# Unit-3 : Fourier Series representation of Periodic Signals.

Sections 3.1 - 3.3
3.4 - Brief discussion
3.5, 3.6 & 3.7 } A. Oppenheim.

## 3.0 - Introduction

## 3.1. Historical Perspective.

Fourier Series was developed by Jean Baptiste Joseph Fourier. (1768-1830) - french mathematician & Physicist.

-> Any periodic phenomena can be expressed as sums of harmonically related sines & cosines or periodic exponentials - concept was started in 1748 by L. Euler. - using Trigonometric series.

-> 1759 - This concept of using trignometric series was opposed by J.L. Lagrange - due to discontinuities in the slope of a signal.

-> Fourier's contribution.
1807 - Presented the Fourier Series - the temperature distribution in a body can be represented as a series of harmonically related sinusoids. (objected by J.L. Lagrange)
- approved by S.F. Lacroix, G. Monge & P.S. de Laplace

-> 1829 - P.L. Dirichlet, German mathematician, provided rigourous mathematical proof (or) conditions under which a periodical signal can be expressed by Fourier series. He derived conditions for convergence of Fourier Series.

-> His contributions were mostly devoted to

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Discrete-Time Fourier Series

-> Investigation on discrete-time events & signals began as early as 1600s.

-> mid-1960 - an algorithm known as Fast Fourier Transform was independently discovered by Cooley & Tukey in 1965

-> Fourier series for discrete time signals are useful in digital implementation.

3.2. The response of LTI systems to complex exponentials.

Any signal that could be represented as the linear combination of some basic signals possess the following properties

(1) The set of basic signals can be used to construct a broad & useful class of signals

(2) The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals

Basic complex exponential signals

Continuous time $\Rightarrow e^{st}$ ; s - complex number

Discrete Time $\Rightarrow z^n$ ; z - complex number

$z \Rightarrow |z|e^{i\theta}$

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Important property of LTI system

The response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude.

Continuous time: $e^{st} \longrightarrow H(s)e^{st}$

Discrete time: $z^n \longrightarrow H(z)z^n$

$H(s)$ or $H(z)$ are complex Amplitude factor in terms of $s$ or $z$.

Definitions:

A signal for which the system output is (possibly complex) constant times the input is referred to as an Eigenfunction of the system.

The amplitude factor $H(s)$ or $H(z)$ is referred to as the system's Eigen value.

All complex exponentials are indeed Eigen functions of LTI system.

Proof

Continuous Time System

Let $h(t)$ be the impulse response of $x(t) = e^{st}$ be the input. By convolution Integral, the output $y(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

Assuming the integral to converge.

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$$y(t) = H(s) e^{st}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$ is a complex constant for some value of $s$.

The constant $H(s)$ for a specific value of $s$ is then the Eigen value associated with the eigen function $e^{st}$ - Hence proved.

Discrete Systems:

Let $h[n]$ be impulse response & $x[n] = z^n$ be the input. Then by convolution Sum, the output $y[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{(n-k)}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[n] \cdot z^{-k}$$

Assuming the summation to converge

$$y[n] = z^n \cdot H(z) \text{ or } H(z) z^n$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

$H(z)$ for a specific value of $z$ is the Eigen value associated with the Eigenfn. $z^n$

Now, consider an input $x(t)$ of the form
$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$

From Eigen function property, individual responses would be
$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t}$$
$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2) e^{s_2 t}$$
$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t}$$

By superposition for an LTI system.

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Specifically if the input is of the form

$$x(t) = \sum_k a_k e^{s_k t}$$
then
$$\boxed{y(t) = \sum_k a_k H(s_k) e^{s_k t}}$$

<u>Discrete-Time System.</u>
If
$$x[n] = \sum_k a_k z_k^n$$

then
$$y[n] = \sum_k a_k H(z_k) z_k^n$$

<u>Inference :-</u>
For both CT & DT systems (LTI), if the input is
→ represented as a linear combination of complex <u>exponential</u> then the output also can be represented as a linear combination of the same complex exponentials.

→ Each coefficient in the output representation

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is obtained as the product of the corresponding coefficient $a_k$ of the input & system's Eigen value $H(s_k)$ or $H(z_k)$ associated with eigen function $e^{s_k t}$ or $z_k^n$.

For further discussions

$s \to j\omega \;\; \to e^{j\omega t}$ $\qquad e^{st}$

$z = e^{j\omega} \to e^{j\omega n}$ $\qquad z^n$

Eg 3.1 Consider an LTI having the following I-O relation:

(a) $$y(t) = x(t-3)$$

If the input to this system is the complex exponential signal $x(t) = e^{j2t}$ then

$$y(t) = e^{j2(t-3)} = e^{-j6} \cdot e^{j2t}$$

Eigen function $= e^{j2t}$

Eigen value $: H(j2) = e^{-j6}$

Verification

The impulse response of the system is

$$h(t) = \delta(t-3)$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau-3) e^{-s\tau} d\tau$$

$$= e^{-3s}$$

$$H(j2) = e^{-j6}$$

(b) Let $x(t) = \cos(4t) + \cos(7t)$

$y(t) = x(t-3)$

Then

$y(t) = \cos(4(t-3)) + \cos(7(t-3))$ — (A)

Expanding $x(t)$ using Euler's Identity.

$x(t) = \cos 4t + \cos 7t$

$= \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$

Then

$y(t) = a_1 H(\Omega_1)e^{j4t} + a_2 H(\Omega_2)e^{-j4t} + a_3 H(\Omega_3)e^{j7t} + a_4 H(\Omega_4)e^{-j7t}$

$= \frac{1}{2}e^{-j12} \cdot e^{j4t} + \frac{1}{2}e^{j12} \cdot e^{-j4t}$
$+ \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t}$

$= \frac{1}{2}e^{j4(t-3)} + \frac{1}{2}e^{-j4(t-3)} + \frac{1}{2}e^{j7(t-3)}$
$+ \frac{1}{2}e^{-j7(t-3)}$

$= \cos(4(t-3)) + \cos(7(t-3))$ — (B)

(A) = (B)

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## 3.3 Fourier Series representation of Continuous-time Periodic Signals

### 3.3.1 Linear Combinations of Harmonically related complex exponentials.

Harmonically related complex exponentials
They are set of periodic complex exponentials, all of which are periodic with common period $T_0$ or $T$

A harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are multiples of a single positive frequency $\omega_0$

$$\phi_k(t) = e^{jk\omega_0 t} \;;\; k = 0, \pm 1, \pm 2, \ldots$$

For $k=0$, $\phi_k(t)$ is a constant
For any other value of $k$, $\phi_k(t)$ is periodic with a fundamental frequency $|k|\omega_0$ & fundamental period $\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$

Periodic Signal :- A signal $x(t)$ is periodic if for some positive $T$ such that

$$x(t+T) = x(t) \quad \text{for all } t$$

$\omega_0 = \frac{2\pi}{T}$ is fundamental frequency

or $T = \frac{2\pi}{\omega_0}$

For $k \geq 2$, fundamental period $= \frac{T_0}{|k|} = \frac{T}{|k|}$

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Thus, a linear combination of harmonically related complex exponentials can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

is also periodic with period T.

→ When $k = 0$, $x(t) =$ a constant $(a_0)$

→ For $k = +1$ or $-1$, the terms have same fundamental frequency $\omega_0$ & are known as fundamental components or first harmonic components.

→ For $k = +2$ or $-2$, the terms have half the period $\left(\frac{T}{|k|} = \frac{T}{2}\right)$ or twice $\omega_0$ & referred as second harmonic components.

→ For $k = +N$ or $-N$, the components have a period of $\left(\frac{T}{|N|}\right)$ & referred as $N^{th}$ harmonic components.

The representation of a periodic signal by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}}$$

is known as Fourier Series representation.

This is also known as Exponential form of Fourier Series representation.

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Eg 3.2 Consider a periodic signal $x(t)$, with a fundamental frequency $2\pi$, that is expressed as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

when $k=0 \Rightarrow a_0 = 1$.

when $k=1 \rightarrow a_1 = a_{-1} = 1/4,\ a_2 = a_{-2} = 1/2,$
$a_3 = a_{-3} = 1/3$ } Given

Then $x(t)$ can be written as

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left[e^{j6\pi t} + e^{-j6\pi t}\right]$$

Using Euler's Identity $e^{j\theta} = \cos\theta + j\sin\theta$

$$x(t) = \underbrace{1}_{x_0(t)} + \underbrace{\frac{1}{2}\cos 2\pi t}_{x_1(t)} + \underbrace{\cos 4\pi t}_{x_2(t)} + \underbrace{\frac{2}{3}\cos 6\pi t}_{x_3(t)}$$

$$x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t).$$

$\boxed{\text{Refer :: Figs - 3.4 of Pg 190.}}$

Figures show, how the signal $x(t)$ is built up from its harmonic component

Let the signal $x(t)$ be real, then its complex

→ conjugate $x^*(t)$ is same as $x(t)$

$$x^*(t) = x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$ ; j - complex conjugate

replacing $k$ by $-k$ in the summation, we have

$$x^*(t) = x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{+jk\omega_0 t}$$

comparing this equation with

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_0 t}$$

we need to have

$$a_k = a_{-k}^*$$

(or)

$$a_k^* = a_{-k}$$

In the example 3.2, we see $a_k = a_{-k}$ ; coefficients are real.

Alternative forms of Fourier Series

$\boxed{\text{Form-1}}$

Consider

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

$$= a_0 + \sum_{k=1}^{\infty} \{a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\}$$

Since the terms inside the summation are complex conjugates of each other, we can express

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

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**Form 2**

If $a_k$ is expressed in Polar form as

$$a_k = A_k e^{j\theta_k}$$

then

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2 \cdot \mathrm{Re}\left\{ A_k e^{j(k\omega_0 t + \theta_k)} \right\}$$

$$= a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

This is frequently encountered form for the Fourier Series of real periodic signals in Continuous time.

**Form 3**

If $a_k$ is represented in rectangular form

$$a_k = B_k + jC_k$$

where $B_k$ & $C_k$ are real.

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} \left\{ B_k \cos k\omega_0 t - C_k \sin k\omega_0 t \right\}$$

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## 3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Consider a periodic signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_0 t}$$ → we need to find the coefficient $a_k$.

multiplying this equation with $e^{-jn\omega_0 t}$, we get

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t} \cdot e^{-jn\omega_0 t}$$

Integrating both sides from 0 to $T = \frac{2\pi}{\omega_0}$ (fundamental period), we have

$$\int_0^T x(t) e^{-jn\omega_0 t} \, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t} \cdot e^{-jn\omega_0 t} \, dt$$

Here T - fundamental period & we are integrating over one period. Interchanging the order of integration & summation yields.

$$\int_0^T x(t) e^{-jn\omega_0 t} \, dt = \sum_{k=-\infty}^{\infty} a_k \left\{ \int_0^T e^{+jk\omega_0 t} \cdot e^{-jn\omega_0 t} \, dt \right\}$$

$$= \sum_{k=-\infty}^{\infty} a_k \left\{ \int_0^T e^{+j(k-n)\omega_0 t} \, dt \right\}$$

Evaluation of the integral in the { }

$$\int_0^T e^{+j(k-n)\omega_0 t} \, dt = \int_0^T \cos(k-n)\omega_0 t \, dt + j \int_0^T \sin(k-n)\omega_0 t \, dt$$

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For $k \neq n$

$\cos(k-n)\omega_0 t$ & $\sin(k-n)\omega_0 t$ are
→ periodic sinusoids with fundamental period $\frac{T}{|k-n|}$

→ Then, we are integrating over any interval

RHS (of length T) that is an integral number of the periods of these signals.

→ Since the integral may be viewed as measuring the total area under the functions over the

Doubt interval, we can observe that for $k \neq n$, both of the integrals are zero

$$\int_0^T \cos(k-n)\omega_0 t \, dt = \left[\frac{\sin(k-n)\omega_0 t}{(k-n)\omega_0}\right]_0^T$$

$$= \frac{1}{(k-n)\omega_0}\left[\sin(k-n)\omega_0 t\right]_0^T$$

$$= \frac{1}{(k-n)\omega_0}\left[\sin(k-n)\omega_0 T - 0\right]$$

$$= \frac{1}{(k-n)\omega_0}\left[\sin(k-n)\frac{2\pi}{T} \times T\right]$$

$$= \frac{1}{(k-n)\omega_0}\left[\sin(k-n)2\pi\right] = 0$$

since $k-n$ = integer

Aliter

Consider $\int_0^T e^{j(k-n)\omega_0 t}\,dt$ ; $k \neq n = p$



$$\int_0^T e^{jp\omega_0 t}\,dt = \left.\frac{e^{jp\omega_0 t}}{jp\omega_0}\right|_0^T = \frac{e^{jp\omega_0 T} - 1}{jp\omega_0}$$

$$= \frac{e^{jp\frac{\omega_0}{1}\times\frac{2\pi}{\omega_0}} - 1}{jp\omega_0} = \frac{e^{j2\pi p} - 1}{jp\omega_0}$$

$$= \frac{1-1}{j\omega_0 p} = 0 \quad \forall,\ \underline{e^{j2\pi p} = 1}.$$

$$\omega_0 \int_0^T t\,dt \quad \omega_0 \left[\frac{t^2}{2}\right]$$

$\boxed{\text{For } k=n}$

$$\int_0^T \cos(k-n)\omega_0 t\,dt = \int_0^T dt = T$$

$$\int_0^T \sin(k-n)\omega_0 t\,dt = 0.$$

Then we have

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T & \text{for } k=n \\ 0 & \text{for } k\neq n \end{cases}$$

Then

$$\sum_{k=-\infty}^{\infty} a_k \left\{\int_0^T e^{+j(k-n)\omega_0 t}\,dt\right\} = a_n T \quad \because k=n$$

(or)

$$\boxed{a_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt}$$

This relation helps in finding the coefficients.

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In general for any length T (for eg $-\frac{T}{2}$ to $+\frac{T}{2}$

$$\int_T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T & \text{for } k=n \\ 0 & \text{for } k\neq n \end{cases}$$

Consequently.

$$a_n = \frac{1}{T}\int_T x(t)\,e^{-jn\omega_0 t}\,dt.$$

Summary

If $x(t)$ has a Fourier Series representation (i.e if it can be expressed a linear combination of harmonically related complex exponentials of the form

$$\sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}\ )$$

co-efficients are given by. $a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}$

① ← $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t}$$

② ← $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(\frac{2\pi}{T})t}\,dt$$

① - Synthesis equation.

② - Analysis equation

$a_k$ = Fourier series co-efficients (or) Spectral co-efficients.

$a_0 = \frac{1}{T}\int_T x(t)\,dt \rightarrow$ DC ~~or~~ constant component.

which is the ~~average~~ average value of $x(t)$ over one period.

Theorem :: Orthogonality of Complex exponentials

$$\frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^{jn\omega_0 t} \cdot e^{jm\omega_0 t^*}\,dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$T_0 = \frac{2\pi}{\omega_0}$ $\qquad f^*(t) \rightarrow$ Complex Conjugate.

Solution

$$\frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^{jn\omega_0 t} \cdot e^{jm\omega_0 t^*}\,dt = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^{j(n-m)\omega_0 t}\,dt$$

Let $n-m = p \neq 0$

$$I = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^{jp\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\left[\frac{e^{jp\omega_0 t}}{jp\omega_0}\right]_{-T_0/2}^{+T_0/2}$$

$$= \frac{1}{T_0}\left\{\frac{e^{jp\pi} - e^{-jp\pi}}{jp\omega_0}\right\} = \frac{1}{T_0}\left[\frac{1-1}{j\omega_0 p}\right] = \underline{0}$$

When $n = m$

$$I = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} e^0\,dt = \frac{1}{T_0} \cdot T_0 = \underline{1}$$

Additional material

$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$
$e^{-j\omega_0 t} = \cos\omega_0 t - j\sin\omega_0 t$
$e^{j\omega_0 t} - e^{-j\omega_0 t} = 2j\sin\omega_0 t$



**Eg 3.3** Consider the signal

$x(t) = \sin\omega_0 t$    Fundamental frequency $\omega_0$

Instead of applying the F.S. eqn

$$\sin\omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing this with

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

we get

$$a_1 = \frac{1}{2j} \quad \& \quad a_{-1} = -\frac{1}{2j}$$

$$a_k = 0 \quad \text{for } k \neq +1 \text{ or } -1$$

**Eg 3.4** Consider

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

Expressing each term in terms of complex exponentials.

$$\sin\omega_0 t = \frac{1}{2j}\left\{e^{j\omega_0 t} - e^{-j\omega_0 t}\right\}$$

$$2\cos\omega_0 t = \left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\}$$

$$\cos\left(2\omega_0 t + \frac{\pi}{4}\right) = \frac{1}{2}\left\{e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right\}$$

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collecting like terms.

$$x(t) = 1 + \left\{1 + \frac{1}{2j}\right\} e^{jw_0 t} + \left\{1 - \frac{1}{2j}\right\} e^{-jw_0 t} + \left\{\frac{1}{2} e^{j(\pi/4)}\right\} e^{j2w_0 t} + \left\{\frac{1}{2} e^{-j(\pi/4)}\right\} e^{-j2w_0 t}$$

Then

$$a_0 = 1 \; ; \; a_1 = 1 + \frac{1}{2j} = 1 - \frac{1}{2}j$$

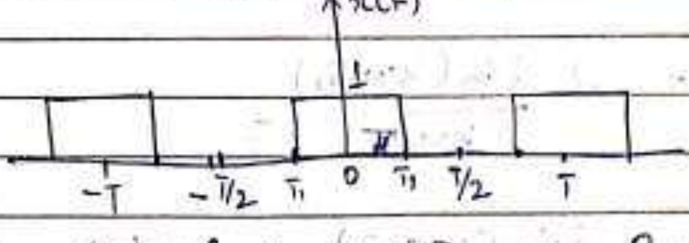
$$a_{-1} = 1 - \frac{1}{2j} = 1 + \frac{1}{2}j \; ; \; a_2 = \frac{1}{2} e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j)$$

$$a_{-2} = \frac{1}{2} e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j)$$

$$a_k = 0 \quad \text{for } |k| > 2$$

→ Note: coefficients can be complex; hence have magnitude & phase angle.

Eg. 3.5 Consider the square wave




$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases} \quad \text{for one period}$$

The given signal has a fundamental period $T$ ; $w_0 = \frac{2\pi}{T}$ fundamental frequency

To determine the fourier co-efficients

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jkw_0 t}\, dt = \frac{1}{T}\int_T x(t)\, e^{-jk\left(\frac{2\pi}{T}\right)t}\, dt$$

→ Because of the symmetry about the origin $t=0$, it is convenient to choose $-\frac{T}{2} \leq t < \frac{T}{2}$ as the interval over which the integration is performed, although any interval of length $T$ is equally valid.

→ $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$

→ $k \neq 0$ we get

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} (1)\, e^{-jkw_0 t} = -\frac{1}{jkw_0 T}\, e^{-jkw_0 t}\Big|_{-T_1}^{+T_1}$$

$$= \frac{2}{kw_0 T}\left\{\frac{e^{jkw_0 T_1} - e^{-jkw_0 T_1}}{2j}\right\}$$

$$= \frac{2\sin(kw_0 T_1)}{kw_0 T}$$

$$= \frac{\sin(kw_0 T_1)}{k\pi}, \quad k \neq 0 \quad \text{since } w_0 = \frac{2\pi}{T}$$

similar to 'sa' function $\left(\frac{\sin x}{x}\right)$

$\boxed{\text{Fig 3.7 Pg - 197}}$ when coefficients are real - one graph
If complex - two graphs are reqd.

case (c) When $T = 4T_1$, $x(t)$ is a square wave, that is unity for half period + zero for the half period.

Then

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$= \frac{2\sin\left(k \cdot \frac{2\pi}{4T_1} \times T_1\right)}{k \frac{2\pi}{\cancel{T}} \cancel{T}}$$

$$= \frac{\cancel{2}\sin\left(k\frac{\pi}{2}\right)}{\cancel{2}k\pi} = \frac{\sin(\pi k/2)}{k\pi} \; ; k \neq 0$$

$a_0 = \frac{1}{2}$ L'Hospital Rule.

From the expression

$a_k = \frac{\sin\left(\frac{\pi k}{2}\right)}{k\pi}$, for k is even & non-zero the corresponding coefficients are zero.

$a_1 = a_{-1} = \frac{1}{\pi}$

$a_3 = a_{-3} = -\frac{1}{3\pi}$

$a_5 = a_{-5} = \frac{1}{5\pi}$

$a_7 = +a_{-7} = -\frac{1}{7\pi}$

$\vdots$

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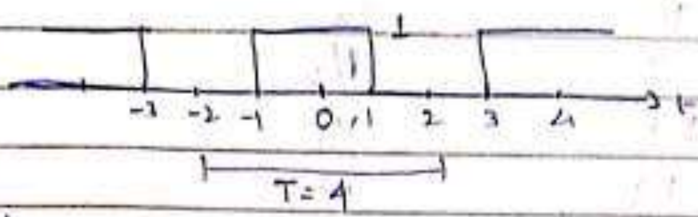
Eg 1.) (12)

Obtain the exponential FS representation of $x(t)$



Solution

Expression for $x(t)$ over 1 period.

$$x(t) = \begin{cases} 0 & \text{for } -2 < t < -1 \\ 1 & \text{for } -1 < t < +1 \\ 0 & \text{for } 1 < t < 2 \end{cases}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \cdot dt$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{1}{2}\pi$$

Then

$$a_k = \frac{1}{4} \int_{-1}^{+1} (1) e^{-jk\frac{\pi}{2}t} \, dt \quad ; \text{ check for the definition of } x(t).$$

$$= \frac{1}{4} \left[ \frac{e^{-jk\frac{\pi}{2}t}}{-jk\pi/2} \right]_{-1}^{+1}$$

$$= \frac{1}{4} \left[ \frac{e^{+jk\pi/2} - e^{-jk\pi/2}}{+jk\pi/2} \right]$$

$$= \frac{1}{2} \times \frac{1}{2} \left[ \frac{e^{jk\pi/2} - e^{-jk\pi/2}}{jk\pi/2} \right] = \frac{1}{2} \left[ \frac{\sin(k\pi/2)}{k\pi/2} \right]$$

Similar to Sa function $\left( \frac{\sin x}{x} \right)$

(21)

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For $k=0$ $a_0 = \frac{1}{2}$ ; L'Hospital's Rule.

| $k$ | $a_k$ |
|---|---|
| 0 | 1/2 |
| $\pm 1$ | $1/\pi$ |
| $\pm 2$ | 0 |
| $\pm 3$ | $-\frac{1}{3\pi}$ |
| $\pm 4$ | 0 |
| $\pm 5$ | $\frac{1}{5\pi}$ |

Fourier Series is then given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \frac{1}{\pi}\left\{ \cdots + \frac{1}{5} e^{-j5\frac{\pi}{2}t} - \frac{1}{3} e^{-j3\frac{\pi}{2}t} + \frac{1}{1} e^{-j\frac{\pi}{2}t} + \frac{\pi}{2} + \frac{1}{1} e^{j\frac{\pi}{2}t} - \frac{1}{3} e^{-j3\frac{\pi}{2}t} + \frac{1}{5} e^{j5\frac{\pi}{2}t} + \cdots \right\}$$

D.C value $a_0 = \frac{1}{2}$

First harmonic

$$h(1) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$

$$= \frac{1}{\pi} e^{j\frac{\pi}{2}t} + \frac{1}{\pi} e^{-j\frac{\pi}{2}t}$$

$$= \frac{2}{\pi} \cos\left(\frac{\pi}{2}t\right)$$

Third harmonic

$$h(3) = -\frac{1}{3\pi} e^{j3\frac{\pi}{2}t} - \frac{1}{3\pi} e^{-j3\frac{\pi}{2}t}$$

$$= -\frac{2}{3\pi} \cos\left(3\frac{\pi}{2}t\right)$$

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3.4 Convergence of Fourier Series.

P.L. Dirichlet (1829) gave the conditions for the convergence of Fourier Series.

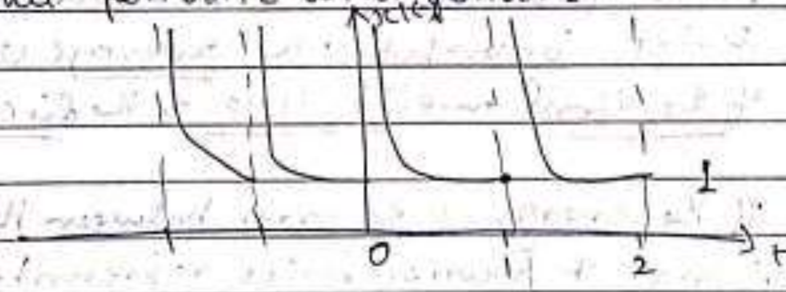
- Condition 1 : Over one period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\,dt < \infty$$

A periodic signal which violates this condition is

$$x(t) = \frac{1}{t} \; ; \quad 0 < t \leq 1$$

This periodic with period 1




Condition 2 :- In any finite interval of time, $x(t)$ is of bounded variation. That is, there are finite number of maxima & minima during any single period.

Eg

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \; ; \quad 0 < t \leq 1$$

This is periodic with a period of 1. Satisfies Condition 1 & violates Condition 2.

$$\int_0^1 |x(t)|\,dt < 1$$

Function has infinite no. of maxima & minima

Fig 3.8 (b)

Condition 3 :- In any finite interval of time, there are only a finite number of discontinuities. Further, each of these discontinuities is finite.

Fig 38 (c)

→ Important Note:

(1) All practical signals obey Dirichlet Condition

(2) For a periodic signal with finite number discontinuities in each period, the Fourier series representation equals the signal everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity

(3) If the energy of the error between the original signal & Fourier series representation is zero, then both are same

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$e_N(t) = x(t) - x_N(t)$$

$$E_N = \text{Energy} = \int_T |e_N(t)|^2 dt$$

(4) Gibbs Phenomenon :- While approximating a discontinuous signal, the approximation signal should have ripples near the point of discontinuity. As N is increased, the ripples got compressed; however the peak overshoot was constant.

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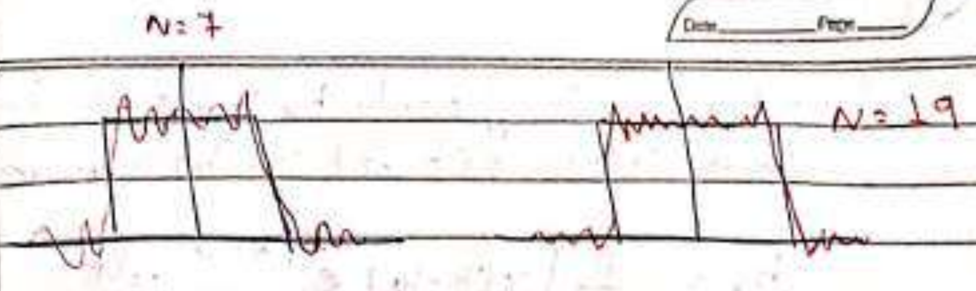


## 3.5 Properties of Continuous Time Fourier Series

**Assumptions**

$x(t)$ is periodic with period $T$ + fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Fourier coefficients $a_k$

$$x(t) \xleftrightarrow{FS} a_k$$

### 3.5.1 Linearity

Consider two signals $x(t)$ + $y(t)$ which are periodic with same period $T$

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \xleftrightarrow{FS} b_k$$

Any linear combination of these two signals will also be periodic with the same period $T$.

If $z(t) = Ax(t) + By(t)$ with $z(t)$ having $c_k$ as Fourier coefficient, then

$$z(t) = Ax(t) + By(t) \longleftrightarrow c_k = Aa_k + Bb_k.$$

This property can be extended to a linear combination of any arbitrary number of signals with period $T$.

### 3.5.1 Time Shifting

When a time shift is applied to a periodic

→ signal with period $T$, its period $T$ is preserved.

The Fourier coefficients $b_k$ of the resulting signal $y(t) = x(t-t_0)$ can be expressed as

$$b_k = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

letting $\tau = t - t_0$ so $t = \tau + t_0$

$\tau$ also range over an interval of duration $T$

$$\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(t_0+\tau)}\, d\tau$$

$$= e^{-jk\omega_0 t_0}\,\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0\tau}\, d\tau$$

$$= e^{-jk\omega_0 t_0}\cdot a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0}\cdot a_k$$

where $a_k$ is the $k^{th}$ F.S. coefficient of $x(t)$

Then

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

while

$$x(t-t_0) \overset{FS}{\longleftrightarrow} e^{-jk\omega_0 t_0}\, a_k$$

$$= e^{-jk\left(\frac{2\pi}{T}\right)t_0}\, a_k$$

<u>Inference</u>

When a periodic signal is shifted in time, the <u>magnitudes</u> of its F.S. coefficients remain unaltered.

i.e. $|b_k| = |a_k|$

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## 3.5.3 Time Reversal

The period T of a periodic signal $x(t)$ also remains unchanged when the signal undergoes time reversal.

let $y(t) = x(-t)$; consider the synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}t}$$

letting $k = -m$, we get

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\frac{2\pi}{T}t}$$

RHS has the form of FS synthesis for $x(-t)$ whose FS coefficients are

$$b_k = a_{-k}$$

That is, if

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

**Inference**

Time reversal applied to a continuous-time signal results in a time reversal of the corresponding sequence of FS coefficients.

→ If $x(t)$ is even, then $x(-t) = x(t)$, then its FS coefficients are also even. i.e $\boxed{a_{-k} = a_k}$

→ If $x(t)$ is odd, $x(-t) = -x(t)$, then so are its F.S coefficients

$$\boxed{a_{-k} = -a_k}$$

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## 3.5.4 Time Scaling - Important

→ Time scaling in general changes the period of the considered signal.

→ If $x(t)$ is periodic with a period $T$ & fundamental frequency $\omega_0 = \frac{2\pi}{T}$, then

$x(\alpha t)$, where $\alpha$ is a positive real number, is also periodic with a period $\frac{T}{\alpha}$ &

fundamental frequency $\alpha\omega_0$.

→ $$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

Inference

→ Fourier coefficients do not change.

→ FS representation has changed due to change in the fundamental frequency.

## 3.5.5 Multiplication

Suppose $x(t)$ & $y(t)$ are both periodic with a period $T$ and that

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \longleftrightarrow b_k$$

The product $x(t)y(t)$ is also periodic with $T$.

$$x(t)\,y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

RHS can be interpreted as

Discrete convolution of $a_k$ & $b_k$.

3.5.6 Conjugation + Conjugate Symmetry.

Complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier Series coefficients.

If

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$x^*(t) \xleftrightarrow{FS} a^*_{-k}$$

Inferences

1) For $x(t)$ being real then $x(t) = x^*(t)$, then from the above equation

$$\boxed{a_{-k} = a_k^*}$$ Fourier Series Co-efficients are Conjugate Symmetric

2) Again for real $x(t)$

$$|a_k| = |a_{-k}|$$

(3) For $x(t)$ being real + even

$$a_k = a_{-k}$$

(4) For $x(t)$ being real + odd : $a_k$ are purely imaginary + odd, $a_0 = 0$

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## 3.5.7 Parseval's Relation for C.T. Periodic Signals

For continuous-time periodic signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

$a_k$ - Fourier co-efficients, $x(t)$ - periodic signal with period T.

LHS - Average power in one period (energy per unit time)

Also

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt$$

$$= |a_k|^2$$

→ $|a_k|^2$ is the Average Power in the $k^{th}$ harmonic component of $x(t)$

→ Parseval's Relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

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Table 3.1

$x(t), y(t)$, periodic signals with period $T$ & $\omega_0 = \frac{2\pi}{T}$ with FS coefficients $a_k$ & $b_k$, respectively.

| | | | |
|---|---|---|---|
| (1) | Linearity | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| (2) | Time Shifting | $x(t-t_0)$ | $a_k e^{-jk\omega_0 t_0}$ |
| (3) | Frequency shifting | $e^{jM\omega_0 t} \cdot x(t) = e^{jM(\frac{2\pi}{T})t} \cdot x(t)$ | $a_{k-M}$ |
| (4) | Conjugation | $x^*(t)$ | $a^*_{-k}$ |
| (5) | Time reversal | $x(-t)$ | $a_{-k}$ |
| (6) | Time scaling | $x(\alpha t); \alpha > 0$ periodic with period $T/\alpha$ | $a_k$ (No change) |
| (7) | Periodic convolution | $\int_T x(\tau) y(t-\tau) d\tau$ | $T a_k b_k$ |
| (8) | multiplication (modulation) | $x(t) y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ (or) $a_k * b_k$ |
| (9) | Differentiation | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$ |
| (10) | Integration | $\int_{-\infty}^{t} x(t) dt$ (finite valued & periodic only if $a_0 = 0$) | $\frac{1}{jk\omega_0} \cdot a_k = \frac{1}{jk(\frac{2\pi}{T})} \cdot a_k$ |

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11) Conjugate Symmetry for Real Signals — $x(t)$ real —
$$\begin{cases} a_k = a^*_{-k} \\ \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \angle a_k = -\angle a_{-k} \end{cases}$$

12) Real + even signal — $x(t)$ real + even — $a_k$ real + even

13) Real + odd signal — $x(t)$ real + odd — $a_k$ is purely imaginary + odd, $a_0 = 0$

14) Even + odd decomposition of Real Signals — $x_e = \mathrm{Ev}\{x(t)\}$, $x_o = \mathrm{Odd}\{x(t)\}$ [$x(t)$ is real] — $\mathrm{Re}\{a_k\}$, $j\,\mathrm{Im}\{a_k\}$

## Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

## Additional Inputs.

(1) Theorem

$$x(t) \xleftrightarrow{FS} a_k \text{ then}$$

$$a_k^* = a_{-k} \text{ for all } k \text{ if } x(t)$$

if + only if $x(t)$ is a real function.

Corollary

Let $a_k = A(k) + j B(k)$

then

$$a_{-k} = a_k^*$$
$$= A(k) - j B(k)$$

Inferences

Since $x(t)$ is an even function

$$|a_k| = |a_{-k}|$$

→ the plot of $|a_k|$ vs $k$ known as Amplitude Spectrum exhibits even Symmetry w.r.t to the vertical axis

→ $\angle$ of $a_k$ = $-\angle$ leg $a_{-k}$, the plot of $\angle$le of $a_k$ w.r.t $k$ known as Phase Spectrum exhibits odd Symmetry about the vertical axis.

(2) Theorem :- let $x(t)$ be a periodic function which is real defined by $x(t) = x_e(t) + x_o(t)$

let its F.S. coefficients defined by

$$a_k = A(k) + j B(k)$$

Then

$$x_e(t) \xleftrightarrow{FS} A(k) \text{ or real part of } a_k$$

$$x_o(t) \longleftrightarrow jB(k) \text{ or } j \text{ Img. part of } a_k$$

Note

$$x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$$

$$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$$

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③ Amplitude + Phase Spectrum of a Periodic Signal.

Let $a_k = A(k) + j B(k)$

$$|a_k| = \sqrt{A^2(k) + B^2(k)}$$

$$\theta_k = \angle a_k = \tan^{-1}\left(\frac{B(k)}{A(k)}\right)$$

Plot $|a_k|$ vs $k$ → Amplitude Spectrum

Plot $\theta_k$ vs $k$ → Phase Spectrum.

Inference

- Spectrum is not continuous, since $a_k$ are discrete values at $k$.

→ Spectrum is referred as Discrete frequency spectra or Line Spectra.

④ Time Domain Convolution.

Let

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

If $z(t) = x(t) \circledast y(t)$

then

$$z(t) = x(t) \circledast y(t) \xleftrightarrow{FS} T a_k b_k$$

where $T = \frac{2\pi}{\omega_0}$ $\circledast$ periodic convolution.

⑤ multiplication or modulation.

Let

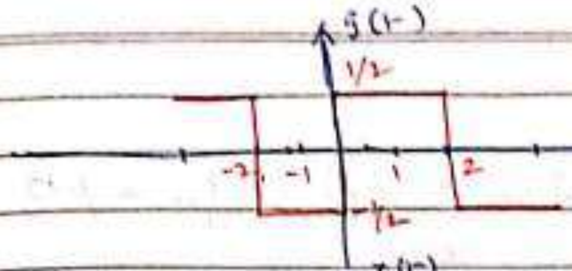
$$x(t) \xleftrightarrow{FS} a_k + y(t) \xleftrightarrow{FS} b_k$$

Then

$$z(t) = x(t)\,y(t) \xleftrightarrow{F(s)} a_k * b_k$$

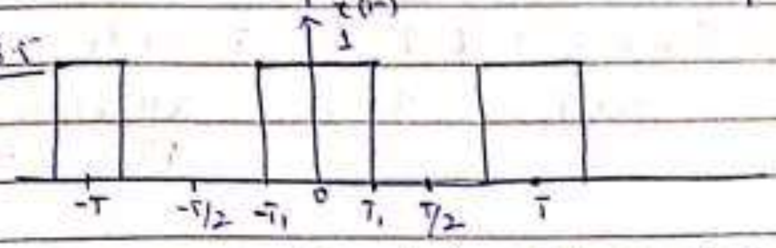
Discrete convolution

Eg 3.6




Fundamental period = 4

Eg 3.5




Comparing $g(t)$ with $x(t)$, with $T=4$ & $T_1=1$
we can express $g(t)$ in terms of $x(t)$ as

$$g(t) = x(t-1) - 1/2$$

Let
$$x(t) \xleftrightarrow{FS} a_k .$$

then
$$x(t-t_0) \xleftrightarrow{FS} a_k e^{-jk\omega_0 t_0} = a_k e^{-jk\left(\frac{2\pi}{T}\right)t_0}$$
$$\longleftrightarrow \quad b_k$$

Then with $T=4$ (no change in periodicity)
$$b_k = a_k e^{-jk\left(\frac{2\pi}{4}\right)t_0} \quad ; \quad t_0 = 1$$

$$b_k = a_k e^{-jk\left(\frac{\pi}{2}\right)}$$

Fourier coefficient of the DC term of $g(t)$ i.e $-1/2$ on the RHS are given by

$$c_k = \begin{cases} 0 & \text{for } k \neq 0 \\ -1/2 & \text{for } k = 0 \end{cases}$$

Applying Linearity property of FS co-efficients.

$$d_k = \begin{cases} a_k e^{-jk\left(\frac{\pi}{2}\right)} & \text{for } k \neq 0 \\ a_0 - \frac{1}{2} & \text{for } k = 0 \end{cases}$$

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From Eq 3.5

$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad \text{for } k \neq 0$$

Since $T_1 = 1$ & $T = 4$, $T = 4T_1$.

$$\omega_0 T_1 = \frac{2\pi}{T} \cdot T_1 = \frac{2\pi}{4T_1} \times T_1 = \pi/2$$

$$a_k = \frac{\sin(k\pi/2)}{k\pi} \;;\; k \neq 0$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} = \frac{2T_1}{4T_1} = \frac{1}{2}.$$

(or)

$$a_k = \frac{1}{2}\,\frac{\sin(k\pi/2)}{k\pi/2} \;;\; \text{L'Hospital's Rule}$$

$$= \frac{1}{2}.$$

Finally

$$d_k = \begin{cases} \frac{\sin(k\pi/2)}{k\pi} \cdot e^{-jk\pi/2} \\ \frac{1}{2} - \frac{1}{2} \end{cases}$$

$$= \begin{cases} \frac{\sin k\pi/2}{k\pi} \cdot e^{-jk\pi/2} & k \neq 0 \\ 0 & k = 0 \end{cases}$$

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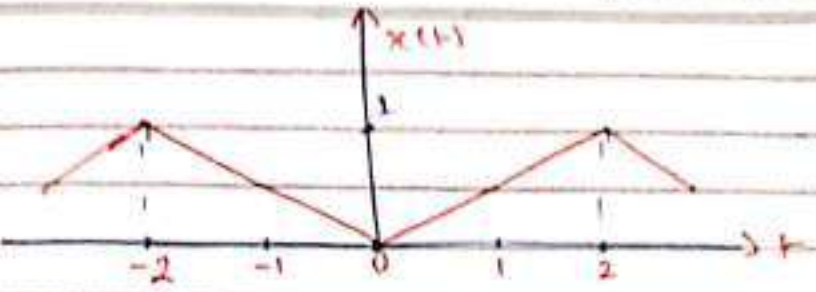
Fig 3.7




Consider x(t), a triangular wave signal having a period $T = 4$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$

Derivative of this signal is g(t) in Fig 3.6.

Denoting the co-efficients of g(t) as $d_k$ & those of x(t) as $e_k$, we have from the property of differentiation

$$d_k = jk\left(\frac{\pi}{2}\right) e_k$$

This equation can be used to get $e_k$ in terms of $d_k$ except when $k = 0$.

$$e_k = \frac{2 d_k}{jk\pi} = \frac{2\sin(k\pi/2)}{j k\pi (k\pi)} ; k \neq 0$$

$$= \frac{2\sin(k\pi/2)}{j(k\pi)^2} ; k \neq 0$$

$k = 0$, $e_0$ can be determined finding the area under one period & dividing by the length of the period.

$$\text{Area} = 2 \times \frac{1}{2}(b)(h) = 2 \times \frac{1}{2}(2)(1) = 2$$

Length of the period $= 4$

Then $c_0 = \frac{2}{4} = \frac{1}{2}$

$$\underbrace{\frac{1}{T}\int_T x(t)dt}_{\text{area/period}}$$

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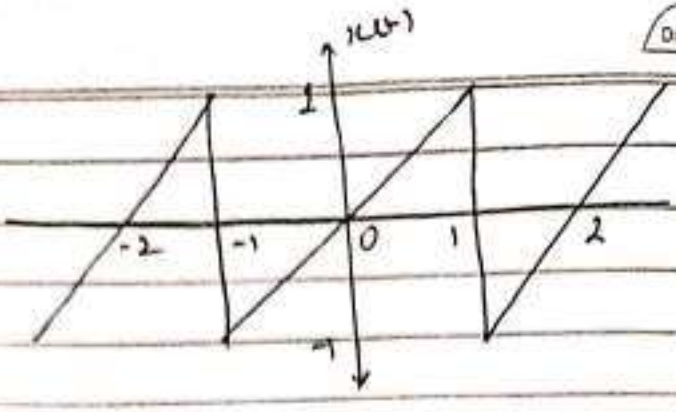
Eg 3.2




Following assumption are made.

(a) Wave form is real & hence $a_{-k} = a^*_k$

(b) Wave form is odd & hence $a_k$ is purely imaginary

i.e if $a_k = A(k) + j B(k)$

$= jB_k$ with $A(k) = 0$.

(c) The imaginary part of $a_k$ i.e $B(k)$ is odd

$B(-k) = -B(k)$

Definition of $x(t)$ over a period

Period $= -1$ to $+1$ $= \boxed{2 = T}$ $\omega_0 = \frac{2\pi}{T} = \pi$

Then

$$\boxed{x(t) = t \text{ for } -1 < t < +1}$$

Then

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt.$$

$$= \frac{1}{2}\int_{-1}^{1} t\{\cos k\omega_0 t - j \sin k\omega_0 t\} dt$$

Since $\omega_0 = \pi$

$$a_k = \frac{1}{2}\int t\{\cos(k\pi t) - j \sin(k\pi t)\} dt$$

t is odd, cos t - even, sin(...) odd.
then
$t\cos(k\pi t) \rightarrow$ odd + limits are
$t\sin(k\pi t) \rightarrow$ even. symmetric

Then

$$a_k = \frac{2}{2}\int_0^1 -jt\sin(k\pi t)\,dt$$

$$= -j\left\{-t\frac{\cos(k\pi t)}{k\pi} + \frac{\sin(k\pi t)}{k^2\pi^2}\right\}_0^1$$

$$= -j\left\{-\frac{\cos(k\pi)}{k\pi}\right\} = j\frac{\cos(k\pi)}{k\pi}$$

$$= \frac{j(-1)^k}{k\pi} \quad \text{for } k \neq 0$$

we need to separately evaluate for $a_0$ i.e $k=0$

when $k=0$ $\quad a_0 = \frac{1}{T}\int_{-1}^{+1} t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_{-1}^{1}$

$$= \frac{1}{4}\left\{(-1)^2 - (1)^2\right\} = \frac{1}{4}(0) = 0$$

Since the given function is odd, $\boxed{a_0 = 0}$

Then

$$a_k = \begin{cases} j\frac{(-1)^k}{k\pi} & ; k \neq 0 \\ 0 & k = 0. \end{cases}$$

| $k$ | $a_k$ | $\theta_k = \angle a_k$ radians |
|---|---|---|
| $-2$ | $-j/2\pi$ | $-\pi/2$ |
| $-1$ | $j/\pi$ | $\pi/2$ |
| $0$ | $0$ | $0$ |
| $1$ | $-j/\pi$ | $-\pi/2$ |
| $2$ | $j/2\pi$ | $+\pi/2$ |

$\tan^{-1}(\alpha)$

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From the table

(i) $a_{-k} = a^*_k \qquad a_{-2} = -j\frac{1}{2\pi} \qquad a_{+2} = \frac{j}{2\pi}$

(ii) $a_k$ = purely imaginary. $j\frac{(-1)^k}{k\pi}$

(iii) Real part of $a_k = 0 = A(k)$

Imaginary part $B(k) = \frac{(-1)^k}{k\pi}$

$= -B(-k)$

or $\boxed{B(-k) = -B(k)}$ odd fun.

$a_{+2} = \frac{j}{2\pi} \qquad a_{-2} = -a_2 = -\frac{j}{2\pi}$

(iv) $a_0 = 0$ since $x(t)$ is odd.

(v) $\theta_k = \angle a_k$ exhibits odd symmetry w.r.t. vertical axis

Ex 3.3 (2.8) (285)

Find the Fourier co-efficients of $x(t)$

$$x(t) = \cos\left(\underbrace{\frac{2\pi}{3}}_{\omega_1}t\right) + 2\cos\left(\underbrace{\frac{5\pi}{3}}_{\omega_2}t\right)$$

Given

Alternate method.

$$x(t) = \cos\frac{2\pi}{3}t + 2\cos\frac{5\pi}{3}t.$$

Different period

$$= \frac{1}{2}e^{j\frac{2\pi}{3}t} + \frac{1}{2}e^{-j\frac{2\pi}{3}t} + e^{j\frac{5\pi}{3}t} + e^{-j\frac{5\pi}{3}t}$$

$$\omega_1 = \frac{2\pi}{3}, \quad \omega_2 = \frac{5\pi}{3}$$

Then $\omega_0 = \text{GCD}[\omega_1, \omega_2] = \text{GCD}\left[\frac{2\pi}{3}, \frac{5\pi}{3}\right] = \frac{\pi}{3}$

So $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/3} = 6$ period

writing $x(t)$ in terms of $\omega_0$ we get

$$x(t) = \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} + e^{j5\omega_0 t} + e^{-j5\omega_0 t}$$

Synthesis eqn.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$a_{-5} = 1, \quad a_{-2} = \frac{1}{2}; \quad a_{+2} = \frac{1}{2}, \quad a_5 = 1.$

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Eg 3.4 (CQ) (28)

Find FS Coefficients of

$$x(t) = \cos(2t) + \sin(\pi t) \qquad \frac{T_1}{T_2} = \frac{\pi}{2} \text{ irrational.}$$

Solution

x(t) is not periodic. Hence no Fourier Series, no $\omega_0$ & $f_0$

Eg 3.5 (CP) (286)

Find FS Co-efficients of

$$x(t) = \frac{1}{4} + \sum_{n=1}^{4} \left\{ \frac{1}{n^2} \cos\left(\frac{nt}{5}\right) + \frac{(-1)^n}{(2n+1)^2} \sin\left(\frac{nt}{5}\right) \right\}$$

Solution

$$\omega_0 = \frac{1}{5} \; ; \; T = \frac{2\pi}{\omega_0} = \frac{2\pi}{1/5} = \underline{\underline{10\pi}}$$

$$x(t) = \frac{1}{4} + \sum_{n=1}^{4} \left\{ \frac{1}{2n^2} e^{\frac{jnt}{5}} + \frac{1}{2n^2} e^{\frac{-jnt}{5}} \right\}$$

$$+ \sum_{n=1}^{4} \left\{ \frac{(-1)^n}{2j(2n+1)^2} e^{\frac{jnt}{5}} - \frac{(-1)^n}{2j(2n+1)^2} e^{\frac{-jnt}{5}} \right\}$$

$$= \frac{1}{4} + \sum_{n=1}^{4} \left\{ \left[ \frac{1}{2n^2} - \frac{j(-1)^n}{2(2n+1)^2} \right] e^{\frac{jnt}{5}} \right\}$$

$$+ \sum_{n=1}^{4} \left\{ \left[ \frac{1}{2n^2} + \frac{j(-1)^n}{2(2n+1)^2} \right] e^{\frac{-jnt}{4}} \right\}$$

Replacing n by k & $\frac{1}{5} = \omega_0$, we get

$$x(t) = \frac{1}{4} + \sum_{k=1}^{4} \left\{ \left[ \frac{1}{2k^2} - \frac{j(-1)^k}{2(2k+1)^2} \right] e^{jk\omega_0 t} \right\}$$

$$+ \sum_{k=1}^{4} \left\{ \left[ \frac{1}{2k^2} + \frac{j(-1)^k}{2(2k+1)^2} \right] e^{-jk\omega_0 t} \right\}$$

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Therefore

$$a_k = \frac{1}{2k^2} - \frac{j(-1)^k}{2(2k+1)^2}$$

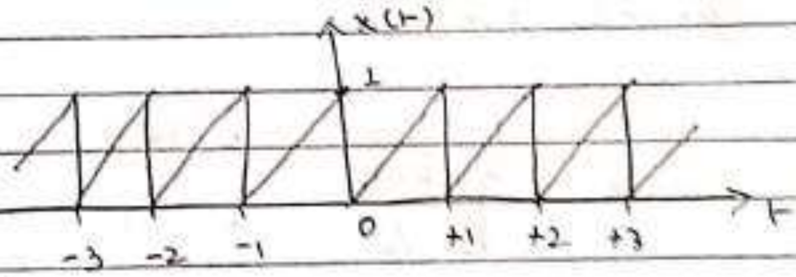
&

$$a_{-k} = \frac{1}{2k^2} + \frac{j(-1)^k}{2(2k+1)^2}$$

$$a_0 = \frac{1}{4} \quad ; \quad k = 1, 2, 3, 4$$

Ej 3.6

GR (287)




Solution

Period : $T = 1$ sec. ; $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$ r/sec

Expression for $x(t)$ for 1 period.

$$x(t) = t \quad 0 < t < 1.$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{1}\int_0^1 t\, e^{-j2\pi k t}\, dt$$

we know

$$\int t e^{at} dt = \frac{e^{at}}{a^2}(at - 1)$$

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Then

$$a_k = \left\{ \frac{e^{-j2\pi kt}}{(-j2\pi k)^2}\left[-j2\pi kt - 1\right] \right\}_0^1$$

$$= -j\frac{2\pi k}{j^2(2\pi k)^2} = \frac{j}{2\pi k} \quad ; k \neq 0$$

To compute $a_0$ ; put $k=0$ in the original expression of $a_k$

$$a_0 = \frac{1}{T}\int_0^1 t\, e^0\, dt = \frac{1}{1}\int_0^1 t\, dt = \left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{2}.$$

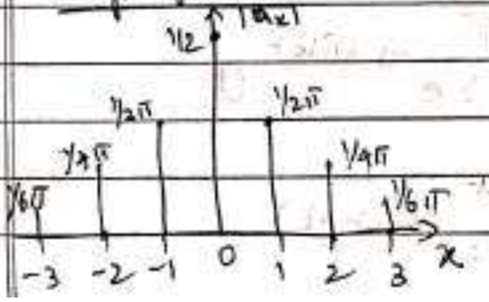
Summarising

$$a_k = \begin{cases} \frac{j}{2\pi k} & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases}$$

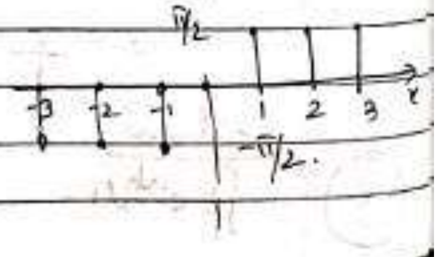
$$|a_k| = \begin{cases} \frac{1}{|2\pi k|} & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases}$$

$$\theta_k = \begin{cases} \pi/2 & k > 0 \\ -\pi/2 & k < 0 \\ 0 & k = 0 \end{cases}$$

Amp Spectrum




Phase Spectrum

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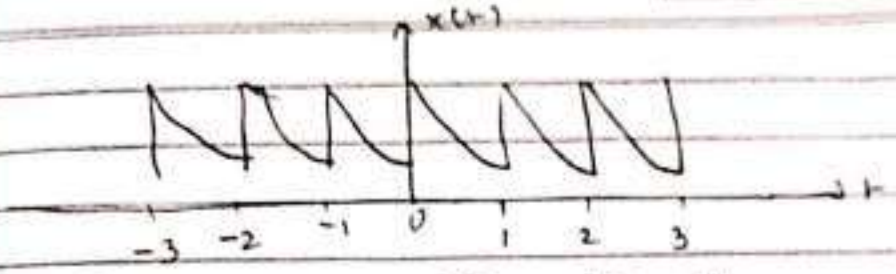
FS 3.7

(a) 259




$T = 1$ sec., $\omega_0 = \frac{2\pi}{T} = 2\pi$ r/sec.

For one period

$x(t) = e^{-t}$ ; $0 < t < 1$.

real Signal.

Blank

## 3.6 Fourier Series Representation of Discrete Time Periodic Signals.

Important Point:- FS representation of D.T periodic signals is a Finite Series. Hence there is no mathematical issues of Convergence.

### 3.6.1 Linear Combination of Harmonically related Complex exponentials.

A discrete time signal $x[n]$ is periodic with a period N if

$$x[n] = x[n+N]$$

-> The fundamental period is the smallest positive ~~Integral~~ Integer N

-> Fundamental frequency $w_0 = \frac{2\pi}{N}$

-> A complex exponential $e^{jw_0 n} = e^{j(\frac{2\pi}{N})n}$ is periodic with a period N.

-> A set of all D.T Complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jkw_0 n} = e^{jk(\frac{2\pi}{N})n} \; ; k = 0, \pm 1, \pm 2, \ldots$$

-> All of these signals have fundamental frequencies that are multiple of $\frac{2\pi}{N}$ & thus harmonically related.

-> Important Point :- There are only N ~~discrete~~ distinct signals in the set given by the above eqn. This is due to the fact that D.T. Complex exponentials

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which differ in frequency by a multiple of $2\pi$ are identical.

Specifically $\phi_0[n] = \phi_N[n]$

$\phi_1[n] = \phi_{N+1}[n]$

In general.

$$\phi_k[n] = \phi_{k+rN}[n]$$

→ when k is changed by any integer multiple of N, identical sequence is generated.

→ This is different from C.T signals, where signals defined by $\phi_k(t)$ are different from one another.

→ we now would like to represent any periodic D.T sequence in terms of linear combination of ~~$\phi_k$~~ $\phi_k[n]$.

→ That is

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n}$$

$$= \sum_k a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$a_k e^{jk\omega_0 n}$ → $k^{th}$ harmonic component of $x[n]$

→ Since the sequences $\phi_k[n]$ are distinct only over a range of N successive values of k, the summation in the equation need only include terms over this range.

→ This range ~~of~~ is expressed as $k = \langle N \rangle$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

→ k takes the value in terms of integers with a range of N

$$k = 0, 1, 2, \ldots (N-1)$$
$$k = 3, 4, \ldots N, (N+1), (N+2)$$

only N values for k, limit of range can be any.

→ The above equation is called D/T Discrete-Time Fourier Series

→ coefficients $a_k$ = F.S. coefficients

## 3.6.2 Determination of the F.S. representation of a Periodic Signal.

**Method 1**

Consider

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- Our aim is to find $a_k$ in terms of $x[n]$.

Then

$$x[0] = \sum_{k=\langle N\rangle} a_k \qquad \text{Here } n=0$$

$$x[1] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}}$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{jk2\pi\frac{(N-1)}{N}}$$

→ The above are set of linearly independent, N simultaneous equations, which can be solved, find the co-efficients over a period N.

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method-2

using this important identity

$$\sum_{n=\langle N\rangle} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & k=0,\pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

k is an integer multiple of N

Hint

Doubt

$$a[k] = \sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)kn} = \begin{cases} N & \text{for } k=0,\pm N, \pm 2N, \dots \\ 0 & \text{when k is not an integer multiple of N.} \end{cases}$$

The above equation states that the sum over one period of the values of a periodic complex exponential is zero, unless the complex exponential is a constant.

→ Now consider the Fourier Series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

→ multiplying both sides with $\sum_{n=\langle N\rangle} e^{-jr\left(\frac{2\pi}{N}\right)n}$ & summing over N terms.

$$\sum_{n=\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k \cdot e^{j(k-r)\frac{2\pi}{N}n}$$

→ Interchanging the summation

$$\sum_{n=\langle N\rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n}$$

→ The inner most sum on 'n' on the R.H.S of the equation is zero, unless $(k-r)$ is zero or integer multiple of N.

→ Therefore, if we choose values of 'r' over the same range as that over which 'k' varies in the outer summation, the inner most sum on the RHS equals to N if $k=r$ or 0 if $k\neq r$.

→ The RHS then reduces to $N \cdot a_r$, we then have

$$a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jr\left(\frac{2\pi}{N}\right)n}$$

Discrete Time Fourier Series Pair:

Synthesis Eqn:
$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)\cdot n}$$

Analysis equation:
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$a_k$ - Spectral coefficients.

→ These coefficients specify a decomposition of $x[n]$ into a sum of N harmonically related complex exponentials.

Note :: Consider

$$x[n] = \sum_{k=\langle N\rangle} a_k\, \phi_k[n]$$

Let k vary from 0 to N-1

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Then

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \cdots + a_{N-1}\phi_{N-1}[n]$$

Similarly if k varies from 1 to N

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \cdots + a_N\phi_N[n]$$

Comparing these two expressions, we see from the relation $\phi_k[n] = \phi_{k+N}[n]$

$\phi_0[n] = \phi_N[n]$ then $a_0 = a_N$

→ Then letting k range over any set of N ~~consecutive~~ consecutive integers, then using the relation

$$\phi_k[n] = \phi_{k+N}[n]$$

$$\boxed{a_k = a_{k+N}} \approx \boxed{a_k = a_{k+N}}$$

→ So even if take arbitrary values of "k", only N successive elements in the sequence will be used in F.S representation.

→ D.T. F.S representation is therefore a finite series with N terms.

Eg 3.10 Consider

$$x[n] = \sin \omega_0 n \quad \text{eqv: } x(t) = \sin \omega_0 t$$

This is periodic only if $\frac{2\pi}{\omega_0}$ is an integer or a ratio of integers (rationals)

→ Assume $\frac{2\pi}{\omega_0} = N$, an integer. Then $\omega_0 = \frac{2\pi}{N}$

→ Then $x[n]$ is periodic with a period $N$.

→ Using Euler's identity

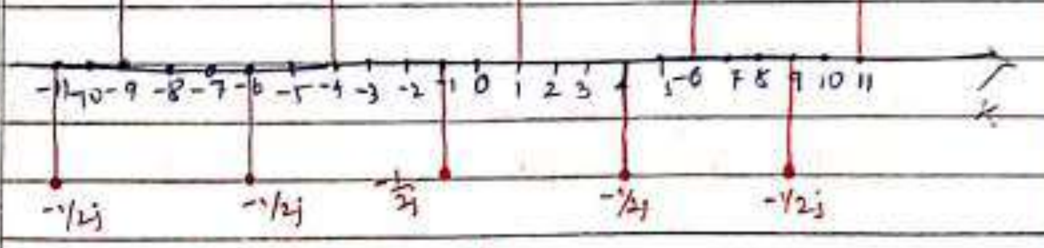
$$x[n] = \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$$

→ This gives $a_k$, $k = 1$ & $-1$. Therefore

$$a_1 = +\frac{1}{2j} \; ; \; a_{-1} = -\frac{1}{2j}$$

→ Remaining coefficients are zero. Since the coefficients repeat with period $N$, then $a_{N+1}$ also $\frac{1}{2j}$ & $a_{N-1} = -\frac{1}{2j}$.

→ For $N=5$ (say), we can observe the repeated nature of the F.S. coefficients. $x[n] = \sin\left(\frac{2\pi}{5}\right)n$




→ We consider only one period

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$$\omega_0 = 2\pi \frac{m}{N}$$

Then $\frac{2\pi}{\omega_0} = \frac{N}{m}$

or $\frac{\omega_0}{2\pi} = \frac{m}{N}$

**Case-ii**

Let $\frac{2\pi}{\omega_0} = \frac{N}{m}$ ; then $\omega_0 = \frac{2\pi m}{N}$

where $\frac{m}{N}$ is a rational number (integers)

Fundamental period = $N$

Let $x[n] = \sin \omega_0 n$ $\quad \omega_0 = \frac{2\pi m}{N}$

then

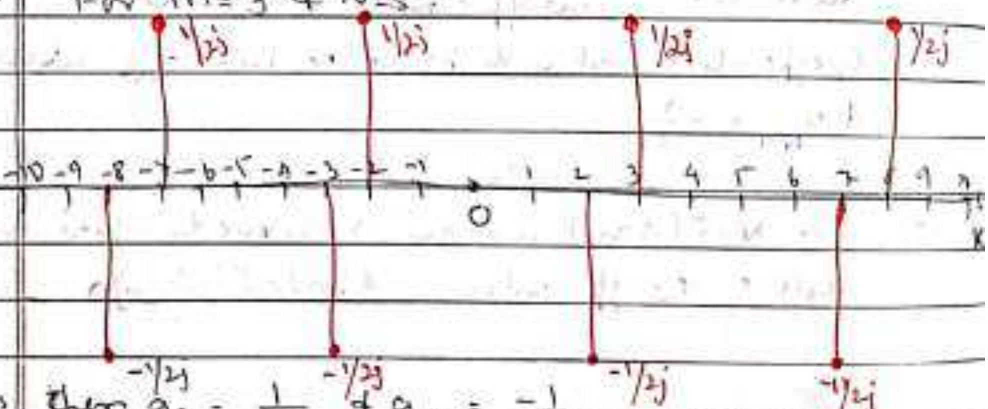
$$x[n] = \frac{1}{2j} e^{jm\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-jm\left(\frac{2\pi}{N}\right)n}$$

then by inspection

$$a_m = \frac{1}{2j} \quad \& \quad a_{-m} = -\frac{1}{2j}$$

remaining co-efficients are zero over the period of length $N$.

→ For $m = 3$ & $N = 5$.




→ then $a_3 = \frac{1}{2j}$ & $a_{-3} = -\frac{1}{2j}$ - plotted

$a_{-3} = a_2$

→ Over any period of $N$, there are only 2 non-zero FS co-efficients.

Eg 3.11 Consider

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left[\frac{4\pi}{N}n + \frac{\pi}{2}\right]$$

The given signal is periodic with period N
Expanding x[n] in terms of complex exponentials

$$x[n] = 1 + \frac{1}{2j}\left\{e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right\}$$

$$+ \frac{3}{2}\left\{e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right\}$$

$$+ \frac{1}{2}\left\{e^{j\left(\frac{4\pi}{N}n + \pi/2\right)} + e^{-j\left(\frac{4\pi n}{N} + \frac{\pi}{2}\right)}\right\}$$

Collecting similar terms

$$x[n] = 1 + \left[\frac{3}{2} + \frac{1}{2j}\right]\left\{e^{j\frac{2\pi}{N}n}\right\} + \left[\frac{3}{2} - \frac{1}{2j}\right]e^{-j\frac{2\pi}{N}n}$$

$$+ \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2\left(\frac{2\pi}{N}\right)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2\left(\frac{2\pi}{N}\right)n}$$

Fourier coefficients are

$a_0 = 1$

$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{j}{2} = \frac{3}{2} - \frac{1}{2}j$$

$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j$$

$$a_2 = \frac{1}{2}e^{j\pi/2} = \frac{1}{2}\left[\cos\frac{\pi}{2} + j\sin\pi/2\right] = \frac{1}{2}j$$

$$a_{-2} = \frac{1}{2}e^{-j\pi/2} = -\frac{1}{2}j$$

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→ $a_k = 0$ for all other values of k in the interval of summation over a period.

→ For any period N, we can generalise the Fourier coefficients to be

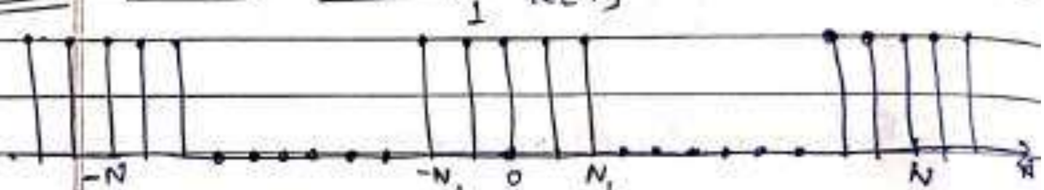
Doubt

$$\begin{cases} a_N = 1 \\ a_{(N-1)} = \frac{3}{2} + \frac{1}{2}j \\ a_{(2-N)} = \frac{1}{2}j \end{cases}$$

→ Since $x[n]$ is real, $\boxed{a_{-k} = a_k^*}$

→ There are two alternate forms of F.S for DT periodic signal; however the exponential form is most common.

Eg 3.12 Consider the signal.




Doubt

$x[n] = 1$ for $-N_1 \le n \le N_1$

Choose an interval N to include $-N_1$ & $N_1$

Then

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} (1) e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Letting $m = n + N_1$, for $n = -N_1$, $m = 0$
$n = +N_1$, $m = 2N_1$

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)}$$

$$= \frac{1}{N} \sum e^{jk\left(\frac{2\pi}{N}\right)N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

The summation in the above equation consists of sum of the first $\underline{2N_1+1}$ terms in a geometric series.

$$a_k = \frac{1}{N} e^{jk\frac{2\pi}{N}N_1} \left\{ \frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)/N}}{1 - e^{-jk\frac{2\pi}{N}}} \right\}$$

$$= \frac{1}{N} \frac{e^{-jk\frac{2\pi}{2N}} \left\{ e^{jk2\pi(N_1+\frac{1}{2})/N} - e^{-jk2\pi(N_1+\frac{1}{2})/N} \right\}}{e^{-jk\frac{2\pi}{2N}} \left\{ e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)} \right\}}$$

$$= \frac{1}{N} \frac{\sin(2\pi k(N_1+\frac{1}{2})/N)}{\sin(\pi k/N)} \; ; \; k \neq 0, \pm N, \pm 2N, \dots$$

Then

$$\boxed{a_k = \frac{2N_1+1}{N}} \; ; \; k = 0, \pm N, \pm 2N, \dots$$

Aliter Try $N = 11$

Trigonometric Form of D.T.F.S. Check Prob. 3.52.

Let $x[n]$ be periodic with period $N$. Let

$a_k = b_k + j c_k$ ; $b_k$ ; $c_k$ - real.

If $x[n]$ is odd

$$x[n] = a_0 + 2 \sum_{k=1}^{\frac{(N-1)}{2}} b_k \cos\left\{\frac{2\pi kn}{N}\right\} - c_k \sin\left\{\frac{2\pi kn}{N}\right\}$$

If $x[n]$ is even

$$x[n] = a_0 + a_{N/2}(-1)^n + 2 \sum_{k=1}^{\frac{(N-2)}{2}} b_k \cos\left\{\frac{2\pi kn}{N}\right\} - c_k \sin\left\{\frac{2\pi kn}{N}\right\}$$

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Eg 3.80 (An) Pg 417

Determine F.S for

$x[n] = \cos\left(\frac{\pi}{4} n\right) \Rightarrow \omega_0 = \frac{\pi}{4}$

$\therefore N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/4} = \underline{8}$ Check for periodicity

Let

$x[n] = \frac{1}{2}\left[e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right]$

$= \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$ ~ $k = 1$.

