UNIT-3 DIFF. EQ OF HIGHER ORDER AND P.D.E

- Method of Variation of Parameters to find PI

- Can be used to find 2^{nd} order LDE when inverse D isn't possible

- Working :

1) Consider LDE $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X$

where a_1 & a_2 are constants

2) Find CF $= y_c = c_1y_1 + c_2y_2$

3) Find PI $= -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

4) $y = CF + PI$

Q. Solve $y'' + 4y = \sec 2x$

$$D^2y + 4y = \sec 2x$$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$CF = c_1 \cos 2x + c_2 \sin 2x$$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$y_1 = \cos 2x$ $\quad$ $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ $\quad$ $y_2' = 2\cos 2x$

$W =$

$$2\cos^2 2x + 2\sin^2 2x$$

$$= 2$$

$$PI = -\cos 2x \int \frac{\sin 2x \times \sec 2x}{2} dx + \sin 2x \int \frac{\cos 2x \times \sec 2x}{2} dx$$

$$= \frac{-\cos 2x}{2} \int \tan 2x \, dx + \frac{x \sin 2x}{2}$$

$$= \frac{+\cos 2x . \log \cos 2x + 2x \sin 2x}{4}$$

$$y = CF + PI$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{\cos 2x . \log \cos 2x + 2x \sin 2x}{4}$$

Q. $y'' + y' = \frac{1}{1+\sin x}$

$$\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$CF = c_1 \cos x + c_2 \sin x$$

$$PI = -y_1 \int \frac{y_2 X\,dx}{W} + y_2 \int \frac{y_1 X\,dx}{W}$$

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$y_1' = -\sin x \qquad y_2' = \cos x$$

$$W = \cos^2 x + \sin^2 x = 1$$

$$PI = -y_1 \int y_2 \cdot \frac{1}{1+\sin x}\,dx + y_2 \int y_1 \frac{1}{1+\sin x}\,dx$$

$$= -\cos x \int \frac{\sin x}{1+\sin x}\,dx + \sin x \int \frac{\cos x}{1+\sin x}\,dx$$

$$= -\cos x \int \left(\frac{\sin x + 1}{\sin x + 1} - \frac{1}{1+\sin x}\right) dx + \sin x \int -\frac{dt}{t}$$

$$= -\cos x \int \left(1 - \frac{\sec^2 \frac{x}{2}}{\left(\tan\frac{x}{2} + 1\right)^2}\right) dx - \sin x \log t$$

$$= -\cos x \left(x - \int \frac{2\,dt}{(t)^2}\right) - \sin x \log(1+\sin x)$$

$$= -\cos x \left(x + \frac{2}{t}\right) - \sin x \log(1+\sin x)$$

$$= -\cos x \left(x + \frac{2}{\left(\tan\frac{x}{2} + 1\right)}\right) - \sin x \log(1+\sin x)$$

$$y = c_1 \cos x + c_2 \sin x - x\cos x - \frac{2\cos x}{\tan\frac{x}{2} + 1} - \sin x \log(1+\sin x)$$

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Q. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x}$

$(D^2 + 6D + 9)y$

$m = -3$

$CF = (c_1 + c_2 x)e^{-3x} = c_1 e^{-3x} + c_2 \cdot x e^{-3x}$

$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X dx}{W}$

$W = y_1 y_2' - y_2 y_1'$

$= e^{-3x} \cdot (\cancel{3x}e^{-3x} + e^{-3x}) - x e^{-3x} \cdot (-3e^{-3x})$

$= \cancel{-3xe^{-6x}} + e^{-6x} + \cancel{3xe^{-6x}}$

$= e^{-6x}$

$PI = -e^{-3x} \int \frac{\cancel{x}\cancel{e^{-3x}} \cdot \cancel{e^{-3x}}}{\cancel{e^{-6x}}\,\cancel{x}} dx + x e^{-3x} \int \frac{\cancel{e^{-3x}} \cdot \cancel{e^{-3x}}}{\cancel{e^{-6x}}\, x} dx$

$= -xe^{-3x} + x e^{-3x} \cdot \log x$

$y = CF + PI$

$= c_1 e^{-3x} + c_2 x \cdot e^{-3x} - xe^{-3x} + xe^{-3x} \log x$

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Q. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

$m^2 + 3m + 2 = 0$

$m = -1, -2$

$CF = c_1 e^{-x} + c_2 e^{-2x}$

$PI = -y_1 \int \frac{y_2 X}{W} + y_2 \int \frac{y_1 X}{W}$

$W = e^{-x} \times -2e^{-2x} + e^{-2x} \cdot e^{-x}$

$= -e^{-3x}$

$PI = -e^{-x} \int \frac{e^{-2x} \cdot e^{e^x}}{-e^{-3x}} + e^{-2x} \int \frac{e^{-x} \cdot e^{e^x}}{-e^{-3x}}$

$= +e^{-x} \int e^{x+e^x} dx - e^{-2x} \int e^{2x+e^x} dx$

$= e^{-x} \cdot e^{e^x} - e^{-2x} \left[(e^x - 1) e^{e^x} \right]$

$= e^{e^x - x} - e^{-2x} \left(e^{e^x + x} - e^{e^x} \right)$

$= \cancel{e^{e^x - x}} - \cancel{e^{e^x - x}} + e^{e^x - 2x} = +e^{e^x - 2x}$

$= +e^{e^x} \cdot e^{-2x}$

$y = CF + PI$

$= c_1 e^{-x} + c_2 e^{-2x} + e^{e^x} \cdot e^{-2x}$

Q. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = \dfrac{1}{1-e^x}$

$m^2 + m - 2 = 0$

$m = 1, -2$

$CF = c_1 e^x + c_2 e^{-2x}$

$W = e^x \cdot -2e^{-2x} - e^x \cdot e^{-2x}$

$= -3e^{-x}$

$PI = -y_1 \int \dfrac{y_2 X}{W} dx + y_2 \int \dfrac{y_1 X}{W} dx$

$= -e^x \int \dfrac{e^{-2x} dx}{-3e^{-x}(1-e^x)} + e^{-2x} \int \dfrac{e^x}{-3e^{-x}(1-e^x)} dx$

$\cancel{= \dfrac{+e^x}{3} \int \dfrac{e^{-2x}}{e^{-x}-1} dx - \dfrac{e^{-2x}}{3} \int \dfrac{e^x}{-1+e^{-x}} dx}$

$\cancel{t = 1 - e^{-x}} \qquad \cancel{t = e^{-x} - 1}$

$\cancel{dt = e^{-x} dx} \qquad \cancel{dt = -e^{-x}}$

$= +\dfrac{e^x}{3} \int \dfrac{e^{-x} dx}{(1-e^x)} - \dfrac{e^{-2x}}{3} \int \dfrac{e^{2x}}{(1-e^x)} dx$

$e^x = t$

$dt = e^x dx$

$= -\dfrac{e^x}{3} \int \dfrac{e^{-x} dx}{e^x - 1} - \dfrac{e^{-2x}}{3} \int \dfrac{t\,dt}{1-t}$

$= -\dfrac{e^x}{3} \int \dfrac{dt}{(t-1)t^2} - \dfrac{e^{-2x}}{3} \int \dfrac{t\,dt}{1-t}$

$= -\dfrac{e^x}{3} \int \left(\dfrac{-1}{t} - \dfrac{1}{t^2} + \dfrac{1}{t-1} \right) dt - e^{-2x} \left[\left(-\ln(e^x - 1) - e^x \right) \right]$

$= -\dfrac{e^x}{3} \left[-\ln(e^x - 1) - e^{-x} + x \right] + e^{-2x} \ln(e^x - 1) + e^{-x}$

$PI = \dfrac{1}{} + e^x \ln(e^x - 1) - \dfrac{xe^x}{} + e^{-2x} \ln(e^x - 1) + e^{-x}$

$$y = CF + PS$$
$$= c_1 e^{x} + c_2 e^{-2x} + \frac{1}{3} + e^{-x} + \frac{e^{x}}{3}\left[\ln(e^{x}-1) - x\right]$$
$$+ e^{-2x} \ln(e^{x} - 1$$

Q. HW

1. $\frac{d^2y}{dx^2} + y = \sec x \tan x$

2. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

3. $y'' - 2y' + y = e^x \log x$

4. $\frac{d^2y}{dx^2} + 4y = \tan 2x$

5. $y'' - y = 2(1-e^{-2x})^{-1/2}$

6. $y'' - y = e^{-2x} \sin(e^{-x})$

7. $y'' + y = \operatorname{cosec} x \cot x$

8. $(D^2 - 1)y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

- Legendre's Linear Differential Equation

- If LDE of the form

$$a_0 (ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} (ax+b) \frac{dy}{dx} + a_n y = X(x)$$

Then,

$$ax+b = e^t \quad \text{(or)} \quad t = \log(ax+b)$$

$$\frac{dt}{dx} = \frac{a}{ax+b}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{a}{ax+b}$$

$$(ax+b)\frac{dy}{dx} = a \cdot \frac{dy}{dt}$$

$$= a\,Dy$$

Therefore, $(ax+b) \dfrac{dy}{dx} = a\,Dy$

Similarly, $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)\,y$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)\,y$$

