Linear Algebra

UNIT-1

MATRICES & GAUSSIAN ELIMINATION

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VIBHA MASTI

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LINEAR ALGEBRA

Linear Equations

- equation in n variables in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where $x_1, x_2 \dots x_n$ are unknown variables,
$a_1, a_2 \dots a_n$ are coefficients and
b is a constant

System of Linear Equations

SYSTEM of LINEAR EQUATIONS

- INCONSISTENT
  no solution exists
- CONSISTENT
  - one unique solution
  - infinitely many solutions

- set of m equations and n variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}$$

MATRIX REPRESENTATION

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \times \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

A — coefficient matrix

X — matrix of unknowns

b — matrix of right-side constants

- a_{ij} : component of i^{th} row and j^{th} column
- if $m=n$, square matrix of n equations and n unknowns
- if all b's are zero, homogeneous system of equations;
  if any one b is nonzero, non-homogeneous system of equations

AUGMENTED MATRIX

$$\text{Matrix } [A:b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : b_2 \\ & \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : b_n \end{bmatrix}_{m\times(n+1)}$$

SINGULAR MATRIX

Matrix whose determinant is 0

Solution to System of Linear Equations

ELEMENTARY ROW TRANSFORMATIONS

1) $R_i \longrightarrow kR_i \quad (k \neq 0)$
   multiply entries of a row by non-zero scalar

2) $R_i \longrightarrow R_i + kR_j \quad (k \neq 0)$
   sum of itself and non-zero scalar multiple of another row

3) $R_i \longleftrightarrow R_j$
   swap two rows

$$AX = BI$$

to transform $\downarrow$

$$IX = BA^{-1}$$

EQUIVALENT MATRICES

- if two matrices A and B are such that each of them can be obtained from the other by a definite number of elementary transformations, they are said to be equivalent
- $A \square B$

Echelon Form of a Matrix

- A rectangular matrix $A_{m \times n}$ is said to be in echelon form if it satisfies the following conditions

1. First non-zero element of each row is called pivot element
2. All entries below the pivot in its column must be 0

3. Each pivot lies right to the pivot of the previous row (produces staircase pattern)

4. Zero rows (if they exist) lie at the bottom of the matrix

- Example:

$$\begin{bmatrix} a & b & c & d \\ 0 & 0 & e & f \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

- if square matrix, Upper Triangular Matrix : determinant $= P_1 \times P_2 \cdots P_n$

ROW REDUCED ECHELON FORM (RREF)

- Every row in echelon form must be divided by its pivot such that the first nonzero element is always 1

convert to 0

denoted by R

$$\begin{bmatrix} 1 & b/a & c/a & d/a \\ 0 & 0 & 1 & f/e \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4} = R$$

all remaining elements in pivot columns are 0

geometry of LE

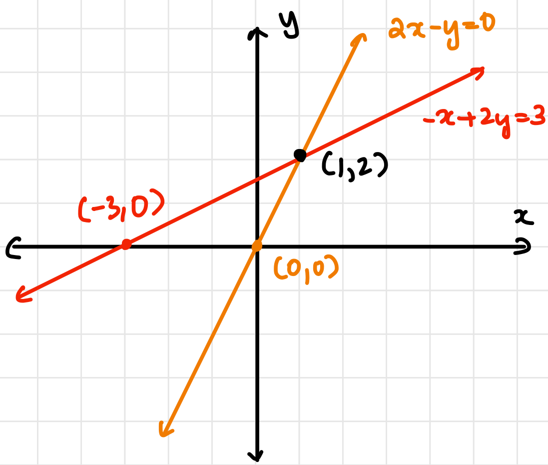
Row Picture

- 2 variables and 2 equations
- 2 straight lines in two dimensions
- solution: unique point of intersection of lines

Q1. Solve, show row picture

$$2x - y = 0$$
$$-x + 2y = 3$$

row picture



$$\underset{A}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

$2x - y = 0$

x	0	1
y	0	2

$-x + 2y = 3$

x	-3	1
y	0	2

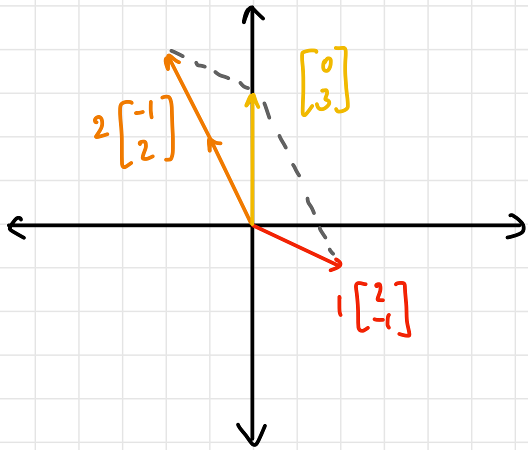
Column Picture

- combination of column vectors on the left side that produces right hand side

Q2. Solve, show column picture

$$2x - y = 0$$
$$-x + 2y = 3$$

column picture



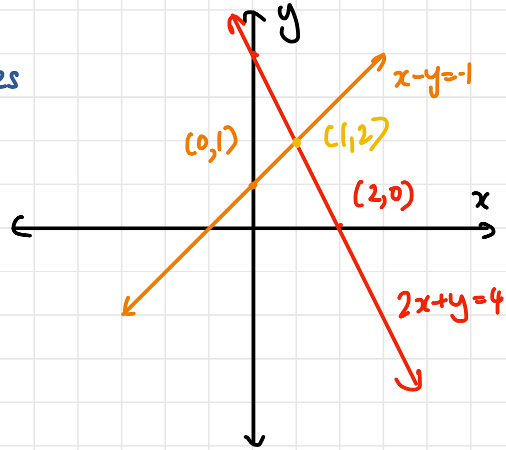
linear combination of columns

$$x \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{column 1}} + y \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{column 2}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

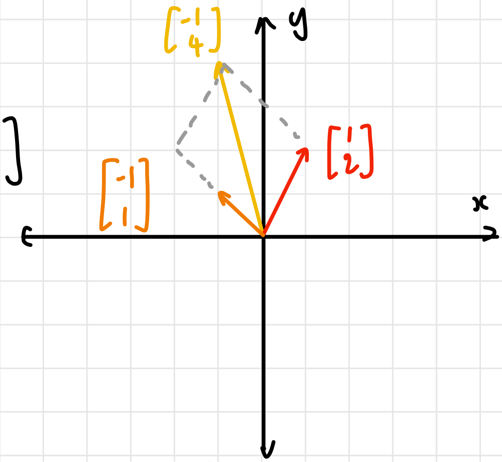
$$1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Q3 Solve and show row & column pictures

$$x - y = -1$$
$$2x + y = 4$$



Row picture: solving, we get (1,2)
unique solutions
two lines intersect at (1,2)

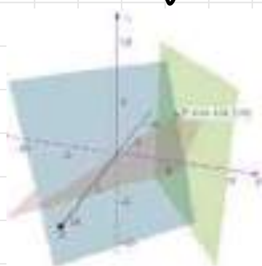


Column picture: linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ gives $\begin{bmatrix}-1\\4\end{bmatrix}$

Three Dimensions

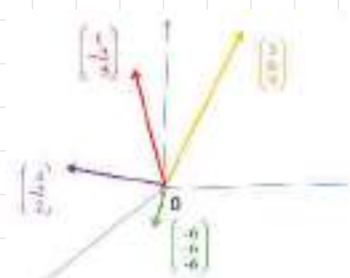
Row Picture

intersection of 3 planes at a point (unique solution)



Column Picture

linear combination of vectors to form parallelopiped

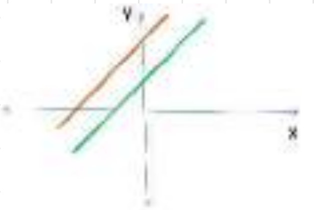


Equation of Line

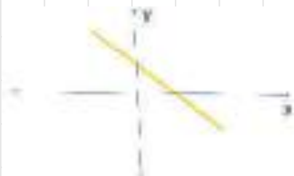
A line in n dimensions requires n-1 equations $(n \geq 2)$

Singular cases in 2-D

1. Two lines are parallel (no solution)

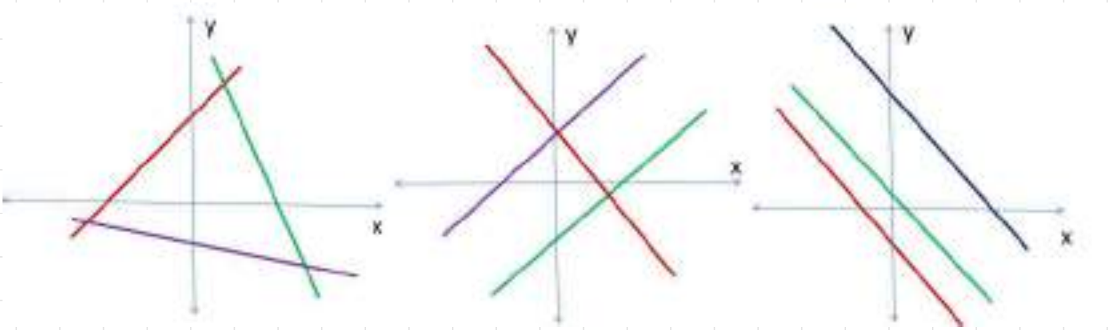


2. Two lines are coincident (same line, infinite solutions)



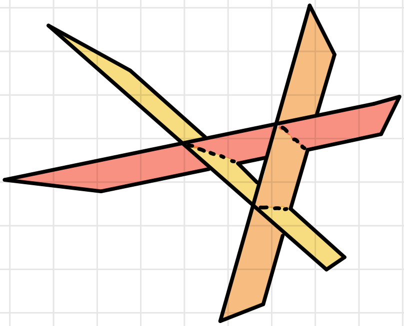
	Dimension	Geometric Description	No. of Solutions	Matrix Condition
1)	2D Row picture	Lines parallel & no intersection	No solution	$\|A\| = 0$
2)	2D Row picture	Lines coincident (lines lie on top of each other)	∞	$\|A\| = 0$
3)	3D Row picture	Each pair intersect in a line but no common intersection	No solution	$\|A\| = 0$
4)	3D Row picture	2 planes intersect along a line, 3rd plane $\|\|^{el}$ to line	No solution	$\|A\| = 0$
5)	3D Row picture	All 3 planes $\|\|^{el}$, no intersection	No solution	$\|A\| = 0$
6)	3D Row picture	All 3 planes overlap perfectly	∞	$\|A\| = 0$
7)	3D Row picture	All 3 planes intersect along single line	∞	$\|A\| = 0$
8)	3D Row picture	2 planes are $\|\|^{el}$, 3rd intersects them	No solution	$\|A\| = 0$
9)	3D Column picture	$\vec{b}$ is not in the plane formed by $\vec{a_1}, \vec{a_2}, \vec{a_3}$	No solution	$\|A\| = 0$
10)	3D Column picture	$\vec{b}$ is in the plane formed by $\vec{a_1}, \vec{a_2}, \vec{a_3}$	∞	$\|A\| = 0$

Three lines

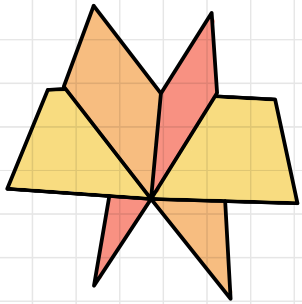


Singular cases in 3-D

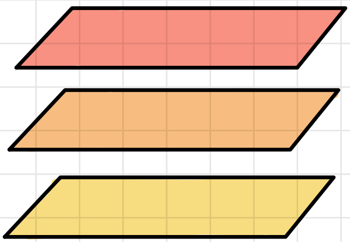
1. Every pair of lines intersect in a line and all those lines are parallel



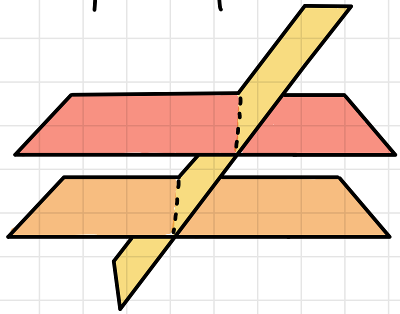
2. Three planes have a line in common



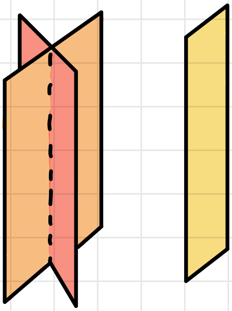
3. All three planes parallel



4. Two planes parallel



5. Two planes intersect in a line and third parallel



6. All three planes overlap

Column Picture

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow x_1\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- each column vector is position vector with origin as a point
- these col vectors lie on a plane (pass through origin)
- every combination of these vectors on LHS lie in the same plane (3 vectors coplanar)
- if vector b is not on the plane, system is singular and has no solution
- if vector b is on the plane, infinite number of solutions

GAUSSIAN ELIMINATION

— rank of matrix

- a square matrix A of order n is said to have rank r if
  - at least one minor of order r does not vanish (sub-determinant not 0)
  - every minor of order r+1 vanishes
- rank of matrix A is denoted by rank(A) = r
- no. of nonzero rows in echelon form of A

Q4. Find the rank of the following

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$ $r = 2$ (b) $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $r = 1$ (c) $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $r = 0$

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $r = 2$ (e) $E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $r = 3$ (f) $F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $r = 1$

(g) $G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $r = 2$

Relationship Between Rank, Consistency and Solution

if $rank(A)=r$, then

1. if $rank(A) = rank(A:b) = r$, system $AX=b$ is consistent and has a solution

2. if $rank(A) = rank(A:b) = r = n$, system $AX=b$ is consistent and has a unique solution

3. if $rank(A) = rank(A:b) = r < n$, system $AX=b$ is consistent and has infinite no. of solutions

4. if $rank(A) \neq rank(A:b)$, system $AX=b$ is inconsistent and has no solution

Gaussian Elimination

- check for consistency and solve linear equations

- for given system of LE $AX=b$ apply elementary row transformations to the augmented matrix $[A:b]$ and reduce it to $[U:c]$ where U is an Upper Triangular matrix

- We get an equivalent system $Ux=c$ which can be solved by backward substitution

- Here A and U are Equivalent matrices and hence solution of $AX=b$ is the same as solution of $Ux=c$

Steps for Elementary Row Transformations in GE

1. No exchange of rows
2. First row should be unaltered
3. First nonzero element in nonzero row is called pivot
4. $Ax=b$ and $Ux=c$ have same solution

example

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

$$[A:b] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \xrightarrow[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1]{R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1} \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & d_{22} & d_{23} & c_2 \\ 0 & d_{32} & d_{33} & c_3 \end{array}\right]$$

$$\xrightarrow{R_3 - \left(\frac{d_{32}}{d_{22}}\right)R_2}$$

$$[U:c] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & d_{22} & d_{23} & c_2 \\ 0 & 0 & e_{33} & c_4 \end{array}\right]$$

Qs: check for consistency and solve if consistent

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$2x_1 + 2x_2 - 3x_3 + x_4 = 3$$
$$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$$

$$[A:b] = \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 4$
$r([A:b]) = 2$
$r(A) = 2$

$$r(A) = r([A:b]) < n$$

$\therefore$ consistent with infinite no. of solutions

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$
$$x_3 - 7x_4 = -7$$

Q6. check for consistency and solve if consistent

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$2x_1 + 3x_2 + 3x_3 - x_4 = 3$$
$$5x_1 + 7x_2 + 4x_3 + x_4 = 5$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - 5R_1]{R_2 - 2R_1} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & -1 & -4 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & -3 & 2 & -5 \end{array}\right]$$

$n = 4 \qquad r(A) = 3 \qquad r([A:b]) = 3$

$\therefore$ consistent with infinite no. of solutions

$+3 - \frac{2}{3}$

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_2 + x_3 - 3x_4 = -5$$
$$-3x_3 + 2x_4 = -5$$

$\frac{-15}{3} \; \frac{-25}{3}$

Let $\boxed{x_4 = k}$

$$-3x_3 + 2k = -25$$
$$-3x_3 = -2k - 25$$

$$\boxed{x_3 = \frac{2}{3}k + \frac{25}{3}}$$

$$x_2 + \frac{2}{3}k + \frac{25}{3} - 3k = -5$$

$$\boxed{x_2 = -\frac{40k}{3} + \frac{7}{3}}$$

$$x_1 - \frac{40k}{3} + \frac{7}{3} + \frac{2}{3}k + \frac{25}{3} + k = 4$$

$$\boxed{x_1 = \frac{-20}{3} + \frac{35}{3}}$$

Q7. check for consistency and solve if consistent

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 5x_2 - x_3 = -4$$
$$3x_1 - 2x_2 - x_3 = 5$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 + 8R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$n = 3$
$r(A) = 3$
$r([A:b]) = 3$

$\therefore$ consistent with unique solution

$$x_1 + 2x_2 + x_3 = 3$$
$$x_2 - 3x_3 = -10$$
$$-28x_3 = -84$$

$\boxed{x_3 = 3}$

$x_2 - 9 = -10$

$\boxed{x_2 = -1}$

$x_1 - 2 + 3 = 3$

$\boxed{x_1 = 2}$

Q8. check for consistency and solve if consistent

$$2x - 3y + 2z = 1$$
$$5x - 8y + 7z = 1$$
$$y - 4z = 3$$

$$[A:b] = \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \\ 0 & 1 & -4 & 3 \end{array}\right] \xrightarrow{R_2 - 5/2 R_1} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 1 & -4 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & -1/2 & 2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$r(A) = 2$$
$$r(A:b) = 2$$
$$n = 3$$

$\therefore$ consistent with infinite no. of solutions

$$2x - 3y + 2z = 1$$
$$-\frac{1}{2}y + 2z = -3/2$$

Let $z = k$

$$-\frac{y}{2} + 2k = -3/2$$
$$-\frac{y}{2} = \frac{-3}{2} - 2k$$
$$y = 3 + 4k$$

$$2x - 3(3 + 4k) + 2k = 1$$
$$2x - 9 - 12k + 2k = 1$$
$$2x = 10 + 10k$$
$$x = 5k + 5$$

$$(x, y, z) = (5k+5, 3+4k, k)$$

Q9. check for consistency and solve if consistent

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 5x_2 - x_3 = -4$$
$$3x_1 - 2x_2 - x_3 = 5$$

$$[A:b] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 5 & -1 & : & -4 \\ 3 & -2 & -1 & : & 5 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 1 & -3 & : & -10 \\ 0 & -8 & -4 & : & -4 \end{bmatrix}$$

$$\xrightarrow{R_3 + 8R_2} \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & 1 & -3 & : & -10 \\ 0 & 0 & -28 & : & -84 \end{bmatrix}$$

$r(A) = 3$
$r(A:b) = 3$
$n = 3$

$\therefore$ consistent with unique solution

$$x_1 + 2x_2 + x_3 = 3$$
$$x_2 - 3x_3 = -10$$
$$-28x_3 = -84$$

$-28x_3 = -84$

$\boxed{x_3 = 3}$

$x_2 - 9 = -10$

$\boxed{x_2 = -1}$

$x_1 - 2 + 3 = 3$

$\boxed{x_1 = 2}$

Q10. check for consistency and solve if consistent

$$x_1 + x_2 - 2x_3 + 3x_4 = 4$$
$$2x_1 + 3x_2 + 3x_3 - x_4 = 3$$
$$5x_1 + 7x_2 + 4x_3 + x_4 = 5$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \xrightarrow[R_3-5R_1]{R_2-2R_1} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

$n = 4$
$r(A) = 2$
$r([A:b]) = 3$

$\therefore$ inconsistent and no solution

Q11. check for consistency and solve if consistent

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2-5)z = a$$

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

if $a \neq \pm 2$

$n = 3$
$r(A) = 3$
$r(A:b) = 3$

$\therefore$ consistent with unique solution

$$x + y - z = 2$$
$$y + 2z = 1$$
$$(a^2 - 4)z = a - 2$$

$$z = \frac{a-2}{a^2-4} = \frac{1}{a+2}$$

$$y + \frac{2}{a+2} = 1$$

$$y = \frac{a+2-2}{a+2} = \frac{a}{a+2}$$

$$x + \frac{a}{a+2} - \frac{1}{a+2} = 2$$

$$x = \frac{2a+4-a+1}{a+2} = \frac{a+5}{a+2}$$

$$(x, y, z) = \left(\frac{a+5}{a+2}, \frac{a}{a+2}, \frac{1}{a+2} \right)$$

If $a = 2$ $\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$r(A) = 2 = r(A:b) < n$

$\therefore$ consistent with infinite no. of solutions

$$x + y - z = 2$$
$$y + 2z = 1$$

Let $z = k$

$$y + 2k = 1$$
$$y = 1 - 2k$$

$$x + 1 - 2k - k = 2$$
$$x + 1 - 3k = 2$$
$$x = 1 + 3k$$

$$(x, y, z) = (1+3k, 1-2k, k)$$

If $a = -2$
$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

$r(A) = 2$
$r(A:b) = 3$ $\quad\therefore$ inconsistent and no solution

Q12.
$$x + z = 1$$
$$x + y + z = 2$$
$$x - y + z = 1$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2}$$

$$[U:C] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$r(A) = 2 \quad r([A:b]) = 3 \quad n = 3$

$\therefore$ inconsistent and no solution

Q13. check for consistency and solve if consistent

$x+y+z=8$
$2x-3y+4z=3$
$3x-y-3z=6$

$$[A:b]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 2 & -3 & 4 & 3 \\ 3 & -1 & -3 & 6 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & -5 & 2 & -13 \\ 0 & -4 & -6 & -18 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{4}{5}R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & -5 & 2 & -13 \\ 0 & 0 & -\frac{38}{5} & -\frac{38}{5} \end{array}\right]$$

$r(A)=3=r([A:b])=n$

$\therefore$ consistent, unique solution

$-\frac{38}{5}z=-\frac{38}{5}$

$\boxed{z=1}$

$-5y+2=-13$
$-5y=-15$

$\boxed{y=3}$

$x+3+1=8$

$\boxed{x=4}$

$(x,y,z)=(4,3,1)$

Breakdown of Elimination

- if a zero appears in pivot position, elimination needs to stop temporarily or permanently
- if problem can be cured & elimination can proceed, system is non-singular
- if breakdown is unavoidable/permanent, system is singular and has no solution/infinitely many solutions
- Non-singular and curable ($|A| \neq 0$)
- Singular and incurable ($|A| = 0$)
- Singular ($|A| = 0$)

Q14. check for consistency and solve if consistent

$$x+y+z=-3$$
$$2x+2y+5z=6$$
$$4x+6y+8z=7$$

pivot is 0

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 2 & 2 & 5 & 6 \\ 4 & 6 & 8 & 7 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & 0 & 3 & 12 \\ 0 & 2 & 4 & 19 \end{array}\right]$$

swap rows

$R_2 \leftrightarrow R_3$

non-singular & curable

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & 2 & 4 & 19 \\ 0 & 0 & 3 & 12 \end{array}\right]$$

$3z = 12$ $\quad$ $\boxed{z = 4}$

$2y + 16 = 19$ $\quad$ $\boxed{y = \frac{3}{2}}$

$x + \frac{3}{2} + 4 = -3$ $\quad$ $\boxed{x = \frac{-17}{2}}$

$$(x, y, z) = \left(\frac{-17}{2}, \frac{3}{2}, 4\right)$$

Qus. check for consistency and solve if consistent

$$x + y + z = 6$$
$$x + y + 3z = 10$$
$$x + 2y + 4z = 12$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 1 & 3 & 10 \\ 1 & 2 & 4 & 12 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

non-singular & curable

$2z = 4$ $\quad$ $\boxed{z = 2}$

$y + 3 \times 2 = 6$ $\quad$ $\boxed{y = 0}$

$x + 2 = 6$ $\quad$ $\boxed{x = 4}$

$$(x, y, z) = (4, 0, 2)$$

Q16. check for consistency and solve if consistent

$$x+y+z=6$$
$$x+y+3z=10$$
$$x+y+4z=13$$

pivot = 0

$$[A:b] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 1 & 3 & : & 10 \\ 1 & 1 & 4 & : & 13 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & 3 & : & 13 \end{bmatrix}$$

$$x+y+z=6$$
$$\left.\begin{aligned} 2z&=4 \\ 3z&=13 \end{aligned}\right] \text{ impossible } (z=2 \ \& \ z=13/2)$$

incurable and singular

Q17. check for consistency and solve if consistent

$$x+y+z=6$$
$$x+y+3z=10$$
$$x+y+4z=12$$

rows consistent

$$[A:b] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 1 & 3 & : & 10 \\ 1 & 1 & 4 & : & 12 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & 3 & : & 6 \end{bmatrix}$$

singular, curable

$$\xrightarrow{R_3 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 0 & 2 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$r(A) = 2 = r(A:b) < n$$

$\therefore$ consistent, infinite no. of solutions

$2z = 4$
$z = 2$

Let $y = k$

$x + k + 2 = 6$
$x = 4 - k$

$$(x, y, z) = (4-k, k, 2)$$

Q18. $u + v + w = -2$
$3u + 3v - w = 6$
$u - v + w = -1$

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 3 & 3 & -1 & 6 \\ 1 & -1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 0 & -4 & 12 \\ 0 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -4 & 12 \end{array}\right]$$

$-4w = 12$
$w = -3$

$-2v = 1$
$v = -1/2$

$u - 1/2 - 3 = -2$
$u = \frac{3}{2}$

$$(u, v, w) = (3/2, -1/2, -3)$$

Q19. For which 3 no.s 'a' will elimination fail?

$$ax + 2y + 3z = b_1$$
$$ax + ay + 4z = b_2$$
$$ax + ay + az = b_3$$

$$[A:b] = \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ a & a & 4 & b_2 \\ a & a & a & b_3 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & a-2 & a-3 & b_3 - b_1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc:c} a & 2 & 3 & b_1 \\ 0 & a-2 & 1 & b_2 - b_1 \\ 0 & 0 & a-4 & b_3 - b_2 \end{array}\right]$$

$$a = 0, \quad a = 2, \quad a = 4$$

Q20. For what values of a and b does the following system have

(i) A unique solution

(ii) Infinitely many solutions

(iii) No solution

$$x + 2y + 3z = 2$$
$$-x - 2y + az = -2$$
$$2x + by + 6z = 5$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ -1 & -2 & a & -2 \\ 2 & 6 & 6 & 5 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & 0 & a+3 & 0 \\ 0 & b-4 & 0 & 1 \end{array}\right]$$

elimination fails

$$[U:C] \quad \left[\begin{array}{ccc:c} 1 & 2 & 3 & 2 \\ 0 & b-4 & 0 & 1 \\ 0 & 0 & a+3 & 0 \end{array}\right] \xleftarrow{R_2 \leftrightarrow R_3}$$

i) Unique solution

$r(A) = 3 \quad r(A:b) = 3 \quad$ or $\quad a \neq -3, \; b \neq 4$

$n = 3$

$(a+3)z = 0$

$z = 0$

$(b-4)y = 1$

$y = \frac{1}{b-4}$

$x + \frac{2}{b-4} = 2$

$x = 2 - \frac{2}{b-4}$

$x = 2\left(\frac{b-5}{b-4}\right)$

ii) Infinite solutions

$r(A) = r(A:b) = 2 \quad , \quad a = -3, \quad b \neq 4$

$(b-4)y = 1$

$y = \frac{1}{b-4}$

Let $x = k$

$k + \frac{2}{b-4} + 3z = 2$

$3z = 2 - k - \frac{2}{b-4}$

$z = \frac{2}{3} - \frac{k}{3} - \frac{2}{3(b-4)}$

iii) No solutions

$r(A) \neq r(A:b) \quad , \quad b = 4 \quad$ and $\quad a = 3$ or $a \neq 3$

rank 2   rank 3

Q21. $x+y+z=1$
$x+y-2z=3$
$2x+y+z=2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 3 \\ 2 & 1 & 1 & 2 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$-3z=2$

$\boxed{z=-\frac{2}{3}}$

$-y-z=0$

$-y+\frac{2}{3}=0$

$\boxed{y=\frac{2}{3}}$

$x+\frac{2}{3}-\frac{2}{3}=1$

$\boxed{x=1}$

$(x,y,z)=(1, 2/3, -2/3)$

Q22. $x+y+2z+3t=13$
$x-2y+z+t=8$
$3x+y+z-t=1$

$$[A:b]=\left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-R_1} \left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & -38 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & 0 & \frac{-13}{3} & \frac{-26}{3} & \frac{-104}{3} \end{array}\right] \xleftarrow{R_3 - \frac{2}{3}R_2}$$

$r(A) = 3 = r(A:b) < n = 4$

$\therefore$ consistent with ∞ solutions

Let $t = k$

$-3y - 8 + \cancel{2k} - \cancel{2k} = -5$

$\frac{-13}{\cancel{3}} z - \frac{26}{\cancel{3}} k = \frac{-104}{\cancel{3}}$

$y = -1$

$-13z = 26k - 104$

$z = -2k + 8$

$x - 1 + 16 - 4k + 3k = 13$

$x + 15 - k = 13$

$x = k - 2$

Q23. $x + z = 1$ what if RHS $= (1, 2, 0)$?