Then

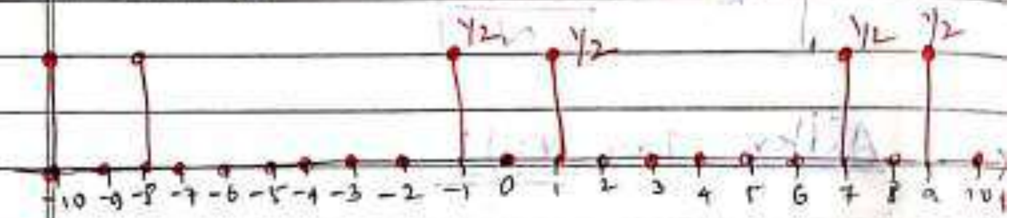
$a_1 = \frac{1}{2}$ & $a_{-1} = \frac{1}{2}$ ; rest all are zero

i.e $a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 0 & k = -3, -2, 0, 2, 3, 4 \end{cases}$

we can observe

~~a(k+N) = a(k) for all k~~

$a_{k+N} = a_k$ ~ $a_{k+8} = a_k$ for all k




Eg 3.82 (An) Pg 421

Given

$x[n] = \sin\left(\frac{4\pi}{21}\right)n + \cos\frac{10\pi}{21}n + 1$

$\omega_1 = \frac{4\pi}{21}$ & $\omega_2 = \frac{10\pi}{21}$

$\left[\frac{2\pi}{\omega_1} = \frac{2\pi}{4\pi/21} = \frac{21}{2} = \frac{N_1}{m_1}\right.$

$\left.\frac{2\pi}{\omega_2} = \frac{2\pi}{10\pi/21} = \frac{21}{5} = \frac{N_2}{m_2}\right]$

$\omega_0 = GCD\left\{\frac{4\pi}{21}, \frac{10\pi}{21}\right\} = \frac{2\pi}{21}$

$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi/21} = 21$

Here $N_1 = N_2 = 21$

$\therefore N = LCM\{21, 21\} = 21$

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Using Euler's Identity

$$x[n] = \frac{1}{2j} e^{j\frac{4\pi}{21}n} - \frac{1}{2j} e^{-j\frac{4\pi}{21}n} + \frac{1}{2} e^{j\frac{10\pi}{21}n} + \frac{1}{2} e^{-j\frac{10\pi}{21}n} + 1$$

Since $N = 21$, we choose

$\cancel{k \in \{-20, -19, \dots 0, 1, \dots +20}$

$$k \in \{-10, -9, -8, \dots 0, 1, 2, 3, \dots 10\}$$

$$\longleftarrow 21 \longrightarrow$$

Comparing the above expression of $x[n]$ with

$$x[n] = \sum_{k=-10}^{+10} a_k e^{jk\omega_0 n}$$

$$= \frac{1}{2j} e^{j2\left(\frac{2\pi}{21}\right)n} - \frac{1}{2j} e^{-j2\left(\frac{2\pi}{21}\right)n} + \frac{1}{2} e^{j5\left(\frac{2\pi}{21}\right)n} + \frac{1}{2} e^{-j5\left(\frac{2\pi}{21}\right)n} + 1$$

$a_0 = 1$

$a_2 = \frac{1}{2j}$ ; $a_{-2} = \frac{-1}{2j}$ ; rest all $a_k$ are zero.

$a_5 = \frac{1}{2}$ ; $a_{-5} = \frac{1}{2}$.

Magnitude Spectrum

$$|a_k| = \begin{cases} 1 & k = 0 \\ 1/2 & k = -5, -2, 2, 5 \\ 0 & \text{for all } k \in (-10, 10) \end{cases}$$

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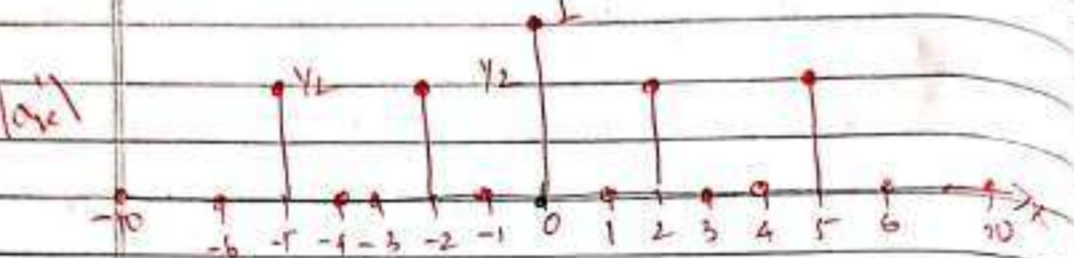
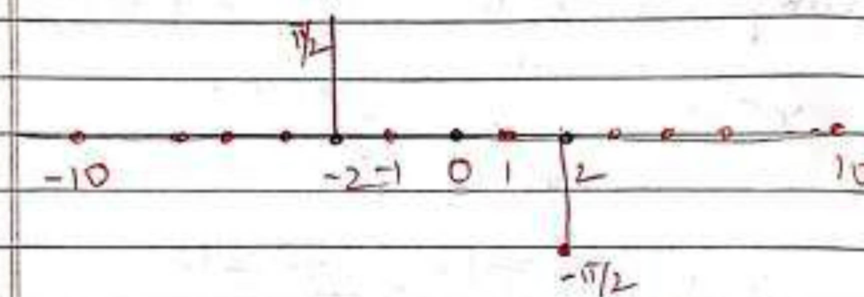
## Phase Spectrum

$$\theta_k = \begin{cases} 0 & k = 0 \\ 0 & k = \pm 5 \\ \pi/2 & k = -2 \\ -\pi/2 & k = +2 \\ 0 & \text{for all other } k \ (-10, 10) \end{cases}$$

$|a_k|$

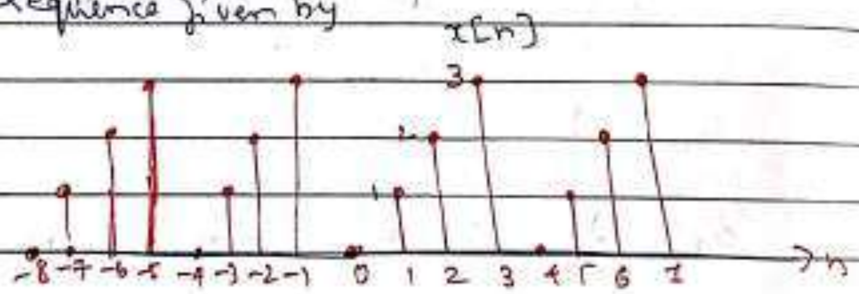


$\theta_k$:



Eg 3.82 (a) Determine Fourier coefficients of a periodic sequence given by

Imp.



(b) Verify Parseval's Identity.

(a) From the figure, we see that $x[n]$ is periodic extension of $(0,1,2,3)$ with fundamental period of $N$. Then $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4}$

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \sum_{k=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

Then

$$a_k = \frac{1}{4}\sum_{k=0}^{3} x[n]\, e^{-jk\omega_0 n} \quad ; k = 0,1,2,3$$

$$= \frac{1}{4}\sum_{k=0}^{3} x[n]\, e^{-jk\left(\frac{2\pi}{4}\right)n}$$

(i) Since k varies all for n = 0, 1, 2, 3. we write

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk\left(\frac{2\pi}{4}\right)n}$$

$$= \frac{1}{4}\left\{ x[0] + x[1]\, e^{-jk\frac{2\pi}{4}} + x[2]\, e^{-j\frac{4\pi}{4}k} + x[3]\, e^{-j\frac{6\pi}{4}k} \right\}$$

$$a_0 = \frac{1}{4}\left[0 + (1)(1) + (2)(1) + (3)(1)\right]$$

$$= \frac{1}{4}\left[0+1+2+3\right] = \frac{6}{4} = \frac{3}{2}$$

$$a_1 = \frac{1}{4}\left[x[0] + (1)\, e^{-j\underbrace{\frac{2\pi}{4}}_{\pi/2}} + (2)\, e^{-j\underbrace{\frac{4\pi}{4}}_{\pi}} + (3)\, e^{-j\underbrace{\frac{6\pi}{4}}_{\frac{3}{2}\pi}}\right]$$

$$\cancel{= \frac{1}{4}[0 +}$$

$$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j \qquad \frac{3\times 180^{90}}{2} = 270^\circ$$

$$e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$e^{-j\frac{3}{2}\pi} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = -j(-1) = +j$$

$$a_1 = \frac{1}{4}\left[0 + (-j) + 2(-1) + 3j\right] = \frac{1}{4}\left[-2+2j\right]$$

$$= -\frac{1}{2} + j\frac{1}{2}$$

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$$a_2 = \frac{1}{4}\left\{x[0] + (1)e^{-j\frac{4\pi}{4}} + (2)e^{-j\frac{8\pi}{4}} + (3)\, e^{-j\frac{12\pi}{4}}\right]$$

$$= \frac{1}{4}\left\{0 + e^{-j\pi} + (2)\, e^{-j2\pi} + (3)\, e^{-j3\pi}\right\}$$

$$= \frac{1}{4}\left\{0 - 1 + 2(1) + (3)(-1)\right\}$$

$$= \frac{1}{4}\left\{0 - 1 + 2 - 3\right\} = \frac{-2}{4} = -\frac{1}{2}$$

$a_3 = -\frac{1}{2} - j\frac{1}{2}$ on similar lines.

$$a_0 = \frac{3}{2},\ a_1 = -\frac{1}{2} + j\frac{1}{2};\ a_2 = -\frac{1}{2};\ a_3 = -\frac{1}{2} - j\frac{1}{2}$$

$$\boxed{a_{-1} = a_3 = a_1^*}$$

(b) Parseval's Identity:

$$P_1 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$= \frac{1}{4}\left[0^2 + 1^2 + 2^2 + 3^2\right] = \frac{1}{4}\left[0 + 1 + 4 + 9\right]$$

$$= \frac{14}{4} = \frac{7}{2}$$

$$P_2 = \sum_{k=0}^{N-1} |a_k|^2 = \left(\frac{3}{2}\right)^2 + \left\{\frac{1}{2}^2 + \frac{1}{2}^2\right\} + \left(\frac{1}{2}\right)^2 + \left\{\frac{1}{2}^2 + \frac{1}{2}^2\right\}$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{9+1+1+1+1+1}{4} = \frac{14}{2} = \frac{7}{2}$$

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Since $P_1 = P_2$ Parseval's Identity is proved.

x ——— x ——— x

Eg 3.8.3 (28)

Determine the F.S. representation of

$$x[n] = \cos^2\left(\frac{\pi}{8}n\right)$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta)$$
$$= \cos^2\theta - 1 + \cos^2\theta$$
$$= 2\cos^2\theta - 1$$

$$\therefore \cos^2\theta = \frac{1}{2}\{1+\cos 2\theta\} \qquad \theta = \frac{\pi}{8}n$$

Then

$$x[n] = \frac{1}{2} + \frac{1}{2}\cos\left(2\cdot\frac{\pi}{8}n\right)$$

$$\omega_0 = \frac{2\pi}{8} \Rightarrow N = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi/8} = \underline{\underline{8}}$$

Then

$$x[n] = \frac{1}{2} + \frac{1}{2}\cos\frac{2\pi}{8}n$$

$$= \frac{1}{2} + \frac{1}{2}\left\{\frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{-j\frac{2\pi}{8}n}\right\}$$

$$= \frac{1}{2} + \frac{1}{4}e^{j\frac{2\pi}{8}n} + \frac{1}{4}e^{-j\frac{2\pi}{8}n}$$

$$= \frac{1}{2} + \frac{1}{4}e^{j\omega_0 n} + \frac{1}{4}e^{-j\omega_0 n}$$

Let $k = -3, -2, -1, 0, 1, 2, 3, 4$
$\underbrace{\qquad\qquad\qquad\qquad}_{N=8}$

Then

$$x[n] = \sum_{k=-3}^{4} a_k e^{-jk\omega_0 n}$$

$a_0 = \frac{1}{2}$; $a_1 = \frac{1}{4}$; $a_{-1} = \frac{1}{4}$; all other coefficients are zero

Note

$a_{-1} = a_{(-1+8)} = a_7 = \frac{1}{4}$ ; coefficients repeat after $N$.
$\phantom{a_{-1} = }\ 1+N$

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⑩



Hence DTFS of $x[n]$ can also be expressed

$$x[n] = \sum_{k=0}^{7} a_k e^{jk\omega_0 n}$$

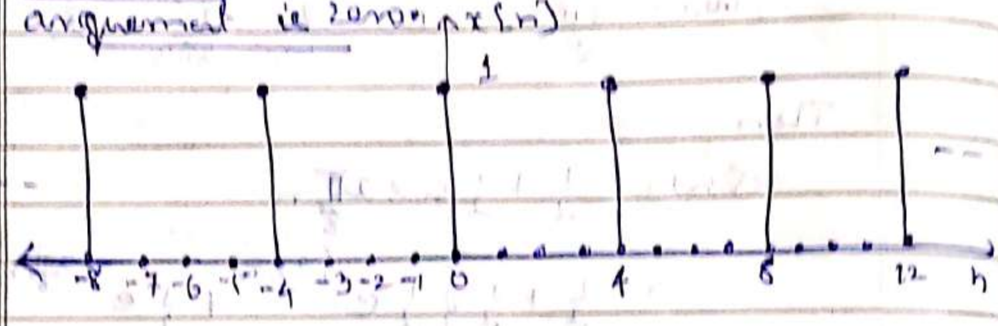
$$= \frac{1}{2} + \frac{1}{4} e^{j\omega_0 n} + \frac{1}{4} e^{j7\omega_0 n}$$

$\boxed{a_k = a_{k+N}}$ $\boxed{a_{-2} = a_{-2+8} = a_6}$

Eg 3.8)
(3.11) (1.23) Consider the sequence

$$x[n] = \sum_{m=-\infty}^{\infty} \delta(n-4m)$$

(a) Sketch $x[n]$ — $x[n]$ exists only when the argument is zero.



(b) Find $a_k$.

Fundamental period $= 4$, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4}$

let $k \in (0,1,2,3)$. Also let $x[n]$ be considered for $n = 0,1,2,3$.

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\left(\frac{2\pi}{4}\right)n}$$

$x[0] = 1, \quad x[1] = x[2] = x[3] = 0$

$$a_k = \frac{1}{4}\left\{ \underset{1}{x[0]} + \underset{0}{x[1]}\, e^{-jk\frac{2\pi}{4}} + \underset{0}{x[2]}\, e^{-jk\frac{4\pi}{4}} + \underset{0}{x[3]}\, e^{-jk\frac{6\pi}{4}} \right\}$$

$$\left.\begin{aligned} a_0 &= \frac{1}{4}\{1\} = \frac{1}{4} \\ a_1 &= \frac{1}{4}\{1\} = \frac{1}{4} \\ a_2 &= \frac{1}{4}\{1\} = \frac{1}{4} \\ a_3 &= \frac{1}{4}\{1\} = \frac{1}{4} \end{aligned}\right\}$$

Also $a_k = a_{k+4}$

Eg. 3.85 (GR) (A30) Evaluate $x(n)$ for the given $a_k$ which is

$$a_k = \cos\left[\frac{6\pi}{17}k\right]$$

From the expression of $a_k$, we can make out

$N = 17$

$$\boxed{\omega_0 = \frac{2\pi}{17}}$$

(*) choose $n, k \in [-8, -7, \cdots 0, 1, 2, \cdots 8]$ — 17

Synthesis eqn:

$$x[n] = \sum_{k=-8}^{8} \cos\left(\frac{6\pi}{17}k\right) e^{jk\omega_0 n}$$

$$= \frac{1}{2}\sum_{k=-8}^{8}\left\{ e^{j\frac{6\pi}{17}k} + e^{-j\frac{6\pi}{17}k}\right\} \cdot e^{jk\omega_0 n}$$

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$$= \frac{1}{2} \sum_{k=-8}^{8} \left\{ e^{j\frac{6\pi}{17}k} + e^{-j\frac{6\pi}{17}k} \right\} e^{jk\frac{2\pi}{17}n}$$

$$= \frac{1}{2} \sum_{k=-8}^{8} \left\{ e^{j\frac{2\pi}{17}k(n+3)} + e^{+j\frac{2\pi}{17}k(n-3)} \right\}$$

Then

$$e^{j\frac{2\pi}{17}k(n+3)} = \begin{cases} 0 & n \neq -3 \\ 17 & n = -3 \\ (N) & \end{cases}$$

$lll^{ly}$

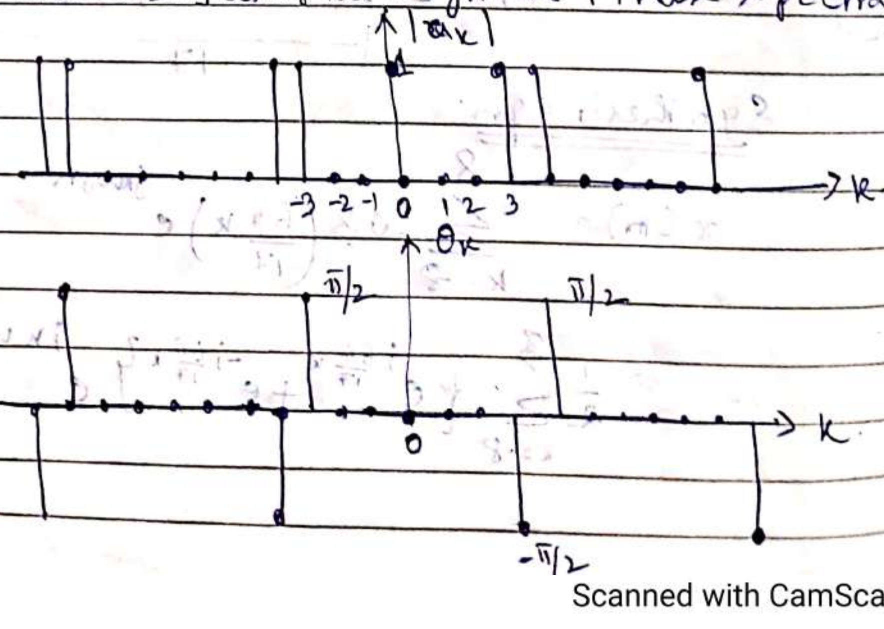
$$e^{j\frac{2\pi}{17}k(n-3)} = \begin{cases} 0 & n \neq 3 \\ 17 & n = 3 \end{cases}$$

Then

$$x[n] = \begin{cases} 0 & n \neq \pm 3 \\ \frac{17}{2} & n = \pm 3 \end{cases}$$

$$n \in \{-8, -7, \ldots 0, 1, 2, \ldots 8\}.$$

Eg 3-86 (431) Find $x[n]$ using the magnitude + Phase Spectra.



from the figure $N = 7 \Rightarrow w_0 = \frac{2\pi}{7}$

$n, k \in \{-3, -2, -1, 0, 1, 2, 3\}$

In the above range.

$$a_k = (1e^{j\pi/2}, 0, 0, 0, 0, 0, 1e^{-j\pi/2})$$

$$= (\underset{k=-3}{j}, 0, 0, 0, 0, 0, \underset{k=3}{-j})$$

Then

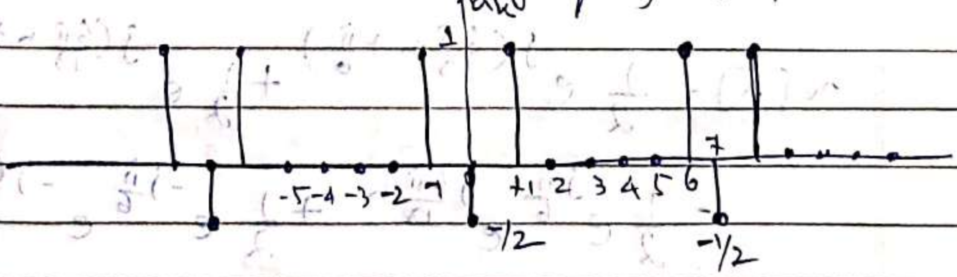
$$x[n] = \sum_{k=-3}^{+3} a_k e^{jkw_0 n}$$

$$= a_{-3} e^{-j\frac{6\pi}{7}n} + a_3 e^{j\frac{6\pi}{7}n}$$

$$= j e^{-j\frac{6\pi}{7}n} - j e^{j\frac{6\pi}{7}n}$$

$$= \underline{\underline{2\sin\left(\frac{6\pi n}{7}\right)}} \; ; \; \underline{\underline{n \in (-3, \dots 0, \dots +3)}}$$

Eg 3.87 (32) Find $x[n]$ using the frequency spectrum shown.




$N = 7$, $w_0 = \frac{2\pi}{7}$ ; $n, k \in \{-3, -2, -1, 0, 1, 2, 3\}$

$$a_k = \{\underset{k=-3}{0}, 0, 1, \underset{k=0}{-\tfrac{1}{2}}, 1, 0, \underset{k=+3}{0}\}$$

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$$x[n] = \sum_{k=-3}^{+3} a_k e^{jk\omega_0 n}$$

$$= a_{-1} e^{-j\omega_0 n} + a_0 + a_1 e^{j\omega_0 n}$$

$$= (1) e^{-j\omega_0 n} + \left(-\frac{1}{2}\right) + (1) e^{j\omega_0 n}$$

$$= 2\cos(\omega_0 n) - \frac{1}{2}$$

$$= 2\cos\left(\frac{2\pi n}{7}\right) - \frac{1}{2}$$

$$n = \{-3, -2, -1, 0, 1, 2, 3\}.$$

Eg 3.88 GR 433 — Evalue $a_k$ + plot the spectra.

$$x[n] = \cos\left(\frac{6\pi}{13} n + \frac{\pi}{6}\right)$$

$$N = 13 \;\&\; \frac{2\pi}{13} = \omega_0$$

Choosing the range

$$n, k \in \{-6, \ldots 0, \ldots +6\}$$

Expanding $x[n]$

$$x[n] = \frac{1}{2} e^{j\left(\frac{6\pi}{13} n + \frac{\pi}{6}\right)} + \frac{1}{2} e^{-j\left(\frac{6\pi}{13} n + \frac{\pi}{6}\right)}$$

$$= \frac{1}{2} e^{j\frac{\pi}{6}} e^{j\frac{6\pi}{13} n} + \frac{1}{2} e^{-j\frac{\pi}{6}} e^{-j\frac{6\pi}{13} n}$$

$$= \frac{1}{2} e^{j\frac{\pi}{6}} e^{j3\omega_0 n} + \frac{1}{2} e^{-j\frac{\pi}{6}} e^{-j3\omega_0 n}$$

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Comparing this with

$$x[n] = \sum_{k=-6}^{6} a_k e^{jk\omega_0 n}$$

$$\underline{a_{-3} = \frac{1}{2} e^{-j\frac{\pi}{6}}} \quad \& \quad \underline{a_{+3} = \frac{1}{2} e^{j\frac{\pi}{6}}}$$

**Magnitude Spectrum**

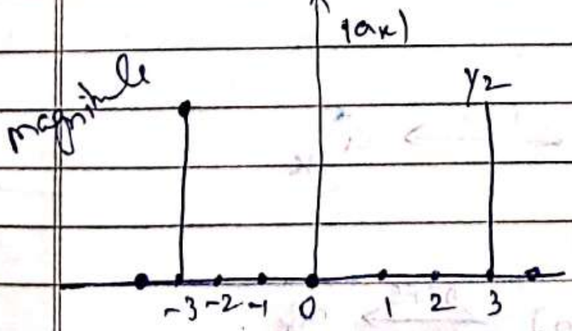
$$|a_k| = \begin{cases} \frac{1}{2} & k = \pm 3 \\ 0 & \text{otherwise between } -6 \,\&\, +6 \end{cases}$$

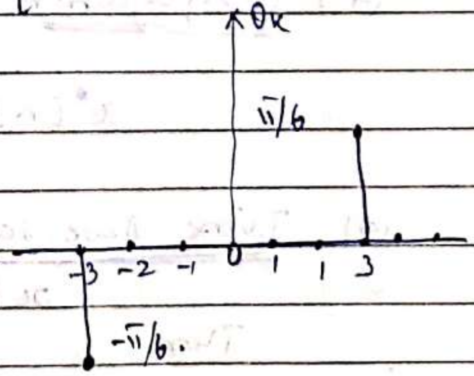
**Phase Spectrum**

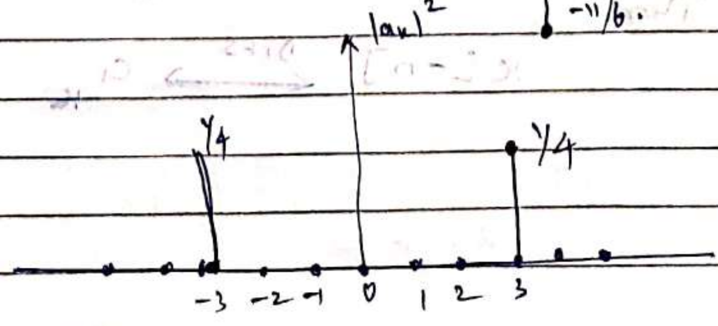
$$\theta_k = \begin{cases} \frac{-\pi}{6} & k = -3 \\ \pi/6 & k = +3 \\ 0 & \text{otherwise between } -6 \,\&\, +6 \end{cases}$$

**Power Spectrum**

$$P_k = |a_k|^2 = \begin{cases} \frac{1}{4} & k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$



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## 3.7 Properties of Discrete Time Fourier Series (DTFS)

Consider $x[n]$ & $y[n]$ with period $N$ & fundamental frequency $\omega_0 = \frac{2\pi}{N}$.

Let

$$x[n] \xleftrightarrow{DTFS} a_k \quad y[n] \xleftrightarrow{DTFS} b_k$$

$a_k$ & $b_k$ are also periodic with period $N$.

1. Linearity ✓

$$A x[n] + B y[n] \xleftrightarrow{DTFS} A a_k + B b_k$$

2. Time Shifting ✓

$$x[n-n_0] \xleftrightarrow{DTFS} a_k \cdot e^{-jk\left(\frac{2\pi}{N}\right)n_0}$$

(3) Frequency Shifting

$$e^{jm\left(\frac{2\pi}{N}\right)n} x[n] \xleftrightarrow{DTFS} a_{k-m}$$

(4) Conjugation ✓

$$x^*[n] \xleftrightarrow{DTFS} a^*_{-k}$$

(5) Time Reversal ✓

$$x[n] \xleftrightarrow{DTFS} a_k$$

Then

$$x[-n] \xleftrightarrow{DTFS} a_{-k}$$

(6) Time Scaling.

$$x_{\langle m \rangle}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$$

periodic with period $mN$

$$x_{\langle m \rangle}[n] \xleftrightarrow{DTFS} \frac{1}{m} a_k.$$

(7) Periodic Convolution.

$$x[n] \circledast y[n] = \sum_{r=\langle N \rangle} x[r]\, y[n-r]$$

$$\xleftrightarrow{DTFS} N a_k b_k.$$

when the summation $(-\infty, +\infty)$ → Aperiodic Convolution

(8) multiplication / modulation

$$x[n]\, y[n] \xleftrightarrow{DTFS} \sum_{l=\langle N \rangle} a_l b_{k-l} = a_k \circledast b_k$$

(9) First Difference - used in conjunction with time shift

$x[n] - x[n-1]$ - First difference.

$$x[n] - x[n-1] \xleftrightarrow{DTFS} a_k - a_k e^{-jk\left(\frac{2\pi}{N}\right)(1)} \quad n_0 = 1$$

$$\xleftrightarrow{DTFS} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k.$$

(10) Running Sum.

$$\sum_{k=-\infty}^{n} x[k]$$ - finite valued & periodic only if $a_0 = 0$

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$$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{DTFS} \left\{ \frac{1}{(1-e^{-jk2\pi/N})} \right\} a_k.$$

(11) <u>Conjugate Symmetry for Real Signals.</u>

$x[n]$ is real then

$$a_k = a^*_{-k}$$

$$\text{Real}(a_k) = \text{Real}(a_{-k})$$

$$\text{Img}(a_k) = -\text{Img}(a_{-k})$$

$$|a_k| = |a_{-k}|$$

$$\theta_k = -\theta_{-k}$$

(12) <u>Real & Even Signals.</u>

$x[n]$ is real & even then

$a_k$ is real & even.

(13) <u>Real & odd Signals.</u>

$x[n]$ is real & odd then

$a_k$ is purely Imaginary & odd.

(14) <u>Even & odd decomposition of real Signals.</u>

$x_e[n] = \text{Even } x[n]$

then $\text{Real}\{a_k\}$

$x_o[n] = \text{odd } x[n]$

then $j\,\text{Imaginary}\{a_k\}$

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(15) Duality

Let $x[n] \xleftrightarrow{DTFS} a_k$

then

$$\frac{1}{N} x[-k] \longleftrightarrow a_n$$

Proof

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

Then by replacing n with −n

$$x[-n] = \sum_{k=\langle N \rangle} a_k e^{-jk\omega_0 n}$$

Replacing n by k & k by n, we get

$$x[-k] = \sum_{n=\langle N \rangle} a_n e^{-jk\omega_0 n}$$

multiplying both sides by $\frac{1}{N}$, we get

$$\frac{1}{N} x[-k] = \frac{1}{N} \sum_{n=\langle N \rangle} a_n e^{-jk\omega_0 n}$$

$$= a_n$$

(16) Parseval's Identity or Theorem or Relation.

Let $x[n] \xleftrightarrow{DTFS} a_k$

then

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Where $\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$ is the Average Power (P) of the periodic sequence $x[n]$ in one period.

The average power in a periodic signal equals the sum of average of powers in all its harmonic components. In Discrete Time, there are only

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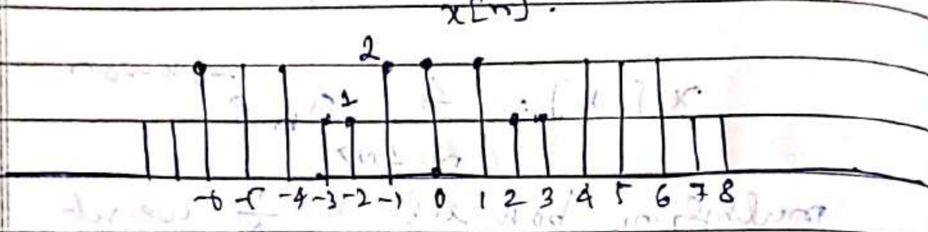
N discrete harmonic components & also $a_k$ are periodic with period N, the sum in the RHS can be taken over any ~~consecutive~~ N consecutive values of k.

→ $|a_k|^2$ is the distribution of Power of frequency k (in terms of k) & is defined as the Power density spectrum of $x[n]$.

Eg 3.13 Consider the sequence $x[n]$. Find $a_k$.

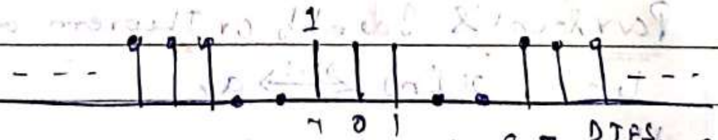



Period = N = 5

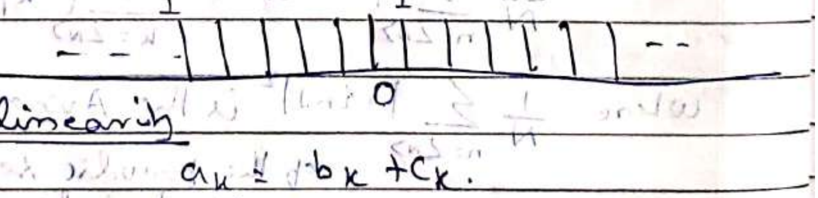
Solution

The given signal can be considered as the sum of the following two sequences.

$x_1[n] \xleftrightarrow{DTFS} b_k$




$x_2[n] \xleftrightarrow{DTFS} c_k$.




By linearity

$a_k = b_k + c_k$.

From Eg 3.12, with $N_1 = 1$ & $N = 5$, Fourier series coefficients $b_k$ can be written as

$$b_k = \begin{cases} \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)} & \text{for } k \neq 0, \pm 5, \pm 10 \\ \frac{3}{5} & \text{for } k = 0, \pm 5, \pm 10 \end{cases}$$

The sequence $x_2[n]$ has only DC value which is captured by $c_0$

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = \frac{1}{5}[1+1+1+1+1] = \underline{\underline{1}}$$

Remaining coefficients are zero.

Then we can write.

$$a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(k3\pi/5)}{\sin(k\pi/5)} ; & k \neq 0, \pm 5, \pm 10 \ldots \\ b_0 + c_0 = \frac{3}{5}+1 = \frac{8}{5} & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

ES3.24. Following information about $x[n]$ are given

(a) $x[n]$ is periodic with $N = 6$

(b) $\sum_{n=0}^{5} x[n] = 2$.

(c) $\sum_{n=2}^{7} (-1)^n x[n] = 1$.

(d) $x[n]$ has the minimum power / period among the set of signals satisfying the preceding 3 conditions.

Find $x[n]$

Solution

Let the Fourier Series coefficients be denoted as $a_k$.

→ From (b) we ~~can~~ conclude $a_0 = 1/3$

since $a_0 = \frac{1}{N}\sum_{n=0}^{5} x[n] = \frac{1}{6} \times 2 = \underline{\underline{1/3}}$

→ $(-1)^n$ can be written $e^{-j\pi n} = e^{-j\left(\frac{2\pi}{6}\right)3n}$

SURYA Gold
Date ______ Page ______

where k = 3 ; then

$$a_3 = \frac{1}{N}\sum_{n=2}^{7} x[n]\cdot e^{-j3\left(\frac{2\pi}{6}\right)n}$$

Any interval of period 6

$$= \frac{1}{6}\times 1 = \frac{1}{6}$$

→ From Parseval's relation, average power in $x[n]$ is

$$P = \sum_{k=0} |a_k|^2$$

Since each nonzero co-efficient contributes a positive amount to P, & since the values of $a_0$ & $a_3$ are prespecified, the value of P is minimised by choosing

$a_1 = a_2 = a_4 = a_5 = 0$

Then $x[n] = \sum_{k=0}^{5} a_k e^{jkw_0 n}$

$$= a_0 + a_3 e^{j3\left(\frac{2\pi}{6}\right)n}$$

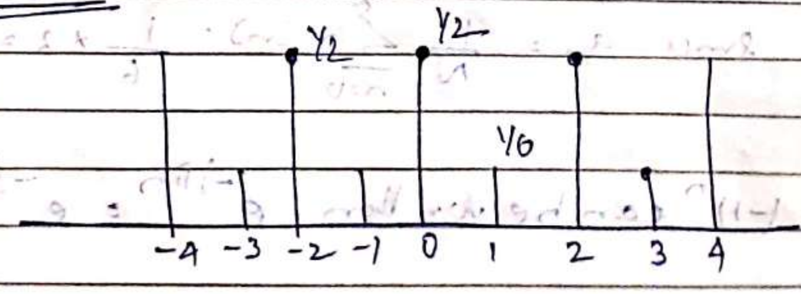
$$= a_0 + a_3 e^{j\pi n}$$

$$= \frac{1}{3} + \frac{1}{6}(-1)^n$$ Doubt

$(-1)^n = e^{j\pi n}$

also $= e^{-j\pi n}$

Sketch

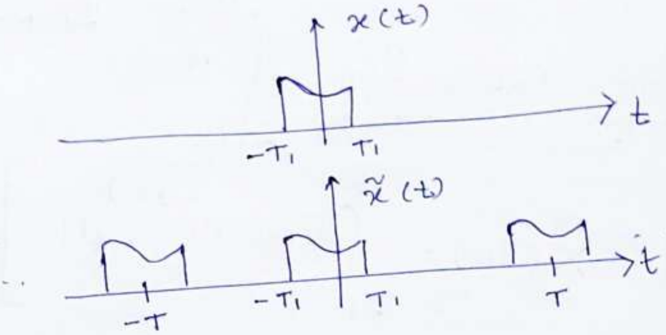


[Ref. $T_1$ - Sec. 4.1 to 4.5
$T_1$ - Sec. 5.1 to 5.3
And sampling theorem of chapter 7]

1

# UNIT - 4

## CONTINUOUS TIME FOURIER TRANSFORM AND DISCRETE TIME FOURIER TRANSFORM




Consider aperiodic signal $x(t)$.

From this aperiodic signal $x(t)$, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period.

We can express $\tilde{x}(t)$ in terms of Fourier series over the interval $-\frac{T}{2} \le t \le T/2$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow (1) \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t} . dt$$

$$T a_k = \int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t} . dt \rightarrow (2)$$

~~As $T \to \infty$ $T a_k$ = $X(jk\omega_0)$ $X(jk\omega_0)$~~

$$\cancel{T a_k = X(jk\omega_0)} = \int_{-T/2}^{T/2} \cancel{\tilde{x}(t)}$$

(2)

Consider (2) $Ta_k = \int_{-\frac{T}{2}}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}$

$T \to \infty$

$\left\{ \begin{array}{l} \omega_0 = \frac{2\pi}{T} \\ k\omega_0 = \omega \end{array} \right.$ $\omega_0 \to 0$

As $T \to \infty$ $\tilde{x}(t) = x(t)$

$k\omega_0 = \omega$ = It becomes a continuous variable

$$\lim_{T \to \infty} Ta_k = \lim_{T \to \infty} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \cdot dt} \rightarrow (3)$$

Equation (3) is known as Fourier transform of $x(t)$. It is also known Fourier Integral of $x(t)$. It is also known as analysis equation.

Consider (2) $Ta_k = \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \rightarrow (4)$

$$Ta_k = X(jk\omega_0) = \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \cdot dt$$

$$a_k = \frac{X(jk\omega_0)}{T} \rightarrow (5)$$

Consider (1) $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Using (5) in (1)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{X(jk\omega_0)}{T} e^{jk\omega_0 t}$$

But $\omega_0 = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0}$

$$\frac{1}{T} = \frac{\omega_0}{2\pi}$$

(3)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{X(jk\omega_0)\, e^{jk\omega_0 t}}{\left(\frac{2\pi}{\omega_0}\right)}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \cdot \omega_0$$

As $T \to \infty$ $\quad k\omega_0 = \frac{k\,2\pi}{T} = \omega \Rightarrow \omega_0 = d\omega$

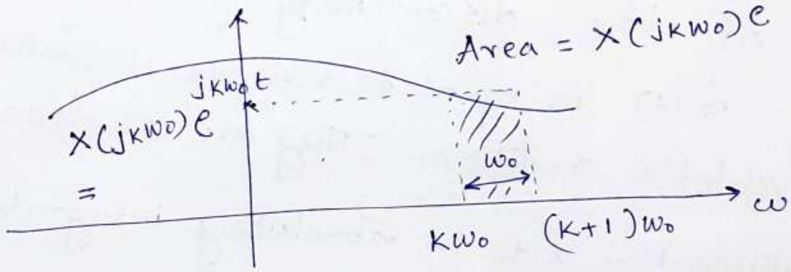
$k\omega_0 \to \omega$ and $\tilde{x}(t) = x(t)$

And ~~Integrand~~ Summation converges to Integration

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega} \rightarrow (4)$$

$x(t)$ is known as synthesis equation

Area $= X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$




Equations (3) and (4) are known as Fourier Transform pair.

# Convergence of Fourier Transform

We know that

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} . dt$$

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} . dW$$

$\hat{x}(t)$ is equal to $x(t)$ for any 't' except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

$\hat{x}(t)$ is equal to $x(t)$ if it Satisfies Dirchlet's conditions. They are as follows

Condition 1 :- $x(t)$ is absolutely integrable

i.e $\int_{-\infty}^{\infty} |x(t)| < \infty$

Condition 2 :- $x(t)$ have a finite number of ~~maxima~~ maxima and minima within any finite interval

Condition 3 :- $x(t)$ have a finite number of discontinuities within any finite interval. Further more each of these discontinuities must be finite.

Example 4.1)

Find the Fourier transform of $x(t) = e^{-at} u(t) \quad a>0$

Sketch $|X(j\omega)|$ and $\angle X(j\omega)$

Solution :

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} . dt$$

$$X(j\omega) = \int_{+0}^{\infty} e^{-at} . e^{-j\omega t} . dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} . dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$= \frac{e^{-\infty} - e^{-(a+j\omega)(0)}}{-(a+j\omega)}$$

$$X(j\omega) = \frac{1}{a+j\omega} \quad a>0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

(6)

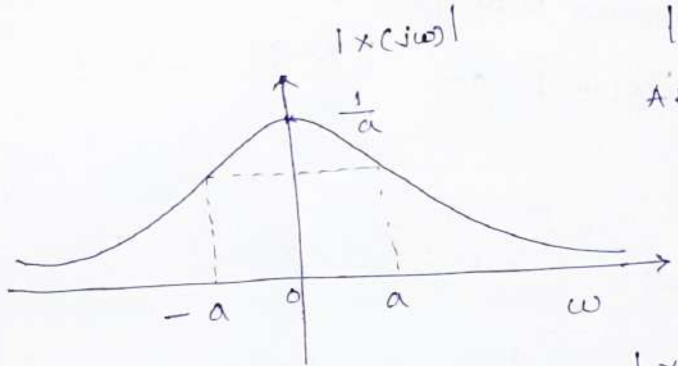
$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \quad \text{for } a>0$$

At $\omega = 0$

$|X(j\omega)| = \frac{1}{a}$

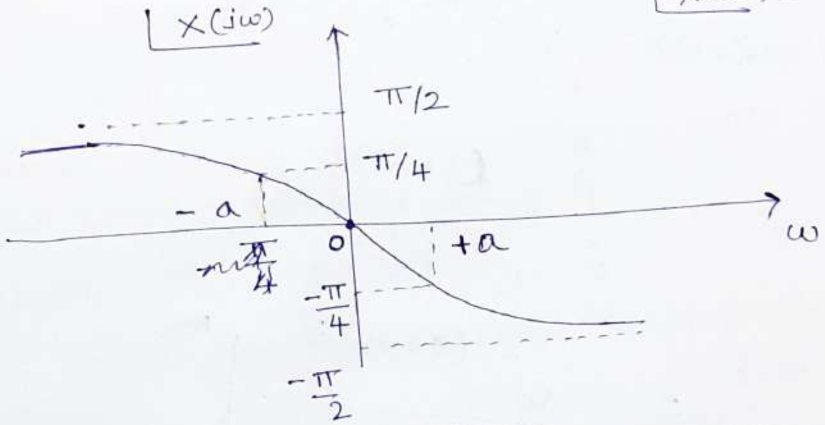
At $\omega = \infty$.

$|X(j\omega)| = 0$




$\omega = 0 \quad 2\pi \quad 4\pi \quad 6\pi \ldots$

$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

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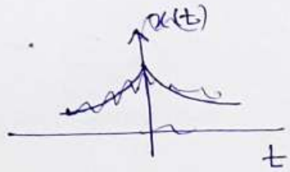
~~Example~~ [illegible]

~~Find the FT of~~ [illegible] $x(t) = e^{-a|t|}$, $a>0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$




$= \int_{-\infty}^{0}$ [illegible] $+ \int_{0}^{\infty}$ [illegible]

$= \int_{-\infty}$ [illegible]

$=$ [illegible]

4.2) Example 4.2

Find the F.T. of $x(t) = e^{-a|t|}$, $a>0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\,.dt$$

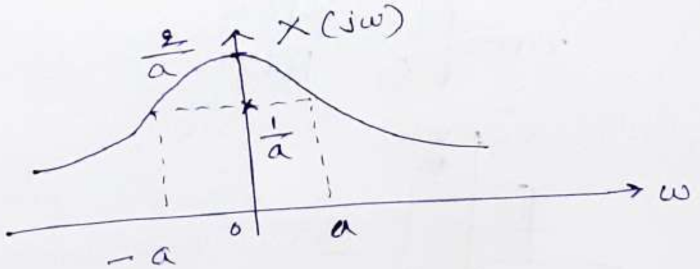
$$= \int_{-\infty}^{0} e^{-a(-t)}.e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}.e^{-j\omega t}.\,dt$$

$$= \int_{-\infty}^{0} e^{+(a-j\omega)t}.\,dt + \int_{0}^{\infty} e^{(-a-j\omega)t}.\,dt$$

$$= \frac{e^{(a-j\omega)t}}{(a-j\omega)}\Bigg|_{-\infty}^{0} + \frac{e^{(-a-j\omega)t}}{(-a-j\omega)}\Bigg|_{0}^{\infty}$$

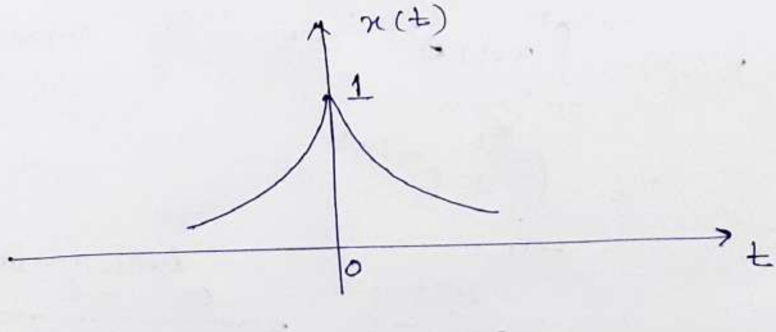
$$= \frac{e^{0}-0}{a-j\omega} + \frac{0-e^{0}}{-a-j\omega}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$$




$$X(j\omega)\Big|_{\omega=0} = \frac{2a}{a^2} = \frac{2}{a}$$

$$X(j\omega)\Big|_{\omega=\infty} = 0$$



4.3) Example 4.3

Find the F.T. of $x(t) = \delta(t)$ 8

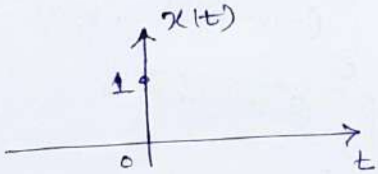
[Find the F.T. of unit impulse]

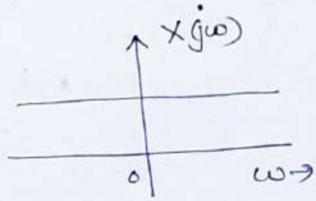
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} \, dt$$

1 at $t=0$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} \, dt$$

$$= \underline{1}$$



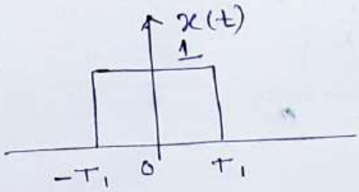




4.4) Example 4.4

Find the F.T. of the rectangular pulse

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

Solution:



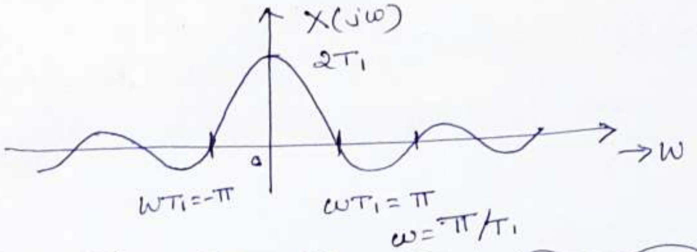

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t} \, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

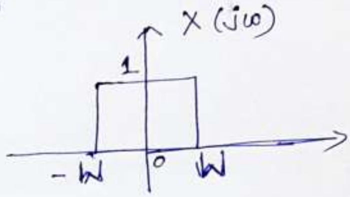
$$= 2T_1 \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j\omega T_1} = 2T_1 \frac{\sin(\omega T_1)}{\omega T_1}$$

$$X(j\omega) = 2T_1 \frac{\sin(\omega T_1)}{\omega T_1}$$




4.5) Example 4.5 Find $x(t)$ if

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$




$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \cdot d\omega$$

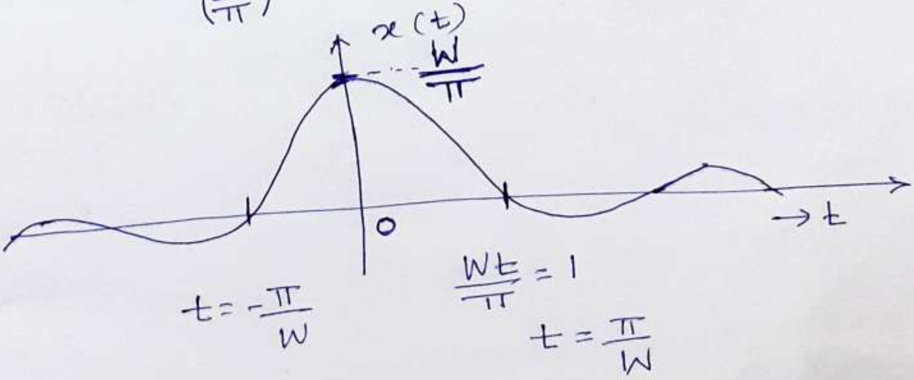
$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} 1 \cdot e^{j\omega t} \cdot d\omega$$

$$x(t) = \frac{1}{2\pi} \times \left. \frac{e^{j\omega t}}{jt} \right|_{-W}^{W} = \frac{1}{2\pi} \left[ \frac{e^{jWt} - e^{-jWt}}{jt} \right]$$

$$x(t) = \frac{1}{\pi t} \times \frac{e^{jWt} - e^{-jWt}}{2j} = \frac{1}{\pi t} (\sin Wt)$$

We know $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

$$\therefore x(t) = \frac{\sin \pi\left(\frac{Wt}{\pi}\right)}{\pi\left(\frac{Wt}{\pi}\right)} \times \left(\frac{W}{\pi}\right) = \frac{W}{\pi} \text{sinc}\left(\frac{Wt}{\pi}\right)$$



## 4.2 Fourier Transform For Periodic Signals

Consider a periodic signal $x(t)$, whose Fourier Transform is $X(j\omega)$

$$X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$$

$$x(t) = IFT[X(j\omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\,e^{j\omega t}\,.d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,.d\omega = e^{j\omega_0 t}$$

$$\therefore \quad x(t) = e^{j\omega_0 t}$$

$\therefore$ If $\boxed{x(t) = e^{j\omega_0 t} \iff X(j\omega) = 2\pi\,\delta(\omega-\omega_0)}$ → (1)

Consider a periodic signal $x(t)$. Then $x(t)$ can be expressed in Fourier series

$$x(t) = \sum_{K=-\infty}^{\infty} a_K\, e^{jK\omega_0 t}$$

Taking F.T. on both sides

$$F.T.[x(t)] = F.T\left[\sum_{K=-\infty}^{\infty} a_K\, e^{jK\omega_0 t}\right]$$

$$X(j\omega) = \sum_{K=-\infty}^{\infty} a_K\, F.T\left[e^{jK\omega_0 t}\right]$$

$$X(j\omega) = \sum_{K=-\infty}^{\infty} a_K \times 2\pi\,\delta(\omega-K\omega_0)$$

$$\therefore \quad X(j\omega) = \sum_{K=-\infty}^{\infty} 2\pi\, a_K\,\delta(\omega-K\omega_0)$$

where $a_K$ = Fourier Series Coefficient and it is not a function of 't' and it is constant.

4.7) Example 4.7

Find the Fourier Transform of

(10)

$x(t) = \sin \omega_0 t$

Sol :- $x(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

Taking F.T. on both sides

$$F.T.[x(t)] = \frac{1}{2j} F.T\left[e^{j\omega_0 t}\right] - \frac{1}{2j} F.T.\left[e^{-j\omega_0 t}\right]$$

$$X(j\omega) = \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - \frac{1}{2j}\left[2\pi\delta(\omega+\omega_0)\right]\right.$$

$$= \frac{\cancel{2}\pi}{\cancel{2}j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

4.7) a) Find the Fourier Transform of $\cos \omega_0 t$

Solution: $x(t) = \cos \omega_0 t$

$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

Taking F.T. on both sides

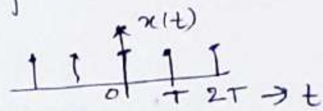
$$F.T.[x(t)] = \frac{1}{2} F.T\left[e^{j\omega_0 t}\right] + \frac{1}{2} F.T.\left[e^{-j\omega_0 t}\right]$$

$$X(j\omega) = \frac{1}{\cancel{2}} \times \cancel{2}\pi\delta(\omega-\omega_0) + \frac{1}{\cancel{2}}\cancel{2}\pi\delta(\omega+\omega_0)$$

$$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

4.8) Find the Fourier transform of impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$




$x(t)$ can be expressed in Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Taking F.T. on both sides

$$F.T.[x(t)] = F.T.\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right]$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \, F.T.\left[e^{jk\omega_0 t}\right]$$

$\boxed{\text{We know} \quad F.T\left[e^{j\omega_0 t}\right] = 2\pi\,\delta(\omega - \omega_0)}$

$$= \sum_{k=-\infty}^{\infty} a_k \, 2\pi\,\delta(\omega - k\omega_0) \qquad \omega_0 = \frac{2\pi}{T}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \, 2\pi\,\delta\left(\omega - \frac{k2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

1 only at $t=0$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}$$

$$\therefore X(j\omega) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{T}\right)}_{a_k} \times 2\pi\,\delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

## 4.3 Properties of the Continuous-Time Fourier Transform

1) Linearity

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$y(t) \xleftrightarrow{F.T} X(j\omega)$$

$$a\,x(t) + b\,y(t) \xleftrightarrow{F.T} a\,X(j\omega) + b\,Y(j\omega)$$

2) Time shifting property

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$x(t - t_0) \xleftrightarrow{F.T.} X(j\omega)\,e^{-j\omega t_0}$$

3) Conjugation and conjugate symmetry

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$x^*(t) \xleftrightarrow{F.T.} X^*(-j\omega)$$

If $x(t)$ = Real function

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

Even part of $x(t) \xleftrightarrow{F.T} \mathrm{Re}[X(j\omega)]$

Odd part of $x(t) \xleftrightarrow{F.T.} j\,\mathrm{Im}[X(j\omega)]$

4) Differentiation and Integration

Differentiation $\quad x(t) \xleftrightarrow{F.T.} X(j\omega)$

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(j\omega)$$

Integration $\quad x(t) \xleftrightarrow{F.T.} X(j\omega)$

$$\int_{-\infty}^{\tau} x(\tau)\,d\tau \xleftrightarrow{F.T.} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

5) Time Scaling and Frequency Scaling

Time scaling $x(t) \xleftrightarrow{F.T} X(j\omega)$

$$x(at) \xleftrightarrow{F.T.} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

~~Frequency scaling~~

6) Time Reversal property

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$x(-t) \xleftrightarrow{F.T.} X(-j\omega)$$

7) Frequency shifting property

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

8) Duality Property

$$x_1(t) = \begin{cases} 1, & |t|<T_1 \\ 0 & |t|>T_1 \end{cases} \xleftrightarrow{F.T.} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{F.T.} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

9) Convolution Property

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$h(t) \xleftrightarrow{F.T} H(j\omega)$$

$$x(t) * h(t) \xleftrightarrow{F.T.} X(j\omega) H(j\omega)$$

10) Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \cdot d\omega$$

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11) Differentiation in Frequency

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$t\,x(t) \xleftrightarrow{F.T.} j\frac{d}{d\omega} X(j\omega)$$

12) Conjugate symmetry for Real Signals

For $x(t) = \text{real}$

$$X(j\omega) = X^*(-j\omega)$$

$$Re\{X(j\omega)\} = Re\{X(-j\omega)\}$$

$$Im\{X(j\omega)\} = -Im\{X(-j\omega)\}$$

$$|X(j\omega)| = |X(-j\omega)|$$

$$\angle X(j\omega) = -\angle X(-j\omega)$$

13) Symmetry for Real and Even Signals

$x(t)$ is Real and even $\quad$ $X(j\omega)$ is Real and even

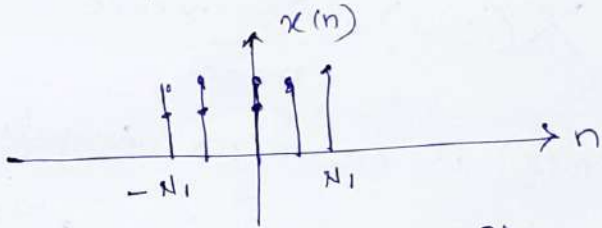
14) Symmetry for Real and Odd Signals

If $x(t)$ is Real and odd $\quad$ $X(j\omega)$ is purely imaginary and odd.

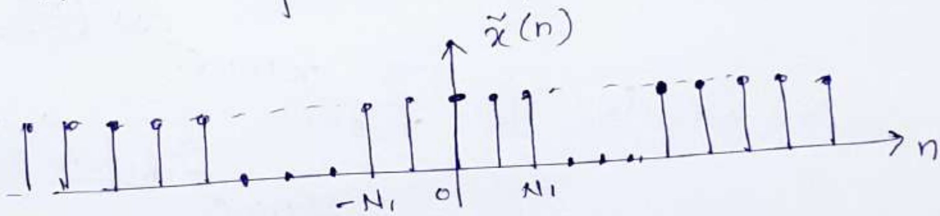
# Discrete Time Fourier Transform

## 5.1 Representation of Aperiodic Signals : The Discrete Time Fourier Transform

Consider a aperiodic signal $x(n)$




Assume a periodic sequence $\tilde{x}(n)$




$\tilde{x}(n)$ can be expressed in terms of Discrete time Fourier Series

$$\tilde{x}(n) = \sum_{k=\langle N \rangle} a_k e^{j\frac{2\pi kn}{N}}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}(n) e^{-j\frac{2\pi kn}{N}}$$

$$N a_k = \sum_{n=\langle N \rangle} \tilde{x}(n) e^{-j\frac{2\pi kn}{N}} \rightarrow (1)$$

From (1) $N.a_K = \sum_{n=\langle N\rangle} \tilde{x}(n) e^{-j\frac{2\pi K n}{N}} = \sum_{n=-N_1}^{N_1} \tilde{x}(n) e^{-j\frac{2\pi K n}{N}}$

As $N\to\infty$ $\tilde{x}(n) = x(n)$ (16)

As $N\to\infty$ $e^{-j\left(\frac{2\pi K}{N}\right)n} \longrightarrow e^{-j\omega n}$ $\lim_{N\to\infty} \frac{2\pi K}{N} \to \omega$

As $N\to\infty$ ~~E~~ Equation (1) becomes

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}} \rightarrow (2)$$

Equation (2) is known as Angalis equation

Consider (1) $N.a_K = \sum_{n=\langle N\rangle} \tilde{x}(n) e^{-j\frac{2\pi K n}{N}}$

$$a_K = \frac{1}{N} \sum_{n=\langle N\rangle} \tilde{x}(n) e^{-jK\left(\frac{2\pi}{N}\right)n}$$

$$a_K = \frac{1}{N} \sum_{n=\langle N\rangle} \tilde{x}(n) e^{-jK\omega_0 n} \Bigg\}$$ where $\omega_0 = \frac{2\pi}{N}$

$$a_K = \frac{1}{N} X\left(e^{jK\omega_0}\right)$$

$$\therefore \tilde{x}(n) = \sum_{K=\langle N\rangle} a_K e^{jK\omega_0 n}$$

$$\tilde{x}(n) = \sum_{K=\langle N\rangle} \left(\frac{1}{N} X\left(e^{jK\omega_0}\right)\right) e^{jK\omega_0 n} \rightarrow (3)$$

As $N\to\infty$ $\tilde{x}(n) = x(n)$ $\omega_0 N = 2\pi$

$\omega_0 = \frac{2\pi}{N}$, $N = \frac{2\pi}{\omega_0}$

As $N\to\infty$ $N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\omega_0} = \frac{2\pi}{d\omega}$

$X\left(e^{jK\omega_0}\right) = X\left(e^{j\omega}\right)$ $K\omega_0 = \omega$

$\omega = k\omega_0$

$$\therefore x(n) = \int_{-\pi}^{\pi} \frac{1}{\frac{2\pi}{d\omega}} X(e^{j\omega}) e^{j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Since $X(j\omega)$ is a continuous function

Summation becomes Integration

$$\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega} \rightarrow (4)$$

Equation (4) is known as synthesis equation.

5.1) Example 5.1 Find the D.T.F.T. of $x(n) = a^n u(n)$ $|a|<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - a[\cos\omega - j\sin\omega]} = \frac{1}{1 - a\cos\omega + j(\sin\omega)a}$$

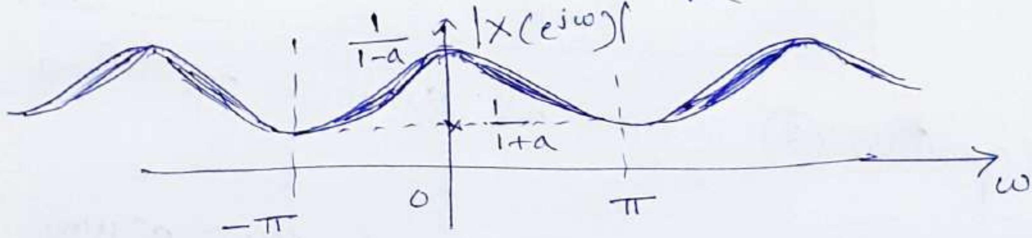
$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + a^2\sin^2\omega}}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + a^2\sin^2\omega}}$$ 18

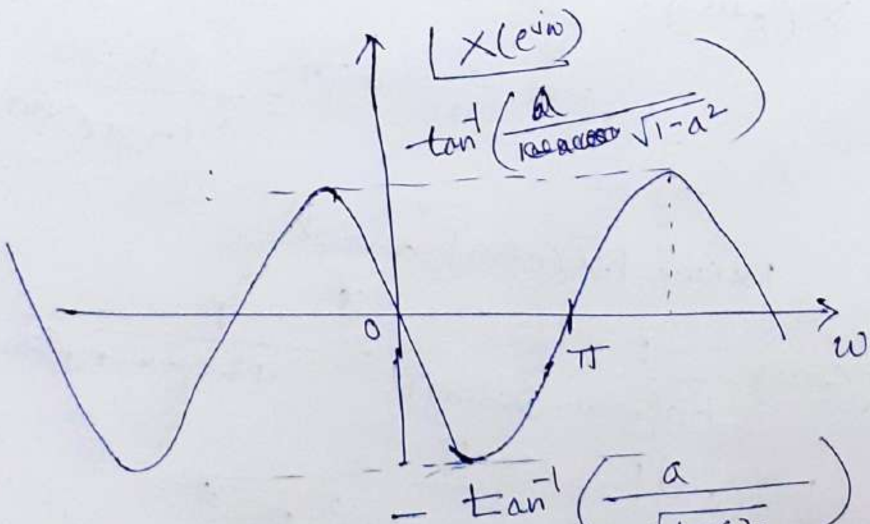
At $\omega = 0$ $\quad |X(e^{j\omega})|_{\omega=0} = \frac{1}{\sqrt{(1-a)^2+0}} = \frac{1}{(1-a)}$

At $\omega = -\pi$ $\quad |X(e^{j\omega})|_{\omega=-\pi} = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a}$

At $\omega = +\pi$ $\quad |X(e^{j\omega})|_{\omega=\pi} = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a}$




$$\angle X(e^{j\omega}) = -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$



5.2) <u>Example 5.2</u> For $x(n) = a^{|n|}$ , $|a|<1$ find DTFT (19)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} \cdot e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} (a^{-1} e^{-j\omega})^{n} + \sum_{n=0}^{\infty} (a e^{-j\omega})^{n}$$

$$= \sum_{n=-1}^{-\infty} (a e^{j\omega})^{-n} + \sum_{n=0}^{\infty} (a e^{-j\omega})^{n}$$

put $n = -r \Rightarrow -n = r$

$$= \sum_{r=1}^{\infty} (a e^{j\omega})^{r} + \sum_{n=0}^{\infty} (a e^{-j\omega})^{n}$$

$$= \frac{a e^{j\omega}}{1 - a e^{j\omega}} + \frac{1}{1 - a e^{-j\omega}}$$

$$\boxed{\begin{aligned}&(1-ae^{j\omega})(1-ae^{-j\omega})\\ &= 1 + a^2 - ae^{-j\omega} - ae^{j\omega}\\ &= 1 + a^2 - 2a\left[\frac{e^{j\omega}+e^{-j\omega}}{2}\right]\\ &= 1 + a^2 - 2a\cos\omega\\ &= 1 + a^2 - 2a\cos\omega\end{aligned}}$$

$$= \frac{a e^{j\omega}(1 - a e^{-j\omega}) + (1 - a e^{j\omega})}{(1 - a e^{j\omega})(1 - a e^{-j\omega})}$$

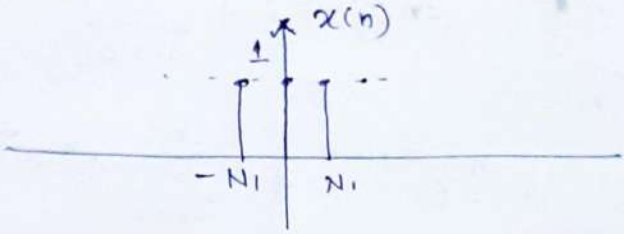
$$X(e^{j\omega}) = \frac{\cancel{a e^{j\omega}} - a^2 + 1 - \cancel{a e^{j\omega}}}{1 - 2a\cos\omega + a^2} = \frac{(1 - a^2)}{(1 - 2a\cos\omega + a^2)}$$

In this case $X(e^{j\omega})$ is real

3) Find DTFT of the pulse



$$x(n) = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$




$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} x(n) e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} 1 \cdot e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{e^{-j\omega(-N_1)} - e^{-j\omega(N_1+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{j\omega N_1} - e^{-j\omega N_1} \cdot e^{-j\omega}}{1-e^{-j\omega}}$$

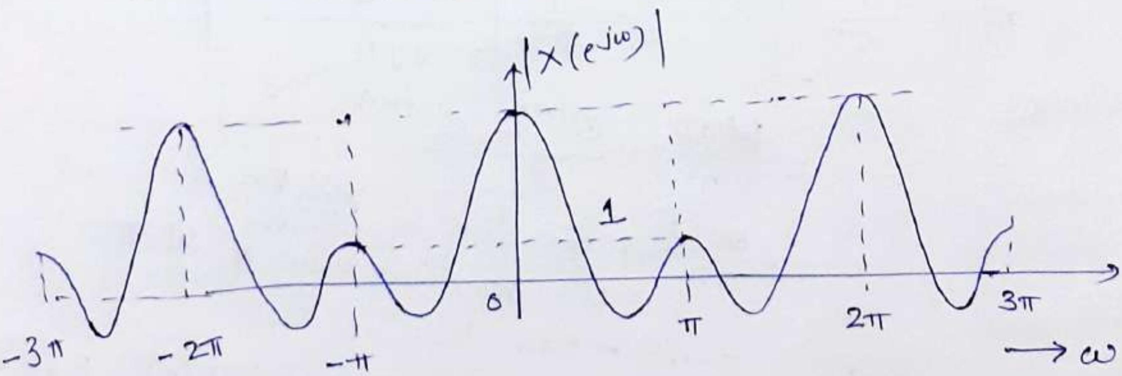
$$= \frac{\cancel{e^{-\frac{j\omega}{2}}}\left[e^{j\omega N_1} \cdot e^{+\frac{j\omega}{2}} - e^{-j\omega N_1} \cdot e^{-\frac{j\omega}{2}}\right]}{\cancel{e^{-\frac{j\omega}{2}}}\left[e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$= \frac{e^{j\omega(N_1+\frac{1}{2})} - e^{-j\omega(N_1+\frac{1}{2})} / 2j}{e^{\frac{j\omega}{2}} - e^{-j\frac{\omega}{2}} / 2j}$$

$$X(e^{j\omega}) = \frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}$$

put $N_1 = 2$

| $\omega$ | $\sin \omega \left(N_1 + \frac{1}{2}\right)$ | $\sin \frac{\omega}{2}$ | $\frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin \omega/2}$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $\frac{N_1 + \frac{1}{2}}{\frac{1}{2}} = 2N_1 + 1 = 5$ |
| $\pi$ | $\sin \pi \left(2 + \frac{1}{2}\right)$ | $\sin \frac{\pi}{2}$ | $1$ |
| $-\pi$ | $\sin\left(-\pi \left(2 + \frac{1}{2}\right)\right)$ | $\sin\left(-\frac{\pi}{2}\right)$ | $1$ |
| $\frac{\pi}{\left(\frac{5}{2}\right)} = \frac{\pi}{\left(2 + \frac{1}{2}\right)}$ | $0$ | $\sin \frac{\pi}{\frac{5}{2}}$ | $0$ |

## 5.1.3 Convergence Issues associated with the Discrete Time Fourier Transform

(22)

(1) For an aperiodic signal $x(n)$, the DTFT exists, if

(i) $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

(ii) or if the sequence has finite energy

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

(1) Find the DTFT of the signal $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X(e^{j\omega}) = 1$$

(2) Find $x(n)$ if $X(e^{j\omega}) = 1$

$$x(n) = \frac{1}{2\pi} \int_{-W}^{W} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$x(n) = \frac{1}{2\pi} \int_{-W}^{W} e^{jwn} = \frac{1}{2\pi} \left[ \frac{e^{jwn} - e^{-jwn}}{jn} \right]$$

$$= \frac{(1)}{\pi n} \times \frac{e^{jwn} - e^{-jwn}}{2j}$$

$$= \frac{(1)}{\pi n} \times \sin Wn = \frac{\sin Wn}{\pi n}$$

The maximum value of $W = \pi$

$x(n) = \delta(n)$ for $W = \pi$

i.e if $W = \pi$ $\quad x(n) = \frac{1}{\pi n} \sin \pi(n) = 1$ at $n = 0$

## 5.2 Fourier Transform for Periodic Signals

If $x(n) = e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)$

Consider a periodic signal $x(n)$ with period $N$

$\therefore$ $x(n)$ can be expressed in terms of DTFT

$$\therefore \quad x(n) = \sum_{K=\langle N\rangle} a_K e^{j\frac{2\pi K n}{N}}$$

~~since it is periodic from -∞ to ∞~~

Taking DTFT on both sides

$$DTFT\left[x(n)\right] = DTFT\left[\sum_{K=\langle N\rangle} a_K e^{j\frac{2\pi K n}{N}}\right]$$

$$X(e^{j\omega}) = \sum_{K=\langle N\rangle} a_K \, DTFT\left[e^{j\frac{2\pi K n}{N}}\right]$$

$$= \sum_{K=\langle N\rangle} a_K \, 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi K}{N} - 2\pi l\right)$$

$$= \sum_{K=\langle N\rangle} \sum_{l=-\infty}^{\infty} 2\pi a_K \, \delta\left(\omega - \frac{2\pi K}{N} - 2\pi l\right)$$

$$\left[ = \sum_{l=-\infty}^{\infty} 2\pi a_l \, \delta\left(\omega - \frac{2\pi l}{N}\right) \right]$$

put $l = K$ Because $x(n)$ is periodic and exists from $-\infty$ to $\infty$

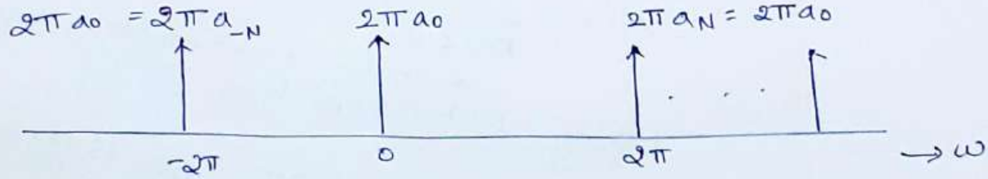
$$= \sum_{K=-\infty}^{\infty} 2\pi a_K \, \delta\left(\omega - \frac{2\pi K}{N}\right)$$

This is because

$$x(n) = a_0 + a_1 e^{j\frac{2\pi}{N}n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} e^{j(N-1)\frac{2\pi}{N}n}$$

$$\text{DTFT}[x(n)] = \text{DTFT}[a_0] + \text{DTFT}\left[a_1 e^{\frac{j2\pi n}{N}}\right] + \text{DTFT}\left[a_2 e^{j2\left(\frac{2\pi}{N}\right)n}\right] + \cdots \text{DTFT}\left[a_{N-1} e^{j(N-1)\frac{2\pi}{N}n}\right]$$

$$\text{DTFT}[1] = 2\pi \Rightarrow \text{DTFT}[a_0] = a_0 2\pi$$

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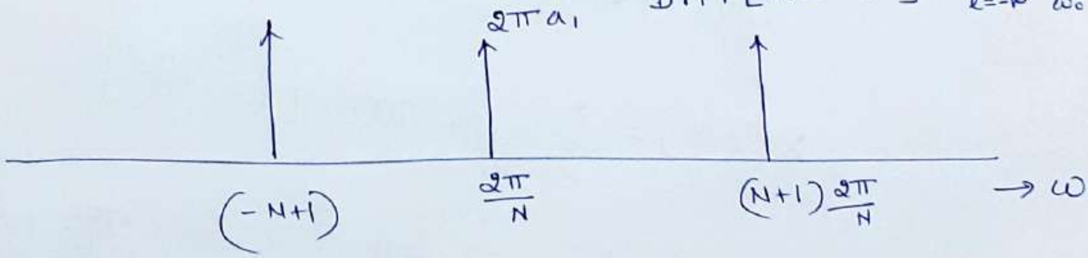
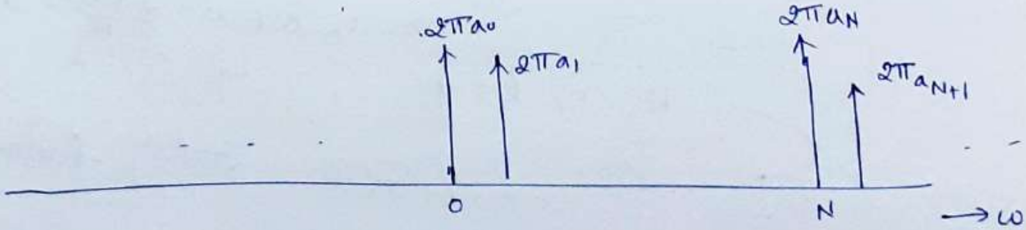
$$\text{DTFT}\left[a_1 e^{j\frac{2\pi}{N}n}\right] = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



5.5) Example 5.5)

Find the DTFT of the periodic signal $x(n) = \cos \omega_0 n$ with $\omega_0 = \frac{2\pi}{5}$

$$x(n) = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$x(n) = e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l)$$

$$\therefore DTFT[x(n)] = \frac{1}{2} DTFT[e^{j\omega_0 n}] + \frac{1}{2} DTFT[e^{-j\omega_0 n}]$$

$$= \frac{1}{2} \times \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l) + \frac{1}{2} \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega + \omega_0 - 2\pi l)$$

$$X(e^{j\omega}) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) \text{ for } -\pi \leq \omega \leq \pi$$

$$X(e^{j\omega}) = \pi \delta\left(\omega - \frac{2\pi}{5}\right) + \pi \delta\left(\omega + \frac{2\pi}{5}\right)$$

5.6) Example 5.6) Find the DTFT of the periodic impulse train

$$x(n) = \sum_{K=-\infty}^{\infty} \delta(n - KN)$$

Solution:

$$x(n) = \sum_{K=-\infty}^{\infty} \delta(n - KN)$$

It is a periodic signal with period N

DTFT of periodic signal $x(n)$ is given by 

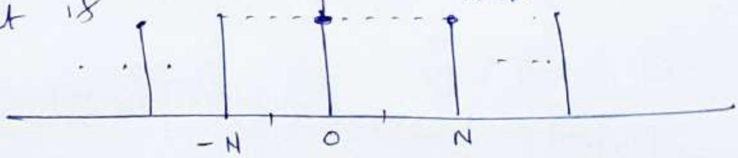
$$X(e^{j\omega}) = \sum_{K=-\infty}^{\infty} 2\pi a_K \delta\left(\omega - \frac{2\pi K}{N}\right)$$

$$a_K = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j\frac{2\pi K n}{N}}$$

5.6) Ex. 5.6

3) Find the DTFT of $x(n) = \sum_{K=-\infty}^{\infty} \delta(n - Kn)$ $x(n) = \sum_{K=-\infty}^{\infty} \delta(n - KN)$

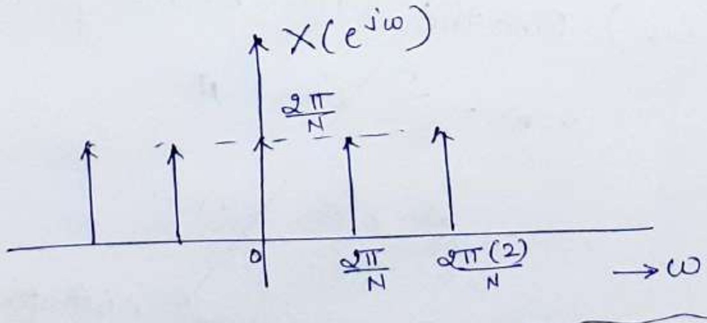
But if



$a_K = \frac{1}{N} \times 1$ $\quad X(e^{j\omega}) = \sum_{K=-\infty}^{\infty} a_K 2\pi \delta\left(\omega - \frac{2\pi K}{N}\right)$

$$\therefore X(e^{j\omega}) = \sum_{K=-\infty}^{\infty} 2\pi \times \frac{1}{N} \delta\left(\omega - \frac{2\pi K}{N}\right)$$

$$= \frac{2\pi}{N} \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi K}{N}\right)$$




$$a_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}}$$

$$a_K = \frac{1}{N} \times (1) = \frac{1}{N}$$

# Properties of Discrete Time Fourier Transform

## (1) Periodicity of DTFT

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

The discrete time Fourier Transform is always periodic in '$\omega$' with period $2\pi$.

