x_1 < x_2$

But error occurs the coordinates of x_1 & x_2 lie on the decision boundary $(x_1 = x_2)$

The decision boundary is perpendicular to the line segment connecting the noiseless signal points s_1 & s_2

→ Define a new Gaussian R.V y such that it represents the nearness to the decision boundary

$$y = x_1 - x_2$$

And as x_1 & x_2 are Gaussian R.V, The conditional mean & variance of Y are given by

$$E[Y|1] = E[x_1|1] - E[x_2|1] = \sqrt{E_b} - 0 = \sqrt{E_b}$$

$$E[Y|0] = E[x_1|0] - E[x_2|0] = 0 - \sqrt{E_b} = -\sqrt{E_b}$$

$$Var[Y] = Var[x_1] + Var[x_2] = \frac{N_0}{2} + \frac{N_0}{2} = N_0$$

→ Suppose symbol 0 was transmitted, Then the conditional density function of the R.V Y is given by

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(\frac{-(y+\sqrt{E_b})^2}{2N_0}\right)$$

→ The probability of the receiver making the decision in favour symbol 1 (Finding $x_1 > x_2 \rightarrow y > 0$) is given by

$$P_{10} = P(y > 0 \mid \text{symbol 0 was sent})$$

$$= \int_0^{\infty} f_Y(y|0)\,dy$$

$$= \frac{1}{\sqrt{2\pi N_0}} \int_0^{\infty} \exp\left(\frac{-(y+\sqrt{E_b})^2}{2N_0}\right) dy$$

→ Now lets introduce an auxillary variable z into P_{10}

Let $z = \dfrac{y+\sqrt{E_b}}{\sqrt{N_0}}$ denote the new variable of integration

Then $$P_{10} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp\left(\frac{-z^2}{2}\right) dz = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The conditional pdf P_{01} can be found similarly

So average probability of error is $P_e = Q\left(\sqrt{\dfrac{E_b}{N_0}}\right)$

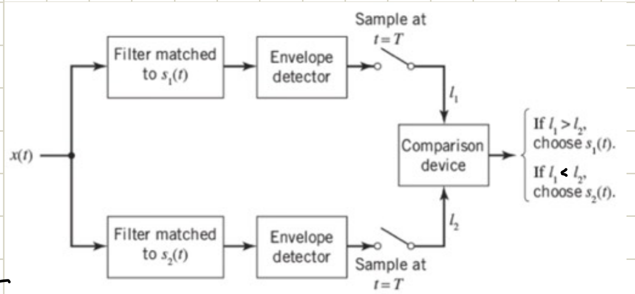
Non coherent Orthogonal Modulation

→ This topic gives the base for DPSK, noncoherent BFSK & many other practical systems where receiver can't track carrier phase

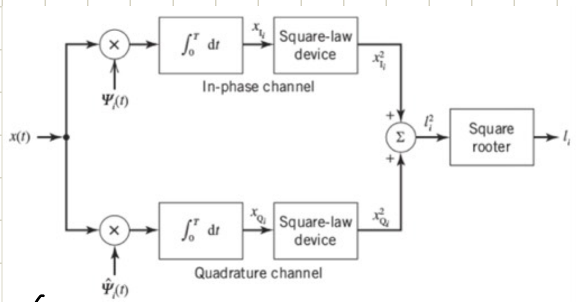
→ Coherent detection requires the receiver to know exact carrier frequency, exact carrier phase & proper synchronization
But real world channels have :
i) rapidly varying carrier phase
ii) expensive phase tracking
iii) unexpectable phase rotation by channel

→ So we need a modulation/detection type that doesn't require carrier phase
This is called Coherent Detection

→ Suppose channel shifts carrier phase by an unknown amount,
Let $s_1(t)$ & $s_2(t)$ denote analog symbols used by transmitter for symbol 1 & 0 respectively
$g_1(t)$ & $g_2(t)$ denote phase shifted signals corresponding to $s_1(t)$ & $s_2(t)$ respectively
$w(t)$ is AWGN having zero mean & PSD of $\frac{N_0}{2}$
The received signal is given by $x(t) = s_i(t) + w(t)$ where $i = 1,2$ & $0 \le t \le T$