$x + y + z = 2$

$x - y + z = 1$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xleftarrow{R_3 + R_2}$$

$r(A) = 2$
$r(A:b) = 3$

$\therefore$ inconsistent

$x + z = 1$
$x + y + z = 2$
$x - y + z = 0$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r(A) = 2 = r(A:b) < n = 3$

$\therefore$ consistent with ∞ solutions

let $z = k$ $\qquad$ $y = 1$

$x + k = 1$
$x = 1 - k$

$(x, y, z) = (1-k, 1, k)$

Q24. Find the values of a and b

$$x + y + az = 2b$$
$$x + 3y + (2+2a)z = 7b$$
$$3x + y + (3+3a)z = 11b$$

i) what is trivial solution (all variables have to be 0)

ii) What is unique non-trivial solution (at least one nonzero solution)

iii) what is infinite set of solutions

iv) No solution

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 1 & 3 & 2+2a & 7b \\ 3 & 1 & 3+3a & 11b \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & -2 & 3 & 5b \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & 0 & 5+a & 10b \end{array}\right]$$

i) $b = 0$, $a \neq -5$

ii) $b \neq 0$, $a \neq -5$

iii) $b = 0$, $a = -5$

iv) $b \neq 0$, $a = -5$

Q25. Let $A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$ and $b = (b_1, b_2, b_3, b_4)$

(i) Find values of b such that $Ax=b$ is consistent

(ii) $b=(2,1,7,1)$ if $(x,0,0,1)$ is solution to $Ax+b$, what is x?

$$[A:b] = \left[\begin{array}{cccc:c} 3 & -6 & 2 & -1 & b_1 \\ -2 & 4 & 1 & 3 & b_2 \\ 0 & 0 & 1 & 1 & b_3 \\ 1 & -2 & 1 & 0 & b_4 \end{array}\right] \xrightarrow[R_4 - 1/3 R_1]{R_2 + 2/3 R_1} \left[\begin{array}{cccc:c} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + \frac{2b_1}{3} \\ 0 & 0 & 1 & 1 & b_3 \\ 0 & 0 & 1/3 & 1/3 & b_4 - \frac{b_1}{3} \end{array}\right]$$

$$\xrightarrow[R_4 - 1/7 R_2]{R_3 - 3/7 R_2} \left[\begin{array}{cccc:c} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + 2/3\, b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3/7\, b_2 - 2/7\, b_1 \\ 0 & 0 & 0 & 0 & b_4 - b_1/3 - b_2/7 - \frac{2b_1}{21} \end{array}\right]$$

(i) $r(A) = r(A:b)$

$$b_3 - \frac{3}{7} b_2 - \frac{2}{7} b_1 = 0$$

$$\boxed{7b_3 - 3b_2 - 2b_1 = 0}$$

$$b_4 - \frac{b_1}{3} - \frac{b_2}{7} - \frac{2b_1}{21} = 0$$

$$21 b_4 - 7b_1 - 3b_2 - 2b_1 = 0$$

$$\boxed{21 b_4 - 9b_1 - 3b_2 = 0}$$

(ii) $\left[\begin{array}{cccc|l} 3 & -6 & 2 & -1 & b_1 \\ 0 & 0 & 7/3 & 7/3 & b_2 + 2/3\, b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3/7\, b_2 - 2/7\, b_1 \\ 0 & 0 & 0 & 0 & b_4 - b_1/3 - b_2/7 - \frac{2b_1}{21} \end{array}\right]$ $b = (2, 1, 1, 1)$

$$\left[\begin{array}{cccc|l} 3 & -6 & 2 & -1 & 2 \\ 0 & 0 & 7/3 & 7/3 & 1 + 4/3 \\ 0 & 0 & 0 & 0 & 1 - 3/7 - 4/7 \\ 0 & 0 & 0 & 0 & 1 - 2/3 - 1/7 - 4/21 \end{array}\right]$$

$$= \left[\begin{array}{cccc|l} 3 & -6 & 2 & -1 & 2 \\ 0 & 0 & 7/3 & 7/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$3x - 6 \times 0 + 2 \times 0 - 1 = 2$$

$$\boxed{x = 1}$$

ELEMENTARY MATRICES

Elementary matrix E_{ij} is obtained from I by performing a single elementary row operation

$$R_i - \ell_{ij} R_j \quad \text{where } \ell_{ij} \text{ is the multiplier}$$

$$\text{i.e. } I \longrightarrow E_{ij}$$

eg: E_{32} : $R_3 - 2R_2$ multiplier

$i=3 \quad j=2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}_{3\times3}$$

at most one nonzero entry off the main diagonal

$$\boxed{E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U} \quad \text{from } A \text{ to } U$$

$$A = E_{21}^{-1} \cdot E_{31}^{-1} \; E_{32}^{-1} \; U \quad \text{from } U \text{ to } A$$

Q26. Write down the elementary matrices associated with the given system of equations

$$2u + v + 3w = -1$$
$$4u + v + 7w = 5$$
$$-6u - 2v - 12w = -2$$

Step 1: convert to matrix A

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Step 2: Convert to Upper Triangular Matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

equivalent to A

$$U = \quad A \sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \text{UTM}$$

Step 3: Identify multipliers

$$E_{21} = -2 \qquad = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for $R_2 = R_2 - 2R_1$

$$E_{31} = 3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = 1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Q27. Which elementary matrices put A into UTM U?

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix}$$

$$\xrightarrow{R_3 + 5/11 R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/11 & 1 \end{bmatrix}$$

$$\boxed{E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U}$$

Q28. Which elementary matrices convert A to UTM U?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2} R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + \frac{2}{3} R_2} \quad A \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4 + \frac{3}{4} R_3} \quad A \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$$

$$\boxed{E_{43} \cdot E_{32} \cdot E_{21} \cdot A = U}$$

Q29. Which elementary matrices convert A to UTM U?

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & -2 & 0 & 8 \\ -1 & -1 & 4 & -2 \\ -2 & -2 & 6 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 + 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & 4 & 1 \\ 0 & -4 & 6 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & -4 & 6 & 3 \end{bmatrix} \xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 + 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$

TRIANGULAR FACTORS

- To undo the steps of Gaussian Elimination and revert back to original matrix A from U
- Instead of subtracting elementary matrices, the inverses are subtracted from A
- E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1} ... should be obtained by changing the signs of elementary matrices
- $\underbrace{E_{21}^{-1}\ E_{31}^{-1}\ E_{32}^{-1}}_{=L}\ U = A$

triangular FACTORISATION – LU

- Any square matrix A can be factorised as $\boxed{A = LU}$ where

asymmetric

  L: lower triangular matrix – 1's on diagonal
  U: upper triangular matrix – u's on diagonal

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

diagonal elements; asymmetric

- Introduced by Alan Turing
- One method: use multiplier coefficients from row transformations ($E_{21} \rightarrow l_{21}$, etc)
- $L\underbrace{Ux}_{z} = b \qquad Lz = b \qquad Ux = z$

Q30. Solve the following system of equations using LU decomposition method.

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$R_3 - 3R_1 \downarrow R_2 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

To find L:

$$\left.\begin{aligned} E_{21} &\longrightarrow -4 \\ E_{31} &\longrightarrow -3 \\ E_{32} &\longrightarrow 2 \end{aligned}\right\} \xrightarrow{\text{change sign}} \begin{aligned} l_{21} &= 4 \\ l_{31} &= 3 \\ l_{32} &= -2 \end{aligned}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\boxed{LZ = b}$$

forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$\boxed{z_1 = 1}$ $\qquad 4 + z_2 = 6$ $\qquad 3 - 4 + z_3 = 4$

$\boxed{z_2 = 2}$ $\qquad \boxed{z_3 = 5}$

$Ux = z$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$-10x_3 = 5$ $\qquad -x_2 + 5/2 = 2$ $\qquad x_1 + 1/2 - 1/2 = 1$

$\boxed{x_3 = -1/2}$ $\qquad \boxed{x_2 = 1/2}$ $\qquad \boxed{x_1 = 1}$

Q31. Solve the following systems of equations using LU decomposition method.

(i) $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$ $\qquad$ (ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$

(iii) $2u + v + 3w = -1$
$4u + v + 7w = 5$
$-6u - 2v - 12w = -2$

(i) $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$

$$Ax = b$$
$$LUx = b \qquad Ux = z$$
$$Lz = b$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 5/3 & 1/3 \end{bmatrix} \xrightarrow{R_3 + 5/11 R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -5/11 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$$

$$\boxed{Lz = b}$$

$\boxed{z_1 = 4}$ $\qquad 8/3 + z_2 = 1 \qquad 4/3 + 25/33 + z_3 = -8$

$\boxed{z_2 = -5/3}$ $\qquad \boxed{z_3 = -111/11}$

$$z = \begin{bmatrix} 4 \\ -5/3 \\ -111/11 \end{bmatrix}$$

$$Ux = z$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & -8/11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5/3 \\ -111/11 \end{bmatrix}$$

$$-\frac{8}{11} z = \frac{-111}{11} \qquad -\frac{11}{\cancel{3}} y \quad -\frac{7}{\cancel{3}} \times \frac{111}{8} = \frac{-5}{\cancel{3}} \qquad 3x - \frac{67}{8} + \frac{222}{8} = 4$$

$$\boxed{z = \frac{111}{8}} \qquad \boxed{y = -\frac{67}{8}} \qquad \boxed{x = -\frac{41}{8}}$$

(ii)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{R_2 - 4R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_2}$$

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$E_{21} \longrightarrow l_{21} = 4$$
$$E_{31} \longrightarrow l_{31} = -2$$
$$E_{32} \longrightarrow l_{32} = 2$$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$\boxed{z_1 = 2}$

$4 \times 2 + z_2 = 6$

$\boxed{z_2 = -2}$

$-4 - 4 + z_3 = -1$

$\boxed{z_3 = 7}$

$$Ux = z$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 7 \end{bmatrix}$$

$-2z = 7$ $\qquad$ $2y - 7/2 = -2$ $\qquad$ $x + 3/4 = 2$

$\boxed{z = -7/2}$ $\qquad$ $\boxed{y = 3/4}$ $\qquad$ $\boxed{x = 5/4}$

(iii) $2u + v + 3w = -1$
$4u + v + 7w = 5$
$-6u - 2v - 12w = -2$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$R_3 + 3R_1$ $\quad \Big\downarrow$ $R_2 - 2R_1$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$\boxed{z_1 = -1}$

$-2 + z_2 = 5$

$\boxed{z_2 = 7}$

$3 - 7 + z_3 = -2$

$\boxed{z_3 = 2}$

$$Ux = z$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$-2z = 2$

$\boxed{w = -1}$

$-y - 1 = 7$

$\boxed{v = -8}$

$2x - 8 - 3 = -1$

$\boxed{u = 5}$

triangular FACTORISATION – LDU

symmetric

- Any square matrix A can be factorised as $\boxed{A = LDU}$ where
  - L: lower triangular matrix – 1's on diagonal
  - D: diagonal matrix – d's on diagonal
  - U: upper triangular matrix – 1's on diagonal

divide row by pivot

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Q32. Solve using LU factorisation

$$2x - 3y = 3$$
$$4x - 5y + z = 7$$
$$2x - y - 2z = 5$$

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\downarrow R_3 - 2R_2$$

$$E_{32} \cdot E_{31} \cdot E_{21} \cdot A = U \qquad U_1 = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$L = [E_{32} \cdot E_{31} \cdot E_{21}]^{-1}$$

$$L = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad U_1 = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$Lz = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$\boxed{z_1 = 3}$ $\qquad 6 + z_2 = 7 \qquad 3 + 2 + z_3 = 5$

$\boxed{z_2 = 1}$ $\qquad \boxed{z_3 = 0}$

$$U_1 x = z$$

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$-4z = 0 \qquad \boxed{y = 1} \qquad 2x - 3 = 3$

$\boxed{z = 0} \qquad \boxed{x = 3}$

Q33. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 + 2R_1 \\ R_4 + R_1}]{R_2 - 1/2 R_1} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_2 \quad R_3 + 2R_2} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix} \xrightarrow{R_4 + 2R_3} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$A = LDU$$

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q34. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2/3\, R_2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \xleftarrow{R_4 + 3/4\, R_3} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

$$A = LU$$

$$\overset{A}{\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \cdot \overset{U}{\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}$$

$$A = LDU$$

$$= \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \cdot \overset{D}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Q35. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix}$$

$$\Big\downarrow R_4 - R_2 \quad R_3 - R_2$$

$$U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

$$A = LDU$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-r & 0 & 0 \\ 0 & 0 & c-s & 0 \\ 0 & 0 & 0 & d-t \end{bmatrix} \begin{bmatrix} 1 & r/a & r/a & r/a \\ 0 & 1 & (s-r)/(b-r) & (s-r)/(b-r) \\ 0 & 0 & 1 & (t-s)/(c-s) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q36. Find $A = LU$ and $A = LDU$ factorisation

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 + 1/3 R_1]{R_2 - 2/3 R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 7/3 & 5/3 \end{bmatrix}$$

$R_3 + \frac{7}{11} R_2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & 2/11 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -11/3 & -7/3 \\ 0 & 0 & 2/11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & -7/11 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -11/3 & 0 \\ 0 & 0 & 2/11 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 1 & 7/11 \\ 0 & 0 & 1 \end{bmatrix}$$

ROW EXCHANGES

- If zero appears in pivot position, row exchanges
- Row exchange is taken care of by Permutation Matrices P
- $A \neq LU$ but $PA = LU$ where P is a Permutation Matrix (identity matrix with rows in different order)
- Inverse of a permutation matrix = permutation matrix
- $P^{-1} = P^T$ $\qquad P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ there are 2! PMs of order 2

Q37. Consider $y = b_1$, $2x + 3y = b_2$

$$Ax = b$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

pivot = 0

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Gaussian elimination fails; row exchange

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = Pb = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

$$PA = LU$$

$$U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}$$

$$A = P^{-1} L U = P^T L U$$

$$\boxed{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}$$

Q38. Factorise $PA = LU$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

elimination fails

PA

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$PA = LU$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & 3 \\ 6 & 9 & 8 \\ 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}}$$

Q39. Factorise into LU and LDU

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

GE fails

$$\downarrow R_2 \leftrightarrow R_3$$

$$U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_2+2R_1]{R_3-2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$PA = LU$$

$$\underset{PA}{\begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

pivots already 1

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = LDU$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

inverses & transposes

- inverse B of a square matrix A is A^{-1}
- $AB = BA = I$ (identity matrix)

Properties

1. A^{-1} is unique for a matrix A
2. $(ABCD)^{-1} = D^{-1} C^{-1} B^{-1} A^{-1}$

   if $A = LU$, $\quad A^{-1} = U^{-1} L^{-1}$

3. A matrix A is invertible if and only if elimination produces n pivots with or without row exchanges (without permanent breakdown)

   $Ax = b \implies x = A^{-1}b$

   elimination solves $Ax = b$ without explicitly finding A^{-1}

Gauss-Jordan Method

- inverse of invertible matrix A is obtained by a set of row operations that transforms A to I and I to A^{-1}
- augmented matrix [A:I]
- convert A to U and reduce I to C.
- reduce U to I and reduce C to A^{-1}

$$\boxed{[A:I] \longrightarrow [U:C] \longrightarrow [I:A^{-1}]}$$

Q40. Compute A^{-1} using Gauss-Jordan Method

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -2 & 2 & 2 \end{bmatrix}$$

pivot

$$[A:I] = \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 & 1 & 0 \\ -2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_1$ | $R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 1 \end{array}\right]$$

pivot

$R_3 \to R_3 - 3R_2$

pivot

$$[U:C] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$R_1 \to R_1 + \tfrac{1}{3} R_3 \qquad R_2 \to R_2 + \tfrac{2}{3} R_3$$

pivot

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 10/3 & -1 & 1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2/3 & 0 & -1/3 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & -3 & 7 & -3 & 1 \end{array}\right]$$

$$R_1 \to \tfrac{1}{2} R_1 \qquad R_3 \to -\tfrac{1}{3} R_3$$

$$[I:A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & -1/6 \\ 0 & 1 & 0 & 8/3 & -1 & 2/3 \\ 0 & 0 & 1 & -7/3 & 1 & -1/3 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1/3 & 0 & -1/6 \\ 8/3 & -1 & 2/3 \\ -7/3 & 1 & -1/3 \end{bmatrix}}$$

Q41. Compute A^{-1} using Gauss-Jordan Elimination

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_1$ | $R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 3/5\, R_2$

$$[U : C] \quad \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$R_1 \to R_1 + 5/3 R_3$ | $R_2 \to R_2 - 5/3 R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 2/3 & 1 & 5/3 \\ 0 & -5 & 0 & -5/3 & 0 & -5/3 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$R_1 \to R_1 + 1/5\, R_2$

$$\downarrow$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 1 & 4/3 \\ 0 & -5 & 0 & -5/3 & 0 & -5/3 \\ 0 & 0 & -3/5 & -1/5 & 3/5 & 1 \end{array}\right]$$

$$R_3 \to -5/3\,R_3 \;\Big\downarrow\; R_2 \to -1/5\,R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/3 & 1 & 4/3 \\ 0 & 1 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 & -1 & -5/3 \end{array}\right]$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - R_1 \;\Big\downarrow\; R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3 \;\Big\downarrow\; R_2 \to R_2 + 5R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & -1 & 0 & -7 & 1 & 5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big|$$

$$\Big\downarrow R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -10 & 2 & 7 \\ 0 & -1 & 0 & -7 & 1 & 5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 \to -R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -10 & 2 & 7 \\ 0 & 1 & 0 & 7 & -1 & -5 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Transpose of a Matrix

- rows & columns interchange

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ -3 & 0 \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 2 & 0 \end{bmatrix}_{2\times 3}$$

Properties

1. $(A^T)^T = A$

2. $(AB)^T = B^T A^T$

3. $(A^{-1})^T = (A^T)^{-1}$

4. $(A \pm B)^T = A^T \pm B^T$

5. $(A^{-1})^T A^T = (AA^{-1})^T = I$

Symmetric MATRIX

- $A^T = A$
- if A is symmetric and A^{-1} exists, A^{-1} is also symmetric
- eg: $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Properties

- $(A^{-1})^T = A^{-1}$
- if A is symmetric and $A = LDU$, then

$$\boxed{A = A^T = LDL^T \quad (\because U = L^T \ \& \ L = U^T)}$$

Q42. Factorise into $A = LDU$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 7 \end{bmatrix} = U$$

$$R_3 \to \tfrac{1}{7}R_3 \,\Big|\, R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U = L^T}$$

Q43. For which 3 no.s 'c' is this matrix not invertible? non-singular

$P_1 \times P_2 \times P_3 = |A|$

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - c/2 R_1} \begin{bmatrix} 2 & c & c \\ 0 & c - c^2/2 & c - c^2/2 \\ 0 & 7-4c & -3c \end{bmatrix} \xrightarrow{R_3 - \frac{7-4c}{c - c^2/2} R_2} \begin{bmatrix} 2 & c & c \\ 0 & c - c^2/2 & c - c^2/2 \\ 0 & 0 & c-7 \end{bmatrix}$$

$$|A| = 2 \times \left(c - \frac{c^2}{2}\right) \times (c-7)$$

$c - 7 \neq 0$

$c - \frac{c^2}{2} \neq 0$

$c\left(1 - \frac{c}{2}\right) \neq 0$

$\boxed{c \neq 7}$ and $\boxed{c \neq 0}$ and $\boxed{c \neq 2}$

Q44. Use Gauss-Jordan Method to find A

$$A = \begin{bmatrix} 1 & a & b \\ 1 & a & 2 \\ 1 & 0 & b \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$[A^{-1} : I] = \left[\begin{array}{ccc:ccc} 1 & a & b & 1 & 0 & 0 \\ 1 & a & 2 & 0 & 1 & 0 \\ 1 & 0 & b & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 0 & 2-b & -1 & 1 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \end{array}\right]$$

$\downarrow R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & a & 0 & \frac{2}{2-b} & \frac{-b}{2-b} & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right] \xleftarrow{R_1 \to R_1 - \frac{b}{2-b} R_3} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

$$\Big\downarrow \; R_1 \longrightarrow R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b/2-b & -b/2-b & 0 \\ 0 & -a & 0 & -1 & 0 & 1 \\ 0 & 0 & 2-b & -1 & 1 & 0 \end{array}\right]$$

$$R_3 \to \frac{R_3}{2-b} \; \Big\downarrow \; R_2 \to -\frac{1}{a} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b/2-b & -b/(2-b) & 0 \\ 0 & 1 & 0 & 1/a & 0 & -1/a \\ 0 & 0 & 1 & -1/2-b & 1/2-b & 0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\frac{b}{2-b} = 1 \Rightarrow b = 2-b$$

$$2b = 2$$

$$\boxed{b = 1}$$

$$1/a = 1 \Rightarrow \boxed{a = 1}$$

Linear Algebra

UNIT-2

VECTOR SPACES

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VIBHA MASTI

Vector Space

- Real vector space V is a non-empty set of objects called vectors satisfying the following axioms

  1. If $u, v \in V$ then $u+v \in V \Rightarrow V$ is closed under vector addition

  2. If $c \in \mathbb{R}$ and $u \in V$, then $cu \in V \Rightarrow V$ is closed under scalar multiplication

PROPERTIES

1. $u+v = v+u$ commutative

2. $u+(v+w) = (u+v)+w$ associative

3. $0+u = u+0 = u$ additive identity

4. For each u there is a unique vector $-u$ such that $u+(-u) = (-u)+u = 0$ inverse

5. $c_1(u+v) = c_1 u + c_1 v$

6. $(c_1+c_2)u = c_1 u + c_2 u$

distributive

7. $1u = u$ multiplicative identity

Examples of vector spaces

1. Set of all real no.s $\mathbb{R}$ associated with addition and scalar multiplication of real no.s (closed under set of all real no.s)

2. Set of all complex no.s $\mathbb{C}$ associated with addition and scalar multiplication of complex no.s

3. Set of all polynomials $R_n(x)$ with real coefficients associated with addition and multiplication of polynomials

4. Set of all vectors of dimension n written as $\mathbb{R}^n$ associated with the addition and scalar multiplication as defined for n-dimensional vectors

5. Set of all matrices of dimension $m \times n$ associated with addition and scalar multiplication as defined for matrices

Q1. Prove that the set of all 2×2 matrices associated with the addition and scalar multiplication of 2×2 matrices is a vector space

1) Addition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$, $A' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}_{2\times2}$ s.t $A, A' \in V_{2\times2}$

$$A + A' = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}_{2\times2} \longrightarrow \text{closed under } 2\times2 \text{ matrices}$$

2) Scalar multiplication

Let $r =$ constant (scalar)

$$rA = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}_{2\times 2}, \quad rA \in V_{2\times 2}$$

3) Commutativity

$$A + A' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$= \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A' + A$$

4) Associativity

$$(A + A') + A'' = A + (A' + A'')$$

$$\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a'+a'' & b'+b'' \\ c'+c'' & d'+d'' \end{bmatrix}$$

$$\begin{bmatrix} a+a'+a'' & b+b'+b'' \\ c+c'+c'' & d+d'+d'' \end{bmatrix} = \begin{bmatrix} a+a'+a'' & b+b'+b'' \\ c+c'+c'' & d+d'+d'' \end{bmatrix}$$

5) Multiplicative associativity

$$r\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = s\left(r\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$r\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix} = s\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

$$\begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix} = \begin{bmatrix} rsa & rsb \\ rsc & rsd \end{bmatrix}$$

6) Zero vector

$$A + 0 = A$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7) Negative vector

$$A + -A = 0$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} a-a & b-b \\ c-c & d-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

8) Distributivity of sum of matrices

$$r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = r\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$= \begin{bmatrix} ra+ra' & rb+rb' \\ rc+rc' & rd+rd' \end{bmatrix} = r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + r\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

9) Distributivity of sums of scalars

$$(r+s)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix}$$

$$= r\begin{bmatrix} a & b \\ c & d \end{bmatrix} + s\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

10) Multiplication by 1

$$1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Q2. Show that the set of all polynomials of degree $n \leq 3$ associated with addition and scalar multiplication form a vector space

the addition of polynomials of degree ≤ 3 results in a polynomial of degree ≤ 3

the multiplication of polynomials of degree ≤ 3 with scalar results in polynomial of degree ≤ 3

$\therefore$ remaining 8 conditions also satisfy

Q3. Show that the set of integers associated with addition and multiplication by a real number is not a vector space

addition of int with real no. is not always an int

eg: $2+\sqrt{5} = 2+\sqrt{5} \notin \mathbb{Z}$

Q4. Verify whether V is a vector space

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; x \geq 0, y \geq 0, x, y \in \mathbb{R} \right\}$$