## (2) Linearity

$$x_1(n) \xleftrightarrow{D.T.F.T} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

$$a\,x_1(n) + b\,x_2(n) \xleftrightarrow{DTFT} a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})$$

## (3) Time Shifting and Frequency Shifting

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x(n-n_0) \xleftrightarrow{DTFT} X\left(e^{j(\omega-\omega_0)}\right)$$

## 4) Conjugation and conjugate Symmetry



$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x^*(n) \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

If $x(n)$ is real then

(i) $X(e^{j\omega}) = X^*(e^{-j\omega})$

(ii) $Re\{X(e^{j\omega})\}$ is even function of $\omega$

(iii) $Im\{X(e^{j\omega})\}$ is an odd function of $\omega$

(iii) The magnitude of $X(e^{j\omega})$ is an even function

(iv) The phase angle of $X(e^{j\omega})$ is an odd function

(v) Even part of $x(n) \xleftrightarrow{DTFT} Re\{X(e^{j\omega})\}$

(Vi) Odd part of $x(n) \xleftrightarrow{DTFT} j\,Im\{X(e^{j\omega})\}$

## 5) Differencing and Accumulation

Differencing: $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$$x(n) - x(n-1) \xleftrightarrow{DTFT} X(e^{j\omega}) - e^{-j\omega} X(e^{j\omega})$$

$$= [1 - e^{-j\omega}] X(e^{j\omega})$$

Accumulation :- $y(n) = \sum_{m=-\infty}^{n} x(m) = \sum_{m=-\infty}^{n} x(m)$

$$y(n) - y(n-1) = \sum_{m=-\infty}^{n} x(m) - \sum_{m=-\infty}^{n-1} x(m)$$

$$y(n) - y(n-1) = x(m)$$

$$DTFT\left[y(n) - y(n-1)\right] = DTFT\left[x(n)\right]$$

$$DTFT\left[y(n)\right] - DTFT\left[y(n-1)\right] = X(e^{j\omega})$$

$$DTFT\left[\sum_{m=-\infty}^{n} x(m)\right] - DTFT\left[\sum_{m=-\infty}^{n-1} x(m)\right] = X(e^{j\omega})$$

$$DTFT\left[\sum_{m=-\infty}^{n} x(m)\right] - e^{-j\omega}\, DTFT\left[\sum_{m=-\infty}^{n} x(m)\right] = X(e^{j\omega})$$

$$\boxed{DTFT\left[\sum_{m=-\infty}^{n} x(m)\right] = \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{K=-\infty}^{\infty} \delta(\omega - 2\pi K)}$$

The term $\pi X(e^{j0}) \sum_{K=-\infty}^{\infty} \delta(\omega - 2\pi K)$ reflects the DC value or average value that can result from summation.

6) Time Reversal

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$$

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## 7) Time Expansion

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x_K(n) \xleftrightarrow{DTFT} X(e^{jK\omega})$$

where $n = rK$

---

## 8) Differentiation in Frequency

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$n\,x(n) \xleftrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$$

---

## 9) Parseval's Relation

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

If $x(n)$ and $X(e^{j\omega})$ are Fourier Transform pair

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \cdot d\omega$$

$|X(e^{j\omega})|^2$ is referred to as energy-density spectrum

## 10) The convolution Property

$$y(n) = x(n) * h(n)$$

$$y(n) = x(n) * h(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})\, H(e^{j\omega}) = Y(e^{j\omega})$$

$$y(n) = x(n) * h(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

~~11) The Multiplication Property~~

$$\cancel{Y(e^{j\omega}) = \frac{1}{2\pi}\left[X_1(e^{j\omega}) * X_2(e^{j\omega})\right]}$$

## 11) The Multiplication property

$$x_1(n) \xleftrightarrow{\text{DTFT}} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{\text{DTFT}} X_2(e^{j\omega})$$

$$x_1(n) \cdot x_2(n) \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

periodic convolution

[Basic Discrete-Time Fourier Transform pairs]

## 5.6 Basic Fourier Transform Pairs



| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$ | $2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\omega-\omega_0-2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$ <br> $a_k = \begin{cases} 1, & k=m, m\pm N, m\pm 2N \\ 0, & \text{otherwise} \end{cases}$ <br> (b) If $\frac{\omega_0}{2\pi}$ = irrational <br> The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{\infty} [\delta(\omega-\omega_0-2\pi l)] + [\delta(\omega+\omega_0-2\pi l)]$ | (a) $\omega_0 = \frac{2\pi m}{N}$ <br> $a_k = \begin{cases} \frac{1}{2}, & k=\pm m, \pm m \pm N \\ 0 & \text{otherwise} \end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ = irrational <br> The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{\infty} [\delta(\omega-\omega_0-2\pi l) - \delta(\omega+\omega_0-2\pi l)]$ | (a) $\omega_0 = \frac{2\pi r}{N}$ <br> $a_k = \begin{cases} \frac{1}{2j}, & k=r, r\pm N, r\pm 2N \ldots \\ -\frac{1}{2j} & k=-r, -r\pm N, -r\pm 2N \\ 0 & \text{otherwise} \end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational <br> $\Rightarrow$ The signal is aperiodic |

| Signal | Fourier Transform | Fourier Series coefficients (if periodic) |
|---|---|---|
| $x(n) = 1$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N \\ 0 & \text{otherwise} \end{cases}$ |
| Periodic square Wave $x(n) = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0 & N_1 < \lvert n\rvert \le \frac{N}{2} \end{cases}$ $x(n+N) = x(n)$ | $2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin\left[\left(\frac{2\pi k}{N}\right)\left(N_1 + \frac{1}{2}\right)\right]}{N \sin\left(\frac{2\pi k}{2N}\right)}$ for $k \ne 0, \pm N, \pm 2N$ <br> $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N$ |
| $\sum_{k=-\infty}^{\infty} \delta(n - kN)$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for a |
| $a^n u(n)$, $\lvert a\rvert < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | – |
| $x(n) = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$ | $\frac{\sin\left[\omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin \frac{\omega}{2}}$ | – |
| $\delta(n)$ | $1$ | — |
| $u(n)$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega - 2\pi k)$ | — |
| $\delta(n - n_0)$ | $e^{-j\omega n_0}$ | — |
| $(n+1) a^n u(n)$ $\lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |

| Signal | Fourier Transform | Fourier Series coefficients (if periodic) |
|---|---|---|
| $\frac{(n+r-1)!}{n!\,(r-1)!}a^n u(n)$ <br> $\lvert a\rvert<1$ | $\frac{1}{(1-ae^{-jw})^2}$ . | — |

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## 5.7 Duality

### 5.7.1 Duality in the Discrete-Time Fourier Series

Consider two periodic sequences with period N related through the summation

$$f(m) = \frac{1}{N} \sum_{r=\langle N \rangle} g(r)\, e^{-jr\left(\frac{2\pi}{N}\right)m} \rightarrow (1)$$

If we let $m = k$ and $r = n$ equation (1) becomes

$$f(k) = \frac{1}{N} \sum_{n=\langle N \rangle} g(n)\, e^{-jn\left(\frac{2\pi}{N}\right)k} \rightarrow (2)$$

From ~~by~~ equations (1) and (2) we can conclude that

$$g(n) \xleftrightarrow{F.S.} f(k)$$

where $g(n)$ is a periodic sequence
$f(k)$ is its fourier series coefficients

~~Consider equation (2)~~

$$\cancel{f(k) = \frac{1}{N} \sum_{n=\langle N \rangle} g(n)\, e^{-jn\left(\frac{2\pi}{N}\right)k}}$$

~~If we let m=n and r=-k~~

Consider (1) $\quad f(m) = \frac{1}{N} \sum_{r=\langle N \rangle} g(r)\, e^{-jr\left(\frac{2\pi}{N}\right)m}$

put $m = n$ and $r = -k$

$$f(n) = \frac{1}{N} \sum_{+k=\langle N \rangle} g(-k)\, e^{j(k)\left(\frac{2\pi}{N}\right)n}$$

$$f(n) = \sum_{k=\langle N \rangle} \frac{1}{N} g(-k)\, e^{jk\left(\frac{2\pi}{N}\right)n} \rightarrow (3)$$

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$$f(n) = \sum_{k=\langle N\rangle} \frac{1}{N} g(-k)\, e^{jk\left(\frac{2\pi}{N}\right)n} \rightarrow (3)$$

The Fourier Series expansion of $f(n)$ is given by

$$f(n) = \sum_{k=\langle N\rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n} \rightarrow (4)$$

Comparing (3) and (4)

$$f(n) \xleftrightarrow{F.S.} a_k = \frac{1}{N} g(-k)$$

$f(n)$ is a periodic sequence

∴ The Fourier Series coefficients of $f(n)$ are

$$a_k = \frac{1}{N} g(-k)$$

Duality implies every property of Discrete Time Fourier Series has a dual.

Example for Duality

(i) $$x(n-n_0) \xleftrightarrow{F.S} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}$$

$$e^{jm\left(\frac{2\pi}{N}\right)n} x(n) \xleftrightarrow{F.S.} a_{k-m}$$

(ii) $$\sum_{r=\langle N\rangle} x(r)\, y(n-r) \xleftrightarrow{F.S.} N a_k b_k$$

$$x(n)\, y(n) \xleftrightarrow{F.S.} \sum_{l=\langle N\rangle} a_l b_{k-l}$$

periodic Convolution

## 5.7.2 Duality between the Discrete-Time Fourier Transform and the Continuous-Time Fourier Series:

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \rightarrow (1)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow (2)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow (3)$$

$$a_k = \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} \, dt \rightarrow (4)$$

Equations (1) and (4) are Similar
Equations (2) and (3) are Similar

We can interpret Equations (1) and (2) as a Fourier series representation of the periodic frequency $X(e^{j\omega})$.

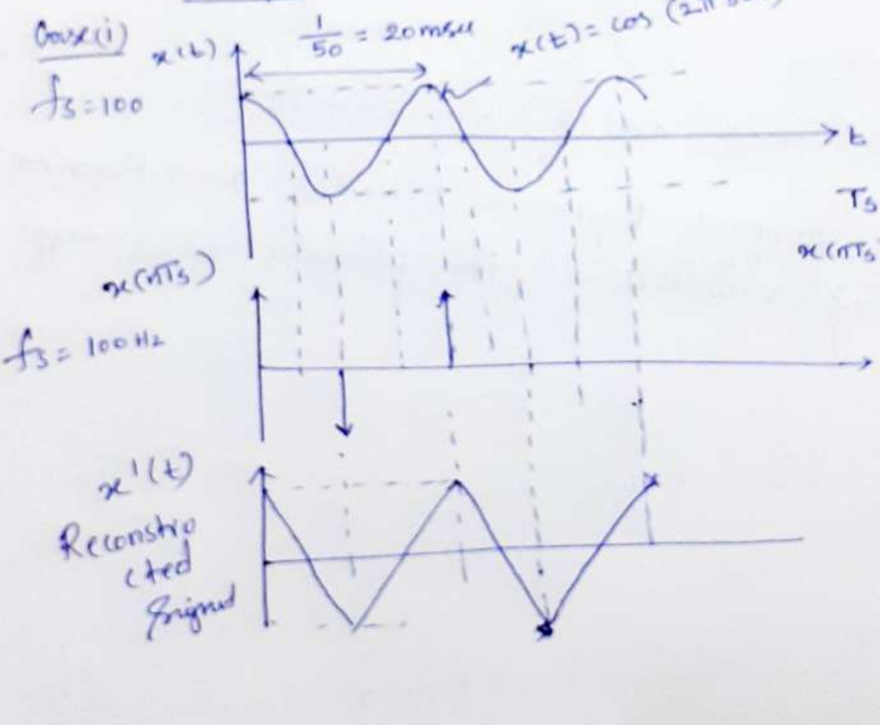
# Introduction to Sampling

**Sampling theorem:** Sampling theorem states that, if "W" is the maximum frequency component present in the signal, then it should be sampled with sampling frequency $f_s \geq 2W$ for proper reconstruction.

**Nyquist Criterion:**

The minimum sampling frequency $f_s = 2W$

**Example:**




$T_s = \frac{1}{100}$

$x(nT_s) = \cos(2\pi 50 n T_s)$

$= \cos\left(2\pi 50 \times \frac{n}{100}\right)$

Case (ii) $f_s = 50\,Hz$

$x(t)$

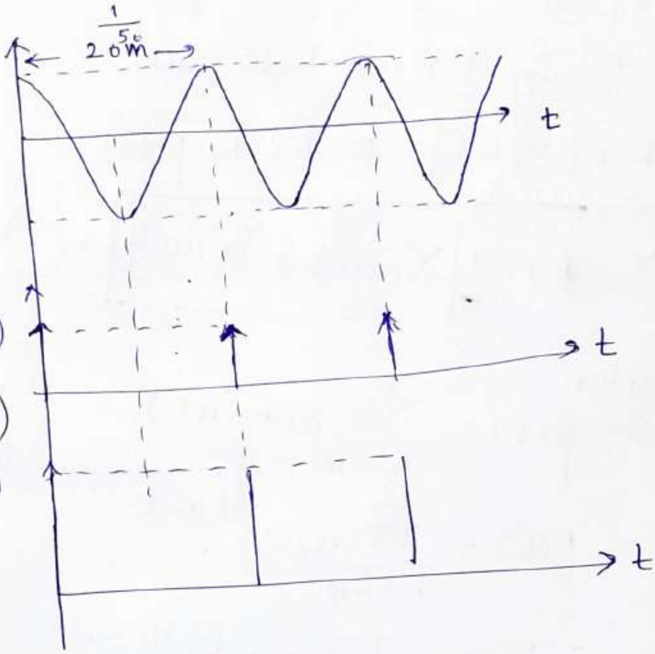
$T_s = \frac{1}{50}$

$x(nT_s)$

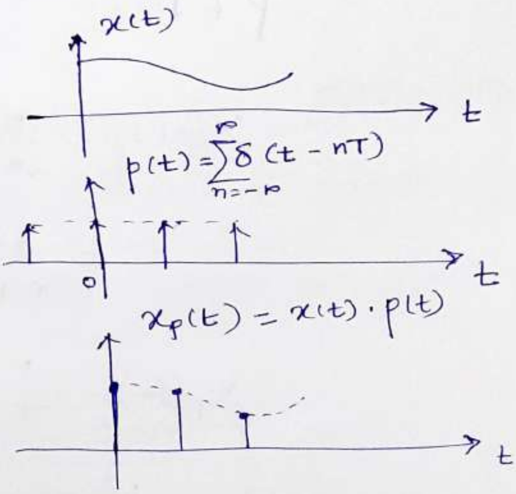
$x(nT_s) = \cos(2\pi\, 50\, nT_s)$

$= \cos\left(2\pi\, \cancel{50}\, \frac{n}{\cancel{50}}\right)$

Reconstructed Signal




## Impulse ~~Train~~ Train Sampling




Consider continuous time Signal $x(t)$. It is ~~morph~~ multiplied with impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

$x_p(t)$ represents Sampled Signal.

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

where $T$ = Sampling period

$f_s = \frac{1}{T}$ $\quad f_s \geq 2W$ where $|W|$ = Maximum frequency component of the signal.

Consider $x_p(t) = x(t) \cdot p(t)$

Taking F.T. on both sides

$$F.T\left[x_p(t)\right] = F.T\left[x(t) \cdot p(t)\right]$$

$$\boxed{X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]} \rightarrow (1)$$

Consider

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$p(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

Taking F.T. on both sides

$$P(j\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

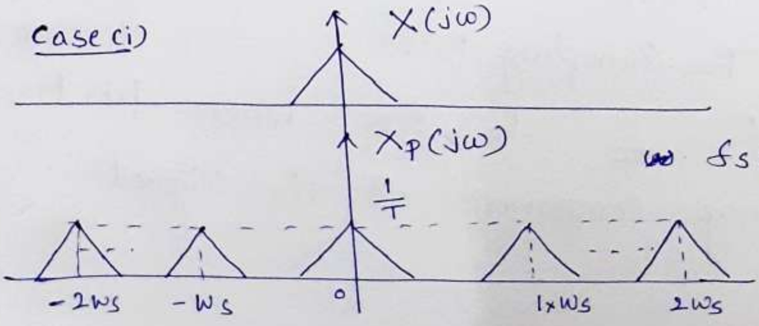
$$X_p(j\omega) = \frac{1}{\cancel{2\pi}} X(j\omega) * \frac{\cancel{2\pi}}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi K}{T}\right)$$

$$= \frac{\cancel{2\pi}}{(\cancel{2\pi})T} \sum_{K=-\infty}^{\infty} X\left(j\left(\omega - \frac{2\pi K}{T}\right)\right)$$

$$X_p(j\omega) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\left(j\left(\omega - K\omega_s\right)\right) \rightarrow (2)$$

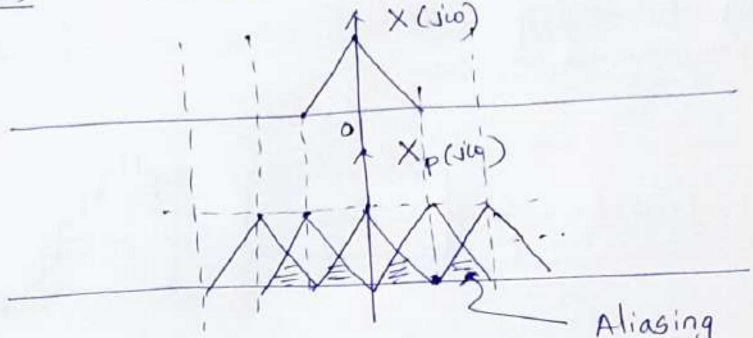
where $\omega_s = \frac{2\pi}{T}$

Case (i)



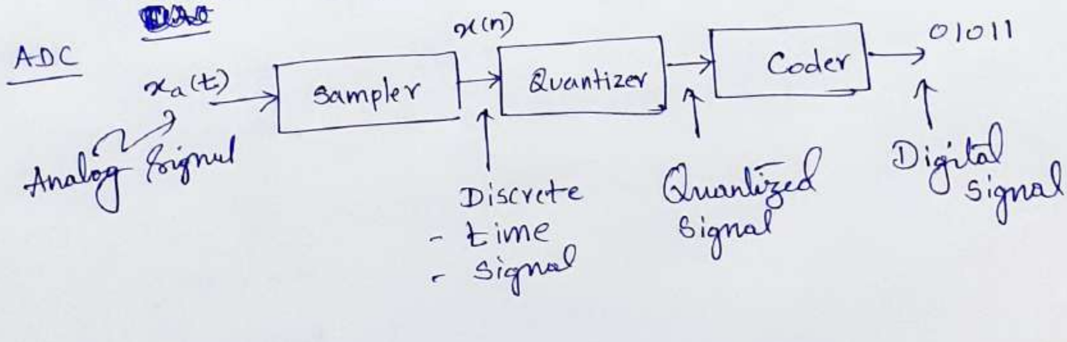

$f_s > 2W$

case (ii): $f_s < 2W$




Aliasing: Suppose a signal whose maximum frequency is W Hz is sampled at a sampling frequency $f < 2W$, the aliasing occures.
Aliasing in time domain implies improper reconstruction of samples.
Aliasing in Frequency domain implies overlapping of frequency spectrum as shown in figure

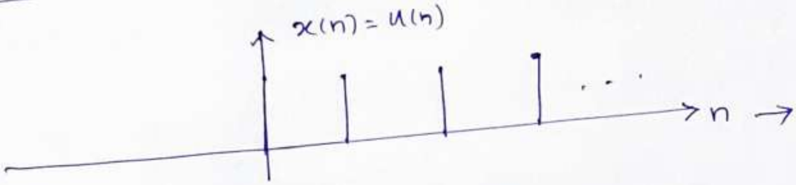
Analog to Digital Converter



# Digital to Analog Converter

1. Find the DTFT of an unit step function

Solution: $x(n) = u(n)$




We know that

$$DTFT[\delta(n)] = 1$$

$$\therefore \delta(n) \xleftrightarrow{DTFT} X(e^{j\omega}) = 1$$

From the property we can write

$$u(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \cdots$$

$$u(n) = \sum_{K=0}^{\infty} \delta(n-K)$$

put $n-K = m$

when $K=0$ $m=n$

when $K=\infty$ $m=-\infty$

$$u(n) = \sum_{m=-\infty}^{n} \delta(m)$$

From the Accumulation property

$$\sum_{m=-\infty}^{n} x(m) \xleftrightarrow{DTFT} \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{K=-\infty}^{\infty} \delta(\omega - 2\pi K)$$

put $x(m) = \delta(m)$

$$\therefore DTFT\left[\sum_{m=-\infty}^{n} x(m)\right] = \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0})[1]$$

for $-\pi < \omega \leq \pi$

But

$$\delta(n) \xleftrightarrow{DTFT} X(e^{j\omega}) = 1$$ (44)

$$\therefore D.T.F.T\left[\sum_{m=-\infty}^{n} \delta(m)\right] = \frac{1}{1-e^{-j\omega}} + \pi X(e^{j0})[1]$$

$$-\pi \leq \omega \leq \pi$$

$$= \frac{1}{1-e^{-j\omega}} + \pi\, \delta(e^{j0})$$

$$= \frac{1}{1-e^{-j\omega}} + \pi\, \delta(\omega)$$

(1) Consider $x(t) = 3\cos(100\pi t)$, determine the minimum Sampling rate required to avoid aliasing.

Solution:

$$x(t) = 3\cos(100\pi t)$$

$$x(t) = 3\cos(2\pi 50 t)$$

$$f_0 = 50\,Hz$$

Minimum Sampling rate $= 2f_0 = 100\,Hz$ to avoid Sampling

(2) Consider $x(t) = 3\cos(100\pi t)$, if the signal is Sampled at $f_s = 200\,Hz$ what is the discrete time signal obtained after Sampling

$$x(t) = 3\cos(100\pi t)$$

$$f_s = 200 \qquad T_s = \frac{1}{200}$$

$$x(n) = x(nT_s) = 3\cos(100\pi n T_s) = 3\cos\left(\frac{100 \times \pi \times n}{200}\right)$$

$$= 3\cos\left(\frac{\pi n}{2}\right)$$

(3) In the above problem, if the signal is sampled at $f_s = 75Hz$ what is the discrete time signal obtained after sampling

$$x(t) = 3\cos 2\pi 50 t \qquad f_s = 75 \quad T_s = \frac{1}{75}$$

$$x(n) = x(nT_s) = 3\cos 2\pi 50 \times nT_s$$

$$= \cos\left(2\pi \frac{\cancel{50}^{2} \times n}{\cancel{75}_{3}}\right)$$

$$= \cos\left(\frac{4\pi n}{3}\right)$$

$$= \cos\left(2\pi - \frac{2\pi}{3}\right)n$$

$$= \cos\left(\frac{2\pi}{3}n\right)$$

(4) what is the frequency $0 < f_0 < \frac{f_s}{2}$ of a sinusoid that yields samples identical to those obtained in the above problem

$$\cos\left(\frac{2\pi}{3}n\right) = \cos\left(2\pi f_0 nT_s\right) \qquad f_s = 75$$

$$T_s = \frac{1}{75}$$

$$\cos\left(\frac{2\pi}{3}n\right) = \cos\left(\frac{2\pi f_0 n}{75}\right)$$

$$\cos\left(\frac{2\pi}{3}n\right) = \cos\left(\frac{2\pi f_0 n}{75}\right)$$

$$\frac{\cancel{2\pi}}{3}\cancel{n} = \frac{\cancel{2\pi} f_0 \cancel{n}}{75}$$

$$25 \; \frac{\cancel{75}}{\cancel{3}} = f_0 \qquad \boxed{f_0 = 25}$$

when $y(t) = \cos(2\pi 25 t)$ is sampled at $f_s = 75Hz$ yields identical samples. Hence $f_0 = 50Hz$ is an alias of $f_0 = 25Hz$ for a sampling rate $f_s = 75Hz$.

(5) consider an analog signal (46)

$x(t) = 3\cos 50\pi t + 10 \sin 300\pi t - \cos 100\pi t$

What is the Nyquist rate?

Solution:

$x(t) = 3\cos 2\pi(25)t + 10 \sin 2\pi(150)t$

$- \cos(2\pi 50 t)$

$f_1 = 25\,Hz \quad f_2 = 150\,Hz \quad f_3 = 50\,Hz$

Highest frequency $= f_m = 150\,Hz$

Nyquist rate $f_s = 2f_m = 2\times150 = 300\,Hz$

(6) a) Consider an analog signal

$x(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$

a) What is the Nyquist rate?

Solution: $f_m = 6000\,Hz$

$f_s = 2f_m = 12000\,Hz$

Sampling frequency should be greater than 12 KHz.

(b) Assume that we sample $x(t)$ at the sampling rate of $f_s = 5000$ samples/sec. What is the expression for Discrete Time Signal?

$x(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$

$x(nT_s) = x(n) = 3\cos 2000\pi n T_s + 5\sin 6000\pi n T_s$

$+ 10\cos 12000\pi n T_s$

$$x(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$$ (47)

$f_s = 5000$ Samples/Sec

$$x(n) = 3\cos \frac{2000}{5000}\pi n + 5\sin \frac{6000\pi n}{5000} + 10\cos \frac{12000\pi n}{5000}$$

$$x(n) = 3\cos \frac{2\pi n}{5} + 5\sin \frac{6\pi n}{5} + 10\cos \frac{12\pi n}{5}$$

$$= 3\cos\left(\frac{2\pi n}{5}\right) + 5\sin\left(\frac{5\pi n + \pi n}{5}\right) + 10\cos\left(\frac{10\pi n + 2\pi n}{5}\right)$$

$$= 3\cos\left(\frac{2\pi n}{5}\right) + 5\sin\left(\pi n + \frac{\pi n}{5}\right) + 10\cos\left(2\pi n + \frac{2\pi n}{5}\right)$$

$$= 3\cos\left(\frac{2\pi n}{5}\right) + 5\sin\left(\pi n + \frac{\pi n}{5}\right) + 10\cos\left(\frac{2\pi n}{5}\right)$$

$$= 13\cos\frac{2\pi 5}{5} + 5\sin\left(\frac{6\pi n}{5}\right)$$

$$= 13\cos\frac{2\pi 5}{5} + 5\sin\left(\frac{10\pi n - 4\pi n}{5}\right)$$

$$= 13\cos\frac{2\pi n}{5} + 5\sin\left(2\pi n - \frac{4\pi n}{5}\right)$$

$$= 13\cos\left(\frac{2\pi n}{5}\right) - 5\sin\frac{4\pi n}{5}$$

$$= 13\cos\ \cos\frac{2\pi n}{5} - 5\sin\left(\frac{2\pi 2n}{5}\right)$$

(c) what is the analog $y(t)$ that we can reconstruct from the sample, if we use an ideal interpolation, where $f_s = 5000$ Hz (48)

$$x(n) = 13\cos 2\pi \frac{1}{5} n - 5\sin 2\pi \left(\frac{2}{5}\right) n$$

$$= 13\cos 2\pi \frac{1}{5} n \times \frac{5000}{5000} - 5\sin 2\pi \left(\frac{2}{5}\right) \times \frac{5000}{5000} n$$

$$= 13\cos\left(2\pi \left(\frac{\overset{1000}{\cancel{5000}}}{\cancel{5}}\right) \frac{n}{5000}\right) - 5\sin 2\pi \left(\frac{2}{\cancel{5}} \times \overset{1000}{\cancel{5000}}\right) \frac{n}{5000}$$

$$= 13\cos\left(2\pi (1000) \underbrace{nT_s}_{t}\right) - 5\sin 2\pi (2000) \underbrace{nT_s}_{t}$$

$$= 13\cos 2\pi 1000 t - 5\sin 2\pi 2000 t$$

1. The impulse response of an LTI System is given by $h(n) = \left(\frac{1}{2}\right)^n u(n)$
Find the response of the System for an input $x(n) = \left(\frac{1}{3}\right)^n u(n)$

$$x(n) = \left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{DTFT} X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{DTFT} H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$H(e^{j\omega}) \cdot X(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)} \times \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

put $e^{-j\omega} = v$

$$= \frac{1}{\left(1-\frac{1}{3}v\right)} \times \frac{1}{\left(1-\frac{1}{2}v\right)}$$

$$= \frac{K_1}{1-\frac{1}{3}v} + \frac{K_2}{1-\frac{1}{2}v}$$

$$K_1 = \frac{1}{1-\frac{1}{2}v}\Bigg|_{\cancel{v=1/3}\ v=3} = \frac{1}{1-\frac{3}{2}} = -2$$

$$K_2 = \frac{1}{1-\frac{1}{3}v}\Bigg|_{v=2} = \underline{3}$$

$$Y(e^{jw}) = \frac{-2}{1-\frac{1}{3}e^{-jw}} + \frac{3}{1-\frac{1}{2}e^{-jw}}$$

— Taking Inverse F.T. transform

$$y(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n)$$

---

2) Find $x(n)$ for the transform

$$X(e^{jw}) = 4\sin 5w + 2\cos w$$

Solution: Using Euler's formula we can write

$$X(e^{jw}) = 4\left[\frac{e^{j5w} - e^{-j5w}}{2j}\right] + 2\left[\frac{e^{jw} + e^{-jw}}{2}\right]$$

$$= -2je^{j5w} + 2je^{-j5w} + e^{jw} + e^{-jw}$$

Comparing this with

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$$

$$x(n) = \begin{cases} -2j & \text{at } n = -5 \\ +2j & \text{at } n = 5 \\ 1 & \text{at } n = 1 \\ 1 & \text{at } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

(3) Find the DTFT of

$$x(n) = a^n \cos \omega_0 n \, u(n)$$

$$= a^n \left[ \tfrac{1}{2} e^{j\omega_0 n} + \tfrac{1}{2} e^{-j\omega_0 n} \right]$$

$$x(n) = \tfrac{1}{2} a^n e^{j\omega_0 n} + \tfrac{1}{2} a^n e^{-j\omega_0 n}$$

$$\sum_{n=0}^{\infty} a^n e^{j\omega_0 n} \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \left( a e^{j\omega_0} \cdot e^{-j\omega} \right)^n$$

$$= \sum_{n=0}^{\infty} \left[ a e^{j\omega_0} \cdot e^{-j\omega} \right]^n$$

$$= \frac{1}{1 - a e^{j\omega_0} \cdot e^{-j\omega}}$$

$$\sum_{n=0}^{\infty} a^n e^{-j\omega_0 n} \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \left( a e^{-j\omega_0} \cdot e^{-j\omega} \right)^n$$

$$= \frac{1}{1 - a e^{-j\omega_0} \cdot e^{-j\omega}}$$

$$\text{DTFT}[x(n)] = \tfrac{1}{2} \, \text{DTFT}\left[ a^n e^{j\omega_0 n} \right] + \tfrac{1}{2} \, \text{DTFT}\left[ a^n e^{-j\omega_0 n} \right]$$

$$= \frac{1}{2} \times \frac{1}{\left[ 1 - a e^{-j\omega_0} \cdot e^{-j\omega} \right]} + \frac{1}{2} \times \frac{1}{1 - a e^{+j\omega_0} \cdot e^{-j\omega}}$$

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$$= \frac{1}{2}\left[\frac{1 - a e^{j\omega_0} e^{-j\omega} + 1 - a e^{-j\omega_0} e^{-j\omega}}{\left(1 - a e^{j\omega_0} e^{-j\omega}\right)\left(1 - a e^{-j\omega_0} e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1 - a e^{-j\omega} 2\cos\omega_0 + 1}{1 - a e^{j\omega_0} e^{-j\omega} - a e^{-j\omega_0} e^{-j\omega} + a^2 e^{-j2\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - 2a e^{-j\omega}\cos\omega_0 + 1}{1 - 2a e^{-j\omega}\cos\omega_0 + a^2 e^{-j2\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2a e^{-j\omega}\cos\omega_0}{1 - 2a e^{-j\omega}\cos\omega_0 + a^2 e^{-j2\omega}}\right]$$

$$= \frac{1 - a e^{-j\omega}\cos\omega_0}{1 - 2a e^{-j\omega}\cos\omega + a^2 e^{-j2\omega}}$$

2) Find the DTFT of
$x(n) = e^{j3n}$

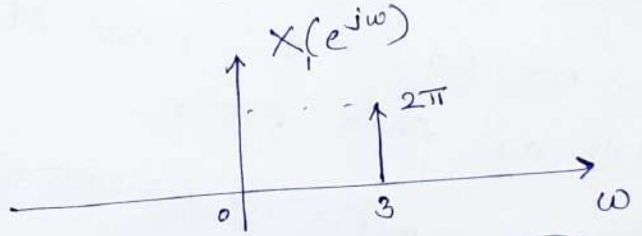
We know $A \xleftrightarrow{DTFT} 2\pi\,\delta(\omega)$

$$x_1(n) = 1 \xleftrightarrow{DTFT} X_1(e^{j\omega}) = 2\pi\,\delta(e^{j\omega})$$

$$e^{j\omega_0 n}x_1(n) \xleftrightarrow{DTFT} X_1(e^{j(\omega-\omega_0)})$$

$$e^{j3n}x_1(n) \xleftrightarrow{DTFT} X_1(e^{j(\omega-3)})$$

$$DTFT\left[e^{j3n}x_1(n)\right] = 2\pi\,\delta(\omega-3)$$



(1) Find the inverse DTFT of

$$X(e^{j\omega}) = 2\pi\delta(\omega-\omega_0)$$



$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\,\delta(\omega-\omega_0)\, e^{j\omega n}\, d\omega$$

1 at $\omega = \omega_0$

$$x(n) = e^{j\omega_0 n}$$

(2) compute the DTFT of the Signal $x(n) = \cos\left(0.2n\pi + \frac{\pi}{4}\right)$ and sketch the amplitude and phase spectra

<u>Solution</u>: $x(n) = \cos\left(0.2n\pi + \frac{\pi}{4}\right)$

$$x(n) = \frac{1}{2}e^{j\left(0.2n\pi + \frac{\pi}{4}\right)} + \frac{1}{2}e^{-j\left(0.2n\pi + \frac{\pi}{4}\right)}$$

We Know that

$$A \xleftrightarrow{DTFT} 2\pi A\,\delta(\omega)$$

$$Ae^{j\frac{\pi}{4}} \xleftrightarrow{DTFT} 2\pi A e^{j\frac{\pi}{4}}\delta(\omega)$$

put $A = \frac{1}{2}$

$$\frac{1}{2}e^{j\frac{\pi}{4}} \xleftrightarrow{DTFT} \cancel{2}\pi \times \frac{1}{\cancel{2}} e^{j\frac{\pi}{4}}\delta(\omega)$$

III'ly $\quad + \frac{1}{2}e^{-j\frac{\pi}{4}} \xleftrightarrow{DTFT} \pi e^{-j\frac{\pi}{4}}\delta(\omega)$

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\delta(\omega-\omega_0)$$

$$-\pi < \omega < \pi$$

$$e^{j0.2\pi n} \xleftrightarrow{DTFT} 2\pi\delta(\omega-0.2\pi)$$

$$\frac{1}{2}e^{\frac{j\pi}{4}}e^{j0.2\pi n} \xleftrightarrow{DTFT} \frac{1}{\cancel{2}}e^{\frac{j\pi}{4}} \times \cancel{2}\pi\delta(\omega-0.2\pi)$$

$$\frac{1}{2}e^{-\frac{j\pi}{4}}e^{-j0.2\pi n} \xleftrightarrow{DTFT} \frac{1}{\cancel{2}}e^{-\frac{j\pi}{4}} \times \cancel{2}\pi\delta(\omega+0.2\pi)$$

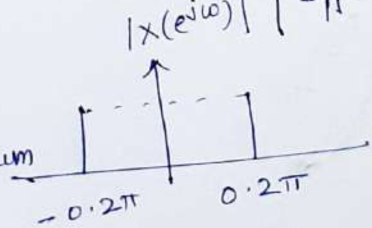
$$\therefore X(e^{j\omega}) = \pi e^{\frac{j\pi}{4}}\delta(\omega-0.2\pi) + \pi e^{-\frac{j\pi}{4}}\delta(\omega+0.2\pi)$$

$$= \pi\left[e^{\frac{j\pi}{4}}\delta(\omega-0.2\pi) + e^{-\frac{j\pi}{4}}\delta(\omega+0.2\pi)\right]$$

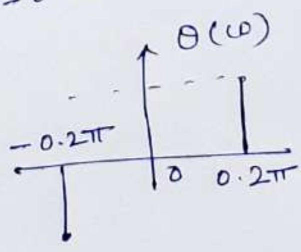
$$|X(e^{j\omega})| = \begin{cases} \pi & \text{at } \omega = 0.2\pi \\ \pi & \text{at } \omega = -0.2\pi \end{cases}$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{4} & \text{at } \omega = 0.2\pi \\ -\frac{\pi}{4} & \text{at } \omega = -0.2\pi \end{cases}$$

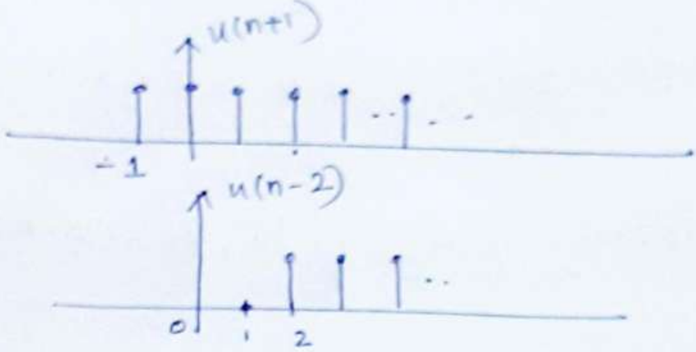
Amplitude Spectrum




phase Spectrum



(3) Compute the DTFT of the signal $x(n) = u(n+1) - u(n-2)$ and sketch the spectrum $X(e^{j\omega})$ over $-\pi \le \omega \le \pi$ 56

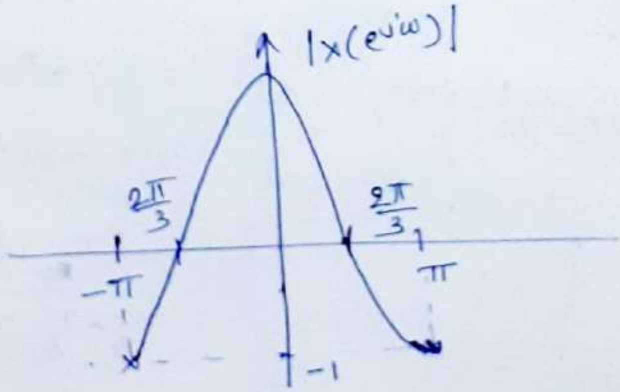



$$x(n) = \delta(n+1) + \delta(n) + \delta(n-1)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= 1 \times e^{+j\omega} + 1 \times e^{-j\omega} + 1$$

$$= 2\cos\omega + 1$$



(4) The DTFT of a real valued signal is $X(e^{j\omega})$. How is the DTFT of the following signal is related to $X(e^{j\omega})$ (57)

(a) $y(n) = x(-n)$

(b) $g(n) = x(n) * x(-n)$

(c) $s(n) = (-1)^n x(n)$

(d) $z(n) = [1 + \cos n\pi] x(n)$

(e) $b(n) = (-1)^{\frac{n}{2}} x(n)$

Solution:

(a) $y(n) = x(-n)$

$$Y(e^{+j\omega}) = X(e^{-j\omega})$$

(b) $g(n) = x(n) * x(-n)$

$$G(e^{j\omega}) = X(e^{j\omega}) \cdot X(e^{-j\omega})$$

$$= |X(e^{j\omega})|^2$$

(c) $s(n) = (-1)^n x(n) = (e^{j\pi})^n x(n)$

$$S(e^{j\omega}) = X(e^{j(\omega - \pi)})$$

From Frequency shifting property

$$e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} X(e^{j(\omega - \omega_0)})$$

(d) $z(n) = [1 + \cos n\pi]\, x(n)$ (58)

$$z(n) = [1 + (-1)^n]\, x(n)$$

$$= [1 + e^{j\pi n}]\, x(n)$$

$$z(n) = x(n) + x(n)e^{j\pi n}$$

$$Z(e^{j\omega}) = X(e^{j\omega}) + X(e^{j(\omega-\pi)})$$

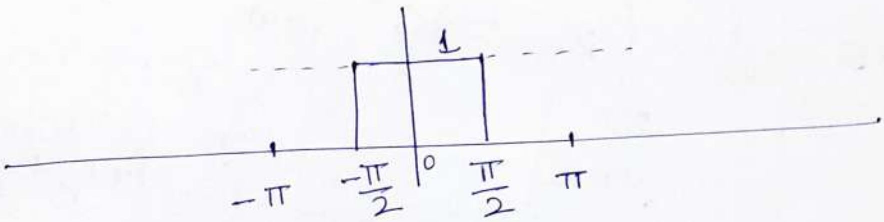
(e) $b(n) = (-1)^{\frac{n}{2}}\, x(n)$

$$b(n) = e^{j\frac{\pi}{2}n}\, x(n)$$

$$B(e^{j\omega}) = X(e^{j(\omega-\frac{\pi}{2})})$$

(5) Find the inverse DTFT of the $X(e^{j\omega})$ described over $|\omega| \le \pi$ (59)

$$X(e^{j\omega}) = \text{rect}\left(\frac{\omega}{\pi}\right)$$




$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\pi/2} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\Big|_{-\frac{\pi}{2}}^{\pi/2}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j}\right]$$

$$= \frac{\sin\frac{\pi}{2}n}{\pi n}$$

6 Find $x(n)$ given (68)

$$X(e^{j\omega}) = e^{-j4\omega} \text{ for } \frac{\pi}{2} < |\omega| < \pi$$

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} e^{-j4\omega} \cdot e^{+j\omega n}\, d\omega + \int_{\frac{\pi}{2}}^{\pi} e^{-j4\omega} \cdot e^{j\omega n}\, d\omega\right]$$

$$x(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} e^{j\omega(n-4)}\, d\omega + \int_{\frac{\pi}{2}}^{\pi} e^{j\omega(n-4)}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-4)}}{j(n-4)}\Bigg|_{-\pi}^{-\pi/2} + \frac{e^{j\omega(n-4)}}{j(n-4)}\Bigg|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{2\pi j(n-4)}\left[e^{j\left(\frac{-\pi}{2}\right)(n-4)} - \cancel{e^{-j\pi(n-4)}} + \cancel{e^{j\pi(n-4)}} - e^{j\frac{\pi}{2}(n-4)}\right]$$

$$= \frac{1}{\pi(n-4)2j}\left[e^{-\frac{j\pi}{2}n} \cdot \cancel{e^{+j2\pi}}^{\,1} - e^{\frac{j\pi}{2}n} \cdot \cancel{e^{-j\frac{\pi}{2}4}}^{\,1}\right]$$

$$= -\frac{1}{\pi(n-4)} \times \sin\left(\frac{n\pi}{2}\right) \text{ for } n \neq 4$$

$e^{-j4\omega} e^{j4\omega}$

For $n = 4$ $\quad x(4) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} 1 \cdot d\omega + \int_{\frac{\pi}{2}}^{\pi} 1 \cdot d\omega\right]$

$$x(4) = \frac{1}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{1}{2}$$

ans

$$x(n) = \begin{cases} -\dfrac{\sin n\pi/2}{\pi(n-4)} & \text{for } n \neq 4 \\ \dfrac{1}{2} & \text{for } n = 4 \end{cases}$$

(61)

(1) ~~Finndurethe~~ The following are the Fourier Transform of discrete time signals. Determine the Signal corresponding to each transform.

Ans. (a) $X(e^{j\omega}) = \cos^2\omega$

(b) $X(e^{j\omega}) = 1 - 2e^{-j3\omega} + 4e^{j2\omega} + 3e^{-j6\omega}$

Solution

(a) $X(e^{j\omega}) = \cos^2\omega$

$$X(e^{j\omega}) = \frac{1+\cos 2\omega}{2}$$

$$= \frac{1}{2} + \frac{1}{2}\left[\frac{e^{+j2\omega} + e^{-j2\omega}}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{4}e^{+j2\omega} + \frac{1}{4}e^{-j2\omega}$$

$$= \frac{1}{2} + \frac{1}{4}e^{+j2\omega} + \frac{1}{4}e^{-j2\omega}$$

$$= \frac{1}{4}e^{+j2\omega} + \frac{1}{2} + \frac{1}{4}e^{-j2\omega}$$

$$x(n) = \frac{1}{4}\delta(n+2) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-2)$$

(b) $X(e^{j\omega}) = 1 - 2e^{-j3\omega} + 4e^{j2\omega} + 3e^{-j6\omega}$ (62)

$$X(e^{j\omega}) = 1 - 2e^{-j3\omega} + 4e^{j2\omega} + 3e^{-j6\omega}$$

$$x(n) = \delta(n) - 2\delta(n-3) + 4\delta(n+2) + 3\delta(n-6)$$

2) Find the DTFT of the Sequence $x(n) = \delta(6-3n)$.
Also, plot the amplitude and phase spectra

Solution:

$$x(n) = \delta(6-3n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

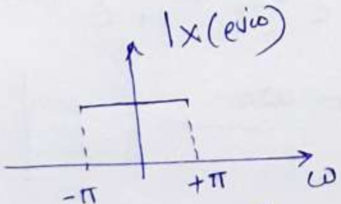
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(6-3n)\, e^{-j\omega n}$$

$$\delta(6-3n) = 1 \text{ iff. } 6-3n=0, \quad 3n=6, \; n=2$$

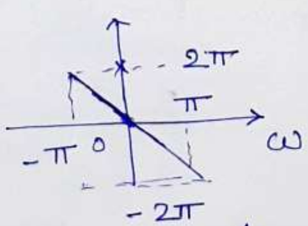
$$X(e^{j\omega}) = 1 \times e^{-j\omega 2} \quad \text{for } -\pi \le \omega \le \pi$$

$$|X(e^{j\omega})| = 1$$

$$\angle X(e^{j\omega}) = \angle e^{-j\omega 2} = -2\omega$$




Amplitude Spectrum of $x(n)$




phase Spectrum of $x(n)$

3) Find the DTFT of

(63)

$x(n) = \cos\omega_0 n \, u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \cos\omega_0 n \, e^{-j\omega n} . u(n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} e^{-j(\omega-\omega_0)n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j(\omega+\omega_0)n}$$

$$= \frac{1}{2} \times \left[\frac{1}{1-e^{-j(\omega-\omega_0)}} + \frac{1}{1-e^{-j(\omega+\omega_0)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(\omega+\omega_0)} + 1-e^{-j(\omega-\omega_0)}}{\left(1-e^{-j(\omega-\omega_0)}\right)\left(1-e^{-j(\omega+\omega_0)}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2 - e^{-j\omega} . e^{-j\omega_0} - e^{-j\omega} . e^{+j\omega_0}}{1 - e^{-j(\omega-\omega_0)} - e^{-j(\omega+\omega_0)} + e^{-j(\omega-\cancel{\omega_0}+\omega+\cancel{\omega_0})}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - e^{-j\omega} \times \frac{1}{2} \times 2\left[\frac{e^{+j\omega_0} + e^{-j\omega_0}}{2}\right]}{1 - e^{-j\omega} . e^{j\omega_0} - e^{-j\omega} . e^{-j\omega_0} + e^{-j2\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2e^{-j\omega}\cos\omega_0}{1 - e^{-j\omega} \times 2\cos\omega_0 + e^{-j2\omega}}\right]$$

$$= \frac{1 - e^{-j\omega}\cos\omega_0}{1 - 2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}$$

4) Find the DTFT of $x(n) = \sin \omega_0 n \, u(n)$

$$x(n) = \sin \omega_0 n \, u(n) = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] u(n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1}{2j} \sum_{n=0}^{\infty} e^{-j(\omega-\omega_0)n} - \frac{1}{2j} \sum_{n=0}^{\infty} e^{-j(\omega+\omega_0)n}$$

$$X(e^{j\omega}) = \frac{1}{2j} \times \left[\frac{1}{1-e^{-j(\omega-\omega_0)}}\right] - \frac{1}{2j}\left[\frac{1}{1-e^{-j(\omega+\omega_0)}}\right]$$

$$X(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1-e^{-j(\omega-\omega_0)}} - \frac{1}{1-e^{-j(\omega+\omega_0)}}\right]$$

$$X(e^{j\omega}) = \frac{1}{2j}\left[\frac{\cancel{1} - e^{-j(\omega+\omega_0)} + e^{-j(\omega-\omega_0)} \cancel{-1}}{\left(1-e^{-j(\omega-\omega_0)}\right)\left(1-e^{-j(\omega+\omega_0)}\right)}\right]$$

$$X(e^{j\omega}) = \frac{1}{\cancel{(2j)}}\left[\frac{\cancel{(2j)} e^{-j\omega}\left[\frac{e^{j\omega_0} - e^{-j\omega_0}}{2j}\right]}{1 - e^{-j\omega} \cdot e^{j\omega_0} - e^{-j\omega} \cdot e^{-j\omega_0} + e^{-j2\omega}}\right]$$

$$= \frac{e^{-j\omega} \sin \omega_0}{1 - e^{-j\omega} \times 2\left[\frac{e^{j\omega_0} - e^{-j\omega_0}}{2}\right] + e^{-j2\omega}}$$

$$X(e^{j\omega}) = \frac{e^{-j\omega} \sin \omega_0}{1 - 2e^{-j\omega} \cos \omega_0 + e^{-j2\omega}}$$

3.22) Consider a discrete-time LTI system described by $y(n) - \frac{1}{2}y(n-1) = x(n) + \frac{1}{2}x(n-1)$. Determine the frequency response $H(e^{j\omega})$

<u>Solution</u>:
$$Y(e^{j\omega}) - \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2}e^{-j\omega}X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 - \frac{1}{2}e^{-j\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{2}e^{-j\omega}\right]$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1}{2}\,\frac{e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

— Taking Inverse DTFT

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

# Problems on Continuous Time Fourier Transform

(1) Find the Fourier Transform of the Signal $x(t)$ defined below

$$x(t) = \int_{-\infty}^{t} e^{-a\tau} u(\tau)\, d\tau$$

Solution:

From the property we know

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

$$e^{-at} u(t) \xleftrightarrow{F.T.} X(j\omega) = \frac{1}{a+j\omega}$$

By applying the property

$$X(j\omega) = \frac{\frac{1}{a+j\omega}}{j\omega} + \pi X(0)\delta(\omega)$$

$$= \frac{1}{j\omega(a+j\omega)} + \frac{\pi}{a}\delta(\omega)$$

(2) For the signal $x(t)$ shown in the figure below, evaluate the following quantities without explicitly computing $X(j\omega)$ (67)

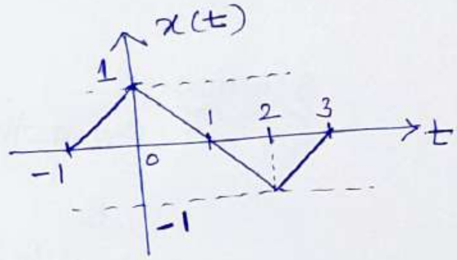
(a) $\int_{-\infty}^{\infty} X(j\omega)\, d\omega$

(b) $\int_{-\infty}^{\infty} |X(j\omega)|^2 . d\omega$

(c) $\int_{-\infty}^{\infty} X(\omega) e^{j2\omega} d\omega$

(d) $\angle X(j\omega)$

(e) $X(0)$




solution

(a) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} . d\omega$

$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, d\omega \Rightarrow \int_{-\infty}^{\infty} X(j\omega) d\omega = 2\pi\, x(0)$
$= 2\pi(1)$
$= 2\pi$

$\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi$

(b) $\int_{-\infty}^{\infty} |x(t)|^2 . dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 . d\omega$

$\int_{-\infty}^{\infty} |X(j\omega)|^2 . d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 . d\omega$

$= 2\pi \left[ \int_{-1}^{0} (t+1)^2 + \int_{0}^{2} (1-t)^2 . dt + \int_{2}^{3} (t-3)^2 dt \right]$

$= 2\pi \left[ \frac{4}{3} \right] = \frac{8\pi}{3}$

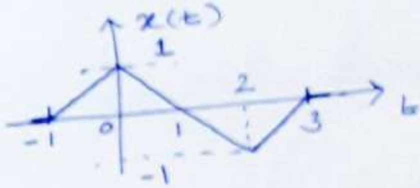
(c) $\int_{-\infty}^{\infty} X(j\omega)\, e^{j2\omega} \cdot d\omega$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} \cdot d\omega$$

$$x(2)\, 2\pi = \int_{-\infty}^{\infty} X(j\omega)\, e^{j2\omega} \cdot d\omega$$

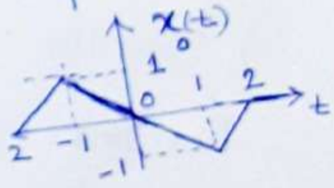
$$-2\pi = \int_{-\infty}^{\infty} X(j\omega)\, e^{j2\omega} \cdot d\omega$$

(d) $x(t)$ is odd and real Signal




$x(t) = x_0(t-1)$

$x_0(t) \xleftrightarrow{FT} X_0(j\omega)$




$X_0(j\omega)$ is purely imaginary ∴

$\angle X_0(j\omega) = \frac{\pi}{2}$ (+j)

$\frac{\pi}{2} \Rightarrow$ is because of "j" term

$x_0(t-1) = X_0(j\omega)\, e^{-j\omega}$

$\therefore \angle X_0(j\omega)\, e^{-j\omega} = \frac{\pi}{2} - \omega$

(e) $X(0)$: $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$X(j0) = X(0) = \int_{-1}^{3} x(t)\, dt = \int_{-1}^{3} x(t) \cdot dt = 0$

3) $X(j\omega) = \dfrac{j\omega}{(2+j\omega)^2}$ find $x(t)$

$$S(j\omega) = \frac{1}{(2+j\omega)^2} \xleftrightarrow{F.T} te^{-2t}u(t)$$

$$j\omega\, S(j\omega) \xleftrightarrow{F.T.} \frac{d}{dt}\left[s(t)\right]$$

$$j\omega\,(S(j\omega)) \xleftrightarrow{F.T} \frac{d}{dt}\left[te^{-2t}\right]u(t)$$

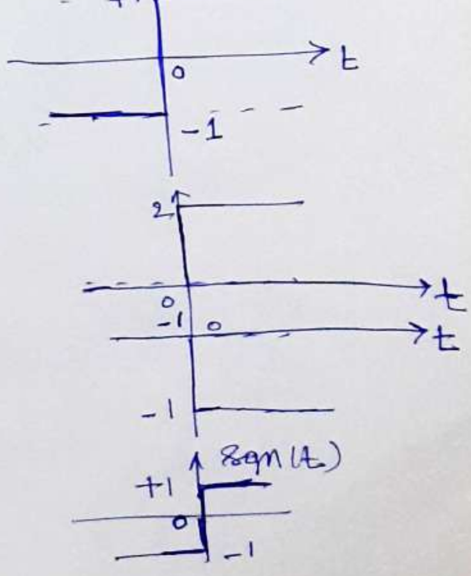
$$= \left[t\,e^{-2t}(-2) + e^{-2t}\right]u(t)$$

$$x(t) = \left[t\,e^{-2t}(-2) + e^{-2t}\right]u(t)$$

---

4) Find the fourier Transform of the Signum function, $x(t) = Sgn(t)$. Also draw the amplitude and phase spectra.