HW Problems

1. $\frac{d^2y}{dx^2} + y = \sec x \tan x$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$CF = c_1 \cos x + c_2 \sin x$

$W = \cos^2 x + \sin^2 x = 1$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -\cos x \int \sin x \cdot \sec x \cdot \tan x \, dx + \sin x \int \cancel{\cos x} \cdot \cancel{\sec x} \cdot \tan x \, dx$$

$$= -\cos x \int \tan^2 x \, dx + \sin x \int \tan x \, dx$$

$$= -\cos x \int (\sec^2 x - 1) dx + \sin x \int \tan x$$

$$= -\cos x (\tan x - x) + \sin x \cdot \log \sec x$$

$$PI = -\sin x + x \cos x + \sin x \cdot \log \sec x$$

$$y = CF + PI = c_1 \cos x + c_2 \sin x - \sin x + x \cos x + \sin x \log \sec x$$

2. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$m = \pm 1 \Rightarrow c_1 e^x + c_2 e^{-x} = CF$

$W = -\cancel{e^x} \cdot \cancel{e^{-x}} - \cancel{e^x} \cdot \cancel{e^{-x}}$

$= -2$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = \frac{+e^x}{+\cancel{2}} \int \frac{e^{-x} \cdot \cancel{2}}{1+e^x} + \frac{e^{-x}}{-\cancel{2}} \int \frac{e^x \cancel{2}}{1+e^x} dx$$

$$= e^x \int \frac{e^{-x} dx}{1+e^x} - e^{-x} \int \frac{e^x}{1+e^x} dx$$

$u = e^x$, $\frac{du}{dx} = e^x$ | $t = 1 + e^x$, $\frac{dt}{dx} = e^x$

$y = c_1 e^x + c_2 e^{-x} - 1 - x e^{x}$

$$= e^x \int \frac{du}{(1+u)u^2} - e^{-x} \int \frac{dt}{t}$$

$$= e^x \int \left(\frac{-1}{u} + \frac{1}{u^2} + \frac{1}{1+u} \right) du - e^{-x} \log(1 + e^x)$$

$$= e^x \left(-\ln(u) - \frac{1}{u} + \log(1+u) \right) - e^x \log(1 + e^x) = -xe^x - 1$$

$e^x (\ldots + \log(1+e^x)) - e^x \log(1+e^x)$

Q 3 $\quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x \qquad \int f(x)g(x) = f(x)\int g(x) - \int \int g(x) . f'(x)$

$m^2 - 2m + 1 \Rightarrow m = +1$

$CF = (c_1 + c_2 x)e^{-x} = c_1 e^{-x} + c_2 e^{-x} x \qquad CF = c_1 e^x + c_2 e^x x$

$W = e^{-x}(e^{-x} - xe^{-x}) + xe^{-2x} \qquad W = e^x(e^x + xe^x) - xe^{2x}$

$= e^{-2x} \qquad = e^{2x}$

$PI = -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W} dx$

$= -e^{-x}\int \frac{x . e^{-x} e^x \log x}{e^{-2x}} dx + xe^{-x}\int \frac{e^{-x} . e^x \log x}{e^{-2x}}$

$= -e^{-x}\int xe^{2x}\log x\, dx + xe^{-x}\int e^{2x}\log x\, dx$

$-e^{-x}\Big[\qquad + xe^{-x}\Big[\frac{e^{2x}\log x}{2} - \int \frac{e^{2x}}{2x}dx\Big]$

$PI = -e^x\int \frac{x . \cancel{e^x} . e^x \log x}{\cancel{e^{2x}}} dx + xe^x\int \frac{\cancel{e^x} . \cancel{e^x}\log x}{\cancel{e^{2x}}} dx$

$= -e^x\int x\log x\, dx + xe^x\int 1 . \log x\, dx$

$= -e^x\Big[\log x . \frac{x^2}{2} - \int \frac{x^{\cancel{2}}}{2} . \frac{1}{\cancel{x}}\Big] + xe^x\Big[\log x . x - \int x . \frac{1}{x}\Big]$

$= -e^x\Big[\frac{x^2}{2}\log x - \frac{x^2}{4}\Big] + xe^x\big[x\log x - x\big]$

$= -\frac{e^x . x^2\log x}{2} + \frac{e^x . x^2}{4} + x^2e^x\log x - x^2e^x$

$= \frac{x^2e^x\log x}{2} + \frac{3x^2e^x}{4}$

4. $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

$m = \pm 2i \Rightarrow CF = c_1\cos 2x + c_2 \sin 2x$

$W = 2\cos^2 2x + 2\sin^2 2x = 2$

$PI = -\cos 2x \int \left(\dfrac{\sin 2x \cdot \tan 2x}{2}\right) + \sin 2x \int \dfrac{\cos 2x \cdot \tan 2x}{2}\,dx$

$u = 2x$

$du = 2dx$

$= \dfrac{-\cos 2x}{4}\int \sin u \tan u\,du - \dfrac{\sin 2x \cos 2x}{4}$

$= \dfrac{-\cos 2x}{4}\int \sec u \cdot \sin^2 u\,du - \dfrac{\sin 2x \cos 2x}{4}$

$= \dfrac{-\cos 2x}{4}\int \sec u(1-\cos^2 u)\,du - \dfrac{\sin 4x}{8}$

$= \dfrac{-\cos 2x}{4}\int (\sec u - \cos u)\,du$

$= \dfrac{-\cos 2x}{4}\left(\ln(\tan u + \sec u) - \sin u\right)$

$= \dfrac{-\cos 2x\left(\ln(\tan 2x + \sec 2x) - \cancel{\sin 2x}\right)}{4} - \cancel{\dfrac{\sin 4x}{8}}$

$= \dfrac{-\cos 2x \ln(\tan 2x + \sec 2x)}{4}$

$y = CF + PI = c_1\cos 2x + c_2 \sin 2x - \dfrac{\cos 2x \ln(\tan 2x + \sec 2x)}{4}$

5. $y'' - y = 2(1-e^{-2x})^{-1/2}$

$m = \pm 1 \Rightarrow c_1 e^{x} + c_2 e^{-x} = CF$

$W = -1 - 1 = -2$

$PI = \cancel{+}2e^{x}\int \frac{e^{-x}(1-e^{-2x})^{-1/2}}{-\cancel{2}}dx + \frac{\cancel{2}e^{-x}}{\frac{-\cancel{2}}{-1}}\int e^{+x}(1-e^{-2x})^{-1/2}dx$

$= e^{x}\int e^{-x}(1-e^{-2x})^{-1/2}dx - e^{-x}\int e^{x}(1-e^{-2x})^{-1/2}dx$

$\downarrow$

$e^{-x} = t$

$\frac{dt}{dx} = -e^{-x}$

$-e^{-x}\int \frac{e^{x}}{\left(1-\frac{1}{e^{2x}}\right)^{1/2}}dx$

$e^{x}\int \frac{-dt}{\sqrt{1-t^2}}$

$-e^{-x}\int \frac{e^{2x}\,dx}{\sqrt{e^{2x}-1}}$

$e^{2x}-1 = t$

$2e^{2x}dx = dt$

$= -e^{x}\sin^{-1}t \quad + \quad \frac{-e^{x}}{2}\int \frac{dt}{\sqrt{t}}$

$= -e^{x}\sin^{-1}(-e^{-x}) - \frac{e^{-x}}{\cancel{2}} \times \cancel{2}\, t^{1/2}$

$PI = -e^{x}\sin^{-1}(-e^{-x}) - e^{-x}\sqrt{e^{2x}-1}$

$y = CF + PI$

$= c_1 e^{x} + c_2 e^{-x} - e^{x}\sin^{-1}(-e^{-x}) - e^{-x}\sqrt{e^{2x}-1}$

6. $y'' - y = e^{-2x}\sin(e^{-x})$

$m^2 = 1 \Rightarrow m = \pm 1$

$CF = c_1 e^{x} + c_2 e^{-x}$

$W = -2$

$$PI = -e^{x}\int \frac{e^{-x}.e^{-2x}\sin(e^{-x})}{-2} + e^{-x}\int \frac{e^{x}e^{-2x}\sin(e^{-x})}{-2}$$

$$= \frac{e^{x}}{2}\int e^{-3x}\sin(e^{-x})dx - \frac{e^{-x}}{2}\int e^{-x}\sin e^{-x}\,dx$$

$u = e^{-x}$ $\qquad$ $u = e^{-x}$

$du = -e^{-x}dx$ $\qquad$ $du = -e^{-x}dx$

$\Rightarrow -du\sin u$

$$= \frac{e^{x}}{2}\int -u^2\sin u\,du - \frac{e^{-x}}{2}\int -\sin u\,du$$

$$= \frac{e^{x}}{2}\left[-u^2\cos u + \int 2u\cos u\,du\right] + \frac{e^{-x}}{2}[-\cos u]$$

$$= \frac{e^{x}}{2}\left[-u^2\cos u + 2\left[u\sin u - \int \sin u\,du\right]\right] + \frac{e^{-x}}{2}[-\cos e^{-x}]$$

$$= \frac{e^{x}}{2}[-u^2\cos u + 2u\sin u + 2\cos u] - \frac{e^{-x}}{2}[+\cos e^{-x}]$$

$$y = c_1e^{x} + c_2e^{-x} + \frac{e^{x}}{2}\left[-e^{-2x}\cos e^{-x} + 2e^{-x}\sin e^{-x} + 2\cos e^{-x}\right] - \frac{e^{-x}}{2}(\cos e^{-x})$$

$$= c_1e^{x} + c_2e^{-x} + \sin e^{-x} + e^{x}\cos e^{-x}$$

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7. $y'' + y = \operatorname{cosec} x \cot x$