1) closure – holds good

2) associative – holds good

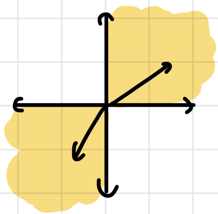
3) zero vector – does not hold

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\notin V$

Q5. Verify if W is a vector space

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} ; xy \geq 0 \right\}$$



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \notin W$$

$\therefore$ not a vector space

Subspace

A nonempty subset of a vector space is called a subspace of V if it itself is a vector space under the same operations (addition, scalar multiplication) as defined in a vector space

example

1) $0 \in W$ (zero vector always belongs to a subspace)

2) if $u, v \in W$, $u+v \in W$

3) if c is scalar and $u \in W$, $cu \in W$

- If U and W are two subspaces of a vector space V, then $U \cap W$ is also a subspace of V
- $0 \in U$ and $0 \in W$ (since they are subspaces)
- Intersection of any number of subspaces of V is a subspace of V

Echelon Form & Row-Reduced Echelon Form

unit 1, pg 4

Pivot Variables and Free Variables

$$Rx = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- columns 1 & 3 have pivots $\Rightarrow$ u & w are pivot variables
- columns 2 & 4 have no pivots $\Rightarrow$ v & y are free variables

RANK OF A MATRIX

- no. of nonzero rows in echelon form U of A, denoted by $\rho(A)$ or r
- unit 1, pg 11

Q6. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} = U$$

$R_2 \to \frac{1}{8} R_2$

$$\downarrow$$

rank $\rho(A) = 2$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

Q7. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3/2 R_1} \begin{bmatrix} 2 & 6 & -2 \\ 0 & -11 & 11 \end{bmatrix} = U$$

$$\Big\downarrow \; R_2 \to -1/_{11} R_2 \;\; R_1 \to 1/2 R_1$$

rank = 2

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xleftarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Q8. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$$

rank = 2

$$\Big\downarrow \; R_3 \to R_3 - R_2$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow[]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow[R_2 \to 1/5 R_2]{R_1 \to 1/2 R_1} \begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Q9. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 3/2 R_1} \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = U \xrightarrow{R_1 \to -1/2 R_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 1$$

Q10. Find (i) echelon form U
(ii) RR echelon form R
(iii) Rank

$$A = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} = U \xrightarrow{R_1 \to -1/2 R_1} \begin{bmatrix} 1 & -3/2 & -1/2 \end{bmatrix}$$

$$\text{rank} = \rho(A) = 1$$

Q11. Solve the following system of LE by identifying pivot and free variables.

$$x + 2y + 3z = 9$$
$$2x - 2z = -2$$
$$3x + 2y + z = 7$$

$$A = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$\rho(A) = \rho(A:b) = 2 < 3$

$\therefore$ consistent, infinite solutions

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free : z

free variables: z

pivot variables: x, y

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ -8 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ 0 \end{bmatrix}$$

Let $z = k$ (free variables)

$-4y - 8k = -20$

$y + 2k = 5$

$y = 5 - 2k$

$x + 2(5 - 2k) + 3k = 9$

$x + 10 - 4k + 3k = 9$

$x = k - 1$

$(x, y, z) = (k-1, 5-2k, k)$, $k \in R$

Q12. Reduce $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix}$ to echelon form

and hence find the special solutions

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

if $c = 1$, $U = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

solving $Rx = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

pivot: x

free: y, z, t

$$x + y + 2z + 2t = 0$$

$$\boxed{x = -y - 2z - 2t}$$

if $t=0$, $y=1$, $z=0$

$$\begin{bmatrix} -y-2z-2t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Q14. Reduce $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ to echelon form, RR echelon

and hence find the special solutions

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_2 \to 1/3 R_2} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$Rx = 0$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

free: y, t
pivot: x, z

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$z + t = 0$$
$$z = -t$$

$$x + 3y + 3z + 2t = 0$$
$$x + 3y - 3t + 2t = 0$$
$$x = t - 3y$$

if $t = 0$, $z = 0$, $x = -3y$

$$\begin{bmatrix} t - 3y \\ y \\ -t \\ t \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

special solutions

Linear COMBINATION

If V is a vector space and $v_1, v_2 \dots v_n$ are vectors in V, then $c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ is a linear combination of the vectors ($c_1, c_2 \dots c_n$ are scalars)

LINEAR INDEPENDENCE of VECTORS

- if the only linear combination of vectors that produces a zero vector is the trivial solution, the vectors are linearly independent
- trivial combination: $0v_1 + 0v_2 + \dots + 0v_n = 0$
- if there exist other nonzero scalars $(c_1, c_2 \dots c_n)$ where at least some constants are nonzero and the linear combination of the vectors produces the zero vector, the vectors are linearly dependent
- vectors are independent if null space = zero vector

Check for dependence

1. Place vectors $\{v_1, v_2 \dots v_n\}$ as columns of matrix A
2. Apply Gaussian elimination on A
3. If pivot exists for every column, the vectors are linearly independent $\text{rank}(A) = n$
4. Else, linearly dependent

Q15. Check if $\begin{bmatrix}1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\7\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$

$$A = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 3\\1 & 2 & 7\\3 & 3 & 1\end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 1 & 6\\0 & 0 & -2\end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 0 & 7\\0 & 0 & -2\end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + \frac{2}{7}R_3} \begin{bmatrix}1 & 1 & 1\\0 & -1 & 1\\0 & 0 & 7\\0 & 0 & 0\end{bmatrix}$$

$\rho(A) = n = 3$

$\boxed{\therefore \text{ linearly independent}}$

Q16. Check whether $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}4\\3\\1\end{bmatrix}$ are independent in $\mathbb{R}^3$

$$A = \begin{bmatrix}1 & 3 & 4\\2 & 1 & 3\\1 & 0 & 1\end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & -3 & -3\end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{3}{5}R_2} \begin{bmatrix}1 & 3 & 4\\0 & -5 & -5\\0 & 0 & 0\end{bmatrix}$$

$\therefore$ linearly dependent

$\rho(A) = 2, \quad n = 3$

note

1. Columns of invertible square matrix are always independent
2. Columns of matrix $A_{m \times n}$ with $m < n$ are always dependent
3. Columns of A are independent when $N(A) = Z$ (null space; Z ← zero)
4. The 'r' nonzero rows of echelon matrix U and reduced matrix R are always independent and so are the 'r' columns that contain the pivots

BASIS

A subset $S = \{v_1, v_2 \dots v_n\}$ of a vector space is called a basis for vector space V if

1. S is a linear independent set
2. S spans the vector space V
3. Every vector in the space can be represented as a linear combination of the basis vectors

The dimension of the vector space is the number of basis vectors

Basis is maximal independent set and minimal spanning set

Unique way to write a vector v as linear combination of basis vectors

examples of basis sets

1. Basis of $\mathbb{R}^2$ —— $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

2. Basis of $\mathbb{R}^3$ —— $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. Basis of 2×2 matrices —— $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

4. Basis of n- degree polynomial —— $\{1, t, t^2, \dots, t^n\}$

Q17. Decide dependence or independence of the following vectors

vectors: $(1,3,2)$, $(2,1,3)$, $(3,2,1)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1/5 R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$$

$-5 + 7/5$

$\rho(A) = n = 3$

independent

Q18. Decide dependence or independence of the following vectors

(1,-3,2), (2,1,-3), (-3,2,1)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 < n = 3$

$\therefore$ linearly dependent

Q19. Decide dependence or independence of the following vectors

(1,1), (2,3), (1,2)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\rho(A) = 2 < n = 3$

$\therefore$ linearly dependent

Q20. If w_1, w_2, w_3 are independent vectors. Show that the differences

$$v_1 = w_2 - w_3$$
$$v_2 = w_1 - w_3 \qquad \text{are dependent}$$
$$v_3 = w_1 - w_2$$

w_1, w_2, w_3 are independent

$$\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \quad \Rightarrow \quad \alpha = \beta = \gamma = 0$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$c_1(w_2 - w_3) + c_2(w_1 - w_3) + c_3(w_1 - w_2) = 0$$

$$w_1(c_2 + c_3) + w_2(c_1 - c_3) + w_3(-c_1 - c_2) = 0$$

$$c_2 + c_3 = 0 \quad \Rightarrow \quad c_3 = -c_2$$

$$c_1 - c_3 = 0 \quad \Rightarrow \quad c_1 = c_3 = -c_2$$

$$-c_1 - c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

$$\boxed{c_1 = -c_2, \quad c_2 = c_2, \quad c_3 = -c_2}$$

$\therefore$ c_1, c_2, c_3 can assume nonzero values also

$\therefore$ v_1, v_2, v_3 are linearly dependent

Q21. Find the basis and hence the dimension of subspaces of R^4

(i) All vectors whose components are equal

(ii) All vectors whose components add up to zero

(i) 4 components define dimension

$$V = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in R^4 \sim \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix}$$

$$V = \left\{ x \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} : x \in R \right\}$$ subspace

basis $= \{ [1 \ 1 \ 1 \ 1]^T \}$

$\therefore$ dimension $= 1$

(ii) $$V = \begin{bmatrix} x \\ y \\ z \\ -x-y-z \end{bmatrix} \in \{x, y, z \in R\}$$

$$V = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} : x, y, z \in R \right\}$$

basic: $\{ [1 0 0 -1]^T, [0 \ 1 0 -1]^T, [0 \ 0 \ 1 -1]^T \}$

$\therefore$ dimension $= 3$

Q22. Let V be a subspace of 4D space R^4 such that

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in R^4 ;\ x_1 - x_2 + x_3 - x_4 = 0 \right\}$$

Find basis and dimensim

$$x_1 = x_2 - x_3 + x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \sim \begin{bmatrix} x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$V = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_2, x_3, x_4 \in R \right\}$$

$$\text{basis} = \{ [1\ 1\ 0\ 0]^T, [-1\ 0\ 1\ 0]^T, [1\ 0\ 0\ 1]^T \}$$

dimensims = 3

Q23. Find a basis for each of the following subspaces of 2x2 matrices

(i) All diagonal matrices
(ii) All symmetric matrices $(A^T = A)$
(iii) All skew symmetric matrices $(A^T = -A)$

2x2 : $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in R$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 4

(i) $V = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, d \in R \right\}$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 2

(ii) $V = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

dimensions = 3

(iii) $V = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

basis $= \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ dimensions $= 1$

Four Fundamental Subspaces of a Matrix

1) $C(A)$ – column space
- subspace of R^m
- linear combinations of column vectors of A
- $\rho(A) = \text{dimensions}(C(A)) = k$
- basis of $C(A)$ corresponds to columns having a pivot in echelon form of A

2) $C(A^T)$ – row space
- subspace of R^n
- linear combination of rows of A
- $\rho(A) = \dim(C(A^T)) = k$
- basis of $C(A^T)$ is set of row vectors in A or in the echelon form, corresponding to the pivots in echelon form

3) $N(A)$ – null space
- all solutions of system $Ax = 0$
- subspace of R^n
- if $\rho(A) = k$ then $\dim(N(A)) = n-k$

4) $N(A^T)$ – left null spaces
- all solutions of system $A^T x = 0$
- subspace of R^n
- if $\rho(A) = k$ then $\dim(N(A^T)) = m-k$
- linear combination of rows that gives 0

1. The row space of Amxn is the column space of A^T.
   It is spanned by the rows of A.

2. The left null space contains all vectors y for which $A^T y = 0$.

3. N(A) and C(AT) are subspaces of R^m

4. N(A^T) and C(A) are subspaces of R^n

5. Dim C(A) = Dim C(AT) = r = rank of A

6. Dim N(A) = n-r and Dim N(AT) = m- r.
7. The dimension of the null space of a matrix is called its nullity.

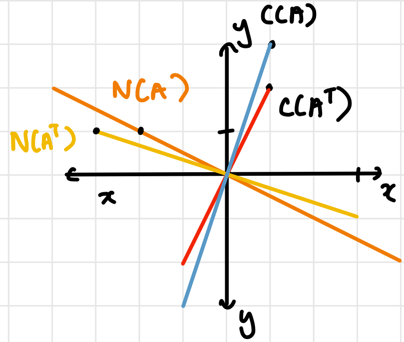
RANK-NULLITY THEOREM

$A_{m\times n}$

- $\dim C(A) + \dim N(A) = n = r + (n-r)$
- $\dim C(A^T) + \dim N(A^T) = m = r + (m-r)$

Q24. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$m = 2, \; n = 2, \; r = 1$



1. C(A) is line through (1,3)

2. $C(A^T)$ is the line through (1,2)

3. $N(A)$ is the line through $(-2,1)$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + 2y = 0$$

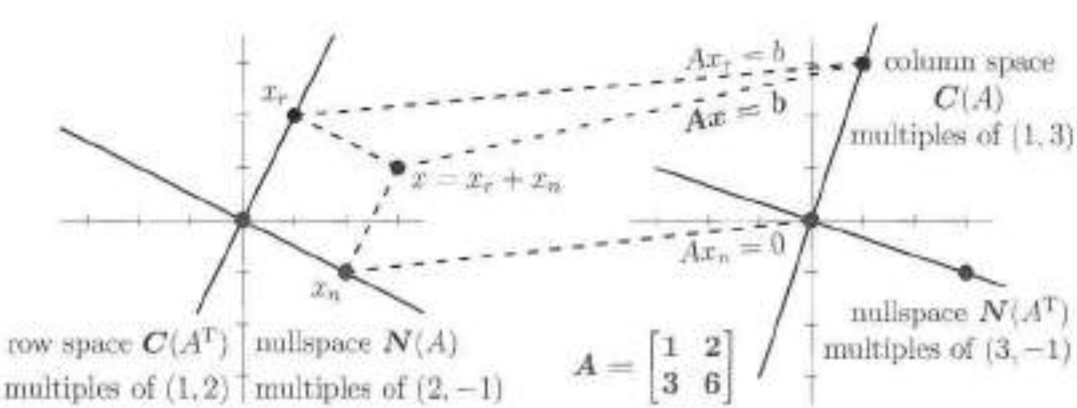
$$x = -2y$$

4. $N(A^T)$ is the line through $(-3,1)$

$$\left[\begin{array}{cc:c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc:c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x + 3y = 0$$

$$x = -3y$$



Q25. Find dimensions and basis for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 rows

$\rho(A) = 2$
$n = 4$

pivot columns 1 & 4

c_1 c_4

1. $C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$

basis: $\{ [1 \;\; 1 \;\; 3]^T, [2 \;\; 3 \;\; 7]^T \}$

dim = 2

2. $C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$

basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$, dim = 2

3. $N(A)$

$$Ax = 0$$

$$[U:0] \quad \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$t = 0$$

$$x + 2y + z = 0$$
$$x = -2y - z$$

special

$$\begin{bmatrix} -2y - z \\ y \\ z \\ 0 \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $y = c_1$ and $z = c_2$

$$N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\dim = 2$$

4. $N(A^T)$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 7 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y + z = 0$ $\qquad$ $x + y + 3z = 0$

let $y = c$ $\qquad$ $x + c - 3c = 0$

$z = -c$ $\qquad$ $x = 2c$

$= (2c, c, -c)$

$$N(A^T) = \left\{ c\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \mid c \in R \right\}$$

$$\text{basis} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

dim $= 1$

Q26. Obtain $N(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1/2 R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$\rho(A^T) = 3$

no zero rows : $N(A^T) = \vec{0}$ (trivial solution)

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$ origin

$$\text{basis} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \dim = 0$$

Q27. Find $N(A^T)$

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

(a) $A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$
$n = 3$

Let $z = k$

$y + 2k = 0$
$y = -2k$

$x - 2k + 2k = 0$
$x = 0$

$$= \begin{bmatrix} 0 \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$N(A^T) = \left\{ k \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \ k \in \mathbb{R} \right\}$$

basis $= \{ [0 \ -2 \ 1]^T \}$, $\dim = 1$

(b) $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$

$$\xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$3y + 6k = 0$ $\qquad$ $x - 4k - k = 0$

$y = -2k$ $\qquad$ $x = 5k$

$$\begin{bmatrix} 5k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \Rightarrow N(B^T) = \left\{ k \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, k \in R \right\}$$

basis $= \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ $\qquad$ dim $= 1$

Q28. Find vector space

(a) $c = [0]$ $\qquad$ (b) $D = [0 \ \ -3]$ $\qquad$ (c) $= \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(a) $C = [0] = Z$ $\qquad$ $[U : b] = [0 \ : \ 0]$

$C^T = [0]$

smallest vector space $C = Z = [0]$

span V

(b) $D = [0 \quad -3]$

$V =$ x-axis in R^1

(c) $= \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $V =$ y-axis in R^1

Q29. Find column space & null space

$$F = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ 3 & 8 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 14 & -1 & -3 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 + 7R_2$$

$$U = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & -24 \end{bmatrix}$$

$$C(F) = \left\{ c_1 \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \middle| \; c_1, c_2, c_3 \in R \right\}, \quad \dim = 3$$

$N(F) \Rightarrow$

$6z - 24t = 0$
$z - 4t = 0$

Let $t = k$

$z = 4k$

$-2y + z - 3t = 0$
$-2y + 4k - 3k = 0$
$-2y + k = 0$
$y = k/2$

$-2x + 2k - 8k - 4k = 0$
$-x + k - 4k - 2k = 0$
$-x - 5k = 0$
$x = -5k$

$$N(F) = \left\{ c \begin{bmatrix} -10 \\ 1 \\ 8 \\ 2 \end{bmatrix}, \; c \in R \right\}, \quad \dim = 1$$

Q30. Find column space and null space. Give an example of a matrix whose col space is same as that of A but the null space is different.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \\ 5 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/7 R_2} \begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} = U$$

column space: columns that have pivots

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \middle| \, c_1, c_2 \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \right\}$$

$$\text{Dim}(A) = 2$$

$N(A) \Rightarrow \quad Ax = 0$

$7y = 0$ $\qquad\qquad$ $x = 0$

$y = 0$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

null space $N(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

origin

column space of B = C(B) = C(A)

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 7 & 14 \\ 5 & 3 & 6 \end{bmatrix}$$

OR

columns

$C_3 = C_1 + C_2$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 7 & 9 \\ 5 & 3 & 8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 3 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3/7 R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

C(B) = C(A)

null space: $Bx = 0$

$7y + 7z = 0$

$x + z = 0$

$y + z = 0$

Let $z = k$

$x = -k$

$y = -k$

$$N(B) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Dim}(B) = 1$$

Q31. Let $V = \{ (a, b, c, d) \mid b + c + d = 0 \}$ and

$W = \{ (a, b, c, d) \mid a + b = 0 \ \& \ c = 2d \}$

be subspaces of R^4. Find $\text{Dim}(V \cap W)$

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \ b + c + d = 0 \right\}$$

$$d = -c - b$$

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ -c-b \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Dim = 3 , 3D plane in R^4

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad a+b=0 \text{ and } c=2d \right\}$$

$$b = -a \qquad c = 2d$$

$$W = \left\{ \begin{bmatrix} a \\ -a \\ 2d \\ d \end{bmatrix} \right\} = \left\{ a\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad a,d \in R \right\}$$

Dim = 2

W is 2D plane in R^4

$V \cap W \Rightarrow$

$$b+c+d=0 \Rightarrow -a+2d+d=0 \Rightarrow a=3d$$
$$a+b=0 \Rightarrow b=-a=-3d$$
$$c=2d$$

$$V \cap W = \left\{ \begin{bmatrix} 3d \\ -3d \\ 2d \\ d \end{bmatrix} \right\} = \left\{ d\begin{bmatrix} 3 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \quad d \in R \right\}$$

$\text{Dim}(V \cap W) = 1$

$V \cap W$ is a 1D line in R^4 spanned by $(3, -3, 2, 1)$

Q32. Find C(A) and N(A) for the following

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 5 & -1 & 6 \\ 2 & 4 & 1 & 2 \\ 2 & 0 & -7 & 11 \end{bmatrix}$

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$

$\Big\downarrow R_3 \to R_3 + R_2$

$U = \begin{bmatrix} -2 & 4 & -2 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & 0 \end{bmatrix}$

$C(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \right\}$

Basis $= \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \right\}$

Dim $= 3$

$N(A) \Rightarrow \quad Ax = 0 \quad$ or $\quad Ux = 0$

$6z = 0 \qquad -2y - 3t = 0 \qquad -2x - 6k + 4k = 0$

$z = 0 \qquad$ let $t = k \qquad -2x - 2k = 0$

$-2y = 3k \qquad x = -k$

$y = \frac{-3}{2}k$

$$N(A) = \left\{ k \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \ k \in \mathbb{R} \right\}$$

$\text{Dim} = 1 \qquad \text{Basis} = \left\{ \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}$

Q33. Reduce to RREF and determine ranks. Identify free & pivot variables.

(i) $A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -\frac{1}{2} & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & \frac{1}{2} & 10 \end{bmatrix} \xrightarrow[R_3 \to 2R_3]{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & -3/2 & -1 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 20 \end{bmatrix}$

$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 + 3/2 R_3} \begin{bmatrix} 1 & 2 & 0 & 29 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 20 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 20 \end{bmatrix}$

rank = 3 free variables = t pivot variables = x, y, z

(ii) $B = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - R_3}$

rank = 3

free variables = z
pivot variables = x, y, t

$$\downarrow R_2 \to R_2 - 7/2 R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q34. Examine if following vectors are linearly independent

$$\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} \right\} \text{ in } M_{2\times 2}(R)$$

linear combination that produces 0 vector

$$c_1 \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 \\ -2c_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & -c_2 \\ c_2 & c_2 \end{bmatrix} + \begin{bmatrix} -c_3 & 2c_3 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 2c_4 & c_4 \\ -4c_4 & 4c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - c_3 + 2c_4 & -c_2 + 2c_3 + c_4 \\ -2c_1 + c_2 + c_3 - 4c_4 & c_1 + c_2 + 4c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} c_1 + 0c_2 - c_3 + 2c_4 &= 0 \\ 0c_1 - c_2 + 2c_3 + c_4 &= 0 \\ -2c_1 + c_2 + c_3 - 4c_4 &= 0 \\ c_1 + c_2 + 0c_3 + 4c_4 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_1 \quad \Big\downarrow \quad R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\Big\downarrow \; R_4 \to R_4 - 3R_3$$

rank $= 3 < 4$

1 free, 3 pivot

$$U = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ vectors are linearly dependent

Q35. Examine if following vectors are linearly independent

$$\{t^2 + t + 2, \; 2t^2 + t, \; 3t^2 + 2t + 2\}$$

$$A = \begin{array}{c} t^2 \\ t \\ 1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -4 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2 < n = 3$

can also do $(2, 1, 1)$

$\therefore$ linearly dependent

existence of inverses

EXISTENCE OF INVERSE FOR RECTANGULAR MATRIX

$[\cdots\cdots]$

- Let $A_{m\times n}$ be a matrix such that $\rho(A) = m$ $(m \le n)$
- Then $Ax = b$ has at least one solution x for every b if and only if the columns span R^m as many
- In this case, A has a right inverse C such that $AC = I_{m\times m}$

$$\boxed{A_{m\times n} C_{n\times m} = I_{m\times m}}$$

$\begin{bmatrix} \vdots \end{bmatrix}$

- Let $A_{m\times n}$ be a matrix such that $\rho(A) = n$ $(n \le m)$
- Then $Ax = b$ has at most one solution x for every b if and only if the columns are linearly independent unique
- In this case, A has a left inverse B such that $BA = I_{n\times n}$

$$\boxed{B_{n\times m} A_{m\times n} = I_{n\times n}}$$

Best Right Inverse

$$C = A^T (AA^T)^{-1}$$

Best Left Inverse

$$B = (A^T A)^{-1} A^T$$

Q36. Find right inverse of A

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$$

$$m = 2 \quad n = 3$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ a & b \end{bmatrix}_{3\times 2} = C \leftarrow \text{infinitely many}$$

$$AC = I_{2\times 2}$$

Q37. Find left/right inverse for the matrix (whichever exists)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Apply GA and obtain rank

Step 2: Check if $\rho = m$ or $\rho = n$

Step 3: if $\rho(A) = m$ then RI exists

$$RI = A^T (AA^T)^{-1}$$

if $\rho(A) = n$ then LI exists

$$LI = (A^T A)^{-1} A^T$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$\rho = 2 = m \Rightarrow \text{RI exists}$$

$$RI = C = A^T (A A^T)^{-1}$$

inverse of 2×2 matrix A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A \cdot A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left(\frac{1}{3}\right)$$

$$A^T (A \cdot A^T)^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}_{2\times 2} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}_{3\times 2}$$

$$C = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

Q38. Find the inverse of A

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$m = 3 \qquad n = 2$

$\rho = 2 = n \Rightarrow$ left inverse exists

$$B = (A^T \cdot A)^{-1} A^T$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left(\frac{1}{3} \right)$$

$$(A^T \cdot A)^{-1} \cdot A^T = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}_{2 \times 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2 \times 3}$$

$$B = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}_{2 \times 3}$$

MATRICES of RANK 1

- every row is a multiple of the first row
- can write matrix as the product of a column vector and a row vector
- $A = uv^T$

Q39. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ as column × row

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

Q40. Which vectors (b_1, b_2, b_3) are in $C(A)$? Which combination of rows of A gives 0? → $N(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\} \qquad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{Dim}(C(A)) = 2$$

$N(A^T) \Rightarrow$ use $[A : b]$

$$[A:b] = \left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \end{array}\right]$$

$$\Big\downarrow R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & -5 & b_3 - 2b_1 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$$

$N(A^T)$ = linear combination of zero rows

$$N(A^T) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad b_2 = 0$$

Q41. Vectors $(1,4,2)$, $(2,5,1)$, $(3,6,0)$

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Find $N(A^T)$, $C(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{from vectors}$$

$$A = \left[\begin{array}{ccc:c} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 2 & 1 & 0 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc:c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -3 & -6 & b_3 - 2b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc:c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right]$$

$$b_3 - b_2 + 2b_1 = 0$$

$$2b_1 - b_2 + b_3 = 0$$

$$N(A^T) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad \text{Dim}(N(A^T)) = 1$$

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \qquad \text{Dim}(C(A^T)) = 2$$

Q42. Reduce the following to RREC and determine their ranks

(a) Identify pivot & free vars

(b) Find special solutions to $Ax = 0$

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & 3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3/2 R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 6 & 0 \end{bmatrix} \xrightarrow[R_2 \to -\frac{1}{2} R_2]{R_1 \to -\frac{1}{2} R_1} \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & -1/2 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 \to \frac{1}{6} R_3$

$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + 1/2 R_3} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} = R$

pivots: x, y, z

free: t

$N(A) \Rightarrow$

$\boxed{z = 0}$

$y + \frac{3}{2} t = 0$

$\boxed{y = -3/2\, t}$

$x \quad + 3t \ + 2t = 0$

$\boxed{x = -5t}$

$N(A) = \left\{ t \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \middle| \ t \in R \right\}$

$N(A) = \left\{ t \begin{bmatrix} -10 \\ -3 \\ 0 \\ 2 \end{bmatrix} \middle| \ t \in R \right\}$