Solution :




$$x(t) = 2u(t) - 1$$

$$\frac{d}{dt}x(t) = 2\delta(t)$$

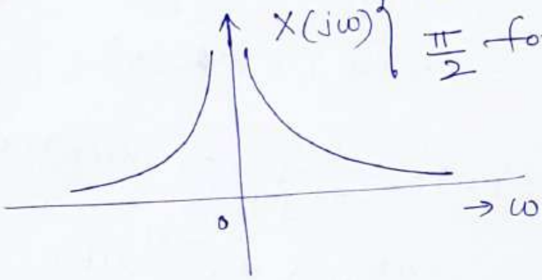
$$j\omega X(j\omega) = 2$$

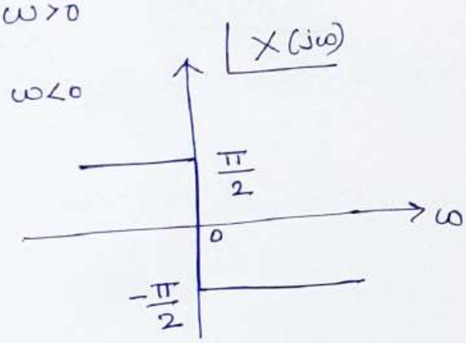
$$X(j\omega) = \frac{2}{j\omega}$$

$$X(j\omega) = \frac{2}{j\omega}$$

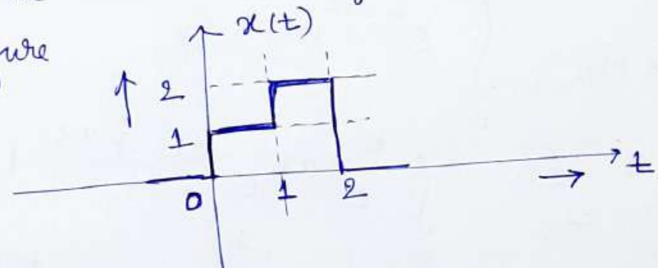
$$|X(j\omega)| = \frac{2}{\omega}$$

$$\angle X(j\omega) = \begin{cases} -\frac{\pi}{2} & \text{for } \omega > 0 \\ \frac{\pi}{2} & \text{for } \omega < 0 \end{cases}$$




5) Find the Fourier Transform of the pulse shown in figure




$$x(t) = \begin{cases} 1 & 0 < t < 1 \\ 2 & 1 < t < 2 \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^1 1\, e^{-j\omega t}\, dt + \int_1^2 2\, e^{-j\omega t}\, dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^1 + \left.\frac{2e^{-j\omega t}}{-j\omega}\right|_1^2$$

$$= \frac{1}{j\omega}\left[1 + e^{-j\omega} - 2e^{-j2\omega}\right]$$

6) Using convolution theorem, find the inverse F.T. of $X(j\omega) = \dfrac{1}{(a+j\omega)^2}$ (71)

Solution:

$$X(j\omega) = \frac{1}{a+j\omega} \times \frac{1}{a+j\omega}$$

$$= X_1(j\omega) \times X_2(j\omega)$$

$$X_1(j\omega) = \frac{1}{a+j\omega} \Rightarrow x_1(t) = e^{-at}u(t)$$

$$x(t) = e^{-at}u(t) * e^{-at}u(t)$$

$$x(t) = \int_0^t e^{-a\tau}u(\tau)\, e^{-a(t-\tau)}u(t-\tau)\, d\tau$$

$$= \int_0^t e^{-a\tau} \cdot e^{-at} \cdot e^{+a\tau}\, 1 . d\tau$$

$$= e^{-at}\int_0^t 1 . d\tau = t e^{-at}$$

---

7) Find the F.T. of the function $x(t) = te^{-2t}u(t)$

Solution:

$$e^{-at} . u(t) \xleftrightarrow{F.T.} \frac{1}{a+j\omega}$$

$$t . e^{-at}u(t) \xleftrightarrow{F.T} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$$

$$j\left[\frac{(a+j\omega)(0) - 1\times(j)}{(a+j\omega)^2}\right]$$

$$= \frac{1}{(a+j\omega)^2}$$

$$\therefore\ te^{-2t}u(t) \xleftrightarrow{F.T.} \frac{1}{(2+j\omega)^2}$$

$$F.T.\left[te^{-2t}u(t)\right] = \frac{1}{(2+j\omega)^2}$$

8) Find the F.T. of the unit step function, $x(t) = u(t)$

Solution:

$$sgn(t) = 2u(t) - 1$$

$$F.T[sgn(t)] = 2\,F.T[u(t)] - F.T[1]$$

$$\frac{2}{j\omega} = 2 \times F.T[u(t)] - 2\pi\delta(\omega)$$

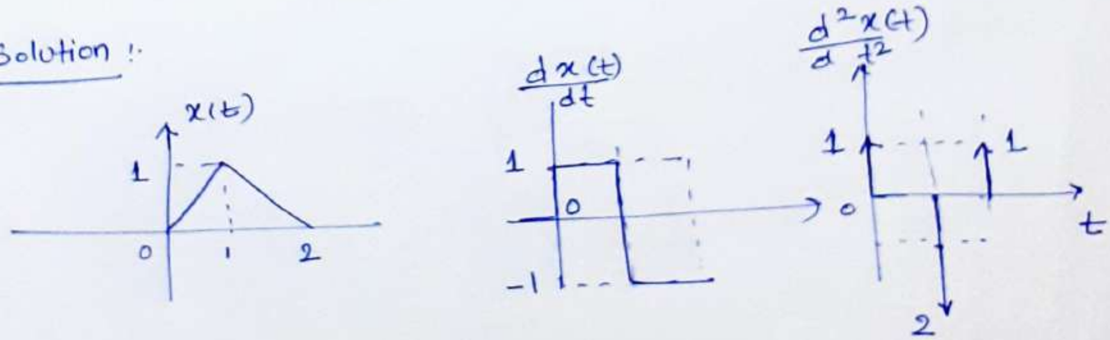
$$2\,F.T[u(t)] = \frac{2}{j\omega} + 2\pi\delta(\omega)$$

$$F.T[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

---

9) Find the F.T. of the triangular pulse $x(t)$


Solution:




$$\frac{d^2x(t)}{dt^2} = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

$$(j\omega)^2 X(j\omega) = 1 - 2e^{-j\omega} + e^{-j2\omega}$$

$$X(j\omega) = \frac{1}{(j\omega)^2}\left[1 - 2e^{-j\omega} + e^{-j2\omega}\right]$$

## 4.4 The Convolution Property

Consider $x(t) \xleftrightarrow{F.T.} X(j\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} X(jK\omega_0)\, e^{jK\omega_0 t}\, \omega_0 \to (1)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, d\omega$$

$$h(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} H(jK\omega_0)\, e^{jK\omega_0 t}\, \omega_0 \to (2)$$

$$\therefore\ y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} X(jK\omega_0)\, H(jK\omega_0)\, e^{jK\omega_0 t}\, \omega_0$$

$$\therefore\ y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, H(j\omega)\, e^{j\omega t}\, .d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, .d\omega \to (3)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \to (4)$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$Y(j\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] e^{-j\omega t}\, .dt \to (5)$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[ \underbrace{\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt}_{} \right] d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[ \int_{-\infty}^{\infty} h(x) e^{-j\omega(x+\tau)} dx \right] d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[ \underbrace{\int_{-\infty}^{\infty} h(x) e^{-j\omega x} dx}_{} \right] e^{-j\omega\tau} d\tau$$

put

$t - \tau = x$

At $t = \infty$

$x = \infty$

At $t = -\infty$

$x = -\infty$

$t = x + \tau$

$dt = dx$

$$\downarrow$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, H(j\omega)\, e^{-j\omega\tau} d\tau$$

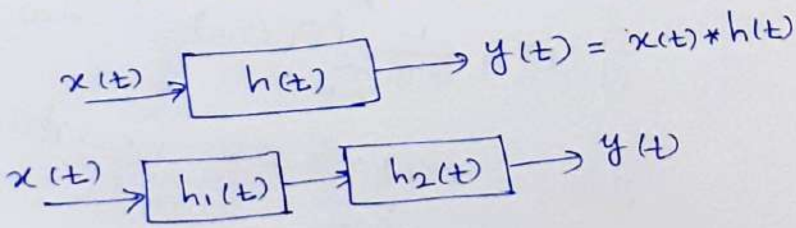
$$Y(j\omega) = H(j\omega) \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau}_{}$$

$$\downarrow$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

$$y(t) = h(t) * x(t) \xleftrightarrow{F.T.} Y(j\omega) = H(j\omega)\, X(j\omega)$$

$H(j\omega)$ = Frequency Response



$$x(t) \rightarrow \boxed{H_1(j\omega)} \rightarrow \boxed{H_2(j\omega)} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{H_1(j\omega)H_2(j\omega)} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$$

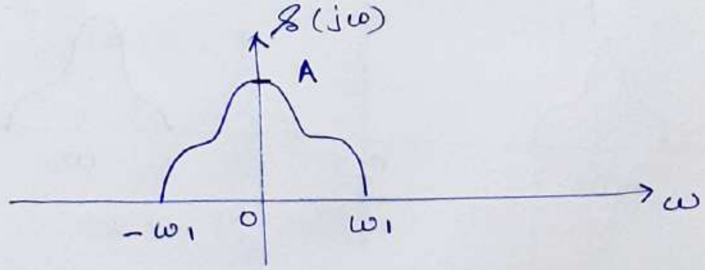
## 4.5 The Multiplication Property

If two signals $s(t)$ and $p(t)$ are multiplied in the time domain, their fourier transforms are convoluted in Frequency domain

$$r(t) = s(t)p(t) \xleftrightarrow{F.T.} R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$$

### Example 4.21

Consider $p(t) = \cos\omega_0 t$, the spectrum $S(j\omega)$ is as shown in figure. The IFT of $S(j\omega)$ is $s(t)$. Find the spectrum of $r(t) = s(t)p(t)$

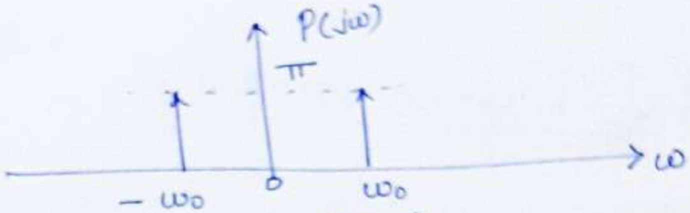


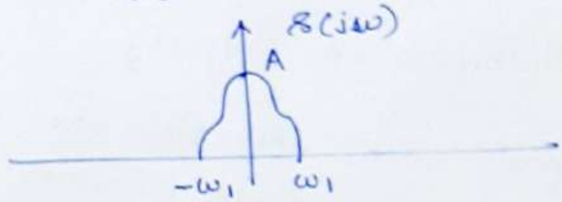
Solution:

$$r(t) = p(t) \cdot s(t)$$

$$p(t) = \cos \omega_0 t$$

$$P(j\omega) = \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$



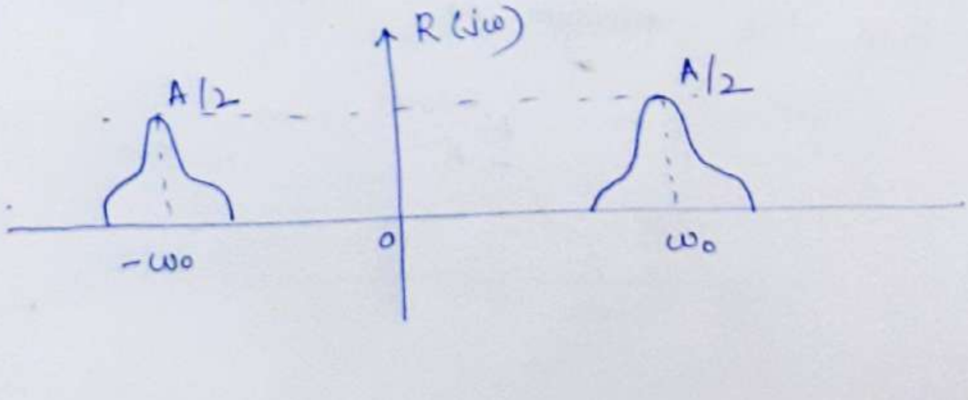




$$p(t)\,s(t) = r(t) \xleftrightarrow{F.T.} R(j\omega) = \frac{1}{2\pi}\left[P(j\omega) * S(j\omega)\right]$$

$$R(j\omega) = \frac{1}{2\cancel{\pi}}\left[\cancel{\pi}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] * S(j\omega)\right]$$

$$= \frac{1}{2}\left[S\left(j(\omega-\omega_0)\right) + S\left(j(\omega+\omega_0)\right)\right]$$

$$= \frac{1}{2} S\left(j(\omega-\omega_0)\right) + \frac{1}{2} S\left(j(\omega+\omega_0)\right)$$

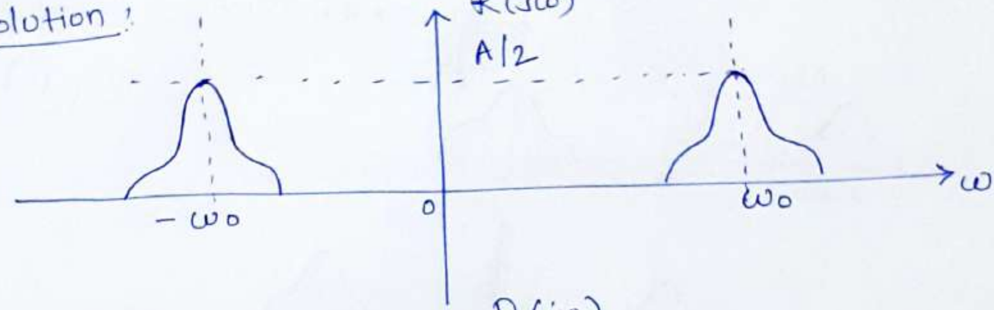


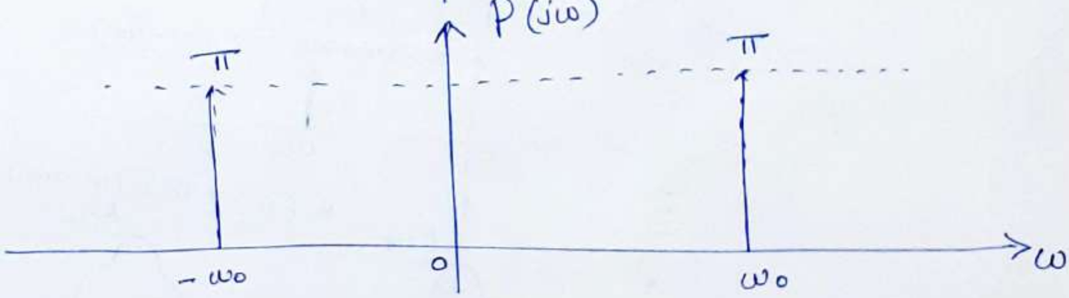
2) Example 4.22

Let $g(t) = r(t)\,p(t)$, where $p(t) = \cos\omega_0 t$

Where $p(t) = \cos\omega_0 t$, $R(j\omega)$ is as shown in figure. Sketch $P(j\omega)$ and $G(j\omega)$

Solution:



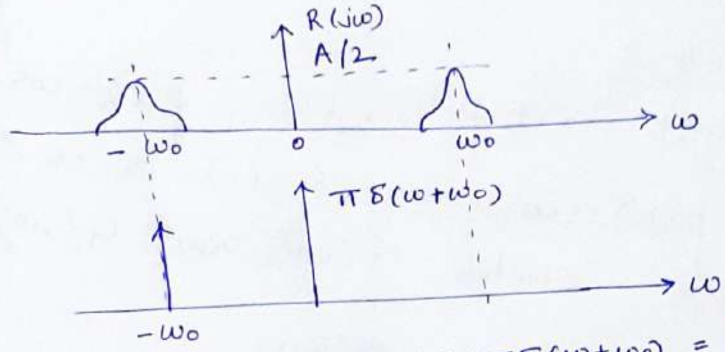

$$p(t) = \cos\omega_0 t$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

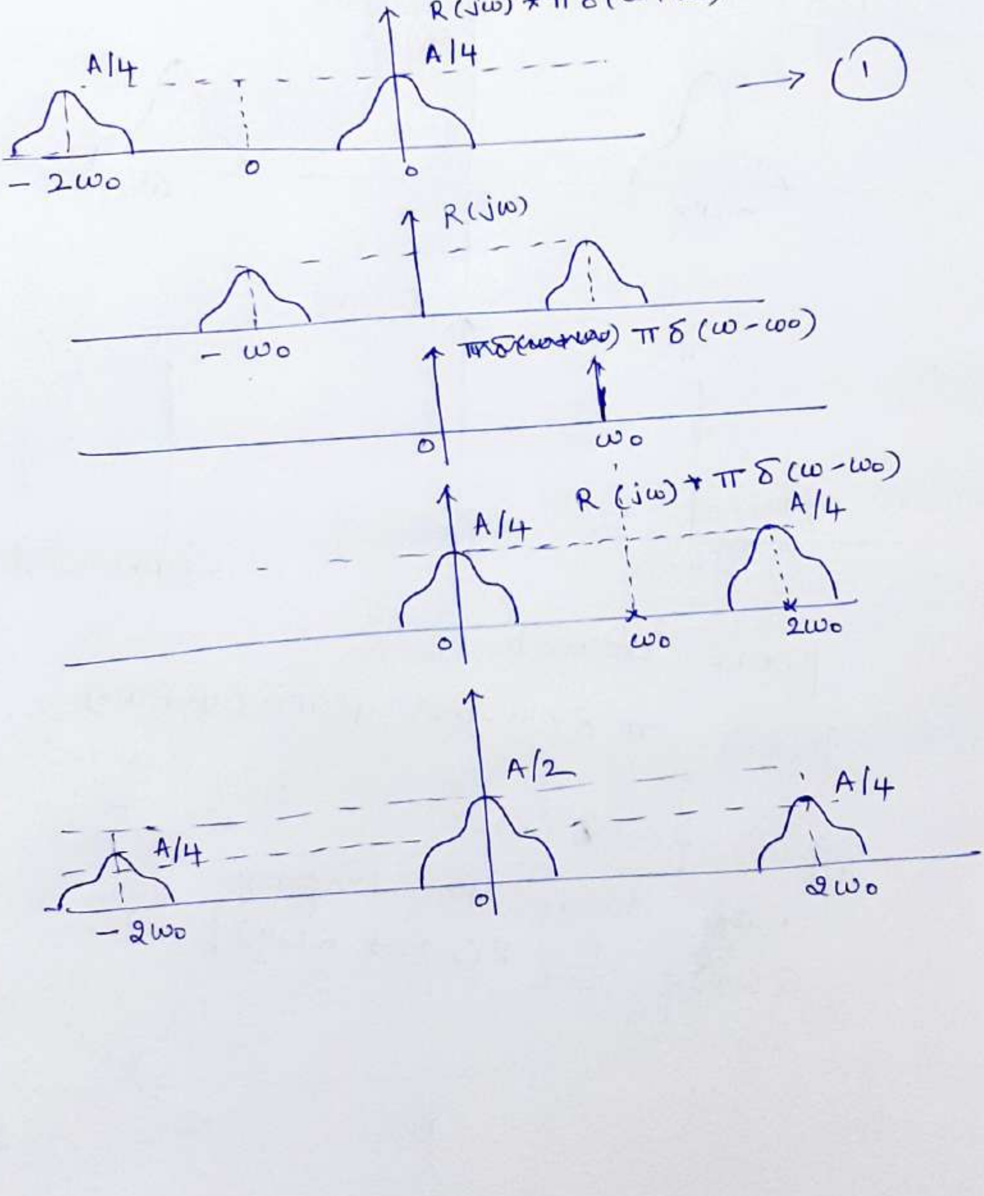
$$g(t) = p(t)\,r(t)$$

Using Multiplication Property

$$G(j\omega) = \frac{1}{2\pi}\left[P(j\omega) * R(j\omega)\right]$$




$R(j\omega) * \pi\delta(\omega+\omega_0) = R(j\omega) * \frac{2\pi}{2}\delta(\omega+\omega_0)$



Example 4.23

3) Find the $X(j\omega)$ if $x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$

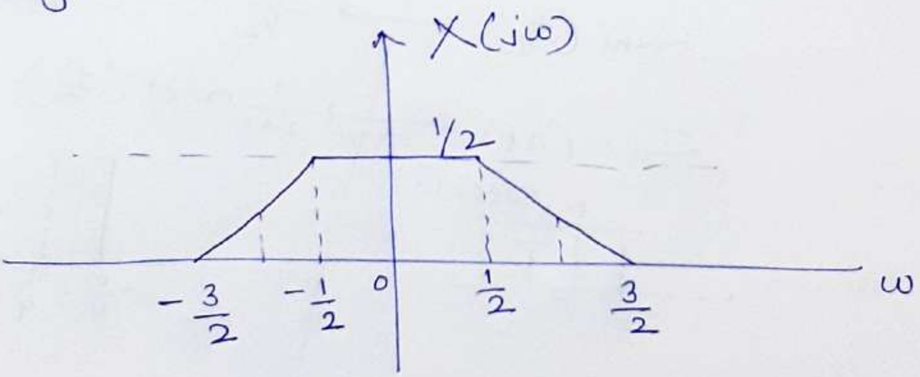
$$\cancel{x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\sin(}$$

$$x(t) = \pi \frac{\sin(t)}{\pi t} \times \frac{\sin(t/2)}{\pi t}$$

Applying Multiplication theorem

$$X(j\omega) = \frac{\cancel{\pi}}{2\cancel{\pi}} F\left[\frac{\sin(t)}{\pi t}\right] * F\left[\frac{\sin(t/2)}{\pi t}\right]$$

Noting that the Fourier transform of each sine function is a rectangular pulse, we can proceed to convolve those pulses to obtain $X(j\omega)$ as shown in figure

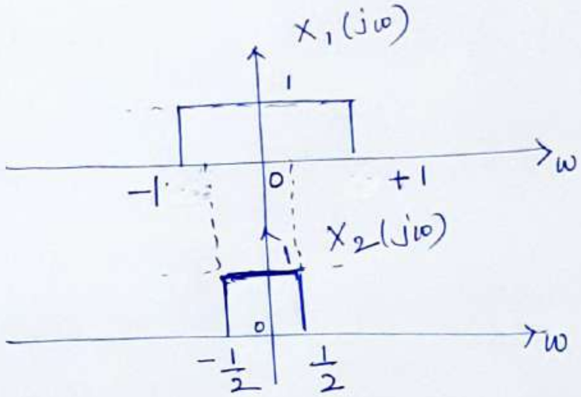


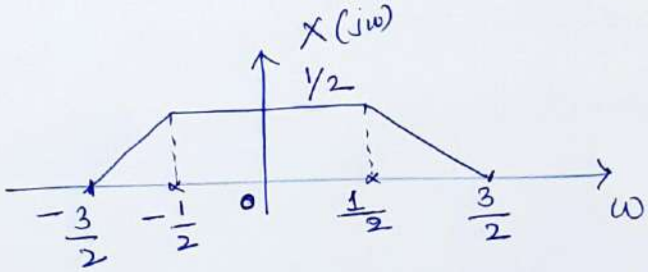
(80)

We know $\dfrac{\sin Wt}{\pi t} \xleftrightarrow{F.T} X_1(j\omega) = \begin{cases} 1 & \text{for } |\omega| < W \\ 0 & \text{otherwise} \end{cases}$

$\dfrac{\sin t}{\pi t} \xleftrightarrow{F.T.} X_2(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 1 \\ 0 \end{cases}$

$\dfrac{\sin \frac{1}{2}t}{\pi t} \xleftrightarrow{F.T.} X(j\omega) = \begin{cases} 1 & \text{for } |\omega| < \frac{1}{2} \\ 0 \end{cases}$



Unit - 4 - Fourier Transforms.

Chapter 4 - Continuous Time FT.

Sections 4.0, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 - Continuous Time FT

A. Oppenheim.

Chapter 5 - Discrete Time Fourier Transform.

Sections 5.0, 5.1, 5.2, 5.3, 5.6, 5.7 } A. Oppenheim
5.4, 5.5

Sampling. (Note along with Unit 1)

x — x — x — x — x —

4.0 Introduction.

-> Periodic signals were represented as linear combination of complex exponentials using Fourier Series representation (Fourier Series coefficients)

-> This can be extended to aperiodic signals as well.

-> For Periodic signals, the building blocks of complex exponentials are harmonically related.

-> whereas, for aperiodic signal, they are infinitesimally close in frequency, the representation in terms of linear combination takes the form of an Integral rather than a sum.

-> The resulting spectrum of co-efficients in this representation is called the Fourier Transform.

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(2)

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→ The synthesis integral itself, which uses these co-efficients to represent the signal as a linear combination of complex ~~coefficients~~ exponentials, is called the Inverse Transform.

→ Representation of ~~aperiodic~~ aperiodic signals in continuous time is one of the Fourier's most important contribution.

→ As per Fourier, an aperiodic signal can be viewed as a periodic signal with infinite period (so that the signal never repeats itself.)

→ As the period (T) increases, the fundamental frequency ($w_0$) decreases (since $w_0 = \frac{2\pi}{T}$), harmonically related components become closer in frequency ($k w_0 \Rightarrow w_0 \downarrow \; k w_0 \downarrow$)

→ As the period (T) becomes infinite, the frequency components form a continuum & the Fourier series sum becomes an Integral.

Recap: For C-T periodic signal

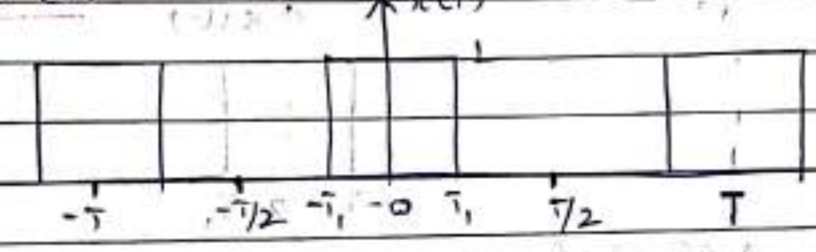
$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jkw_0 t}\, dt = \frac{1}{T}\int_T x(t)\, e^{-jk\left(\frac{2\pi}{T}\right)t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

.1 Representation of Aperiodic Signals :- The Continuous-Time Fourier Transform

2.1 Development of the F.T. representation of an aperiod signal.

Consider the following periodic signal with period (T).




$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

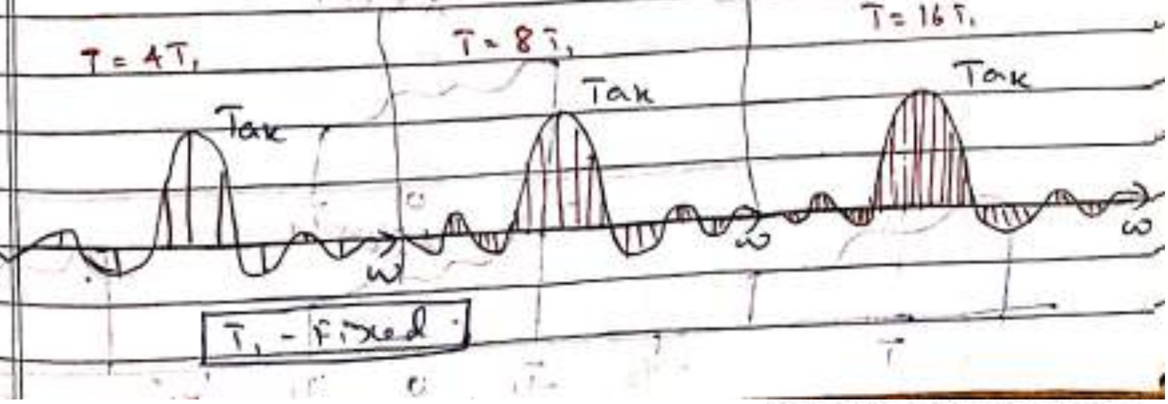
The Fourier Series co-efficients were derived as

$$a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T} \quad ; \; \omega_0 = \frac{2\pi}{T}$$

→ ~~with~~

→ An alternative way of interpreting the above relation is through an envelope function

$$T a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0} = \frac{2 \sin \omega T_1}{\omega} \Bigg|_{\omega = k\omega_0}$$



(1X) → with $\omega = k\omega_0$, it can be thought of as a continuous variable, the fn. $\frac{2 \sin \omega T_1}{\omega}$ represents the envelope of $Ta_k$.

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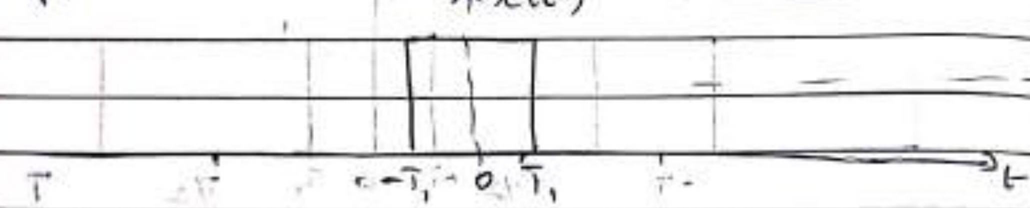
From the figures, we observe that as

→ T increases ; $\omega_0 = \frac{2\pi}{T}$ decreases, the envelope is sampled ~~closer~~ with a closer & closer spacing

→ As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

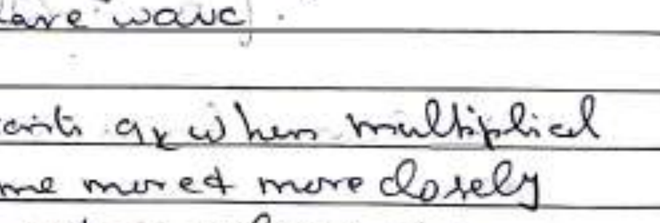


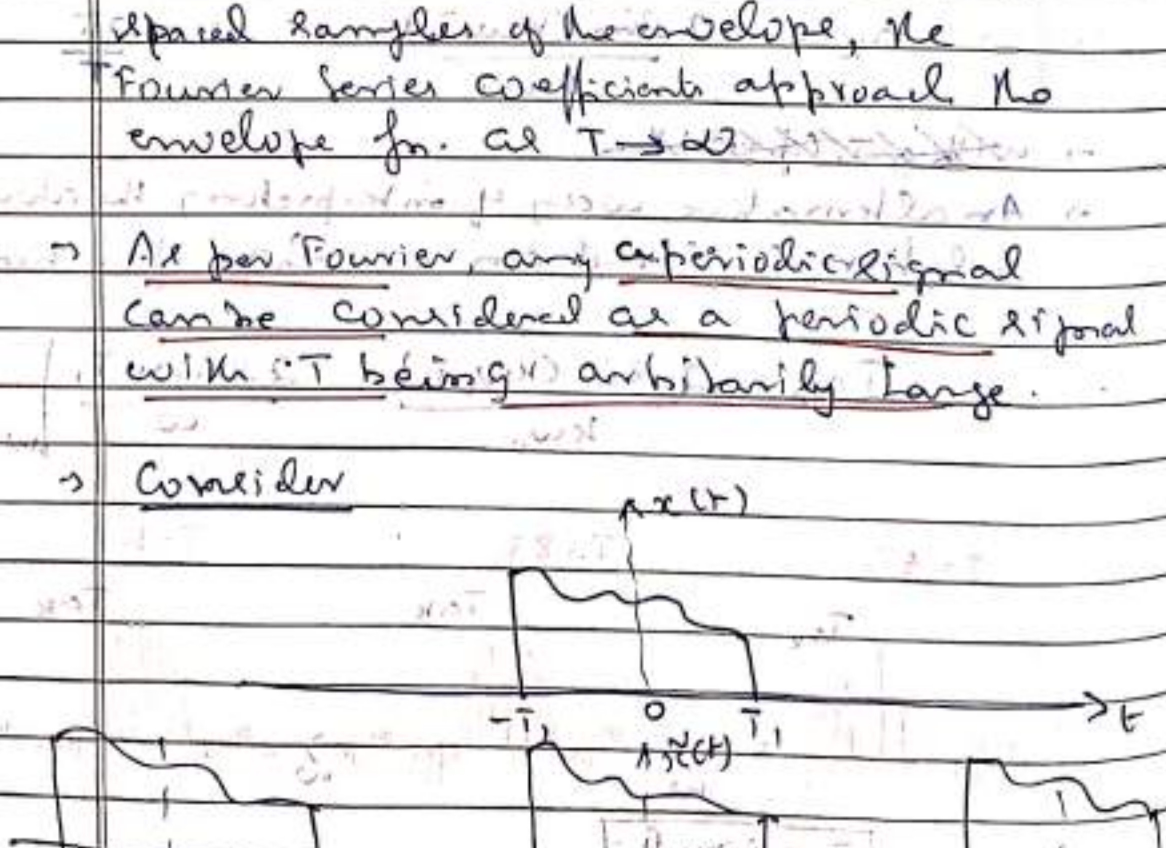
→ All that remains in the time domain is an aperiodic signal corresponding to one period of the square wave.

→ Fourier co-efficients $a_k$ when multiplied with T, become more & more closely spaced samples of the envelope, the Fourier series coefficients approach the envelope fn. as $T \to \infty$

→ As per Fourier, any aperiodic signal can be considered as a periodic signal with T being arbitrarily large.

→ Consider





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For some $T_1$,

$x(t) = 0$ if $|t| > T_1$

→ we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period.

→ If we choose $T$ to be arbitrarily large, $\tilde{x}(t)$ is identical to $x(t)$; as $T \to \infty$ $\tilde{x}(t) = x(t)$ for any $t$.

→ Consider the F.S representation of $\tilde{x}(t)$ over the interval $-T/2$ to $T/2$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \quad ; \omega_0 = \frac{2\pi}{T}$$

→ Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$ & $x(t) = 0$ outside the interval, we can re-write.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt$$

More generally

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} \, dt.$$

→ Defining the envelope $\boxed{X(j\omega) \text{ as } Ta_k}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

Note $\omega = k\omega_0$

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Date_____ Page_____

Then

$$a_k = \frac{1}{T} X(j\omega) = \frac{1}{T} X(jk\omega_0)$$

We can now express $\tilde{x}(t)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

→ Since $\frac{2\pi}{T} = \omega_0 \Rightarrow T = \frac{2\pi}{\omega_0}$ or $\frac{1}{T} = \frac{\omega_0}{2\pi}$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$$

→ As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$

→ ~~As~~ $\omega_0 \to 0$ as $T \to \infty$, the R.H.S of the above eqn. passes to an integral.

→ Hence as $T \to \infty$ & $\tilde{x}(t) \to x(t)$ then

Inv. F.T

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \cdot d\omega}$$

~~dw~~ $\omega_0 \to d\omega$ since $\omega_0$ is infinitesimally small.

→ Also

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

Fourier Integral

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$\boxed{\text{Aliter}}$ As for CTFS pg 326-327

$$\lim_{T\to\infty} T\cdot X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

As $T\to\infty$, $\omega_0 = \frac{2\pi}{T} \to 0$

The line spectrum becomes a continuous spectrum

→ If we replace $\omega_0$ by an infinitesimally small quantity $d\omega \to 0$, the discrete frequency $k\omega_0$ may be replaced by continuous frequency $\omega$.

Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

Now

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} T\cdot a_k\, e^{jk\omega_0 t}\cdot \frac{1}{T}$$

$$= \sum_{k=-\infty}^{\infty} T a_k\, e^{jk\omega_0 t}\cdot \frac{\omega_0}{2\pi}$$

As $T\to\infty$, the summation becomes an integral. $\omega_0 = d\omega$, $k\omega_0 = \omega$. then

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\cdot d\omega$$

fnally

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If we consider $\omega = 2\pi f$, then

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Also

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df.$$

$X(f) = X(2\pi f)$ & not $\omega$ replaced by $f$ in $X(j\omega)$ ∴

Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

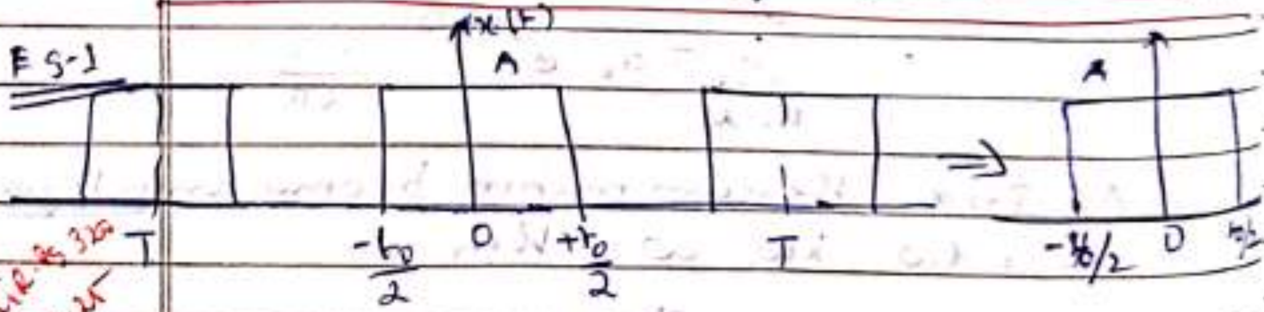
Fourier Transform or Fourier Integral.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Inverse Fourier Transform or Synthesis eqn

ES-1

GR-3.326
ES-3.25




$$a_k = \frac{1}{T} A t_0 \,\text{sinc}\left\{\frac{k\omega_0 t_0}{2\pi}\right\}$$

Recall $\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$

Then

$$X(j\omega) = T \cdot a_k$$

$$= A t_0 \,\text{sinc}\left\{\frac{k\omega_0 t_0}{2\pi}\right\}$$

since $k\omega_0 = \omega$

$$X(j\omega) = A t_0 \,\text{sinc}\left(\frac{\omega t_0}{2\pi}\right)$$

**Laplace Transform Pair**

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(s)\, e^{st}\, ds.$$

**Important Inference:**

Consider an aperiodic signal $x(t)$ & its Fourier transform is $X(j\omega)$. If this considered to be periodic ($\tilde{x}(t)$) then

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0 = \omega}$$

→ $X(j\omega)$ - Fourier Spectrum of $x(t)$

Fourier coefficients of $\tilde{x}(t)$ are proportional to samples of the Fourier Transform of one period of $\tilde{x}(t)$.

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## 4.1.2 Convergence of Fourier Transforms -

As in CTFS, we have the conditions formulated by Dirichlet

① Signals of finite Total Energy i.e

$$E_{\infty} = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

② **Dirichlet Conditions:**

(a) $x(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

(b) $x(t)$ have a finite number of maxima & minima within any finite interval.

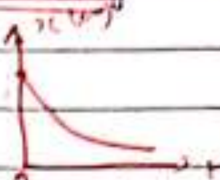
(c) $x(t)$ have a finite number of discontinuities within any finite interval.

4.1.3

Eg-1

## 4.1.3 Examples of Continuous-Time Fourier Transforms.

Eg. 1 Given $x(t) = e^{-at} \cdot u(t)$ $a > 0$




$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} \, dt \quad \text{since } u(t) = 1 \; t \geq 0$$

$$= \int_0^\infty e^{-(a+j\omega)t} \, dt$$

$$= -\frac{1}{(a+j\omega)} e^{-(a+j\omega)t} \Big|_0^\infty$$

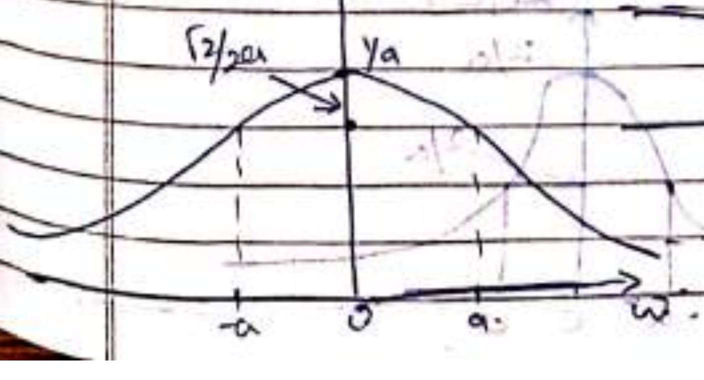
$$= -\frac{1}{(a+j\omega)} [0 - 1] = \frac{1}{a+j\omega} \; ; a > 0$$

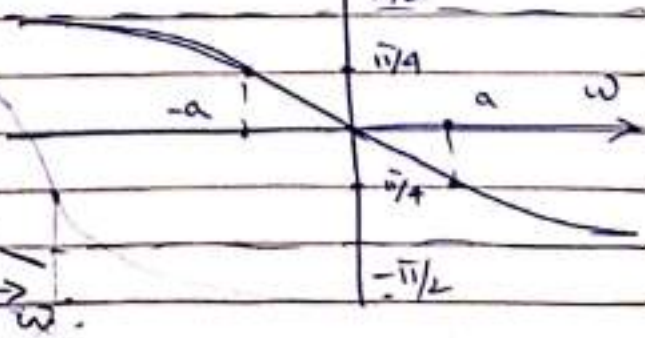
$X(j\omega)$ is a complex quantity.

we can plot $X(j\omega)$ as a fn. of $\omega$; for which we need its magnitude & Phase.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \; ; \; \angle X(j\omega) = -\tan^{-1}(\omega/a)$$

| $\omega$ | $\lvert X(j\omega)\rvert$ | $\angle X(j\omega)$ |
|---|---|---|
| 0 | $1/a$ | 0 |
| $+a$ | $\sqrt{2}/2a$ | $-\pi/4$ |
| $-a$ | $\sqrt{2}/2a$ | $+\pi/4$ |



②

Frequency Spectrum

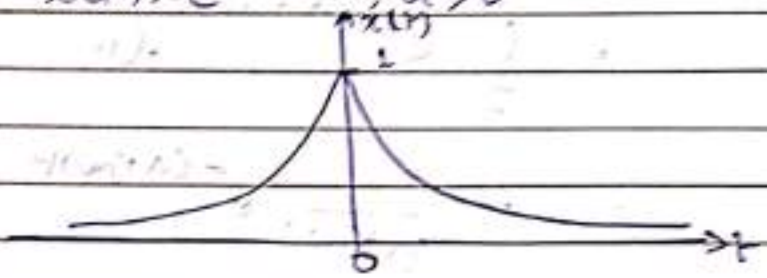
magnitude Spectrum → Even symmetry

Phase Spectrum → Odd symmetry

Note: If a is complex, then x(t) is absolutely integrable as long as $Re\{a\} > 0$ & then

$$X(j\omega) = \frac{1}{a+j\omega} \quad ; \quad Re\{a\} > 0$$

Eg 4.2 Let $x(t) = e^{-a|t|}$ ; $a > 0$



The F.T of the above signal.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

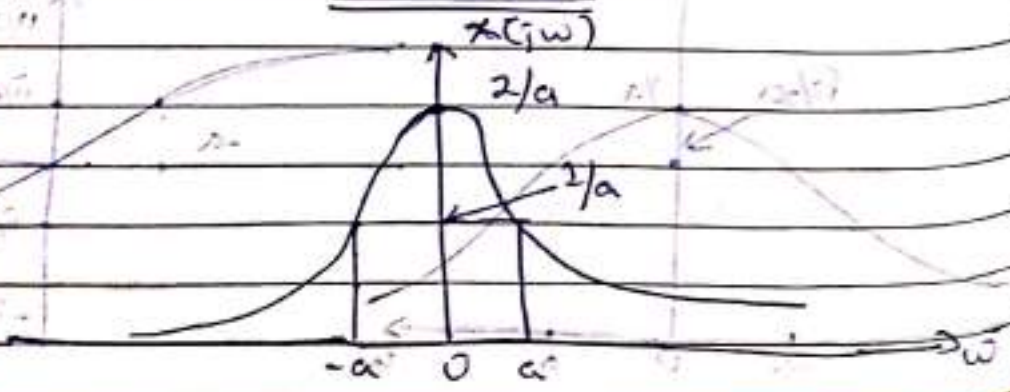
$$= \int_{-\infty}^{0} e^{at} \cdot e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[ \frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_{-\infty}^{0} + \left[ \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty}$$

$$= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)} = \frac{2a}{a^2+\omega^2}$$

In this case $X(j\omega)$ is real.



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**Eg. 1.3** Let $x(t) = \delta(t)$ - unit impulse.

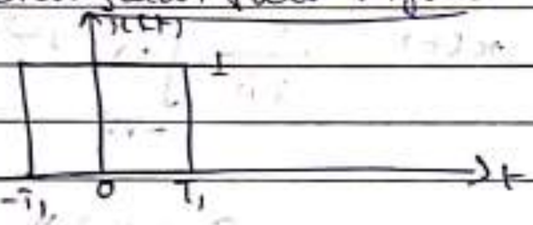
Then

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = 1 \quad \text{since } \delta(0) = 1.$$

This means the unit impulse has a F.T consisting of equal contributions at all frequencies.

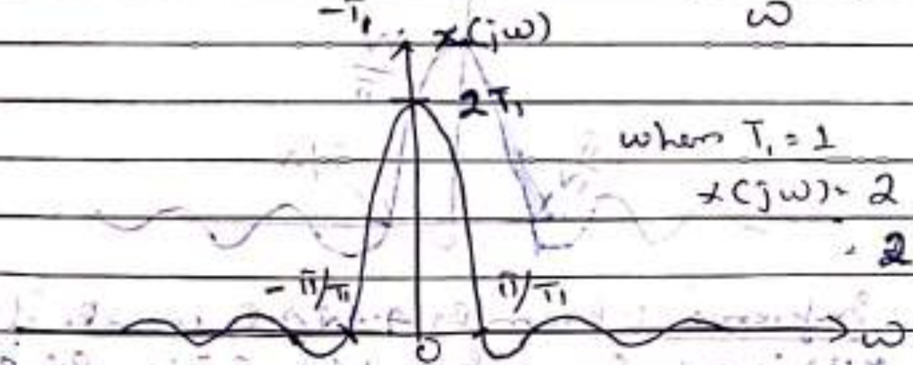
**Eg. 1.4** Consider the rectangular pulse signal.




$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

Then

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \frac{2 \sin \omega T_1}{\omega}$$




when $T_1 = 1$

$$X(j\omega) = 2\, Sa(\omega) = \frac{2 \sin(\omega)}{\omega}$$

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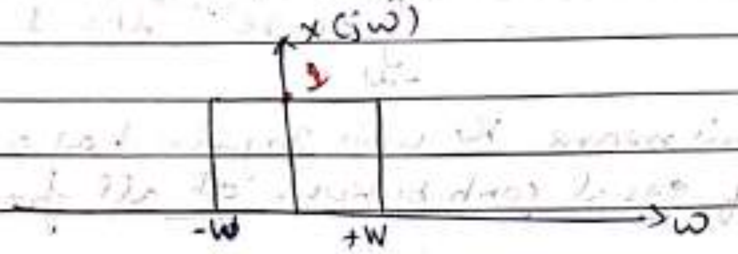
Eg 4.5 Consider a signal $x(t)$ whose F.T is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

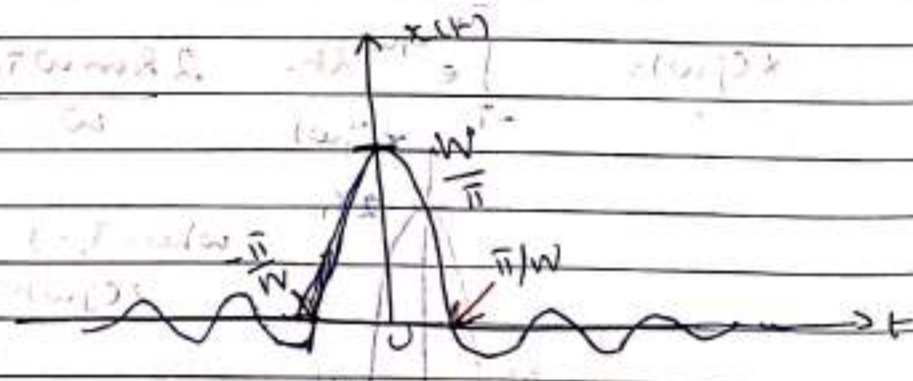



Using the synthesis eqn.

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} (1)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W}$$

$$= \frac{1}{jt}\cdot\frac{1}{2\pi}\left[e^{j\omega t}\right]_{-W}^{W} = \frac{1}{j2\pi t}\left[e^{jWt} - e^{-jWt}\right]$$

$$= \frac{\sin Wt}{\pi t}$$




Inference :- From Eq 4.4 & 4.5, we see if the $x(t)$ is a pulse, its F.T is $\frac{\sin a\theta}{b\theta}$

& if F.T is a pulse, then $x(t)$ is of the form $\frac{\sin a\theta}{b\theta}$

→ This is due to Duality Property.

**Recap**

$$\text{Sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

**Consider**

$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega} = 2T_1\,\text{Sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$||^{ly}$

$$x(t) = \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\text{Sinc}\left(\frac{Wt}{\pi}\right)$$

**Important Inference**

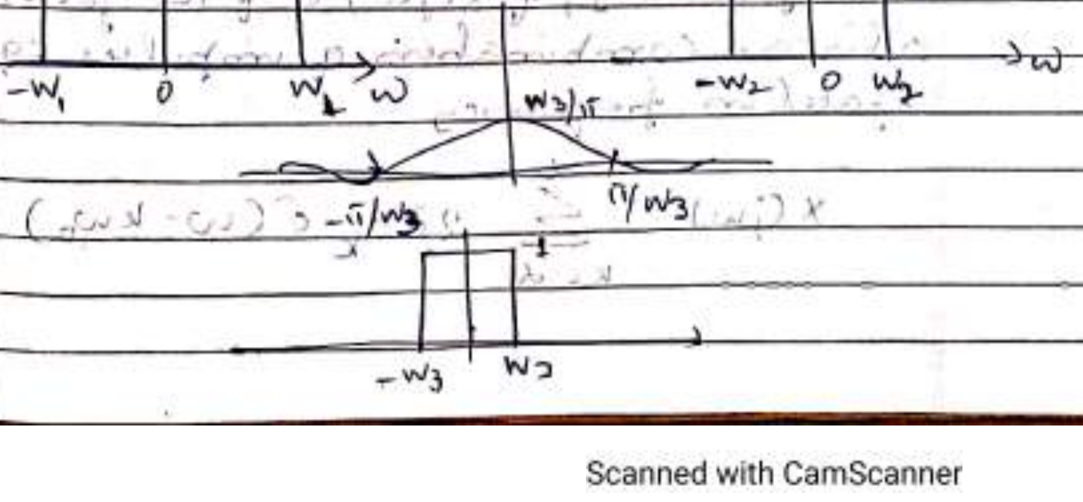
→ As $W$ becomes broader, the first lobe of $x(t)$ becomes narrower.

→ When $W \to \infty$, $x(t)$ tends to become an Impulse.

$||^{ly}$

→ As $T_1$ is increased, $X(j\omega)$ becomes more sharper.

Inverse relation between Time & frequency domain.



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## 1.2 The Fourier Transform for Period Signals.

- F. Transform of periodic signals are constructed from their F.S. co-efficients.

- The resulting transform consists of train of impulses in the frequency domain.

→ The areas of these impulses are proportional to the Fourier Series co-efficients.

Consider a signal $x(t)$ whose F.T is $X(j\omega)$ that is a single impulse of area $2\pi$ at $\omega = \omega_0$

$$X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$$

To determine $x(t)$ we use I.F.T relation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$= e^{j\omega_0 t}$$

→ More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega - k\omega_0)$$

Then applying inverse F.T.

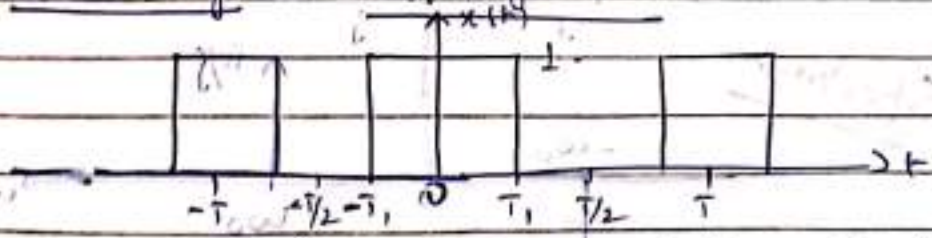
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

→ Therefore, the F.T. of a periodic signal with F.S. co-efficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies & for which the area of the impulse at the $k^{th}$ harmonic frequency $k\omega_0$ is $2\pi$ times the $k^{th}$ Fourier Series co-efficient $a_k$.

→ Then, if we know the F.S. coefficients $a_k$ of a periodic signal $x(t)$ having a fundamental frequency $\omega_0$, then its F.T is given by

$$\boxed{\begin{aligned} X(j\omega) &= 2\pi \sum_{k=-\infty}^{+\infty} a_k \, \delta(\omega - k\omega_0) \\ &\text{or} \\ &= \sum_{k=-\infty}^{+\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0) \end{aligned}}$$

Eg. 4.6 Consider the following Signal.




For this signal the Fourier series coefficients were derived as

$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k} = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$\omega_0 = \frac{2\pi}{T}$$

F.T. of this signal would be

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\cancel{\pi} \cdot \frac{\sin k\omega_0 T_1}{\cancel{\pi} k} \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$

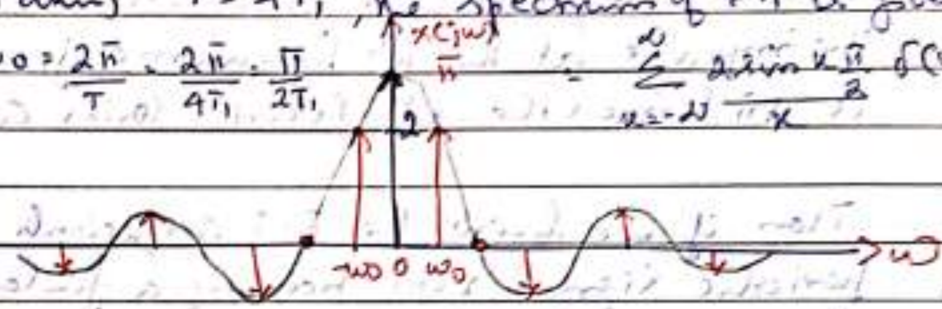
Eg 4.8

$$= \sum_{k=-\infty}^{\infty} \frac{2\sin k\frac{\pi}{2T_1} \cdot T_1}{k} \delta(\omega - k\omega_0)$$

Taking $T = 4T_1$, the spectrum of F.T. is given

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4T_1} = \frac{\pi}{2T_1}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\sin k\frac{\pi}{2}}{\pi k} \delta(\omega - k\omega_0)$$




Eg 4.7 Let $x(t) = \sin \omega_0 t$

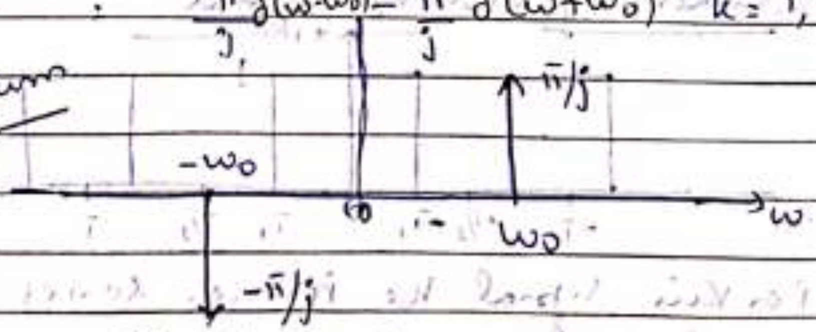
F.S. coefficients

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j} \; ; \; a_k = 0 \text{ for } k \neq 1 \text{ or } -1$$

$$X(j\omega) = \sum 2\pi \cdot \frac{1}{2j} \delta(\omega - \omega_0) + 2\pi\left(\frac{-1}{2j}\right)\delta(\omega + \omega_0)$$

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0) \quad k = 1, -1$$

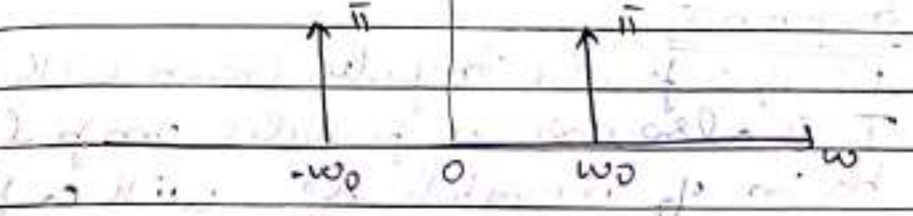
Spectrum




If $x(t) = \cos \omega_0 t$

$$a_1 = a_{-1} = \frac{1}{2} \; ; \; a_k = 0 \text{ for } k \neq 1 \text{ or } -1$$

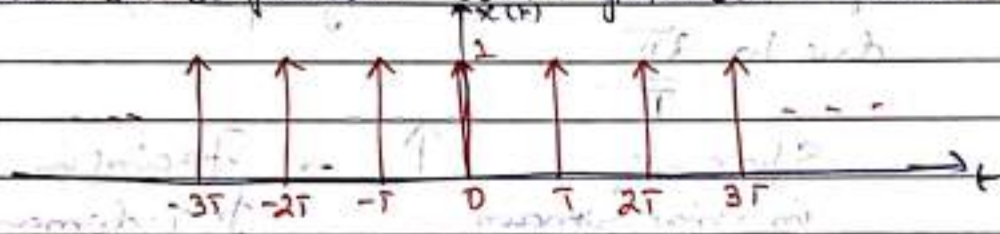
$$X(j\omega) = 2\pi \times \frac{1}{2} = \pi$$




Eg 4.8 Consider an impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

which is periodic with period T as indicated.




<u>Fourier Series co-efficients</u>

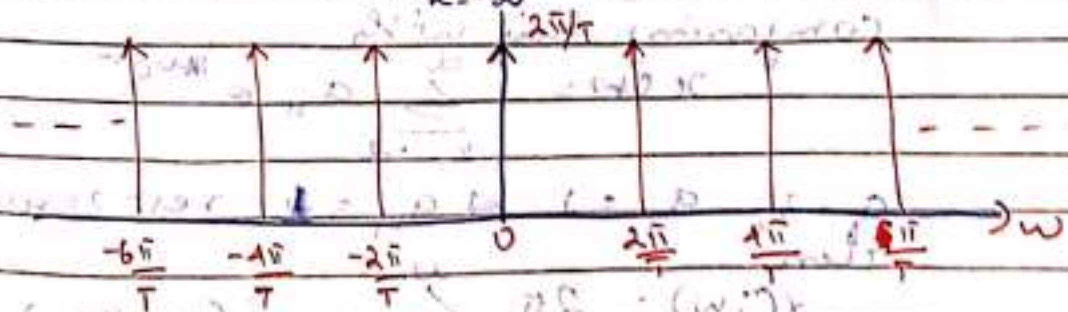
$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}$$

Every F.S. co-efficient for this periodic impulse train has the same value $\frac{1}{T}$.

Then

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0) \quad ; \omega_0 = \frac{2\pi}{T}$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right).$$




<u>Inference</u>

<u>Inference</u>

→ F.T of a periodic Impulse Train with period T is also ~~a~~ a periodic impulse train of magnitude $\frac{2\pi}{T}$ with a period of $\frac{2\pi}{T}$.

→ As T increases in time domain, the period reduces in frequency domain due to $\frac{2\pi}{T}$

Spacing in Time domain ↑ ⇒ Spacing in freq. domain ↓

→ <u>Alternate nomenclature</u>

F.T Dirac Comb is also Dirac Comb

Dirac Comb → A periodic Impulse train

Eg 3.68 Pg 386 (A.O) Consider a periodic signal

$$x(t) = 3 + 2\cos(10\pi t).$$

<u>Solution</u>

$$x(t) = 3 + 2\left\{\frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{-j10\pi t}\right\}$$

$$\Rightarrow 3 + e^{j\omega_0 t} + e^{-j\omega_0 t}$$

where $\omega_0 = 10\pi$ r/sec.

comparing this with

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$a_0 = 3$, $a_1 = 1$ & $a_{-1} = 1$ rest zero.

Then

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

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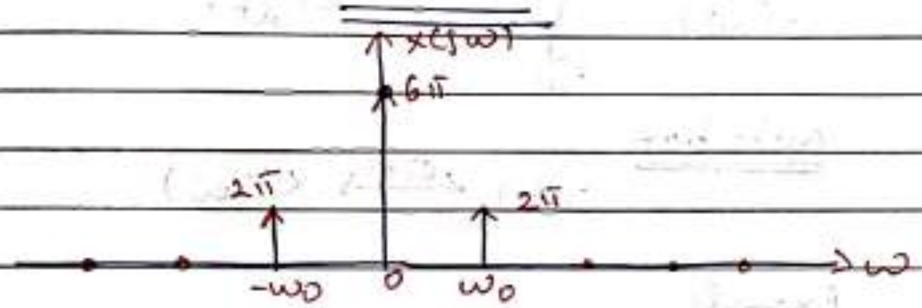
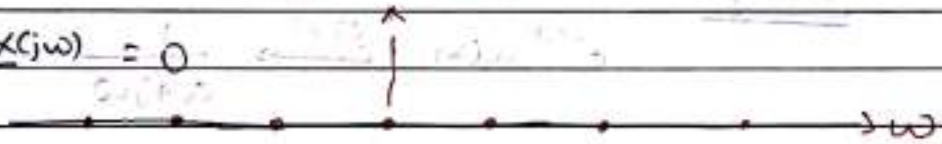
$$= 2\pi \sum_{k=-1}^{+1} a_k \, \delta(\omega - k\omega_0)$$

$$= 2\pi \left\{ a_{-1} \delta(\omega+\omega_0) + a_0 \delta(\omega) + a_1 \delta(\omega - \omega_0) \right\}$$

$$= 2\pi \delta(\omega+\omega_0) + 6\pi \delta(\omega) + 2\pi \delta(\omega - \omega_0)$$




$\angle X(j\omega) = 0$



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## 4.3 Properties of Continuous-Time Fourier Transform

Recap

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$$

Notation

$$x(t) \xleftrightarrow{CTFT} X(j\omega)$$

Eq 4.9

Example

$$e^{-at} u(t) \xleftrightarrow{CTFT} \frac{1}{a+j\omega}$$

### 4.3.1 Linearity

If

$$x(t) \xleftrightarrow{CTFT} X(j\omega) \quad \& \quad y(t) \xleftrightarrow{CTFT} Y(j\omega)$$

then

$$a\,x(t) + b\,y(t) \xleftrightarrow{CTFT} a\,X(j\omega) + b\,Y(j\omega)$$

### 4.3.2 Time Shifting

If

$$x(t) \xleftrightarrow{CTFT} X(j\omega)$$

then

$$x(t-t_0) \xleftrightarrow{CTFT} e^{-j\omega t_0} X(j\omega)$$

Proof

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

replacing $t$ with $t - t_0$

$$x(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega (t-t_0)} d\omega$$

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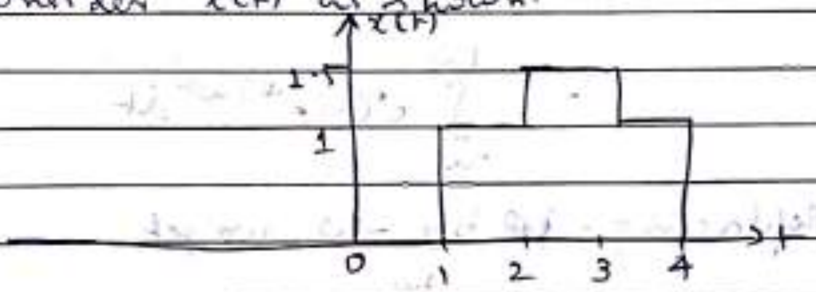
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}\,d\omega$$

Recognising this as the synthesis equation for $x(t-t_0)$, we infer

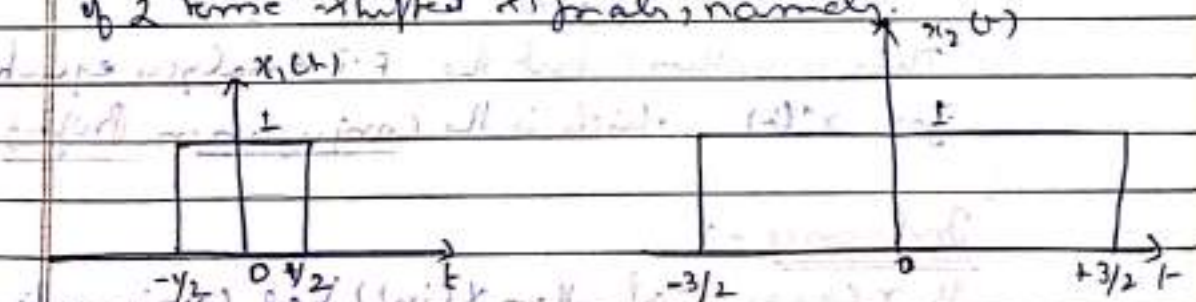
$$\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega)$$

- Time shift does not alter the magnitude of $X(j\omega)$

Eg 4.9 Consider $x(t)$ as shown.



This can be represented as a linear combination of 2 time shifted signals, namely



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$x_1(t)$ & $x_2(t)$ are rectangular pulses. We can then compute

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \quad \& \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

$T_1 = 1/2$ $\qquad$ $T_1 = 3/2$

Using linearity & time shifting,

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

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## 4.38 Conjugation & Conjugate Symmetry.

**Conjugation Property**

If

$$x(t) \xleftrightarrow{CTFT} X(j\omega)$$

then

$$\boxed{x^*(t) \xleftrightarrow{CTFT} X^*(-j\omega)}$$

**Proof**

$$X^*(j\omega) = \left\{ \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \right\}^*$$

$$= \int_{-\infty}^{+\infty} x^*(t) e^{+j\omega t} dt$$

Replacing $j\omega$ by $-\omega$, we get

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\omega t} dt$$

This is nothing but the F.T Analysis equation for $x^*(t)$, which is the conjugation Property

**Inference - 1**

If $x(t)$ is real, then $X(j\omega)$ has conjugate symmetry

$$X(-j\omega) = X^*(j\omega) \quad \text{if } x(t) \text{ is real}$$

When $x(t)$ is real, then $x^*(t) = x(t)$. Then

$$X^*(-j\omega) = \int_{-\infty} x(t) e^{j\omega t} dt = X(j\omega)$$

$$\boxed{\begin{aligned} X(-j\omega) &= X^*(j\omega) \\ X^*(-j\omega) &= X(j\omega) \end{aligned}}$$

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Eg.
If $x(t) = e^{-at}u(t)$ → real
then
$$X(j\omega) = \frac{1}{a+j\omega}$$
and
$$X(-j\omega) = \frac{1}{a-j\omega} = X^*(j\omega)$$

Inference 2 : If we express $X(j\omega)$ in rectangular form as
$$X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\}$$
then if $x(t)$ is real
$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$
&
$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$

Real part of the F.T is an even fn. of frequency
Img. part of the F.T is an odd fn of freq.

Inference 3 : If we express $X(j\omega)$ in the polar form
$$X(j\omega) = |X(j\omega)|e^{j\theta} \quad ; \theta = \angle X(j\omega)$$
then for $x(t)$ being real
$|X(j\omega)|$ → An even function of $\omega$
$\theta$ → An odd function of $\omega$

Inference - 4 - If $x(t)$ is both real & even, then $X(j\omega)$ will also be real & even.
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$
By substituting $\tau = -t$
$$X(-j\omega) = \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}d\tau$$

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Since $x(-\tau) = x(\tau)$ even function

then
$$X(-j\omega) = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega t} d\tau$$

$$= X(j\omega)$$

Also
$$X^*(j\omega) = X(j\omega)$$

If $x(t)$ is real & odd i.e $x(t) = -x(-t)$

then $X(j\omega)$ is purely imaginary & odd.

Inference - 5

If $x(t)$ can be expressed as a sum of odd & even fn.

$x(t) = x_e(t) + x_o(t)$

then

$F(x(t)) = F(x_e(t)) + F(x_o(t))$ → by linearity

Then following relations hold good:

$$x(t) \xleftrightarrow{CTFT} X(j\omega)$$
$$Ev\{x(t)\} \xleftrightarrow{CTFT} Real\{X(j\omega)\}$$
$$Od\{x(t)\} \xleftrightarrow{CTFT} j\,Im\{X(j\omega)\}$$

Eg 4.10 Consider $x(t) = e^{-a|t|}$ ; $a > 0$

we know

$$e^{-at}u(t) \xleftrightarrow{CTFT} \frac{1}{a+j\omega}$$

For $t > 0$, $x(t) = e^{-at}u(t)$ & for $t < 0$, $x(t)$ takes on mirror image values

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left\{\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right\}$$

$$= 2\,Ev\{e^{-at}u(t)\}$$

Since $e^{-at}u(t)$ is real valued, the symmetry properties of F.T lead us to conclude that

$$Ev\{e^{-at}u(t)\} \xleftrightarrow{CTFT} Re\left\{\frac{1}{a+jw}\right\}$$

Then

$$X(jw) = 2\,Re\left\{\frac{1}{a+jw}\right\} = \frac{2a}{a^2+w^2}$$

Consider

$$\frac{1}{a+jw} = \frac{a-jw}{(a+jw)(a-jw)} = \frac{a-jw}{a^2+w^2}$$

$$= \frac{a}{a^2+w^2} - j\frac{w}{a^2+w^2}$$

4.3.4 **Differentiation & Integration**

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{+jwt}\,dw$$

Differentiating both sides w.r.t 't'

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(jw)X(jw)e^{+jwt}\,dw$$

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Then

$$\frac{dx(t)}{dt} \xleftrightarrow{CTFT} j\omega \cdot X(j\omega)$$

Inference
Differentiation in time domain is equivalent to multiplication by $j\omega$ in frequency domain.

Integration → division by $j\omega$ in freq. domain

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{CTFT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

The term $\pi X(0)\delta(\omega)$ - reflects the DC or average value that can result from integration.

Eg 4.21 Important - CTFT of unit step
Let $x(t) = u(t)$ then.

$$\frac{d\,u(t)}{dt} = \delta(t) \quad \text{or} \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau.$$

Let

$$g(t) = \delta(t) \xleftrightarrow{CTFT} G(j\omega) = 1$$

Since

$$x(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

Taking F.T on both sides, we get

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Since $G(j\omega) = 1$, we get

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \quad ; \quad \underline{\underline{G(0) = 1}}$$

$$\boxed{u(t) \xleftrightarrow{CTFT} \frac{1}{j\omega} + \pi\delta(\omega)}$$

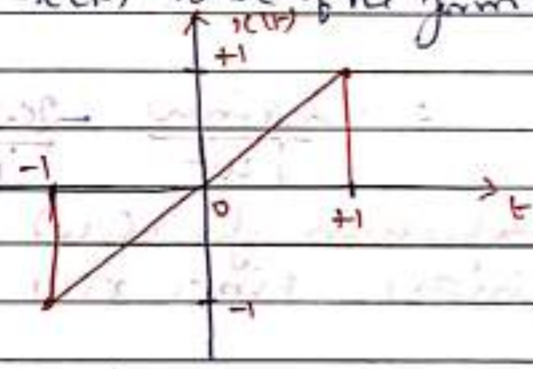
**Also**

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{CTFT} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= 1$$

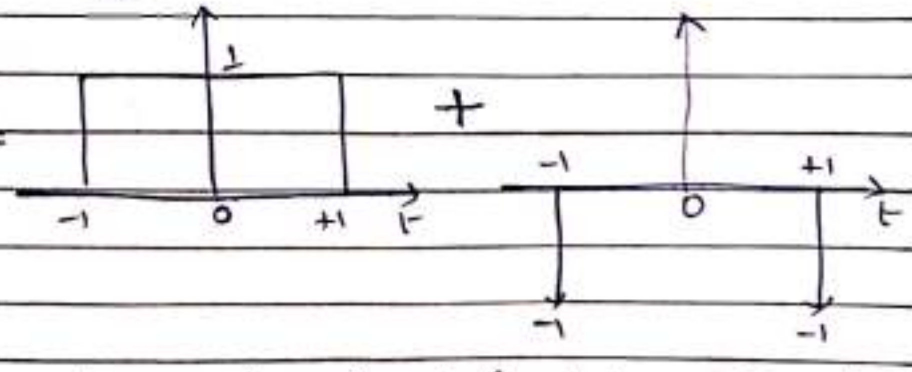
result as $\underline{\omega\delta(\omega) = 0}$ since $\underline{\delta(0) = 1 \text{ with } \omega = 0}$

Eg 4.22 Consider $x(t)$ to be of the form.




Let $g(t) = \frac{d}{dt} \cdot x(t)$

$g(t) = \frac{dx(t)}{dt} =$




$g(t)$ - sum of a rectangular pulse + two impulses

By linearity

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$$G(j\omega) = \left\{ \frac{2\sin\omega}{\omega} \right\} - \underbrace{e^{j\omega} - e^{-j\omega}}_{\text{time shifting}}$$

We can observe $G(0) = 2(1) - 1 - 1 = 0$

Now using Integration property, we get
$\int g(t)\,dt = x(t)$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\,\delta(\omega)$$

Since $G(0) = 0$, then

$$X(j\omega) = \frac{1}{j\omega}\left\{ \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega} \right\}$$

$$= \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

This expression for $X(j\omega)$ is purely imaginary, since $x(t)$ is real & odd.

4.3.5 Time of frequency Scaling.

If $x(t) \xleftrightarrow{CTFT} X(j\omega)$,

then

$$x(at) \xleftrightarrow{CTFT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

where 'a' is a non-zero real number.

Proof:-

$$F\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t} dt$$

using $at = \tau$, we get

$$F\{x(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau} d\tau & ; a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau} . d\tau & ; a < 0 \end{cases}$$

$$\boxed{\begin{aligned} at &= \tau \\ a.dt &= d\tau \\ dt &= \frac{1}{a}.d\tau \end{aligned}}$$

Inference

Aside from the amplitude factor $\frac{1}{|a|}$, a linear scaling in time by a factor of 'a' corresponds to a linear scaling in frequency $\frac{1}{a}$ & vice versa.

→ If $a = -1$, then from the CTFT relation, we get

$$x(-t) \xleftrightarrow{CTFT} \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right) = X(-j\omega)$$

Reversing a signal in time also reverses its Fourier Transform.

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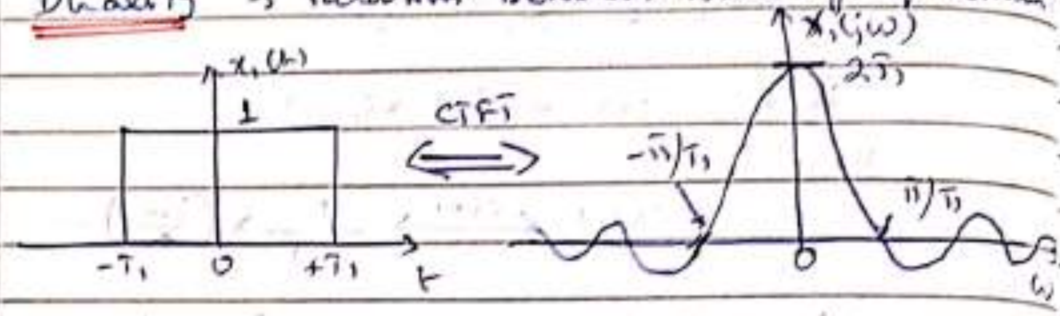
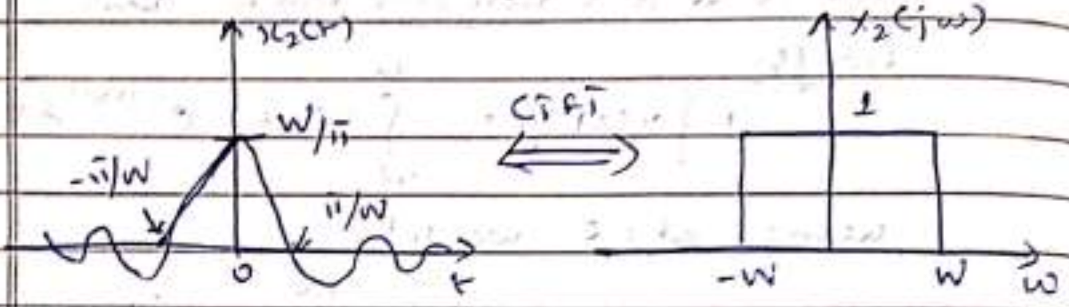
**4.3.6** **Duality** → Relation between time + freq. domain




Ex 4.4 $x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{CTFT} X_1(j\omega) = \dfrac{2\sin\omega T_1}{\omega}$

$x_2(t) = \dfrac{\sin Wt}{\pi t} \xleftrightarrow{CTFT} X_2(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$

→ Because of a symmetry between the Analysis + Synthesis equations, for any transform pair, there is a dual pair with time + frequency variables interchanged.