$m = \pm i$

$CF = c_1 \cos x + c_2 \sin x$

$W = 1$

$$PI = -\cos x \int \frac{\sin x \cdot \frac{1}{\sin x} \cdot \cot x}{1} dx + \sin x \int \frac{\cos x \cdot \operatorname{cosec} x \cdot \cot x}{1} dx$$

$$= -\cos x \int \cot x \, dx + \sin x \int \cot^2 x \, dx$$

$$= -\cos x \int \cot x \, dx + \sin x \int (\operatorname{cosec}^2 x - 1) dx$$

$$= -\cos x (\ln \sin x) + \sin x (-\cot x - x)$$

$$= -\cos x \ln \sin x - \cos x - x \sin x$$

$$= + \cos x \ln \operatorname{cosec} x - \cos x - x \sin x$$

$y = CF + PI$

$$= c_1 \cos x + c_2 \sin x + \cos x \ln \operatorname{cosec} x - \cos x - x \sin x$$

8. $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos e^{-x}$

$m = \pm 1 \Rightarrow CF = c_1 e^{x} + c_2 e^{-x}$

$W = -2$

$$PI = -e^{x}\int \frac{e^{-x}.(e^{-x}\sin e^{-x} + \cos e^{-x})}{-2} + e^{-x}\int \frac{e^{x}(e^{-x}\sin e^{-x} + \cos e^{-x})}{-2}$$

$$= -e^{+x}\int \frac{e^{-2x}\sin e^{-x} + e^{-x}\cos e^{-x}}{-2} - \frac{e^{-x}}{2}\int (\cancel{e^{x}e^{-x}}\sin e^{-x} + e^{x}\cos e^{-x})\,dx$$

$$= \frac{+e^{x}}{2}\int(-u\sin u - \cos u)\,du - \frac{e^{-x}}{2}\left[e^{x}\cos(e^{-x}) - \int \sin e^{-x}\,dx + \int \sin e^{-x}\right]$$

$$= \frac{e^{x}}{2}\left[-\left(-u\cos u - \int -\cos u\right)du - \int \cos u\, du\right] - \frac{e^{-x}}{2}\left[e^{x}\cos e^{-x}\right]$$

$$= \frac{e^{x}}{2}\left[-\sin u + u\cos u - \sin u\right] - \frac{\cos e^{-x}}{2}$$

$$= \frac{e^{x}}{2}\left[-2\sin u + u\cos u\right] - \frac{\cos e^{-x}}{2}$$

$$= \frac{e^{x}}{2}\left[-2\sin e^{-x} + e^{-x}\cancel{\cos e^{-x}}\right] - \cancel{\frac{\cos e^{-x}}{2}}$$

$PI = -e^{x}\sin e^{-x}$

$y = CF + PI$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - e^{x}\sin e^{-x}}$$

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Legendre's LDE for 2nd order

$$\Rightarrow a_0(ax+b)^2 \cdot \frac{d^2y}{dx^2} + a_1(ax+b)\frac{dy}{dx} + a_2y = X(x)$$

$(ax+b) = e^t \Rightarrow t = \log(ax+b)$

$$\frac{dt}{dx} = \frac{a}{ax+b}$$

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)^2\frac{d^2y}{dy^2} = a^2D(D-1)y$$

$$(ax+b)^3\frac{d^3y}{dy^3} = a^3D(D-1)(D-2)y$$

Q. $(2x-1)^2\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2-2x+3$

$\cancel{4D(D-1)y + 2Dy = 8x^2 - 2x + 3x}$

$(2x-1) = e^t \Rightarrow \log(2x-1) = t$

$$\frac{dt}{dx} = \frac{2}{2x-1} \qquad x = \frac{e^t+1}{2}$$

If $D = \frac{d}{dt}$, $4D(D-1)y + 2Dy - 2y = 8x^2-2x+3$

$$4D(D-1)y + 2Dy - 2y = \cancel{8}^{2}\left(\frac{e^t+1}{\cancel{2}}\right)^2 - \cancel{2}\left(\frac{e^t+1}{\cancel{2}}\right) + 3$$

$$4D^2y - 4Dy + 2Dy - 2y = 2(e^{2t}+1+2e^t) - e^t - 1 + 3$$

$$(4D^2 - 2D - 2)y = 2e^{2t} + 2 + 4e^t - e^t + 2$$

$$(2D^2 - D - 1)y = \underline{2e^{2t} + 3e^t + 4}$$

$m_1 = 1 \quad , \quad m_2 = -\frac{1}{2}$

$CF = c_1 e^{t} + c_2 e^{-t/2}$

$$PI = \frac{e^{2t} + \frac{3}{2} e^{t} + 2e^{0t}}{2D^2 - D - 1}$$

$$= \frac{e^{2t}}{5} + \frac{x e^{t}}{2} - 2$$

$y = CF + PI$

$$= c_1 e^{t} + c_2 e^{-t/2} + \frac{e^{2t}}{5} + \frac{t e^{t}}{2} - 2$$

$$= c_1(2x-1) + c_2 \frac{1}{\sqrt{2x-1}} + \frac{(2x-1)^2}{5} + \frac{t(2x-1)}{2} - 2$$

$$= c_1(2x-1) + \frac{c_2}{\sqrt{2x-1}} + \frac{(2x-1)^2}{5} + \frac{(2x-1)}{2} \log(2x-1) - 2$$

Q. $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$

$(2x+1) = e^t$

$t = \log(2x+1)$ $\qquad x = \frac{e^t - 1}{2}$

$\frac{dt}{dx} = \frac{2}{2x+1}$

If $D = \frac{d}{dt}$,

$4 \times D(D-1)y - 6 \times 2 \times Dy + 16y = 8\left(e^t - \cancel{1} + \cancel{1}\right)^2$

$= 8e^{2t}$

$(4D^2 \cancel{-4D} \cancel{-12D} + 16)y = 8e^{2t}$ $\quad -16D$

$(D^2 - 4D + 4)y = 2e^{2t}$

$CF = c_1 e^{2t} + c_2 t e^{2t}$

$PI = \frac{2 \cdot e^{2t}}{D^2 - 4D + 4} = \frac{2 t^2 e^{2t}}{2} = t^2 e^{2t}$.

$= (\log(2x+1))^2 \cdot (2x+1)^2$

$y = CF + PI$

$= c_1(2x+1)^2 + c_2 \log(2x+1)(2x+1)^2$

$+ (\log(2x+1))^2 (2x+1)^2$

$= (2x+1)^2 \left(c_1 + c_2 \log(2x+1) + (\log(2x+1))^2\right)$

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Q. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)]$

$1+x = e^t$ | $x = e^t - 1$

$t = \log(1+x)$

$\frac{dt}{dx} = \frac{1}{1+x}$

$1 \times D(D-1)y + Dy + y$

$(D^2 - \cancel{D} + \cancel{D} + 1)y \Rightarrow (D^2+1)y = \sin[2\log(1+x)]$

$= \sin[2\log e^t$

$= \sin 2t$

$\Rightarrow CF = c_1 \cos t + c_2 \sin t$

$PI = \frac{\sin 2t}{D^2+1} = \frac{\sin 2t}{-3}$

$= \frac{\sin 2(\log(1+x))}{-3}$

$y = c_1\cos(\log(1+x)) + c_2\sin(\log(1+x)) - \frac{\sin(2\log(1+x)}{3}$

Q. $(2x+5)^2 \frac{d^2y}{dx^2}$

$t = \log(2x+5)$

$x = \frac{e^t - 5}{2}$

$\cancel{D(D-1) \quad -6D + 8)y}$

$\cancel{(4D^2 - 4D - 24D + 8)y}$

$\cancel{(D^2 - 7D + 8)y}$

$\cancel{4D^2 - 28D + 8}$

$\cancel{D^2 - 7D + 2}$

$4D(D-1)y - 12Dy + 8y = \frac{3}{}(e^t - 5)$

$4D^2 - 4D - 12D + 8$

$4(D^2 - 4D + 2)y = 3(e^t - 5)$

$m = (2 \pm \sqrt{2})$

$CF = c_1 e^{(2+\sqrt{2})t} + c_2 e^{(2 - \sqrt{2})t}$

$PI = \frac{1}{D^2 - 4D + 2} \cdot \frac{3e^t - 15}{4}$

$= \frac{3e^t}{\cancel{1-4+2} \; -4} - \frac{15e^0}{8}$

$= -\frac{3}{4}e^t - \frac{15}{8}$

$= -\frac{3}{4}e^{\log(2x+5)} - \frac{15}{8}$

$= -\frac{3}{4}(2x+5) - \frac{15}{8} = -\frac{3x}{2} - \frac{15}{4} - \frac{15}{8}$