(ii) $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 4 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\downarrow R_3 \to R_3 - R_1$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -3/2 & -7/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_3 \to R_3 - 3/2 R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_3 \to -2/3 R_3 \downarrow R_2 \to 1/2 R_2$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 1 & 7/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 7/2 R_4 \\ R_1 \to R_1 - R_4}]{R_3 \to R_3 - 7/3 R_4} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 \to R_1 - R_3 \downarrow R_2 \to R_2 - 3/2 R_3$

pivots: x, y, z, t
no free

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\downarrow R_1 \to R_1 - R_2$

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (origin)

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{4 \times 4}$

Q43. For which vectors $b = (a \; b \; c)$ does the following system $Ax = b$ has a solution

$$x + 2y = a$$
$$x + y + 2z = b$$
$$3x - 4z = c$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 1 & 1 & 2 & b \\ 3 & 0 & -4 & c \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 0 & -1 & 2 & b-a \\ 0 & -6 & -4 & c-3a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 6R_2} \left[\begin{array}{ccc:c} 1 & 2 & 0 & a \\ 0 & -1 & 2 & b-a \\ 0 & 0 & -16 & 3a-6b+c \end{array}\right]$$

$$\rho(A) = 3 = n$$

consistent, unique

for all values of a, b, c solution exists

Q44. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $\quad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Determine if w is in $C(A)$, $N(A)$

$$A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} -6 & 12 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} -6 \\ -3 \end{bmatrix}, \; c_1 \in R \right\}$$

for $c_1 = -1/3$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = w$

$\therefore w \in C(A)$

$N(A) \Rightarrow$ $-6x + 12y = 0$

$$x = 2y$$

$$N(A) = \left\{ \begin{bmatrix} 2y \\ y \end{bmatrix} \right\}$$

$$= \left\{ y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$N(A) = \left\{ k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

for $k = 1$, $V = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = w$

$\therefore w \in N(A)$

Linear Algebra

UNIT-3

LINEAR TRANSFORMATIONS & ORTHOGONALITY

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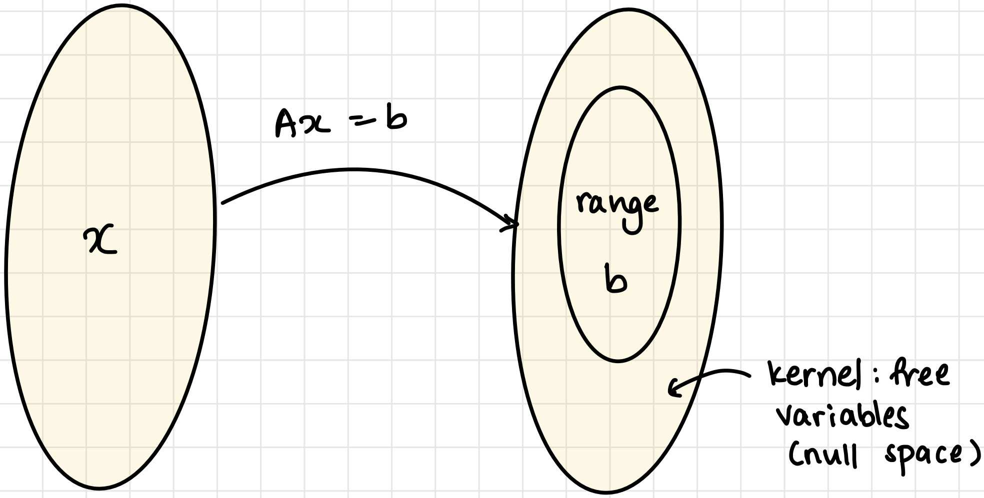
VIBHA MASTI

LINEAR TRANSFORMATIONS

- $f : A \to B$ defined by $f(x) = y$
- $A : \mathbb{R}^n \to \mathbb{R}^m$ (mapping / function)



domain codomain

$C(A)$ = range

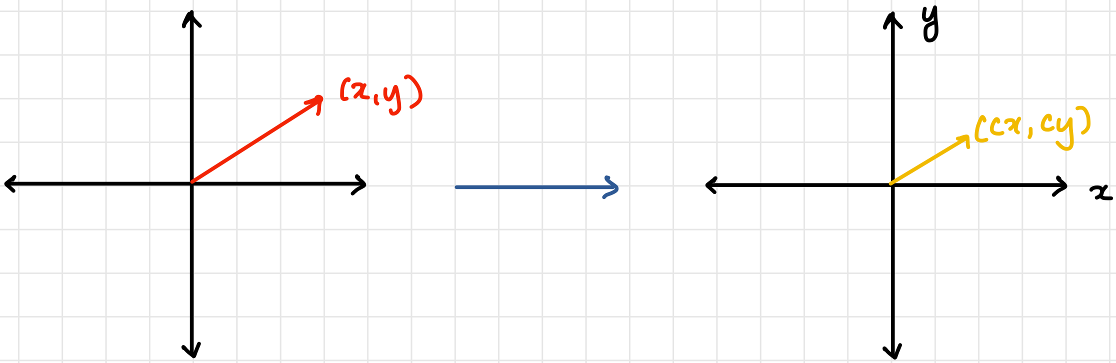
$N(A)$ = free variables / kernel area

Examples

1. $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ $\quad x = (x, y)$ stretching

$$Ax = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

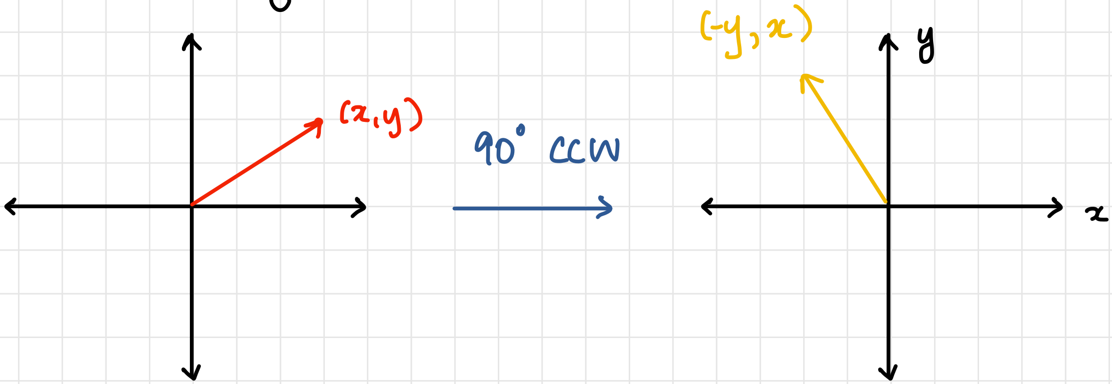
- A multiple of identity matrix $A = cI$ stretches every vector by the scale factor c
- Whole vector space expands or contracts



2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $x = (x, y)$ Rotation

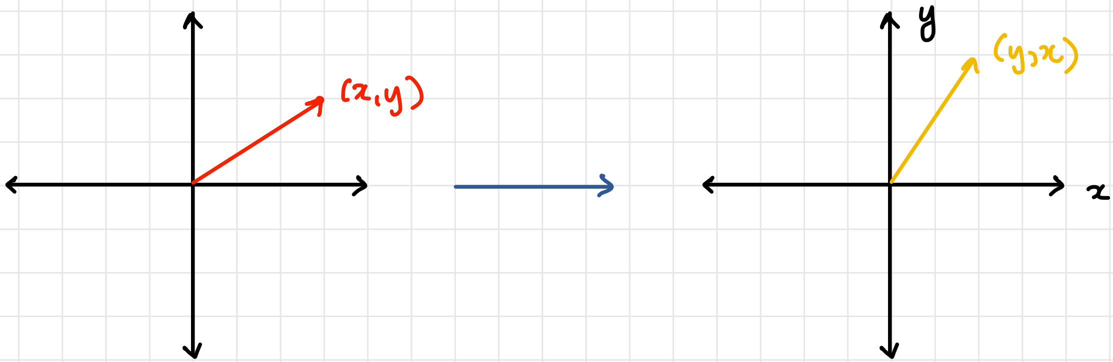
$$Ax = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

- Rotate by $90°$ CCW



3. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$ Reflection

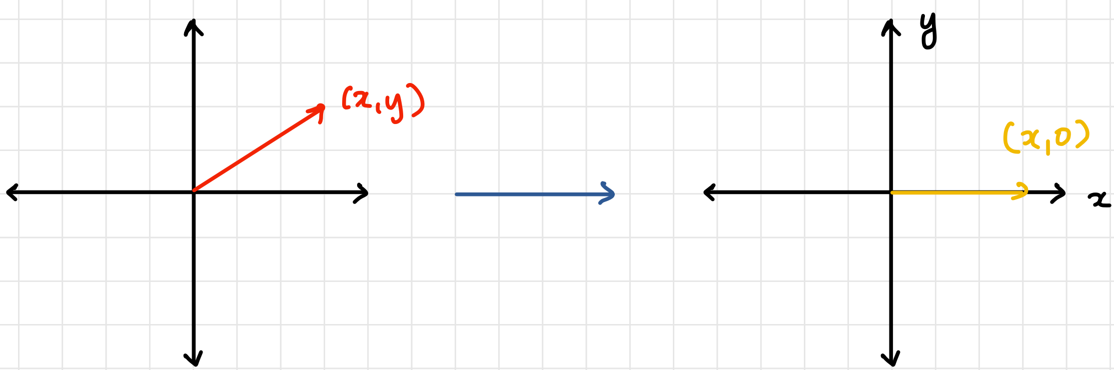
$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



- Reflection over $y=x$ ($45°$ mirror)

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $x = (x \quad y)$ Projection

$$Ax = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$



- Projection on x-axis

General Matrix to Rotate by Angle θ

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

1. $\theta = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $\theta = \pi/2$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3. $\theta = \pi$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

RULE of LINEARITY

- A transformation T on R^n is said to be linear if

$$T(cx + dy) = c\,T(x) + d\,T(y)$$

- Preserves origin

Polynomial Space

Space of all polynomials in t of degree n is a vector space denoted by P_n

$$P_n = c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n \mid c_i \in R\}$$

Basis = $(1 \quad t \quad t^2 \quad \ldots \quad t^n)$

Dimension = $n+1$

Examples

1. Differentiation

$A = \frac{d}{dt}$ is linear

- $P_{n+1} \longrightarrow P_n$
- $C(A)$ = all of P_n (n-D area)
- $N(A) = P_0$ (1-D space of all constants)

2. Integration

$A = \int_0^t$ is linear

- $P_n \longrightarrow P_{n+1}$
- $C(A)$ = subspace of P_{n+1}
- $N(A) = Z$

3. Multiplication by Fixed Polynomial

- $A = (3+4t)$
- $A P_n = (3+4t) P_n$

Representation of Polynomial Transformations in Matrix Form

Q1. Construct a matrix associated with differentiation of a polynomial

$$P_3 \rightarrow P_2$$

Basis $(P_3) = (1 \quad t \quad t^2 \quad t^3)$

Basis $(P_2) = (1 \quad t \quad t^2)$

$$\frac{d}{dt}(1) = 0 = 0(1) + 0(t) + 0(t^2)$$

$$\frac{d}{dt}(t) = 1 = 1(1) + 0(t) + 0(t^2)$$

$$\frac{d}{dt}(t^2) = 2t = 0(1) + 2(t) + 0(t^2)$$

$$\frac{d}{dt}(t^3) = 3t^2 = 0(1) + 0(t) + 3(t^2)$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\frac{d}{dt}1 . \quad 0 \rightarrow 0 1 + 0t + 0t^2$$

$$\begin{array}{c} A \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4} \\ \begin{array}{cccc} 1 & t & t^2 & t^3 \end{array} \end{array} \overset{x}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}_{4\times 1} = \overset{b}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

$$P(t) = \sqrt{7} - 2\sqrt{3}\,t + 1.78\,t^2 + \sqrt{5}\,t^3$$

$$x = \begin{bmatrix} \sqrt{7} \\ -2\sqrt{3} \\ 1.78 \\ \sqrt{5} \end{bmatrix}$$

free variable (1)

$$\frac{d}{dt}(P(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{7} \\ -2\sqrt{3} \\ 1.78 \\ \sqrt{5} \end{bmatrix} = \begin{bmatrix} -2\sqrt{3} \\ 3.56 \\ 3\sqrt{5} \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$\text{Basis}(C(A)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

$$\text{Dim}(C(A)) = 3$$

$$N(A) = \left\{ k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad \text{Dim}(N(A)) = 1 \qquad n - r = 1$$

The multiplication of the matrix A by the polynomial P(t) always yields the derivative of the polynomial.

A is called the differentiation matrix

Q2. Construct a matrix associated with the integration of a polynomial

$$P_2 \longrightarrow P_3$$

$$\text{Basis}(P_2) = \{1, t, t^2\}$$

$$\text{Basis}(P_3) = \{1, t, t^2, t^3\}$$

$$A: \int_0^t \quad dt$$

$$\int_0^t 1\, dt = t = 0(1) + 1(t) + 0(t^2) + 0(t^3)$$

$$\int_0^t t\, dt = \frac{t^2}{2} = 0(1) + 0(t) + \frac{1}{2}(t^2) + 0(t^3)$$

$$\int_0^t t^2\,dt = \frac{t^3}{3} = 0(1) + 0(t) + 0(t^2) + \frac{1}{3}(t^3)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}$$

$$P(t) = 3 + 4t - 6t^2$$

$$\overset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}} \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ -2 \end{bmatrix}_{4\times 1}$$

$$C(A) = \left\{ c_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \right\}$$

$$\text{Dim}(C(A)) = 3$$

$$N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad Dim(N(A)) = 0 \qquad n - r = 0$$

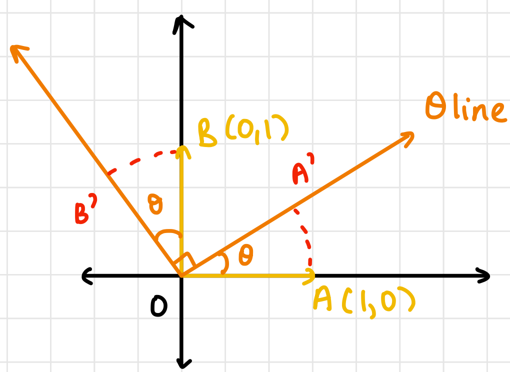
no free variable

Note:
Differentiation is the left inverse of integration

Representation of Transformations in Matrix Form

1. Rotation Q_θ in R^2

- $OA(1,0)$ and $OB(0,1)$ are basis vectors
- Consider rotation Q_θ of the basis vectors by an angle θ in the CCW direction
- Let $A(1,0)$ and $B(0,1)$ be moved to A' and B' respectively
- The new basis vectors are now OA' and OB'
- The coordinates of new bases wrt old bases



$A' = (OA' \cos\theta,\ OA' \sin\theta)$

$= (\cos\theta, \sin\theta)$

$B' = (OB' \cos(90+\theta),\ OB' \sin(90+\theta))$

$= (-\sin\theta\ ,\ \cos\theta)$

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$|Q_\theta| = 1$

inverse of 2x2 matrix A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Q_θ is non-singular and hence, invertible

$$Q_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(Q_\theta^{-1})^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = Q_\theta$$

Rotation by same angle twice

- $Q_\theta \cdot Q_\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = Q_{2\theta}$$

Rotation by 2 angles

- $Q_\theta \cdot Q_\phi$

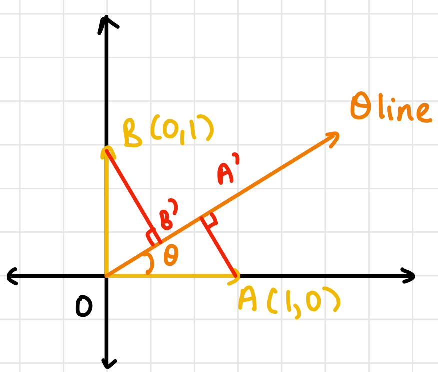
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

2. Projection P in R^2

- Consider projection of R^2 onto θ-line
- Let $A(1,0)$ and $B(0,1)$ get projected onto the theta line as A' and B' respectively



$A' = (OA' \cos\theta, OA' \sin\theta)$ $\quad$ $B' = (OB' \cos\theta, OB' \sin\theta)$

$= (\cos^2\theta, \cos\theta \sin\theta)$ $\quad$ $= (\sin\theta\cos\theta, \sin^2\theta)$

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$|P| = 0$

- P is singular and non-invertible
- There is no way to get original coordinates from the projection (infinitely many)

Projection followed by projection onto same θ line

- $P \cdot P$

Let $c = \cos\theta$
$s = \sin\theta$

$$\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix}$$

$$= \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix}$$

$c^2 + s^2 = 1$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

- $P^n = P \quad n = 1, 2, 3, \ldots$

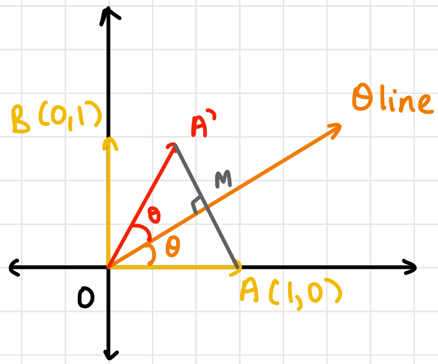
  Projecting any number of times = projecting once

Transpose of P

- $P^T = P$
- $\therefore C(P) = C(P^T)$
- Matrix P is symmetric

3. Reflection H in R^2

- Consider reflection in R^2 on θ-line
- Let A' be the reflection of A on θ line
- Let M be the midpoint of AA'. It is the projection of A on the θ-line



- Consider ΔOAM

$$\vec{OA} + \vec{AM} = \vec{OM} \quad \text{———} \quad (1)$$

- Consider $\Delta OA'M$

$$\vec{OA'} + \vec{A'M} = \vec{OM} \quad \text{———} \quad (2)$$

(1) + (2)

$$\vec{OA} + \vec{OA'} + \underbrace{\vec{AM} + \vec{A'M}}_{\vec{AM} + \vec{A'M} = \vec{0}} = 2\vec{OM}$$

(same magnitude, different directions)

$$\vec{OA} + \vec{OA'} = 2\vec{OM}$$

$\vec{OM}$ is projection of OA on the θ-line

$$\vec{x} + H \cdot \vec{x} = 2P \cdot \vec{x}$$

$$\vec{x}\,(I + H \cdot I) = \vec{x}\,(2P \cdot I)$$

Drop $\vec{x}$

$$I + H = 2P$$

$$H = 2P - I$$

$$H = 2\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$|H| = -1$$

- H is non-singular and $\therefore$ invertible

Double Reflection

- $H \cdot H$

$$(2P - I)(2P - I)$$

$$= 4P^2 - 4PI + I^2$$

$$= (4P - 4P + I)$$

$$H^2 = I$$

$$H^{2n} = I$$

Q3. Suppose T is the reflection about 45° line and S is the reflection about Y axis, find in general ST and TS

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$T = \begin{bmatrix} \cos(2\times45^\circ) & \sin(2\times45^\circ) \\ \sin(2\times45^\circ) & -\cos(2\times45^\circ) \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} \cos(2\times90^\circ) & \sin(2\times90^\circ) \\ \sin(2\times90^\circ) & -\cos(2\times90^\circ) \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$ST = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$TS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(ST)^T = TS$$

Q4. Find the matrix of the linear transformation T on $\mathbb{R}^3$ defined by $T(x,y,z) = (2y+z, x-4y, 3x)$ wrt

i) The standard basis

ii) The basis $\{(1,1,1), (1,1,0), (1,0,0)\}$

i) $T(1,0,0) = (0,1,3)$

$T(0,1,0) = (2,-4,0)$

$T(0,0,1) = (1,0,0)$

$$T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

ii) $T(1,1,1) = (3,-3,3)$ with standard bases

$$\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ using new bases

$c_1 + c_2 + c_3 = 3$

$c_1 + c_2 = -3$

$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -6} \Rightarrow \boxed{c_3 = 6} = \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}$

$T(1,1,0) = (2,-3,3)$ with standard bases

$$\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + c_2 + c_3 = 2$$

$$c_1 + c_2 = -3$$

$$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -6} \Rightarrow \boxed{c_3 = 5} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$$

$T(1,0,0) = (0,1,3)$ wrt standard bases

$$\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ wrt new basis

$$c_1 + c_2 + c_3 = 0$$

$$c_1 + c_2 = 1$$

$$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -2} \Rightarrow \boxed{c_3 = -1} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$T = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

NOTE

$$T \neq \begin{bmatrix} 3 & 2 & 0 \\ -3 & -3 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$ as we did not change final coordinates to new basis

Q5. For each of the following LTs T, find the bases and dimension of the range and kernel of T

range ↳ column space

kernel ↳ null space

(i) $T: R^2 \longrightarrow R^3$ given by

$T(x,y) = (x+y,\ x-y,\ y)$

$(1,0) \to (1,1,0)$

$(0,1) \to (1,-1,1)$

(ii) $T: R^2 \longrightarrow R^2$ given by

$T(x,y) = (y, 0)$

$x \quad y$

$(1, 0) \to (0,0)$

$(0,1) \to (1,0)$

i) bases of $R^2 = \{ (1,0), (0,1) \}$

bases of $R^3 = \{ (1,0,0), (0,1,0), (0,0,1) \}$

$$T(1,0) = (1, 1, 0)$$

$$T(0,1) = (1, -1, 1)$$

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 1/2 R_2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} = U$$

$$\rho(T) = 2 = n$$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{basis}(C(T)) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\mathrm{Dim}(C(T)) = 2$ D plane in R^3

$$N(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \because \; n - r = 0$$

$$D(N(T)) = 0$$

(ii) $T(1,0) = (0,0)$

$T(0,1) = (1,0)$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad\downarrow\quad$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad \rho(T) = 1 \qquad n = 2$

$C(T) = \left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ c \in R \right\}$

basis $(C(T)) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\text{Dim}(C(T)) =$ 1-D line in R^2

$N(T) = Tx = 0$

$y = 0$

$N(A) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ $\qquad \text{Dim}(N(A)) = 1$

$N(A) = \left\{ k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ k \in R \right\}$

Q6. Construct a matrix that transforms (1,0) to (3,5) and (0,1) to (2,4). Also find the matrix that helps to come back to the original bases (inverse)

$T(1,0) = (3,5)$

$T(0,1) = (2,4)$

$$T = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \Rightarrow T^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}$$

$(1\ 0\ 0) \to (1,0)$
$(0\ 1\ 0) \to (1,1)$
$(0\ 0\ 1) \to (0,-1)$

Q7. For each of the following LTs find a basis and dimension of the range and kernel of T

range $\to$ C(A)   kernel $\to$ N(A)

(i) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x+y, y-z)$

(ii) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+2y, 2x-y)$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(i) $T(1,0,0) = (1,0)$
$T(0,1,0) = (1,1)$
$T(0,0,1) = (0,-1)$

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

free: 1
pivot: 2

$\rho(T) = 2$ $n = 3$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{Dim}(C(T)) = 2$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$N(T) = \quad Tx = 0$$

$$y - z = 0 \qquad x + y = 0$$

$$y = z \qquad x = -z$$

$$N(T) = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = \left\{ z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mid z \in R \right\}$$

$$\text{Dim}(N(T)) = 1$$

$$\text{Basis}(N(T)) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$C(T^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \,\middle|\, c_1, c_2 \in R \right\}$$

$$N(T^T) = \quad T^T x = 0$$

$$T^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\rho(T^T) = 2 \qquad n = 2$$

$$\therefore N(T^T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(ii) $T(x,y) = (x+2y, 2x-y)$

$T(1,0) = (1, 2)$
$T(0,1) = (2, -1)$

$$T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$$

$$\rho(T) = 2 = n$$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{Basis} : \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\text{Dim}(C(T)) = 2$$

$$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Dim}(N(T)) = 0$$

Q8. Find the matrix of LT T on R^3 defined by

$T(x, y, z) = (x + 2y + z,\ 2x - y,\ 2y + z)$ wrt

i) Standard basis vectors

ii) Basis: $\{(1,0,1), (0,1,1), (0,0,1)\}$

1 0 0

(i) $T(1,0,0) = (1, 2, 0)$

$T(0,1,0) = (2, -1, 2)$

$T(0,0,1) = (1, 0, 1)$

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(ii) $T(1,0,1) = (2,2,1)$ using old basis

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 2}$

$c_1 + c_2 + c_3 = 1$

$\boxed{c_2 = 2}$

$\boxed{c_3 = -3}$

$T(0,1,1) = (3,-1,3)$

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 3}$ $\boxed{c_2 = -1}$ $c_1 + c_2 + c_3 = 3$

$\boxed{c_3 = 1}$

$T(0, 0, 1) = (1, 0, 1)$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 1}$ $\boxed{c_2 = 0}$ $c_1 + c_2 + c_3 = 1$

$\boxed{c_3 = 0}$

$$T = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

Q9. Let T be LT that sends each matrix x to Ax where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $V \rightarrow$ set of all 2×2 real matrices.