**Ex 4.13** Consider a signal

$$g(t) = \frac{2}{1+t^2}$$

From Ex. 4.2

$$x(t) = e^{-a|t|} \xleftrightarrow{CTFT} \frac{2a}{a^2+\omega^2} = X(j\omega)$$

letting $a = 1$

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then

$$x(t) = e^{-|t|} \xleftrightarrow{CTFT} X(j\omega) = \frac{2}{1+\omega^2}$$

This looks similar to $g(t)$.
Using the synthesis equation, we get

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}\cdot d\omega$$

multiplying both sides with $2\pi$ & replacing $t$ with $-t$

$$2\pi\, e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{(1+\omega^2)}\right)\cdot e^{-j\omega t}\, d\omega$$

Now interchanging the variables $t$ & $\omega$, we get

$$2\pi\, e^{-|\omega|} = \int_{-\infty}^{\infty}\frac{2}{(1+t^2)}\cdot e^{-j\omega t}\cdot dt$$

we can now say that

$$F\left\{\frac{2}{1+t^2}\right\} \xleftrightarrow{CTFT} 2\pi e^{-|\omega|}$$

Other dual ~~rela~~ relations.
Differentiating the Analysis eqn. w.r.t $\omega$,

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jt\cdot x(t)\, e^{-j\omega t}\cdot dt$$

means.

$$\boxed{-jt\cdot x(t) \xleftrightarrow{CTFT} \frac{dX(j\omega)}{d\omega}}$$

Similarly

$$\boxed{e^{j\omega_0 t}\, x(t) \xleftrightarrow{CTFT} X(j(\omega-\omega_0))}$$

and:

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$$\boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\text{CTFT}} \int_{-\infty}^{\omega} x(\eta)\,d\eta}$$

## 4.3.7 Parseval's Relation:

If $x(t)$ & $X(j\omega)$ are the Fourier Transform pairs then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

**Proof**

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)\,x^*(t)\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\,e^{-j\omega t}\,d\omega\right\}dt$$

Reversing the order of Integration.

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\underbrace{\left\{\int_{-\infty}^{+\infty} x(t)\,e^{-j\omega t}\,dt\right\}}_{\text{F.T of } x(t)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\cdot X(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

L.H.S $\rightarrow$ Total energy in the signal $x(t)$

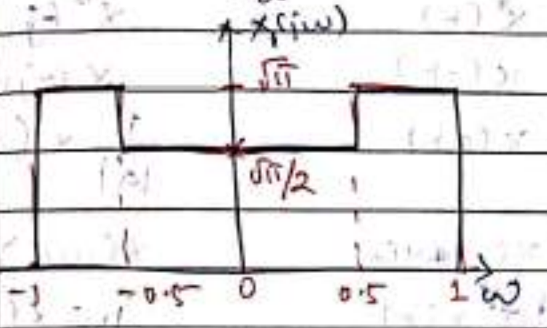
$|x(t)|^2 \rightarrow$ Energy per unit time

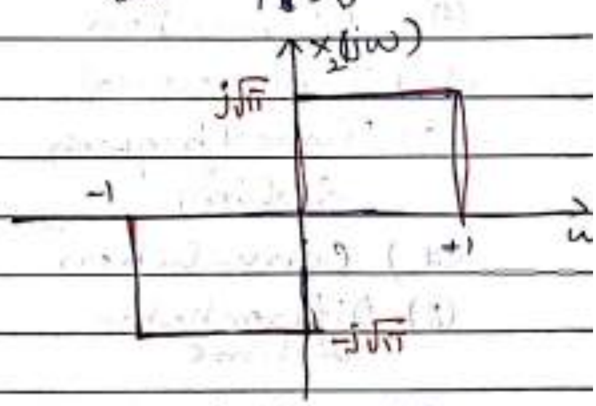
$\frac{|X(j\omega)|^2}{2\pi} \rightarrow$ Energy per unit frequency

$|X(j\omega)|^2 \rightarrow$ Energy density Spectrum of $x(t)$

Eg 4.14 For the given Fourier Transform, we need to evaluate

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \& \quad D = \left.\frac{dx(t)}{dt}\right|_{t=0}$$




Sol$^n$

To evaluate E, we use the Parseval's relation.

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

$$E_1 = \frac{5}{8} \quad \& \quad E_2 = 1$$

To evaluate D in frequency domain, we first use the differentiation property to get;

$$g(t) = \frac{dx(t)}{dt} \xleftrightarrow{CTFT} j\omega X(j\omega) = G(j\omega)$$

Noting that

$$D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega) d\omega$$

We conclude

$$D = \int_{-\infty}^{+\infty} j\omega X(j\omega) d\omega$$

which evaluates to

$$D_1 = 0 \quad \& \quad D_2 = \frac{-1}{2\sqrt{\pi}}$$

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Table (3b) 4.1

## Properties of Fourier Transform

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| | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| 1. | Linearity | $ax(t)+by(t)$ | $aX(j\omega)+bY(j\omega)$ |
| 2) | Time Shifting | $x(t-t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| (3) | Freq. Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega-\omega_0))$ |
| (4) | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| (5) | Time reversal | $x(-t)$ | $X(-j\omega)$ |
| (6) | Time & frequency scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$ |
| (7) | Convolution | $x(t)*y(t)$ | $X(j\omega)Y(j\omega)$ |
| (8) | Differentiation in time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| (9) | multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$ |
| (10) | Integration | $\int_{-\infty}^{t}x(t)dt$ | $\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$ |
| (11) | Differentiation in frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| (12) | Conjugate Symmetry for Real Signals | $x(t)$ real | $X(j\omega)=X^*(-j\omega)$; $Re\{X(j\omega)\}=Re\{X(-j\omega)\}$; $Im\{X(j\omega)\}=-Im\{X(-j\omega)\}$; $\lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert$; $\angle X(j\omega)=-\angle X(-j\omega)$ |
| (13) | Symmetry for Real & Even Signals | $x(t)$ real & even | $X(j\omega)$ real & even |
| (14) | Symmetry for Real & odd signals | $x(t)$ real & odd | $X(j\omega)$ purely imaginary & odd |
| (15) | Even-odd decomposition of Real Signals | $x_e(t)=Ev\{x(t)\}$; $x_o(t)=Od\{x(t)\}$ [$x(t)$ - Real] | $Re\{X(j\omega)\}$; $Odd\{X(j\omega)\}$ |

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## Parseval's Relation for Aperiodic Signals:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

## Table 4.2 Basic Fourier Transforms.

| Signal | Fourier Transform | Fourier series coefficients (if periodic) |
|---|---|---|
| (1) $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| (2) $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ | $a_1 = 1$, $a_k = 0$ otherwise |
| (3) $\cos\omega_0 t$ | $\pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}$ | $a_1 = a_{-1} = \frac{1}{2}$, $a_k = 0$ otherwise |
| (4) $\sin\omega_0 t$ | $\frac{\pi}{j}\{\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\}$ | $a_1 = -a_{-1} = \frac{1}{2j}$, $a_k = 0$ otherwise |
| (5) $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1$, $a_k = 0$ for $k \neq 0$ (this is the F-S representation for any choice of $T > 0$) |
| (6) Periodic square wave $x(t) = \begin{cases} 1 & \lvert t\rvert < T_1 \\ 0 & T_1 < \lvert t\rvert \le T/2 \end{cases}$ & $x(t+T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| (7) $\sum_{n=-\infty}^{+\infty}\delta(t-nT)$ | $\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| (8) $x(t) = \begin{cases} 1 & \lvert t\rvert < T_1 \\ 0 & \lvert t\rvert > T_1 \end{cases}$ | $\frac{2\sin\omega T_1}{\omega}$ | |

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(9) $\frac{\sin Wt}{\pi t}$ $\quad X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$

(10) $\delta(t)$ $\quad 1$

(11) $u(t)$ $\quad \frac{1}{j\omega} + \pi\delta(\omega)$

(12) $\delta(t-t_0)$ $\quad e^{-j\omega t_0}$

(13) $e^{-at}u(t)$ $\quad \frac{1}{a+j\omega}$
$Re\{a\} > 0$

(13) $te^{-at}u(t)$ $\quad \frac{1}{(a+j\omega)^2}$
$Re\{a\} > 0$

(14) $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ $\quad \frac{1}{(a+j\omega)^n}$
$Re\{a\} > 0$

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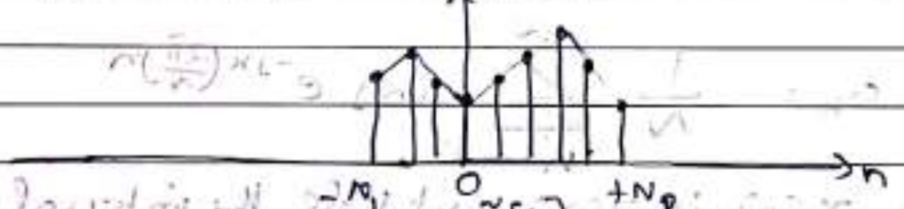
# Discrete-Time Fourier Transform

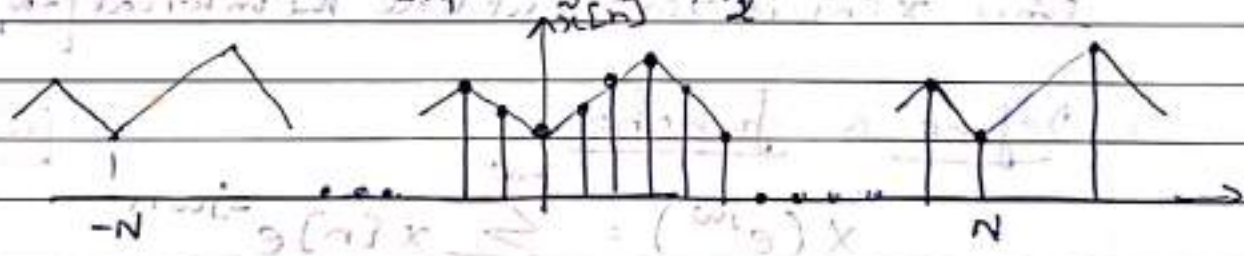
Sections: 5.0, 5.1, 5.2, 5.3, 5.6, 5.7 } A.O. Oppenheim
$\boxed{5.4\ 5.5}$

## 5.1 Representation of Aperiodic Signals - Discrete-Time Fourier Transform.

### 5.1.1. Development of the Discrete-Time Fourier Transform (DTFT)

Consider ~~of~~ an aperiodic sequence $x[n]$ that is for finite duration




$x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$

→ We have a periodic sequence $\tilde{x}[n]$ such that we have $x[n]$ ~~as~~ in one period.

→ when we choose the period of $N$ of $\tilde{x}[n]$ to be large enough, then $\tilde{x}[n]$ is identical to $x[n]$ (or) as $N \to \infty$ $x[n] = \tilde{x}[n]$ for any finite value of '$n$'

→ we now examine the F.S representation of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

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where $\frac{2\pi}{N} = \omega_0$ is the fundamental frequency.

also

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\omega_0 n}$$

→ Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval in the Analysis equation to include this interval, we can replace $\tilde{x}[n]$ by $x[n]$

i.e

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

or

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

since $x[n]$ is zero outside the interval $\{-N_1, N_2\}$

→ Define a function.

$$\boxed{\omega = k\omega_0}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Then

$$a_k = \frac{1}{N} X(e^{jk\omega_0}) \quad \text{since } \omega_0 = \frac{2\pi}{N}$$

Then

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) \cdot e^{jk\omega_0 n}$$

Since

$$N = \frac{2\pi}{\omega_0} \Rightarrow \boxed{\frac{1}{N} = \frac{\omega_0}{2\pi}}$$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \cdot \omega_0$$

→ As N increases, $\omega_0$ decreases & as $N \to \infty$, the summation passes to an integral. ($\omega_0$ is too small)

→ Also $\tilde{x}[n] = x[n]$ as $N \to \infty$

→ Now consider

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ Proof next page.

is periodic in $\omega$ with a period $2\pi$ & so is $e^{j\omega n}$. The summation is carried out over N consecutive intervals of width $\omega_0 = \frac{2\pi}{N}$, the total interval of integration will always have width of $2\pi$.

$$N \cdot \omega_0 = N \cdot \frac{2\pi}{N} = 2\pi$$

→ Then,

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \cdot d\omega$$

→ Interval can be taken arbitrarily but width of $2\pi$ length.

Fourier Transform Pairs.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ - Analysis DTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$ - Synthesis - INVDTFT

Continuous case

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \Big| \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

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## Important Inferences:

→ $X(e^{j\omega})$ is referred as the spectrum of $x[n]$
- gives information on how $x[n]$ is composed of complex exponentials at different frequencies.

→ unlike C.T.F.T, D.T.F.T differs in
- periodicity of D.T.F.T $X(e^{j\omega})$
- Finite interval of integration.

This is due to the fact that Discrete Time complex exponentials that differ in frequency by a multiple of $2\pi$ are identical.

→ For periodic D.T signals, F.S. co-efficients are periodic & F.S. representation is a finite sum. (N)

→ For aperiodic D.T signals, $X(e^{j\omega})$ is periodic with a period of $2\pi$.

consider

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

then

$$X(e^{j\omega+2\pi k}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n} \cdot e^{-j(2\pi k)n}$$

As $e^{-j(2\pi kn)} = \cos(2\pi kn) - j\sin(2\pi kn)$
$= 1$

$$X(e^{j\omega+2\pi k}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

$$= X(e^{j\omega})$$

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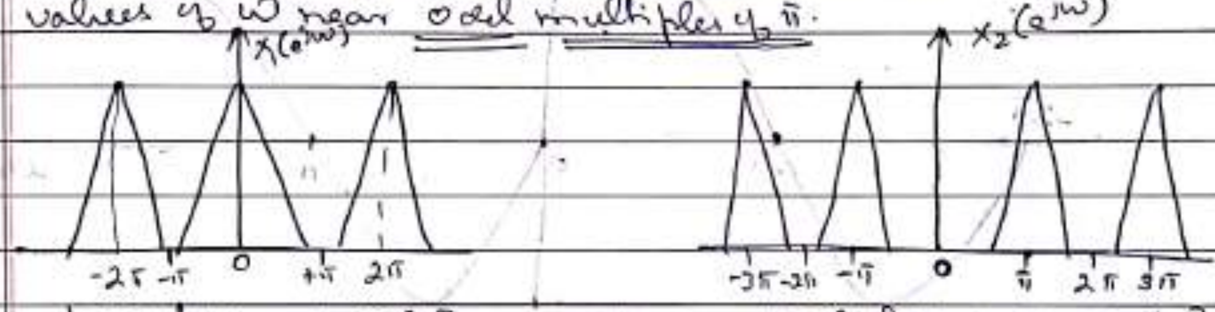
→ In D.T.F.T, Synthesis equation, the integration is over the interval $2\pi$ only

→ Since the term $e^{j\omega n}$ is periodic as a function of $\omega$, when $\omega = 0$ or $2\pi$, the signals are same. Signals near these values or any other even multiples of $\pi$ are slowly varying + deemed as low-frequency signals.

→ Similarly, the high frequencies in discrete-time are the values of $\omega$ near odd multiples of $\pi$.




5.1.2 Examples of D.T.F.T

Eg.1 Consider $x[n] = a^n u[n]$ ; $|a| < 1$

Then the D.T.F.T would be

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ Note $u[n]=1$ for $n \geq 0$

then

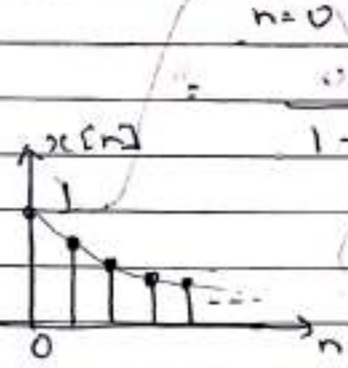
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n \underbrace{u[n]}_{1} e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$ ; summation.

D.T.F.T is a fn of $\omega$.

Formula

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} ; |a|<1$$



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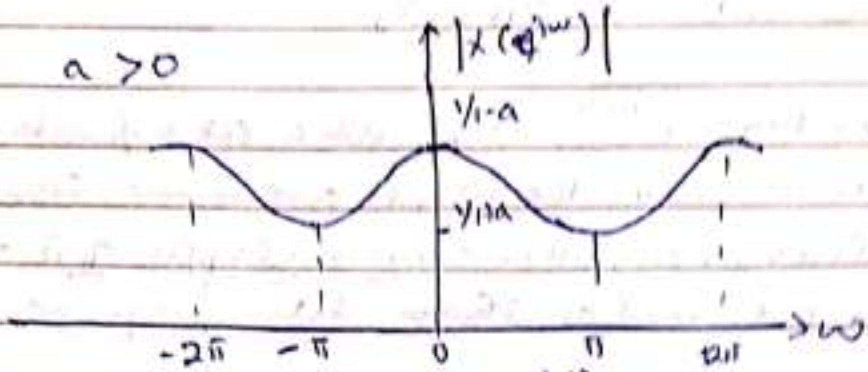
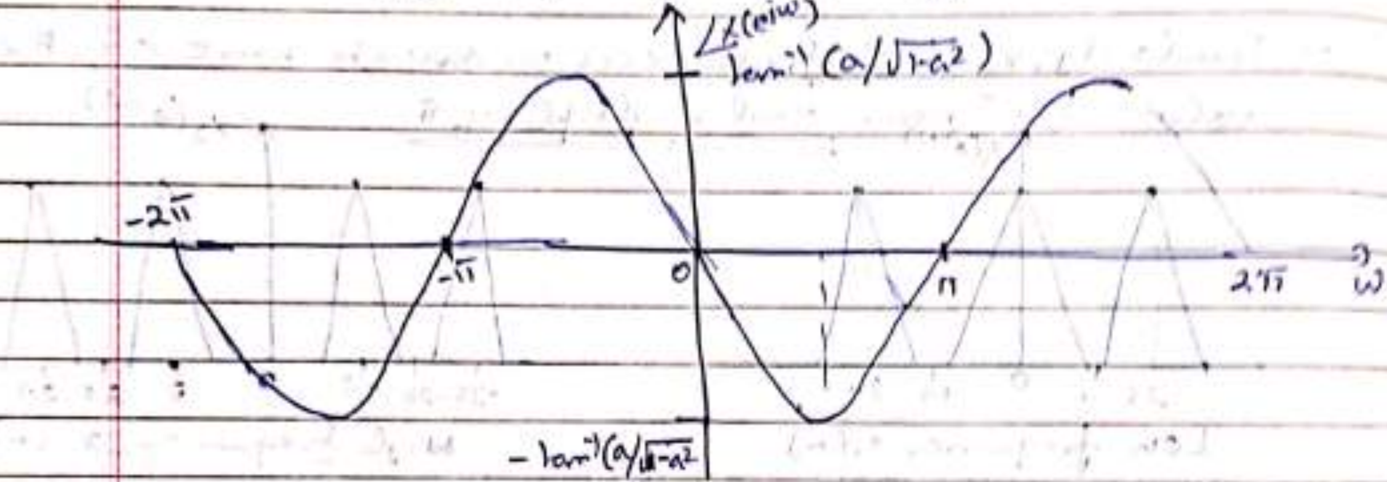
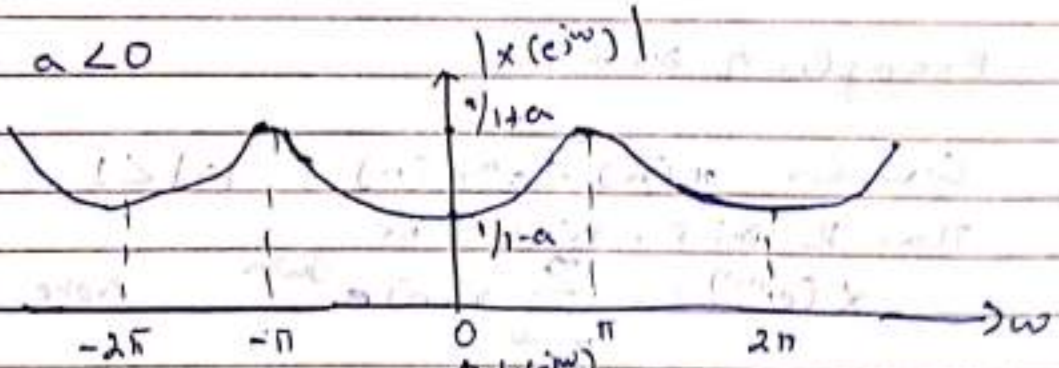
we now plot $|X(e^{j\omega})|$ & $\angle X(e^{j\omega})$ as a function of $\omega$

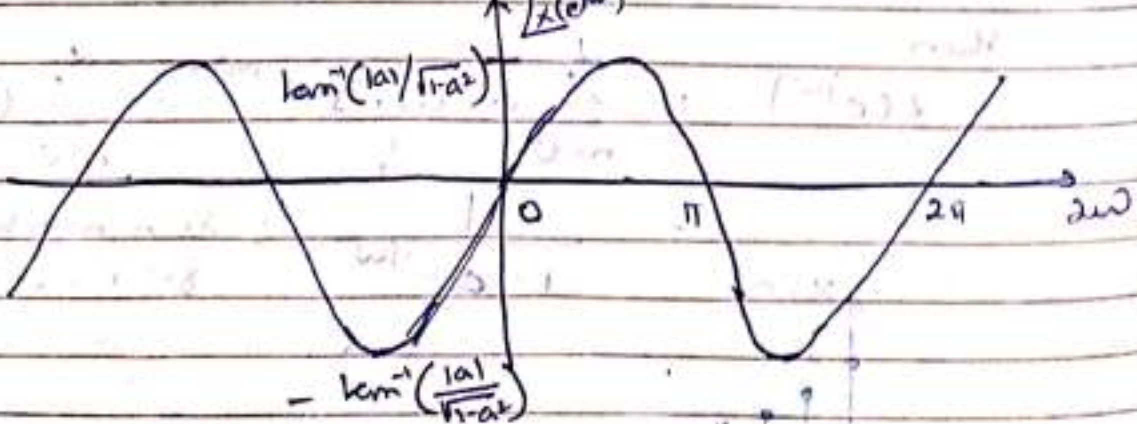
(i) For $a > 0$



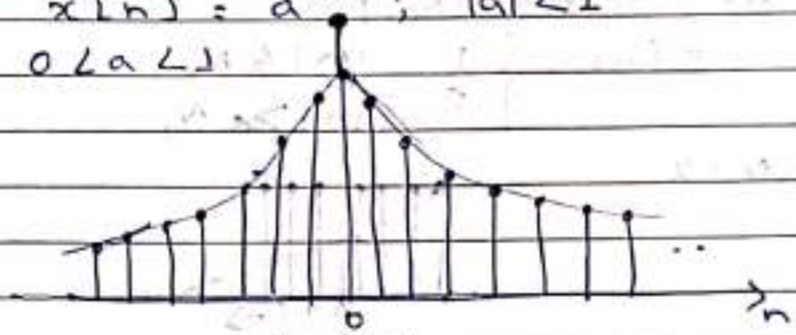

(ii) For $a < 0$



Eg 5.2 let $x[n] = a^{|n|}$ ; $|a| < 1$

For $0 < a < 1$:




Then its D.T.F.T would be

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{-\infty}^{-1} a^{-n} e^{-j\omega n}$$

For the second term, using $m = -n$, we get

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^{n} + \sum_{m=1}^{\infty} (ae^{j\omega})^{m}$$

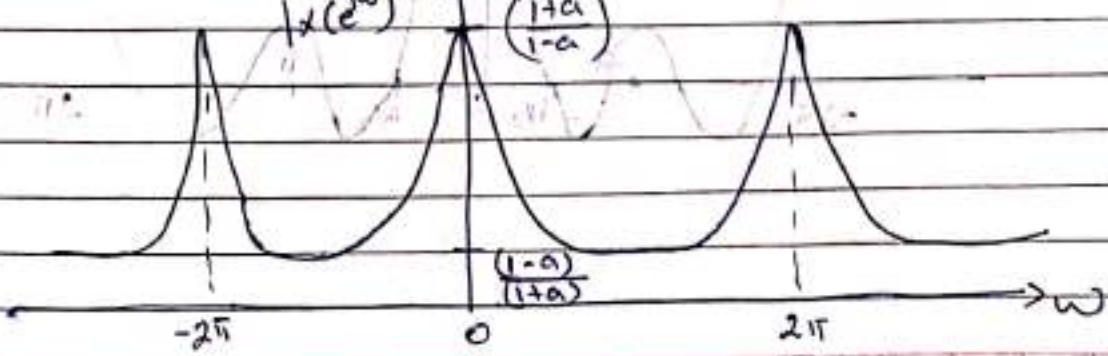
which yields

Formula: $\sum_{n=n_0}^{\infty} a^n = \frac{a^{n_0}}{1-a}$ for $|a| < 1$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-a^2}{1-2a\cos\omega + a^2} \rightarrow \text{real.}$$

No phase plot

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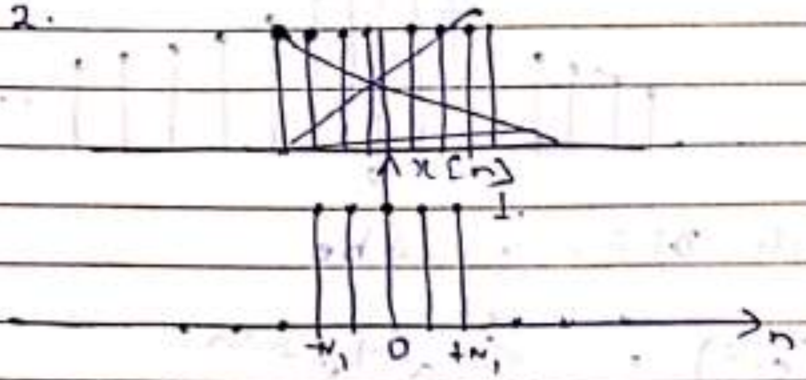
Eg. 5.3 Consider a rectangular pulse

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$$

Let $N_1 = 2$.




In this case

$$X(e^{j\omega}) = \sum_{n=-N_1}^{+N_1} x[n] e^{-j\omega n}$$

since $x[n] = 1$ in this interval

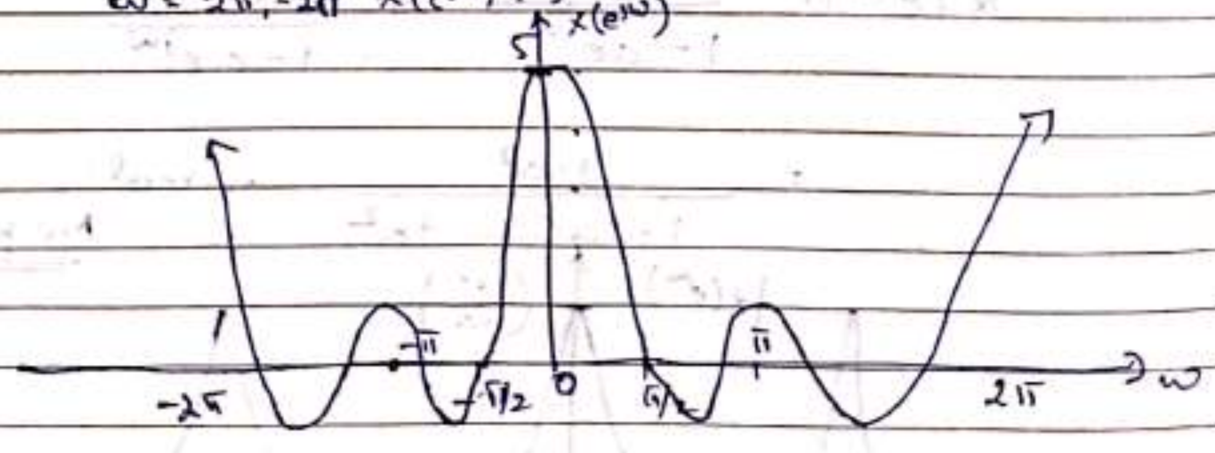
$$= \sum_{n=-N_1}^{+N_1} e^{j\omega n}$$

$$= e^{-j2\omega} + e^{-j\omega} + e^0 + e^{j\omega} + e^{j2\omega}$$

For $\omega = 0$ $X(e^{j\omega}) = 1+1+1+1+1 = 5$

$\omega = \pi, -\pi$ $X(e^{j\omega}) = 1$

$\omega = 2\pi, -2\pi$ $X(e^{j\omega}) = 5$



For any $N_1$,

$$X(e^{j\omega}) = \frac{\sin \omega (N_1 + 1/2)}{\sin(\omega/2)}$$

The function above is the discrete-time counterpart of the sinc function, which appears in the F.T of the continuous-time rectangular pulse (Fig. 4.4)

→ Important difference between these 2 functions is that the function above is periodic with period $2\pi$, whereas the sinc function is aperiodic.

5.1.3 Convergence Issues associated with D.T.F.T

→ If $x[n]$ is for finite or infinite duration (fig 5.1 & 5.2), F.T pairs are valid.

→ Condition for Convergence

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

(or)

the sequence $x[n]$ has finite energy.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

→ No Gibbs Phenomenon.

→ Consider an aperiodic signal $x[n]$ approximated by an integral of complex exponentials with frequencies taken over an interval $|\omega| \leq W$ i.e.

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$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})\, e^{j\omega n} d\omega \quad -(A)$$

then $\hat{x}[n] = x[n]$ for $\underline{\underline{W = \pi}}$

Eg. 5.A. Let $x[n]$ be an unit impulse.

$$x[n] = \delta[n]$$

Then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n}$$

$$= \underline{\underline{1}} \qquad \text{as } \delta[0] = 1$$

Then we infer that unit impulse has a Fourier Transform

→ consisting of equal contribution at all frequencies

→ If we apply (A) for this situation, then

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} (1) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \times \frac{1}{jn}\left[ e^{j\omega n} \right]_{\omega=-W}^{W}$$

$$= \frac{1}{2\pi j n}\left[ e^{jWn} - e^{-jWn} \right]$$

$$= \frac{1}{2\pi j n}\left[ \cos(Wn) + j\sin(Wn) - \cos(Wn) + j\sin(Wn) \right]$$

$$= \frac{1}{2\pi j n} 2j \sin(Wn) = \underline{\underline{\frac{\sin Wn}{\pi n}}}$$

When $W = \pi$, then $\boxed{\hat{x}[n] = \frac{\sin \pi n}{\pi n}}$

for $\hat{x}[0]$ then $\frac{\sin \pi n}{\pi n} = \underline{\underline{1 = x[n]}}$

5.2 **The Fourier Transform for Periodic Signals.**

Consider a periodic D.T signal $x[n]$ of period $N$.

Then $\omega_0 = \frac{2\pi}{N}$ → Fundamental frequency.

$\omega_0 = \frac{2\pi}{T}$

The F.Transform is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{+\infty} 2\pi\, a_k\, \delta\left(\omega - k\frac{2\pi}{N}\right) \quad \to \text{A.O.}$$

CTFT of periodic $x(t)$: $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)$

Since DTFT is periodic with fundamental period $2\pi$, we can write

$$\left.\begin{aligned} X(e^{j\omega}) &= \sum_{k=0}^{N-1} 2\pi \cdot a_k\, \delta(\omega - k\omega_0) \\ &= \sum_{k=0}^{N-1} 2\pi\, a_k\, \delta\left(\omega - k\frac{2\pi}{N}\right) \end{aligned}\right\} \to \text{QR}$$

**Eg. 5.5** Consider $x[n] = \cos\omega_0 n$ ; $\omega_0 = \frac{2\pi}{5}$ ; $N = 5$

$$= \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

$a_{-1} = a_{+1} = \frac{1}{2}$. Rest all zeros.

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{+\infty} a_k\, \delta(\omega - k\omega_0)$$

Here $k$ can be $-1$ & $+1$

$$= 2\pi\left\{a_{-1}\,\delta(\omega + \omega_0) + a_{+1}\,\delta(\omega - \omega_0)\right\}$$

$$= 2\pi \times \frac{1}{2}\,\delta(\omega + \omega_0) + 2\pi \times \frac{1}{2}\,\delta(\omega - \omega_0)$$

$$= \pi\,\delta\left(\omega + \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega - \frac{2\pi}{5}\right)$$

$$= \pi\,\delta(\omega + \omega_0) + \pi\,\delta(\omega - \omega_0)$$

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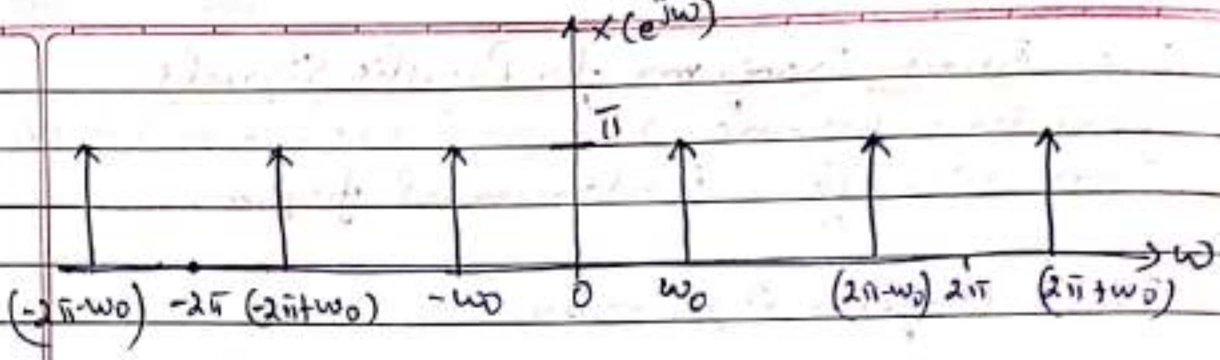
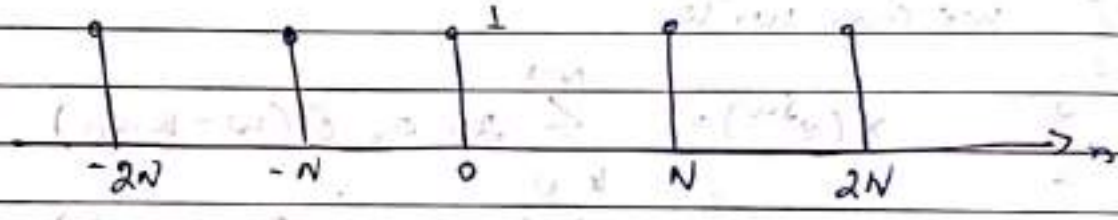



Repeats periodically after $2\pi$

Eg 5.6 Consider a discrete-time impulse train

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$$




Fourier Series $a_k$ can be computed as

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Choosing an interval of summation $0 \le n \le N-1$

$$a_k = \frac{1}{N} \qquad \text{Refer Eg 3.8A (GR)}$$

Then the F.T would be

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{+\infty} a_k\, \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$= 2\pi \sum_{k=-\infty} \frac{1}{N}\, \delta\left(\omega - \frac{2\pi k}{N}\right)$$

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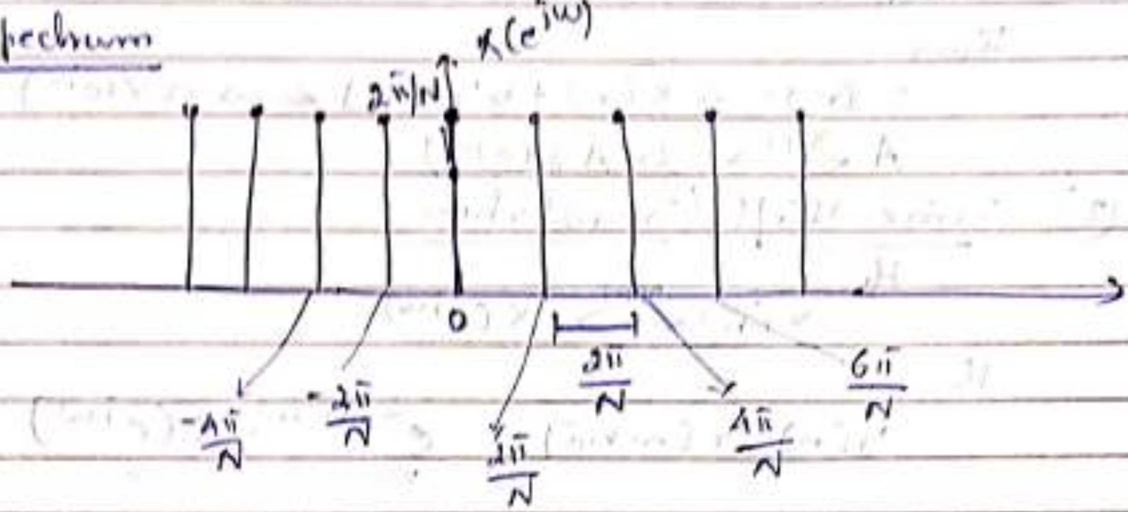
$$= \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{k 2\pi}{N}\right)$$

Spectrum




$X(e^{j\omega})$ is again a train of impulses of amplitude $\frac{2\pi}{N}$.

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Eg 3.89 / GR 437 — Find the D.T.F.T of unit impulse sequence $x[n] = \delta[n]$

Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega_0 n}$$

Since $\delta[0] = 1$

then

$$X(e^{j\omega}) = (1) e^{-j\omega n}\Big|_{n=0} = 1$$

we can express

$$x[n] = \delta[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = 1.$$

Eg 3.90 / GR-438 — Evaluate DTFT of $x[n] = a^n u[n]$

Solved as Eg. 5.1 of A.O (previous page)

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## Properties of DTFT (Ref. G.R)

1. **Linearity**

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ & $y[n] \xleftrightarrow{DTFT} Y(e^{j\omega})$

then

$$z[n] = a\,x[n] + b\,y[n] \longleftrightarrow a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$$

$$A \xleftrightarrow{DTFT} 2\pi A\,\delta(e^{j\omega})$$

(2) **Time Shift / Translation**

If

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

then

$$y[n] = x[n-n_0] = e^{-j\omega n_0}\,X(e^{j\omega})$$

(3) **Frequency Shift / Translation**

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then

$$y[n] = e^{j\beta n}\cdot x[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j(\omega-\beta)})$$

(4) **Scaling**

$$x(at) \xleftrightarrow{CTFT} \frac{1}{|a|}X\left(\frac{\omega}{a}\right) \quad \text{– continuous time}$$

In discrete time case, 'a' has to be an integer, otherwise $x[an]$ is not a sequence. If 'a' is an integer say $a=4$, then $x[4n]$ consists of only even samples of $x[n]$, which means loss of information. Hence the valid time-scaled discrete sequence is defined as

$$x_a[n] = \begin{cases} x\left(\frac{n}{a}\right) = x[k] & \text{if } \frac{n}{a} = \text{integer} = k \quad \boxed{n = ka} \\ 0 & \text{if } n \neq ka \end{cases}$$

Then

$$x_a[n] \longleftrightarrow X(e^{jaw})$$

⑤ **Frequency Differentiation**

Let $x[n] \xleftrightarrow{DTFT} X(e^{jw})$

then $(-jn)\,x[n] \xleftrightarrow{DTFT} \dfrac{d\,X(e^{jw})}{dw}$

⑥ **Summation or Accumulation**

15.142 Let $x[n] \xleftrightarrow{DTFT} X(e^{jw})$

then

$$\sum_{k=-\infty}^{\infty} x[k] \xleftrightarrow{DTFT} \pi X(e^{j0})\,\delta(e^{jw}) + \frac{1}{1-e^{-jw}} X(e^{jw})$$

⑦ **Time Domain Convolution**

Let $x[n] \xleftrightarrow{DTFT} X(e^{jw})$ & $y[n] \xleftrightarrow{DTFT} Y(e^{jw})$

then

$$z[n] = x[n] * y[n] = X(e^{jw}) \cdot Y(e^{jw})$$

⑧ **multiplication / modulation**

$$z[n] = x[n]\,y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X(e^{jw}) \circledast Y(e^{jw})\right]$$

periodic convolution

(5A)

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(9) **Parseval's Theorem**

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

$|X(e^{j\omega})|^2$ = Energy density spectrum

E - Total energy content in the sequence $x[n]$

(10) **Conjugation**

If $x[n] \longleftrightarrow X(e^{j\omega})$

then

$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$

(11) Time Reversal

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$

(12) **Symmetry Property.**

(a) If $x[n]$ is purely real + even, then its DTFT is purely real.

(b) If $x[n]$ is real + odd, then its DTFT is purely Imaginary.

PQ 3.97 GP-448 Find DTFT of $x[n] = [1, 2, \underset{\uparrow}{3}, 2, 1]$. Also evaluate $X(e^{jw})$ at $w = 0$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-2}^{2} x[n] e^{-jwn}$$

$$= (1)e^{+j2w} + (2)e^{+jw} + (3)(1) + 2 \cdot e^{-jw} + 1 e^{-j2w}$$

Consider $e^{j2w} + e^{-j2w} = \cos 2w + j \sin 2w$
$\cos 2w - j \sin 2w$
$= \underline{\underline{2 \cos 2w}}$

IIIly $e^{jw} + e^{-jw} = \underline{\underline{2 \cos w}}$

$$= 3 + (2)(2\cos w) + 2\cos 2w$$

$$= \underline{3 + 4 \cos w + 2 \cos 2w}$$

For $w = 0$

$$X(e^{j0}) = 3 + 4(1) + 2(1) = 9.$$

ES 4.92 GP-449 Find the DTFT

(a) $x[n] = (0.5)^{n+2} u[n]$ ; $u[n] = 1$ for $n \geq 0$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (0.5)^{n+2} e^{-jwn}$$ ; since $u[n]$

$$= (0.5)^2 \sum_{n=0}^{\infty} (0.5 e^{-jw})^n$$

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$$= \frac{1}{4} \cdot \frac{1}{1 - 0.5e^{-jw}} = \frac{1/4}{1 - \frac{1}{2} e^{-jw}}$$

(b) $x[n] = n(0.5)^{2n} u(n)$

$$X(e^{jw}) = \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^{2n} e^{-jwn} \quad \text{; due to } u[n]$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{4}\right)^{n} e^{-jwn}$$

$$= \sum_{n=0}^{\infty} n \left\{ \frac{1}{4} \cdot e^{-jw} \right\}^{n}$$

Use the formula $\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}$

then

$$X(e^{jw}) = \frac{1/4 \cdot e^{-jw}}{\left\{1 - \frac{1}{4} e^{-jw}\right\}^2}$$

Eg 3.93 / AR-450 Find DTFT $x[n] = -a^n u[-n-1]$, a is real.

Note

$u[m] = 1$ for $m \geq 0$ then $-n-1$ has to be +ve or zero

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

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$$= \sum_{n=-\infty}^{-1} -a^{n} e^{-j\omega n}$$

let $n = -m$, then

$$X(e^{j\omega}) = \sum_{m=+\infty}^{+1} -a^{-m} e^{j\omega m}$$

$$= -1 \left\{ \sum_{m=1}^{\infty} a^{-m} e^{j\omega m} + 1 - 1 \right\}$$

$$= 1 - \sum_{m=0}^{\infty} a^{-m} e^{j\omega m}$$

$$= 1 - \sum_{m=0}^{\infty} \left(a^{-1} e^{j\omega}\right)^{m} .$$

$$= 1 - \frac{1}{1 - a^{-1} e^{j\omega}} \qquad \text{using } \sum_{k=0}^{\infty} a^{k} = \frac{1}{1-a}$$

$$= \frac{\cancel{1} - a^{-1} e^{j\omega} - \cancel{1}}{1 - a^{-1} e^{j\omega}} = \frac{-e^{j\omega}/a}{1 - \frac{e^{j\omega}}{a}}$$

$$= -\frac{e^{j\omega}}{\cancel{a}} \times \frac{\cancel{a}}{a - e^{j\omega}} = -\frac{\cancel{e^{j\omega}}}{\cancel{e^{j\omega}}\left(a e^{-j\omega} - 1\right)}$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

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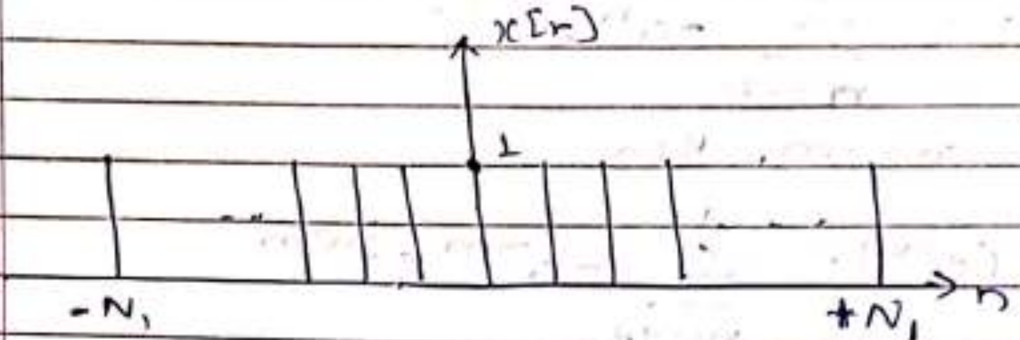
E 3.94

GN-450

Find DTFT of the rectangular pulse.



Solution

$x[n]$ can be expressed as.

$$x[n] = \begin{cases} 1 & -N_1 \le n \le +N_1 \\ 0 & \text{otherwise.} \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} (1) e^{-j\omega n}$$

Let $l = N_1 + n$, then for $n = -N_1$, $l = 0$
for $n = +N_1$, $l = 2N_1$

$$X(e^{j\omega}) = \sum_{l=0}^{2N_1} e^{-j\omega(l-N_1)}$$

$$= \sum_{l=0}^{2N_1} e^{-j\omega l} \cdot e^{+j\omega N_1}$$

$$= e^{+j\omega N_1} \sum_{l=0}^{2N_1} e^{-j\omega l}$$

Note $$\sum_{n=0}^{m} a^n = \frac{a^{m+1}-1}{a-1} = \frac{1-a^{m+1}}{1-a} \quad \text{for } a \ne 1$$

$$x(e^{j\omega}) = e^{j\omega N_1}\left\{\frac{e^{-j\omega(2N_1+1)}-1}{e^{-j\omega}-1}\right\}$$

$$= \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin(\omega/2)} ;\ \omega \neq 0, \pm 2\pi, \pm 4\pi \ldots$$

To find $x(e^{j\omega})$ at $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, put $\omega = 0$ in

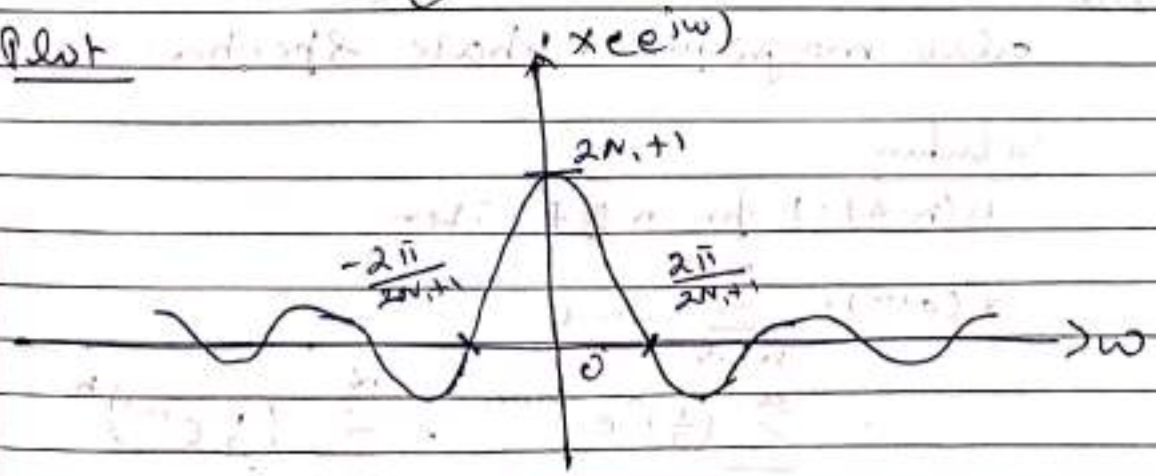
$$x(e^{j\omega}) = e^{j\omega N_1}\sum_{l=0}^{2N_1} e^{-j\omega l}$$

$$x(e^{j0}) = (1)\sum_{l=0}^{2N+1}(1) = \underline{\underline{2N_1+1}}$$

Summarizing

$$x(e^{j\omega}) = \begin{cases} \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin(\omega/2)} & ;\ \omega \neq 0, \pm 2\pi, \pm 4\pi \ldots \\ 2N_1+1 & ;\ \omega = 0, \pm 2\pi, \pm 4\pi + \ldots \end{cases}$$

Plot



To find zero-crossings, equate $X(e^{jw}) = 0$

$$\frac{\sin\left[\frac{w}{2}(2N_1+1)\right]}{\sin(w/2)} = 0$$

this can be possible if

$$\frac{w}{2}(2N_1+1) = \pm m\pi$$

where $m = 1, 2, 3, \ldots$

(or)

$$w = \pm \frac{2m\pi}{(2N_1+1)}$$

For the first zero crossing $m = 1$

$$w_1 = \pm \frac{2\pi}{(2N_1+1)}$$

Eg. 3.95 / GII-452 Find DTFT of $x[n] = \left(\frac{1}{2}\right)^n u(n-4)$

also magnitude + phase spectra.

Solution

$u(n-4) = 1$ for $n \geq 4$. Then

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n e^{-jwn} = \sum_{n=4}^{\infty} \left(\frac{1}{2} e^{jw}\right)^n$$

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Recall

$$\sum_{n=n_0}^{\infty} a^n = \frac{a^{n_0}}{1-a} \; ; |a|<1 \cdot \text{ Since } \left|\tfrac{1}{2}e^{jw}\right| < 1$$

$\boxed{\text{Here } n_0 = 4}$

$$X(e^{jw}) = \frac{\left(\frac{1}{2}\right)^4 e^{-j4w}}{1-\frac{1}{2}e^{-jw}}$$

$$= \left(\frac{1}{2}\right)^4 \frac{e^{-j4w}}{1-\frac{1}{2}\{\cos w - j\sin w\}}$$

$$= \left(\frac{1}{2}\right)^4 \frac{e^{-j4w}}{\underbrace{\left(1-\frac{1}{2}\cos w\right)}_{u} + \underbrace{j\left(\frac{1}{2}\sin w\right)}_{+jv}}$$

Magnitude Spectrum

$$|X(e^{jw})| = \left(\frac{1}{2}\right)^4 \frac{1}{\left\{\left(1-\frac{1}{2}\cos w\right)^2 + \left(\frac{1}{2}\sin w\right)^2\right\}^{1/2}}$$

as $|e^{-j4w}| = 1.$

Phase Spectrum $\qquad e^{\pm j\theta} = 1\angle \pm\theta$

$\theta(e^{jw}) = \angle X(e^{jw})$ No contribution by $(1/2)^4$

$$= \tan^{-1}\left(\frac{-\sin 4w}{\cos 4w}\right) - \tan^{-1}\left\{\frac{1/2 \sin w}{\left(1-\frac{1}{2}\cos w\right)}\right\}$$

$$= -4w - \tan^{-1}\left\{\frac{1/2 \sin w}{1-\frac{1}{2}\cos w}\right\}$$

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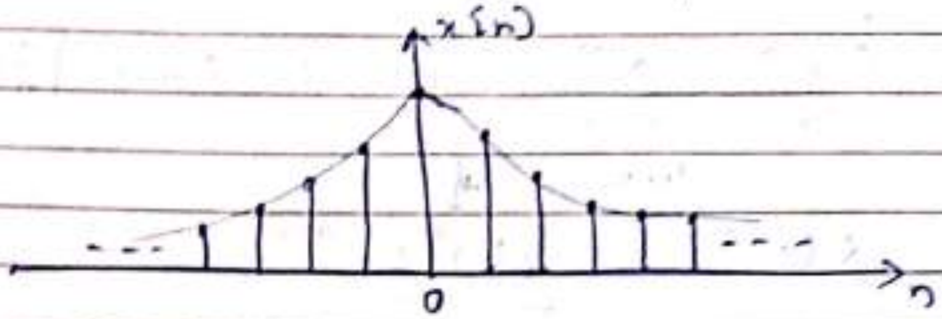
(2)

Eg 3.9b

GO-ASM

Find DTFT of $x[n] = a^{|n|}$; Refer Eg 5.2 of A.O




$$X(e^{j\omega}) = \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}}$$

$$= \frac{1-a^2}{1+a^2-2a\cos\omega} \quad ; \ |a|<1$$

| $\omega$ | $X(e^{j\omega})$ | $\lvert X(e^{j\omega})\rvert$ | $\theta(e^{j\omega})$ |
|---|---|---|---|
| $-\pi$ | $\frac{1-a}{1+a} = +ve$ | $\frac{1-a}{1+a}$ | 0 |
| $-\pi/2$ | $\frac{1-a^2}{1+a^2} = +ve$ | $\frac{1-a^2}{1+a^2}$ | 0 |
| 0 | $\frac{1+a}{1-a} = +ve$ | $\frac{1+a}{1-a}$ | 0 |
| $\pi/2$ | $\frac{1-a^2}{1+a^2} = +ve$ | $\frac{1-a^2}{1+a^2}$ | 0 |
| $\pi$ | $\frac{1-a}{1+a} = +ve$ | $\frac{1-a}{1+a}$ | 0 |

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Eg 3.97

Doubt

Find DTFT of a unit step sequence

$$x[n] = u[n]$$

Since $\sum_{n=-\infty}^{\infty} |x[n]| \not< \infty$, we cannot evaluate

$X(e^{j\omega})$ from Analysis eqn.

Solution :- method-1.

$$\delta[n] = u[n] - u[n-1]$$

or

$$= x[n] - x[n-1]$$

Taking DTFT on both sides.

$$1 = X(e^{j\omega}) - X(e^{j\omega})e^{-j\omega} \quad ; n_0 = 1.$$

$$= [1 - e^{-j\omega}]X(e^{j\omega})$$

When $\omega = 0$, the above equation is not satisfied.
Hence to satisfy we need to have

$$X(e^{j\omega}) = A\,\delta(e^{j\omega}) + \left(\frac{1}{1-e^{-j\omega}}\right)$$

where A is to be evaluated.

→ The even part of the unit step sequence is given by

$$u_e[n] = \frac{1}{2}\{u[n] + u[-n]\}$$

$$= \frac{1}{2} + \frac{1}{2}\delta[n] \quad \rightarrow \text{How?}$$

Then the odd of the unit step sequence is given by

$$u_o[n] = u[n] - u_e[n]$$

$$= x[n] - \frac{1}{2} - \frac{1}{2}\delta[n]$$

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Taking DTFT on both sides.

$$u_o[n] \xleftrightarrow{DTFT} X(e^{jw}) - 2\pi\left(\tfrac{1}{2}\right)\delta(e^{jw}) - \tfrac{1}{2}$$

since DTFT of $\delta[n] = 1$.

$$= A\,\delta(e^{jw}) + \frac{1}{1-e^{-jw}} - \pi\,\delta(e^{jw}) - \frac{1}{2}.$$

As $u_o[n]$ is real & odd, its DTFT should be purely Imaginary. This is possible only if $A = \pi$

Therefore

$$X(e^{jw}) = \pi\,\delta(e^{jw}) + \frac{1}{1-e^{-jw}}$$

Then

$$u[n] \xleftrightarrow{DTFT} \pi\,\delta(e^{jw}) + \frac{1}{1-e^{-jw}}$$

method-2.

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

Let $n-k = m$, then

$$u[n] = \sum_{m=n}^{-\infty} \delta[m]$$

$$= \sum_{m=-\infty}^{n} \delta[m].$$

(55)

Since m is a dummy variable, it can be replaced by k.

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad \text{how?}$$

Also $\delta[n] \xleftrightarrow{FT} 1.$

By Accumulation Property.

$$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{FT} \pi X(e^{j0})\,\delta(e^{j\omega}) + \frac{1}{1-e^{-j\omega}}\,X(e^{j\omega}).$$

Using $x(k) = \delta[k]$, noting that

$$x[n] = \delta[n] \xleftrightarrow{FT} X(e^{j\omega}) = 1.$$

&

$$X(e^{j0}) = 1.$$

we get

$$\sum_{k=-\infty}^{n} \delta[k] = u[n] \xleftrightarrow{DTFT} \pi\cdot(1)\cdot\delta(e^{j\omega}) + \frac{1}{1-e^{-j\omega}}(1)$$

Then

$$\boxed{u[n] = \pi\,\delta(e^{j\omega}) + \frac{1}{1-e^{-j\omega}}}$$

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Eg 3.98 / Qn-458 Find the DTFT of $x[n] = \delta[6-3n]$. Plot amplitude & phase spectra.

Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta[6-3n]e^{-j\omega n}$$

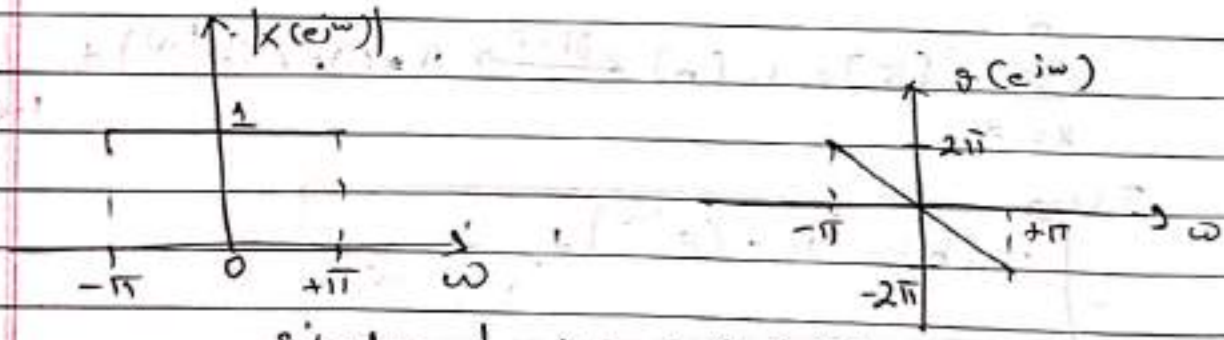
$$= e^{-j\omega n}\Big|_{6-3n=0}$$

$$= e^{-j\omega n}\Big|_{n=2}$$

$$= \underline{\underline{e^{-j2\omega}}}$$

$$|X(e^{j\omega})| = |e^{-j2\omega}| = 1.$$

$$\theta(e^{j\omega}) = \tan^{-1}\left(-\frac{\sin 2\omega}{\cos 2\omega}\right) = -2\omega.$$




Spectra for 1 period ($2\pi$)

Eg-3.99 / Qn-458 Find the Inv. DTFT of

$$X(e^{j\omega}) = 2\pi\delta\left(e^{j(\omega-\omega_0)}\right)$$

Solution

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\cdot e^{j\omega n}\, d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(e^{j(\omega-\omega_0)}).e^{j\omega n}\,d\omega$$

Doubt

$$= \frac{2\pi}{2\pi}.e^{j\omega n}\Big|_{\omega=\omega_0} = \underline{\underline{e^{j\omega_0 n}}}$$

Therefore

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} 2\pi\,\delta(e^{j(\omega-\omega_0)})$$

Important Inference.

(7.) If $\omega_0 = 0$; we get $1 \xleftrightarrow{DTFT} 2\pi\,\delta(e^{j\omega})$

Also, from the property of linearity.

(*) $$A \xleftrightarrow{DTFT} 2\pi A.\delta(e^{j\omega})$$

ES 3.100, GP-459: Find DTFT of $x[n] = \cos(0.2n\pi + \frac{\pi}{4})$. Sketch Amp. + Phase Spectra.

Solution

$$x[n] = \cos(0.2n\pi + \frac{\pi}{4})$$

Property: Using Euler's form.

$$x[n] = \frac{1}{2}e^{j(0.2n\pi+\pi/4)} + \frac{1}{2}e^{-j(0.2n\pi+\pi/4)}$$

$$= \frac{1}{2}e^{j\pi/4}.e^{j0.2n\pi} + \frac{1}{2}e^{-j\pi/4}.e^{-j0.2n\pi}$$

we know

$$A \xleftrightarrow{DTFT} 2\pi A\,\delta(e^{j\omega})$$

Considering $\frac{1}{2}e^{j\pi/4} = A$

(9)

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$$\frac{1}{2}e^{j\pi/4} \xleftrightarrow{DTFT} 2\pi\left(\frac{1}{2}e^{j\pi/4}\right)\delta(e^{j\omega})$$

$$= \pi e^{j\pi/4}\cdot\delta(e^{j\omega})$$

IIIly

$$\frac{1}{2}e^{-j\pi/4} \xleftrightarrow{DTFT} = \underline{\underline{\pi e^{-j\frac{\pi}{4}}\cdot\delta(e^{j\omega})}}$$

<u>Shifting Property</u>

$$e^{j\beta n}\cdot y[n] \xleftrightarrow{DTFT} Y(e^{j(\omega-\beta)})$$

Hence:

$$e^{j0.2\pi n}\cdot\left(\frac{1}{2}e^{j\pi/4}\right) \xleftrightarrow{DTFT} \pi\cdot e^{j\frac{\pi}{4}}\cdot\delta\left(e^{j(\omega-0.2\pi)}\right)$$

&

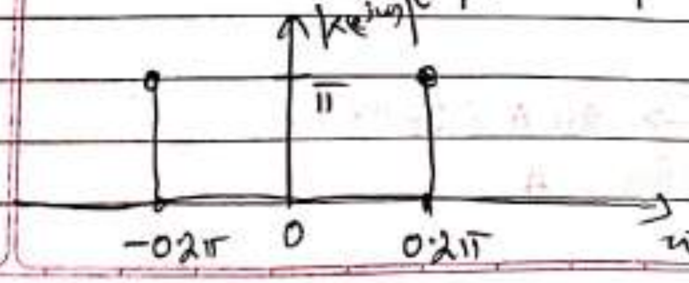
$$e^{-j0.2\pi n}\left(\frac{1}{2}e^{-j\pi/4}\right) \xleftrightarrow{DTFT} \pi\cdot e^{-j\pi/4}\cdot\delta\left(e^{j(\omega+0.2\pi)}\right)$$

<u>Finally, by linearity property</u>

$$x[n] = \cos\left(0.2n\pi+\frac{\pi}{4}\right) \xleftrightarrow{DTFT} \pi\left\{e^{j\frac{\pi}{4}}\delta(e^{j(\omega-0.2\pi)}) + e^{-j\frac{\pi}{4}}\delta\left[e^{j(\omega+0.2\pi)}\right]\right\}$$

<u>Amplitude Spectrum</u>

$$|X(e^{j\omega})| : \begin{cases} |\pi e^{j\pi/4}| = \pi \text{ at } \pi = 0.2\pi \\ |\pi e^{-j\pi/4}| = \pi \text{ at } \pi = -0.2\pi \end{cases}$$

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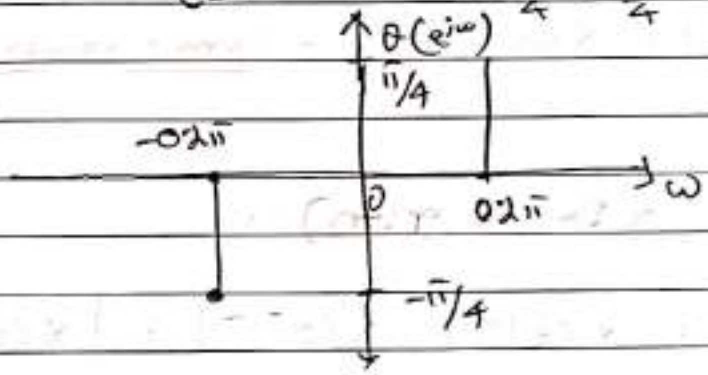
**Phase Spectrum.**

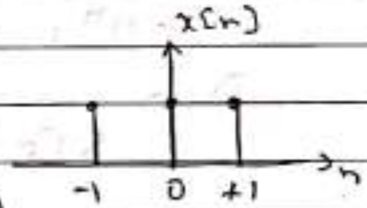
$$\theta(e^{j\omega}) = \begin{cases} \angle \pi e^{j\pi/4} = 0 + \frac{\pi}{4} = \frac{\pi}{4} & \text{at } \omega = 0.2\pi \\ \angle \pi e^{-j\frac{\pi}{4}} = 0 - \frac{\pi}{4} = -\frac{\pi}{4} & \text{at } \omega = -0.2\pi \end{cases}$$




Eg 3.101 (Q-4b) Compute DTFT of $x[n] = u[n+1] - u[n-2]$ (−1). Sketch the spectra over $-\pi \le \omega < \pi$.

**Solution**

$$x[n] = u[n+1] - u[n-2] \quad (-1)$$



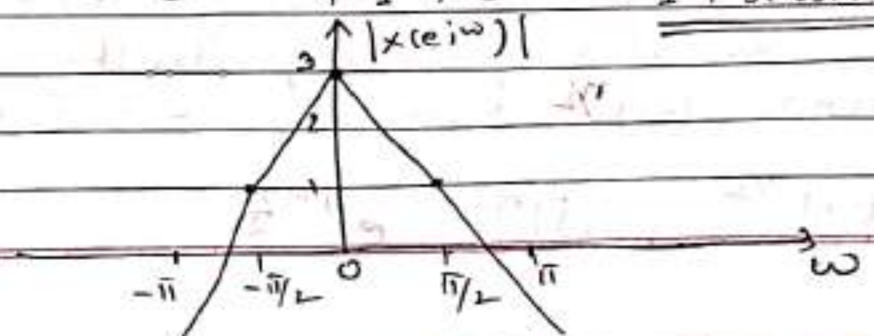

$$= \delta[n+1] + \delta[n] + \delta[n-1]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \{\delta[n+1] + \delta[n] + \delta[n-1]\}\, e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=-1} + e^{-j\omega n}\Big|_{n=0} + e^{-j\omega n}\Big|_{n=+1}$$

$$= e^{j\omega} + 1 + e^{-j\omega} = \underline{1 + 2\cos\omega}$$



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Eg 3.102 Gn-461

The DTFT of real signal $x[n]$ is $X(e^{j\omega})$. How is the DTFT of the following signals is related to $X(e^{j\omega})$

(a) $y[n] = x[-n]$

A.Oppen

$Y(e^{j\omega}) = X(e^{-j\omega})$ - Time reversal.

(b) $g[n] = x[n] * x[-n]$

$G(e^{j\omega}) = X(e^{j\omega}) \cdot X(e^{-j\omega}) = |X(e^{j\omega})|^2$

(c) $s[n] = (-1)^n x[n]$

$(-1)^n = e^{j\pi n}$

Then

$S(e^{j\omega}) = X(e^{j(\omega-\pi)})$ - frequency Shift.

(d) $z[n] = [1 + \cos n\pi] x[n]$

$\cos n\pi = (-1)^n = e^{j\pi n}$

Then

$z[n]$ ~~$Z(e^{j\omega})$~~ $= \{1 + e^{j\pi n}\} x[n]$

$= x[n] + e^{j\pi n} x[n]$

Therefore

$Z(e^{j\omega}) = X(e^{j\omega}) + X(e^{j(\omega-\pi)})$

Linearity + frequency Shift.

(e) $h[n] = (-1)^{n/2} x[n] = (-1)^{n/2} x[n]$

$(-1)^{n/2} = (j)^n = e^{jn\frac{\pi}{2}}$

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Then

$$b[n] = e^{jn\frac{\pi}{2}} \cdot x[n]$$

Therefore

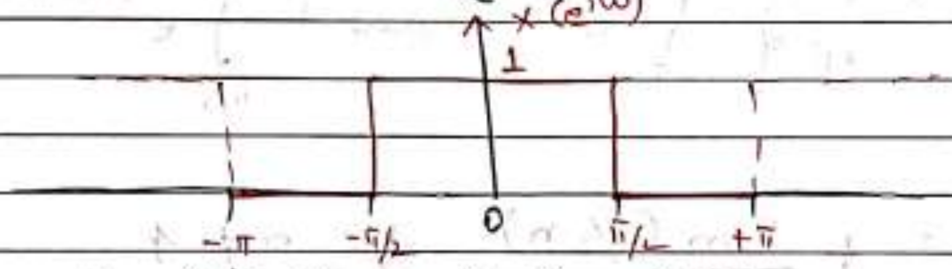
$$B(e^{j\omega}) = X\left(e^{j(\omega - \pi/2)}\right)$$

Frequency Shift.