$= -3x \quad 45$

$\boxed{y = c_1(2x+5)^{2+\sqrt{2}} + c_2(2x+5)^{2-\sqrt{2}} - \frac{3x}{2} - \frac{45}{8}}$

LRC circuit

$V = iR$

$i = \frac{dq}{dt}$

$V_R = R\frac{dq}{dt}$ | $V_L = L\frac{di}{dt}$ | $V_C = \frac{q}{C}$

$E = L\frac{di}{dt} + iR + \frac{q}{C}$

$= L\frac{d^2q}{dt^2} + i\frac{dq}{dt} + \frac{1}{C}\cdot q$

$= LD^2q + iDq + \frac{1}{C}\cdot q$

$E = \left(LD^2 + iD + \frac{1}{C}\right)q$

Q. $L = 0.25\,H$ $R = 250\,\Omega$ $C = 2\times10^{-6}$

$\cancel{L\int\frac{d^2q}{dt^2} + R\int\frac{dq}{dt} + \int\frac{q}{C} = 0}$

$\cancel{L\cdot\frac{dq}{dt} + R\cdot q + Ct = K}$

$\cancel{\text{at } t = 0,}$

$\cancel{L(0) + R\times0.002 + C(0) = K}$

$\cancel{250\times0.002 = K}$

$\cancel{500\times10^{-3} = K}$

$\cancel{\therefore K = 1/2}$

$\cancel{Lq + Rqt + \frac{Ct^2}{2} = \frac{1}{2} + K'}$

$0.25\times$

$0.25\frac{d^2q}{dt^2} + 250\frac{dq}{dt} + \frac{1}{2\times10^{-6}} =$

$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + [illegible] =$

$\left(D^2 + 1000D + \frac{[illegible]}{2\times10^{6}}\right)\cdot$

$=$

$\cancel{m_1 = -20.41}$

$\cancel{m_2 = -979.58}$

Q. $L = 0.25H$ $R = 250\,\Omega$ $C = 2\times10^{-6}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$0.25\frac{d^2q}{dt^2} + 250\frac{dq}{dt} + \frac{q}{2\times10^{-6}} = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + \frac{q}{0.5\times10^{-6}} = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + 2\times10^{+6}q = 0$$

$$m^2 + 1000m + 2\times10^{+6} = 0$$

$$m = -500 \pm 1322.87i$$

$$q = e^{-500t}(c_1\cos 1323t + c_2\sin 1323t)$$

$$i = \frac{dq}{dt} = e^{-500t}(-1323c_1\sin 1323t + 1323c_2\cos 1323t)$$
$$- 500e^{-500t}(c_1\cos 1323t + c_2\sin 1323t)$$

$\frac{dq}{dt} = 0$ at $t = 0$

$$\Rightarrow 0 = e^0(-\cancel{1323c_1\sin 0} + 1323c_2\cos 0)$$
$$- e^0(c_1\cos 0 + \cancel{c_2\sin 0})$$

$$= 1323c_2 - 500c_1$$

and $q = 0.002$, at $t = 0$

$$0.002 = e^0(c_1) \Rightarrow c_1$$

$$500\times0.002 = 1323c_2$$

$$c_2 = 7.55\times10^{-4}$$

$$q = e^{-500t}(0.002\cos 1323t + 7.55\times10^{-4}\sin 1323t)$$

Q. At $t > 0$, $E = E_0(1-\cos t)$

at $L = R = C = 1$, where $q_{\text{initially}} = 0$

$\hookrightarrow t = 0$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$\frac{d^2q}{dt^2} + \frac{dq}{dt} + q = E$$

$$m^2 + m + 1 = 0$$

$$i = \frac{-1 \pm \sqrt{3}i}{2}$$

$$CF = e^{-1/2}\left(c_1\cos\frac{\sqrt{3}}{2} + c_2\sin\frac{\sqrt{3}}{2}\right)$$

$$PI = \frac{E_0(1-\cos t)}{D^2+D+1} = \frac{E_0 e^0 - E_0\cos t}{D^2+D+1}$$

$$= \frac{E_0}{1} - \frac{E_0\cos t}{\cancel{D^2+D+1}\ D}$$

$$= E_0 - \frac{E_0 \cos t}{D} \times \frac{D}{D} = E_0 - \frac{E_0(D \cos t)}{D^2}$$

$$= E_0 - \left(\frac{0 - E_0}{0-1}\right) = E_0 - \frac{E_0}{1} = E_0\sin$$

$$q = e^{-t/2}\left(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right) + E_0(1-\sin t)$$

$$\frac{dq}{dt} = -\frac{1}{2}e^{-t/2}\left(c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right) + e^{-t/2}\left(-\frac{\sqrt{3}}{2}c_1\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}c_2\cos\frac{\sqrt{3}}{2}t\right) - E_0\cos t$$

$$0 = (c_1) + E_0 \Rightarrow -E_0 = c_1$$

$$0 = -\frac{1}{2}(c_1 + 0) + \left(+\frac{\sqrt{3}}{2}c_2\right) - E_0 \Rightarrow E_0 - \frac{E_0}{2} = \frac{\sqrt{3}}{2}c_2$$

$$\frac{E_0}{2} = \sqrt{3}c_2$$

$$c_2 = \frac{E_0}{\sqrt{3}}$$

$$\boxed{q = e^{-t/2}\left(-E_0\cos\frac{\sqrt{3}}{2}t + \frac{E_0}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) + E_0(1-\sin t)}$$

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Simple Harmonic Motion

$$mg - Kx = F_R = 0 \text{ (at eq)}$$

$$\cancel{m\frac{d^2}{dt^2} \; Ks}$$

$$\Rightarrow mg - K(s+x) = ma$$

$$mg - Ks - Kx = m\frac{d^2x}{dt^2}$$

$$0 = m\frac{d^2x}{dt^2} + Kx$$

$$\frac{d^2x}{dt^2} + \frac{Kx}{m} = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

angular frequency

$$m^2 + \omega^2 = 0$$

$$m = \pm \omega i$$

$$x = (C_1 \cos\omega t + C_2 \sin\omega t)$$

$$C_1 = A\cos\phi \qquad C_2 = -A\sin\phi$$

$$= A\cos(\phi + \omega t) \qquad \text{where, } A = \sqrt{C_1^2 + C_2^2}$$

$$\phi = -\tan^{-1}\left(\frac{C_2}{C_1}\right)$$

$$T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{m}{K}}$$

$$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{\omega}{2\pi}$$

$$V_{max} = A \times \omega$$

Q. $m = 10kg$. A pull of 20kg stretches spring 10cm. The body is pulled down to 20cm below static eq & released. Find displacement of body from eq pos^n at t sec, Max velocity & period of osc.

A. $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega = \sqrt{\frac{k}{m}}$

Given that, a pull of 20 kg stretches to 10 cm

$mg - ks = 0$

$mg = ks$

$20 = k \times 0.1 \Rightarrow k = 200\ Nm^{-1}$

$F = 10N$

$mg = 10$

$m \times 9.8 = 10 \Rightarrow m = 1.02$

$\omega^2 = \frac{k}{m} = \frac{200}{1.02} = 196 \Rightarrow \omega = 14$

$V_{max} = A\omega$

$\frac{d^2x}{dt^2} + \omega^2 x = 0$

$m^2 + 196 = 0 \Rightarrow m = \pm 14i$

$x = c_1 \cos 14t + c_2 \sin 14t$

$x = 0.2$ when $t = 0$

$\frac{dx}{dt} = 0$ "

$\frac{dx}{dt} = -14c_1 \cancel{\sin 14t} + 14c_2 \cos 14t$

$0 = 14c_2 \Rightarrow c_2 = 0$

① $0.2 = c_1 \Rightarrow$

$x(t) = 0.2\cos 14t$

$V_{max} = 0.2 \times 14$
$= 2.8\ ms^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{14} = \frac{\pi}{7}$
$= 0.45s$

2.

$$\frac{d^2x}{dt} + \omega^2 x = 0$$

$$mg = Ks$$

$$10 = K \times 0.05$$

$$K = 200\ Nm^{-1}$$

$$F = 49N$$

$$mg = 4.9$$

$$m = 0.5$$

$$\omega^2 = \frac{K}{m} = \frac{200}{0.5} = 400$$

$$\omega = 20\ s^{-1}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$0.06 = c_1$$

$$x = c_1 \cos 20t + c_2 \sin 20t$$

$$\frac{dx}{dt} = -20c_1 \sin 20t + 20c_2 \cos 20t$$

$$0 = 20C_2$$

$$C_2 = 0$$

$$x(t) = 0.06 \cos 20t$$

$$V_{max} = A\omega = 0.06 \times 20$$

$$= 1.2\ ms^{-1}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = \frac{\pi}{10} = 0.314s$$

Partial Differential Equation

→ Involves 2 or more independent variables x, y &

1 independent variable & it partial derivatives

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$ etc.,

ex: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial y}\right)^3$$

Applications :

→ Wave prop

→ Fluid Flow $\Longrightarrow$ weather

→ Vibration

→ Mechanics of Solids etc.,

→ Heat flow & distribution

$\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$; $\frac{\partial^2 z}{\partial x^2} = r$; $\frac{\partial^2 z}{\partial y^2} = t$; $\frac{\partial^2 z}{\partial x \partial y} = s$

General form of 1st order pde

$f(x, y, z, p, q) = 0$ ——— (1)

PDE can have many sol^n

but ODE has only 1 unique sol^n

Complete solution:

- $f(x, y, z, a, b) = 0$ having 2 arbitary const^n's satisfying PDE (1) is known as complete sol^n/integral

ex: PDE $z = pq$ obtained by eliminating a & b from $z = (x+a)(y+b)$

So $z = (x+a)(y+b)$ is a complete solution of $z = pq$

Particular Sol^n

→ A sol^n obtained by giving particular values in place of arbitary constant gives particular sol^n/integral.