$x \in V$

Find the matrix that represents T

$T(x) = Ax$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in R \right\}$$

$T:\ V$ to V

$$\text{Basis} = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$x = AV = Av_1 + Av_2 + Av_3 + Av_4$$

$$1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = (1, 0, 1, 0)$$

$$Av_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = (0, 1, 0, 1)$$

$$Av_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = (1, 0, 1, 0)$$

$$Av_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = (0, 1, 0, 1)$$

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

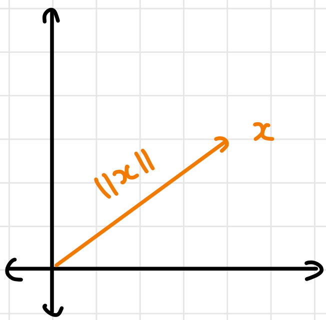
ORTHOGONAL VECTORS

– 2 vectors x & y are orthogonal in the plane spanned by x & y if they form right triangle

(1) Norm – length of vector
(2) Inner Product
(3) Orthogonal Subspaces

(1) NORM

- Length of a vector
- Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$



- $\|x\| \longrightarrow$ norm x

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2} \geq 0$$

$$\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 \geq 0$$

$$= x_1 \cdot x_1 + x_2 \cdot x_2 + \dots + x_n \cdot x_n \geq 0$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$(\|x\|)^2 = x^T x$$

Note:

- $||x||$ is the distance of the point from the origin
- $||x|| = 0$ iff $x = \vec{0}$

(2) INNER PRODUCT

- Let $x = (x_1 \quad x_2 \quad \ldots \quad x_n)$
- Let $y = (y_1 \quad y_2 \quad \ldots \quad y_n)$

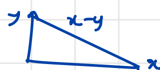
$\Big\}$ 2 n-dim vectors

- $\langle x, y \rangle = x^T y = y^T x = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

Properties of Inner Product

1. if $\langle x, y \rangle > 0$, angle between x & y is acute
2. if $\langle x, y \rangle < 0$, angle between x & y is obtuse
3. if $\langle x, y \rangle = 0$, x and y are orthogonal

Can prove from norm



$||x-y||^2 = ||x||^2 + ||y||^2$

(a) $\vec{0}$ is the only vector orthogonal to itself

$\langle 0, 0 \rangle = 0$ or $0^T 0$

(b) $\vec{0}$ is the only vector orthogonal to every other vector

$\langle 0, x \rangle = 0 \quad \forall x$ or $0^T x = 0$

(c) $\vec{0}$ is the only vector whose length is $\vec{0}$

(3) ORTHOGONAL SUBSPACES

- Let V be a vector space and S and T be subspaces of V
- We can say that S and T are orthogonal to each other if

$$x^T y = 0 \quad \begin{array}{l} \forall x \in S \\ \forall y \in T \end{array}$$

- In other words, every vector s in S is orthogonal to every vector t in T

__Examples__

(a) $V = \{0\}$, $S = \{0\}$, $T = \{0\}$

$$\langle S, T \rangle = 0$$

(b) $V = \mathbb{R}^1$, $S = \{0\}$, $T =$ subspace of $\mathbb{R}^1$

$$\langle S, T \rangle = 0$$

Note:

1. If $\text{Dim}(V) = n$, then $\text{Dim}(S) + \text{Dim}(T) \leq n$
2. If S and T are orthogonal, then $S \cap T = \vec{0} = Z$

THEOREM 1

If nonzero vectors $v_1, v_2, v_3 \ldots v_n$ are mutually orthogonal, then these vectors are linearly independent

Mutually orthogonal: $v_i^T v_j = 0 \;\; \forall \; i \neq j$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = 0 \longrightarrow (1)$$

To prove: $c_i = 0 \;\; \forall \; i$

Multiply (1) by v_1^T

$$c_1 v_1^T v_1 + \cancelto{0}{c_2 v_1^T v_2} + \cancelto{0}{c_3 v_1^T v_3} + \ldots + \cancelto{0}{c_n v_1^T v_n} = 0$$

$$c_1 \; \|v_1\| = 0$$

$$\|v_1\| \neq 0 \quad \text{(nonzero vector)}$$

$$\therefore c_1 = 0$$

Similarly, for all v_i^T,

$$c_i = 0$$

Generally,

$$\therefore \; c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n = 0$$

$$\text{iff} \quad c_i = 0 \;\; \forall \; i$$

THEOREM 2 : FUNDAMENTAL THEOREM OF ORTHOGONALITY

Let A be a matrix of order $m \times n$, then

1) $C(A^T)$ and $N(A)$ are orthogonal subspaces in R^n

2) $C(A)$ and $N(A^T)$ are orthogonal subspace in R^m

Proof

1) Suppose x is a vector in the null space. Then $Ax = 0$ and system of m equations can be written as:

$$Ax = \begin{bmatrix} \cdots & \text{row } 1 & \cdots \\ \cdots & \text{row } 2 & \cdots \\ & \vdots & \\ \cdots & \text{row } m & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Row 1 is orthogonal to x (inner product = 0)
- Every row is orthogonal to x
- x orthogonal to every combination of rows
- Each x in the null space is perpendicular to each vector in the row space

$$N(A) \perp C(A^T)$$

2) Suppose y is a vector in the left null space. Then $A^T y = 0$ system of m equations can be written as:

$$A^T y = \begin{bmatrix} \cdots \text{column } 1 \cdots \\ \vdots \\ \cdots \text{column } n \cdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = 0$$

- Every column is orthogonal to y
- y is orthogonal to every combination of columns

$$N(A^T) \perp C(A)$$

Orthogonal Complement

Let V be a vector space. The set of all vectors orthogonal to every vector in V is called orthogonal complement of V

$$V^\perp \rightarrow V \text{ perp}$$

$\therefore$ the largest set of vectors becomes the orthogonal complement

eg: XOY is the complement of z

THEOREM 3: FUNDAMENTAL THEOREM OF LINEAR ALGEBRA, PT 2

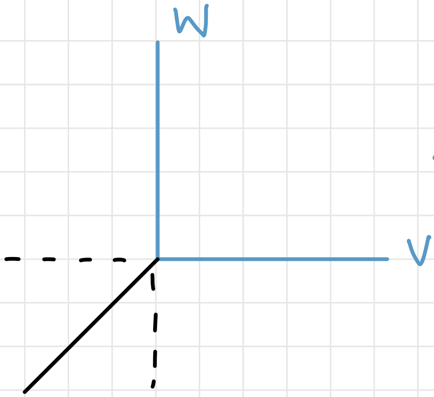
Let A be a matrix of order $m \times n$

1) $C(A^T)$ = complement of $N(A)$ in R^n

2) $C(A)$ = complement of $N(A^T)$ in R^m

They are orthogonal and complementary subspaces

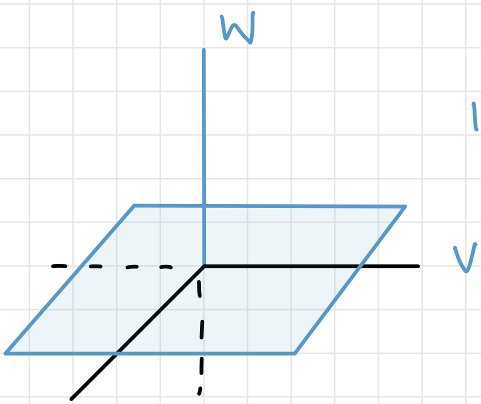
ORTHOGONAL BUT NOT ORTHOGONAL COMPLEMENTS



$W \perp V$ in $\mathbb{R}^3$

two orthogonal axes

ORTHOGONAL COMPLEMENTS



line $W \perp$ to plane V in $\mathbb{R}^3$

$V^\perp = W$ and $W^\perp = V$

Properties

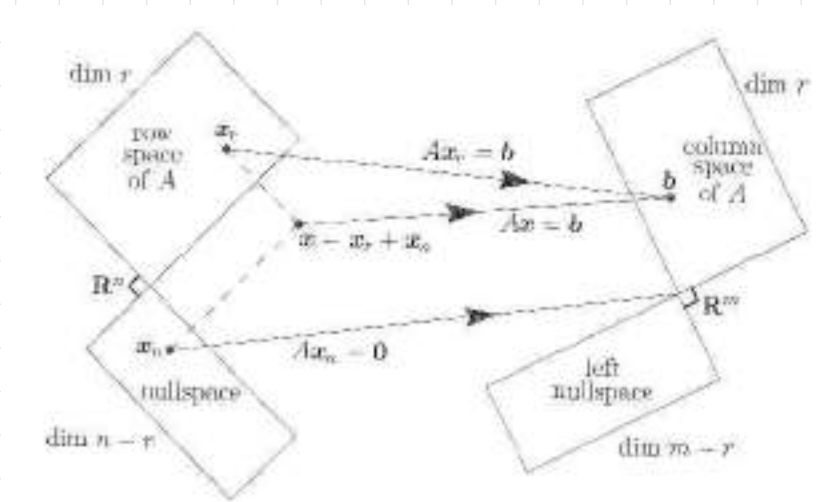
(i) $(V^{\perp})^{\perp} = V$

(ii) If $V^{\perp} = W$, $W^{\perp} = V$

(iii) If V and W are orthogonal complements in $\mathbb{R}^n$, then

$$\dim(V) + \dim(W) = n = \dim \mathbb{R}^n$$

The following figure summarises the effect of matrix multiplication



- Everything from row space goes to column space
- Everything from null space goes to origin
- Splitting $\mathbb{R}^n$ into orthogonal parts V and W will split every vector into $x = v + w$
  - vector v is projection of x onto subspace V
  - orthogonal component w is the projection of x onto W

- The true effect of matrix multiplication is that
  - every Ax is in column space
  - null space goes to 0
  - row space component goes to column space
  - nothing is carried to left null space

- Every Ax transforms row space to column space

Q10. Find the lengths and inner product of $x = (1,4,0,2)$ and $y = (2,-2,1,3)$

$$\|x\| = \sqrt{1+16+4} = \sqrt{21} \qquad \|y\| = \sqrt{4+4+1+9} = \sqrt{18}$$

$$\langle x, y\rangle = x^T y = 2-8+0+6 = 0$$

Q11. Which pairs of vectors are orthogonal?

$v_1 = (1,2,-2,1)$ $v_2 = (4,0,4,0)$ $v_3 = (1,-1,-1,-1)$ $v_4 = (1,1,1,1)$

$\langle v_1 v_2\rangle = 4+0-8+0 = -4$
$\langle v_1 v_3\rangle = 1-2+2-1 = 0$ ✓
$\langle v_1 v_4\rangle = 1+2-2+1 = 2$
$\langle v_2 v_3\rangle = 4+0-4+0 = 0$ ✓
$\langle v_2 v_4\rangle = 4+0+4+0 = 0$
$\langle v_3 v_4\rangle = 1-1-1-1 = -2$

Q12. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$ be a matrix.

(a) Find a vector x which is orthogonal to row space of A

(b) Find a vector y which is orthogonal to column space of A

(c) Find a vector z which is orthogonal to null space of A

(a) Null space : $Ax = 0$ (Null space $\perp$ row space)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = 0$ $\qquad x + 2y = 0$

let $y = k$ $\qquad x = -2k$ $\qquad z = 0$

$$N(A) = \left\{ \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} \middle| k \in R \right\}$$

$$N(A) = \left\{ k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \middle| k \in R \right\}$$

$\dim(A) = 1$ $\qquad$ basis $= \{ (-2 \quad 1 \quad 0) \}$

(b) Left null space (Column space ⊥ Left null space)

$$A:b = \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 4 & 3 & b_2 \\ 3 & 6 & 4 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]$$

$$N(A^T) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

or

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$y + z = 0$

Let $z = k$

$y = -k$

$x + 2y + 3z = 0$

$x - 2k + 3k = 0$

$x + k = 0$

$x = -k$

$$N(A^T) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(c) Vectors with pivot variables ⇒ row space

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{or} \quad z = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

ORTHONORMALITY

Set of nonzero vectors are said to be orthonormal if

(i) $v_i^T v_j = 0 \;, \; i \neq j$

(ii) $\|v_i\| = 1 \quad \forall \, i$

$$v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In other words, $v_i^T v_j = 0$

Eg:

(i) $(1/\sqrt{2}, 1/\sqrt{2}), \; (1/\sqrt{2}, -1/\sqrt{2})$

Note:

The coordinate vectors i.e. the vectors that lie on the x-axis, are orthonormal in R^n.

In particular, if $e_1 = (1,0), e_2 = (0,1)$ are orthonormal in R^2

If the vectors are rotated through θ, then the the new vectors $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$ are also orthonormal

Q13. Find all vectors in R^3 that are orthogonal to a (1,1,1) and b(1,-1,0). Construct an orthonormal basis from these vectors

- 2 vectors form plane
- Find line perpendicular to plane

Let u be a vector in R^3 $\perp$ to (1,1,1) and (1,-1,0).

$u = (x \quad y \quad z)$

$u^T a = 0 = u^T b$

$x + y + z = 0$ and $x - y = 0$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

$-2y - z = 0$

Let $y = k$ $\qquad x + k - 2k = 0$

$z = -2k \qquad x = k$

$$\therefore u = \begin{bmatrix} k \\ k \\ -2k \end{bmatrix} = \left\{ k \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \mid k \in R \right\}$$

$v_1 = a = (1, 1, 1)$ $\quad$ $v_2 = b = (1, -1, 0)$

$v_3 = (1, 1, -2)$

(orthonormal to v_1 & v_2)

From independent orthonormal vectors, produce basis by dividing each vector by its norm to make unit vectors

Normalising v_1, v_2, v_3 vectors will get orthonormal bases

$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$(1/\sqrt{2}, 1/\sqrt{2}, 0)$

$(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$

Q14. Let P be the plane in R^4 $\quad x - 2y + 3z - t = 0$

(i) Find a vector $\perp$ to P
(ii) What matrix has the plane P as its null space?
(iii) What is the basis for P?

(i) $P = x - 2y + 3z - t = 0$ is a 3D plane in R^4

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix}}_{P^T} \underbrace{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_{v} = 0$$

$(1, -2, 3, -1)$ is dir ratio $\perp$ to plane

$$\therefore\ v = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} \;\middle|\; k \in R \right\} \text{ is } \perp \text{ to } P$$

(ii) Let the matrix A have null space $= P$

Let $u \in P$ such that $u = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$

u is left null space

$$u^T P = \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix}$$

$x - 2y + 3z - t = 0$ is the solution to $Ax = 0$

(iii) Basis for P : basis of null space

null space: solutions to P

$$\begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y + 3z - t = 0$$

$$x = 2y - 3z + t$$

$$N(P) = \begin{bmatrix} 2y-3z+t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis = $\{ (2,1,0,0), (-3,0,1,0), (1,0,0,1) \}$

Q15. Suppose S is spanned by $(1,2,2,3)$ and $(1,3,3,2)$. Find the basis for $S^{\perp}$

Let v be a vector in $S^{\perp}$. Let $v = (x,y,z,t)$

$v^T(1,2,2,3) = 0$ and $v^T(1,3,3,2) = 0$

or

Let S = row space of matrix. $S^{\perp}$ = null space of matrix

$$A:b = \left[\begin{array}{cccc:c} 1 & 2 & 2 & 3 & 0 \\ 1 & 3 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc:c} 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$R = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

Let $t = k_1$, $z = k_2$

$$x + 5k_1 = 0$$

$$x = -5k_1$$

$$y + z - t = 0$$

$$y + k_2 - k_1 = 0$$

$$y = k_1 - k_2$$

$$N(V) = \begin{bmatrix} -5k_1 \\ k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix} = \left\{ k_1 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; k_1, k_2 \in \mathbb{R} \right\}$$

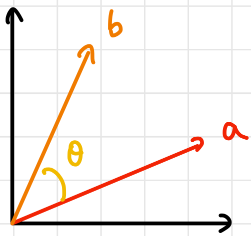
$$\therefore \text{Basis for } S^{\perp} = \left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

COSINES & PROJECTIONS

If $a = (a_1, a_2)$, $b = (b_1, b_2)$ angled θ apart, then

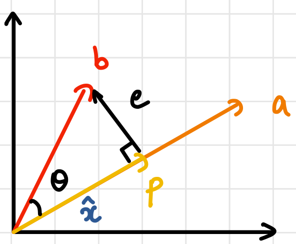
$$\cos\theta = \frac{a_1 b_1 + a_2 b_2}{\|a\| \, \|b\|} = \frac{a^T b}{\|a\| \, \|b\|}$$

Applies to R^n



Projections onto Line

Projection of $\vec{b}$ onto line a



$e = b - p$

$\vec{p}$ is a multiple of a, point closest to $\vec{b}$ on a

$$p = \hat{x} a$$

multiple (scalar)

$$a \perp e \quad \text{or} \quad a^T(b - \hat{x}a) = 0$$

$$\hat{x} a^T a = a^T b$$

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

$$p = a\hat{x}$$

$$\boxed{p = a\,\frac{a^T b}{a^T a}}$$

$$p = Pb$$

P: projection matrix

$$\boxed{P = \frac{a a^T}{a^T a}}$$

→ $n \times n$ matrix, symmetric $P^T = P$

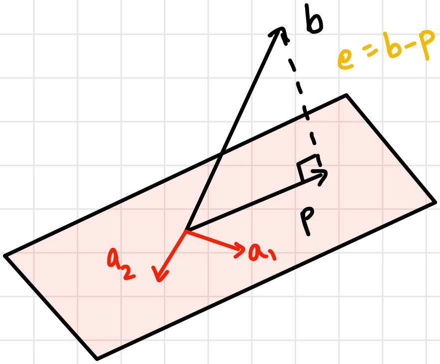
$C(P)$ = line through a

PROJECTION MATRIX IS OF RANK 1

$r(P) = 1$ (column vector x row vector)

- Note: $P^n = P$ (property)

Project vector onto space



plane of a_1, a_2

= column space of

$$\begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix}$$

$e \perp$ plane spanned by a_1 & a_2

P is some multiple of a_1 and a_2

$$P = \hat{x}_1 a_1 + \hat{x}_2 a_2$$

$$\boxed{P = A\hat{x}}$$

$$P = \begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$e = b - A\hat{x} \perp$ plane $\Rightarrow \perp$ to a_1 & $\perp$ to a_2

$$a_1^T (b - A\hat{x}) = 0 \quad \text{and} \quad a_2^T (b - A\hat{x}) = 0$$

Writing equations into matrix form

$$\underbrace{\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}}_{2\times 1} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T (\underbrace{b - A\hat{x}}_{e}) = 0 \longrightarrow (1)$$

e is in $N(A^T)$

$e \perp C(A) \longrightarrow$ plane

Rewrite eq (1)

$$\boxed{A^T A\hat{x} = A^T b} \longrightarrow (2)$$

Solve for $\hat{x}$

$$\hat{x} = (A^T A)^{-1} A^T b$$

Projection p

$$p = A\hat{x}$$

if b is in $C(A)$, $p = b$ and if b is in $N(A^T)$ then $p = 0$

$$\boxed{p = A(A^T A)^{-1} A^T b}$$

projection vector closest to b

$$\boxed{P = A(A^T A)^{-1} A^T}$$

projection matrix

In 1-D

$$p = \frac{aa^T}{a^T a} b$$

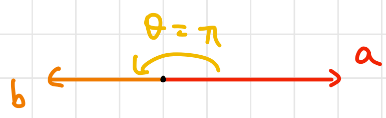
note: A is not square; cannot do $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$ as A^{-1} does not exist

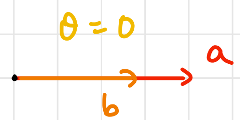
SCHWARZ INEQUALITY

All vectors a and b in $\mathbb{R}^n$

$$|a^T b| \leq \|a\| \|b\| \quad \text{or} \quad |\cos\theta| \leq 1$$

If $\theta = 0$ or $\theta = \pi$, equality holds (dependent vectors) and b = projection on a, $e = 0$





Note

1. P is symmetric
2. $P^n = P$ for $n = 1, 2, 3 \ldots$
3. $r(P) = 1$
4. Trace of $P = 1$
5. If a is n-dimensional vector of order n, P is square matrix of order n
6. If a is a unit vector, $P = a a^T$ $(a^T a = 1)$

Q16. What multiple of $a(1,1,1)$ is closest to the point $b(2,4,4)$? Find the point which is closest to a on the line through b.

$$P_a = \hat{x} a \quad \text{where} \quad \hat{x} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$

$$\hat{x} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{10}{3} \text{ multiple}$$

$$P_a = \hat{x} a = 10/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_b = \hat{x}\, b \qquad \hat{x} = \frac{b^T a}{b^T b} = \frac{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}$$

$$P_b = \frac{10}{36}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20/36 \\ 40/36 \\ 40/36 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

Q17. Find the projection of b onto a

(i) $a = (1,0)$, $b = (c, s)$

(ii) $a = (1,-1)$, $b = (1,1)$

(i) $a = (1,0)$, $b = (\cos\theta, \sin\theta)$

$$P_a = \hat{x}\, a = \frac{a^T b}{a^T a}\, a = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P_a = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

(ii) $a = (1,-1)$, $b = (1,1)$

$$P_a = a\hat{x} = \frac{a^T b}{a^T a}\, a = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q18. If P is a plane of vectors in R^4

$$P \equiv u+v+w+t=0$$

Find P and $P^\perp$ (null space of P)

(i) P

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$u+v+w+t=0$$

$$u=-v-w-t$$

$$P=N(A)=\left\{ \begin{bmatrix} -v-w-t \\ v \\ w \\ t \end{bmatrix} \right\}$$

$$=\left\{ v\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| v, w, t \in R \right\}$$

(ii) $P^\perp$. Row space is $(\text{Null space})^\perp$

$$P^\perp = \left\{ k\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \middle| k \in R \right\}$$

Q19. Let S be a 2D subspace in $\mathbb{R}^3$ spanned by $a = (1,2,1)$, $b = (1,-1,1)$. Write the vector $v = (-2,2,2)$ as the sum of a vector in S and a vector orthogonal to S.

vector in $S \in$ column space of S
vector in $S^\perp \in$ left null space of S

find row space

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to -\tfrac{1}{3} R_2$$

$$R^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \text{row space} = \left\{ k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, k_1, k_2 \in \mathbb{R} \right\}$$

$S^\perp$ = left null space (solution to $A^T x = 0 = R^T x$)

$$x + z = 0 \qquad\qquad y = 0$$
$$x = -z$$

left null space $S^\perp = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, k \in \mathbb{R} \right\}$ = line $\perp$ to plane

$$\therefore\ v = \overbrace{c_1 v_1 + c_2 v_2}^{\text{vector in } S} + \overbrace{c_3 v_3}^{\text{vector in } S^\perp}$$

where v_1, v_2 are bases of $C(A^T)$ and v_3 is basis of $N(A)$

$$v = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - c_3 \\ c_2 \\ c_1 + c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

OR

Projection of v onto line (left null space)

Let P lie on $S^\perp$

$$P = \frac{a^T v}{a^T a} \cdot a = \frac{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{4}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore$ orthogonal component in $S = v - P$

$$v - P = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore \; v = (0, 2, 0) + (-2, 0, 2)$

Q20. Project $b = (1, 0, 0)$ onto the lines

(i) $a_1 = (-1, 2, 2)$

(ii) $a_2 = (2, 2, -1)$

(iii) $a_3 = (2, -1, 2)$

Add the three points of projections and explain what the sum is and why it is.

(i) $$P_1 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -1/9 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

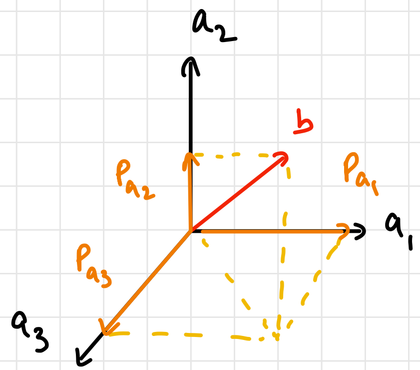
(ii) $$P_2 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 2/9 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

(iii) $$P_3 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 2/9 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 + 4 + 4 \\ -2 + 4 - 2 \\ -2 - 2 + 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b$$

Since a_1, a_2 and a_3 are mutually orthogonal,
$P_1 + P_2 + P_3 = (1, 0, 0) = b$

We bring the original vector back



Q21. V is a subspace of R^5 spanned by $a = (1, 2, 3, -1, 2)$ and $b = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal comp. $V^\perp$.

Let $s \in V^\perp = (v, w, x, y, z)$. $As^T = 0$

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$R = \begin{bmatrix} 1 & 2 & 0 & -13 & 17 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix}$$

Let $y = k_1$, $z = k_2$ Let $v = k_3$

$x + 4y - 5z = 0$ $u + 2v - 13k_1 + 17k_2 = 0$

$x = -4k_1 + 5k_2$ $u = -2k_3 + 13k_1 - 17k_2$

$$\therefore V^{\perp} = \left\{ \begin{bmatrix} -2k_3 + 13k_1 - 17k_2 \\ k_3 \\ -4k_1 + 5k_2 \\ k_1 \\ k_2 \end{bmatrix} \right\}$$

$$V^{\perp} = \left\{ k_1 \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

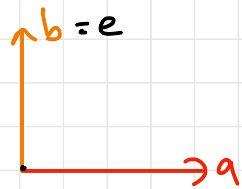
$$\text{Basis}(V^{\perp}) = \left\{ \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\text{Dim}(V^{\perp}) = 3$

Q22. Project $b = (1,2,2)$ onto the line through $a = (2,-2,1)$.
Check if e is perpendicular to a.

$$P = \frac{a^T b}{a^T a} \cdot a = \frac{\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore b \perp a$



$$e = b - P = b$$

$$\langle e, a \rangle = 2 - 4 + 2 = 0$$

Q23. Project $b = (1,2,2)$ onto the line through $a = (1,1,1)$.
Check if $e \perp a$

$$P = \frac{a^T b}{a^T a} \cdot a = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$e = b - P = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\langle e, a \rangle = \frac{-2}{3} + \frac{1}{3} + \frac{1}{3} = 0 \quad \therefore \ e \perp a$$

PROJECTIONS & LEAST SQUARES

Failure of Gaussian elimination with multiple equations and one variable (b not in $C(A)$)

$$\begin{aligned} a_1 x &= b_1 \\ a_2 x &= b_2 \\ a_3 x &= b_3 \end{aligned} \qquad \text{or} \qquad Ax = b$$

Solvable if $a_1 : a_2 : a_3 = b_1 : b_2 : b_3$

If system is inconsistent, choose value of x that minimises average error E in the m equations.

Sum of squares $= E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$

If there is exact solution, $E = 0$. If not, $\frac{dE^2}{dx} = 0$

Solving for x

$$\frac{dE^2}{dx} = \sum_{i=1}^{m} 2(a_i x - b_i) a_i = 2 \sum_{i=1}^{m} a_i^2 x - 2 \sum_{i=1}^{m} a_i b_i = 0$$

$$\sum_{i=1}^{m} a_i^2 x = \sum_{i=1}^{m} a_i b_i$$

$$a^T a (x) = a^T b$$

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

$\hat{x} = a^{-1} p$

Least Squares WITH Multiple Variables

- Consider an inconsistent system of linear equations

$$A_{m \times n} X_{n \times 1} = b_{m \times 1}$$

- We look for best possible approximate solution.