↳ Generalized Binary Receiver for noncoherent orthogonal modulation



↳ Quadrature Receiver equivalent to either one of the 2 matched filters

In the receiver, $\varphi_i(t) = m(t) \cos 2\pi f_i t$, $i = 1,2$
$\hat{\varphi}_i(t) = m(t) \sin 2\pi f_i t$

Let the in-phase & quadrature of each path be denoted by x_{Ii} & x_{Qi} $(i = 1,2)$
These 4 o/ps are statistically independant & identically distributed

→ Now we calculate avg. prob of error, which occurs when R.V L_2 produced by lower path is greater than random variable L_1 produced by upper path
Due to orthogonality, L_2 is contributed by only noise
Now, sample value $L_2 = l_2$ is given by corresponding quadrature receiver as $l_2 = \sqrt{x_{I2}^2 + x_{Q2}^2}$

X_{I2} & X_{Q2} are Gaussian R.V mean zero & variance of $N_0/2$

→ The density function corresponding to random variables X_{I2} & X_{Q2} are :

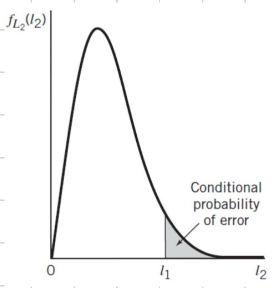
$$f_{X_{I2}}(x_{I2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{I2}^2}{N_0}\right)$$

$$f_{X_{Q2}}(x_{Q2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{Q2}^2}{N_0}\right)$$

The envelope of Gaussian process represented in polar form is Rayleigh distributed & independant of the phase θ

→ Thus, the density function of L_2 is

$$f_{L_2}(l_2) = \begin{cases} \frac{2 l_2}{N_0} \exp\left(-\frac{l_2^2}{N_0}\right) & , l_2 > 0 \\ 0 & , \text{elsewhere} \end{cases}$$



Conditional probability of error, that $l_2 > l_1$ given l_1

$$P(l_2 > l_1 \mid l_1) = \int_{l_1}^{\infty} f_{L_2}(l_2)\, dl_2 = \exp\left(-\frac{l_1^2}{N_0}\right)$$

$$f_{X_{I1}}(x_{I1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_{I1} - \sqrt{E})^2}{N_0}\right)$$

$$f_{X_{Q1}}(x_{Q1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{Q1}^2}{N_0}\right)$$

and avg prob. of error,

$$P_e = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{P}(\text{error} \mid x_{I1}, x_{Q1})\, f_{X_{I1}}(x_{I1}) f_{X_{Q1}}(x_{Q1})\, dx_{I1}\, dx_{Q1} = \frac{1}{2}\exp\left(\frac{-E}{2N_0}\right)$$

Differential Phase Shift Keying

→ We have seen about how BPSK requires coherent detection :
- Requires exact carrier phase
- Must use PLL (Phase - locked loop)
- Must track fast phase changes in mobile / fading channels

→ Differential PSK solves this by eliminating the need to know absolute phase

→ Information is encoded in the change in phase from one symbol to the next, thus, receiver only needs to compare 2 consecutive symbol phases, not know the carrier phase

→ DPSK combines differential encoding & PSK

→ Differential Encoding

→ If input bit sequence $b_k \in \{0,1\}$

DPSK constructs a transmitted phase sequence d_k such that

If bit 1 → change phase by 180°

bit 0 → no phase change

→ $d_k = b_k \oplus d_{k-1} = b_k d_{k-1} + \bar{b}_k \bar{d}_{k-1}$

b_k : current bit

d_{k-1} : Prev. transmitted phase bit

b_k	d_{k-1}	d_k	Notes
0	0	0	No Change
0	1	1	No Change
1	0	1	Change
1	1	0	Change

Illustration of DPSK

Q. Consider the input binary sequence, denoted $\{b_k\}$, to be 10010011, which is used to derive the generation of a DPSK signal. The differentially encoded process starts with the reference bit 1. Let $\{d_k\}$ denote the differentially encoded sequence starting in this manner and $\{d_{k-1}\}$ denote its delayed version by one bit. The complement of the modulo-2 sum of $\{b_k\}$ and $\{d_{k-1}\}$ defines the desired $\{d_k\}$, as illustrated in the top three lines of Table 7.6. In the last line of this table, binary symbols 1 and 0 are represented by phase shifts of 1 and π radians.