Q. $z = (x-a)^2 + (y-b)^2$

$\frac{\partial z}{\partial x} = p = 2(x-a) \Rightarrow x - a = \frac{p}{2}$

self learning component

$\frac{\partial z}{\partial y} = q = 2(y-b) \Rightarrow y - b = \frac{q}{2}$

$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \Rightarrow 4z = p^2 + q^2$

$4z = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

Lagrange's Linear PDE

$Pp + Qq = R$ where P, Q, R are functions of x, y, z

p & q are $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

Working:

1) compare $Pq + Qq = R$ & write P, Q, R

2) Form auxillary eq $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

3) Solve auxillary eq by method of grouping / method of multiplers (or) both to get independant sol^{n}s (where a, b are arbitrary constants)

4) $\phi(u, v) = 0$ (or) $u = f(v)$ is general sol^n of $Pp + Qq = R$

Type I :

→ Obtained by taking 2 members of auxillary eq $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ at a time & integrating to have 2 independant sol^{n}s in variables, whose differentials are involved in equation

Q. $2yzp + zxq = 3xy$

$P = 2yz \qquad Q = zx \qquad R = 3xy$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{2yz} = \frac{dy}{zx} = \frac{dz}{3xy}$$

$$\int\frac{dx}{2yz} = \int\frac{dy}{zx} \Rightarrow \frac{x}{2y\cancel{z}} = \frac{y}{\cancel{z}x}$$

$$x^2 = 2y^2$$

$$\int\frac{dy}{zx} = \int\frac{dz}{3xy} \Rightarrow \frac{y}{z\cancel{x}} = \frac{z}{3\cancel{x}y} \Rightarrow 3y^2 = z^2$$

$$\frac{x^2}{2} = y^2 = \frac{z^2}{3}$$

$$\frac{dx}{2y\cancel{z}} = \frac{dy}{\cancel{z}x} \Rightarrow xdx = 2dyy$$

$$\frac{x^2}{2} = \frac{\cancel{2}y^2}{\cancel{2}} + c_1$$

$$x^2 - 2y^2 = 2c_1$$

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$$\frac{dy}{2x} = \frac{dz}{3xy}$$

$$3y\,dy = 2dz$$

$$\frac{z^2}{2} = \frac{3y^2}{2} + c_2$$

$$z^2 - 3y^2 = 2c_2$$

General sol^n,

$$\phi(x^2 - 2y^2,\ 3y^2 - z^2) = 0$$

Q. $p\cot x + q\cot y = \cot z$

$P = \cot x \qquad Q = \cot y \qquad R = \cot z$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\int \tan x\,dx = \int \tan y\,dy$$

$$\log \sec x = \log \sec y + \log c_1$$

$$\frac{\sec x}{\sec y} = c_1$$

$$\int \tan y\,dy = \int \tan z\,dz$$

$$\frac{\sec z}{\sec y} = c_2$$

General Sol^n,

$$\phi\left(\frac{\sec x}{\sec y}, \frac{\sec z}{\sec y}\right) = 0$$

Q. $y^2zp = x^2(zq + y)$

$P = y^2z \qquad Q = -x^2z \qquad R = x^2y$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{y^2\cancel{z}} = \frac{dy}{-x^2\cancel{z}}$

$-x^2dx = y^2dy$

$-\frac{x^3}{3} = \frac{y^3}{3} + c_1$

$+x^3 + y^3 = 3c_1$

$\frac{dy}{-\cancel{x^2}z} = \frac{dz}{\cancel{x^2}y}$

$-\int y\,dy = \int z\,dz$

$-\frac{y^2}{2} = \frac{z^2}{2} + c_2$

$y^2 + z^2 = -2c_2$

$\phi\left(+x^3 + y^3,\ +y^2 + z^2\right) = 0$

Q. $\frac{y^2z}{x}p + xzq = y^2$

$\int \frac{x\,dx}{y^2\cancel{z}} = \int \frac{dy}{x\cancel{z}}$

$\int x^3dx = \int y^2dy$

$\frac{x^4}{4} = \frac{y^3}{3} + c_1$

$\frac{x^4}{4} - \frac{y^3}{3} = c_1$

$3x^4 - 4y^3 = 12c_1$

$\frac{x\,dx}{y^2z} = \frac{dz}{y^2}$

$x\,dx = z\,dz$

$x^2 - z^2 = c_2$

$\phi\left(3x^4 - 4y^3,\ x^2 - z^2\right) = 0$

Q. $y^2p - xyq = (xz - 2xy)$

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\int x\,dx = -\int y\,dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c_1$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

$$x^2 + y^2 = 2c_1$$

$$\frac{dy}{xy} = \frac{dz}{\cancel{x(z-2y)}}$$

$$\frac{dy}{xy} = \frac{dz}{2y - z}$$

$$dy(2y - z) = y\,dz$$

$$z\,dy + y\,dz = 2y\,dy$$

$$\int d(yz) = \int 2y\,dy$$

$$yz = y^2 + c_2$$

$$yz - y^2 = c_2$$

$$\phi(x^2 + y^2,\ y(z - y)) = 0$$

Type 2: Sol^n obtained by taking 2 member of auxillary eq & integrate to have an eq (1 independant sol^n) in the variables whose differentials are involved & another independant sol^n obtained by making use of first solution (integral)

Q. $2p + q = \sin(x - 2y)$

$P = 2 \quad Q = 1 \quad R = \sin(x - 2y)$

$$\frac{dx}{2} = \frac{dy}{1} = \frac{dz}{\sin(x - 2y)}$$

$$dx = 2dy$$

$$x = 2y + c_1$$

$$x - 2y = c_1$$

$$dy = \frac{dz}{\sin(x - 2y)}$$

$$\sin c_1\, dy = dz$$

$$y \sin c_1 = z + c_2 \Rightarrow y\sin(x - 2y) - z = c_2$$

$$\phi(x - 2y,\ y\sin(x - 2y) - z) = 0$$

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Q. $xzp + yzq = xy$

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\frac{x}{y} = c_1$$

$$\frac{dy}{yz} = \frac{dz}{xy}$$

$$c_1 \int y\,dy = \int z\,dz$$

$$\frac{c_1 y^2}{2} - \frac{z^2}{2} = c_2$$

$$c_1 y^2 - z^2 = 2c_2$$

$$xy - z^2 = 2c_2$$

$$\phi\left(\frac{x}{y},\ xy - z^2\right) = 0$$

Q. $p + 3q = 5z - \tan(3x - y)$

$$dx = \frac{dy}{3}$$

$$3x - y = c_1$$

$$dx = \frac{dz}{5z - \tan c_1}$$

$$\cancel{\frac{5x^2}{2} - x\tan c_1 = z + c_2}$$

$$\cancel{\frac{5x^2}{2} - x\tan(3x - y) - z = c_2}$$

$$\cancel{\phi\left(3x - y,\ \frac{5x^2}{2} - x\tan(3x - y) - z\right) = 0}$$

$$\int \frac{dz}{(5z - \tan c_1)}$$

$$x = \frac{\log(5z - \tan c_1)}{5}$$

$$= \frac{\log(5z - \tan c_1)}{5}$$

Q. $(p - q)(x + y) = z$

$$(x+y)p - (x+y)q = z$$

$$\int \frac{dx}{x+y} = \frac{\int dy}{-(x+y)}$$

$$x = -y + c_1$$

$$x + y = c_1$$

$$\int \frac{dx}{x+y} = \int \frac{dz}{z}$$

$$\log z + c_2 = \frac{x}{c_1}$$

$$c_2 = \frac{x}{c_1} - \log z$$

$$c_2 = \frac{x}{x+y} - \log z$$

$$\phi\left(x+y,\ \frac{x}{x+y} - \log z\right) = 0$$

Note : by property & proportion $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

$$= \frac{k_1 a_1 + k_2 a_2 + k_3 a_3}{k_1 b_1 + k_2 b_2 + k_3 b_3}$$

Type 3 : Method of Multipliers

i) Given PDE of form $Pp + Qq = R$

form AE, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

ii) Find multipliers, k_1, k_2, k_3 ; k_1', k_2', k_3'

such that $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{k_1 dx + k_2 dy + k_3 dz}{k_1 P + k_2 Q + k_3 R}$

$$= \frac{k_1' dx + k_2' dy + k_3' dz}{k_1' P + k_2' Q + k_3' R}$$

iii) Integrating 2 new expressions we obtain 2 relations connecting x, y, z

iv) $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ obtained

then $\phi(u,v) = 0$ is general sol^n of PDE $Pp + Qq = R$

Set of multiplies

1)	x	y	z	6)	l	m	n
2)	$+x$	$-y$	$-z$	7)	$-x$	$-y$	$-z$
3)	$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{z}$	8)	1	-1	-1
4)	$\frac{1}{x^2}$	$\frac{1}{y^2}$	$\frac{1}{z^2}$	9)	1	-1	0
5)	1	1	1	10)	1	0	-1
[illegible]	x	$-y$	[illegible]	11)	0	1	-1