Ex 103 / Q. No 462

Find the Inverse DTFT of $X(e^{j\omega})$ described over $|\omega| \le \pi$ as

$$X(e^{j\omega}) = \text{rect}\left(\frac{\omega}{\pi}\right)$$




From the definition of Inv. DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi j n} \left[ e^{j\frac{\pi}{2} n} - e^{-j\frac{\pi}{2} n} \right] = \frac{1}{\cancel{2\pi}\,\pi n} \left[ \frac{e^{j\frac{\pi}{2} n} - e^{-j\frac{\pi}{2} n}}{2j} \right]$$

$$= \frac{1}{\pi n} \cdot \sin\left(\frac{\pi}{2} n\right)$$

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E 3.104
Rn - 463

Find $x[n]$ if

$$X(e^{j\omega}) = e^{-j4\omega} \quad ; \quad \frac{\pi}{2} < |\omega| < \pi$$

Solution

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} . d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\pi/2} e^{-j4\omega} . e^{j\omega n} . d\omega + \frac{1}{2\pi} \int_{\pi/2}^{\pi} e^{-j4\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\pi/2} e^{j\omega(n-4)} . d\omega + \int_{\pi/2}^{\pi} e^{j\omega(n-4)} . d\omega \right\}$$

$$= + \frac{\sin(\pi/2\, n)}{\pi(n-4)} \quad \text{for } n \neq 4$$

For $n = 4$

$$x[4] = \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\pi/2} e^{0} d\omega + \int_{\pi/2}^{\pi} e^{0} d\omega \right\}$$

$$= \frac{1}{2}$$

Summarizing

$$x[n] = \begin{cases} + \frac{\sin(\frac{\pi}{2} n)}{\pi(n-4)} & \text{for } n \neq 4 \\ 1/2 & \text{for } n = 4 \end{cases}$$

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Eg 3.205
GR-469

Find Inv. DTFT of

$$X(e^{j\omega}) = \frac{3 - \frac{5}{4}e^{-j\omega}}{\frac{1}{8}e^{-j2\omega} - \frac{3}{4}e^{-j\omega} + 1}$$

Solution

Let $e^{-j\omega} = v$

Then

$$X(e^{j\omega}) = \frac{3 - \frac{5}{4}v}{\frac{1}{8}v^2 - \frac{3}{4}v + 1}.$$

$$= \frac{3 - \frac{5}{4}v}{\left(\frac{v}{4} - 1\right)\left(\frac{v}{2} - 1\right)}$$

$$= \frac{A}{\left(\frac{v}{4} - 1\right)} + \frac{B}{\left(\frac{v}{2} - 1\right)}$$

Solving for the residues.

$$A = \left.\frac{3 - \frac{5}{4}v}{\left(\frac{v}{2} - 1\right)}\right|_{v=4} = -2.$$

$$B = \left.\frac{3 - \frac{5}{4}v}{\left(\frac{v}{4} - 1\right)}\right|_{v=2} = -1.$$

Therefore

$$X(e^{j\omega}) = \frac{-2}{\left(\frac{v}{4} - 1\right)} - \frac{1}{\left(\frac{v}{2} - 1\right)}$$

$$= \frac{2}{1 - \frac{v}{4}} + \frac{1}{1 - \frac{v}{2}}$$

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or:

$$X(e^{j\omega}) = \frac{2}{1-\frac{1}{4}e^{-j\omega}} + \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

we know

(X)

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

Then

$$x[n] = 2\left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n]$$

Eg 3.106 / GN-AKT Find Inv. DTFT of

$$X(e^{j\omega}) = \frac{3-\frac{1}{4}e^{-j\omega}}{-\frac{1}{16}e^{-j2\omega}+1}$$

Solution

Let $e^{-j\omega} = v$

then

$$X(e^{j\omega}) = \frac{3-\frac{1}{4}v}{-\frac{1}{16}v^2+1} = \frac{3-\frac{1}{4}v}{(1-\frac{v}{4})(1+\frac{v}{4})}$$

$$= \frac{A}{(1-v/4)} + \frac{B}{(1+v/4)}$$

Then

$$A = \left.\frac{3-\frac{1}{4}v}{1+v/4}\right|_{v=4} = 1.$$

$$B = \left.\frac{3-\frac{1}{4}v}{1-v/4}\right|_{v=-4} = 2$$

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Therefore

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{4}v} + \frac{2}{1-(-\frac{1}{4}v)}$$

$$= \frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{2}{1-(-\frac{1}{4})e^{-j\omega}}$$

Then

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{-1}{4}\right)^n u[n]$$

Eg 3.10 GP-466 Using Convolution Theorem, find Inv. DTFT of

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} \; ; \; |a|<1$$

Rofah

Solution

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})^2} = \frac{1}{(1-ae^{-j\omega})} \times \frac{1}{(1-ae^{-j\omega})}$$

We know

$$a^n u[n] \xleftrightarrow{DTFT} \frac{1}{(1-ae^{-j\omega})}$$

Therefore

$$x[n] = a^n u[n] * a^n u[n]$$

$$= \sum_{k=-\infty}^{\infty} a^k u[k] \cdot a^{n-k} u[n-k] \quad ; \; a^k \cdot a^{n-k} = a^n$$

$$= a^n \sum_{k=-\infty}^{\infty} u[k]\, u[n-k]$$

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**Also**

$$u[k]u[n-k] = \begin{cases} 1, & 0 \le k \le n;\ n \geq 0 \\ 0, & n < 0 \end{cases}$$

Hence for $n \geq 0$

$$x[n] = a^n \sum_{k=0}^{n} 1 = a^n.(n+1)$$

For $n < 0$

$$x[n] = a^n \sum_{k=0}^{n} 0 = 0$$

**Summarizing**

$$x[n] = \begin{cases} (n+1)a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(or)

$$x[n] = (n+1)a^n u[n].$$

Eg 3.108 Pn-467. Find the DTFT of the sequences given below.

(a) $x_1[n] = \alpha^n \cos(\omega_0 n + \phi)u[n]$

$$= \alpha^n \left\{ \frac{e^{j(\omega_0 n+\phi)} + e^{-j(\omega_0 n+\phi)}}{2} \right\} u[n]$$

$$= \frac{1}{2} e^{j\phi} \alpha^n e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\phi} \alpha^n e^{-j\omega_0 n} u[n]$$

$$= \frac{1}{2} e^{j\phi} [\alpha e^{j\omega_0}]^n u[n] + \frac{1}{2} e^{-j\phi} [\alpha e^{-j\omega_0}]^n u[n]$$

Considering the existence of $x_1[n]$

Then

$$X_1(e^{j\omega}) = \frac{1}{2} e^{j\phi} \sum_{n=0}^{\infty} (\alpha e^{j\omega_0})^n e^{-j\omega n}$$

$$+ \frac{1}{2} e^{-j\phi} \sum_{n=0}^{\infty} (\alpha e^{-j\omega_0})^n e^{-j\omega n}$$

$$= \frac{1}{2} e^{j\phi} \cdot \frac{1}{1-\alpha e^{j\omega_0} e^{-j\omega}} + \frac{1}{2} e^{-j\phi} \frac{1}{1-\alpha e^{-j\omega_0} e^{-j\omega}}$$

$$= \frac{1}{2} \left\{ \frac{e^{j\phi} - e^{j\phi} \alpha e^{-j\omega_0} e^{-j\omega} + e^{-j\phi} - \alpha e^{+j\omega_0} e^{-j\omega}}{1 - \alpha e^{-j\omega_0} e^{-j\omega} - \alpha e^{j\omega_0} e^{-j\omega} + \alpha^2 e^{-j2\omega}} \right\}$$

$$= \frac{\cos\phi - \alpha \cos\phi \cos\omega_0 e^{-j\omega}}{1 - 2\alpha \cos\omega_0 e^{-j\omega} + \alpha^2 e^{-2j\omega}}$$

(b) $x_2[n] = n\alpha^n u[n] \quad ; |\alpha| < 1$

Let $s[n] = \alpha^n u[n] \xleftrightarrow{DTFT} S(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}}$

Then

$x_2[n] = n\, s[n]$

Differentiation property

$$(-jn)\, x[n] \xleftrightarrow{DTFT} \frac{d X(e^{j\omega})}{d\omega}$$

In the present case

$$X_2(e^{j\omega}) = j \cdot \frac{d}{d\omega} S(e^{j\omega}) = j \frac{d}{d\omega} \left\{ \frac{1}{1-\alpha e^{-j\omega}} \right\}$$

$$= \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

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Eg 3.109
GR - 468

Find Inv. DTFT of
$$H_2(e^{jw}) = 1 + 2\cos w + \frac{3}{2}(1+\cos 2w)$$

Solution

$$H_2(e^{jw}) = 1 + 2\cos w + \frac{3}{2} + \frac{3}{2}\cos 2w$$

$$2\cos w = e^{jw} + e^{-jw}$$

$$\frac{3}{2}\cos 2w = \frac{3 \times 2\cos 2w}{4} : \underline{\underline{e^{j2w} + e^{-j2w}}}$$

$$= \frac{3}{4}\{e^{j2w} + e^{-j2w}\}$$

$$H_2(e^{jw}) = 1 + \frac{3}{2} + e^{jw} + e^{-jw} + \frac{3}{4}e^{j2w} + \frac{3}{4}e^{-j2w}$$

$$= \frac{3}{4}e^{-j2w} + e^{-jw} + \frac{5}{2} + e^{jw} + \frac{3}{4}e^{j2w}$$

It is a length-5 sequence. From the co-efficient.

$$H_2[n] = \{0.75, 1, \underset{\uparrow}{2.5}, 1, 0.75\}, \quad -2 \le n \le +2$$

Eg 3.110
GR - 468

Using Parseval's Relation, evaluate
$$\int_0^{\pi} \frac{4}{5+4\cos w}\, dw$$

Property

Solution: Let $x[n] = -\alpha^n u[-n-1]$, then
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{-1} -\alpha^n e^{-jwn}$$

$$= -\sum_{n=1}^{\infty} \alpha^{-n} e^{jwn}$$

$$= -\alpha^{-1} e^{jwn} \sum_{n=0}^{\infty} \left\{\frac{e^{jw}}{\alpha}\right\}^n$$

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For $|\alpha|>1$ $\quad X(e^{j\omega}) = -\alpha^{-1}e^{j\omega}\dfrac{1}{1-\left(\frac{e^{j\omega}}{\alpha}\right)} = \dfrac{1}{1-\alpha e^{-j\omega}}$

$\therefore \left|X(e^{j\omega})\right|^2 = \dfrac{1}{1+\alpha^2-2\alpha\cos\omega}$

By Parseval's relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty}|x[n]|^2$$

$$\Rightarrow \int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega = 2\pi\sum_{n=-\infty}^{\infty}|x[n]|^2$$

Let

$$|X(e^{j\omega})|^2 = \frac{1}{5+4\cos\omega} \Rightarrow \alpha = -2.$$

Hence $x[n] = -(-2)^n u[-n-1]$

Now

$$\int_0^{\pi}\frac{4}{5+4\cos\omega}d\omega = 4\int_0^{\pi}|X(e^{j\omega})|^2 d\omega$$

$$= 2\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega$$

$$= 4\pi\sum_{n=-\infty}^{\infty}|x[n]|^2$$

$$= 4\pi\sum_{n=-\infty}^{-1}|(-2)^n|^2$$

$$= 4\pi\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= \pi\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= \frac{4\pi}{3}$$

Eg 3.111 / GD-170 Without computing DTFT, determine which one of the following sequences have a real-valued DTFTs + which ones have imaginary DTFTs.

(a) $y_1[n] = \begin{cases} |n| & -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$

Since $|-n| = |n|$, $y_1[n]$ is an even sequence + hence $Y_1(e^{j\omega})$ is real-valued.

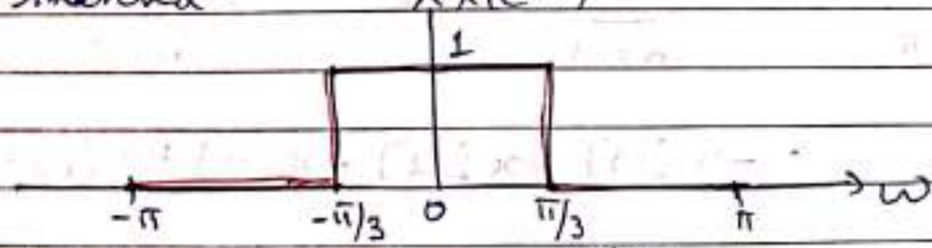
(b) $y_2[n] = \begin{cases} n^3 & -N \le n \le N \\ 0 & \text{otherwise} \end{cases}$

Since $(-n)^3 = -n^3$, $y_1[n]$ is an odd sequence + hence $Y_2(e^{j\omega})$ is imaginary valued.

(c) $y_3[n] = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2}{\pi n} & \text{for } n \text{ odd} \end{cases}$

Since $y_3[n]$ is an odd sequence, $Y_3(e^{j\omega})$ is an imaginary valued DTFT.

Eg 3.112 A sequence $x[n]$ has a zero-phase DTFT $X(e^{j\omega})$ as sketched.




Sketch the sequence $x[n]e^{\frac{-j\pi n}{3}}$

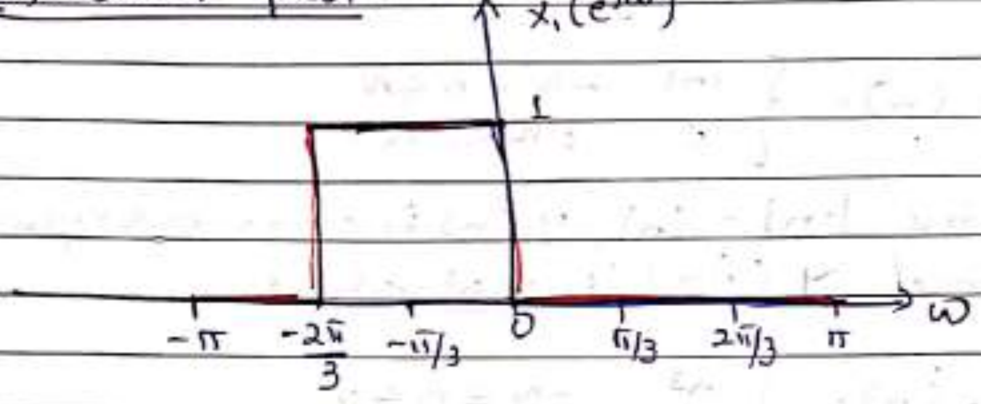
Solution

Let $x_1[n] = x[n]e^{\frac{-j\pi n}{3}}$

Applying Shifting property (frequency)

$$X_1(e^{j\omega}) = X(e^{j(\omega+\frac{\pi}{3})})$$

Resultant plot




Eg 3.113 GR-471

Consider the discrete sequence

$$x[n] = \{4, -1, 3, \underset{\uparrow}{-2}, 3, -1, 4\}$$

(-3) (+3)

Without explicitly finding $X(e^{j\omega})$ evaluate the following functions of the DTFT:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-3}^{+3} x[n]e^{-j\omega n}$$

(a) $\underset{\omega=0}{X(e^{j0})} = \sum_{n=-3}^{+3} x[n] = 4-1+3-2+3-1+4 = 10$

(b) $\underset{\omega=\pi}{X(e^{j\pi})} = \sum_{n=-3}^{+3} x[n]e^{-j\pi n} = \sum_{n=-3}^{+3} (-1)^n x[n]$

$= -x[-3] + x[-2] - x[-1] + x[0] - x[1] + x[2] - x[3]$

$= -4 + (-1) - (+3) + (-2) - (3) + (-1) - (4)$

$= -4 - 1 - 3 - 2 - 3 - 1 - 4 = \underline{\underline{-18}}$

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(c) $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cdot e^{j\omega n}\, d\omega$$

with $n=0$ gives

$$\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega = 2\pi\cdot x[0] = 2\pi(-2) = \underline{\underline{-4\pi}}$$

(d) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

Solution

By Parseval's relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega$$

~~Hence~~ $\cancel{\int_{-\pi}^{\pi} |x[n]|}$

Hence

$$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi\cdot \sum_{n=-\infty}^{\infty} |x[n]|^2$$

For the present problem

$$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-3}^{+3} |x[n]|^2$$

$$= 2\pi\{4^2 + (-1)^2 + 3^2 + (-2)^2 + 3^2 + (-1)^2 + 4^2\}$$

$$= 2\pi\{\underbrace{16+1+9}_{26} + 4 + \underbrace{9+1+16}_{26}\} = 2\pi\{56\}$$

$$= \underline{\underline{112\pi}}$$

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(e) $Arg[X(e^{jw})]$

Solution

$x[-n] = x[n]$, the sequence is an even sequence.

The DTFT of a real + even sequence is purely real.

$$\theta(e^{jw}) = Arg[X(e^{jw})] = \begin{cases} 0 & X(e^{jw}) \geq 0 \\ \pi & X(e^{jw}) < 0,\ w < 0 \\ -\pi & X(e^{jw}) < 0,\ w > 0 \end{cases}$$

Note: $\theta(e^{jw})$ must exhibit odd symmetry about $w = 0$.

Eg 3.24 / GE-473 Using the properties of DTFT, find the DTFT of the following:

(a) $y[n] = (n+1)a^n u[n]$

Solution: We are aware that;

$$x_1[n] = a^n u[n] \xleftrightarrow{DTFT} X_1(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

important

By frequency differentiation property.

$$-jn\, x[n] \xleftrightarrow{DTFT} \frac{dX(e^{jw})}{dw}$$

Then $n\, x[n] \xleftrightarrow{DTFT} j\dfrac{dX(e^{jw})}{dw}$

Therefore applying differentiation property.

$$x_2[n] = n a^n u[n] = n\, x_1[n] \xleftrightarrow{DTFT} X_2(e^{jw}) = j\frac{dX_1(e^{jw})}{dw}$$

Hence $X_2(e^{jw}) = j\frac{d}{dw}\left\{\frac{1}{1-ae^{-jw}}\right\}$

$$= \frac{ae^{-jw}}{(1-ae^{-jw})^2}.$$

Finally applying the linearity property.

$y[n] = x_1[n] + x_2[n] \Rightarrow Y(e^{jw}) = X_1(e^{jw}) + X_2(e^{jw})$

$$Y(e^{jw}) = \frac{1}{1-ae^{-jw}} + \frac{ae^{-jw}}{(1-ae^{-jw})^2}$$

$$= \underline{\underline{\frac{1}{(1-ae^{-jw})^2}}}$$

$$Y(e^{jw}) = \frac{1}{(1-ae^{-jw})^2}$$

(b) $z[n] = \frac{(n+2)(n+1)}{2} a^n u[n]$

Rewriting

$z[n] = \frac{1}{2}n(n+1)a^n u[n] + (n+1)a^n u^n$

$= \frac{1}{2} z_2[n] + z_1[n]$

From part (a)

$z_1[n] = (n+1)a^n u[n] \xleftrightarrow{DTFT} Z_1(e^{jw}) = \frac{1}{(1-ae^{-jw})^2}$

Applying Differentiation Property.

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$z_2[n] = n(n+1)a^n u[n] = n\, z_1[n] \xleftrightarrow{DTFT} Z_2(e^{jw}) = j\frac{d}{dw}\{X_1(e^{jw})\}$

Hence

$$Z_2(e^{jw}) = j\left\{\frac{(-2)(-a)(-j)e^{-jw}}{(1-ae^{-jw})^2}\right\}$$

$$= \frac{2ae^{-jw}}{(1-ae^{-jw})^3}$$

Applying Linearity Property, we have

$$z[n] = \frac{1}{2}z_2[n] + z_1[n] \xleftrightarrow{DTFT} \frac{1}{2}Z_2(e^{jw}) + Z_1(e^{jw})$$

or

$$Z(e^{jw}) = \frac{ae^{-jw}}{(1-ae^{-jw})^3} + \frac{1}{(1-ae^{-jw})^2}$$

$$= \frac{1}{(1-ae^{-jw})^3}$$

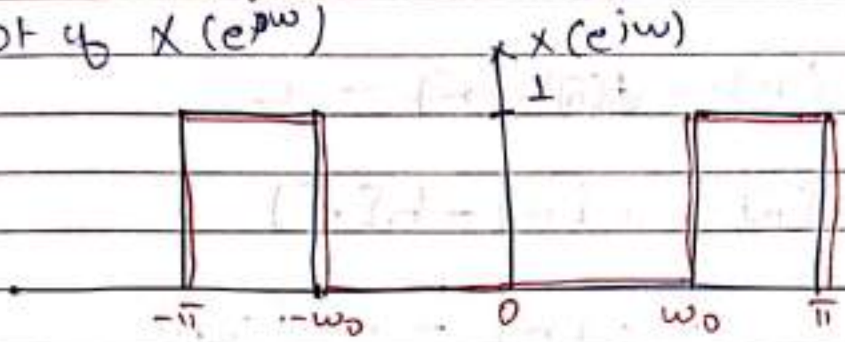
By Induction

$$\therefore \frac{(n+N-1)(n+N-2)\cdots(n+1)}{(N-1)!}a^n u[n] \xleftrightarrow{DTFT} \frac{1}{(1-ae^{-jw})^N}$$

Eg 3.115 GP-475 Evaluate the inverse DTFT of ;

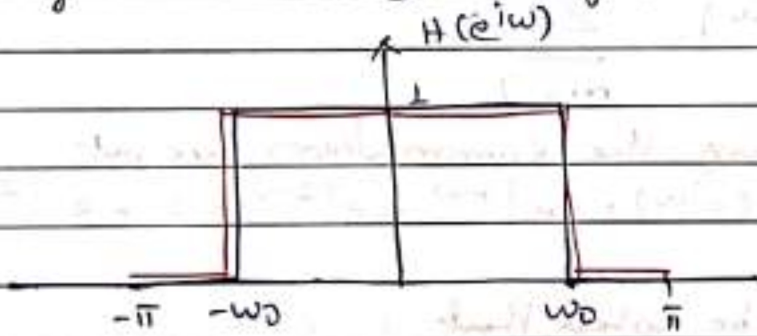
(a) $X(e^{j\omega}) = \begin{cases} 1 & \pi \geq |\omega| \geq \omega_0 \\ 0 & |\omega| < \omega_0 \end{cases}$

Plot of $X(e^{j\omega})$




Solution

Let us find the Inv. DTFT of $H(e^{j\omega})$ as shown




$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left\{\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{jn}\right\} = \frac{1}{\cancel{2}\pi \cancel{j} n}\cancel{2}\cancel{j}\cdot \sin(\omega_0 n)$$

$$= \frac{\sin(\omega_0 n)}{\pi n}$$

(2b)

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we can also write

$$X(e^{j\omega}) = 1 - H(e^{j\omega})$$

Taking Inv. DTFT on both sides.

$$\cancel{x[n] = \delta[n] h[n] = 1}$$

$$x[n] = \delta[n] - h[n]$$

$$= \delta[n] - \frac{\sin(\omega_0 n)}{\pi n}$$

(b) $X(e^{j\omega}) = \sum_{m=-2}^{2} e^{-j2m\omega}$

Expanding the summation, we get

$$X(e^{j\omega}) = e^{j4\omega} + e^{j2\omega} + 1 + e^{-j2\omega} + e^{-j4\omega}$$

It may be noted that

$$\delta[n-n_0] \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-j\omega n} = e^{-j\omega n_0}$$

Hence

$$x[n] = \delta[n+4] + \delta[n+2] + 1 + \delta[n-2] + \delta[n-4]$$

$$= \begin{cases} 1, & n = 0, \pm 2, \pm 4 \\ 0 & \text{elsewhere} \end{cases}$$

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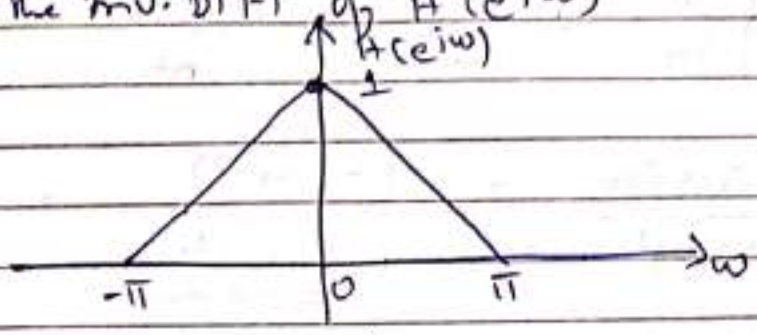
Eg 3.226
ia-477

Find the Inv. DTFT of $H(e^{j\omega})$




Solution : The analytical expression for $H(e^{j\omega})$ is

$$H(e^{j\omega}) = \begin{cases} 1+\frac{\omega}{\pi} & ; -\pi < \omega < 0 \\ 1-\frac{\omega}{\pi} & ; 0 < \omega < \pi \end{cases}$$

By definition

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\left\{\left(\frac{\omega+\pi}{\pi}\right)\right\} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{\pi-\omega}{\pi}\right) e^{j\omega n} d\omega$$

Let $m = \omega+\pi$ & $s = \pi-\omega$

$$\therefore h[n] = \frac{1}{2\pi^2}\int_{0}^{\pi} m\, e^{j(m-\pi)} dm - \frac{1}{2\pi^2}\int_{\pi}^{0} s\, e^{(\pi-s)n} ds$$

$$= \frac{(-1)^n}{2\pi^2}\int_{0}^{\pi} m e^{jmn} dm + \frac{(-1)^n}{2\pi^2}\int_{0}^{\pi} s e^{-jsn} ds$$

Integration by part results in

$$h[n] = \frac{(-1)^n}{2\pi^2}\left\{\frac{\pi e^{j\pi n}}{jn} + \frac{e^{j\pi n}-1}{n^2} - \frac{\pi e^{-j\pi n}}{jn} + \frac{e^{-j\pi n}-1}{n^2}\right\}$$

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$$= \frac{1}{2\pi^2}\left\{\frac{\pi}{jn} - \frac{\pi e^{-j2\pi n}}{jn} + \frac{1-e^{-j\pi n}}{n^2} + \frac{1-e^{-j\pi n}}{n^2}\right\}$$

$$= \frac{1-\cos \pi n}{(\pi n)^2} = \frac{1-(-1)^n}{(\pi n)^2}$$

Eg 3.227 / RP-478

The impulse response of an LTI system is given by

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Find the response of the system for an input

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

Solution :- We are aware that

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

Hence $x(n) = \left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{DTFT} X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{3}e^{-j\omega}}$

IIIly

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{DTFT} H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Applying Convolution Property.

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Hence

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}$$

Let $e^{-j\omega} = v$, we have

$$Y(e^{j\omega}) = \frac{1}{(1-\frac{1}{3}v)(1-\frac{1}{2}v)} = \frac{k_1}{(1-1/3v)} + \frac{k_2}{(1-1/2v)}$$

$$k_1 = \left.\frac{1}{(1-\frac{1}{2}v)}\right|_{v=3} = 3$$

$$k_2 = \left.\frac{1}{(1-\frac{1}{3}v)}\right|_{v=2} = -2.$$

Then

$$Y(e^{j\omega}) = \frac{3}{(1-\frac{1}{3}e^{-j\omega})} - \frac{2}{(1-\frac{1}{2}e^{-j\omega})}$$

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Taking Inv. DTFT on both sides.

$$y[n] = 3\left(\frac{1}{3}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n]$$

ES3118

GR-A78

Find $x[n]$ for the transforms

(a) $X(e^{j\omega}) = 4\sin 5\omega + 2\cos\omega$

using Euler's formula

$$X(e^{j\omega}) = \frac{4\{e^{j5\omega} - e^{-j5\omega}\}}{2j} + \frac{2\{e^{j\omega} + e^{-j\omega}\}}{2}$$

$$= -j2e^{j5\omega} + j2e^{-j5\omega} + e^{j\omega} + e^{-j\omega}$$

Comparing this with

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$ we get

$$x[n] = \begin{cases} -j2 & \text{for } n = 5 ✓ \\ j2 & \text{for } n = +5 ✓ \\ 1 & \text{for } n = -1 ✓ \\ 1 & \text{for } n = +1 ✓ \\ 0 & \text{for otherwise} \end{cases}$$

Alternate form:

$$x[n] = -j2\,\delta[n+5] + 1\,\delta[n+1] + 1\,\delta[n-1] + j2\,\delta[n-5]$$

(b) $$X(e^{j\omega}) = \frac{4}{[e^{-j\omega}-2]^2}$$

Rewriting $X(e^{j\omega})$ as

$$X(e^{j\omega}) = 2e^{j\omega}\left\{\frac{\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right\}$$

we know

$$n\,a^n u[n] \xleftrightarrow{DTFT} \frac{a\,e^{-j\omega}}{1-a e^{-j\omega}}$$

Let

$$x_1[n] = n\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow X_1(e^{j\omega}) \left\{\frac{\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right\}$$

Applying time-shift & linearity properties.

$$x_2[n] = 2\,x_1(n+1) \xleftrightarrow{DTFT} X_2(e^{j\omega}) = 2e^{j\omega}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\}$$

Therefore

$$x_2[n] = 2(n+1)\left(\frac{1}{2}\right)^{n+1} \cdot u[n+1]$$

2

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9A

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FS3-229
GR-480
Properties

Using Properties of DTFT, find the DTFT of
$$y[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi n}{2}\right) u[n]$$

Solution :: To proceed further, let us remember the following

1. $a^n u[n] \longleftrightarrow \dfrac{1}{1-ae^{-jw}}$

2. $a^n \cos(w_0 n) u[n] \longleftrightarrow \dfrac{1 - a\cos w_0\, e^{-jw}}{1 - 2a\cos w_0\, e^{-jw} + a^2 e^{-j2w}}$

3. $a^n \sin(w_0 n) u[n] \longleftrightarrow \dfrac{a \sin w_0\, e^{-jw}}{1 - 2a\cos w_0\, e^{-jw} + a^2 e^{-j2w}}$

Let $y[n] = x_1[n] * x_2[n]$, where
$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$ + $x_2[n] = \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{2} n\right) u[n]$

Using the above listed FT identities, we have
$$X_1(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}}$$
$$X_2(e^{jw}) = \frac{1 - \frac{1}{3}\cos(\pi/2)\, e^{-jw}}{1 - 2\left(\frac{1}{3}\right)\cos(\pi/2)\, e^{-jw} + \frac{1}{9} e^{-j2w}}$$
$$= \frac{1}{1 + \frac{1}{9} e^{-j2w}}$$

Applying Convolution property
$$y[n] = x_1[n] * x_2[n] \longleftrightarrow Y(e^{jw}) = X_1(e^{jw}) X_2(e^{jw})$$

Hence $$Y(e^{jw}) = \frac{1}{\left(1+\frac{1}{9}e^{-j2w}\right)\left(1-\frac{1}{2}e^{-jw}\right)}$$

Let $e^{-jw} = v$, we have

$$Y(e^{jw}) = \frac{1}{\left(1+\frac{1}{9}v^2\right)\left(1-\frac{v}{2}\right)}$$

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$$= \frac{K_1 + K_2 v}{\left(1+\frac{1}{9}v^2\right)} + \frac{K_3}{(1-v/2)}$$

The residues can be found out as

$$K_1 = \frac{4}{13}\ ;\ K_2 = \frac{2}{13}\ \ \& \ K_3 = 9/13$$

Hence

$$Y(e^{j\omega}) = \frac{9}{13}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\} + \frac{2}{13}\left\{\frac{e^{-j\omega}}{1+\frac{1}{9}e^{-2j\omega}}\right\} + \frac{4}{13}\left\{\frac{1}{1+\frac{1}{9}e^{-j2\omega}}\right\}$$

For finding the Inv. DTFT of $Y(e^{j\omega})$, we need the following

$$\left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{2}n\right)u(n) \longleftrightarrow \frac{1}{1+\left(\frac{1}{3}\right)^2 e^{-j2\omega}}$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{2}n\right)u[n] \longleftrightarrow \frac{1/3\, e^{-j\omega}}{1+\left(\frac{1}{3}\right)^2 e^{-j2\omega}}$$

Rewriting $Y(e^{j\omega})$ as.

$$Y(e^{j\omega}) = \frac{9}{13}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\} + \frac{6}{13}\left\{\frac{\frac{1}{3}e^{-j\omega}}{1+\left(\frac{1}{3}\right)^2 e^{-j2\omega}}\right\} + \frac{4}{13}\left\{\frac{1}{1+\left(\frac{1}{3}\right)^2 e^{-j2\omega}}\right\}$$

Taking Inv. DTFT, we have

$$y[n] = \frac{9}{13}\left(\frac{1}{2}\right)^n u(n) + \frac{6}{13}\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{2}n\right)u[n] + $$
$$+ \frac{4}{13}\left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{2}n\right)$$

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ES 3.120
GR-482
(a)

Find DTFT of

$x[n] = a^n \cos(\omega_0 n) u[n]$

**Solution:** Using Euler's Identity:

$$x[n] = \frac{1}{2} a^n e^{j\omega_0 n} u[n] + \frac{1}{2} a^n e^{-j\omega_0 n} u[n]$$

$$= \frac{1}{2} \{a e^{j\omega_0}\}^n u[n] = \frac{1}{2} \{a e^{-j\omega}\}^n u[n]$$

Since $b^n u[n] \longleftrightarrow \dfrac{1}{1-be^{-j\omega}}$

we have

$$x_1[n] = [a e^{j\omega_0}]^n u[n] \longleftrightarrow X_1(e^{j\omega}) = \frac{1}{1-ae^{j\omega_0}e^{-j\omega}} = \frac{1}{1-ae^{-j(\omega-\omega_0)}}$$

IIIly

$$x_2[n] = [a e^{-j\omega_0}]^n u[n] \longleftrightarrow X_2(e^{j\omega}) = \frac{1}{1-ae^{-j\omega_0}e^{-j\omega}}$$

$$= \frac{1}{1-ae^{-j(\omega+\omega_0)}}$$

**Applying Linearity Property.**

$$x[n] = \frac{1}{2} x_1[n] + \frac{1}{2} x_2[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{2}\left[ X_1(e^{j\omega}) + X_2(e^{j\omega}) \right]$$

**Therefore**

$$X(e^{j\omega}) = \frac{1}{2}\left\{ \frac{1}{1-ae^{-j(\omega-\omega_0)}} + \frac{1}{1-ae^{-j(\omega+\omega_0)}} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1-ae^{-j\omega}e^{-j\omega_0} + 1 - ae^{-j\omega}e^{j\omega_0}}{1-ae^{-j\omega}e^{-j\omega_0} - ae^{-j\omega}e^{j\omega_0} + a^2 e^{-j2\omega}} \right\}$$

$$= \frac{1 - a\cos\omega_0 \, e^{-j\omega}}{1-2a\cos\omega_0 \, e^{-j\omega} + a^2 e^{-j2\omega}} \quad //$$

(b) $x[n] = e^{j3n}$

Solution : Recall $A \longleftrightarrow 2\pi\delta(e^{j\omega})$

Hence

$$x_1[n] = 1 \longleftrightarrow X_1(e^{j\omega}) = 2\pi\delta(e^{j\omega})$$

Applying frequency shift property.

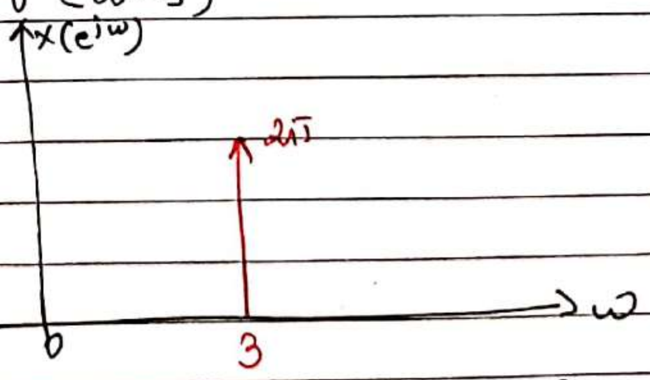
$$x[n] = e^{j3n}x_1[n] \longleftrightarrow X(e^{j\omega}) = X_1(e^{j(\omega-3)})$$

Hence

$$X(e^{j\omega}) = 2\pi\delta(e^{j(\omega-3)})$$

For plotting the spectrum of $X(e^{j\omega})$, it is convenient to express.

$$X(e^{j\omega}) = 2\pi\delta(\omega-3)$$




Spectrum for one period, with fundamental period $= 2\pi$.

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Fourier Representation:

| Time-domain (t) or (n) | Frequency domain ($\omega$) |
|---|---|
| (1) Continuous | Non-periodic |
| (2) Discrete | Periodic |
| (3) Periodic | Discrete |
| (4) Non-periodic | continuous. |

Properties of Continuous Time Fourier Transforms (CTFT) (Continued) Alan. Oppenheim

4.4 The Convolution Property

$$y(t) = h(t) * x(t) \xleftrightarrow{CTFT} Y(j\omega) = H(j\omega)X(j\omega)$$

Convolution $\longleftrightarrow$ multiplication.

Proof: Informal Proof. By the definition of CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{\omega_0 \to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \quad —(1)$$

where $\omega = k\omega_0$.

The response of a linear system with an impulse response $h(t)$ to a complex exponential $e^{jk\omega_0 t}$ is $H(jk\omega_0)e^{jk\omega_0 t}$, where

$$H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-jk\omega_0 t}dt.$$

We can recognise the frequency response $H(j\omega)$ as defined by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

as the Fourier Transform of the system impulse response. In other words, the FT of the impulse response (evaluated at $\omega = k\omega_0$) is the complex scaling factor that the LTI system applies to the Eigenfunction $e^{jk\omega_0 t}$.

→ From Superposition & referring to the equation

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

we have

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \longrightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

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and thus from Eqn. ①, the response of the linear system to $x(t)$ is

$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) H(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega \quad ②$$

Since $y(t)$ and its FT $Y(j\omega)$ are related by

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(j\omega) e^{j\omega t} d\omega \quad - ③$$

From ② & ③ $\boxed{Y(j\omega) = X(j\omega) H(j\omega)}$

Formal Proof.

Consider the convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

we need $Y(j\omega)$, which is $Y(j\omega) = F\{y(t)\} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right\} e^{-j\omega t} dt$

Interchanging the order of integration & noting that $x(\tau)$ does not depend on t, we have.

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left\{ \int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right\} d\tau$$

By time shift property, the bracketed term is $e^{-j\omega \tau} H(j\omega)$.
Substituting this yields.

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} \cdot H(j\omega) d\tau = H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

The integral is $X(j\omega)$. Hence

$$\boxed{Y(j\omega) = H(j\omega) X(j\omega)}$$

$$\boxed{y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega) X(j\omega)}$$

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$s \to j\omega$
$z \to e^{j\omega}$





$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt$$ - frequency response of a system in the continuous time case.

$h(t) \to$ Impulse response.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n}$$ $\to$ frequency response of a system in the Discrete Time case.

Inferences

(1) $x(t) \to \boxed{H_1(j\omega)} \to \boxed{H_2(j\omega)} \xrightarrow{y(t)}$

(or)

$x(t) \to \boxed{H_1(j\omega) H_2(j\omega)} \xrightarrow{y(t)}$

(or)

$x(t) \to \boxed{H_2(j\omega)} \to \boxed{H_1(j\omega)} \xrightarrow{y(t)}$

(2) Existence of Fourier Transform $H(j\omega)$ of Impulse response $h(t)$ is guaranteed if

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$ - Dirichlet condition

(3) All stable LTI systems have Frequency response.

(4) Continuous Time Fourier Transform $\to$ Laplace Transform.

Applications of Convolution Property

4.A.1 Examples.

BJ4.15 Consider an LTI system with impulse response

$h(t) = \delta(t - t_0)$.

Frequency response of this system is the F.T of $h(t)$ & is given by

$H(j\omega) = e^{-j\omega t_0}$

Thus for any input $x(t)$ with a F.T $X(j\omega)$, the F.T of the output is

$Y(j\omega) = H(j\omega) X(j\omega)$

$= e^{-j\omega t_0} \cdot X(j\omega)$

This amounts to the time-shift property. Specifically, a system with an impulse response $\delta(t-t_0)$ applies a time shift of $t_0$ to any input.

Then

$$\boxed{y(t) = x(t-t_0)}$$

Note

$$|e^{-j\omega t_0}| = 1 \text{ at all frequencies}$$

Phase $= -\omega t_0$ - a linear function of $\omega$

Eg 4.16 Let the input & output of an LTI system be related by

$$y(t) = \frac{d\,x(t)}{dt}$$

By differentiation property.

$$Y(j\omega) = j\omega\, X(j\omega)$$

From convolution property, we infer

$$\boxed{H(j\omega) = j\omega}$$

Frequency response of a differentiator.

Eg 4.17 Consider an Integrator.

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

The impulse response for this system is the unit step. Therefore the frequency response of this system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \quad \text{From Eg 4.11}$$

Then $Y(j\omega) = H(j\omega)\, X(j\omega)$

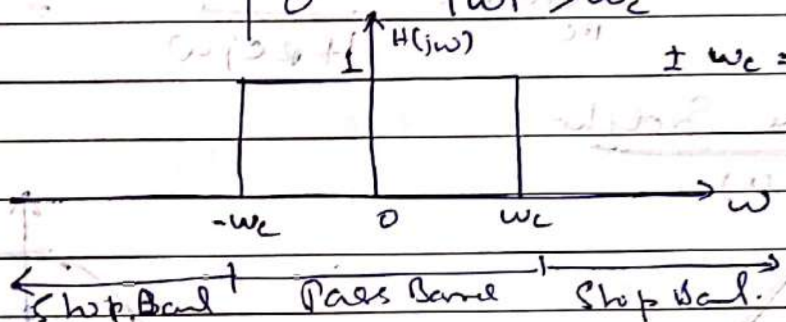
$$= \frac{1}{j\omega} X(j\omega) + \pi X(j\omega)\delta(\omega)$$

$$= \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$ - Integration Property

Eq 4.18 **Ideal Low-pass filter.**

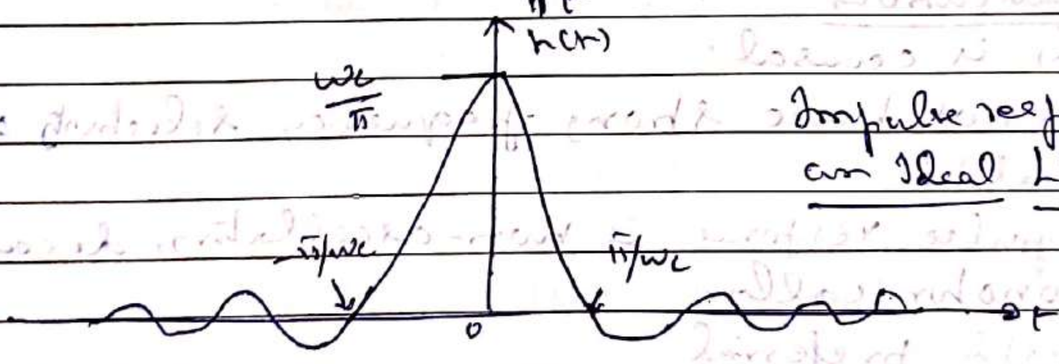
$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$\pm \omega_c$ = cut-off freq.




From Eq. 4.5, we know

$$h(t) = \frac{\sin \omega_c t}{\pi t}$$




Impulse response of an Ideal LP filter.

**Implications comparing $H(j\omega)$ & $h(t)$**

(1) $h(t) \neq 0$ for $t < 0$

(2) Ideal L.P filters are not causal.

(3) Ideal LP filters cannot be employed for causal systems

(4) Ideal LPF cannot be implemented easily.

(5) Oscillating Impulse response is generally undesirable.

Consider the LTI system with impulse response

$$h(t) = e^{-t}u(t)$$

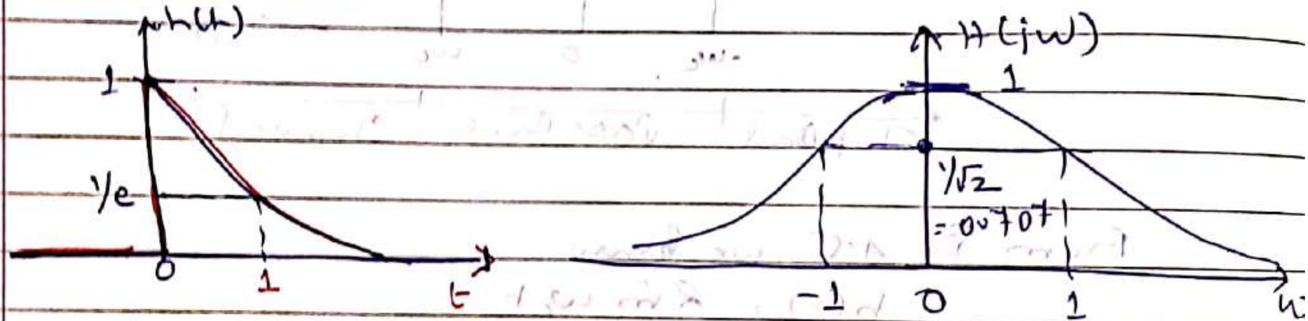
Then the frequency response is

$$H(j\omega) = \frac{1}{j\omega+1} \; ; \; a = 1$$

If we compare this equation with that of an RC filter (LPF)

$$H_{RC}(j\omega) = \frac{1}{1+RCj\omega} \Rightarrow \underline{RC = 1}$$

Response Graph:




Implication

→ h(t) is causal.

→ Does not have strong frequency selectivity of an Ideal LPF

→ Impulse response is non-oscillatory, decays monotonically.

→ mostly preferred.

Eg 1.19 Consider the response of an LTI system with Impulse response

$$h(t) = e^{-at} u(t) \quad ; a > 0$$

to the input signal

$$x(t) = e^{-bt} u(t) \; ; \; b > 0$$

Instead of computing

$$y(t) = x(t) * h(t),$$

we will tackle this problem in frequency domain

Then

$$X(j\omega) = \frac{1}{b+j\omega} \quad \& \quad H(j\omega) = \frac{1}{a+j\omega}$$

By convolution

$$x(t) * h(t) \overset{CTFT}{\longleftrightarrow} X(j\omega) H(j\omega) = Y(j\omega)$$

Therefore

$$Y(j\omega) = \frac{1}{(a+j\omega)} \cdot \frac{1}{(b+j\omega)}$$

$$y(t) = \cancel{F_0} \; F^{-1}\{Y(j\omega)\}$$

Use partial fractions. Case (i)

$$\text{Let } Y(j\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} \qquad \boxed{\text{with } a \neq b}$$

solving for the residues, we get

$$A = \frac{1}{b-a} \quad \& \quad B = \frac{-1}{b-a}$$

$$\text{Then } Y(j\omega) = \frac{1}{b-a} \left\{ \frac{1}{a+j\omega} - \frac{1}{b+j\omega} \right\}$$

Taking Inv. F.T

$$y(t) = \frac{1}{b-a} \left\{ e^{-at} u(t) - e^{-bt} u(t) \right\}$$

Case (ii) when $a = b$, partial fraction approach will not work.

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Then $Y(j\omega) = \frac{1}{(a+j\omega)^2}$ since $a = b$

Noting

$$\frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$$

and

$$e^{-at}u(t) \xleftrightarrow{CTFT} \frac{1}{a+j\omega}$$

$$te^{-at}u(t) \xleftrightarrow{CTFT} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] = \frac{1}{(a+j\omega)^2}$$

we get

$$\boxed{y(t) = t\cdot e^{-at}u(t)} \text{ for } a = b.$$

**Eg 4.20** Response of an Ideal Low Pass filter.

Let $x(t) = \frac{\sin \omega_i t}{\pi t}$ → sort of sinc function.

The impulse response would be of similar structure;

$$h(t) = \frac{\sin \omega_c t}{\pi t}.$$

$$y(t) = x(t) * h(t)$$

Instead, we work in the frequency domain while using the convolution property. Then

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

where

$$X(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_i \\ 0 & \text{elsewhere} \end{cases},$$

&

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{elsewhere} \end{cases}$$

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Therefore

$$Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\omega_0$ is the small of the two numbers $\omega_i$ & $\omega_c$.

The final output $y(t)$ is given by taking Inv.F.T whereby

$$y(t) = \begin{cases} \frac{\sin \omega_c t}{\pi t} & \text{if } \omega_c \leq \omega_i \\ \frac{\sin \omega_i t}{\pi t} & \text{if } \omega_i \leq \omega_c \end{cases}$$

Depending upon which of $\omega_i$ & $\omega_c$ is smaller, the output is equal to either $x(t)$ or $h(t)$.

4.5. multiplication Property (or) modulation Property

convolution (time domain) $\longrightarrow$ multiplication (freq domain)

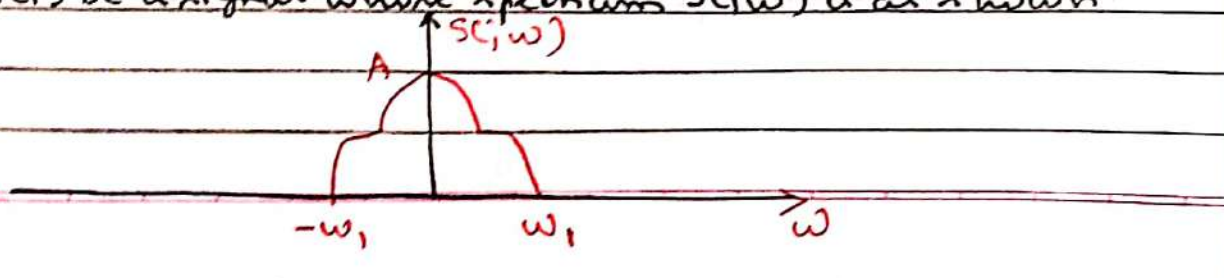
multiplication (time domain) $\longrightarrow$ convolution (freq domain)

$$\boxed{r(t) = s(t)p(t) \xleftrightarrow{CTFT} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))\,d\theta}$$

$$= \frac{1}{2\pi}\{S(j\omega) * P(j\omega)\}$$

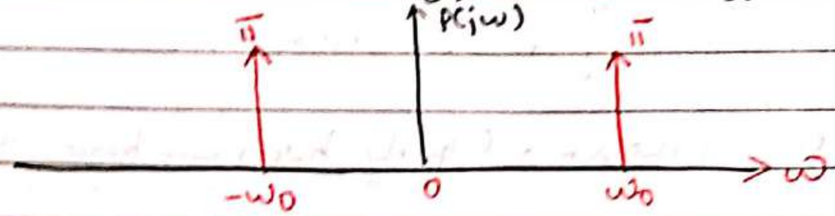
→ multiplication of one signal by another - Amplitude modulation

E.5.4.21 Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is as shown.



Also, consider the signal $p(t) = \cos\omega_0 t$. Then

$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ as sketched below

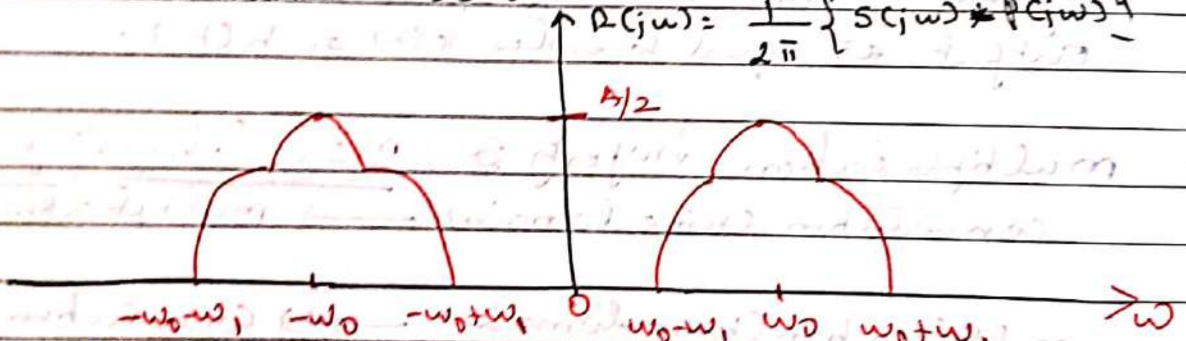



The spectrum of $R(j\omega)$ of $r(t) = s(t)\,p(t)$ is obtained by application of the multiplication property as

$$R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)\,P(j(\omega-\theta))\,d\theta$$

$$= \frac{1}{2} S(j(\omega-\omega_0)) + \frac{1}{2} S(j(\omega+\omega_0))$$

which is sketched below.



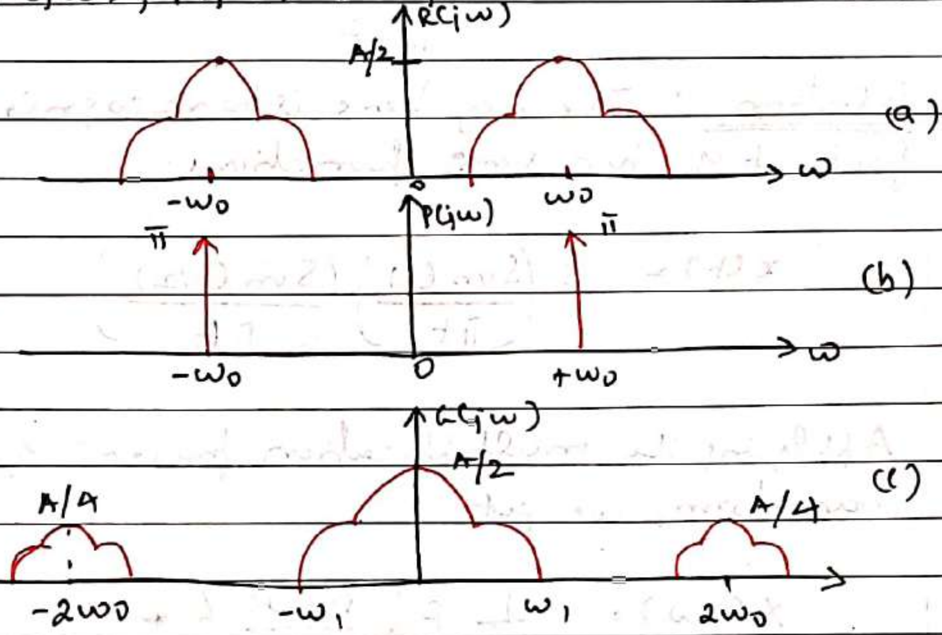

→ Here we have assumed that $\omega_0 > \omega_1$, so that the two non-zero portions of $R(j\omega)$ do not overlap.

→ Clearly, the spectrum of $r(t)$ consists of the sum of two shifted & scaled versions of $S(j\omega)$.

→ All the information of $s(t)$ is preserved when we multiply this signal by a sinusoidal signal, although the information has been shifted to higher frequencies.

→ This fact forms the basis of sinusoidal amplitude modulation systems for communications.

Fig 4·22. Let we consider the $r(t)$ obtained in Eg. 4·21. Let $g(t) = r(t)\,p(t)$, where again $p(t) = \cos\omega_0 t$. Then $R(j\omega)$, $P(j\omega)$ & $G(j\omega)$ are as shown below.




From Fig (c) & the linearity of FT, we see that $g(t)$ is the

→ sum of $\frac{1}{2}s(t)$ + a signal with a spectrum that is non-zero only at higher frequencies (centered around $\pm 2\omega_0$).

→ Suppose then that we apply the signal $g(t)$ as the input to a frequency-selective low pass filter with frequency response $H(j\omega)$ that is constant at low frequencies (say, for $|\omega| < \omega_1$) & zero at higher frequencies (for $|\omega| > \omega_1$). Then the output of this system will have as its spectrum $H(j\omega)G(j\omega)$, which, because of the particular choice of $H(j\omega)$, will be a scaled replica of $S(j\omega)$.

→ Therefore the output itself will be a scaled version of $s(t)$.

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Eg. 4.23 Consider the case wherein we need to evaluate the FT of

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}$$

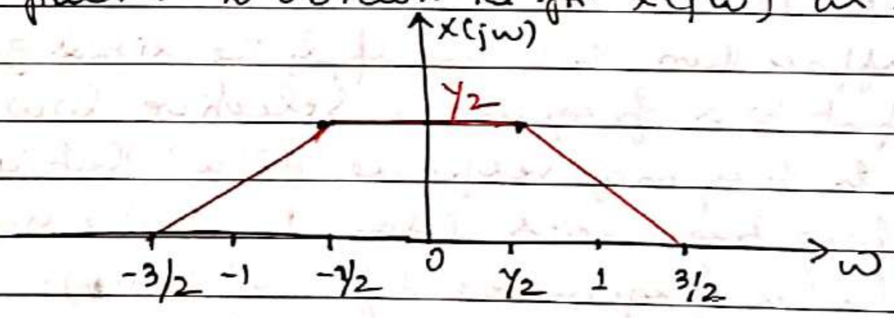
Solution :- The key here is to recognise $x(t)$ as the product of two sinc functions.

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

Applying the multiplication property of the Fourier transform, we get

$$X(j\omega) = \frac{1}{2} F\left\{\frac{\sin(t)}{\pi t}\right\} * F\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

Noting that the FT of each sinc function is a rectangular pulse, we can proceed to convolve these pulses to obtain the fn. $X(j\omega)$ as shown below



$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ $\qquad$ $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $\qquad$ 111

Analysis Eqn.

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## 5.3 Properties of Discrete-Time Fourier Transform

Allan Oppenheim.

### 5.3.1 Periodicity of the DTFT

DTFT is always periodic in $\omega$, with a period $2\pi$.

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

This is in contrast with CTFT, which in general is not periodic.

### 5.3.2 Linearity

If $\quad x_1[n] \xleftrightarrow{DTFT} X_1(e^{j\omega})$

&

$x_2[n] \xleftrightarrow{DTFT} X_2(e^{j\omega})$

Then

$$a x_1[n] + b x_2[n] \xleftrightarrow{DTFT} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

### 5.3.3 Time Shifting & Frequency Shifting.

**Time Shifting**

If $\quad x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

**Frequency Shifting.**

If $\quad x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then

$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$$

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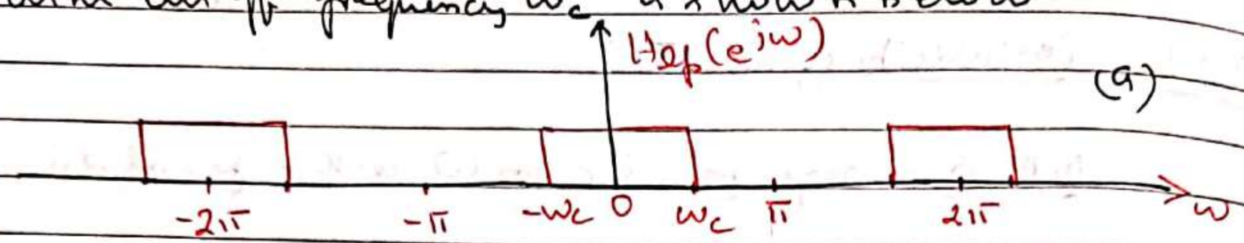
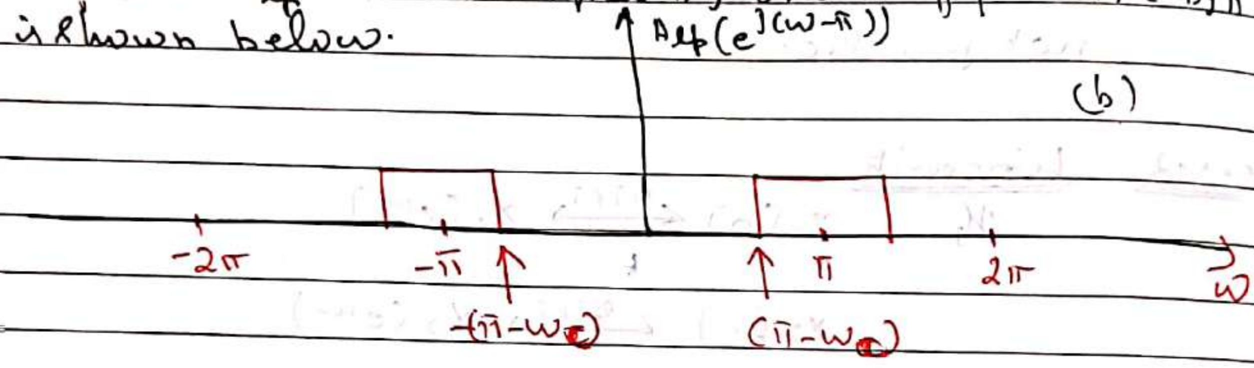
Eg 5.7 The frequency response $H_{lp}(e^{j\omega})$ of a low-pass filter with cut-off frequency $\omega_c$ is shown below.




while the response $H_{lp}(e^{j(\omega-\pi)})$ - that is, the frequency response $H_{lp}(e^{j\omega})$ shifted by one-half period i.e by $\pi$ is shown below.




→ Since the high frequencies in DT are concentrated near $\pi$ (and other odd multiples of $\pi$), the filter in Fig (b) is an ideal highpass filter with cut off frequency $\pi - \omega_c$.

→ That is

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) \quad -(1)$$

→ As the frequency response of an LTI system is the FT of the impulse response of the system, if $h_{lp}[n]$ & $h_{hp}[n]$, respectively denote the impulse responses of the LP & HP filters shown in Figs (a) & (b), then Eqn (1) & frequency shifting property of DTFT imply that

$$\boxed{\begin{aligned} h_{hp}[n] &= e^{j\pi n} h_{lp}[n] \\ &= (-1)^n h_{lp}[n] \end{aligned}}$$

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5.3.4 Conjugation & Conjugate Symmetry.

If

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

then

$$x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

If $x[n]$ is real-valued, its $X(e^{j\omega})$ is conjugate symmetric i.e

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad \text{if } x[n] - \text{real}$$

Inferences

Real $\{X(e^{j\omega})\}$ - Even function of $\omega$
Img $\{X(e^{j\omega})\}$ → Odd function of $\omega$
also
$|X(e^{j\omega})|$ → Even function.
$\angle X(e^{j\omega})$ → Odd function.

Further

$$\text{Even}\{x[n]\} \xleftrightarrow{DTFT} \text{Real}\{X(e^{j\omega})\}$$

&

$$\text{Odd}\{x[n]\} \xleftrightarrow{DTFT} j\cdot\text{Img}\{X(e^{j\omega})\}$$

If $x[n]$ is real & even, its DTFT is also real + even.
Refer: Eg. 5.2.

5.3.5 Differencing & Accumulation (Eqt. to Diff. & Integration)

Difference

If $x[n] \xrightarrow{DTFT} X(e^{j\omega})$

Then for the first difference using linearity & time-shift property:

$$x[n] - x[n-1] \xleftrightarrow{DTFT} \{1 - e^{-j\omega}\} X(e^{j\omega})$$

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Accumulation

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{\text{DTFT}} \frac{1}{(1-e^{-j\omega})} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

The impulse train on the RHS of the above equation reflects the DC or Average value as result of accumulation / summation.

Eg.5.8 To get DTFT of unit step $x[n] = u[n]$

we know

$$g[n] = \delta[n] \xleftrightarrow{\text{DTFT}} G(e^{j\omega}) = 1.$$

Also

$$x[n] = \sum_{m=-\infty}^{+\infty} g[m]$$

i.e unit step is the running sum of unit Impulse

Using accumulation property.

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{(1-e^{-j\omega})}(1) + \pi(1) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{(1-e^{-j\omega})} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

-X-

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5.3.6 Time Reversal-

If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

then

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

Let $y[n] = x[-n]$, then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n] e^{-j\omega n}$$

Letting $m = -n$, then

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m] e^{-j(-m)\omega} = \sum_{m=-\infty}^{+\infty} x[m] e^{-j(-\omega)m}$$

$$= X(e^{-j\omega})$$

5.3.7 Time Expansion : $\boxed{\text{Special case.}}$

We know

$$x(at) \xleftrightarrow{CTFT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

If we consider $x[an]$, we cannot choose $a<1$. 'a' has to be an integer other than $\pm 1$. The argument [.] has to be an integer.

Assuming 'k' be a +ve integer & define a signal.

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{if } n \text{ is not a multiple of } k \end{cases}$$

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For example if $k=3$, then $x_{(k)}[n]$ is obtained from $x[n]$ by choosing the samples at points where $n$ is a multiple of k i.e 3 ; like $0, \pm 3, \pm 6, \ldots$

-> We can say $x_{(k)}[n]$ exists for $\underline{\underline{n = rk}}$, then

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n] e^{-j\omega n}$$

$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk] e^{-j\omega rk}$$

Since $x_{(k)}[rk] = x[r]$, we infer.

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r] e^{-j(k\omega)r} = \underline{\underline{X(e^{jk\omega})}}$$

(or)

$$x_{(k)}[n] \xleftrightarrow{DTFT} X(e^{jk\omega})$$

Inferences:

1) When $k > 1$, the $x[n]$ becomes spread out & slowed down.

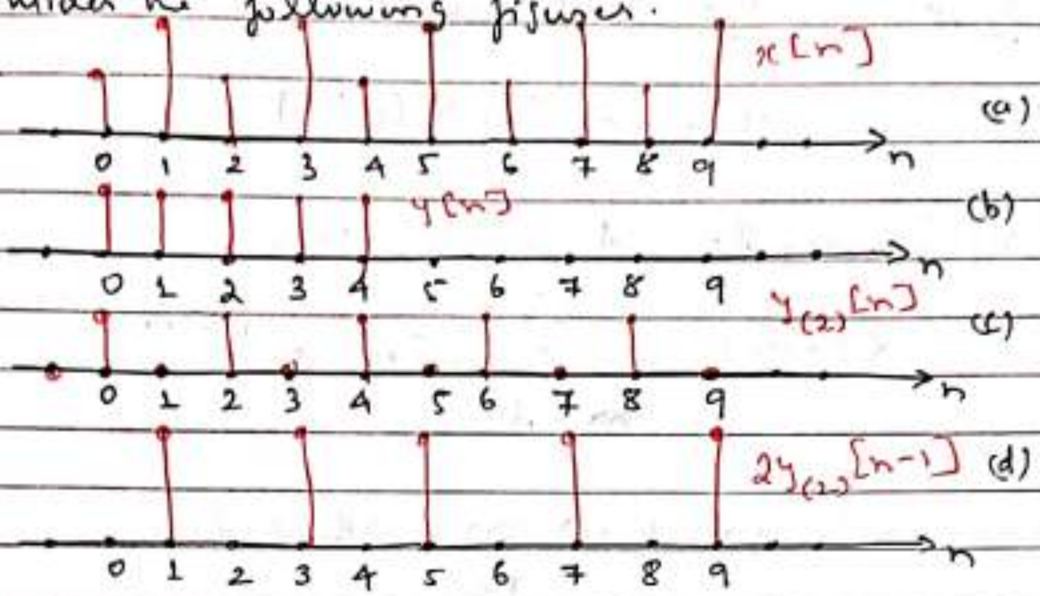
(2) However the corresponding DTFT gets Compressed.

(3) Inverse relation between Time & Freq. domain.

(4) Since $X(e^{j\omega})$ is periodic with period $2\pi$, $X(e^{jk\omega})$ is also periodic with period $\frac{2\pi}{k}$

Fig 5.14 Pg 381.

Eg 5.9 Consider the following figures.



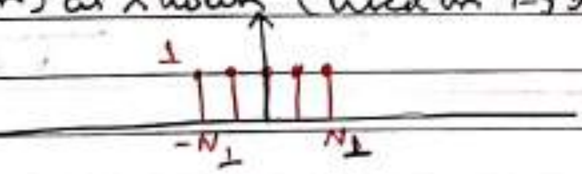

x[n] is depicted in (a). This sequence can be generated using y[n] shown in (b) & using (c) & (d)

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

where

$$y_2[n] = \begin{cases} y[n/2] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Consider $g[n]$ as shown (used in Eg 5.3) with $N_1 = 2$




Then $y[n] = g[n-2]$. From Eg 5.3 + Time shifting property

$$Y(e^{jw}) = e^{-j2w}\frac{\sin(5w/2)}{\sin(w/2)}$$

Using Time-expansion property, we have

$$y_{(2)}[n] \xleftrightarrow{DTFT} e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

Using linearity + time shifting properties, we get

$$2y_{(2)}[n-1] \xleftrightarrow{DTFT} 2e^{-j5w}\frac{\sin(5w)}{\sin(w)}$$

Therefore

$$X(e^{jw}) = e^{-j4w}\left\{1+2e^{-jw}\right\}\left\{\frac{\sin(5w)}{\sin(w)}\right\}$$

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5.3.8 **Differentiation in frequency.**

Let $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$

Considering the Analysis eqn.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Differentiating w.r.t $\omega$ on both sides.

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jn\, x[n] e^{-j\omega n}$$

RHS of the above eqn. is DTFT of $-jn\,x[n]$. Hence multiplying both sides with $(+j)$, we get

$$\boxed{n\,x[n] \xleftrightarrow{DTFT} j \cdot \frac{dX(e^{j\omega})}{d\omega}}$$

5.3.9 **Parseval's Relation**

If $x[n]$ & $X(e^{j\omega})$ are DTFT pair, then

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy of } x[n]} = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Parseval's relation states this total energy can be determined by integrating the energy per unit freq.

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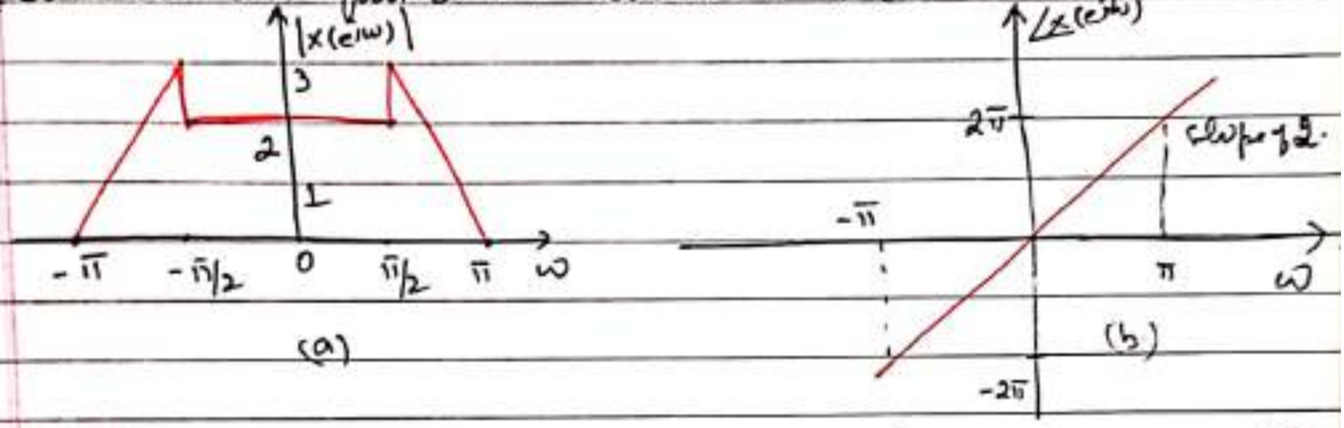
$\frac{|X(e^{j\omega})|^2}{2\pi}$ over a full $2\pi$.