Table 7.6 Illustrating the generation of DPSK signal

$\{b_k\}$		1	0	0	1	0	0	1	1
$\{d_{k-1}\}$		1	1	0	1	1	0	1	1
reference									
Differentially encoded sequence $\{d_k\}$	1	1	0	1	1	0	1	1	1
Transmitted phase (radians)	0	0	π	0	0	π	0	0	0

→ Error Probability of DPSK

→ The DPSK signal is represented for 2 bit intervals $(0 \le t \le 2T_b)$

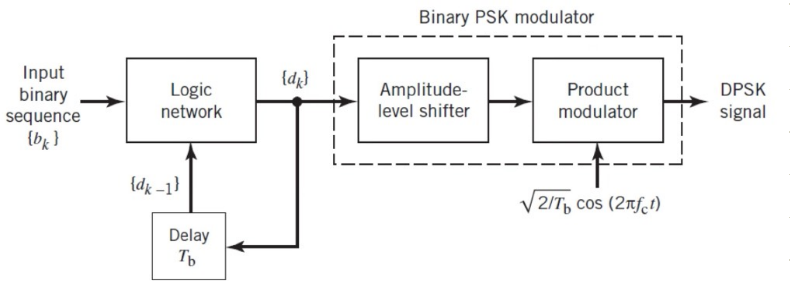
If the 1st bit is 1 & 2nd bit is 1, no change occurs

$$s_1(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_c t), & 0 \le t \le T_b \\ \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_c t), & T_b \le t \le 2T_b \end{cases}$$

If the 1st bit is 1 & 2nd bit is 0, 180° phase change occurs

$$s_2(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_c t), & 0 \le t \le T_b \\ \sqrt{\dfrac{2E_b}{T_b}} \cos(2\pi f_c t + \pi), & T_b \le t \le 2T_b \end{cases}$$

→ Here, logic n/w converts binary sequence into differentially encoded sequence



→ In summary,

Scheme	Synchronization Needed	BER Performance	Notes
BPSK	Coherent (phase needed)	Best (most power-efficient)	Optimal binary
DPSK	**Noncoherent** (no phase req.)	3 dB worse than BPSK	Simpler RX
Coherent BFSK	Coherent	Worse than BPSK	Uses frequency difference
Noncoherent BFSK	Noncoherent	Worst among these	Simplest, robust

Modulation Scheme	Detection Type	BER / SER Formula	Notes
BPSK	Coherent	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	*Best binary performance (antipodal)*
QPSK	Coherent	$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	*Same BER as BPSK, 2 bits/symbol*
Coherent BFSK (Orthogonal)	Coherent	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	*3 dB worse than BPSK*
Noncoherent BFSK	Noncoherent	$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$	*Worst among binary schemes*
DPSK	Noncoherent Differential	$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$	*3 dB worse than BPSK, no carrier sync required*
M-PSK (M ≥ 4)	Coherent	$P_s \approx 2Q\left(\sqrt{2\frac{E_s}{N_0}}\sin\frac{\pi}{M}\right)$ $P_b \approx \frac{P_s}{\log_2 M}$	*Phase-only modulation, poor BER for large M*
Square M-QAM	Coherent	$P_s \approx 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$ $P_b \approx \frac{P_s}{\log_2 M}$	*Better power efficiency than M-PSK*
Coherent BFSK (Orthogonal, Symbol Error)	Coherent	$P_s = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	*Where $E_s = E_b$ for binary*
M-FSK (Orthogonal)	Coherent	$P_s = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	*Not in-depth in Unit 4 but useful formula*
Noncoherent M-FSK	Noncoherent	$P_s = \sum_{k=1}^{M-1}\frac{1}{k+1}\binom{M-1}{k}(-1)^{k+1}e^{-kE_s/(2N_0)}$	*General result (not normally needed in ISA)*

Scheme	Bits/Symbol	Distance Efficiency	Coherent Needed?	Power Efficiency
BPSK	1	Max (antipodal)	Yes	★★★★★
QPSK	2	Max (two BPSK branches)	Yes	★★★★★
Coherent BFSK	1	Medium	Yes	★★★★
DPSK	1	Medium–Low	No	★★★
Noncoherent BFSK	1	Low	No	★★
M-PSK	$\log_2 M$	Falls rapidly for large M	Yes	★–★★★
M-QAM	$\log_2 M$	Best for large M	Yes	★★★★