Q. $(y^2+z^2)p + (xy)q = xz$

$P = y^2+z^2$

$Q = xy$

$R = xz$

$$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz} = \frac{x\,dx - y\,dy - z\,dz}{(xy^2+xz^2) - xy^2 - xz^2}$$

$$\boxed{\frac{y}{z} = c_1}$$

$$\cancel{\frac{dx}{y^2+z^2}} \cdot \frac{dy}{xy} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$x^2y\,dx - xy^2dy - \cancel{xy}z\,dz = 0$$

$$\frac{dz}{xz} = \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$x^2z\,dx - xyz\,dy - xz^2dz = 0$$

$$\frac{c_1y^2 + y^2}{c_1} = \left(\frac{c_1+1}{c_1}\right)y^2$$

$$\frac{dx}{\frac{(c_1+1)}{c_1}\cdot y^2} = \frac{dy}{xy}$$

$$x\,dx = \left(\frac{c_1+1}{c_1}\right)y\,dy$$

$$\frac{x^2}{2} = \left(\frac{c_1+1}{c_1}\right)\frac{y^2}{2} + c_2$$

$$\frac{x^2}{2} - \left(1+\frac{z}{y}\right)\frac{y^2}{2} = c_2$$

Q. $(y+z)p + (z+x)q = x+y$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x}$$

$$\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log c_1$$

$$\log(x+y+z) = -2\log(x-y) + 2\log c_1$$

$$\boxed{(x+y+z)(x-y)^2 = c_1^2}$$

$$\frac{dx+dy+z}{2(x+y+z)} = \frac{dy-dz}{z-y}$$

$$\boxed{(x+y+z)(y-z)^2 = c_2^2}$$

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dz-dx}{x-z}$$

$$\boxed{(x+y+z)(z-x)^2 = c_3^2}$$

$$\frac{dy-dz}{z-y} = \frac{dz-dx}{x-z}$$

$$\boxed{\left(\frac{y-z}{z-x}\right) = c_4}$$

3.

6. $(x+2z)p + (4zx-y)q = 2x^2+y$

$$\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y} = \frac{y\,dx + x\,dy - 2z\,dz}{xy+2yz+4zx^2-xy-4x^2z-2yz} = \frac{2x\,dx - dy - dz}{2x^2+4xz-4zx \ldots}$$

multipliers are $(y, x, -2z)$ & $(2x, -1, -1)$

$y\,dx + x\,dy - 2z\,dz = 0$

$xy - z^2 = c_1$

$\int 2x\,dx - \int dy - \int dz = 0$

$x^2 - y - z = c_2$

$\phi\left(xy - z^2,\ x^2 - y - z\right) = 0$

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2. $(mz - ny)\cancel{\frac{\partial z}{\partial x}}\,p + (nx - lz)q = ly - mx$

4. (l, m, n) $(x, -y, -z)$

$\frac{dx}{\ }$

Q. $x^2 p + y^2 q = 0$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y}$$

$u = XY$

where $X = X(x)$, $Y = Y(y)$

$$\frac{\partial (XY)}{\partial x} = Y \frac{\partial X}{\partial x} = Y \frac{dX}{dx}$$

$$\frac{\partial (XY)}{\partial y} = X \frac{\partial Y}{\partial y} = X \frac{dY}{dy}$$

$$\Rightarrow x^2 Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

$$\frac{x^2}{X} \frac{dX}{dx} + \frac{y^2}{Y} \frac{dY}{dy} = 0$$

$$\frac{x^2}{X} \frac{dX}{dx} = -\frac{y^2}{Y} \frac{dY}{dy} = K$$

$$\Rightarrow \int \frac{dX}{X} = \int K \frac{dx}{x^2} \qquad \Bigg| \qquad \int \frac{dY}{Y} = -K \int \frac{dy}{y^2}$$

$$\log X = -\frac{K}{x} + c_1 \qquad \Bigg| \qquad \log Y = +\frac{K}{y} + c_2$$

$$\boxed{X = e^{-\frac{K}{x} + c_1}} \qquad \boxed{Y = e^{\frac{K}{y} + c_2}}$$

$$X = e^{-\frac{K}{x}} e^{c_1} \qquad Y = e^{+\frac{K}{x}} e^{c_2}$$

$$X = e^{c_1} e^{-\frac{K}{x}} \qquad Y = e^{c_2} e^{\frac{K}{y}}$$

$u = XY$

$$= e^{c_1 + c_2} e^{K\left(\frac{1}{y} - \frac{1}{x}\right)}$$

$$= C e^{K\left(\frac{1}{y} - \frac{1}{x}\right)}$$

Q. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ $\quad u(0,y) = 2e^{5y}$

$u = XY$

$4 \cdot Y \frac{dX}{dx} + X \frac{dY}{dy} = 3XY$

$\frac{4}{X} \cdot \frac{dX}{dx} + \frac{1}{Y} \cdot \frac{dY}{dy} = 3$

$\frac{4}{X} \cdot \frac{dX}{dx} = 3 - \frac{1}{Y} \cdot \frac{dY}{dy} = K$

$\frac{4}{dx} \cdot \frac{dX}{X} = K$ $\quad 3 - \frac{dY}{Y} \cdot \frac{1}{dy} = K$

$\int \frac{dX}{X} = \int \frac{K \cdot dx}{4}$ $\quad dy(3-K) = \frac{dY}{Y}$

$\log X = \frac{K}{4} \cdot x + c_1$ $\quad c_2 + (3-K)y = \log(Y)$

$X = e^{\frac{Kx}{4}} e^{c_1}$ $\quad Y = e^{(3-K)y} \cdot e^{c_2}$

$u = XY$

$u = e^{\frac{Kx}{4}} \cdot e^{(3-K)y} \cdot e^{c_1+c_2}$

$$\boxed{u = c\, e^{\frac{Kx}{4} + (3-K)y}}$$

$u = c\, e^{0 + (3-K)y} = 2e^{5y}$

$$\boxed{K = -2, \quad c = 2}$$

$$\boxed{u = 2e^{\frac{-x}{2} + 5y}}$$

$$u(x,0) = 6e^{-3x}$$

Q. $$\frac{\partial u}{\partial x} = \frac{2\partial u}{\partial t} + u$$

$$u = X.T$$

$$\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial t} = u$$

$$T\frac{dX}{dx} - 2X\frac{dT}{dt} = XT$$

$$\frac{1}{X}\cdot\frac{dX}{dx} - \frac{2}{T}\cdot\frac{dT}{dt} = 1$$

$$\frac{1}{X}\cdot\frac{dX}{dx} = 1 + \frac{2}{T}\cdot\frac{dT}{dt} = K$$

$$\int\frac{dX}{X} = \int K\,dx$$

$$\log X = Kx + C_1$$

$$X = e^{Kx}\cdot e^{C_1}$$

$$\frac{2}{T}\cdot\frac{dT}{dt} = K-1$$

$$\int\frac{dT}{T} = \int\frac{K-1}{2}\,dt$$

$$T = e^{\frac{K-1}{2}t}\cdot e^{C_2}$$

$$u = XT$$

$$u = e^{Kx}\cdot e^{\frac{K-1}{2}t}\cdot C$$

$$u(x,0) = 6e^{-3x}$$

$$K = -3, \quad C = 6$$

$$u = 6e^{-3x-2t}$$

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Q. $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

$u = XY$

$\frac{\partial u}{\partial x} = Y\frac{dX}{dx}$ $\qquad$ $\frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2} = Y\frac{d^2X}{dx^2}$

$\frac{\partial u}{\partial y} = X\frac{dY}{dy}$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2}$

$$Y\frac{d^2X}{dx^2} - 2.Y.\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

$$\frac{1}{X}.\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} + \frac{1}{Y}.\frac{dY}{dy} = 0$$

$$\frac{1}{X}.\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = -\frac{1}{Y}.\frac{dY}{dy} = K$$

$\frac{d^2X}{dx^2} - \frac{2dX}{dx} - KX = 0$ $\qquad$ | $\frac{dY}{dy} + KY = 0$

$(D^2 - 2D - K)X = 0$ $\qquad$ | $m = -K$

$m^2 - 2m - K = 0$

$m = 1 \pm \sqrt{1+K}$

$X = c_1 e^{(1+\sqrt{1+K})x} + c_2 e^{(1-\sqrt{1+K})x}$ $\qquad$ | $Y = c_3 e^{-Ky}$

$$u = XY = \left(c_1 e^{(1+\sqrt{1+K})} + c_2 e^{(1-\sqrt{1+K})}\right)\left(c_3 e^{-Ky}\right)$$

Q. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$

$u = XY$

$\frac{\partial u}{\partial x} = Y\frac{dX}{dx}$ $\qquad$ $\frac{\partial u}{\partial y} = X\frac{dY}{dy}$