- The vector b lies outside C(A) and we need to project it onto C(A) to get the point p in C(A) closest to b

- The system is reduced to $A\hat{x} = p$

From pages 49,50

$$\boxed{A^T A \hat{x} = A^T b} \rightarrow \text{normal equation}$$

solve for $\hat{x}$ (estimate)

- The equation $A^T A \hat{x} = A^T b$ is called the normal equation

Q24. Find $\|E\|^2 = \|Ax - b\|^2$ and set to zero its derivatives wrt the unknowns u and v. Compare the resulting equation with the normal equation

$$A^T \cdot A\hat{x} = A^T \cdot b$$

(i) Find the solution $\hat{x}$ and the projection $p = A\hat{x}$
(ii) Why is $p = b$?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} u \\ v \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Using least squares method

$$\|E\|^2 = \|Ax - b\|^2$$

$$Ax - b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$Ax - b = \begin{bmatrix} u-1 \\ v-3 \\ u+v-4 \end{bmatrix}$$

$$\|E\|^2 = \left\| \begin{matrix} u-1 \\ v-3 \\ u+v-4 \end{matrix} \right\|^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

Derivative wrt u

$$\frac{\partial \|E\|^2}{\partial u} = 2(u-1) + 2(u+v-4) = 0$$

$$u+1 + u+v-4 = 0$$

$$2u + v - 3 = 0$$

$$2u+v=3 \longrightarrow (1)$$

Derivative wrt v

$$\frac{\partial \|E\|^2}{\partial v} = 2(v-3) + 2(u+v-4) = 0$$

$$v-3 + u+v-4 = 0$$

$$2v + u - 7 = 0$$

$$2v+u=7 \longrightarrow (2)$$

Using geometry

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$2u+v=5 \quad \text{and} \quad u+2v=7$$

$\therefore$ the equations are the same

(i) Solution $\hat{x}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A = \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - 1/2 R_1} \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 0 & 3/2 & 9/2 \end{array}\right]$$

$$\frac{\cancel{3}}{\cancel{2}}v = \frac{\cancel{9}^{3}}{\cancel{2}} \quad \Rightarrow \quad \boxed{v=3}$$

$$2u+3=5 \quad \Rightarrow \quad \boxed{u=1}$$

Solution: $\hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(ii) Projection $p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$p = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$$

$p = b \Rightarrow b$ is in column space of A

Q25. Let $A = [3 \quad 1 \quad -1]$. Let $V = N(A)$. Find

(i) A basis for V, basis for $V^\perp$
(ii) Projection matrix P_1 onto $V^\perp$
(iii) Projection matrix P_2 onto V

$V = N(A)$ = solution to $Ax = 0$ where $x = (x, y, z)$

$3x + y - z = 0 \qquad [3 \quad 1 \quad -1] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0]$

$$x = \frac{z - y}{3}$$

$$N(A) = \left\{ \begin{bmatrix} (z-y)/3 \\ y \\ z \end{bmatrix} \right\}$$

$$V = N(A) = \left\{ y \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, y, z \in R \right\}$$

(i) Basis for $V = \left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $V^\perp$ = basis for row space

$$= \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(ii) P_1 onto $V^\perp$

$$P = A(A^T A)^{-1} A^T$$

$$P_1 = V^\perp ((V^\perp)^T V^\perp)^{-1} (V^\perp)^T$$

$$P_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 11 \end{bmatrix}^{-1} \right) \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\frac{1}{11} \right) \underset{1\times 1}{\begin{bmatrix} 1 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 3 & 1 & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\frac{1}{11}\right) \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$P_1 = \frac{1}{11} \begin{bmatrix} 9 & 3 & -3 \\ 1 & 3 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

(iii) P_2 onto V

$$V = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_2 = V (V^T V)^{-1} V^T$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10/9 & -1/9 \\ -1/9 & 10/9 \end{bmatrix}^{-1} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{9}{11}\right) \underset{2\times 2}{\begin{bmatrix} 10/9 & 1/9 \\ 1/9 & 10/9 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} -1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}}$$

$$= \frac{9}{11} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -11/27 & 10/9 & 1/9 \\ -11/27 & 1/9 & 10/9 \end{bmatrix}$$

$$= \frac{9}{11} \begin{bmatrix} 0 & -1/3 & 1/3 \\ -11/27 & 10/9 & 1/9 \\ -11/27 & 1/9 & 10/9 \end{bmatrix}$$

Q26. Find projection of b onto the C(A)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Split b into p+q such that p is in C(A) and q is ⊥ to that space. Which of the four subspaces contains q?

Column space

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 6 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ x \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Projection p

$$(A^T A)\,\hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 6 & -8 & -11 \\ -8 & 18 & 27 \end{array}\right] \xrightarrow{R_2 \to R_2 + 8/6\, R_1} \left[\begin{array}{cc:c} 6 & -8 & -11 \\ 0 & 22/3 & 37/3 \end{array}\right]$$

$$22y = 37 \qquad\qquad 6x - \frac{148}{11} = -11$$

$$y = \frac{37}{22} \qquad\qquad x = \frac{9}{22}$$

$$\hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$p = \begin{bmatrix} 9/22 + 37/22 \\ 9/22 + -37/22 \\ -9/11 + 74/11 \end{bmatrix}$$

$$p = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

$$q = b - p = \begin{bmatrix} 1 - 23/11 \\ 2 + 14/11 \\ 7 - 65/11 \end{bmatrix}$$

$$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

q is in null space of A^T

Linear Algebra

UNIT-4

ORTHOGONALISATION, EIGEN VALUES & EIGEN VECTORS

feedback/corrections: vibha@pesu.pes.edu

VIBHA MASTI

ORTHOGONAL BASES

BS Grewal : 2.13 to 2.15, 28.9
Gillbert: 3.4, 5.1, 5.2

A basis consisting of mutually orthogonal vectors

Orthonormal BASIS

A basis consisting of unit length, mutually orthogonal vectors

Q1. Is $\{(0,2), (2,0)\}$ orthonormal basis?

$$\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0 \longrightarrow \text{orthogonal}$$

length $\neq 1 \Rightarrow$ not orthonormal

ORTHOGONAL MATRIX

- A matrix with orthonormal columns is called Q ($m \geq n$)
- If $m = n$, the matrix is orthogonal

Properties of Q

1. If Q (square or rectangular) has orthonormal columns, then

$$Q^T Q = I$$

Let $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$ $\quad Q^T = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix}$

$$Q^TQ = \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ \vdots & & & \\ q_n^Tq_1 & q_n^Tq_2 & \cdots & q_n^Tq_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

2. An orthogonal matrix is square matrix with orthonormal columns,

$$Q^T = Q^{-1}$$

3. If Q is a tall matrix,

$$Q^T = \text{left inverse of } Q$$
$$Q^TQ = I$$

4. Multiplication by any Q preserves the length

$$\|x\| = \|Qx\|$$

5. Q preserves inner products and angles

$$(Qx)^T(Qy) = x^TQ^TQy = x^Ty$$

6. If $q_1, q_2 \dots q_n$ are orthonormal bases of R^n then any vector b in R^n can be expressed as

$$b = x_1 q_1 + x_2 q_2 + \dots + x_n q_n \longrightarrow (1)$$

To solve for x_1, multiply (1) by q_1^T

$$q_1^T b = x_1 q_1^T q_1$$

$$x_1 = \frac{q_1^T b}{q_1 q_1^T} = q_1^T b$$

$$x_1 = q_1^T b \leftarrow \text{projection of } b \text{ onto } q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2 + \dots + (q_n^T b) q_n$$

Rectangular Matrices with Orthonormal Columns

If Q has orthonormal columns, least squares solution becomes easier

$$Qx = b \quad \text{where} \quad b \notin C(Q)$$

Recall least squares solution:

$$Q^T Q \hat{x} = Q^T b$$

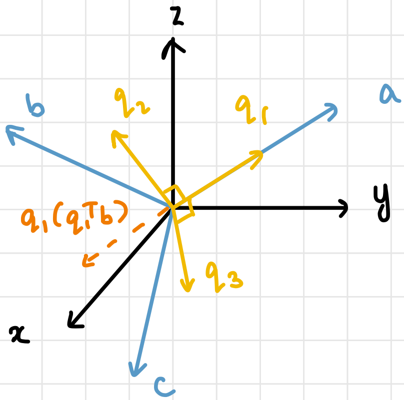
$$I \hat{x} = Q^T b$$

$$\boxed{\hat{x} = Q^T b}$$

GRAM-SCHMIDT PROCESS

Process of converting linearly independent vectors into a set of orthonormal vectors

Consider three linearly independent vectures a, b, c. The set of orthonormal vectors: q_1, q_2, q_3



$$q_1 = \frac{a}{\|a\|}$$

$$B = b - \underline{(q_1^T b) q_1}$$

projection of b on q_1

$$q_2 = \frac{B}{\|B\|}$$

$$C = c - (q_2^T c) q_2 - (q_1^T c) q_1$$

$$q_3 = \frac{C}{\|C\|}$$

Projection of b onto a

$$p = a \frac{a^T b}{a^T a}$$

If b is not $\perp$ to a, the projection of b onto a must be subtracted to form an orthogonal vector to a

Q-R FACTORISATION

If $A_{m\times n}$ is a matrix with linearly independent columns, then A can be factorised as

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

where Q is a matrix with orthonormal vectors (constructed using Gram-Schmidt Process) and R is an upper triangular and invertible matrix)

If $A = \begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix}$

We express a, b, c as linear combinations of q_1, q_2 and q_3

$$a = (q_1^T a) q_1$$

$$b = (q_1^T b) q_1 + (q_2^T b) q_2 \quad \leftarrow q_3 \perp ab \text{ plane}$$

$$c = (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}}_{Q_{m\times n}} \underbrace{\begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}}_{R_{n\times n}}$$

In general,

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} q_1^T a_1 & q_1^T a_2 & \cdots & q_1^T a_n \\ 0 & q_2^T a_2 & \cdots & q_2^T a_n \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & q_m^T a_n \end{bmatrix}$$

System is Inconsistent - Least Squares Method

$$Ax = b \quad \text{where} \quad b \notin C(A)$$

$$A^T A \hat{x} = A^T b$$

$$(QR)^T QR \hat{x} = (QR)^T b$$

$$R^T \underbrace{Q^T Q}_{I} R \hat{x} = R^T Q^T b$$

$$R^T R \hat{x} = R^T Q^T b$$

R is square matrix $\Rightarrow$ multiply by $(R^T)^{-1}$

$$\boxed{R\hat{x} = Q^T b}$$

Q2. Let $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$

Verify that

(i) $Q^T Q = I$

(ii) $\|Qx\| = \|x\|$, $\|Qy\| = \|y\|$ or Q preserves length

(iii) $((Qx)^T Qy) = x^T y$

(i) $Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$

$$Q^T Q = \begin{bmatrix} 1/2 + 1/2 & 0 \\ 0 & 4/9 + 4/9 + 1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) $Qx = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$$\|Qx\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|x\| = \sqrt{2+9} = \sqrt{11}$$

$$Qy = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{bmatrix} -3+4 \\ -3-4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$$

$$\|Qy\| = \sqrt{1+49+4} = \sqrt{54}$$

$$\|y\| = \sqrt{18+36} = \sqrt{54}$$

(iii) $((Qx)^T Qy) = x^T y$

$$(Qx)^T Qy = x^T y$$

$$\begin{bmatrix} 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$$

$$[3+7+2] = [-6+18]$$

$$[12] = [12]$$

Q3. Find the orthogonal basis spanned by a set of vectors $a = (2,-5,1)$, $b = (4,-1,5)$

$$q_1 = \frac{a}{\|a\|} = \frac{(2,-5,1)}{\sqrt{4+25+1}} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{\sqrt{30}} (8+5+5) \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{30}(18)\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} - 3/5 \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 14/5 \\ 2 \\ 22/5 \end{bmatrix} \qquad \|B\| = \sqrt{(14/5)^2 + 4 + (22/5)^2}$$

$$= \sqrt{\frac{156}{5}}$$

$$\|B\| = \frac{2\sqrt{195}}{5}$$

$$b = \frac{B}{\|B\|} = \frac{(14/5,\ 2,\ 22/5)}{2\sqrt{195}} \times 5$$

$$q_2 = \frac{1}{2\sqrt{195}} \begin{bmatrix} 14 \\ 10 \\ 22 \end{bmatrix}$$

Q4. Apply GS Process of Orthogonalisation to the vectors $a = (1,0,1)$, $b = (1,0,-1)$, $c = (0,3,4)$ to obtain an orthonormal basis q_1, q_2, q_3

$$q_1 = \frac{a}{\|a\|} = \frac{(1,0,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = b - (q_1^T b)\, q_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 0$$

$$q_2 = \frac{(1, 0, -1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$C = c - (q_1^T c)\, q_1 - (q_2^T c)\, q_2$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Qs. $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$. Find q_1, q_2, q_3 orthonormal basis from a, b, c (columns of A).

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ a & b & c \end{array}$$

Then write A as QR

$a = (1, 0, 0)$

$$q_1 = \frac{a}{\|a\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \Rightarrow q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = QR$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Q6. Use GS Process to find a set of orthonormal vectors from the independent vectors

$a_1 = (1,0,1)$, $a_2 = (1,0,0)$, $a_3 = (2,1,0)$

Also find $A = QR$

Let orthonormal vectors be q_1, q_2, q_3

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{(1,0,1)}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$A_2 = a_2 - (q_1^T a_2) q_1$$

$$= (1,0,0) - 1/\sqrt{2}\,(1/\sqrt{2}, 0, 1/\sqrt{2})$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$q_2 = \frac{(1/2, 0, -1/2)}{\sqrt{1/4 + 1/4}} = \sqrt{2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$A_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - (2/\sqrt{2})\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} - (2/\sqrt{2})\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_3 = \frac{A_3}{\|A_3\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \qquad Q^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = QR$$

$$Q^T A = R$$

$$R = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 2/\sqrt{2} \\ 0 & 1/\sqrt{2} & 2/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Q7. Find a third column so that the matrix Q

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \_\_ \\ 1/\sqrt{3} & 2/\sqrt{14} & \_\_ \\ -1/\sqrt{3} & 3/\sqrt{14} & \_\_ \end{bmatrix}$$ is orthogonal

Assume (x, y, z) is third column

null space $\perp$ row space

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Bigg\downarrow \quad \begin{aligned} R_1 &\to \sqrt{3} R_1 \\ R_2 &\to \sqrt{14} R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

one free variable $z \longrightarrow$ infinite solutions

can assume any value for z

let $z = 1$

$$x + y - 1 = 0 \Rightarrow x = 1 - y \longrightarrow (1)$$

$$x + 2y + 3 = 0 \Rightarrow x = -3 - 2y \longrightarrow (2)$$

(1) & (2)

$$1 - y = -3 - 2y$$

$$2y - y = -3 - 1$$

$$y = -4 \Rightarrow x = 5$$

$$\therefore (x, y, z) = \frac{(5, -4, 1)}{\sqrt{25 + 16 + 1}}$$

$$= \frac{1}{\sqrt{42}} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & 5/\sqrt{42} \\ 1/\sqrt{3} & 2/\sqrt{14} & -4/\sqrt{42} \\ -1/\sqrt{3} & 3/\sqrt{14} & 1/\sqrt{42} \end{bmatrix}$$

Q5. Find an orthonormal set q_1, q_2, q_3 for which q_1 & q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

(i) Which fundamental subspace contains q_3?

(ii) What is the least square solution of $Ax = b$ if $b = (0,3,0)$?

$$q_1 = \frac{(1,2,2)}{\sqrt{9}} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$B = b - (q_1^T b)\, q_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - (1/3 + 2 + 2/3) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{(0,1,-1)}{\sqrt{2}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

ii) Left null space

$$A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \qquad A^T y = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \; R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Let $z = 1$ (free var)

$$x + 2y + 2 = 0$$

$$y - 1 = 0 \Rightarrow y = 1$$

$$x + 2 + 2 = 0 \Rightarrow x = -4$$

$$q_3 = \frac{(-4,1,1)}{\sqrt{18}} = \begin{bmatrix} -4/\sqrt{18} \\ 1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

(ii) least square solution

$b = (0,3,0)$

$$A\hat{x} = b$$

$$R\hat{x} = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3/\sqrt{2} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3/\sqrt{2} \end{bmatrix}$$

$$\Big\downarrow R_2 \to \sqrt{2} R_2$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$2y = 3 \Rightarrow y = 3/2$$

$$3x + 9/2 = 2$$

$$3x = -5/2$$

$$x = -5/6$$

$$\hat{x} = \begin{bmatrix} -5/6 \\ 3/2 \end{bmatrix}$$

Q9. If W is the subspace spanned by the orthogonal vectors (2,5,-1), (-2,1,1), find the point in W closest to (1,2,3)

Let W = column space of A

$$A = \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A = QR$$

$$R = Q^T A$$

$$R\hat{x} = Q^T b \qquad p = A\hat{x}$$

$$q_1 = \frac{(2,5,-1)}{\sqrt{4+25+1}} = \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ -1/\sqrt{30} \end{bmatrix}$$

$$B = b - (q_1^T b) q_1$$

$$= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} - 0 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{(-2,1,1)}{\sqrt{6}} = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

$$R\hat{x} = Q^T b$$

$$\begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{30} & 0 \\ 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix}$$

$$\left[\begin{array}{cc:c} \sqrt{30} & 0 & 9/\sqrt{30} \\ 0 & \sqrt{6} & 3/\sqrt{6} \end{array}\right]$$

$R_1 \to \sqrt{30}\, R_1$
$R_2 \to \sqrt{6}\, R_2$

$$\downarrow$$

$$\left[\begin{array}{cc:c} 30 & 0 & 9 \\ 0 & 6 & 3 \end{array}\right]$$

$30x = 9 \qquad 6y = 3$

$x = 3/10 \qquad y = 1/2$

$$\hat{x} = \begin{bmatrix} 3/10 \\ 1/2 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 2 & -2 \\ 5 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3/10 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6/10 & -1 \\ 15/10 & +1/2 \\ -3/10 & +1/2 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

Q10. Find an orthonormal set q_1, q_2, q_3 for which q_1 & q_2 span the column space of A

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

(a) Which fundamental subspace contains q_3?

(b) What is the least square solution of $A\hat{x} = b$ if $b = (1, 2, 7)$

$$q_1 = \frac{(1,2,-2)}{\sqrt{4+4+1}} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \overset{q^T}{\begin{bmatrix} 1/3 & 2/3 & -2/3 \end{bmatrix}} \overset{a_2}{\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}} \overset{q}{\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \left(1/3 - 2/3 - 8/3 \right) \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$q_2 = \frac{(2,1,2)}{\sqrt{9}} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

(a) q_3 is in left null space; let $q_3 = (x, y, z)$

$$\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_1 \to 3R_1 \\ R_2 \to 3R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 6 \end{bmatrix}$$

free

z is free: let $z=1$

$$-3y=-6$$
$$y=2$$

$$x+4-2=0$$
$$x=-2$$

$$q_3=\frac{(-2,2,1)}{3}=\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$$

(b) $A=\begin{bmatrix}1&1\\2&-1\\-2&4\end{bmatrix}$ $\quad b=\begin{bmatrix}1\\2\\7\end{bmatrix}$

$$A\hat{x}=b, \quad R\hat{x}=Q^Tb$$

$$R=Q^TA=\begin{bmatrix}1/3&2/3&-2/3\\2/3&1/3&2/3\end{bmatrix}\begin{bmatrix}1&1\\2&-1\\-2&4\end{bmatrix}=\begin{bmatrix}3&-3\\0&3\end{bmatrix}$$

$$Q^Tb=\begin{bmatrix}1/3&2/3&-2/3\\2/3&1/3&2/3\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix}=\begin{bmatrix}-3\\6\end{bmatrix}$$

$$R\hat{x}=Q^Tb$$

$$\begin{bmatrix}3&-3\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3\\6\end{bmatrix}$$

$3y=6 \Rightarrow y=2$ $\qquad$ $3x-6=-3 \Rightarrow 3x=3$

$$x=1$$

$$\hat{x}=\begin{bmatrix}1\\2\end{bmatrix}$$

Q11. What multiple of $a_1 = (2, 2)$ should be subtracted from $a_2 = (4, 0)$ for the result to be orthogonal to a_1? Factor $A = QR$ with orthonormal vectors in Q.

can do $(a_2 - ka_1) \cdot a_1 = 0$

$$a_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \overbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}^{\text{vector} \parallel \text{ to } a_1}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - 2\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - (1)\, a_1$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$\therefore$ multiple $= 1$

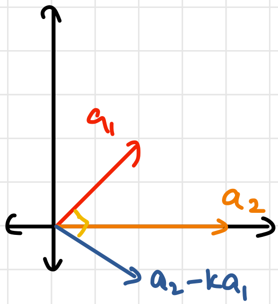
$$q_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad R = Q^TA = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}$$

OR

$$(a_2 - ka_1)^T a_1 = 0$$



$$\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} - k \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4-2k \\ -2k \end{bmatrix}^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$2(4-2k) - 4k = 0$$

$$8 - 4k - 4k = 0$$

$$8k = 8$$

$$k = 1$$

EIGEN VALUES & EIGEN VECTORS

Let A be any square matrix of order n, then all the values of λ (real or complex) which satisfy the equation $|A-\lambda I| = 0$ are called the eigenvalues of A

$|A-\lambda I| = 0 \longrightarrow$ characteristic equation

All the vectors 'x' that satisfy the equation $Ax = \lambda x$ or $(A-\lambda I)x = 0$ are called the eigen vectors corresponding to the eigen value λ

$$Ax = \lambda x \quad \text{or} \quad (A-\lambda I)x = 0$$

Note:

1. If A is a square matrix of order n, then there are exactly n eigenvalues of A

eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$ $\quad A-\lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$

$|A| = (1-\lambda)^2 - 1 = 0 \Rightarrow 1-\lambda = \pm 1$

$\lambda = 1 \pm 1$

$\lambda = 0, 2 \longrightarrow 2$ values

2. λ is an eigenvalue of A iff $A-\lambda I$ is singular, or $|A-\lambda I| = 0$

If $|A|$ is already 0, then $\lambda = 0$ is always an eigenvalue of A

3. If A is invertible, i.e $|A| \neq 0$, $\lambda = 0$ is never an eigenvalue of A

4. $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$

5. If $Ax = \lambda x$ and $\lambda = 0$, then $Ax = 0$ and $x \in N(A)$

PROPERTIES OF EIGEN VALUES & EIGEN VECTORS

1. Given an eigen vector x of a matrix, corresponding eigen value λ is unique

2. Given an eigenvalue of a matrix, there are infinitely many eigenvectors

3. The eigenvalues of a square matrix and its transpose are equal

4. The eigen values of an idempotent matrix $(A^2 = A = A^n)$ are either 0 or 1

5. If λ is an eigenvalue of A with x as the corresponding eigen vector, then λ^2 is an eigen value of A^2 with the same eigen vector x

6. The trace of a matrix is equal to the sum of its eigenvalues (trace = sum of principal diagonal entries)

7. The product of all eigenvalues of A is the determinant of A

8. The eigenvalues of a triangular / diagonal matrix are the principal diagonal elements of the matrix

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9. If λ is an eigenvalue of A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}

10. If A is an orthogonal matrix, then if λ is an eigenvalue of A, $1/\lambda$ is also an eigenvalue of A

11. Cayley-Hamilton Theorem

Every square matrix A satisfies the characteristic equation

$$|A - \lambda I| = 0$$

Procedure

Step 1: Calculate $|A - \lambda I|$ (polynomial in λ of order n)

Step 2: Find roots of equation (eigenvalues)

Step 3: For each value of λ, solve the equation

$$(A - \lambda I)x = 0$$

Non-zero values of $x \rightarrow$ eigen vectors

Q12. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Verify that

(i) Trace = sum of eigenvalues

(ii) Determinant of A equals the product of eigenvalues

(iii) If we shift A to A-7I

(a) what are the eigenvalues of $A-\lambda I$?

(b) How are they related to those of A?

Characteristic eq.

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \qquad \lambda^2 - (\text{trace})\lambda + \det = 0$$

$$\lambda = 2 \qquad \lambda = 3$$

Eigenvectors:

(a) $\lambda = 2$

$$(A-2I)x = 0$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 + 2R_1$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $y = k$

$$-x - k = 0$$
$$x = -k$$

$$x = \begin{bmatrix} -k \\ k \end{bmatrix} = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

(b) $\lambda = 3$

$$(A-3I)x = 0$$

$$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$$

Let $y = k$

$$-2x - k = 0$$

$$x = -k/2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \end{bmatrix} = \left\{ k/2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

$$x = \left\{ c \begin{bmatrix} -1 \\ 2 \end{bmatrix} \mid c \in R \right\}$$

(i) Trace of A = sum of principal diagonals

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

eigen values

$$1+4 = 2+3 = 5$$

(ii) $|A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4+2 = 6$

product of λ's $= 2 \times 3 = 6$

(iii) $A \longrightarrow A - 7I$

$$A - 7I = \begin{bmatrix} 1-7 & -1 \\ 2 & 4-7 \end{bmatrix} = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix} = B$$

(a) Eigenvalues of B

$$|B-\lambda I| = 0$$

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$+(6+\lambda)(3+\lambda) + 2 = 0$$

$$\lambda^2 + 9\lambda + 18 + 2 = 0$$

$$\lambda^2 + 9\lambda + 20 = 0$$
$$(\lambda+4)(\lambda+5) = 0$$

$$\lambda = -4 \qquad \lambda = -5$$

(b) $\lambda_{A_1} = 2 \qquad \lambda_{A_2} = 3$

$\lambda_{B_1} = -5 \qquad \lambda_{B_2} = -4$

$$\lambda_{A_1} - \lambda_{B_1} = 2+5 = 7$$

$$\lambda_{A_2} - \lambda_{B_2} = 3+4 = 7$$

$$\therefore \lambda_A - \lambda_B = 7$$

$\therefore$ if $A \longrightarrow A + kI$ then $\lambda_k = k + \lambda$

Q13. Find the eigenvalues of the matrices A, A^2, A^{-1} and $A+4I$ given

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(i) $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$2-\lambda = \pm 1$$
$$\lambda = 2 \pm 1$$

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

(ii) Property: $\lambda \to A \Rightarrow \lambda^2 \to A^2$

$$\lambda_1 = 9 \qquad \lambda_2 = 1$$

verify: $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

$$|A^2 - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)^2 = 16$$
$$5-\lambda = \pm 4 \Rightarrow \lambda = 1, 9$$

(iii) $1/\lambda$ is eigenvalue of A^{-1}

$\lambda_1 = 1 \quad \lambda_2 = 1/3$

(iv) $\lambda_1 = 5 \quad \lambda_2 = 7$

Q14. Write the 3 different 2x2 matrices for which eigenvalues are 4, 5 and $|A| = 20$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{aligned} ad - bc &= 20 \\ a + d &= 9 \end{aligned}$$

eg 1: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$

eg 2: $\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

eg 3: $\begin{bmatrix} 4 & 3 \\ 0 & 5 \end{bmatrix}$

Q15. Find the eigenvalues and eigenvectors for the given matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$

$$= (2-\lambda)\big((3-\lambda)(2-\lambda)-2\big) - 2\big((2-\lambda)-1\big) + \big(2-(3-\lambda)\big)$$

$$= (2-\lambda)(\lambda^2-5\lambda+6-2) - 2(1-\lambda) + (\lambda-1)$$

$$= (2-\lambda)(\lambda^2-5\lambda+4) + 2\lambda - 2 + \lambda - 1$$

$$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = 1 \quad \lambda_3 = 1$$

(i) $\lambda = 5$

$$(A - 5I)x = 0$$

$$\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 1/3 R_1$
$R_3 \to R_3 + 1/3 R_1$

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -4/3 & 4/3 \\ 0 & 8/3 & -8/3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4/3 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$$-\frac{4}{3}y + \frac{4}{3}k = 0 \qquad\qquad -3x + 2k + k = 0$$

$$y = k \qquad\qquad x = k$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 1$

$$\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + 2k_1 + k_2 = 0$$

$$x = -2k_1 - k_2$$

$$x = \left\{ \begin{bmatrix} -2k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}, k_1, k_2 \in R \right\}$$

$$x = \left\{ k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid k_1, k_2 \in R \right\}$$

Q16. Find the eigenvalues and eigenvectors for the given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$A = A^T \Rightarrow |A| = 0$ (product of λ's $= 0$)