$|X(e^{j\omega})|^2 \rightarrow$ Energy density spectrum of $x[n]$

Eg 5.10 Consider a sequence $x[n]$ whose FT $X(e^{j\omega})$ is depicted below




It is desired to examine if $x[n]$ is periodic, real, even, and/or of finite energy.

Solution :- For the periodicity in Time domain, the FT is zero except possibly for impulses located at various integer

→ multiples of $\omega_0$, the fundamental frequency. From (a), it is not so & hence $x[n]$ is not periodic

→ From Symmetry properties of FT, for a real valued sequence the FT of even magnitude & a phase fn. that is odd. This is true from (a) & (b) & hence $x[n]$ is real.

→ If $x[n]$ is an even function, then, by the symmetry properties for real signals $X(e^{j\omega})$ must be real & even. Since $X(e^{j\omega}) = |X(e^{j\omega})|e^{-j2\omega}$, $X(e^{j\omega})$ is not a real-valued fn. Hence $x[n]$ is not even.

→ Using Parseval's relation the $\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2 d\omega$

between the limits $-\pi$ & $\pi$, from fig (a) will yield a finite quantity. Hence $x[n]$ has finite energy

5.4 The Convolution Property.

If $y[n] = x[n] * h[n]$

then

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

Inferences :- The frequency response of a DT LTI systems is the F.T of the impulse response $h[n]$ of the system.

-> The convolution property relates the convolution of aperiodic sequences using DTFT.

-> This property is useful to design frequency-selective filtering of signals using $H(jw)$.

PS 5.11 Consider an LTI system with an impulse response

$$h[n] : \delta[n-n_0]$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n} = \underline{e^{-j\omega n_0}}$$

Thus for any input $x[n]$ with F.T $X(e^{j\omega})$, the F.T of the output is given by

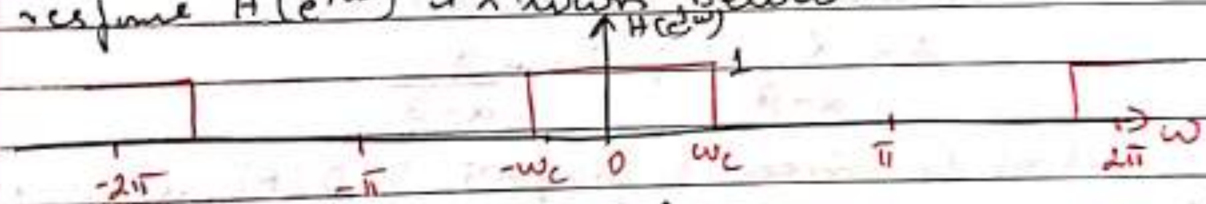
$$\underline{\underline{Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})}}$$

In this case with Time shifting property

$$y[n] = x[n-n_0]$$

The frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ has unity magnitude at all frequencies & a phase characteristic of ~~$-\omega t_0$~~ $-\omega n_0$, which is linear with frequency.

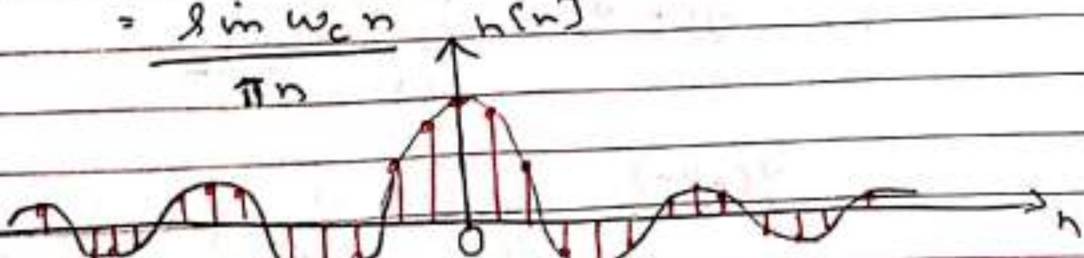
PS 5.12 Consider a DT ideal low pass filter, whose frequency response $H(e^{j\omega})$ is shown below.




Since the Impulse response & frequency response of an LTI system are a Fourier Transform pair, we can determine the impulse response of an ideal LP filter from the frequency response using inv. F.T. Therefore

$$h[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$



Eg 5.23 Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$ ; $|\alpha| < 1$ & suppose the input to this system is $x[n] = \beta^n u[n]$ with $|\beta| < 1$. Find $y[n]$.

Solution :- The individual F.T of $x[n]$ & $h[n]$ are

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \ \& \ X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

Using Convolution property of DTFT

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

$$= \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

Using the partial fraction approach with $\boxed{\alpha \neq \beta}$

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

The residues are calculated as

$$A = \frac{\alpha}{\alpha-\beta} \ ; \ B = -\frac{\beta}{\alpha-\beta}$$

Using the linearity property of DTFT, we get $y[n]$ as

$$y[n] = \left\{\frac{\alpha}{\alpha-\beta}\right\}\alpha^n u[n] - \left\{\frac{\beta}{\alpha-\beta}\right\}\beta^n u[n]$$

$$= \frac{1}{\alpha-\beta}\left\{\alpha^{n+1}u[n] - \beta^{n+1}u[n]\right\} \quad \text{—} (1)$$

When $\alpha = \beta$ ; (1) is invalid. In such case

$$Y(e^{j\omega}) = \left[\frac{1}{(1-\alpha e^{-j\omega})}\right]^2$$ which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left\{\frac{1}{1-\alpha e^{-j\omega}}\right\}$$

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Using the differentiation property & observing

$$\alpha^n u(n) \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}$$

we have $\quad n\alpha^n u[n] \longleftrightarrow j\frac{d}{d\omega}\left\{\frac{1}{1-\alpha e^{-j\omega}}\right\}$

To account for $e^{j\omega}$, we use time shifting property.

$$(n+1)\alpha^{n+1} u[n+1] \longleftrightarrow j e^{j\omega}\frac{d}{d\omega}\{\ldots\}$$

Finally utilising $1/\alpha$, we get

$$y[n] = (n+1)\alpha^n u[n+1] \quad -(2)$$

(*) Since $u[n+1]=0$ at $n=-1$, we can re-write this as

$$\underline{\underline{y[n] = (n+1)\alpha^n u[n]}}$$

Eg 5.24

5.5 ## The multiplication Property.

If $y[n] = x_1[n]\, x_2[n]$, then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

P.5.25 $$= \frac{1}{2\pi}\left[X_1(e^{j\omega}) \circledast X_2(e^{j\omega})\right]$$

Evaluate $X(e^{j\omega})$ of a signal $x[n]$ defined by
$x[n] = x_1[n] \cdot x_2[n]$ ; $x_1[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$ $\rightarrow \left(\dfrac{3\pi n}{4}\right)$

$x_2[n] = \dfrac{\sin(\pi n/2)}{\pi n}$

Solution :- From the multiplication property of DTFT, it is clear that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ & $X_2(e^{j\omega})$, where the limits of integral can be over a length of $2\pi$. Now, choosing an interval $-\pi < \theta \leq \pi$, we have

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta \quad -①$$

Eqn ① resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \leq \pi$. We can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \text{for } -\pi < \omega \leq \pi \\ 0 & \text{elsewhere.} \end{cases}$$

Then replacing $X_1(e^{j\theta})$ in ① with $\hat{X}_1(e^{j\theta})$ & using the fact that $\hat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we get

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

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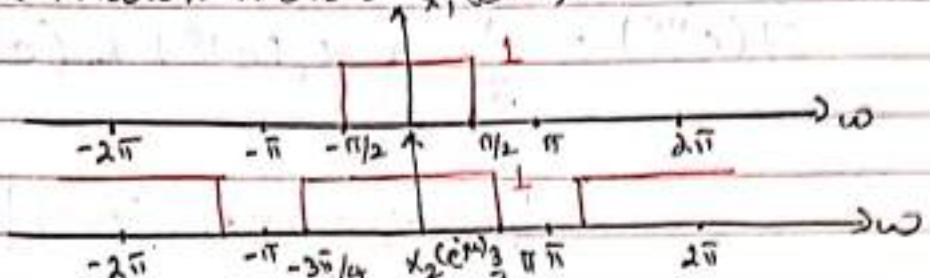
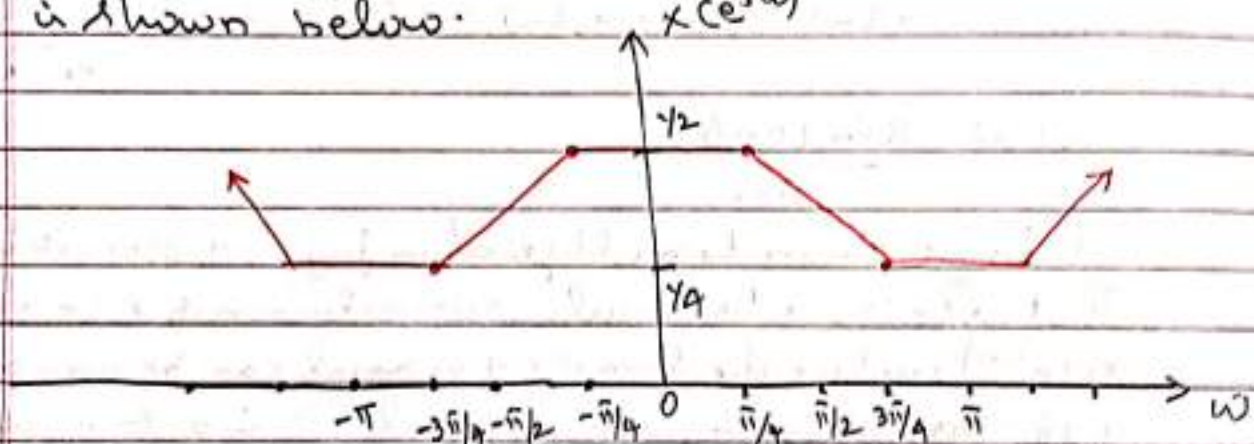
$$\Rightarrow \; = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}_1(e^{j\theta})\, x_2(e^{j(\omega-\theta)})\, d\theta .$$

Thus $X(e^{j\omega})$ is $\frac{1}{2\pi}$ times the aperiodic convolution of the rectangular pulse $\hat{x}_1(e^{j\omega})$ & periodic square wave $x_2(e^{j\omega})$, both are shown below.




The result of this convolution is the F.T $X(e^{j\omega})$ which is shown below.




## 5.7 Duality.

In CTFT → Duality exists.

In DTFT → No such duality exists.

However

In DTFS → Duality exists.

### 5.7.1 Duality in DTFS

Any periodic D.T signal $x[n]$ can be represented by Fourier series $a_k$ which themselves are periodic. Hence $a_k$ can be expanded in a Fourier series.

The property of Duality of D.T.F.S implies that the Fourier series co-efficients for a periodic sequence $a_k$

are the values of $\frac{1}{N}\{x[-n]\}$ i.e proportional to the values of original signal, reversed in time.

Proof:-

consider two periodic sequences with common period N are related by

$$f[m] = \frac{1}{N} \sum_{r=\langle N\rangle} g[r] e^{-jr\left(\frac{2\pi}{N}\right)m}$$

letting $m = k$ and $r = n$, we get

$$f[k] = \frac{1}{N} \sum_{n=\langle N\rangle} g[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Comparing this with

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

we see the sequence $f[k]$ corresponds to the F.S. coefficients of the signal $g[n]$. Then, if we adopt the notation

$$x[n] \xleftrightarrow{FS} a_k$$

then we can write

$$\boxed{g[n] \xleftrightarrow{FS} f[k]}$$

Alternatively, if we let $m = n$ & $r = -k$, we get

$$f[n] = \sum_{k=\langle N\rangle} \frac{1}{N} g[-k] e^{jk\left(\frac{2\pi}{N}\right)n}$$

comparing this with

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

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we see that $\frac{1}{N} g[-k]$ corresponds to the sequence of F.S. co-efficients of $f[n]$. That is

$$\boxed{f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]}$$

Inference
As in C.T case, every property of D.T.F.S has a dual

Examples of Dual properties of DTFS.

(a) $x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}$

&

Dual $e^{jm\left(\frac{2\pi}{N}\right)n} \cdot x[n] \xleftrightarrow{FS} a_{k-m}$

(b) $$\sum_{r=\langle N\rangle} x[r]\, y[n-r] \xleftrightarrow{FS} N a_k b_k$$

Convolution

&

$$x[n]\, y[n] \xleftrightarrow{FS} \sum_{l=\langle N\rangle} a_l\, b_{(k-l)}$$

Convolution of FS

Eg 5.16 Consider the following periodic signal with a period of $N=9$

$$x[n] = \begin{cases} \frac{1}{9} \cdot \frac{\sin(5\pi n/9)}{\sin(\pi n/9)} & ; n \neq \text{multiple of } 9. \\ 5/9 & n = \text{multiple of } 9 \end{cases}$$

in chapter 3, we found that a rectangular square wave has FS co-efficients in the same form as above.

Duality suggests that the coefficients of $x[n]$ must be in the form of a rectangular square wave.
Let $g[n]$ be a rectangular square wave with period $N=9$ such that

$$g[n] = \begin{cases} 1 & |n| \leq 2 \\ 0 & 2 < |n| \leq 4 \end{cases}$$

The FS coefficients $b_k$ for $g[n]$ from Eq 3.12 as

$$b_k = \begin{cases} \frac{1}{9} \cdot \frac{\sin(5\pi k/9)}{\sin(\pi k/9)} & ; k \neq \text{multiple of } 9 \\ 5/9 & k = \text{multiple of } 9 \end{cases}$$

From the FS Analysis eqn for $g[n]$ can be written as

$$b_k = \frac{1}{9} \sum_{n=-2}^{2} (1) e^{-j2\pi nk/9}$$

Interchanging the names of the variables $k$ & $n$ + noting that $x[n] = b_n$, we see.

$$x[n] = \frac{1}{9} \sum_{k=-2}^{2} (1) e^{-j2\pi nk/9}$$

Letting $k' = -k$ in the RHS we have.

$$x[n] = \frac{1}{9} \sum_{k'=-2}^{2} e^{+j2\pi nk'/9}$$

Finally moving $1/9$ inside the summation, we see that RHS. has the form of the Synthesis Equation for $x[n]$. Hence the FS coefficients of $x[n]$ can be given as.

$$a_k = \begin{cases} 1/9 & ; |k| \leq 2 \\ 0 & ; 2 < |k| \leq 4 \end{cases}$$

with a period $N=9$.

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## 5.7.2 Duality between DTFT & CTFS.

DTFT pair

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \rightarrow (A)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n} \quad \rightarrow (B)$$

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \quad - (C)$$

$$a_k = \frac{1}{T}\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt \quad - (D)$$

Eqns (A) + (D) are similar + so are (B) + (C)

### Inferences

1. We can interpret Eqns (A) + (B) as a Fourier Series representation of the periodic frequency response $X(e^{j\omega})$. Since $X(e^{j\omega})$ is periodic fn. of $\omega$ with a period $2\pi$, it has a F.S. representation as a weighted sum of harmonically related periodic exponential fn. of $\omega$, all of which have a common period $2\pi$.
2. $X(e^{j\omega})$ can be represented in a F.S as a weighted sum of the signals $e^{j\omega n}$, $n = 0, \pm 1, \pm 2, \ldots$
3. From (B) we see the $n^{th}$ Fourier coefficient in this expansion is $x[-n]$. Since the period of $X(e^{j\omega})$ is $2\pi$, (A) can be interpreted as FS analysis equation for the F.S. co-efficient $x[n]$ i.e. for the co-efficient multiplying $e^{-j\omega n}$ in the expression for $X(e^{j\omega})$ in (B).

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Table 5.3 Summary of F.S & Transform expressions.

| | Continuous Time | | Discrete Time | |
|---|---|---|---|---|
| | Time Domain | frequency domain | Time Domain | frequency Domain |
| Fourier Series | $x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $a_k=\frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\,dt$ | $x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$ | $a_k=\frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$ |
| | Continuous time periodic in time | Discrete frequency aperiodic in frequency | Discrete in time Periodic in time $\xleftrightarrow{\text{Dual}}$ | Discrete frequency Periodic in frequency |
| | | Dual | | |
| Fourier Transform | $x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(e^{j\omega}) e^{j\omega t}\,d\omega$ | $X(j\omega)=\int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\,dt$ | $x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\,d\omega$ | $X(e^{j\omega})=\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ |
| | Continuous time aperiodic in time $\xleftrightarrow{\text{Dual}}$ | Continuous frequency aperiodic in frequency | Discrete Time aperiodic in time | continuous frequency periodic in frequency |

(3)

Table 5.1 Properties of DTFT

| Property | Aperiodic Signal | DTFT |
|---|---|---|
| | $x[n]$ | periodic $\{X(e^{j\omega})$ |
| | $y[n]$ | with period $2\pi$ $[Y(e^{j\omega})$ |
| 1. Linearity | $a x[n] + b y[n]$ | $a X(e^{j\omega}) + b Y(e^{j\omega})$ |
| 2. Time shifting | $x[n-n_0]$ | $e^{-j\omega n_0} X(e^{j\omega})$ |
| (3) Freq. shifting | $e^{j\omega_0 n} x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| (5) Time reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| (6) Time expansion | $x_{(k)}[n] = \begin{cases} x[n/k] & n = \text{multiple of } k \\ 0 & n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| (7) Convolution | $x[n] * y[n]$ | $X(e^{j\omega}) Y(e^{j\omega})$ |
| (8) Difference in time | $x[n] - x[n-1]$ | $(1-e^{-j\omega}) X(e^{j\omega})$ |
| (9) Accumulation | $\sum_{k=-\infty} x[k]$ | $\frac{1}{(1-e^{-j\omega})} \cdot X(e^{j\omega})$ |
| (10) Differentiation in frequency | $n x[n]$ | $j \cdot \frac{dX(e^{j\omega})}{d\omega}$ |
| (11) Conjugate symmetry for real signal | $x[n]$ - real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\} \\ Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega}) \rvert = \lvert X(e^{-j\omega}) \rvert \\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}) \end{cases}$ |
| (12) Symmetry for real, even signal | $x[n]$ - real + even | $X(e^{j\omega})$ - real + even |

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(13) Symmetry for real & odd signals — $x[n]$ - real & odd — $X(e^{j\omega})$ is purely imaginary & odd

(14) Even-odd decomposition of Real signals. — $x_e[n] = Ev\{x[n]\}$, $x_o[n] = odd\{x[n]\}$, $x[n]$ - real — $Re\{X(e^{j\omega})\}$, $j\,Img\{X(e^{j\omega})\}$

(15) Parseval's Relation }
$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Table 5.2 DTFT Pairs

| Signal | DTFT | F-Series coefficients, if periodic |
|---|---|---|
| (1) $\sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left[\omega - \frac{2\pi k}{N}\right]$ | $a_k$ |
| (2) $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega-\omega_0-2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} 1 ; & k = m, m\pm N, m\pm 2N \\ 0 & \text{Otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ - irrational, the signal is aperiodic. |
| (3) $\cos\omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega-\omega_0-2\pi l) + \delta(\omega+\omega_0-2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} \frac{1}{2} ; & k = \pm m, \pm m\pm N, \pm m\pm 2N, \dots \\ 0 & \text{Otherwise}\end{cases}$ (b) $\frac{\omega_0}{2\pi}$ - irrational. The signal is aperiodic. |

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(13A)

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(4) $\sin \omega_0 n$ : $\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\left\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0+2\pi l)\right\}$ (a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases}\frac{1}{2j} & k=r, r\pm N, r\pm 2N, \dots \\ -\frac{1}{2j} & k=-r, -r\pm N, -r\pm 2N, \dots \\ 0 & \text{Otherwise}\end{cases}$

(b) $\frac{\omega_0}{2\pi}$ - is rational, signal is aperiodic

(5) $x[n] = 1$ $2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$ $a_k = \begin{cases}1 & k=0, \pm N, \pm 2N \\ 0 & \text{Otherwise}\end{cases}$

(6) Periodic Square wave.

$x[n] = \begin{cases}1, & |n| \le N_1 \\ 0 & N_1 < |n| \le N/2\end{cases}$

&

$x[n+N] = x[n]$

$2\pi\sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$

$$a_k = \frac{\sin\left\{\frac{2\pi k}{N}\left(N_1+\frac{1}{2}\right)\right\}}{N\sin\left(\frac{2\pi k}{2N}\right)} ; k \ne 0, \pm N, \pm 2N \dots$$

$$a_k = \frac{2N_1+1}{N} ; k = 0, \pm N, \pm 2N, \dots$$

(7) $\sum_{k=-\infty}^{\infty}\delta[n-kN]$ $\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$ $a_k = \frac{1}{N}$ for all $k$

(8) $a^n u[n]; |a|<1$ $\frac{1}{1-ae^{-j\omega}}$

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(9) $x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}$ $\quad \dfrac{\sin\{\omega(N_1 + \frac{1}{2})\}}{\sin(\omega/2)}$ —

(10) $\dfrac{\sin Wn}{\pi n} = \dfrac{W}{\pi}\,\text{sinc}\left(\dfrac{Wn}{\pi}\right)$ $\quad X(\omega) = \begin{cases} 1 & 0 \le |\omega| \le W \\ 0 & W < |\omega| \le \pi \end{cases}$ —

$X(\omega)$ is periodic with period $2\pi$

(11) $\delta[n]$ $\quad 1$ —

(12) $u[n]$ $\quad \dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k)$ —

(13) $\delta[n-n_0]$ $\quad e^{-j\omega n_0}$ —

(14) $(n+1)a^n u[n]$, $|a| < 1$ $\quad \dfrac{1}{(1-ae^{-j\omega})^2}$ —

(15) $\dfrac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $|a| < 1$ $\quad \dfrac{1}{(1-ae^{-j\omega})^r}$ —

# Unit 5 - Z-Transform

Reference :- Signals & Systems → A.V. Oppenheim.

Sections : 10.1, 10.2, 10.3, 10.5, 10.6, 10.7, 10.9.

10.4, 10.8 → not included for S&S . UEL7EC204

## 10.1. The Z-Transform (Zee-Transform)

A discrete LTI system with impulse response h[n], has an output response y[n], when the input is complex exponential of the form $z^n$ is

$$y[n] = H(z)\, z^n$$

$$\text{where } H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n}$$

letting $\boxed{z = e^{j\omega}}$ with $\omega$ being real.

→ meaning $|z| = 1$, the above summation corresponds to D.T.F.T of h[n]

→ When $|z|$ is not restricted to unity, summation is referred to as Z-transform of h[n]

→ General definition :- The Z-transform of a general D.T signal is defined as x[n]

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} \qquad ; z = e^{j\omega}$$

where z is a complex variable.

$$x[n] \xleftrightarrow{z} Z(x)$$

Bilateral Z-transform :- when n varies from $-\infty$ to $+\infty$

Unilateral Z-transform :- when n varies from 0 to $+\infty$ (causal systems)

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Relationship between Z-transform & Fourier Transform (D.T.)

Let the complex variable z be represented in the Polar form

$$z = re^{j\omega}$$

r - magnitude ω - phase angle of z. In terms of r & ω the equation

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \{x[n] r^{-n}\} e^{-j\omega n}$$

Then we infer that $X(re^{j\omega})$ is the Fourier transform of the sequence $x[n]$ multiplied by $r^{-n}$ i.e

$$X(re^{j\omega}) = F\{x[n] r^{-n}\}$$

Inferences

-> $r^{-n}$ can be decaying or growing with increasing n, dependent on whether r is greater or less than 1

-> In particular for r = 1 or |z| = 1, we get

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = F\{x[n]\}$$

-> Therefore, Z-transform is the same as F.Transform for |z| = 1 or r = 1.

-> In C.T.wise case, L.Transform is the same as F.T when the real part of $s = u + j\omega$ ($\sigma + j\omega$) is zero.

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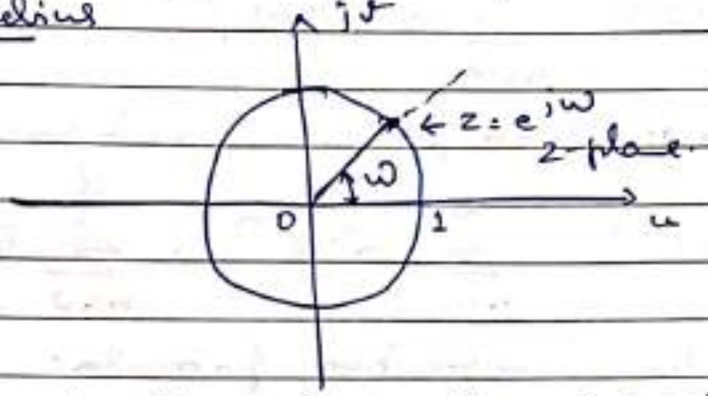
→ Z-transform reduces to the Fourier transform on the contour in the complex Z-plane corresponding to a circle of unit radius




→ As jω axis in s-plane plays an important role in Laplace Transform; unit circle also plays an important role in Z-transform.

→ Actually jω-axis in s-plane maps to the Unit circle in Z-plane.

Region of Convergence. (ROC)

Consider

$$X(re^{j\omega}) = F\{x[n]r^{-n}\} \quad (\text{or}) \quad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since Z transform is a summation series, we need
→ the F.Transform of $x[n]r^{-n}$ to converge.

→ The range of z for which X(z) converges is referred to as Region of Convergence (ROC)

→ If the ROC includes unit circle, then the DTFT also converges.

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Eg. 10.1 consider $x[n] = a^n u[n]$ $|a| < 1$.

Then
$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n}$$

Since u[n] exists for $n \geq 0$, we have
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Using the Summation formula:
$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

Now for X(z) to converge, we require
$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

This implies that $|az^{-1}| < 1$ or
$|a| < |z|$ or $|z| > |a|$

we then write
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{z}{z-a} \; ; \; |z| > |a|$$

For example when $a = 1$, $x[n] = u[n]$ - unit step sequence
then
$$X(z) = \frac{z}{z-1} \; ; \; |z| > 1$$ z-transform of unit step sequence

→ Consider $X(z) = \frac{z}{z-a}$; It is a rational function
$|a| < 1$

Pole = +a, zero = 0

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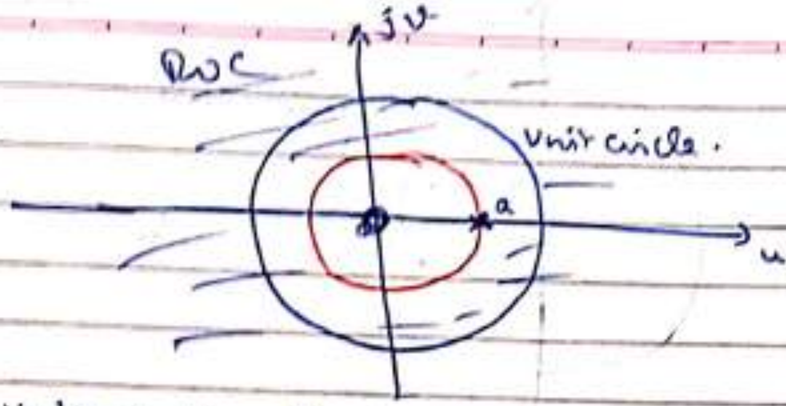



∴ For $|a|>1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, F.T of $a^n u[n]$ does not converge.

Eg 10.2 Let $x[n] = -a^n u[-n-1]$

then

$$X(z) = \sum_{n=-\infty}^{+\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

Looking at $u[-n-1]$, we get

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

changing the limits

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

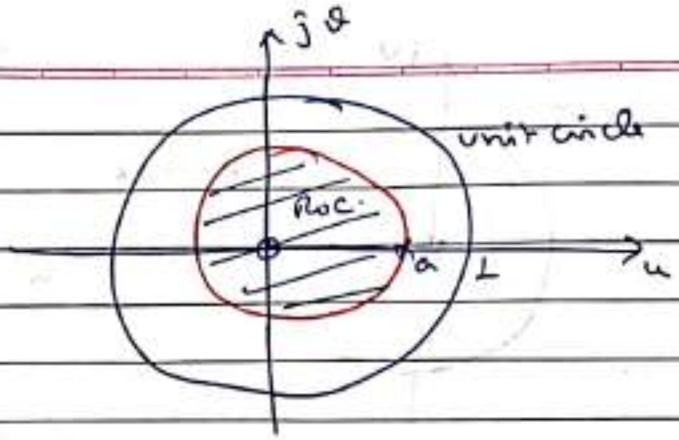
If $|a^{-1}z| < 1$ or $|z| < |a|$, the summation converges.

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = 1 - \frac{a}{a-z} = \frac{a-z-a}{a-z} = \frac{-z}{a-z}$$

$$= \frac{z}{z-a} \;; \quad |z| < |a|$$

⑥






Note

→ z-transforms need to be represented together with ROC.

Eg 10.3 Consider a signal that is a sum of two real exponentials

$$x[n] = 7\left(\tfrac{1}{3}\right)^n u[n] - 6\left(\tfrac{1}{2}\right)^n u[n]$$

Then

$$X(z) = \sum_{n=\infty}^{+\infty} \left\{ 7\left(\tfrac{1}{3}\right)^n u[n] - 6\left(\tfrac{1}{2}\right)^n u[n] \right\} z^{-n}$$

$$= 7 \sum_{n=0}^{\infty} \left(\tfrac{1}{3}z^{-1}\right)^n - 6 \sum_{n=0}^{\infty} \left(\tfrac{1}{2}z^{-1}\right)^n$$

since $u[n]$ exists for $n \geq 0$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{z\left(z-3/2\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

For convergence

$$\left|\tfrac{1}{3}z^{-1}\right| < 1 \quad \& \quad \left|\tfrac{1}{2}z^{-1}\right| < 1$$

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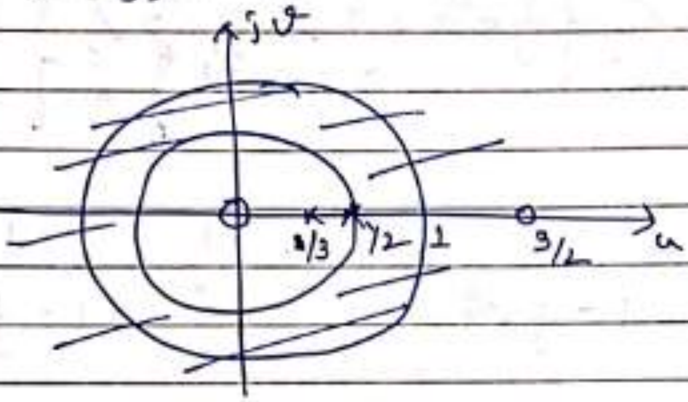
or $|z| > \frac{1}{3}$ or $|z| > \frac{1}{2}$. Both would be satisfied when $|z| > \frac{1}{2}$

$$X(z) = \frac{z(z-3/2)}{(z-\frac{1}{3})(z-\frac{1}{2})} \; ; \; |z| > 1/2$$




<u>Aliter</u>

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \; ; \; |z| > 1/3$$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}} \; ; \; |z| > 1/2$$

Then.

$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} \; ; \; |z| > \frac{1}{2}$$

<u>Eg 10.4</u> Let $x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$

$$= \frac{1}{2j}\left[\frac{1}{3}^n e^{j\frac{\pi n}{4}}\right]u[n] - \frac{1}{2j}\left[\frac{1}{3}^n e^{-j\frac{\pi n}{4}}\right]u[n]$$

$$= \frac{1}{2j}\left[\frac{1}{3}e^{\frac{j\pi}{4}}\right]^n u[n] - \frac{1}{2j}\left[\frac{1}{3}e^{-\frac{j\pi}{4}}\right]^n u[n]$$

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Then z transform would be

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2j}\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n u(n) - \frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u(n) \right\} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left\{\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}\right\}^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left\{\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}\right\}^n$$

$$= \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}} - \frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}$$

This can be simplified to

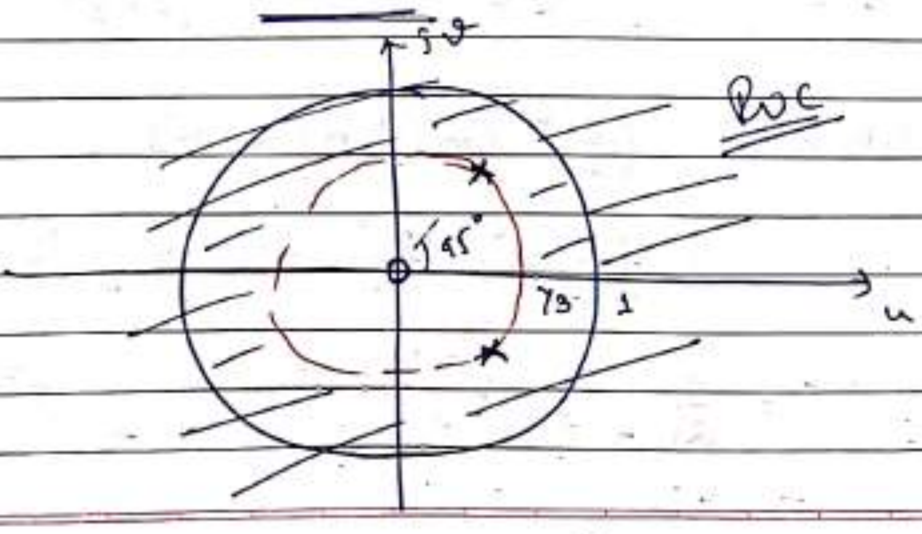
$$X(z) = \frac{\frac{1}{3\sqrt{2}}z}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}$$

Convergence

$$\left|\frac{1}{3}e^{j\pi/4}z^{-1}\right| < 1 \quad \& \quad \left|\frac{1}{3}e^{-j\pi/4}z^{-1}\right| < 1$$

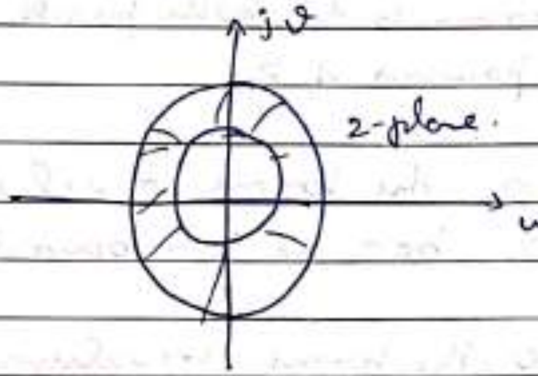
boils down to

$$|z| > 1/3$$



## 10.2 The Region of Convergence for the z-transform.

**Property-1** :- The ROC of $X(z)$ consists of a ring in the z-plane centred about the origin.




**Inferences**

(1) ROC consists of those values of $z = re^{j\omega}$ for which $x[n]r^{-n}$ has a F.T that converges.

(2) The ROC of the z-transform of $x[n]$ consists of the values of $z$ for which $x[n]r^{-n}$ is absolutely summable

i.e $\sum_{n=-\infty}^{+\infty} |x[n]|r^{-n} < \infty$

(3) The convergence is dependent only on $r = |z|$ & not on $\omega$

**Property-2** :- The ROC does not contain any poles.
As seen from previous examples, ROC did not contain any poles.

**Property-3** : If $x[n]$ is of finite duration, then ROC is the entire z-plane, except possibly $z=0$ and/or $z=\infty$

**Explanation** :- A finite duration signal between $N_1$ & $N_2$ has only finite number of nonzero values.

Then

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

→ For $z$ not equal to zero or infinity, each term in the sum will be finite & then $X(z)$ will converge.

→ If $N_1$ is negative & $N_2$ positive, then the summation includes terms with both positive powers of $z$ & negative powers of $z$.

→ As $|z| \to 0$, the terms involving negative powers of $z$ become unbounded.

→ As $|z| \to \infty$, the terms involving positive powers of $z$ become unbounded.

→ Consequently, for $N_1$ negative & $N_2$ positive, the ROC does not include $z=0$ & $z=\infty$

→ If $N_1$ is zero, or positive, there are only –ve powers of $z$ in the above equation & consequently, the ROC includes $z=\infty$

→ If $N_2$ is zero or negative, there are only positive powers of $z$ in the above equation & consequently the ROC includes $z=0$.

Ex 10.5 Consider the unit impulse signal $\delta[n]$. Its z-transform is given by

$$\delta[n] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1.$$

with an ROC consisting of the entire z-plane, including $z=0$ & $z=\infty$

On the other hand, consider the delayed unit impulse $\delta[n-1]$

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$$\delta[n-1] \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1} = \frac{1}{z}$$

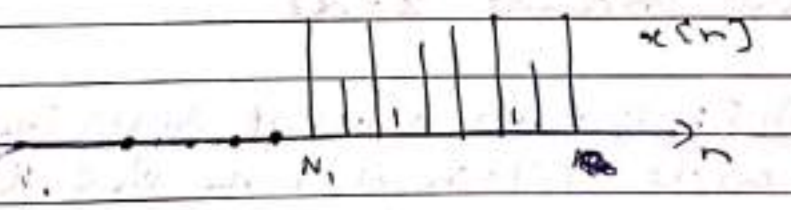
The Z-transform is defined except at $z=0$, where there is a pole. Thus the ROC consists of the entire z-plane including $z=\infty$ except $z=0$

Now consider an advanced impulse $\delta[n+1]$. Then

$$\delta[n+1] \xleftrightarrow{z} \sum_{n=-\infty}^{+\infty} \delta[n+1] z^{-n} = z^{+1}$$

which is well defined for all finite values of $z$. Thus the ROC consists of the entire finite z-plane including $z=0$. There is a pole at infinity.

Property-4 : If $x[n]$ is a right-handed sequence, & if the circle $|z| = r_0$ is in the ROC, then for all finite values of $z$ for which $|z| > r_0$ will also be in the ROC.




Explanation

- A right-handed sequence is zero prior to some value of n, say $N_1$
- If the circle $|z| = r_0$ is in the ROC, then $x[n] r_0^{-n}$ is absolutely summable.
- Consider $|z| = r_1$ with $r_1 > r_0$, so that $r_1^{-n}$ decays more quickly than $r_0^{-n}$ for increasing n.
- The fast decay ensures faster convergence & also since $x[n]$ is right-sided, $x[n] z^{-n} = 0$ for $n < N_1$

R.S x[n] ROC.




→ Consequently $x[n]r_1^{-n}$ is summable.

→ For right-sided sequences in general, the Z.T is given by

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n}$$

where $N_1$ is finite & may be +ve or -ve.

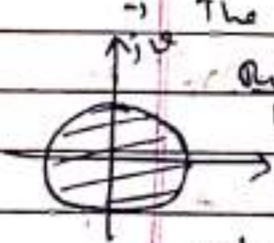
→ If $N_1$ is negative, then the summation includes terms with positive powers of z, which become unbounded as $|z| \to \infty$. Consequently, for right-sided sequence in general, the ROC will not include infinity.

→ For particular case like causal sequences, that are zero for $n<0$, $N_1$ will be +ve & ROC will include $z=\infty$

Property 5: If x[n] is a left-sided sequence, & if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

→ Explanation based on Property 4.

→ The Z-transform of left-sided sequence would be




ROC for LS sequence

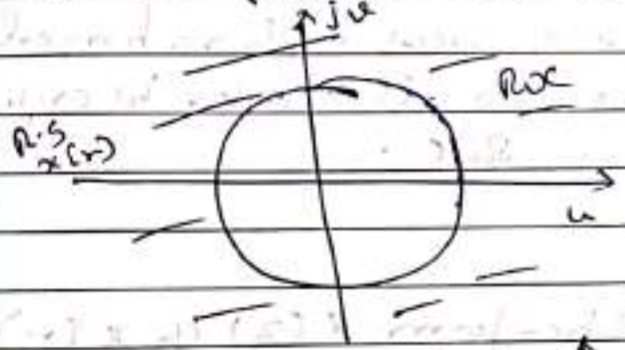
$$X(z) = \sum_{n=-\infty}^{N_2} x[n] z^{-n}$$

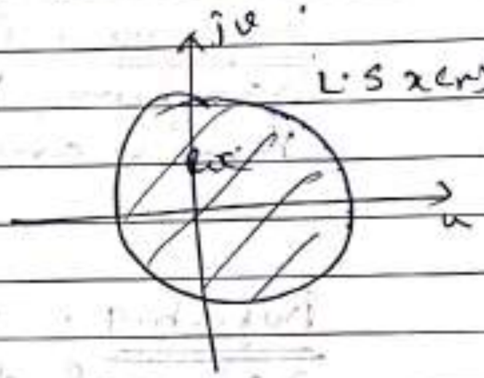
where $N_2$ can be ±. If $N_2$ is ~~negative~~ positive then the summation includes negative powers of z, which may become unbounded as $|z| \to 0$

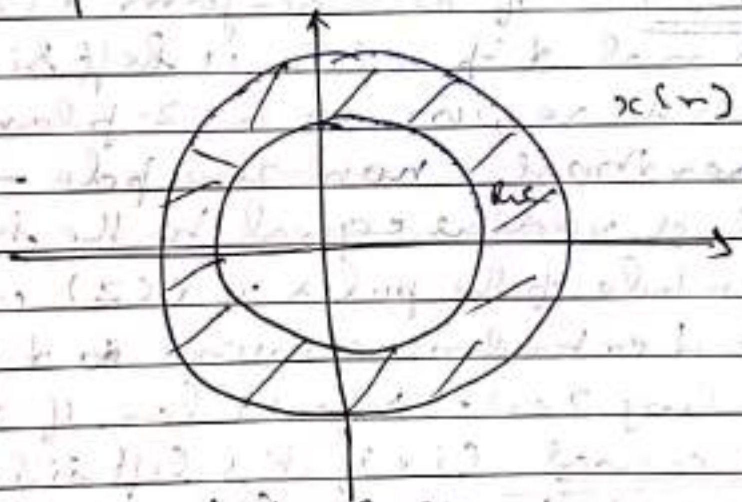
→ consequently, for 2-sided sequences the ROC will not in general include $z=0$.

→ If $n_2 \le 0$ (so that $x(n)=0$ for all $n>0$), the ROC will include $z=0$

**Property 6:** If $x[n]$ is two sided & if the circle $|z|=r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_0$




Intersection of both ROCs.

**Property-7:** If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

→ Refer previous examples.

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Property 8 :- If the z-transform $X(z)$ of $x[n]$ is rational, & if $x[n]$ is right-sided, then the ROC is the region in the z-plane outside the outermost pole i.e outside the circle of radius equal to the largest magnitude of poles of $X(z)$. Further, if $x[n]$ is causal (i.e if it is right sided & equal to zero for $n<0$), then ROC includes $z=\infty$.

Inference :- For R.S sequences with rational transforms, the poles are all closer to origin than is any point in ROC.

Property 9 :- If the z transform $X(z)$ of $x[n]$ is rational & if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost non-zero pole - i.e, inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z=0$ & extending inward to & possibly including $z=0$. In particular, if $x[n]$ is anti-causal (i.e if it is left sided & equal to 0 for $n>0$), then the ROC also includes $z=0$

Inference :- For L.S sequences, the poles of $X(z)$ other than any at $z=0$ are farther from origin than ~~origin~~ is any point in the ROC.

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Q 10.6 Consider

$$x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a>0 \\ 0 & \text{Otherwise} \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n}$$

- Since $x[n]$ is of finite length from Property 3, the ROC includes the entire z-plane except origin and/or infinity.

$$= \sum_{n=0}^{N-1} (az^{-1})^{n}$$

- Since $x[n]=0$ for $n<0$, the ROC will extend to infinity.
- However, since $x[n]$ is non-zero for some positive values of $n$, the ROC will not include origin; also due to poles at origin.

$$= \frac{1-(az^{-1})^{N-1+1}}{1-az^{-1}} = \frac{1-(az^{-1})^{N}}{1-az^{-1}}$$

$$= \frac{1}{z^{N-1}} \cdot \left[\frac{z^N - a^N}{z-a}\right]$$

From the above expression, we infer

i) there $(N-1)$ poles at origin, one pole at '$a$'

ii) The numerator polynomial is of $N^{th}$ order.

The roots of numerator polynomial

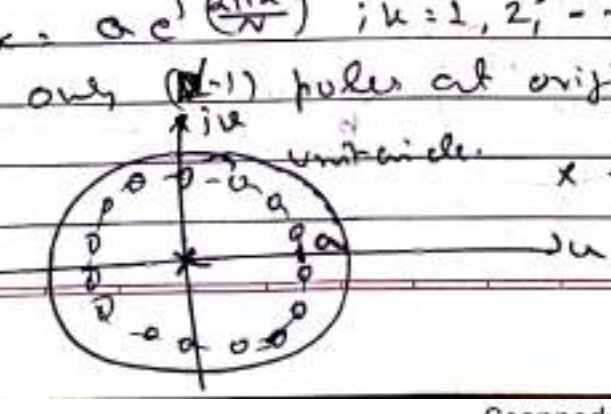
$$z_k = a e^{j\left(\frac{2\pi k}{N}\right)} ;\ k=0,1,2,\dots N-1.$$

When $k=0$, the one root of numerator is $(z-a)$, which cancels with the pole $(z-a)$

Hence the left over zeros are

$$z_k = a e^{j\left(\frac{2\pi k}{N}\right)} ;\ k=1,2,\dots N-1$$

Also there are only $(N-1)$ poles at origin.




x - N-1 poles at origin.

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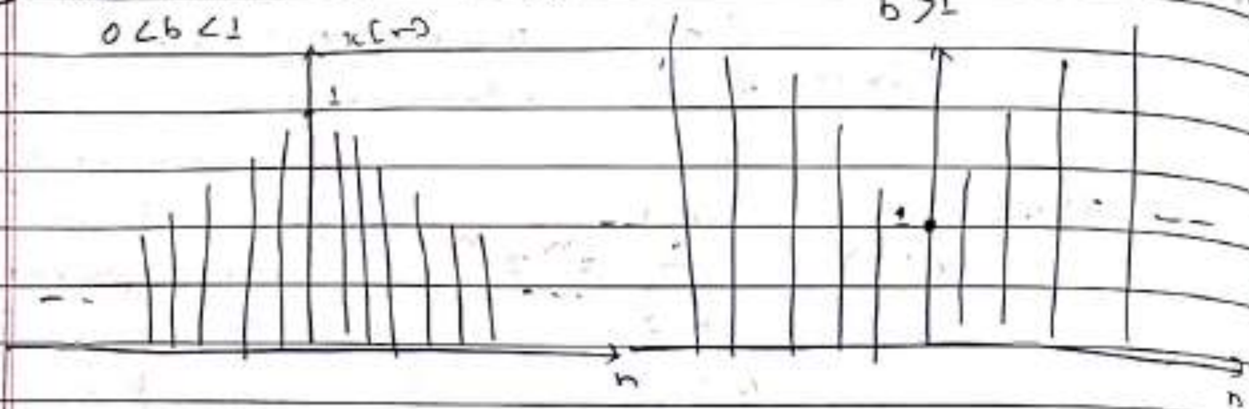
**Eg 10.7** Let $x[n] = b^{|n|}$ ; $b > 0$




we consider two options for b:

(1) $0 < b < 1$ & (2) $b > 1$

$\approx b < 1$

we can express $x[n]$ as the sum of R.S & L.S sequence.

Then

$$x[n] = b^n u[n] + b^{-n} u[-n-1] \quad -(1)$$

from Eg. 10.1 we have $x[n] = a^n u[n]$ ; $|a| < 1$

R.S $\quad b^n u[n] \xleftrightarrow{Z} \dfrac{1}{1 - bz^{-1}}$ ; $|z| > b \quad -(2)$

from Eg 10.2, we have $x[n] = -a^n u[-n-1]$

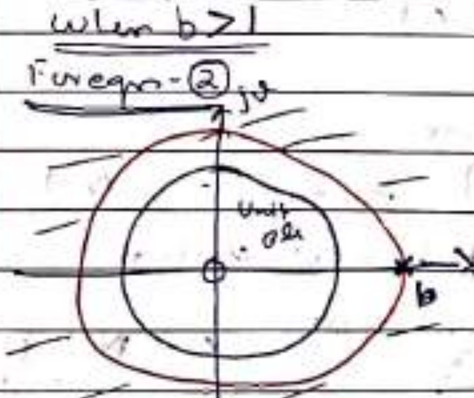
L.S $\quad b^{-n} u[-n-1] \xleftrightarrow{Z} \dfrac{-1}{1 - b^{-1} z^{-1}}$ ; $|z| < \dfrac{1}{b} \quad (3)$

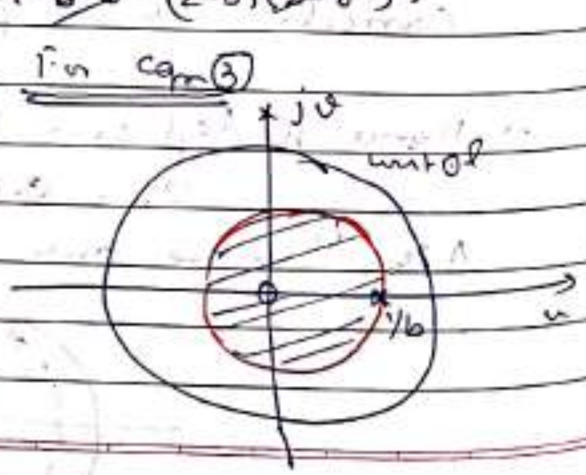
$= \cancel{\left\{\dfrac{b^2-1}{b}\right\} \dfrac{z}{(z-b)(z-b^{-1})}}$

Pole-zero pattern & ROC when $b > 1$

For eqn-(2)

For eqn (3)



So when $b>1$, there is no common ROC. Thus the sequence ① will not have a Z-transform, even though the R.S.+ L.S components do have individually.

when $0 < b < 1$

fig ② $|z|>b$

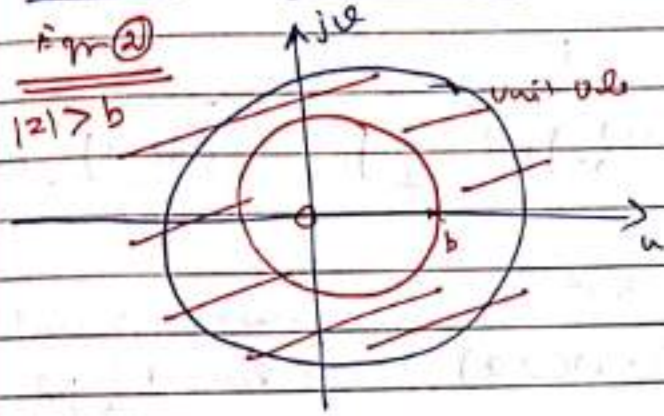
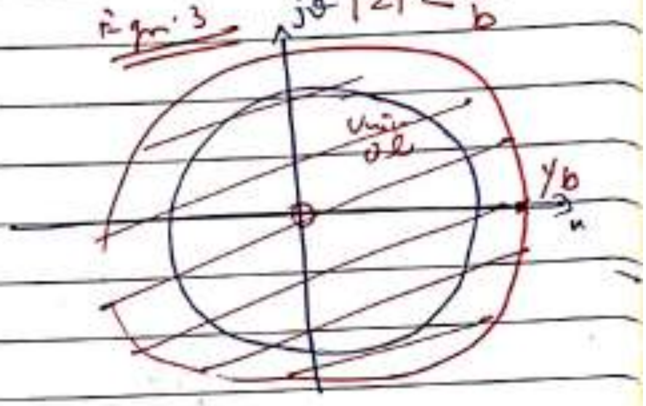
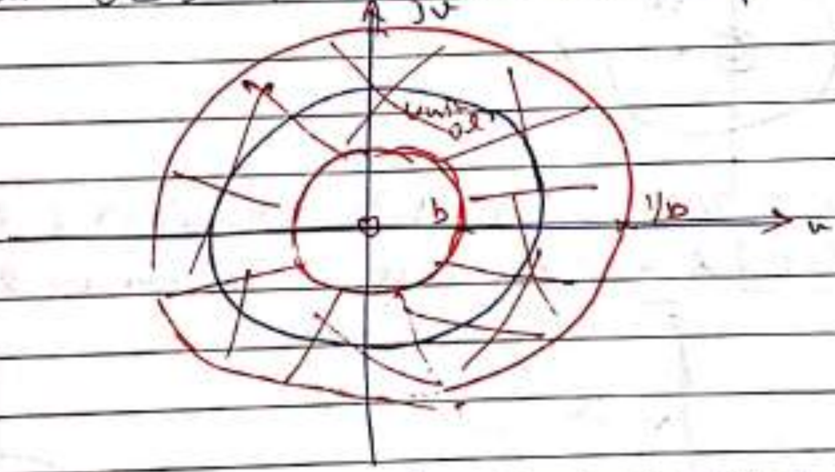



fig 3 $|z| < \frac{1}{b}$




So when $0<b<1$ there is a overlap.




Hence the Z-transform of the composite sequence

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}; \quad b < |z| < \frac{1}{b}$$

or

$$X(z) = \frac{b^2-1}{b} \cdot \frac{z}{(z-b)(z-b^{-1})}; \quad b < |z| < \frac{1}{b}$$

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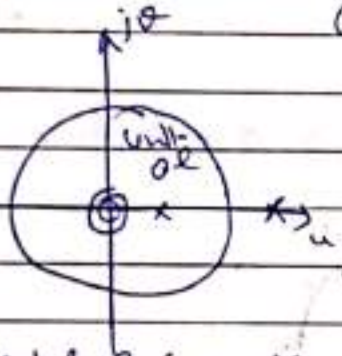
E3.10.8 Let us consider all of the possible ROCs connected to the function

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

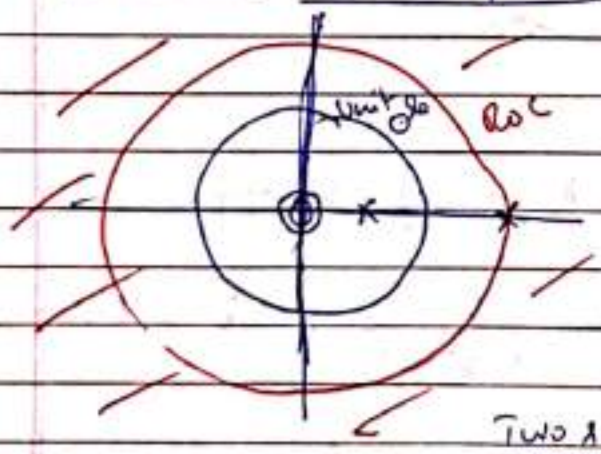
$$= \frac{1}{(1-\frac{1}{3z})(1-\frac{2}{z})} = \frac{1}{\left(\frac{3z-1}{3z}\right)\cdot\left(\frac{z-2}{z}\right)}$$

$$= \frac{3z^2}{(3z-1)(z-2)}$$ ; Zeros: origin (twice)
Poles: $+\frac{1}{3}, 2$.

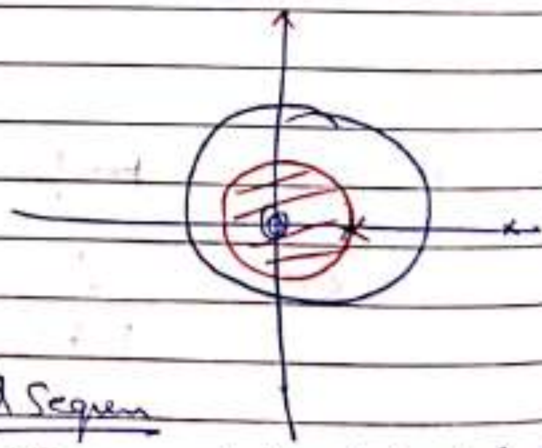



3 possible ROCs associated with $X(z)$ & Pole-zero pattern

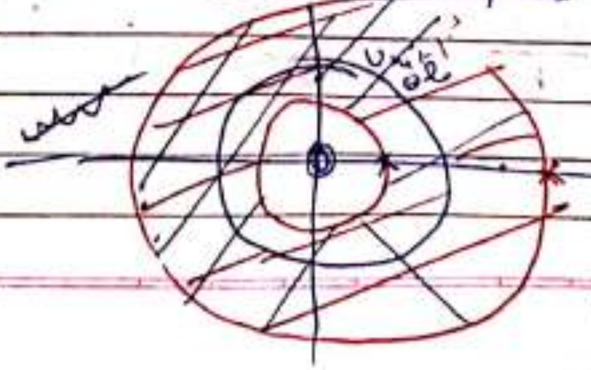
(i) For Right sided sequence.




(ii) For LS Sequence



Two sided Sequen




Since this includes the unit circle, the sequence corresponding to this choice of ROC is the only one of the 3 for which F.T converges

10.3 The Inverse Z-Transform.

$$X(re^{j\omega}) = F\{x[n]r^{-n}\}$$

for any value of r so that $z = re^{j\omega}$ inside the ROC.
Applying Inv. F.T

$$x[n]r^{-n} = F^{-1}\{X(re^{j\omega})\}$$

$$\therefore \quad x[n] = r^n F^{-1}\{X(re^{j\omega})\}$$

$$= r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega}) (re^{j\omega})^n d\omega$$

→ we can recover $x[n]$ from its z-transform evaluated along a contour $z = re^{j\omega}$ in the RoC, with r fixed and ω varying over a $2\pi$ interval.

→ Changing the variable of integration from ω to z, with r fixed

$$z = re^{j\omega}$$

$$dz = rje^{j\omega}d\omega \Rightarrow dz = jz\,d\omega$$

$$\Rightarrow d\omega = \frac{1}{j} z^{-1} dz$$

→ The interval of integration is over $2\pi$, which in terms of z corresponds to one traversal around the circle $|z| = r$

→ Hence

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz}$$

$\oint$ → integration around a ccw closed circular contour centered at origin with radius r.

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→ Generally the I.Z.T is obtained using partial fractions

Eg 10.9 $$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \quad ; \quad |z| > 1/3$$

For the given X(z), there are two poles

$1 - \frac{1}{4} \times \frac{1}{z} = 0$ $\qquad 1 - \frac{1}{3}z^{-1}$

$-\frac{1}{4z} = -1$ $\qquad 1 - \frac{1}{3z} = 0$

$z = \frac{1}{4}$ $\qquad z = \frac{1}{3} = \underline{0.333}$

$= 0.25$

→ For this problem, ROC is outside $\frac{1}{3}$, i.e outside the pole with greater magnitude (0.33)

→ From Property-4, for this condition, the IZT or x[n] is Right-sided sequence.

→ We now split X(z) into partial fractions.

Let $X(z) = \dfrac{3 - \frac{5}{6}x}{\left(1-\frac{1}{4}x\right)\left(1-\frac{1}{3}x\right)} = \dfrac{A}{\left(1-\frac{1}{4}x\right)} + \dfrac{B}{\left(1-\frac{1}{3}x\right)}$

$$A = \left.\frac{3 - \frac{5}{6}x}{\left(1-\frac{1}{3}x\right)}\right|_{x=4} = \frac{3 - \frac{5}{6}(4)}{1 - \frac{1}{3}(4)} = \frac{\frac{18-20}{6}}{\frac{3-4}{3}} = \frac{-2/6}{-1/3}$$

$$= \frac{-2}{\cancel{6}_2} \times \frac{-\cancel{3}}{1} = \underline{1}$$

$$B = \left.\frac{3 - \frac{5}{6}x}{\left(1-\frac{1}{4}x\right)}\right|_{x=3} = \frac{3 - \frac{5(3)}{6}}{1 - 3/4} = \frac{(18-15)/6}{(4-3)/4} = \frac{3/6}{1/4}$$

$$= \frac{\cancel{3}}{\cancel{6}_2} \times \cancel{4}^2 = 2$$

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{3}z^{-1}\right)} \quad - (A)$$

→ Inorder to find $x[n]$, we must associate the given ROC correctly.

→ $x[n] = x_1[n] + x_2[n]$

when

$$x_1[n] \xleftrightarrow{z} \frac{1}{1 - \frac{1}{4}z^{-1}} ; \; |z| > 1/4$$

$$x_2[n] \xleftrightarrow{z} \frac{2}{1 - 1/3 z^{-1}} ; \; |z| > 1/3$$

→ Since the ROC of $X(z)$ is outside the outermost pole, the ROC of each individual term in (A) must also be outside the pole associated with that term.

→ ~~F~~ That is, the ROC for each term consists of all points with magnitude greater than the magnitude of the corresponding pole.

→ From Ex 10.1, we can identify

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n]$$

$$x_2[n] = 2\left(\frac{1}{3}\right) u[n]$$

→ Hence

$$x[n] = x_1[n] + x_2[n]$$

$$= \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

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E10.10 Consider

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} \quad ; \quad \boxed{\tfrac{1}{4} < |z| < \tfrac{1}{3}}$$

Eqn (1)

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$ is still a valid

partial fraction expansion

→ ROC associated with each term will change, now

$$x_1[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{4}z^{-1}} \; ; \; |z| > \tfrac{1}{4}$$

is acceptable.

$$x_2[n] \longleftrightarrow \frac{2}{1-\frac{1}{3}z^{-1}} \; ; \; |z| > \tfrac{1}{3}$$ violates our requirement.

To meet the requirement $\frac{1}{4} < |z| < \frac{1}{3}$, we need

$$x_2[n] = -2\left(\tfrac{1}{3}\right)^n u[-n-1]$$ from Eq 10.2.

$|z| < \frac{1}{3}$

→ Therefore

$$\boxed{x[n] = \left(\tfrac{1}{4}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[-n-1]}$$

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Consider

$$x(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} \quad ; \quad \boxed{|z| < \frac{1}{4}}$$

→ In this case ~~all~~ the RoC is inside both poles 1/3 & 1/4.

→ Consequently the RoC for each ~~term~~ term in the partial fraction expansion must also lie inside the corresponding pole.

→ The resultant total z-T pair is given by.

$$x_1[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{4}z^{-1}} \; ; \; |z| < \frac{1}{4}$$

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1] \quad \text{from Fig 10.2}$$

$\underline{\text{IIIly}}$

$$x_2[z] \xleftrightarrow{z} \frac{2}{1-\frac{1}{3}z^{-1}} \; ; \; |z| < \frac{1}{3}$$

$$x_2[n] = -2\left(\frac{1}{3}\right)^n u[-n-1]$$

Therefore

$$\boxed{x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]}$$

<u>main Inferences</u>

→ Partial fraction expansion is useful when $x(z)$ is expressed in terms of <u>rational functions</u>.

$$\rightarrow x(z) = \sum_{i=1}^{m} \frac{A_i}{1-a_i z^{-1}}$$

→ If ~~|z|>|a_i|~~ the RoC is such that $|z| > a_i$, then

$$x_i[n] = A_i (a_i)^n u[n]$$

→ If the RoC is such that $|z| < a_i$ then

$$x_i[n] = -A_i (a_i)^n u[-n-1]$$

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## Inverse Z-transform using Power Series.

Ex10.12 Consider

$$X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty$$

Solution

we know that

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

comparing the two.

$$x[n] \begin{cases} 4 & ; n = -2 \\ 2 & ; n = 0 \\ 3 & ; n = 1 \\ 0 & ; \text{elsewhere.} \end{cases}$$

Then

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Formula:

$$\boxed{\delta[n+n_0] \xleftrightarrow{z} z^{+n_0}}$$

Ex10:13

$$X(z) = \frac{1}{(1-az^{-1})} \quad ; \quad |z| > |a|$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1-az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & \cancel{az^{-1}} - a^2z^{-2} \\ \hline & +a^2z^{-2} \\ & \cancel{a^2z^{-2}} - a^3z^{-3} \\ \hline & +a^3z^{-3} \end{array}$$

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Hence

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

This series converges since $|z| > |a|$ or $|az^{-1}| < 1$.

Comparing the above with

$$x(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

we see $x(n) = 0$ for $n < 0$. Then

$x[0] = 1$, $x[1] = a$, $x[2] = a^2$ ; $x[3] = a^3$

$\Rightarrow$ $\boxed{x[n] = a^n u[n]}$

If the ROC is $|z| < |a|$ or $|az^{-1}| > 1$; the above power series will not converge.

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 + \cdots \\ \hline -az^{-1}+1 & \cancel{1} \\ & \cancel{1} - a^{-1}z \\ \hline & +a^{-1}\cancel{z} \\ & \cancel{a^{-1}z} - a^{-2}z^2 \\ \hline & +a^{-2}z^2 \end{array}$$

Then

$$\frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots$$

In this case $x[n] = 0$ for $n \geq 0$. Then

$x(-1) = -a^{-1}$, $x(-2) = -a^{-2}$, $\cdots$

$$\boxed{x[n] = -(a^n)\, u[-n-1]}$$

Inv. Z-T ~~[illegible]~~ by Power series expansion is useful when $X(z)$ is not in the rational form.

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Eg10:4 Consider
$$x(z) = \log(1+az^{-1}) \; ; \; |z|>|a|$$

Since $|z|>|a|$ or $|az^{-1}|<1$.
using Taylor's series expansion

$$\log(1+v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v^n}{n} \; ; \; |v|<1.$$

Then
$$x(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

which results in

$$x[n] = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n} \; ; & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Equivalently

$$x[n] = \frac{-(-a)^n}{n} u[n-1]$$

10.5 **Properties of the z-Transform:**

10.5.1 **Linearity**

If

$x_1[n] \xleftrightarrow{z} X_1(z)$ with ROC = $R_1$

&

$x_2[n] \xleftrightarrow{z} X_2(z)$ with ROC = $R_2$

then

$$a x_1[n] + b x_2[n] \xleftrightarrow{z} a X_1(z) + b X_2(z)$$

$$\text{ROC} \rightarrow R_1 \cap R_2$$

**Note:-**

1) If $aX_1(z) + bX_2(z)$ does not contain any pole-zero cancellation, then the ROC will be the exact overlap of individual ROC.

2) For pole-zero cancellation, ROC will be larger.

(3) If $x_1[n] = a^n u[n]$ & $x_2[n] = a^n u[n-1]$ with $|z| > |a|$ then if $x_3[n] = x[n] - x_2[n] = \delta[n]$ has ROC entire z-plane.

2. **Time Shifting**

If $x[n] \xleftrightarrow{z} X(z)$ with ROC = R

then

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z) \quad \text{with ROC = R}$$

except for the possible addition or deletion of origin or infinity.

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Inferences

(1) Because of the multiplication of $z^{-n_0}$ for $n_0 > 0$, poles will be introduced at $z = 0$ which may cancel corresponding zeros of $X(z)$. Consequently $z = 0$ may be a pole of $z^{-n_0}X(z)$ while it may not be a pole of $X(z)$. In this case the ROC for $z^{-n_0}X(z)$ equals of that of $X(z)$ but with origin deleted.

(2) If $n_0 < 0$, zeros will be introduced in $z^{-n_0}X(z)$ which may cancel corresponding poles of $X(z)$ at $z = 0$. In this case $z = \infty$ is a pole of $z^{-n_0}X(z)$, & thus ROC of $z^{-n_0}X(z)$ equals ROC of $X(z)$ but with $z = \infty$ deleted.

3. Scaling in the z-domain:

If

$$x[n] \xleftrightarrow{z} X(z) \text{ with ROC} = R$$

then

$$\boxed{z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right) \text{ with ROC} = |z_0|R}$$

where $|z_0|R$ is the scaled version $R$.

-> If $z$ is a point in the ROC of $X(z)$, then the point $|z_0| z$ is in the ROC of $X\left(\frac{z}{z_0}\right)$

-> If $X(z)$ has a pole (or zero) at $z = a$, then $X\left(\frac{z}{z_0}\right)$ has a pole (or a zero) at $z = z_0 a$

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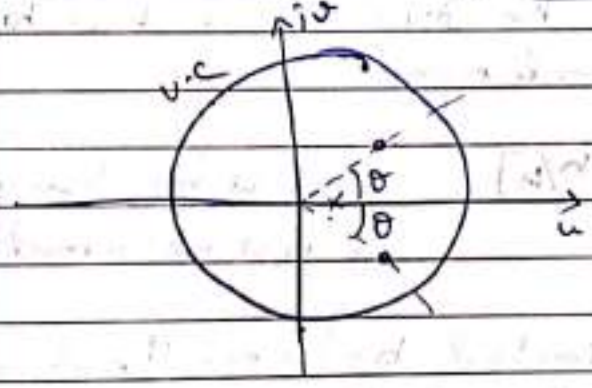
Important Special case:

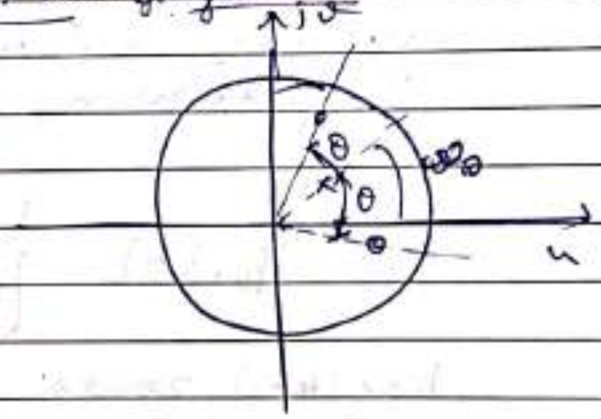
when $z_0 = e^{j\omega_0}$ then

$$\boxed{e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(e^{-j\omega_0} z) \quad ROC = R}$$

Inferences

→ LHS of the equation corresponds to multiplication by a complex exponential sequence.
RHS can be interpreted as a rotation in the ~~z-plane~~ z-plane, that is, all pole-zero locations rotate in the z-plane by an angle $\omega_0$.