$Y\frac{dX}{dx} + X\frac{dY}{dy} = 2(x+y)XY$

$\frac{1}{X}\cdot\frac{dX}{dx} + \frac{1}{Y}\cdot\frac{dY}{dy} = 2(x+y)$

$\frac{dX}{X}\cdot\frac{1}{dx} - 2x = 2y - \frac{dY}{Y}\cdot\frac{1}{dy} = K$

$\int\frac{dX}{X} = \int(2x+K)\,dx$ $\quad\Big|\quad$ $\int\frac{dY}{Y} = \int(2y-K)\,dy$

$\log X = x^2 + Kx + c_1$ $\quad\Big|\quad$ $\log Y = y^2 - Ky + c_2$

$X = e^{x^2+Kx}\cdot e^{c_1}$ $\quad\Big|\quad$ $Y = e^{y^2-Ky}\,e^{c_2}$

$u = XY = Ce^{x^2+Kx+y^2-Ky}$

$= Ce^{x^2+y^2+K(x-y)}$

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x

3. Su

→ Various possible sol^n of 1D heat equation

$u_t = c^2 u_{xx}$ by method of seperation of variabl

$$\Rightarrow \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$U = XT$

$$\frac{\partial U}{\partial x} = T \frac{\partial X}{\partial x} = T \frac{dX}{dx}$$

$$\frac{\partial^2 U}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2} = T \cdot \frac{d^2 X}{dx^2}$$

$$\frac{\partial U}{\partial t} = X \frac{\partial T}{\partial x} = X \frac{d}{d}$$

Traditional Method

4.

$$X \frac{dT}{dt} = c^2 \cdot T \frac{d^2 X}{dx^2}$$

$$\frac{1}{c^2} \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = K$$

$$\Rightarrow \int \frac{dT}{T} = \int K c^2 dt$$

$$\log T = Kc^2 t + c_1$$

$$T = e^{Kc^2 t} e^{c_1}$$

$$\frac{1}{X} \cdot \frac{d^2 X}{dx^2} = K$$

$$\frac{d^2 X}{dx^2} - KX = 0$$

$$(D^2 - K)X = 0$$

$$m = \pm\sqrt{K}$$

$$CF = c_2 e^{\sqrt{K}x} + c_3 e^{-\sqrt{K}x} = X$$

$$\Rightarrow X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2}$$

$$\frac{X}{c^2}\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

$$\frac{1}{T}\cdot\frac{1}{c^2}\cdot\frac{dT}{dt} = \frac{1}{X}\cdot\frac{d^2X}{dx^2} = K$$

$$\Rightarrow \frac{dT}{dt} = Kc^2T$$

$$\frac{dT}{dt} - Kc^2T = 0$$

$$DT - Kc^2T = 0$$

$$\frac{d^2X}{dx^2} - KX = 0$$

$$D^2X - KX = 0$$

case (i) : $K = 0$

$$(D - 0)T = 0$$

$$m = 0$$

$$T = c_3 e^{0t}$$

$$D^2X = 0$$

$$m^2 = 0$$

$$m = 0, 0$$

$$X = (c_1 + c_2 x)e^{0x}$$

$$\boxed{U = (c_1 + c_2 x)\, c_3}$$

case ii) K is +ve $K = p^2$

$$(D^2 - p^2)X = 0$$

$$m^2 - p^2 = 0$$

$$m = \pm p$$

$$X = c_1 e^{px} + c_2 e^{-px}$$

$$(D - p^2c^2)T = 0$$

$$m = p^2c^2$$

$$T = c_3 e^{p^2c^2t}$$

$$\Rightarrow \boxed{U = (c_1 e^{px} + c_2 e^{-px})c_3 e^{p^2c^2t}}$$

case (iii) : K is -ve $K = -p^2$

$$(D^2 + p^2)X = 0$$

$$m^2 + p^2 = 0$$

$$m = \pm ip$$

$$X = c_1\cos px + c_2\sin px$$

$$(D + p^2c^2)T = 0$$

$$m = -p^2c^2$$

$$T = c_3 e^{-p^2c^2t} \Rightarrow$$

$$\boxed{U = (c_1\cos px + c_2\sin px)c_3 e^{-p^2c^2t}}$$

$$(A\cos px + B\sin px)\cdot e^{-p^2c^2t}$$

$A = c_1c_3$
$B = c_2c_3$

Of these 3 sol^n, case 3 is most suitable, cuz acc. to heat eq Temp↓ as time.

• Various possible sol^n for Laplace eq $U_{xx} + U_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

↳ steady state heat eq

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\cdot\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = K$$

$\frac{d^2X}{dx^2} - KX = 0$	$\frac{d^2Y}{dy^2} + KY = 0$
$(D^2 - K)X = 0$	$(D^2 + K)Y = 0$

case (i) $K = 0$

$D^2X = 0$	$D^2Y = 0$
$m^2 = 0$	$m^2 = 0$
$m = 0, 0$	$m = 0, 0$
$X = (c_1 + c_2x)$	$Y = (c_3 + c_4y)$

case (ii) $K = p^2$

$(D^2 - p^2)X = 0$	$(D^2 + p^2)Y = 0$
$m^2 = p^2$	$m = \pm pi$
$m = \pm p$	$Y = c_3\cos py + c_4\sin py$
$X = c_1e^{px} + c_2e^{-px}$	

case (iii) $K = -p^2$

$(D^2 + p^2)X = 0$	$(D^2 - p^2)Y = 0$
$X = c_1\cos px + c_2\sin px$	$Y = c_3e^{py} + c_4e^{-py}$

case 2 & 3 o

• Solution of homogeneous linear PDE with const^ant^ coeff

$$F(x,y) = a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + \dots$$

→ homogeneous linear PDE

For convenience,

$$D \Rightarrow \frac{\partial}{\partial x} \quad \& \quad D' \Rightarrow \frac{\partial}{\partial y}$$

So, $(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} (D')^2 + \dots + a_{n-1} D (D')^{n-1} + a_n (D')^n) \dots = F(x,y)$

$$\boxed{F(D, D') z = F(x,y)}$$

consider, $$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 + a_1 DD' + a_2 (D')^2) z = 0$$

$$m^2 + a_1 m + a_2 = 0 \qquad m = \frac{D}{D'}$$

Case (i) If roots are real & distinct

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

ex: $(D_x^2 - D_y^2) z = 0$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$z = f_1(y+x) + f_2(y-x)$$

Case ii) $m_1 = m_2$

$z = f_1(y+m_1x) + x f_2(y+m_1x)$

ex: $(4D^2 + 12DD' + 9D'^2)z = 0$

$4m^2 + 12m + 9 = 0$

$f(x)$, sin/cos, e^x

$m^2 + 3m + \frac{9}{4} = 0$

$m = -3/2, -3/2$

$z = f_1\left(y - \frac{3x}{2}\right) + x f_2\left(y - \frac{3x}{2}\right)$

$= f_1(2y - 3x) + x f_2(2y - 3x)$

Q. $D_x^3 - 6D_x^2 D_y + 11 D_x D_y^2 - 6D_y^3$

$(m^3 - 6m^2 + 11m - 6) = 0$

$m = 1, 3, 2$

$z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

Rules for finding PI

- $(D^2 + a_1 DD' + a_2 (D')^2) z = F(x, y)$

$$F(D, D') z = F(x, y)$$

$$PI = \frac{1}{F(D, D')} \cdot F(x, y)$$

Case (i) : $F(x, y) = e^{ax + by}$

$$PI = \frac{1}{F(D, D')} \cdot e^{ax+by}$$

$$= \frac{e^{ax+by}}{F(a, b)}$$

Q. $(D_x^2 - D_x D_y - 6D_y^2) = e^{2x - 3y}$

$$m^2 - m - 6 = 0$$

$$m_1 = 3, -2$$

$$CF = f_1(y + 3x) + f_2(y - 2x)$$

$$PI = \frac{1}{D_x^2 - D_x D_y - 6D_y^2} e^{2x - 3y}$$

$$= \frac{e^{2x-3y}}{4 + 6 - 54} = \frac{e^{2x-3y}}{-44}$$

$$z = f_1(y + 3x) + f_2(y - 2x) - \frac{e^{2x-3y}}{44}$$

Q. $(D_x^3 - 3D_x D_y + 4D_y$ [illegible]

$m^3 - 3m^2 + 4 = 0$

$m = -1, 2, 2$

$CF = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

$PI = \dfrac{e^{x+2y}}{1 - 6 + 32} = \dfrac{e^{x+2y}}{+27}$

$z = CF + PI = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \dfrac{e^{x+2y}}{27}$

Q $m^2 - 4m + 4 = e^{2x+y}$

$m = 2, 2$

$CF = f_1(y+2x) + x f_2(y+2x)$

$PI = \dfrac{e^{2x+y}}{D_x^2 - 4D_x D_y + 4D_y^2}$

$= \dfrac{e^{2x+y}}{4 - 8 + 4}$

$= x \cdot \dfrac{e^{2x+y}}{\underset{4-4}{2D_x - 4D_y}} = x^2 \cdot \dfrac{e^{2x+y}}{2}$