$\therefore \lambda_1 = 0$

Eigenvalues

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\left((7-\lambda)(3-\lambda) - 16\right) + 6\left(-6(3-\lambda) + 8\right) + 2\left(24 + 2(\lambda - 7)\right) = 0$$

$$(8-\lambda)(\lambda^2 - 10\lambda + 21 - 16) + 6(-18 + 6\lambda + 8) + 2(24 + 2\lambda - 14) = 0$$

$$(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$$

$$\underline{8\lambda^2} - \underline{80\lambda} + 40 - \lambda^3 + \underline{10\lambda^2} - \underline{5\lambda} + \underline{36\lambda} - 60 + \underline{4\lambda} + 20 = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda + 0 = 0$$

$$\lambda(-\lambda^2 + 18\lambda - 45) = 0$$

$$-\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$-\lambda(\lambda - 15)(\lambda - 3) = 0$$

$$\lambda = 0$$
$$\lambda = 3$$
$$\lambda = 15$$

shortcut if $|A| = 0$

$$\frac{x}{\underbrace{y_1 z_2 - y_2 z_1}}, \quad \frac{y}{\underbrace{z_1 x_2 - z_2 x_1}}, \quad \frac{z}{\underbrace{x_1 y_2 - x_2 y_1}}$$

special sol.

$$\text{when } A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(i) $\lambda = 0$

$$\frac{x}{24 - 14}, \quad \frac{y}{-12 + 32}, \quad \frac{z}{56 - 36}$$

$$\frac{x}{10}, \quad \frac{y}{20}, \quad \frac{z}{20}$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\frac{x}{24-8}, \frac{y}{-12+20}, \frac{z}{20-36}$$

$$\frac{x}{16}, \frac{y}{8}, \frac{z}{-16}$$

$$x = \left\{ k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \mid k \in R \right\}$$

(iii) $\lambda = 15$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$\frac{x}{24+16}, \frac{y}{12-28}, \frac{z}{56-36}$$

$$\frac{x}{40}, \frac{y}{-40}, \frac{z}{20}$$

$$x = \left\{ k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

Q17. Find the eigenvalues and eigenvectors for the given matrices

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad |A| = 2+0+0-0-1-1 = 0$$

$$|A - \lambda I| = 0$$

$$\begin{array}{cc|ccc|cc} & & 1-\lambda & -1 & 0 & & \\ & -1 & -1 & 2-\lambda & -1 & -1 & \\ -1 & 1-\lambda & 0 & -1 & 1-\lambda & 0 & -1 \end{array} = 0$$

$$(1-\lambda)(2-\lambda)(1-\lambda) + (0) + (0) - (1-\lambda)(-1)(-1) - (-1)(-1)(1-\lambda) + 0 = 0$$

$$(1-\lambda)(\lambda^2 - 3\lambda + 2) - (1-\lambda) - (1-\lambda) = 0$$

$$\lambda^2 - 3\lambda + 2 - \lambda^3 + 3\lambda^2 - 2\lambda - 1 + \lambda - 1 + \lambda = 0$$

$$-\lambda^3 + 4\lambda^2 - 3\lambda + 0 = 0$$

$$-\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$-\lambda(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 0$$

Eigenvectors

(i) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 1/2\, R_1$$

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$-\frac{1}{2}y - k = 0$ $\qquad\qquad$ $-2x + 2k = 0$

$y = -2k$ $\qquad\qquad$ $x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle| \, k \in R \right\}$$

(ii) $\lambda = 1$

$$\begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$-y = 0$

$y = 0$

$-x - k = 0$

$x = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ k \in R \right\}$$

(iii) $\lambda = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y - k = 0$
$y = k$

$x - k = 0$
$x = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \middle| \ k \in R \right\}$$

Q18. Find the eigenvalues and eigenvectors for the given matrices

(i) $A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(i) $A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $\qquad |A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2-6\lambda+5-1) - 1(1-\lambda-3) + 3(1-3(5-\lambda)) = 0$$

$$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+2+\lambda+3-9(5-\lambda) = 0$$

$$-\lambda^3+7\lambda^2+0\lambda-36 = 0$$

$\lambda_1 = -2$
$\lambda_2 = 6$
$\lambda_3 = 3$

Eigenvectors

(a) $\lambda = -2$

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 1/3 R_1} \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y = 0$

$$3x + 3k = 0$$
$$x = -k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| \, k \in R \right\}$$

(b) $\lambda = 6$

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/5 R_1]{R_2 \to R_2 + 1/5 R_1} \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 8/5 & -16/5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$$-\frac{4}{5} y + \frac{8}{5} k = 0$$

$$y = 2k$$

$$-5x + 2k + 3k = 0$$

$$x = k$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid k \in R \right\}$$

(c) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $z = k$

$y = -k$

$$-2x - k + 3k = 0$$
$$-2x + 2k = 0$$
$$x = k$$

$$x = \left\{ k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(ii) $A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

Eigenvalues $|A_2 - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ -1 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-\lambda(-1-\lambda)) + 1(-1-\lambda) + 1(1-\lambda) = 0$$

$$(1-\lambda)(\lambda+\lambda^2) \cancel{-1} - \lambda \cancel{+1} - \lambda = 0$$

$$\lambda + \lambda^2 - \lambda^2 - \lambda^3 - 2\lambda = 0$$

$$-\lambda^3 - \lambda = 0$$

$$-\lambda(\lambda^2+1) = 0$$

$\lambda = 0$ $\qquad\qquad$ $\lambda^2 + 1 = 0$

$$\lambda = \pm i$$

Eigenvectors

i) $\lambda = 0$

$$Ax = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \begin{array}{c} \\ \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

free

let $z = k$

$$y - k = 0$$
$$y = k$$

$$x - k + k = 0$$
$$x = 0$$

$$x = \left\{ k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; k \in R \right\}$$

(ii) $\lambda = i$

$$(A - iI)x = 0$$

$$\begin{bmatrix} 1-i & -1 & 1 \\ 1 & -i & 0 \\ -1 & 1 & -1-i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

RAYLEIGH'S POWER METHOD

To find numerically largest / dominant eigenvalue and the corresponding eigenvector of a given matrix

Procedure

1. Start with the initial approximation

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

then

$$A x_0 = \lambda_1 x_1$$

$$A x_1 = \lambda_2 x_2$$

$$A x_2 = \lambda_3 x_3$$

$$\vdots$$

Repeat until $x_n - x_{n-1}$ becomes negligible

$(\lambda_n \approx \lambda_{n-1})$

Q19. Calculate 5 iterations of the power method to find the dominant eigenvalue of A.

Use $x_0 = (1, 0, 0)$ as initial approximation

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

(i) $Ax_0 = \lambda_1 x_1$

Let $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax_0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \lambda_1 x_1$$

numerically largest value $= 4 = \lambda_1$

$$Ax_0 = 4 \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \lambda_1 x_1 \implies x_1 = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

(ii) $Ax_1 = \lambda_2 x_2$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} \implies \lambda_2 = 5$$

$$Ax_1 = 5 \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix} \implies x_2 = \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix}$$

(iii) $Ax_2 = \lambda_3 x_3$ $\qquad\qquad$ $4 + 4/5 + 4/5$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4/5 \\ -4/5 \end{bmatrix} = \begin{bmatrix} 28/5 \\ 28/5 \\ -26/5 \end{bmatrix} = \frac{28}{5} \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix}$$

$$\lambda_3 = \frac{28}{5} = 5.6 \qquad x_3 = \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.93 \end{bmatrix}$$

(iv) $Ax_3 = \lambda_4 x_4$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -13/14 \end{bmatrix} = \begin{bmatrix} 83/14 \\ 83/14 \\ -79/14 \end{bmatrix} = \frac{83}{14} \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix}$$

$$\lambda_4 = \frac{83}{14} = 5.93 \qquad x_4 = \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.95 \end{bmatrix}$$

(v) $Ax_4 = \lambda_5 x_5$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -79/83 \end{bmatrix} = \begin{bmatrix} 494/83 \\ 494/83 \\ -478/83 \end{bmatrix} = \frac{494}{83} \begin{bmatrix} 1 \\ 1 \\ -478/494 \end{bmatrix}$$

$$\lambda_5 = \frac{494}{83} = 5.95 \qquad x_5 = \begin{bmatrix} 1 \\ 1 \\ -478/494 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -0.97 \end{bmatrix}$$

The largest eigenvalue is 6 and corresponding eigenvector is $x = (1, 1, -1)$

Q20. Obtain the numerically smallest eigenvalue of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ starting with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We know that if λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}

$\therefore$ smallest eigenvalue of A = reciprocal of largest eigenvalue of A^{-1}

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad A^{-1} = \frac{1}{6-5}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

(i) $A^{-1}x_0 = \lambda_1 x_1$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -5/3 \end{bmatrix}$$

$$\lambda_1 = 3 \qquad x_1 = \begin{bmatrix} 1 \\ -5/3 \end{bmatrix}$$

(ii) $A^{-1}x_1 = \lambda_2 x_2$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -5/3 \end{bmatrix} = \begin{bmatrix} 14/3 \\ -25/3 \end{bmatrix} = \frac{14}{3}\begin{bmatrix} 1 \\ -25/14 \end{bmatrix}$$

$$\lambda_2 = \frac{14}{3} = 4.67 \qquad x_2 = \begin{bmatrix} 1 \\ -25/14 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

(iii) $A^{-1}x_2 = \lambda_3 x_3$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -25/14 \end{bmatrix} = \begin{bmatrix} 67/14 \\ -60/7 \end{bmatrix} = \frac{67}{14}\begin{bmatrix} 1 \\ -120/67 \end{bmatrix}$$

$$\lambda_3 = \frac{67}{14} = 4.79 \qquad x_3 = \begin{bmatrix} 1 \\ -120/67 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

(iv) $A^{-1}x_3 = \lambda_4 x_4$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -120/67 \end{bmatrix} = \begin{bmatrix} 321/67 \\ -575/67 \end{bmatrix} = \frac{321}{67}\begin{bmatrix} 1 \\ -575/321 \end{bmatrix}$$

$$\lambda_4 = 4.79 \qquad x_4 = \begin{bmatrix} 1 \\ -1.79 \end{bmatrix}$$

$\therefore$ smallest eigenvalue of $A = 1/\lambda_4 = 0.208$

verify:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \qquad |A - 0.21I| \approx 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 5 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 - 5 = 0$$

$$\lambda^2 - 5\lambda + 1 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-4}}{2} \qquad \frac{5-\sqrt{21}}{2} = 0.209$$

DIAGONALISATION of A MATRIX

- Suppose $A_{n \times n}$ has n linearly independent eigenvectors (not a defective matrix — order ≠ independent vectors)

- If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ (lambda)

- The eigenvalues of A are on the diagonal of Λ

- Λ is called the eigenvalue matrix and S is called the eigenvector matrix

- S is not unique

- Any matrix with distinct eigenvalues can be diagonalised

Proof

Let $x_1, x_2, \ldots, x_n$ be the independent eigenvectors of A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$\text{let } \quad S = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

$$AS = A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

$$AS = \begin{bmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{bmatrix}$$

We know $Ax_1 = \lambda x_1$ as $(A-\lambda)x = 0$

$$AS = \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \lambda_n \end{bmatrix}$$

$$\boxed{AS = S\Lambda} \qquad \boxed{S^{-1}AS = \Lambda}$$

Note:

- $A = S\Lambda S^{-1}$
- $A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$

  all $\lambda \to \lambda^2$
  same x

  $A^n = S\Lambda^n S^{-1} \quad \forall\, n \in Z^+$
- $A^k = S\Lambda^k S^{-1} \longrightarrow 0$ as $k \to \infty$ if $|\lambda_i| < 1$

Q21. Show that $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalisable

Eigenvalues $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2-4\lambda+3) - 3(0) + 0 = 0$$

$$(1-\lambda)(1-\lambda)(3-\lambda) = 0$$

$\lambda = 1 \qquad \lambda = 3$

$\therefore$ only 2 independent eigenvectors

(defective matrix)

Q22. Check if $A = \begin{bmatrix} -8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalisable. If yes, find S.

eigenvalues

$$\begin{vmatrix} -8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(-8-\lambda)\Big((\lambda^2-10\lambda+21)-16\Big) + 6\Big(6(\lambda-3)+8\Big) + 2\Big(24+2(\lambda-7)\Big) = 0$$

$$-(\lambda+8)(\lambda^2-10\lambda+5) + 6(6\lambda-10) + 2(2\lambda+10) = 0$$

$$-\lambda^3 + \underline{10\lambda^2} - \underline{5\lambda} - \underline{8\lambda^2} + \underline{80\lambda} - 40 + \underline{36\lambda} - 60 + \underline{4\lambda} + 20 = 0$$

$$-\lambda^3 + 2\lambda^2 + 115\lambda - 80 = 0$$

$$\left.\begin{aligned} \lambda_1 &= -10.13 \\ \lambda_2 &= 11.44 \\ \lambda_3 &= 0.69 \end{aligned}\right\} \text{approx}$$

Eigenvectors

(i) $\lambda = -10.13$

$$x = \left\{ k \begin{bmatrix} -20.36 \\ -6.90 \\ 1 \end{bmatrix} \right\}$$

(ii) $\lambda = 11.44$

$$x = \left\{ k \begin{bmatrix} 0.65 \\ -1.78 \\ 1 \end{bmatrix} \right\}$$

(iii) $\lambda = 0.69$

$$x = \left\{ k \begin{bmatrix} -0.13 \\ 0.51 \\ 1 \end{bmatrix} \right\}$$

$$S = \begin{bmatrix} -20.36 & 0.65 & -0.13 \\ -6.90 & -1.78 & 0.51 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -10.13 & 0 & 0 \\ 0 & 11.44 & 0 \\ 0 & 0 & 0.69 \end{bmatrix}$$

Eigenvectors from https://www.emathhelp.net/calculators/linear-algebra/eigenvalue-and-eigenvector-calculator/?i=%5B%5B-8%2C-6%2C2%5D%2C%5B-6%2C7%2C-4%5D%2C%5B2%2C-4%2C3%5D%5D

Cayley-Hamilton Theorem

Every square matrix A satisfies the characteristic equation

$$|A-\lambda I| = 0$$

Replace λ with A in polynomial, solve for A^{-1}

Q23. Find the matrix A whose eigenvalues are 2 & 5 and eigenvectors are (1, 0) and (1, 1) using $S \Lambda S^{-1}$

$$A = S \Lambda S^{-1}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

Q24. Factor $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ into $S \Lambda S^{-1}$ and hence compute A^{85}

Eigenvalues

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3 \qquad \lambda = 1$$

Eigenvectors

(i) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$y = k \qquad \begin{aligned} -x + k &= 0 \\ x &= k \end{aligned}$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 1$

$$(A-I)x = 0$$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$y = k$ $\qquad$ $x + k = 0$
$\qquad\qquad\qquad x = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$S \Lambda S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^{85} = S \Lambda^{85} S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{85} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3^{85} & -1 \\ 3^{85} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^{85}+1 & 3^{85}-1 \\ 3^{85}-1 & 3^{85}+1 \end{bmatrix}$$

Q25. Find $S \Lambda S^{-1}$ for given matrix $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -6 \\ 2 & -6-\lambda \end{vmatrix} = 0$$

$$(\lambda+6)(\lambda-1) + 12 = 0$$

$$\lambda^2 + 5\lambda - 6 + 12 = 0$$
$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+2)(\lambda+3) = 0$$

$$\lambda = -2 \qquad \lambda = -3$$

(i) $\lambda = -2$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1+2 & -6 \\ 2 & -6+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2/3 R_1} \begin{bmatrix} 3 & -6 \\ 0 & 0 \end{bmatrix}$$

$$y = k$$

$$3x - 6k = 0$$
$$x = 2k$$

$$x = \left\{ k \begin{bmatrix} 2 \\ 1 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = -3$

$$(A-\lambda I)x = 0$$

$$\begin{bmatrix} 1+3 & -6 \\ 2 & -6+3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} 4 & -6 \\ 0 & 0 \end{bmatrix}$$

$y = k$

$$4x - 6k = 0$$
$$x = \frac{3}{2}k$$

$$x = \left\{ c\begin{bmatrix} 3 \\ 2 \end{bmatrix}, c \in R \right\}$$

$$S = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -9 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -8+9 & 12-18 \\ -4+6 & 6-12 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = A$$

Q26. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ compute A^6

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-4) + 2 = 0$$

$$\lambda^2 - 5\lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda = 3 \qquad \lambda = 2$$

(i) $\lambda = 3$

$$(A - 3I)x = 0$$

$$\begin{bmatrix} 1-3 & 1 \\ -2 & 4-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2x + y = 0$$
$$x = \frac{y}{2}$$

$$x = \left\{ k \begin{bmatrix} 1 \\ 2 \end{bmatrix}, k \in R \right\}$$

(ii) $\lambda = 2$

$$\begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 1/2 R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-x + y = 0$$
$$x = y$$
$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad S^{-1} = -1 \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\Lambda^2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Lambda^6 = \begin{bmatrix} 3^6 & 0 \\ 0 & 2^6 \end{bmatrix}$$

$$S\Lambda^6 S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3^6 & 0 \\ 0 & 2^6 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^6 & 2^6 \\ 2\times 3^6 & 2^6 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2^7 - 3^6 & 3^6 - 2^6 \\ 2^7 - 2\times 3^6 & 2\times 3^6 - 2^6 \end{bmatrix}$$

$$= \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}$$

Q27. Find all eigenvalues & eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and write 2 diff diagonalising matrices S.

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2-2\lambda+1-1) - 1(1-\lambda-1) + 1(1-1+\lambda) = 0$$

$$(1-\lambda)(\lambda^2-2\lambda) + \lambda + \lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda^2 - 2\lambda + 2\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda - 3) = 0$$

$$\lambda = 0 \qquad \lambda = 3$$

only 2 eigenvalues

$\therefore$ cannot be diagonalised

Q28. Find the characteristic equation and hence find the inverse of A using Cayley-Hamilton Theory

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\lambda^3 - \text{trace}(A)\,\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A) = 0$$

$$\text{trace}(A) = 1 + 2 + 1 = 4$$

$$M_{11} = 2 - 6 = -4$$

$$M_{22} = 1 - 1 = 0$$

$$M_{33} = 2 - 12 = -10$$

$$\begin{aligned} \det(A) &= 1(2-6) - 3(4-3) + 1(8-2) \\ &= -4 - 3 + 6 \\ &= -1 \end{aligned}$$

eq: $\lambda^3 - 4\lambda^2 - 14\lambda + 1 = 0$

$= A^3 - 4A^2 - 14A + I = 0$

multiply by A^{-1} on the right

$$A^2 - 4A - 14I + A^{-1} = 0$$

$$A^{-1} = -A^2 + 4A + 14I$$

$$A^2 = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 15 & 22 & 13 \\ 10 & 9 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -14 & -11 & -11 \\ -15 & -22 & -13 \\ -10 & -9 & -8 \end{bmatrix} + \begin{bmatrix} 4 & 12 & 4 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 4 & 1 & -7 \\ 1 & 0 & -1 \\ -6 & -1 & 10 \end{bmatrix}$$

Q29. $A = \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$

eq: $\lambda^3 - \text{trace}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A)$

$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

$\text{trace}(A) = 1 + 2 + 3 = 6$

$$= \begin{bmatrix} 20 & 3 & 8 \\ 27 & 13 & 18 \\ 20 & 5 & 19 \end{bmatrix}$$

$M_{11} = 6 - 0 = 6$

$M_{22} = 3 - 10 = -7$

$M_{33} = 2 - 9 = -7$

$M_{11} + M_{22} + M_{33} = -8$

$$\det(A) = 1(6-0) - 1(27-0) + 2(0-10)$$

$$= 6 - 27 - 20 = -41$$

eq: $A^3 - 6A^2 - 8A + 41I = 0$

multiply by A^{-1} to the right

$$A^2 - 6A - 8I + 41A^{-1} = 0$$

$$A^{-1} = \frac{-1}{41}\left(A^2 - 6A - 8I\right)$$

$$= \frac{-1}{41}\left(\begin{bmatrix} 20 & 3 & 8 \\ 27 & 13 & 18 \\ 20 & 5 & 19 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{-1}{41}\left(\begin{bmatrix} 6 & -3 & -4 \\ -27 & -7 & 18 \\ -10 & 5 & -7 \end{bmatrix}\right) = \frac{1}{41}\begin{bmatrix} -6 & 3 & 4 \\ 27 & 7 & -18 \\ 10 & -5 & 7 \end{bmatrix}$$

Q30. Use CH Theorem to calculate A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

eq: $\lambda^3 - \text{trace}(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - \det(A)$

$$0 = \lambda^3 - 7\lambda^2 + (9 + 6 + 0)\lambda - \left(1(9) - 2(3) + 2(1+2)\right)$$

$$0 = \lambda^3 - 7\lambda^2 + 15\lambda - 9$$

$$0 = A^3 - 7A^2 + 15A - 9I$$

multiply by A^{-1} to the right

$$A^2 - 7A + 15I - 9A^{-1} = 0$$

$$A^{-1} = \frac{1}{9}\left(A^2 - 7A + 15I\right)$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 8 \\ 4 & 5 & -4 \\ -4 & 4 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} 1 & 8 & 8 \\ 4 & 5 & -4 \\ -4 & 4 & 13 \end{bmatrix} - 7\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix} + 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} 9 & -6 & -6 \\ -3 & 6 & 3 \\ 3 & -3 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2/3 & -2/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & -1/3 & 0 \end{bmatrix}$$

Linear Algebra

UNIT-5

SINGULAR VALUE DECOMPOSITION

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VIBHA MASTI

POSITIVE DEFINITENESS

- A symmetric matrix $A \Rightarrow A = A^T$ (real symmetric matrix)
- Eigenvalues of symmetric matrices are always real and eigenvectors can be chosen to be perpendicular
- For symmetric cases, eigenvector matrix S is orthogonal (orthonormal columns)

$$A = S\underbrace{\Lambda}_{\text{spectrum}}S^{-1} = S\Lambda S^T = Q\Lambda Q^T$$

- Symmetric matrix: signs of pivots are signs of eigenvalues

POSITIVE DEFINITE MATRIX

- A symmetric real square matrix A is a positive definite matrix if

1. All eigenvalues are positive
2. All pivots are positive
3. * $x^T A x > 0$ (except origin)
4. All subdeterminants are positive (upper left)

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 5 & 4 \\ 1 & 4 & 2 \end{bmatrix} \rightarrow |1| > 0, \quad \begin{vmatrix} 1 & 4 \\ 4 & 5 \end{vmatrix} > 0, \quad \begin{vmatrix} 1 & 4 & 1 \\ 4 & 5 & 4 \\ 1 & 4 & 2 \end{vmatrix} > 0$$

if all 3 are true, matrix is positive definite

POSITIVE DEFINITE MATRICES & LEAST SQUARES

- A symmetric matrix A is positive definite iff there is a matrix R with independent columns such that

$$A = R^T R$$

- LDU decomposition Cholesky Decomposition

$$A = LDU = LDL^T = R^T R$$

$$R^T R = L\sqrt{D}\,\sqrt{D}\,L^T$$

$$\boxed{R = \sqrt{D}\,L^T} \qquad R^T = \sqrt{D}\,L \qquad \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}\begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{bmatrix}$$

- Diagonalisation

$$A = Q\Lambda Q^T = R^T R$$

$$\boxed{R = \sqrt{\Lambda}\,Q^T}$$

Semi Definite MATRICES

- Subdeterminants can be 0 (singular)
- Eigenvalues ≥ 0
- Pivot test not relevant (insufficient pivots)

- $x^T A x \geq 0$

- $A = R^T R$; R can have dependent columns

Q1. Find the symmetric matrix A which gives the quadratic form

$$3x^2 - 2y^2 + 8xy - 5yz + xz + z^2$$

Assume $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $\qquad x^T A x > 0$

$$x^T A x = [x \quad y \quad z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= ax^2 + hxy + gzx + hxy + by^2 + fzy + gxz + fyz + cz^2$$

$$= ax^2 + 2hxy + 2gxz + by^2 + 2fzy + cz^2$$

$a = 3$
$b = -2$
$c = 1$

$2h = 8$
$h = 4$

$2g = 1$
$g = 1/2$

$2f = -5$
$f = -5/2$

$$A = \begin{bmatrix} 3 & 4 & 1/2 \\ 4 & -2 & -5/2 \\ 1/2 & -5/2 & 1 \end{bmatrix}$$

Q2. Find the symmetric matrix A which gives the quadratic form

$$5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3 + 0x_1x_3$$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$eq = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$a = 5$ $\quad$ $2f = 8$ $\quad$ $2g = 0$ $\quad$ $2h = -1$

$b = 3$ $\quad$ $f = 4$ $\quad$ $g = 0$ $\quad$ $h = -1/2$

$c = 2$

$$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Q3. Find the symmetric matrix A which gives the quadratic form

$$8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$$eq: ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2$$

$a=8$ $\quad$ $2f=-2$ $\quad$ $2g=4$ $\quad$ $2h=-6$

$b=7$ $\quad$ $f=-1$ $\quad$ $g=2$ $\quad$ $h=-3$

$c=-3$

$$A=\begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

Q4. Find the symmetric matrix A which gives the quadratic form

$$10x_1^2-6x_1x_2-3x_2^2$$

$$A=\begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

$$x^TAx$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$=\begin{bmatrix} ax+hy & hx+by \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$=ax^2+2hxy+by^2$$

$a=10$ $\quad$ $2h=-6$

$b=-3$ $\quad$ $h=-3$

$$A=\begin{bmatrix} 10 & -3 \\ -3 & -3 \end{bmatrix}$$

Q5. Compute the quadratic form x^TAx for A

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 4x+3y & 3x+2y+z & y+z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= 4x^2+3xy + 3xy+2y^2+ yz + yz+z^2$$

$$= 4x^2+ 2y^2+ z^2+ 6xy + 2yz$$

Q6. Check if A is positive definite or positive semi definite

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

1. All eigenvalues pos
2. All pivots positive
3. $x^TAx > 0$
4. Subdeterminants > 0

1. pivots

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

not positive

value of 3^{rd} pivot $= 0$

$\therefore$ cannot be positive definite

may be positive semi definite

2. Eigenvalues: $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$$

$$-\lambda^3 + 6\lambda^2 - (3+3+3)\lambda + 2(3) + (-2-1) - 1(1+2)$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$-\lambda(\lambda^2 - 6\lambda + 9) = 0$$
$$-\lambda(\lambda-3)(\lambda-3) = 0$$

$\lambda = 0$ and $\lambda = 3$

3. Subdeterminants

$D_1 = 2 \qquad D_2 = 3 \qquad D_3 = 0$

4. $x^T A x > 0$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x - y - z & -x + 2y - z & -x - y + 2z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$2x^2 - yx - zx \quad -xy + 2y^2 - yz \quad - xz - yz + 2z^2$$

$$(x-y)^2 + (y-z)^2 + (z-x)^2 > 0$$

if $x = y = z$, $x^T A x = 0$

$\therefore$ semi definite matrix

Q7. Check if positive definite

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1. Pivots

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2/3 R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

2. Eigenvalues

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - (3+4+3)\lambda + 2(3) + 1(-2)$$

$$-\lambda^3 + 6\lambda^2 - 10\lambda + 4 = 0$$

$$\lambda_1 = 2 - \sqrt{2}, \quad \lambda_2 = 2 + \sqrt{2}, \quad \lambda_3 = 2$$

3. Subdeterminants

$$D_1 = 2 \qquad D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \qquad D_3 = 4$$

4. $x^T A x > 0$ $\qquad A = LDL^T$

$$(x^T L)\, D\, (L^T x)$$
$$(L^T x)^T\, D\, (L^T x)$$

$$U = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix} \xrightarrow[R_2 \to 2/3 R_2,\ R_3 \to 3/4 R_3]{R_1 \to 1/2 R_1} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L^T = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}$$

$$(L^T x) = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y/2 \\ y - 2/3 z \\ z \end{bmatrix}$$

$$(L^T x)^T\, D\, (L^T x)$$

$$= \begin{bmatrix} x - y/2 & y - 2/3 z & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x - y/2 \\ y - 2/3 z \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x - y/2 & y - 2/3 z & z \end{bmatrix} \begin{bmatrix} 2(x - y/2) \\ 3/2(y - 2/3) \\ 4/3(z) \end{bmatrix}$$

$$2(x-y/2)^2 + 3/2\,(y-2/3\,z)^2 + 4/3\,(z)^2 > 0$$

$\therefore$ positive definite

Q8. Check for positive definiteness and semi definiteness

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+c \end{bmatrix}$$

1. Pivot

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+c \end{bmatrix} \xrightarrow[R_3 \to R_3 + 1/2 R_1]{R_2 \to R_2 + 1/2 R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2+c \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & c \end{bmatrix}$$

for $c < 0$, not semi definite

for $c \leq 0$, not positive definite

2. Subdeterminants

$D_1 = 2 \qquad D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$$D_3 = 2(4+2C-1) + 1(-2-C-1) - 1(1+2)$$
$$= 6 + 4C - 3 - C - 3$$
$$= 3C$$

not semi definite if $c < 0$

not positive definite if $c \leq 0$

3. $x^T A x > 0$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+C \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + (2+c)z \end{bmatrix}$$

$$= 2x^2 - xy - xz - xy + 2y^2 - yz - xz - yz + (2+c)z^2$$
$$= 2x^2 + 2y^2 + (2+c)z^2 - 2xy - 2yz - 2xz$$

$$= (x-y)^2 + cz^2 + (x-z)^2 + (y-z)^2 > 0$$

if $c > 0$, positive definite

if $c \geq 0$, semi definite

4\. Eigenvalues $\quad |A-\lambda I| = 0$

$$-\lambda^3 + (6+c)\lambda^2 - (4+2c-1+4+2c-1+3)\lambda + 3c = 0$$

$$-\lambda^3 + (6+c)\lambda^2 - (4c+9)\lambda + 3c = 0$$

$$-(\lambda-3)(-(3+c)\lambda + c + \lambda^2)$$

$$\lambda_1 = 3$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{c^2+2c+9} + c + 3\right)$$

$$\lambda_3 = \frac{1}{2}\left(-\sqrt{c^2+2c+9} + c + 3\right)$$

if $c > 0$, $\lambda_3 > 0$

$\therefore$ positive definite if $c > 0$
semi definite if $c \geq 0$

Q9. Test for positive definiteness and write the corresponding quadratic forms

(i) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

(i) $\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

1. Pivot

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}$$

negative

$\therefore$ not +ve def

(ii) $\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$

1. Pivot

$\begin{bmatrix} -1 & 2 \\ 2 & -8 \end{bmatrix}$ $\therefore$ not positive definite

(iii) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ 1. Subdeterminants: $D_1 = 1$

$D_2 = 0$

$\therefore$ not +ve def

(iv) $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

1. Pivot

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$P_1 = |A_1| = 2$$

$$P_2 = \frac{|A_2|}{|A_1|} = \frac{4-1}{P_1} = \frac{3}{2}$$

$$P_3 = \frac{|A_3|}{|A_2|} = \frac{2(3)+1(-2-1)-1(1+2)}{3/2}$$

$$= 0$$

$\therefore$ not +ve definite

(v) $\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

1. Pivot

$$P_1 = |A_1| = 3$$

$$P_2 = \frac{|A_2|}{|A_1|} = \frac{2}{3}$$

$$P_3 = \frac{|A_3|}{|A_2|} = \frac{3(-2)-2(2)}{2} = -5$$

$\therefore$ not +ve definite

(vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

1. Eigenvalues

$$|B| = \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 2 & 1 & -\lambda \end{vmatrix}$$

$$-\lambda^3 + 0\lambda^2 - (-1-4-1)\lambda + 0 - 1(-2) + 2 = 0$$

$$-\lambda^3 + 6\lambda + 4 = 0$$

$\lambda = -2 \longrightarrow \lambda_B = 4$

$\lambda = 1+\sqrt{3} \longrightarrow \lambda_B = 4+2\sqrt{3}$

$\lambda = 1-\sqrt{3} \longrightarrow \lambda_B = 4-2\sqrt{3}$

2. Pivots

$P_1 = 0 \longrightarrow$ not +ve def

Singular Value Decomposition

- Any $m \times n$ matrix A can be factored into

eigenvalues (diagonal)

$$A = U \Sigma V^T$$

orthogonal   orthogonal

symmetric

- The columns of $U_{m \times m}$ are eigenvectors of AA^T

$$AA^T = U \Sigma \boxed{V^T V} \Sigma^T U^T = U \Sigma \Sigma^T U^T = U \Sigma \Sigma^T U^T$$

- The columns of $V_{n \times n}$ are eigenvectors of $A^T A$

$$A^T A = V \Sigma^T \boxed{U^T U} \Sigma V^T = V \Sigma^T \Sigma V^T = V \Sigma^T \Sigma V^T$$

- For positive definite matrices, Σ is Λ

- U & V give orthonormal bases for four fundamental subspaces of A
  - Column space: first r columns of U
  - Row space: first r columns of V
  - Left null space: last $m-r$ columns of U
  - Null space: last $n-r$ columns of V

- For detailed explanation on cases, read LA unit 5- UE18.pdf, pg 4

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^T_{n \times n}$$

orthonormal   r singular values on diagonal   orthonormal

COMPUTE SVD for a MATRIX

Case 1: Matrix A is a short matrix (eg: 2x3)

1. Find AA^T (2x2)

2. Find eigenvalues of AA^T : λ_1 & λ_2

3. Find corresponding eigenvectors x_1 & x_2

4. Normalise them to get u_1 & u_2 such that

$$U = [u_1 \quad u_2]_{2\times 2}$$

5. Find singular values $\sigma_1 = \sqrt{\lambda_1}$ & $\sigma_2 = \sqrt{\lambda_2}$ such that

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix}_{2\times 3} \qquad (\lambda_1 > \lambda_2)$$

6. No need to find A^TA. Eigenvalues of A^TA are λ_1, λ_2 and 0

7. Use formula $v_i = \dfrac{A^T u_i}{\sigma_i}$ or $v_i^T = \dfrac{u_i^T A}{\sigma_i}$ and find v_1 & v_2

8. Find v_3 using orthogonality (v_3 is orthogonal to v_1 & v_2)

$$\begin{bmatrix} \text{---} & v_1^T & \text{---} \\ \text{---} & v_2^T & \text{---} \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

V_3 = normalised null space

9. Matrix $V = [v_1 \quad v_2 \quad v_3]_{3\times 3}$

10. Write $A = U \Sigma V^T$

Eg: SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}_{2\times 3}$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Eigenvectors of AA^T are columns of U

Eigenvalues: $\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 34\lambda + 225 = 0$$
$$(\lambda - 25)(\lambda - 9) = 0 \Rightarrow \lambda_1 = 25 \quad \lambda_2 = 9$$

(i) $\lambda = 25$

$$\begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -8 & 8 \\ 0 & 0 \end{bmatrix}$$

$$-8x + 8y = 0 \Rightarrow x = y$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(ii) $\lambda = 9$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2}$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}_{2\times 3}$$

$$v_1 = \frac{A^T u_1}{5} = \frac{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{5} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{3} = \frac{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{3} = \begin{bmatrix} -1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for v_3 (Null space)

$$v_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 0 & -4/3\sqrt{2} & 1/3 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

Case 2: Matrix A is a tall matrix (eg: 3x2)

1. Find A^TA (2x2)

2. Find eigenvalues of A^TA : λ_1 & λ_2

3. Find corresponding eigenvectors x_1 & x_2

4. Normalise them to get v_1 & v_2 such that

$$V = [v_1 \quad v_2]_{2\times2}$$

5. Find singular values $\sigma_1 = \sqrt{\lambda_1}$ & $\sigma_2 = \sqrt{\lambda_2}$ such that

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix}_{3\times2} \qquad (\lambda_1 > \lambda_2)$$

6. No need to find AA^T. Eigenvalues of AA^T are λ_1, λ_2 and 0

7. Use formula $u_i = \dfrac{Av_i}{\sigma_i}$ and find u_1 & u_2

8. Find u_3 using orthogonality

9. Find $U = [u_1 \ u_2 \ u_3]$

10. Write $A = U \Sigma V^T$

Case 3: Matrix A is a square matrix

1. Find if $A = A^T$ (symmetric) and then check if A is positive definite

2. If A is positive definite, SVD is same as diagonalisation

$$A = U \Sigma V^T = \underbrace{Q}_{\text{eigenvector matrix (orthonormal)}} \Lambda Q^T$$

eg: $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

(using Gram-Schmidt process)

3. If A is not positive definite, find eigenvectors of AA^T (columns of U) and A^TA (columns of V).

4. Follow the steps to find U, Σ and V from case 1 & case 2

Q10. Find SVD for $A = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$

large $\longrightarrow$ small eigenvalues

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^T \Sigma V^T$$

$$A^T A = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix} = V \Sigma^T \Sigma V^T$$

eigenvectors $A^T A \longrightarrow$ columns of V

$$|A^T A - \lambda I| = 0$$

$$\begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 100\lambda + 1924 - 324 = 0$$

$$\lambda^2 - 100\lambda + 1600 = 0$$

$$(\lambda - 80)(\lambda - 20) = 0$$

$$\lambda_1 = 20 \qquad \lambda_2 = 80$$

eigenvectors

(i) $\lambda = 20$

$$(A^TA - 20I)x = 0$$

$$\begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 6 & 18 \\ 0 & 0 \end{bmatrix}$$

let $y = k$

$$6x + 18k = 0$$
$$x = -3k$$

$$\vec{x} = \left\{ \begin{bmatrix} -3k \\ k \end{bmatrix}, k \in R \right\} = \left\{ k \begin{bmatrix} -3 \\ 1 \end{bmatrix}, k \in R \right\}$$

unit vector $\vec{x} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

(ii) $\lambda = 80$

$$\begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 + 1/3 R_1$$

$$\begin{bmatrix} -54 & 18 \\ 0 & 0 \end{bmatrix}$$ let $y = k$

$$-54x + 18k = 0$$

$$x = \frac{1}{3}k$$

$$\vec{x} = \left\{ k\begin{bmatrix} 1 \\ 3 \end{bmatrix}, k \in R \right\}$$

$$\text{unit vector} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$AV = U\Sigma$$

$$\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = U\Sigma$$

$$= \begin{bmatrix} -10/\sqrt{10} & 20/\sqrt{10} \\ 10/\sqrt{10} & 20/\sqrt{10} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} -10 & 20 \\ 10 & 20 \end{bmatrix}$$

$$= \sqrt{10}\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \sqrt{10}\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{20} & 0 \\ 0 & 2\sqrt{20} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & 2\sqrt{20} \end{bmatrix}$$

$$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Q11. Find SVD for $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

$$AV = U\Sigma$$

symmetric positive / semi definite

$$\underline{A^TA} = V\Sigma\Sigma^T V^T \qquad S\Lambda S^{-1} = Q\Lambda Q^T$$

eigenvectors of A^TA : V

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16+64 & 12+48 \\ 12+48 & 9+36 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 80-\lambda & 60 \\ 60 & 45-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 125\lambda + 3600 - 3600 = 0$$

$$\lambda(\lambda - 125) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 125$$

eigenvectors

(i) $\lambda = 0$

$$\begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - 3/4 R_1$$

$$\begin{bmatrix} 80 & 60 \\ 0 & 0 \end{bmatrix} \qquad \text{let } x_2 = k$$

$$80x_1 + 60k = 0$$

$$x_1 = -\frac{3}{4}k$$

$$x = \left\{ k\begin{bmatrix} -3 \\ 4 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$

(ii) $\lambda = 125$

$$\begin{bmatrix} -45 & 60 \\ 60 & -80 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow R_2 \to R_2 + 4/3 R_1$$

$$\begin{bmatrix} -45 & 60 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = k$

$$-45x_1 + 60k = 0$$

$$x_1 = \frac{4}{3}k$$

$$x = \left\{ k \begin{bmatrix} 4 \\ 3 \end{bmatrix}, k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$V = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$AV = U\Sigma$$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5\sqrt{5} \end{bmatrix}$$

COVARIANCE MATRIX

- Principle component analysis
- Sample mean : M

Mean Deviation Form

Let $[x_1 \ x_2 \ \dots \ x_n]$ be a matrix

$$M = \frac{1}{N}(x_1 + x_2 + \dots + x_n)$$

$$\hat{x}_k = x_k - M$$

$$B = [\hat{x}_1 \ \hat{x}_2 \ \dots \ \hat{x}_n]$$

$$S = \frac{1}{N-1} \underbrace{BB^T}$$ positive semi definite

Q12. Find covariance of the following. Compute sample mean and the covariance matrix

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} \quad x_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} \quad x_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

Sample mean

$$M = \frac{1}{4}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}\right)$$

$N = 4$

$$= \frac{1}{4}\left(\begin{bmatrix} 20 \\ 16 \\ 20 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

$$M = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$ sample mean

Subtract M from all vectors

$$B = \begin{bmatrix} 1-5 & 4-5 & 7-5 & 8-5 \\ 2-4 & 2-4 & 8-4 & 4-4 \\ 1-5 & 13-5 & 1-5 & 5-5 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$S = \frac{1}{4-1} BB^T$$

$$S = \frac{1}{3} \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

variance of x

$$S = \text{covariance matrix} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

$S_{ij}\ \forall\ i \neq j$ is called covariance of x_i and x_j

app: correlation b/w sets of vectors

Q13. Find covariance matrix for the given dataset

$$a_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 8 \\ 4 \\ 26 \end{bmatrix} \quad a_3 = \begin{bmatrix} 14 \\ 16 \\ 2 \end{bmatrix} \quad a_4 = \begin{bmatrix} 16 \\ 8 \\ 10 \end{bmatrix}$$

$$M = \frac{1}{4}\begin{bmatrix} 2+8+14+16 \\ 4+4+16+8 \\ 2+26+2+10 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 40 \\ 32 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 2-10 & 8-10 & 14-10 & 16-10 \\ 4-8 & 4-8 & 16-8 & 8-8 \\ 2-10 & 26-10 & 2-10 & 10-10 \end{bmatrix}$$

$$B = \begin{bmatrix} -8 & -2 & 4 & 6 \\ -4 & -4 & 8 & 0 \\ -8 & 16 & -8 & 0 \end{bmatrix} \quad B^T = \begin{bmatrix} -8 & -4 & -8 \\ 2 & -4 & 16 \\ 4 & 8 & -8 \\ 6 & 0 & 0 \end{bmatrix}$$

$$S = \frac{1}{3}\begin{bmatrix} -8 & -2 & 4 & 6 \\ -4 & -4 & 8 & 0 \\ -8 & 16 & -8 & 0 \end{bmatrix}\begin{bmatrix} -8 & -4 & -8 \\ 2 & -4 & 16 \\ 4 & 8 & -8 \\ 6 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 120 & 72 & 0 \\ 72 & 96 & -96 \\ 0 & -96 & 384 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 24 & 0 \\ 24 & 32 & -32 \\ 0 & -32 & 128 \end{bmatrix}$$

Q14. Find covariance matrix for the vectors

$$x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$M = \frac{1}{2}\begin{bmatrix} 3+7 \\ 2+4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3-5 & 7-5 \\ 2-3 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \quad B^T = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$S = \frac{1}{1}\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+4 & 2+2 \\ 2+2 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$$

Q15. The following table lists weights and heights of 5 boys

Boy	#1	#2	#3	#4	#5
Weight (lbs)	120	125	125	135	145
Height (in)	61	60	64	68	72

Find covariance matrix of the following data

Sample mean

$$= \frac{1}{5}\left(\begin{bmatrix}120\\61\end{bmatrix}+\begin{bmatrix}125\\60\end{bmatrix}+\begin{bmatrix}125\\64\end{bmatrix}+\begin{bmatrix}135\\68\end{bmatrix}+\begin{bmatrix}145\\72\end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix}650\\325\end{bmatrix} = \begin{bmatrix}130\\65\end{bmatrix}$$

$$B = \begin{bmatrix}120-130 & 125-130 & 125-130 & 135-130 & 145-130\\ 61-65 & 60-65 & 64-65 & 68-65 & 72-65\end{bmatrix}$$

$$B = \begin{bmatrix}-10 & -5 & -5 & 5 & 15\\ -4 & -5 & -1 & 3 & 7\end{bmatrix} \quad B^T = \begin{bmatrix}-10 & -4\\ -5 & -5\\ -5 & -1\\ 5 & 3\\ 15 & 7\end{bmatrix}$$

$$S = \frac{1}{4}\begin{bmatrix} -10 & -5 & -5 & 5 & 15 \\ -4 & -5 & -1 & 3 & 7 \end{bmatrix}\begin{bmatrix} -10 & -4 \\ -5 & -5 \\ -5 & -1 \\ 5 & 3 \\ 15 & 7 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 100+25+25+25+225 & 40+25+5+15+105 \\ 40+25+5+15+105 & 16+25+1+9+49 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 400 & 190 \\ 190 & 100 \end{bmatrix} = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

$$S = \begin{bmatrix} 100 & 47.5 \\ 47.5 & 25 \end{bmatrix}$$

Q16. Construct SVD of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

A is a short matrix

$$A = U \Sigma V^T \Rightarrow AV = U\Sigma$$

$$AA^T = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 333-\lambda & 81 \\ 81 & 117-\lambda \end{vmatrix} = 0$$

$$(\lambda-333)(\lambda-117)-81^2=0$$

$$\lambda^2-450\lambda+32400=0$$

$$\lambda_1=360 \qquad \lambda_2=90$$

eigenvectors

(i) $\lambda_1=360$

$$\begin{bmatrix} 333-360 & 81 \\ 81 & 117-360 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -27 & 81 \\ 81 & -243 \end{bmatrix} \xrightarrow{R_2 \to R_2+3R_1} \begin{bmatrix} -27 & 81 \\ 0 & 0 \end{bmatrix}$$

$$-27x+81y=0$$
$$x=3y$$

$$u_1=\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

(ii) $\lambda_2=90$

$$\begin{bmatrix} 333-90 & 81 \\ 81 & 117-90 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 243 & 81 \\ 81 & 27 \end{bmatrix} \xrightarrow{R_2 \to R_2-1/3R_1} \begin{bmatrix} 243 & 81 \\ 0 & 0 \end{bmatrix}$$

$$243x+81y=0$$
$$x=-\frac{1}{3}y$$

$$u_2 = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$U = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$v_1 = \frac{A^T u_1}{\sqrt{360}} = \frac{\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{360}} = \frac{\begin{bmatrix} 2\sqrt{10} \\ 4\sqrt{10} \\ 4\sqrt{10} \end{bmatrix}}{6\sqrt{10}} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{\sqrt{90}} = \frac{\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}\begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}}{3\sqrt{10}} = \frac{\begin{bmatrix} 2\sqrt{10} \\ \sqrt{10} \\ -2\sqrt{10} \end{bmatrix}}{3\sqrt{10}} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

To find v_3:

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & -1 & -2 \end{bmatrix}$$

$$-y-2z=0$$
$$y=-2z$$

$$x-4z+2z=0$$
$$x=2z$$

$$x=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$v_3=\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$V=\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \quad V^T=\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$U=\begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \quad \Sigma=\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$A=\underbrace{\begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}}_{V^T}$$

Q17. Compute SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

A is tall matrix

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 18\lambda = 0$$

$$\lambda = 0 \qquad \lambda = 18$$

eigenvectors

(i) $\lambda = 18$

$$\begin{bmatrix} 9-18 & -9 \\ -9 & 9-18 \end{bmatrix} = \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} -9 & -9 \\ 0 & 0 \end{bmatrix}$$

let $x_2 = k \Rightarrow -9x_1 - 9k = 0 \Rightarrow x_1 = -k$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = v_1$$

(ii) $\lambda = 0$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 9 & -9 \\ 0 & 0 \end{bmatrix}$$

Let $x_2 = k \Rightarrow 9x_1 - 9k = 0 \Rightarrow x_1 = k$

$$x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = v_2$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{18} = 3\sqrt{2}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0} = 0$$

$$\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A v_1 = u_1 \sigma_1 \quad \text{or} \quad u = \frac{A v_1}{\sigma_1}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 2\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = 3\sqrt{2}\, U_1$$

$$U_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

find U_2 & U_3 through orthogonality

$$-\frac{1}{\cancel{3}}x + \frac{2}{\cancel{3}}y - \frac{2}{\cancel{3}}z = 0 \implies x = \begin{bmatrix} 2y-2z \\ y \\ z \end{bmatrix}$$

$$NS = \left\{ y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Using G.S process

$$q_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - q_1^T \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} q_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$c = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$- \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{5}} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

$$V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Q18. Compute $x^T A x$ for the following

(a) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

(a) $x^T A x$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4x \\ 3y \end{bmatrix}$$

$$= 4x^2 + 3y^2$$

(b) $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3x - 2y \\ -2x + 7y \end{bmatrix}$$

$$= 3x^2 - 2xy - 2xy + 7y^2$$

$$= 3x^2 + 7y^2 - 4xy$$

Q19. For x in R^3, Let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$

Write quadratic form as $x^T A x$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2fx_2x_3 + 2gx_3x_1$$

$a = 5$ $\quad 2h = -1$ $\quad 2f = 8$ $\quad g = 0$

$b = 3$ $\quad h = -1/2$ $\quad f = 4$

$c = 2$

$$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

Q20. Let $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.
Check if A is positive definite.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$a = 3$ $\quad h = 2$

$b = 2$ $\quad f = 2$

$c = 1$ $\quad g = 0$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(i) pivots

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2/3 R_1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2/3 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2/3 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$-ve$

$\therefore$ not positive definite

Q21. Let $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

Compute $Q(x)$ for

i) $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ (ii) $x = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (iii) $x = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

i) $Q(x) = (-3)^2 - 8(-3)(1) - 5(1)^2$

$= 9 + 24 - 5 = 28$

(ii) $Q(x) = (2)^2 - 8(2)(-2) - 5(-2)^2$

$= 4 + 32 - 20 = 16$

(iii) $Q(x) = (1)^2 - 8(1)(-3) - 5(-3)^2$

$= 1 + 24 - 45 = -20$

Q22. Compute $Q(x)$ form x^TAx for

$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(a) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (b) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ (c) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

(a) $[x_1 \quad x_2 \quad x_3] \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [4x_1 + 3x_2 \quad 3x_1 + 2x_2 + x_3 \quad x_2 + x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 4x_1^2 + 3x_1x_2 + 3x_1x_2 + 2x_2^2 + x_2x_3 + x_2x_3 + x_3^2$$

$$= 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$$

(b) $x = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$$Q(x) = 4(4) + 2(1) + 25 + 6(-2) + 2(-5)$$

$$= 16 + 2 + 25 - 12 - 10$$

$$= 21$$

(c) $x = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$Q(x) = 4(1/3) + 2(1/3) + (1/3) + 6(1/3) + 2(1/3)$$

$$= 5$$

Q23. Find SVD of the following

(i) $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$A = U \Sigma V^T$$

(i) $A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 16+9 & 16-9 \\ 16-9 & 16+9 \end{bmatrix}$

$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix} = 0$

$\lambda^2 - 50\lambda + 625 - 49 = 0$

$\lambda^2 - 50\lambda + 576 = 0$

$\lambda_1 = 32 \quad \lambda_2 = 18$ descending order

eigenvectors

(a) $(A - 32I)x = 0$

$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix}$

Let $x_2 = k$

$-7x_1 + 7k = 0 \Rightarrow x_1 = k \to x = \left\{ k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in R \right\}$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(b) $(A - 18I)x = 0$

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 7 & 7 \\ 0 & 0 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$\sigma_1 = \sqrt{32} = 4\sqrt{2}$ $\quad$ $\sigma_2 = \sqrt{18} = 3\sqrt{2}$

$$AV = U\Sigma$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

A is tall matrix

$$A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

eigenvalues

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$\lambda = 2 \quad \lambda = 3$

eigenvectors

(a) $\lambda = 3$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{aligned} -x_1 + 0 &= 0 \\ x_1 &= 0 \end{aligned} \quad \Rightarrow x = \left\{ k \begin{bmatrix} 0 \\ 1 \end{bmatrix}, k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(b) $\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_2 = 0 \quad \Rightarrow \quad x = \left\{ k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \sqrt{3} \qquad \sigma_2 = \sqrt{2}$$

$$AV = U\Sigma$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$-y - 2z = 0 \qquad\qquad x - 2z + z = 0$$
$$y = -2z \qquad\qquad x = z$$

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$U_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Q24. check if orthogonally diagonalisable

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} \qquad A = UDU^T$$

yes, $\because$ it is symmetric

characteristic equation

$$-\lambda^3 + 17\lambda^2 - (29 + 29 + 32)\lambda + 144 = 0$$

$$-\lambda^3 + 17\lambda^2 - 90\lambda + 144 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 8 \qquad \lambda_3 = 6$$

eigenvectors

(a) $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 1/2 R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} -2 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -5/2 \end{bmatrix}$$

$$-\frac{5}{2} z = 0 \Rightarrow \quad z = 0$$

$$-2x - 2k = 0$$
$$x = -k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

(b) $\lambda = 6$

$$\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1/2 R_1}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 \to R_3 - 1/2 R_2} \begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ 0 & -1 & -1/2 \end{bmatrix}$$

$$-2y - k = 0 \quad \Rightarrow \quad y = -\frac{1}{2} k$$

$$-2x - k = 0 \Rightarrow x = \frac{-1}{2}k$$

$$x = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix},\ k \in R \right\}$$

$$\hat{x} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\begin{matrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{matrix}$$

(c) $\lambda = 3$

$$\begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

null space: $x = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ k \in R \right\}$

$$\hat{x} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$S = U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$S^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$SDS^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$