$z = CF + PI$

$= f_1(y+2x) + x \cdot f_2(y+2x) + \dfrac{x^2}{2} e^{2x+y}$

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Case(ii) $F(x+y) = \sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{1}{F(D^2, DD', (D')^2)} \sin(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2)}$$

$D^2 = -a^2$

$DD' = -ab$

$(D')^2 = -b^2$

Q. $D^2 - DD' = \cos(x+2y)$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$CF = f_1(y) + f_2(y+x)$

$$PI = \frac{\cos(x+2y)}{D^2 - DD'}$$

$$= \frac{\cos(x+2y)}{-1-(-2)} = \cos(x+2y)$$

$z = f_1(y) + f_2(y+x) + \cos(x+2y)$

Q $D^2 + 2DD' + (D')^2 = \sin(2x+3y)$

$m = -1, -1$

$CF = f_1(y-x) + x f_2(y-x)$

$$PI = \frac{\sin(2x+3y)}{-4-12-9}$$

$$= -\frac{\sin(2x+3y)}{25}$$

$$z = f_1(y-x) + x f_2(y-x) - \frac{\sin(2x+3y)}{25}$$

Q. $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2)z = 2\sin(3x+2y)$

$m^3 - 4m^2 + 4m = 0$

$m = 2, 2, 0$

$CF = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

$$PI = \frac{2\sin(3x+2y)}{D_x^3 - 4D_x^2 D_y + 4D_x D_y^2}$$

$$= \frac{2\sin(3x+2y)}{D_x(D_x^2 - 4D_x D_y + 4D_y^2)}$$

$$= \frac{2\sin(3x+2y)}{D_x(-9+24-16)} = \frac{2\sin(3x+2y) \cdot D_x}{D_x^2(-1)}$$

$$= \frac{-6\cos(3x+2y)}{-9} = \frac{2}{3}\cos(3x+2y)$$

$$y = f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{2}{3}\cos(3x+2y)$$

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Q. $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

$m^2 + m - 6$

$2, -3$

$CF = f_1(y+2x) + f_2(y-3x)$

$PI = \dfrac{\cos(2x+y)}{\cancel{-4-2+6}}$

$= \dfrac{\cos(2x+y)}{2D+D'}$

$= \dfrac{\cos(2x+y)\,(2D-D')}{4D^2-(D')^2}$

$= \dfrac{\cos(2x+y)\,(2D-D')}{-16+1}$

$= \dfrac{+2\sin(2x+y) + (-\sin(2x+y))}{+15} \cdot x$

$= + \dfrac{3x\sin(2x+y)}{\cancel{+5}\ 5}$

$z = f_1(y+2x) + f_2(y-3x) + \dfrac{x\sin(2x+y)}{5}$

Q. $(2D_{xx} - 5D_{xy} + 2D_{yy})z = e^{x+2y} + \sin(2x+y)$

$2m^2 - 5m + 2 = 0$

$m = 2, \frac{1}{2}$

$CF = f_1(y+2x) + f_2(y + \frac{x}{2})$

$PI = \cancel{\frac{e^{x+2y} + \sin(2x+y)}{2D_{xx} - 5D_{xy} + 2D_{yy}}}$

$= \cancel{\frac{e^{x+2y}}{2-10+8}}$

$= \frac{e^{x+2y} + \sin(2x+y)}{2D^2 - 5DD' + 2(D')^2}$

$= \frac{x e^{x+2y}}{\cancel{4D-5D'}\ -6} + \frac{x[\sin(2x+y) \times (4D+5D')]}{\cancel{2(-4)+5(2)-2}\ \cancel{16}D^2 - 25(D')^2}$

$-64 + 25 = -39$

$= -\frac{xe^{x+2y}}{6} - \frac{x[\cancel{8\cos(2x+y) + 5\cos(2x+y)}]}{\cancel{39}\ 3}$ $\cos(2x+y)$

$PI = -\frac{xe^{x+2y}}{6} - \frac{x\cos(2x+y)}{3}$

$z = f_1(y+2x) + f_2(y+\frac{x}{2}) - \frac{xe^{x+2y}}{6} - \frac{x\cos(2x+y)}{3}$

Case (iii)

When $F(x,y) = x^m y^n$, m & n being constants

$$PI = \frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} x^m y^n$$

a) if $n<m$, $\frac{1}{F(D,D')}$ is expanded in powers of $\frac{D'}{D}$

b) if $m<n$, $\frac{1}{F(D,D')}$ is expanded in powers of $\frac{D}{D'}$

Q. Solve $D^2 + 3DD' + 2(D')^2 = x+y$

$m^2 + 3m + 2 = 0$

$m = -1, -2$

$CF = f_1(y-x) + f_2(y-2x)$

$$PI = \frac{x+y}{D^2 + 3DD' + 2(D')^2}$$

$$= \frac{x+y}{D^2\left(1 + \frac{3D'}{D} + 2\left(\frac{D'}{D}\right)^2\right)}$$

$$= \frac{x+y}{D^2}\left[1 + \frac{3D'}{D} + 2\left(\frac{D'}{D}\right)^2\right]^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 - x^3$

$$= \frac{x+y}{D^2}\left[1 - \frac{3D'}{D} + 2\left(\frac{D'}{D}\right)^2\right]$$

$$= \left[x+y - \frac{3(1)}{D} + 0\right] \cdot \frac{1}{D^2}$$

$$= \frac{1}{D^2}\left[x+y-3\int 1\,dx\right]$$

$$= \frac{1}{D^2}\left[x+y-3x\right] = \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{1}{D}\int (y-2x)\,dx$$

$$= \frac{1}{D}\left[yx - x^2\right]$$

$$= \int (yx - x^2)\,dx$$

$$\boxed{PI = \frac{yx^2}{2} - \frac{x^3}{3}}$$

$$z = CF + PI$$

$$= f_1(y-x) + f_2(y-2x) + \frac{yx^2}{2} - \frac{x^3}{3}$$

Q. $(D_x^3 - 2D_x^2 D_y)\,z = 3x^2y$

$$z = f_1(y+2x) + f_2(y) + x f_3(y) + \frac{x^5 y}{20} + \frac{x^6}{60}$$

$m = 2, 0, 0$

$CF = f_1(y+2x) + f_2(y) + x f_3(y)$

$$PI = \frac{3x^2y}{D_x^3\left(1 - \frac{2D_y}{D_x}\right)}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{3x^2y}{D_x^3\left(1 - \frac{2D'}{D}\right)} = \frac{3x^2y}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1}$$

$$= \frac{3x^2y}{D^3}\left(1 + \frac{2D'}{D}\right)$$

$$= \frac{\left(3x^2y + \frac{6x^2}{D}\right)}{D^3} = \frac{3x^2y + 2x^3}{D^3}$$

$$= \frac{x^3y + \frac{x^4}{2}}{D^2} = \frac{\frac{x^4y}{4} + \frac{x^5}{10}}{D} = \boxed{\frac{x^5y}{20} + \frac{x^6}{60}} = \frac{x^5}{20}\left(y + \frac{x}{3}\right)$$

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Case iv Exponential shift

$$F(x,y) = e^{ax+by} V(x,y)$$

$$PI = \frac{1}{F(D,D')} e^{ax+by} V(x,y)$$

$D \to D+a$, $D' \to D'+b$

$$= e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} V(x,y)$$

Q. $D^2 - DD' - 2(D')^2 = (y-1)e^x$

$CF = f_1(y+2x) + f_2(y-x)$

$$= e^x \frac{1}{F(D+1, D')}(y-1)$$

$$= e^x \frac{1}{D^2+2D+1-DD' - D' - 2(D')^2}(y-1)$$

$$= e^x \frac{1}{D^2 - 2(D')^2 + 2D - DD' - D' + 1}(y-1)$$

$$= e^x \frac{1}{D(D+2) - D'(2D'+D-1) + 1}(y-1)$$

$$= e^x \frac{(y-1)}{1 + (D^2 + 2D - DD' - D' - 2(D')^2)}$$

$$= e^x (y-1)(1 - D^2 - 2D + DD' + D' + 2(D')^2)$$

$$= e^x(y-1+1) = ye^x = PI$$

$$z = f_1(y+2x) + f_2(y-x) + ye^x$$

Q. $(D - D')^2 z = e^{x+y} \sin(x+2y)$

$(D^2 + (D')^2 - 2DD')z = e^{x+y} \sin(x+2y)$

m^2 [illegible] $- 2m$

$m = 1, 1$

$CF = f_1(y+x) + x f_2(y+x)$

$$PI = \frac{1}{(D-D')^2} \cdot e^{x+y} \sin(x+2y)$$

$$= \frac{e^{x+y}}{((D-1)-(D'-1))^2} \cdot \sin(x+2y)$$

$$= \frac{e^{x+y}}{(D-D')^2} \cdot \sin(x+2y)$$

$D^2 = -1 \qquad (D')^2 = -4 \qquad DD' = -2$

$$= \frac{e^{x+y} \cdot \sin(x+2y)}{-1 - \cancel{4} + \cancel{4}} = -e^{x+y} \cdot \sin(x+2y)$$

$z = f_1(y+x) + x f_2(y+x) - e^{x+y} \cdot \sin(x+2y)$