→ If $X(z)$ has a factor of the form $1 - az^{-1}$, then $X(e^{-j\omega_0} z)$ will have a factor $1 - ae^{+j\omega_0} z^{-1}$ & thus, a pole or zero at $z = a$ in $X(z)$ will become a pole or zero at $z = ae^{j\omega_0}$

→ For a general case when $z_0 = r_0 e^{j\omega_0}$, then the pole-zero locations are rotated by $\omega_0$ & scaled in magnitude $r_0$

4. Time Reversal.

If

$$x[n] \xleftrightarrow{z} X(z) \quad \text{with ROC} = R$$

then

$$\boxed{x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right) \quad \text{with ROC} = \frac{1}{R}}$$

Inference

If $z_0$ is in the ROC of $x[n]$, then $\frac{1}{z_0}$ is in the ROC of $x[-n]$.

5. Time Expansion.

Unlike C.T signals, the D.T. signal has to obey the following condition:

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{if } n \text{ is not a multiple of } k \end{cases}$$

has $(k-1)$ zeros inserted between the successive values of the original signal.

Then, if

$$x[n] \xleftrightarrow{z} X(z), \quad \text{with ROC} = R.$$

then

$$\boxed{x_{(k)}[n] \xleftrightarrow{z} X(z^k) \quad \text{with ROC} = R^{1/k}}$$

Inference

1) If $z$ is in the ROC of $X(z)$, the point $z^{1/k}$ is in the ROC of $X(z^k)$

2) If $X(z)$ has a pole (or zero) $z = a$, then $X(z^k)$ has a pole (or zero) at $z = a^{1/k}$

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6. Conjugation

If

$$x[n] \xleftrightarrow{z} X(z) \quad \text{with ROC} = R$$

then

$$\boxed{x^*[n] \xleftrightarrow{z} X^*(z^*) \quad \text{with ROC} = R}$$

Consequently, if $x[n]$ is real, we can conclude

$$\boxed{X(z) = X^*(z^*)}$$

Thus if $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$

7. The convolution Property

If

$$x_1[n] \xleftrightarrow{z} X_1(z) \quad \text{with ROC} = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z) \quad \text{with ROC} = R_2$$

then

$$\boxed{x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)X_2(z) \quad \text{with ROC} = R_1 \cap R_2}$$

Inference

The convolution property states that when two polynomials or power series $X_1(z)$ & $X_2(z)$ are multiplied, the co-efficients in the polynomial representing the product are the convolution of the co-efficients in the polynomials $X_1(z)$ & $X_2(z)$

Es 10.15 Consider an LTI System such that

$$y[n] = h[n] * x[n]$$

where

$$h[n] = \delta[n] - \delta[n-1]$$

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Note

→ Time shifting

$$\delta[n] - \delta[n-1] \xleftrightarrow{Z} 1 - z^{-1} = \frac{z-1}{z}$$

with ROC equal to the entire z-plane except the origin. The z-transform has a zero = 1.

If

$$x[n] \xleftrightarrow{Z} X(z) \text{ with ROC} = R$$

Then

convolution property

$$y[n] \xleftrightarrow{Z} (1-z^{-1})X(z)$$

with ROC equal to R, with possible deletion of z = 0 and/or addition of z = 1.

Further

$$y[n] = \{\delta[n] - \delta[n-1]\} * x[n]$$

$$= x[n] - x[n-1]$$

y[n] is the first difference sequence x[n], which is equivalent to the differentiation in C-T domain

Eg 10.26 We consider the inverse of first-differencing, namely accumulation / summation. Let w[n] be the running sum of x[n]

$$w[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]$$

then by convolution property

$$w[n] = \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{Z} \frac{1}{1-z^{-1}} X(z)$$

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as $u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}}$

with ROC including at least the intersection of R with $|z|>1$.

8. Differentiation in the z-Domain:

If

$$x[n] \xleftrightarrow{z} X(z) \text{ with ROC} = R$$

then

$$\boxed{n x[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz} \text{ with ROC} = R}$$

E/10'17

If $X(z) = \log(1+az^{-1}) \quad |z|>|a|$

then

$$n x[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz} = -z \cdot \frac{1}{(1+az^{-1})} (-1) a z^{-2}$$

$$= \frac{az^{-1}}{(1+az^{-1})} \; ; \; |z|>|a| \quad \text{—} (1)$$

we know from Eg - 10.2

$$a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}} \; ; \; |z|>|a|$$

using Time shifting property.

$$a(-a)^{n-1} u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{(1+az^{-1})} \; ; \; |z|>|a| \quad -(2)$$

From (1) & (2)

$$n x[n] = a(-a^{n-1}) u[n-1] = (-1)(-a)(-a^{n-1}) u[n-1]$$

$$= \frac{-1(-a)^n}{n} u[n-1]$$

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Eg 10.18 Consider

$$X(z) = \frac{a z^{-1}}{(1-az^{-1})^2} \quad ; \; |z| > |a|$$

From Eg 10.1

$$a^n u[n] \xleftrightarrow{z} \frac{1}{(1-az^{-1})} \quad ; \; |z| > |a|$$

Hence

$$n a^n u[n] \xleftrightarrow{z} -z \frac{d}{dz}\left\{\frac{1}{(1-az^{-1})}\right\} = \underline{\underline{\frac{a z^{-1}}{(1-az^{-1})^2}}} \; ; |z|>|a|$$

Hence I.Z.T $\underline{\underline{n a^n u[n]}}$

9. <u>The Initial Value Theorem:</u>

If $x[n] = 0$ for $n < 0$, then <u>Causal Sequence</u>

$$x[0] = \lim_{z \to \infty} X(z)$$

with this constraint

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$
whereas for $n = 0$, $z^{-n} = 1$ thus

$$\underline{\underline{x[0] = \lim_{z \to \infty} X(z)}}$$

Inference

(1) For a causal sequence, if $x[0]$ is finite, then $\lim_{z \to \infty} X(z)$ is also finite.

(2) If $X(z)$ is expressed a ratio of polynomials, the order of the numerator cannot be greater than the order of denominator.

(3) The no. of finite zeros of $X(z)$ cannot be greater than the finite number of poles.

Ex 10.19 Consider Eg 10.3

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{6}\right)^n u[n]$$

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})} = \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$x[0] = 7(1)(1) - 6(1)(1) = 7-6 = 1.$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{(1-0)}{(1-0)(1-0)} = 1$$

Table 10.1 Properties of z-transform. pg-782.

10.6 Some Common z-transform Pairs.

| | Signal | Transform | ROC |
|---|---|---|---|
| ① | $\delta[n]$ | $1$ | All z |
| ② | $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| ③ | $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |

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| | | | |
|---|---|---|---|
| (4) | $\delta[n-m]$ | $z^{-m}$ | All z, except 0 (if $m>0$) or $\infty$ (if $m<0$) |
| (5) | $\alpha^n u[n]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\|z\| > \|\alpha\|$ |
| (6) | $-\alpha^n u[-n-1]$ | $\frac{1}{1-\alpha z^{-1}}$ | $\|z\| < \|\alpha\|$ |
| (7) | $n\alpha^n u[n]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z\| > \|\alpha\|$ |
| (8) | $-n\alpha^n u[-n-1]$ | $\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z < \|\alpha\|$ |
| (9) | $[\cos w_0 n]u[n]$ | $\frac{1-[\cos w_0]z^{-1}}{1-[2\cos w_0]z^{-1}+z^{-2}}$ | $\|z\|>1$ |
| (10) | $[\sin w_0 n]u[n] =$ | $\frac{[\sin w_0]z^{-1}}{1-[2\cos w_0]z^{-1}+z^{-2}}$ | $\|z\|>1$ |
| (11) | $[r^n \cos w_0 n]u[n] =$ | $\frac{1-[r\cos w_0]z^{-1}}{1-[2r\cos w_0]z^{-1}+r^2z^{-2}}$ | $\|z\|>r$ |
| (12) | $[r^n \sin w_0 n]u[n] =$ | $\frac{[r\sin w_0]z^{-1}}{1-[2r\cos w_0]z^{-1}+r^2z^{-2}}$ | $\|z\|>r$ |

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## 10.7 Analysis & characterisation of LTI Systems using Z-transforms.

### 10.7.1 Causality

Consider a discrete-time LTI system. For any input $x[n]$ the output $y[n]$ is given by

$$y[n] = h[n] * x[n]$$

$$\Rightarrow \quad Y(z) = H(z) \cdot X(z) \quad \text{multiplication}$$

$H(z)$ - System function or Transfer fn of the system

$h[n]$ - Impulse response of the system.

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$ — for a causal system since $h[n] = 0 ; n < 0$

Principle 1.

A discrete-time LTI system is causal if & only if the ROC of its system fn. $H(z)$ is exterior of a circle including infinity.

Principle 2:

A discrete-time LTI system with rational system function $H(z)$ is causal if & only if

(a) The ROC is exterior of a circle outside the outermost pole and

(b) with $H(z)$ expressed as a ratio of polynomials in $z$, the order of the numerator cannot be greater than the order of the denominator.

Ex 10:20 Consider $H(z) = \dfrac{z^3 - 3z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$ → Non-causal since order of N > order of D.

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Eg 10.21 Consider a system with system function

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} \quad |z| > 2$$

then

$$H(z) = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}$$ ⇒ Causal
order(N) = order(D)

Impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n].$$

10.7.2 Stability

Stability of a discrete-time LTI system is equivalent
- to its impulse response h[n] being absolutely summable.
- In this case Fourier Transform of h[n] converges & consequently, the ROC of H(z) must include unit circle.

Principle-3

A discrete-time LTI system is stable if & only if the ROC of its system function H(z) includes the unit circle, $|z| = 1$.

Eg 10.22 Consider

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

Case (i)

Poles: $+\frac{1}{2}, +2$

Since the ROC includes the region for $|z| > 2$, which does not include unit circle $|z| = 1$, the system is not stable. However it is causal.

Also

$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$ is not absolutely summable.

Case(ii)

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} \qquad \frac{1}{2} < |z| < 2$$

The ROC is a region between $\frac{1}{2} < |z| < 2$, hence includes unit circle $|z| = 1$. However it is non-causal. There fore the system is Non-causal & Stable

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1] \rightarrow \text{summable.}$$

Case(iii)

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} ; \quad |z| < \frac{1}{2}$$

ROC is not outside the outermost pole - Non-causal

ROC does not include unit circle → unstable

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2^n u[-n-1]$$

- not summable.

System is non-causal & unstable

Principle-4

A causal System (LTI) with rational system fn. $H(z)$ is stable if & only if all the poles of $H(z)$ lie inside the unit circle - they must have magnitudes less than 1.

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**Eg 10.23** Consider a causal system with system function

$$H(z) = \frac{1}{1 - az^{-1}}$$

Pole $= a$

$\boxed{\text{For stability } |a| < 1.}$ The same can be understood since

$h[n] = a^n u[n]$ - summable for $|a| < 1$

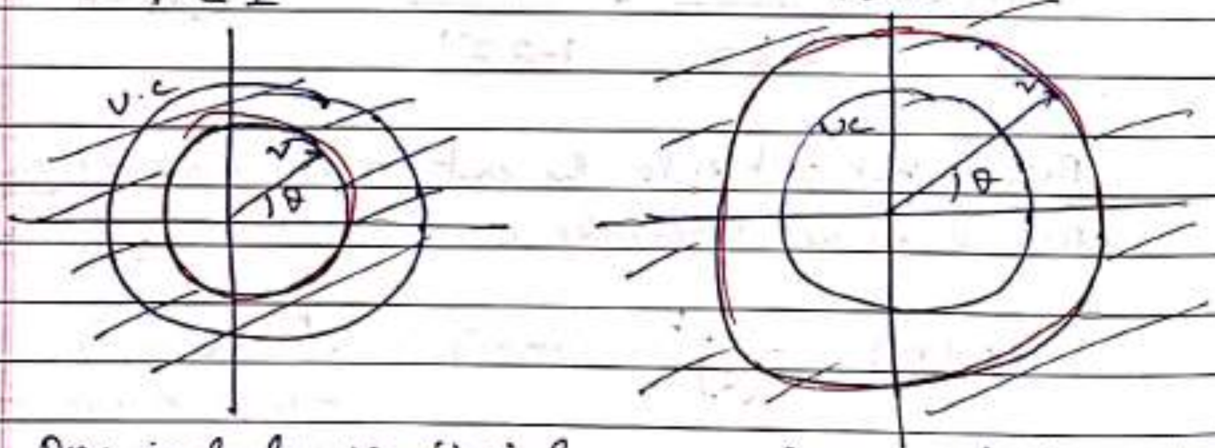
**Eg 10.24** Consider a 2nd order system with

$$H(z) = \frac{1}{1 - (2r\cos\theta) z^{-1} + r^2 z^{-2}}$$

Poles $z_1 = re^{j\theta}$ & $z_2 = re^{-j\theta}$

Assuming it to be causal, then the ROC is outside the outermost pole $\rightarrow |z| > |r|$

$r < 1$      $r > 1$




ROC includes unit circle
Poles inside unit circle
$\boxed{\text{Stable}}$

ROC is outside unit circle
Poles outside unit circle
$\boxed{\text{Unstable}}$

10.7.3 LTI Systems characterized by Linear Constant Coefficient Difference equations (at initial rest) (%)

31:10:25 Consider a system whose i/p & o/p are related by

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

Applying Z.T on both sides using time-shifting property, we get

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$\therefore \frac{Y(z)}{X(z)} = H(z) = \frac{(1+\frac{1}{3}z^{-1})}{(1-\frac{1}{2}z^{-1})} = \frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{1}{3}\frac{z^{-1}}{(1-\frac{1}{2}z^{-1})}$$

Since the RoC is not defined, we will have 2 different choices of RoC.

If $|z| > \frac{1}{2}$ then $h[n]$ is right sided.
If $|z| < \frac{1}{2}$ then $h[n]$ is left-sided (*)

Consider $|z| > 1/2$, we have

$$H(z) = \left(1+\frac{1}{3}z^{-1}\right) \cdot \frac{1}{(1-\frac{1}{2}z^{-1})}$$

Using the property of linearity + time shifting, we get

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

causal & stable

For the other case where $|z| < \frac{1}{2}$, using linearity + time shifting

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

System is anti causal ($h[n] = 0$ for $n > 0$) + unstable

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Concluding remarks:

→ System satisfying a linear constant coefficient difference eqn is always rational.

→ Since ROC is not specified; conditions like causality & stability provide the information on ROC.

→ If the system is causal, ROC will be outside the outermost pole. If it is also ~~stable~~ stable, the ROC would include unit circle.

10.7.4 Examples Relating System Behaviour to the System function.

BS10.26 Following information are available for an LTI system

(a) If the input is $x_1[n] = \left(\frac{1}{6}\right)^n u[n]$, the output is

$$y_1[n] = \left\{ a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n \right\} u[n]$$

where a is a real number.

(b) If $x_2[n] = (-1)^n$, the output is

$$y_2[n] = \left(\frac{7}{4}\right)(-1)^n$$

Find $H(z)$, value of 'a' & other properties of the system.

Solution

From (a)

$$X_1(z) = \frac{1}{1 - \frac{1}{6} z^{-1}} \quad ; \quad |z| > 1/6$$

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$$Y_1(z) = \frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{(a+10) - (5+\frac{a}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} ; \quad |z| > \frac{1}{2}$$

since $\frac{1}{2} > \frac{1}{3}$

System function / Transfer function $H(z)$ is then given by

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{\{(a+10)-(5+\frac{a}{3})z^{-1}\}\{1-\frac{1}{6}z^{-1}\}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

From (b) when $x_2[n] = (-1)^n$, then $y_2[n] = \frac{7}{4}(-1)^n$

Then the response to $x_2[n]$ must be equal to $(-1)^n$ multiplied by the systems fn. $H(z)$ evaluated at $z=-1$. Thus from (b), we get

Doubt?

$$\frac{7}{4} = H(-1) = \frac{[(a+10)+5+\frac{a}{3}][7/6]}{(\frac{3}{2})(\frac{4}{3})}$$

Solving we get $a=-9$, hence

$$H(z) = \frac{(1-2z^{-1})(1-\frac{1}{6}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{1-\frac{13}{6}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$= \frac{z^2-\frac{13}{6}z+\frac{1}{3}}{z^2-\frac{5}{6}z+\frac{1}{6}}$$

Inferences

→ From convolution property, we know that the ROC of $Y_1(z)$ must include at least the intersections of the ROCs of $X_1(z)$ & $H(z)$.

→ Examining the three possible ROCs of $H(z)$ namely

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$|z| < \frac{1}{3}$, $\frac{1}{3} < |z| < \frac{1}{2}$ & $|z| > 1/2$, we can see that the only choice that is consistent with the ROC of $X_1(z)$ & $Y_1(z)$ is $|z| > 1/2$

-> Since the ROC for the systems includes the unit circle, the given system is ~~causal~~ stable

-> Also the ration of $H(z)$ is rational, the system is also causal.

-> From the relation

$$\frac{Y_1(z)}{X_1(z)} = \frac{1 - \frac{13}{6} z^{-1} + \frac{1}{3} z^{-2}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$\left\{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right\} Y_1(z) = \left\{1 - \frac{13}{6} z^{-1} + \frac{1}{3} z^{-2}\right\} X_1(z)$$

$$y[n] - \frac{5}{6} y[n-1] + \frac{1}{6} y[n-2] = x[n] - \frac{13}{6} x[n-1] + \frac{1}{3} x[n-2]$$

Difference equation

E9.10-27 Consider a stable & causal system with impulse response $h[n]$ & rational system function $H(z)$. It is known that $H(z)$ contains a pole at $z = 1/2$ & a zero somewhere on the unit circle. Locations of all other poles and zeros are unknown. Check the correctness of the following statements

(a) $F\{(\frac{1}{2})^n h[n]\}$ converges.

Doubt

True. $F\{(\frac{1}{2})^n h[n]\}$ corresponds to the value of the z-transform of $h[n]$ at $z = 2$. Thus, it's converges

$$\frac{1}{1 - \frac{1}{2} z^{-1}} \Rightarrow z = 2$$

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is equivalent to the point $z = 2$ being in the ROC. Since the system is stable & causal, all the poles of $H(z)$ are inside the unit circle, & the ROC includes all points outside the unit circle including $z = 2$

(b) $H(e^{j\omega}) = 0$ for some $\omega$

True. There is a zero on the unit circle.

(c) $h[n]$ has finite duration.

False: A finite-duration sequence must have an ROC that includes the entire z-plane, except possibly $z = 0$ and/or $z = \infty$. This is not consistent with having a pole at $z = 1/2$.

(d) $h[n]$ is real.

This requires $H(z) = H^*(z^*)$. This implies that if there is a pole (zero) at a nonreal location $z = z_0$, there must also be a pole (zero) at $z_0 = z_0^*$.

Insufficient information available to make a conclusion

(e) $g[n] = n\{h[n] * h[n]\}$ is the impulse response of a stable system

True :- Since the given system is causal, $h[n] = 0$ for $n < 0$. Then $h[n] * h[n] = 0$ for $n < 0 \rightarrow$ The system with $h[n] * h[n]$ as its impulse response is causal. The same is then true for $g[n] = n\{h[n] * h[n]\}$. Further by convolution & Differentiation property, with $h[n] * h[n]$ is $H^2(z)$, the system function for $g[n]$

$$G(z) = -z\frac{dH^2(z)}{dz} = -2zH(z)\left\{\frac{dH(z)}{dz}\right\}$$

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From the above equations, we can conclude that poles of $G(z)$ are at the same locations as those of $H(z)$, with the possible exception of origin. Therefore, since $H(z)$ has all its poles inside the unit circle, so must $G(z)$. It follows that $g[n]$ is the impulse response of a causal & stable system.

10.9 The Unilateral Z-Transform.

Characteristics.

(1) Useful in analysing LTI systems which are not initially at rest.

(2) Defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Limits of summation is from 0 to $\infty$.

(3) Convention

$$x[n] \xleftrightarrow{UZ} X(z) = UZ\{x[n]\}$$

(4) Whether $x[n]$ is zero for $n<0$ is immaterial.

(5) Can be considered as bilateral Z-T of $x[n]u[n]$

(6) Any sequence $x[n]=0$ for $n<0$, both B.Z.T & U.Z.T are identical.

(7) Since $x[n]u[n]$ is a right-sided sequence, the ROC in UZT is always exterior to a circle.

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10.9.1 Examples of Unilateral z-Transforms & Inverse Transforms.

Eg 10.32 Consider $x[n] = a^n u[n]$

Since $x[n] = 0$ for $n<0$, UZT & BZT are identical

$$X(z) = \frac{1}{1-az^{-1}} \; ; \; |z|>|a|$$

Eg 10.33 Consider $x[n] = a^{n+1} u[n+1] = a a^n u[n+1]$

BZT Note $x[-1] = a^0 u[0] = 1 \neq 0$

From Eg 10.1 & time shifting property, BZT is given by

$$\boxed{X(z) = \frac{z}{1-az^{-1}} \; ; \; |z| > |a|}$$

UZT

$$\boxed{X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^{n+1} z^{-n}}$$

$$\boxed{= \frac{a}{1-az^{-1}} \; ; \; |z|>|a|}$$

Eg 10.34 Consider UZT

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

The Inverse BZT is

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

In the case of UZT, ROC must be exterior to the circle having radius greatest among the poles; here it is $1/3$ $(\frac{1}{3} > 1/4)$ i.e $|z| > \frac{1}{3}$. Hence in this case

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] \; ; \text{ for } n \geq 0$$

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→ Another approach to Inverse UZT is by power-series expansion.

→ Caution

Since $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$, the power-series expansion of the transform cannot contain terms with positive powers of z.

Consider

$$X(z) = \frac{1}{1-az^{-1}} \; ; \; |z| > |a|$$

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots \quad \text{— (A)}$$

$$X(z) = \frac{1}{1-az^{-1}} \; ; \; |z| < |a|$$

$$= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - \cdots \quad \text{— (B)}$$

Therefore if X(z) obtained by UZT, then (A) is the only choice.

→ If X(z) is expressed as ratio of polynomials of z & not $z^{-1}$

$$X(z) = \frac{p(z)}{q(z)}$$

then X(z) to be UZT, the degree of p(z) should not be greater than the degree of q(z)

Eg 10.35 Consider $X(z) = \frac{z}{1-az^{-1}} = \frac{z^2}{(z-a)}$

Two BZT can be associated with X(z) for |z|>|a| & |z|<|a|. The choice |z|>|a| corresponds to a right-sided sequence, but not to a signal that is

zero for all $n<0$, since the Inv. BZT is
$x[n] = a^{n+1} u[n+1]$ → Eq 10.33.
& $x[-1] = 1$. Hence UZT does not exist for this.

## 10.9.2. Properties of Unilateral z-transform.

| Property | Signal. | UZT |
|---|---|---|
| | $x[n]$ | $X(z)$ |
| | $x_1[n]$ | $X_1(z)$ |
| | $x_2[n]$ | $X_2(z)$ |
| (1) Linearity | $ax_1[n]+bx_2[n]$ | $aX_1(z)+bX_2(z)$ |
| (2) Time delay | $x[n-1]$ | $z^{-1}X(z)+x[-1]$ |

$x[n-m] \Rightarrow z^{-m}X(z)+x[-1]z^{-m+1}+x[-2]z^{-m+2}+\cdots+x[-m]$

| | | |
|---|---|---|
| (3) Time advance | $x[n+1]$ | $zX(z)-zx[0]$ |
| (4) Scaling in z-domain | $e^{j\omega_0 n}x[n]$ | $X(e^{-j\omega_0}z)$ |
| | $z_0^{n}x[n]$ | $X(z/z_0)$ |
| | $a^n x[n]$ | $X(a^{-1}z)$ |
| (5) Time expansion | $x_k[n] = \begin{cases} x[m], & n=mk \\ 0 & n\neq mk \text{ for any } m \end{cases}$ | $X(z^k)$ |
| (6) Conjugation | $x^*[n]$ | $X^*(z^*)$ |
| (7) Convolution (assuming $x_1[n]$ & $x_2[n]$ are identically equal to zero for $n<0$) | $x_1[n]*x_2[n]$ | $X_1(z)X_2(z)$ |

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(8) First difference $x[n]-x[n-1]$ $(1-z^{-1})X(z) - x[-1]$

(9) Accumulation $\sum_{k=0}^{n} x[k]$ $\frac{1}{(1-z^{-1})} X(z)$

(10) Differentiation in z-domain $n x[n]$ $-z \frac{d X(z)}{dz}$.

(11) Initial value Theorem

$$x[0] = \lim_{z \to \infty} X(z)$$

(12) Summation. If $x[n] = 0$ for $n<0$

$$\sum_{k=0}^{n} x[k] = x[n] * u[n] \xleftrightarrow{UZ} X(z)U(z) = X(z) \cdot \frac{1}{1-z^{-1}}$$

Ex 10.36 Consider a causal LTI system described by
$y[n] + 3y[n-1] = x[n]$
with condition of initial rest.

Solution

$$Y(z) + 3z^{-1} Y(z) = X(z)$$

$$Y(z)[1+3z^{-1}] = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+3z^{-1}}$$

Suppose if $x[n] = \alpha u[n]$, where $\alpha$ - a constant.
Then the BZT & UZT of the output $y[n]$

$$Y(z) = H(z) \cdot X(z)$$

$$= \frac{1}{(1+3z^{-1})} \cdot \frac{\alpha}{(1-z^{-1})}$$

$$= \frac{(3/4)\alpha}{(1+3z^{-1})} + \frac{(1/4)\alpha}{(1-z^{-1})}$$

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Taking I ZT, we get

$$y[n] = \frac{3\alpha(-3)^n u[n]}{4} + \frac{\alpha}{4}u[n]$$

Shifting Property of UZT.

Consider $y[n] = x[n-1]$

then

$$Y(z) = \sum_{n=0}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n}$$

$$= x[-1] + \sum_{n=0}^{\infty} x[n]z^{-(n+1)}$$

$$= x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n}$$ – Time delay

Hence $Y[z] = x[-1] + z^{-1}X(z)$

By repeated application, the UZT of

$$w[n] = y[n-1] = x[n-2]$$

then

$$W(z) = x[-2] + x[-1]z^{-1} + z^{-2}X(z)$$

Continuing we can the UZT of $x[n-m]$ for any +ve m.

10.9.3 Solving Difference Equations using UZT

consider

$$y[n] + 3y[n-1] = x[n]$$

while $x[n] = \alpha u[n]$ & $y[-1] = \beta$

Taking UZT:

$$Y[z] + 3z^{-1}Y(z) + 3y[-1] = X(z)$$

$$Y(z) + 3z^{-1}Y(z) + 3\beta = \frac{\alpha}{1-z^{-1}}$$

$$Y[z] = \underbrace{-\frac{3\beta}{(1+3z^{-1})}}_{A} + \underbrace{\frac{\alpha}{(1+3z^{-1})(1-z^{-1})}}_{B}$$

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(A) → zero-input response – only due to I.C.; I/p = 0
(B) → zero-state response → due to input with condition of initial rest.

If $\alpha = 8$ & $\beta = 1$, then

$$Y(z) = \frac{3}{(1+3z^{-1})} + \frac{2}{(1-z^{-1})}$$

Therefore

$$y[n] = 3(-3)^n u[n] + 2u[n] \quad \text{for } n \geq 0$$

$$= \{3(-3)^n + 2\} u[n] \quad \text{for } n \geq 0$$

(13) Final Value Theorem → Applicable for UZT only.

Let $X(z)$ be the UZT of $x[n]$ & $(z-1)X(z)$ has all the poles inside the unit circle, then the ~~the~~ final value of $x[n]$ i.e. $x[\infty]$ is given by

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} [(z-1)X(z)]$$

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# Unit - 5 Z - Transform

**Eg. 3**

Consider a signal that is a sum of two real exponentials.

$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

**Solution:**

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{ 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \right\} z^{-n}$$

$$= 7\sum_{n=0}^{\infty} \left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n \text{; due to } u[n]$$

$$= \frac{7}{\left(1-\frac{1}{3}z^{-1}\right)} - \frac{6}{\left(1-\frac{1}{2}z^{-1}\right)} = \frac{\left(1-\frac{3}{2}z^{-1}\right)}{\left[1-\frac{1}{3}z^{-1}\right]\left[1-\frac{1}{2}z^{-1}\right]}$$

$$= \frac{z\,(z-3/2)}{(z-1/3)(z-1/2)}$$

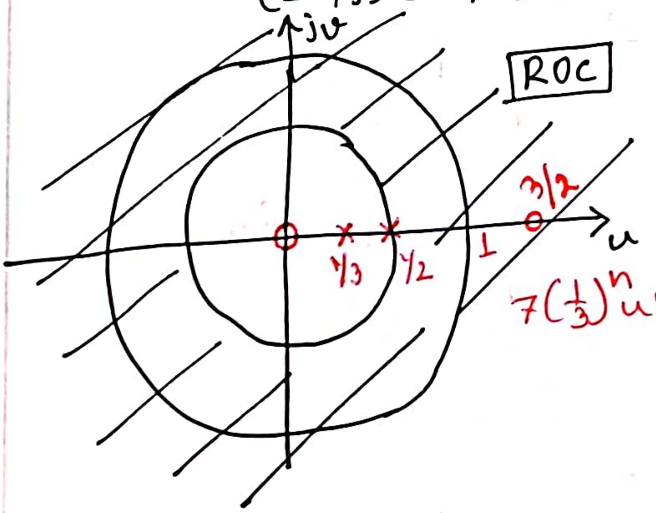
To evaluate the ROC.

$$\left|\frac{1}{3}z^{-1}\right| < 1 \quad \& \quad \left|\frac{1}{2}z^{-1}\right| < 1.$$

(or)

$|z| > \frac{1}{3}$ or $|z| > \frac{1}{2}$. Both would get satisfied when $|z| > \frac{1}{2}$; intersection of ROC.

$$X(z) = \frac{z\,(z-3/2)}{(z-1/3)(z-1/2)} \;;\; |z| > \frac{1}{2}.$$




$\left(\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{3}z^{-1}} \;;\; |z| > \frac{1}{3}$

$\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} \;;\; |z| > \frac{1}{2}$

$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \overset{z}{\longleftrightarrow}$

$$\frac{7}{\left(1-\frac{1}{3}z^{-1}\right)} - \frac{6}{\left(1-\frac{1}{2}z^{-1}\right)} \;;\; |z| > \frac{1}{2}$$

Eg-4

(1A)

Let $x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n]$

**Solution**: Using Euler's Identity:

$$x[n] = \frac{1}{2j}\left\{\left(\frac{1}{3}\right)^n e^{j\frac{\pi n}{4}}\right\}u[n] - \frac{1}{2j}\left\{\left(\frac{1}{3}\right)^n e^{-j\frac{\pi n}{4}}\right\}u[n]$$

$$= \frac{1}{2j}\left[\frac{1}{3}e^{j\frac{\pi}{4}}\right]^n u[n] - \frac{1}{2j}\left[\frac{1}{3}e^{-j\frac{\pi}{4}}\right]^n u[n]$$

The Z-Transform would be;

$$X(z) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{2j}\left[\frac{1}{3}e^{j\pi/4}\right]^n u[n] - \frac{1}{2j}\left[\frac{1}{3}e^{-j\frac{\pi}{4}}\right]^n u[n]\right\}z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left\{\frac{1}{3}e^{j\pi/4}z^{-1}\right\}^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left\{\frac{1}{3}e^{-j\pi/4}z^{-1}\right\}^n; \text{ due to } u[n]$$

$$= \frac{1}{2j}\cdot\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}} - \frac{1}{2j}\,\frac{1}{1-\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}}$$

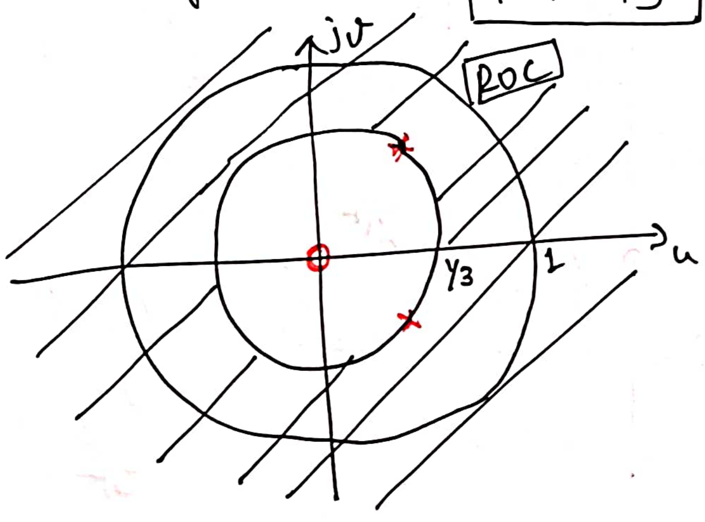
This can be simplified as

$$X(z) = \frac{\frac{1}{3\sqrt{2}}\,z}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}$$

**ROC**

$$\left|\frac{1}{3}e^{j\frac{\pi}{4}}z^{-1}\right| < 1 \quad \text{and} \quad \left|\frac{1}{3}e^{-j\frac{\pi}{4}}z^{-1}\right| < 1$$

can be satisfied when $\boxed{|z| > 1/3}$



## Explanation for some Important properties of ROC associated with z-transforms. ②

**Property-3** :- If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z=0$ and/or $z=\infty$.

**Explanation** :- Consider a finite duration signal between $N_1$ & $N_2$ has only finite number of non-zero values. Then

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

→ For z not equal to zero ~~or~~ infinity, each term in the summation will be finite & then $X(z)$ will converge.

→ If $N_1$ is negative & $N_2$ is positive, then the summation includes terms with both positive & negative powers of z.

→ As $|z| \to 0$, the terms involving negative powers of z become unbounded.

→ As $|z| \to \infty$, the terms involving positive powers of z become unbounded.

→ As a result, for $N_1$ being negative & $N_2$ being positive, the ROC does not include $z=0$ & $z=\infty$

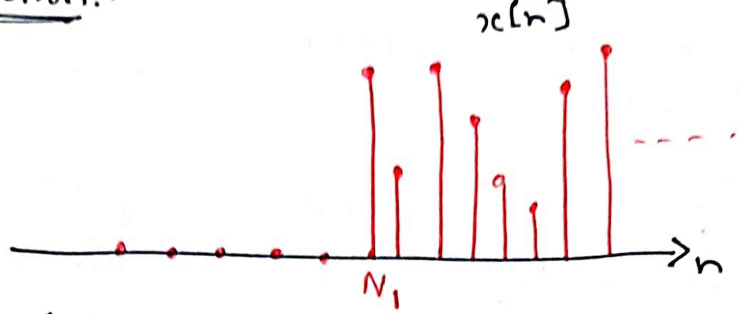
→ If $N_1$ is zero or positive, there are only negative powers of z in the above equation & consequently the ROC includes $z=\infty$.

→ If $N_2$ is zero or negative, there are only positive powers of z in the above equation & consequently the ROC includes $z=0$.

Self Study - Example 10.5

**Property 4** : If $x[n]$ is a right-sided sequence & if the circle $|z| = r_0$ is in the ROC, then for all finite values of $z$ for which $|z| > r_0$, will also be in the ROC. (2)

**Explanation**:




→ A right-sided sequence is zero prior to some value of $n$, say $N_1$.

→ If the circle $|z| = r_0$ is in the ROC, then $x[n]\, r_0^{-n}$ is absolutely summable.

→ Consider $|z| = r_1$ with $r_1 > r_0$, so that $r_1^{-n}$ decays more quickly than $r_0^{-n}$, for increasing $n$.

→ The fast decay ensures faster convergence & also since $x[n]$ is right-sided, $x[n] z^{-n} = 0$ for $n < N_1$.

→ Consequently $x[n]\, r_1^{-n}$ is summable.

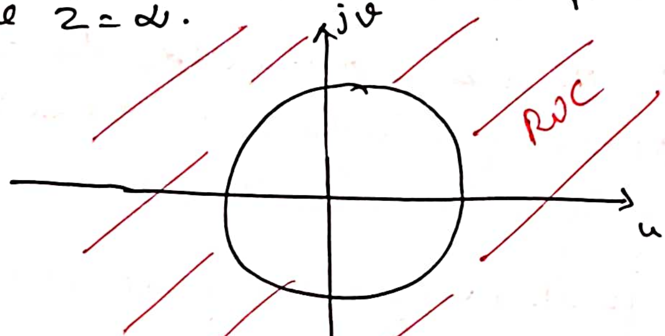
→ For RS sequences, in general, the ZT is given by

$$X(z) = \sum_{n=N_1}^{\infty} x[n] z^{-n}$$

where $N_1$ is finite & may be positive or negative!

→ If $N_1$ is negative, then the summation includes terms with positive powers of $z$, which become unbounded as $|z| \to \infty$. Consequently, for RS sequences, in general, the ROC will not include infinity

→ However, for particular cases like causal sequences, that are zero for $n < 0$, $N_1$ will be positive & ROC will include $z = \infty$.



Property 5 :: If $x[n]$ is a 'left-sided' sequence, and (3)
if the circle $|z| = r_0$ is in the ROC, then all values of $z$ for which $0 < |z| < r_0$ will also be in the ROC.

Explanation

→ Based on the argument put forward for Property 4.

→ The ZT of LS sequence would be

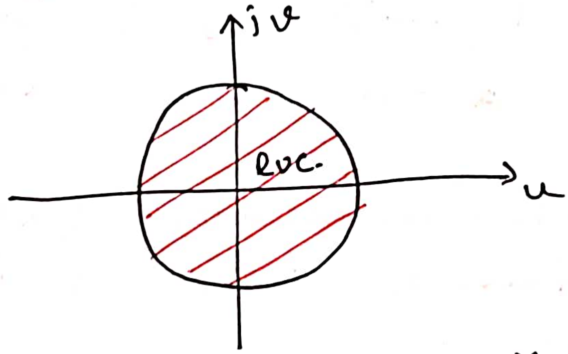
$$X(z) = \sum_{n=-\infty}^{N_2} x[n] z^{-n}$$

where $N_2$ can be positive or negative.

→ If $N_2$ is positive, then the summation includes negative powers of $z$, which may become unbounded as $|z| \to 0$.

→ Consequently, for LS sequences, the ROC will not, in general, include $z = 0$

→ If $N_2 \leq 0$ (so that $x[n] = 0$ for all $n > 0$), the ROC will include $z = 0$.




x ——— x ——— x ——— x ——— x ——— x

Property 6 :: If $x[n]$ is 2 sided & if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

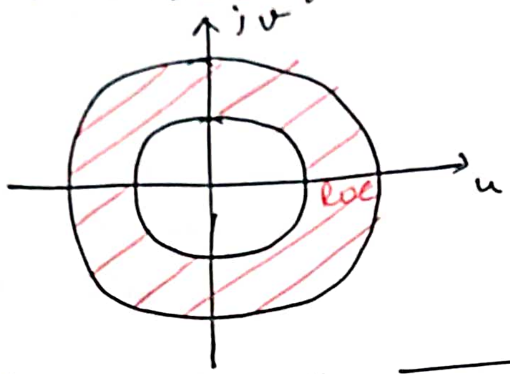
Explanation

→ The signal $x[n]$ can be considered as the sum of two sequences, one RS & the other being LS.

→ The ROC for the RS component is a region bounded on the inside by a circle & extending outward to (and possibly including) infinity.

→ The ROC of the LS component is a region bounded on the outside by a circle & extending inward to, possibly including, the origin.

→ The ROC for the composite signal includes the intersection of the ROCs of the individual components.

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**Eg-5** Consider $x[n] = \begin{cases} a^n & ; 0 \le n < N-1 \; ; a > 0 \\ 0 & ; \text{elsewhere.} \end{cases}$

Then
$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (a z^{-1})^{+n}$$

$$= \frac{1-(az^{-1})^{N-1+1}}{1-az^{-1}} = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \cdot \left\{ \frac{(z^N - a^N)}{(z-a)} \right\}$$

-> From the above expression of $X(z)$, we infer

(i) There are $(N-1)$ poles at origin, one pole at 'a'.

(ii) The numerator polynomial is of $N^{th}$ order.

-> The roots of the numerator polynomial
$$z_k = a e^{j\left(\frac{2\pi k}{N}\right)} \; ; k = 0, 1, \ldots N-1.$$

-> When $k=0$, one root of numerator is $(z-a)$, which cancels with the pole $(z-a)$.

-> Hence the left over zeros are
$$z_k = a e^{j\left(\frac{2\pi k}{N}\right)} \; ; k = 1, 2, \ldots N-1.$$

→ Also there are only $(N-1)$ poles at origin.

→ Since $x[n]$ is of finite length, from Property 3, the RoC includes the entire z-plane except origin and/or infinity.

→ Since $x[n] = 0$ for $n < 0$, the RoC will extend to infinity.

→ The RoC will not include origin due to the poles at origin.

→ For $N = 16$, & $0 < a < 1$, the region of convergence consists of all values of z except $z = 0$.

→ Refer slide.

---

Eg-6 Consider $x[n] = b^{|n|}$ ; $b > 0$

→ The plots when $0 < b < 1$ & $b > 1$ are shown in the slides. Therefore we will consider these two options for b.

→ $x[n]$ can be expressed as the sum of LS & RS sequences as:

$$x[n] = b^n u[n] + b^{-n} u[-n-1] \quad — ①$$

→ For the RS component.

$$b^n u[n] \xleftrightarrow{ZT} \frac{1}{1-bz^{-1}} \; ; |z| > b \quad — ②$$

→ For the LS component

$$b^{-n} u[-n-1] \xleftrightarrow{ZT} \frac{-1}{1-b^{-1}z^{-1}} \; ; |z| < \frac{1}{b} \quad -③$$

→ Consider $b > 1$, the pole-zero pattern & the RoC are as shown for Eqns. ② + ③

→ So when $b > 1$, there is no common ROC. Thus sequence (1) will not have a Z-transform, even though the RS + LS components do have ZT individually. (4A)

→ Considering the other case of $0 < b < 1$, the pole-zero pattern + the ROC are as shown.

→ Hence when $0 < b < 1$, there is a overlap.

→ Hence the z transform of the given $x[n]$ is

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}} \;;\; b < |z| < 1/b$$

(or)

$$= \frac{b^2-1}{b} \cdot \frac{z}{(z-b)(z-b^{-1})} \;;\; b < |z| < 1/b$$

---

Example 10.8 → Self Study

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## Inverse Z-transform by Partial fractions

Eg.1

Consider $X(z) = \dfrac{3 - \frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$ ; $\boxed{|z| > 1/3}$ ← ROC

Solution: For the given $X(z)$, there two poles

$1 - \frac{1}{4}\cdot\frac{1}{z} = 0$ ; $1 - \frac{1}{3z} = 0$

$-\frac{1}{4z} = -1$ $\quad$ $-\frac{1}{3z} = -1$

or $\boxed{z = \frac{1}{4}}$ $= 0.25$ $\quad$ $\boxed{z = \frac{1}{3}}$ $= 0.33$

→ For this problem, ROC is outside $\frac{1}{3}$ i.e outside the pole with greater magnitude (0.33)

→ From property 4, for this condition, the Inv. ZT or $x[n]$ is a Right-sided sequence.

(5)

→ we now split $X(z)$ into partial fractions.
Let $z^{-1} = x$. Then.

$$X(x) = \frac{3 - \frac{5}{6}x}{(1-\frac{1}{4}x)(1-\frac{1}{3}x)} = \frac{A}{(1-\frac{1}{4}x)} + \frac{B}{(1-\frac{1}{3}x)}$$

$$A = \left.\frac{3-\frac{5}{6}x}{(1-\frac{1}{3}x)}\right|_{x=4} = 1.$$

$$B = \left.\frac{3-\frac{5}{6}x}{(1-\frac{1}{4}x)}\right|_{x=3} = 2.$$

$$\therefore \; X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})} \quad \text{— (A)}$$

→ In order to find $x[n]$, we must associate the given ROC, correctly.

→ Let $x[n] = x_1[n] + x_2[n]$
then $x_1[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{4}z^{-1}}$ ; $|z| > \frac{1}{4}$

$x_2[n] \xleftrightarrow{z} \frac{2}{1-\frac{1}{3}z^{-1}}$ ; $|z| > \frac{1}{3}$

→ Since the ROC of $X(z)$ is outside the outermost pole, the ROC of each individual term in (A), must also be outside the pole associated with that term.

→ That is, the ROC for each term consists of all points with magnitude greater than the magnitude of the corresponding pole.

→ Then $x_1[n] = \left(\frac{1}{4}\right)^n u[n]$ & $x_2[n] = 2\left(\frac{1}{3}\right)^n u[n]$

→ Hence

$$\boxed{x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]}$$

RS RS

**Eg-2**

Consider $X(z) = \dfrac{3 - \frac{5}{6} z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$ ; $\boxed{\frac{1}{4} < |z| < \frac{1}{3}}$

**Solution** :- Equation (A) from Eg-1 is still valid!

Then

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

→ ROC associated with each term;

$$x_1[n] \xleftrightarrow{Z} \frac{1}{1-\frac{1}{4}z^{-1}} \; ; \; |z| > \frac{1}{4}$$ – Acceptable.

$$x_2[n] \xleftrightarrow{Z} \frac{2}{1-\frac{1}{3}z^{-1}} \; ; \; |z| > \frac{1}{3}$$ – Not acceptable as it violates our requirement.

To meet the given conditions, we need

$$X_2[z] \xleftrightarrow{ZT} -2\left(\frac{1}{3}\right)^n u[-n-1] \; ; \; |z| < \frac{1}{3}$$

Therefore

$$\boxed{x[n] = \underbrace{\left(\frac{1}{4}\right)^n u[n]}_{RS} - \underbrace{2\left(\frac{1}{3}\right)^n u[-n-1]}_{LS}}$$

**Eg-3**

Consider $X(z) = \dfrac{3 - \frac{5}{6} z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$ ; $\boxed{|z| < \frac{1}{4}}$ ← ROC

**Solution**

→ In this case the ROC is inside both poles $\frac{1}{3}$ & $\frac{1}{4}$.

→ Consequently, the ROC for each term in the partial fraction expansion must also lie inside the corresponding pole.

→ The resultant ZT pair is given by

$$x_1[n] \xleftrightarrow{ZT} \frac{1}{1-\frac{1}{4}z^{-1}} \; ; \; |z| < \frac{1}{4}$$

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1]$$

→ Similarly

$$x_2[n] \xleftrightarrow{Z} \frac{2}{1-\frac{1}{3}z^{-1}} \; ; \; |z| < \frac{1}{3}$$

or

$$x_2[n] = -2\left(\frac{1}{3}\right)^n u[-n-1]$$

Therefore

$$\boxed{x[n] = \underbrace{-\left(\frac{1}{4}\right)^n u[-n-1]}_{LS} \underbrace{-2\left(\frac{1}{3}\right)^n u[-n-1]}_{LS}}$$

---

Main Inferences.

→ Partial fraction expansion method is useful when $X(z)$ is expressed in terms of rational fn.

→ Let $X(z) = \sum_{i=1}^{m} \frac{A_i}{1-a_i z^{-1}}$

→ If the ROC is such that $|z| > a_i$, then

$$x_i[n] = A_i (a_i)^n u[n] \rightarrow RS.$$

→ If the ROC is such that $|z| < a_i$, then

$$x_i[n] = -A_i (a_i)^n u[-n-1] \rightarrow LS.$$

---

Inverse Z-Transform using Power Series.

Eg.1

Consider $X(z) = 4z^2 + 2 + 3z^{-1}$ ; $0 < |z| < \infty$

Solution:. We are aware that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Comparing this with the given $X(z)$, we have

$$x[n] = \begin{cases} 4 & ; n = -2 \\ 2 & ; n = 0 \\ 3 & ; n = 1 \\ 0 & ; \text{elsewhere} \end{cases}$$

6.

We can therefore express

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

We also infer $\boxed{\delta[n+n_0] \xleftrightarrow{ZT} z^{+n_0}}$

x ——— x ——— x ——— x ——— x ——— x —

**Eg. 2**

$$X(z) = \frac{1}{(1-az^{-1})} \; ; \; |z| > |a|$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1-az^{-1} & 1 \\ & 1 - az^{-1} \\ \hline & \cancel{a z^{-1}} \\ & a z^{-1} - a^2 z^{-2} \\ \hline & +\cancel{a^2 z^{-2}} \\ & a^2 z^{-2} - a^3 z^{-3} \\ \hline & +a^3 z^{-3} \\ & \vdots \end{array}$$

Hence

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

This series converges since $|z| > |a|$ or $|az^{-1}| < 1$.

Comparing the above series with

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

We observe $x[n] = 0$ for $n < 0$. Then $x[0] = 1$, $x[1] = a$, $x[2] = a^2$; $x[3] = a^3$ ....

$$\Rightarrow \boxed{x[n] = a^n u[n]}$$

x ——— x ——— x ——— x ——— x —

If the ROC is $|z| < |a|$ or $|az^{-1}| > 1$, the above series will not converge. We then proceed to carry out;

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 - \cdots \\ \hline -az^{-1}+1 & 1 \\ & 1 - a^{-1}z^{+1} \\ \hline & +a^{-1}z^{+1} \\ & a^{-1}z^{+1} - a^{-2}z^2 \\ \hline & +a^{-2}z^2 \cdots \end{array}$$

Then

$$\frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^{2} - a^{-3}z^{3} - \cdots$$

In this case $x[n] = 0$ for $n \geq 0$. Then

$x[-1] = -a^{-1}$, $x[-2] = -a^{-2}$, ......

or $\boxed{x[n] = -(a^{n})u[-n-1]}$

→ Inv. Z.T by Power Series expansion is useful when $X(z)$ is not in the rational form.

$\boxed{\text{Self Study Example 10.14.}}$

x ———x ———x ———x ———x—

Examples based on Properties of Z-Transform

Eg.1

Consider a DT LTI system such that

$y[n] = h[n] * x[n]$ where

$h[n] = \delta[n] - \delta[n-1]$.

Solution : By convolution property

$Y(z) = H(z)\,X(z)$.

Consider now $h[n]$, then by Time-shifting property

$$\delta[n] - \delta[n-1] \xleftrightarrow{z} 1 - z^{-1} = \frac{z-1}{z} = H(z)$$

with ROC equal to the entire z plane except at origin. $H(z)$ has a zero at 1.

If $x[n] \xleftrightarrow{z} X(z)$ with ROC $= R$.

Then $y[n] \xleftrightarrow{z} (1-z^{-1})X(z)$ with ROC equal to R, with possible deletion of $z=0$ and/or addition of $z=1$.

Therefore

$y[n] = h[n] * x[n] = \{\delta[n] - \delta[n-1]\} * x[n]$

$= x[n] - x[n-1]$

**Eg-2** Find $x[n]$ if $X(z) = \log(1+az^{-1})$ ; $|z|>|a|$ (7.

**Solution**:: First we convert $X(z)$ into a rational function. Using the differentiation property of ZT, we have.

$$n x[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz} = (-z)\cdot\frac{1}{(1+az^{-1})}(-1)az^{-2}$$

$$= \frac{az^{-1}}{(1+az^{-1})} \quad ; |z|>|a| \quad —(1).$$

we are also aware that

$$a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{(1+az^{-1})} \quad ; |z|>|a|$$

Using Time Shifting property

$$a(-a)^{n-1}u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{(1+az^{-1})} \quad ; |z|>|a| \quad -(2)$$

From (1) & (2)

$$n x[n] = a(-a)^{n-1}u[n-1] = (-1)(-a)(-a)^{n-1}u[n-1]$$

$$= \underline{(-1)(-a)^n u[n-1]}$$

Therefore $\boxed{x[n] = \dfrac{(-1)(-a)^n u[n-1]}{n}}$

---

**Eg-3** Find $x[n]$, if $X(z) = \dfrac{az^{-1}}{(1-az^{-1})^2}$ ; $|z|>|a|$

**Solution**:: We are aware that

$$a^n u[n] \xleftrightarrow{z} \frac{1}{(1-az^{-1})} \quad ; |z|>|a|$$

Then

$$n a^n u[n] \xleftrightarrow{z} -z\frac{d}{dz}\left\{\frac{1}{(1-az^{-1})}\right\} = \frac{az^{-1}}{(1-az^{-1})^2} \quad ; |z|>|a|$$

Hence Inv. ZT $\left\{\dfrac{az^{-1}}{(1-az^{-1})^2}\right\} = \underline{\underline{n a^n u[n]}}$

# Unilateral Z-Transforms.

## Important Properties of UZT

1. Time delay $x[n-1] \xleftrightarrow{UZT} z^{-1}X(z) + x[-1]$

In general

$$x[n-m] \xleftrightarrow{UZT} z^{-m}X(z) + x[-1]z^{-m+1} + x[-2]z^{-m+2} + \dots + \dots x[-m]$$ for positive m

2. **Convolution** If $x_1[n]$ & $x_2[n]$ are identically equal to zero for $n < 0$ then

$$x_1[n] * x_2[n] \xleftrightarrow{UZT} X_1(z)\,X_2(z)$$

3. **First difference** $x[n] - x[n-1] \xleftrightarrow{UZT} (1-z^{-1})X(z) - x[-1]$

4. **Differentiation in z-domain.**

$$n\,x[n] \xleftrightarrow{UZT} -z\frac{dX(z)}{dz}.$$

**Eg. 1** Consider a causal DT LTI systems described by

$$y[n] + 3y[n-1] = x[n]$$ with condition of

initial rest (zero IC).

**Solution** : Given $y[n] + 3y[n-1] = x[n]$

Taking ZT on both sides.

$$Y(z) + 3z^{-1}Y(z) = X(z)$$

$$Y(z)\{1+3z^{-1}\} = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+3z^{-1}}$$

Suppose if $x[n] = \alpha u[n]$ ; $\alpha$ - a constant the BZT & UZT of the output $y[n]$ is given by

$$Y(z) = H(z)X(z) = \frac{1}{(1+3z^{-1})} \cdot \frac{\alpha}{(1-z^{-1})};$$

$$= \frac{\left(\frac{3}{4}\right)\alpha}{(1+3z^{-1})} + \frac{(1/4)\alpha}{(1-z^{-1})}$$

Taking Inv. ZT, we have

$$\boxed{y[n] = \left(\frac{3}{4}\right)\alpha\,(-3)^n u[n] + \frac{\alpha}{4} u[n]}$$

---

ES.2 For the same example consider $x[n] = \alpha u[n]$ & $y[-1] = \beta$.

Solution :- Given $y[n] + 3y[n-1] = x[n]$

Taking UZT on both sides.

$$y[z] + 3z^{-1}y(z) + 3y[-1] = x(z)$$

$$y[z]\{1+3z^{-1}\} + 3\beta = \frac{\alpha}{(1-z^{-1})}$$

$$\therefore Y(z) = \underbrace{\frac{-3\beta}{(1+3z^{-1})}}_{A} + \underbrace{\frac{\alpha}{(1+3z^{-1})(1-z^{-1})}}_{B}$$

A → Zero-Input response; only due to IC as i/p = 0

B → Zero-State response; due to input with condition of Initial rest.

If $\alpha = 8$ & $\beta = 1$, then $Y(z)$ can be simplified into

$$Y(z) = \frac{3}{(1+3z^{-1})} + \frac{2}{(1-z^{-1})}$$

Hence

$$\boxed{y[n] = 3(-3)^n u[n] + 2u[n]}$$

(87) 9

# Practice Problems in Z-Transforms.

P.1 Consider $x[n] = [\underset{\uparrow}{1}, 2, 2, 1]$. compute $X(z)$ & indicate the ROC.

Solution: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{3} x[n] z^{-n}$

$= x[0](1) + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$

$= (1)(1) + (2)z^{-1} + (2) z^{-2} + (1) z^{-3}$

$= \underline{1 + 2z^{-1} + 2z^{-2} + z^{-3}}$

Since $X(z)$ is absolutely summable except for $z=0$,

ROC $= \underline{|z| > 0}$

PP.2 Consider $y[n] = [1, 1, 2, \underset{\uparrow}{2}]$. compute $Y(z)$ & its ROC.

Solution: $Y(z) = \sum_{n=-\infty}^{+\infty} y[n] z^{-n} = \sum_{n=-3}^{0} y[n] z^{-n}$

$= y[-3] z^{+3} + y[-2] z^{+2} + y[-1] z^{+1} + y[0](1)$

$= (1) z^3 + (1) z^2 + 2(z) + 2(1)$

$= \underline{z^3 + z^2 + 2z + 2}$

As $Y(z) \to \infty$ when $z = \infty$, ROC $= \underline{|z| < \infty}$

PP.3 Consider $p[n] = [2, 1, \underset{\uparrow}{1}, 2]$: find $P(z)$ & its ROC.

Solution: $P(z) = \sum_{n=-\infty}^{+\infty} p[n] z^{-n} = \sum_{n=-2}^{+1} p[n] z^{-n}$

$= \underline{2z^2 + z + 1 + 2z^{-1}}$

$P(z)$ is absolutely summable except at $z = 0$ & $\infty$.

Hence its ROC $\Rightarrow$ $\underline{0 < |z| < \infty}$

**PP-4** Evaluate $X(z)$ & its ROC if $x[n] = \alpha^{-n} u[-n-1]$ (9)

**Solution**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (\alpha z)^{-n}$$

Letting $n = -m$, then

$$X(z) = \sum_{m=+\infty}^{1} (\alpha z)^{m} = \sum_{m=1}^{\infty} (\alpha z)^{m}$$

$$= \sum_{m=0}^{\infty} (\alpha z)^{m} - 1.$$

As $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ ; $|a| < 1$.

we have

$$X(z) = \frac{1}{1-\alpha z} - 1 = \frac{\alpha z}{1-\alpha z} \quad \text{or} \quad \frac{-z}{z - 1/\alpha}$$

**ROC**

$|\alpha z| < 1$ or $|z| < \frac{1}{|\alpha|}$

---

**H.W.1** If $x[n] = \{5, 2, \underset{\uparrow}{-2}, 1, 1, -3\}$. Find $X(z)$ & its ROC.

**Ans** :. $X(z) = 5z^2 + 2z - 2 + z^{-1} + z^{-2} - 3z^{-3}$ ; ROC $0 < |z| < \infty$

---

**PP-5** Evaluate $X(z)$ if $x[n] = 3\left(-\frac{1}{2}\right)^n u[n] - 2\{3^n u[-n-1]\}$.

**Solution**:: Let $x_1[n] = \left(\frac{-1}{2}\right)^n u[n] \Rightarrow X_1(z) = \frac{z}{z+\frac{1}{2}}$ ; $|z| > |-1/2|$ $> |1/2|$

$x_2[n] = -3^n u[-n-1] \Rightarrow X_2(z) = \frac{z}{z-3}$ ; $|z| < 3$

Applying the Linearity Property.

$$X(z) = 3 \cdot \frac{z}{(z+\frac{1}{2})} + 2\left(\frac{z}{z-3}\right)$$ ; RoC is the intersection of the RoC of $X_1(z)$ & $X_2(z)$

Hence

$$X(z) = \frac{3z(z-3) + 2z(z+1/2)}{(z+1/2)(z-3)}$$

$$= \frac{3z^2 - 9z + 2z^2 + \cancel{z}}{(z+1/2)(z-3)}$$

$$= \frac{5z^2 - 8z}{(z+1/2)(z-3)}$$ ; RoC $\Rightarrow$ $\frac{1}{2} < |z| < 3$

**H.W.2** Find $X(z)$ if $x[n] = 2^n u[n] + 3\left(\frac{1}{2}\right)^n u[n]$

Answer: $X(z) = \frac{z(z-1/2) + 3z(z-2)}{(z-2)(z-1/2)}$ ; RoC $\Rightarrow |z| > 2$.

**H.W.3** Prove that $X(z)$ does not exist if

$x[n] = 3^{n+1} u[n] - 2\left(\frac{1}{2}\right)^n u[-n-1]$

Answer: The RoC of the individual terms do not intersect & hence $X(z)$ does not exist. $|z| > 3$ & $|z| < 1/2$.

**P.P.6** Evaluate the Inverse Z.T. of $X(z) = \frac{z}{3z^2 - 4z + 1}$

for the following RoC; $|z| > 1$, $|z| < \frac{1}{3}$; $\frac{1}{3} < |z| < 1$.

Solution: Let $\frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z-1/3)(z-1)}$

$$= \frac{k_1}{(z-1/3)} + \frac{k_2}{(z-1)}$$

Solving for the residues $k_1$ & $k_2$, we have

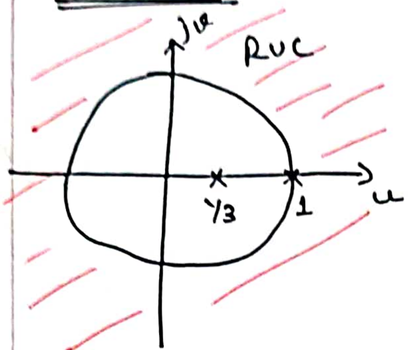
$$\frac{X(z)}{z} = -\frac{1}{2}\frac{1}{(z-1/3)} + \frac{1}{2}\frac{1}{(z-1)}$$

or

$$\boxed{X(z) = -\frac{1}{2}\frac{z}{(z-1/3)} + \frac{1}{2}\frac{z}{(z-1)}}$$

Poles $\frac{1}{3}$ & 1.

(a) $|z| > 1$

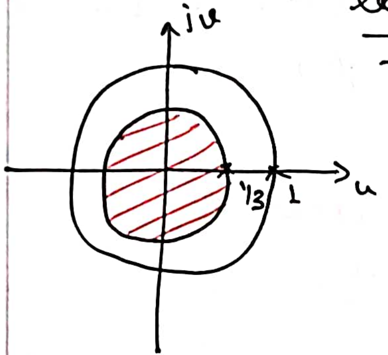



For this condition, $x[n]$ has to be right-sided sequence.

Therefore

$$\boxed{x[n] = -\frac{1}{2}\left(\frac{1}{3}\right)^n u[n] + \frac{1}{2}u[n]}$$

(b) $|z| < \frac{1}{3}$

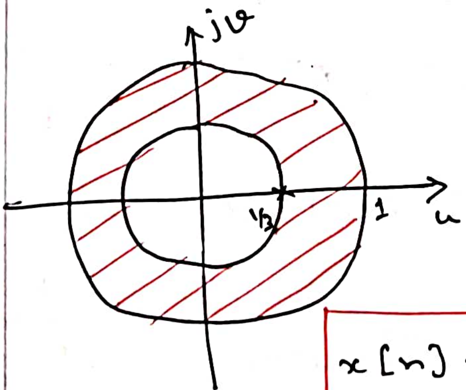



For this condition, $x[n]$ has to be a left sided sequence.

Therefore

$$\boxed{x[n] = \frac{1}{2}\left(\frac{1}{3}\right)^n u[-n-1] - \frac{1}{2}u[-n-1]}$$

(c) $\frac{1}{3} < |z| < 1$




∴ For this condition $x[n]$ has to be Two sided sequence.

$|z| > \frac{1}{3} \rightarrow$ RS sequence

$1 > |z| \rightarrow$ LS sequence.

Therefore

$$\boxed{x[n] = -\frac{1}{2}\left(\frac{1}{3}\right)^n u[n] - \frac{1}{2}u[-n-1]}$$

10A

11

Compute $x[n]$ if $X(z) = \dfrac{z+1}{3z^2-4z+1}$ ; ROC $\Rightarrow |z|>1$.

Solution Let $\dfrac{X(z)}{z} = \dfrac{z+1}{z(3z^2-4z+1)} = \dfrac{1}{3}\left[\dfrac{K_1}{z} + \dfrac{K_2}{(z-1/3)} + \dfrac{K_3}{(z-1)}\right]$

Then

$$\frac{X(z)}{z} = \frac{1}{z} - \frac{2}{(z-1/3)} + \frac{1}{(z-1)}$$

or

$$X(z) = 1 - 2\frac{z}{(z-1/3)} + \frac{z}{(z-1)}$$

Since the given ROC is $|z|>1$, then it is a causal or RS sequence. Therefore

$$\boxed{x[n] = \delta[n] - 2\left(\tfrac{1}{3}\right)^n u[n] + u[n]}$$

H.W.4 Prove that $h[n] = r^n \dfrac{\sin\{(n+1)\theta\}}{\sin\theta} u[n]$ if

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

H.W.5 Evaluate $x[n]$ by long division method for

$X(z) = \dfrac{z}{2z^2-3z+1}$ for ROC $|z|<1/2$ & $|z|>1$

Ans :. For $|z|<\frac{1}{2}$ ; $x[n] = \{\ldots, 15, 7, 3, 1, \underset{\uparrow}{0}\}$

For $|z|>1$ ; $x[n] = \{\underset{\uparrow}{0}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\}$.

H.W.6 Solve ~~H.W.S~~ H.W.5 by partial fractions.

Ans For $|z|<\frac{1}{2}$ ; $x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[-n-1]$

For $|z|>1$ ; $x[n] = u[n] - \left(\frac{1}{2}\right)^n u[n]$

P.P. 8

Solve the difference equation for $y[n]$ using $x[n]$ & the Initial conditions given. (11A)

$$y[n] = \frac{1}{2} y[n-1] + x[n] \;;\; n \geq 0$$

$$y[-1] = 1/4 \quad \& \quad x[n] = \left[\frac{1}{2}\right]^n u[n].$$

Solution :: Taking UZT on both sides

$$Y(z) = \frac{1}{2}\left\{Y(z)z^{-1} + y[-1]\right\} + X(z)$$

$$= \frac{1}{2}\left\{\frac{1}{4} + z^{-1}Y(z)\right\} + \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z)\left\{1-\frac{1}{2}z^{-1}\right\} = \frac{1}{8} + \frac{z}{z-1/2}$$

$$\Rightarrow Y(z) = \frac{1}{8} \times \frac{1}{1-\frac{1}{2}z^{-1}} + \left\{\frac{1}{1-1/2 z^{-1}}\right\}\left\{\frac{z}{z-1/2}\right\}$$

$$= \frac{1}{8}\frac{1}{1-1/2 z^{-1}} + \left\{\frac{z}{z-1/2}\right\}\left\{\frac{z}{z-1/2}\right\}$$

$$= \frac{1}{8}\frac{1}{1-\frac{1}{2}z^{-1}} + z\left\{\frac{z}{(z-1/2)^2}\right\}$$

Taking Inv. Z.T

$$y[n] = \frac{1}{8}\left(\frac{1}{2}\right)^n u[n] + n\left(\frac{1}{2}\right)^{n-1} u[n] \;\Big|\; n \to n+1$$

$$= \frac{1}{8}\left(\frac{1}{2}\right)^n u[n] + (n+1)\left(\frac{1}{2}\right)^n u[n+1]$$

Since $n \geq 0$, we need to replace $u[n+1]$ by $u[n]$

Therefore

$$\boxed{\begin{gathered} y[n] = \frac{1}{8}\left(\frac{1}{2}\right)^n u[n] + (n+1)\left(\frac{1}{2}\right)^n u[n] \\ \text{(or)} \\ = \frac{1}{8}\left(\frac{1}{2}\right)^n + (n+1)\left(\frac{1}{2}\right)^n \;;\; n \geq 0 \end{gathered}}$$

---

PP9 The output of a DT LTI system is given as $y[n] = 2\left(\frac{1}{3}\right)^n u[n]$ when the input $x[n]$ is $u[n]$.

(a) Find the Impulse response $h[n]$ of the system.

**Solution** :: Given $y[n] = 2\left(\frac{1}{3}\right)^n u[n]$

$\Rightarrow$ $Y(z) = 2 \cdot \frac{z}{z - 1/3}$ ; ROC $\Rightarrow |z| > 1/3$

& $x[n] = u[n] \Rightarrow X(z) = \frac{z}{z-1}$ ; ROC $\Rightarrow |z| > 1$.

Hence, the system function or Pulse Transfer function

$H(z) = \frac{Y(z)}{X(z)} = \frac{2(z-1)}{(z-1/3)}$ ; ROC $\Rightarrow |z| > 1/3$

Let

$$\frac{H(z)}{z} = \frac{2(z-1)}{z(z-1/3)} = \frac{k_1}{z} + \frac{k_2}{(z-1/3)}$$

Solving for $k_1$ & $k_2$, we get

$$\frac{H(z)}{z} = \frac{6}{z} - \frac{4}{(z-1/3)} \Rightarrow H(z) = 6 - \frac{4z}{(z-1/3)}$$

Taking Inv. Z.T

$$\boxed{h[n] = 6\delta[n] - 4\left(\tfrac{1}{3}\right)^n u[n]}$$

---

**H.W.7** For PP.9, evaluate $y[n]$ if $x[n] = \left(\frac{1}{2}\right)^n u[n]$ & $h[n]$ as computed in PP.9.

**Answer** :: $y[n] = -6\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$.

---

**H.W.8** Given $H(z) = \frac{z+1}{z^2 - 2z + 3}$ representing a causal system. Find the difference equation realisation relating $y[n]$ & $x[n]$. Determine its frequency response.

**Answer** :: $y[n] - 2y[n-1] + 3y[n-2] = x[n-1] + x[n-2]$

$$H(z)\Big|_{z = e^{jw}} = \frac{(1 + \cos w) + j\sin w}{(\cos 2w - 2\cos w + 3) + j(\sin 2w - 2\sin w)}$$

(12)

**P.P. 10** Given $H(z) = \frac{z}{z^2+1}$ ; ROC : $|z| > 1$.

(a) Is the system causal? Yes, the system is causal. The order of the numerator is less than the order denominator & the poles of the system function $H(z)$ which are $\pm j$ are interior to the ROC $: |z| > 1$.

(b) Is the system BIBO Stable? No, as the ROC $|z|>1$ does not include / encompass the unit circle, it is not BIBO stable.

(c) Find the unit sample (unit impulse) response of the system

$$H(z) = \frac{z}{z^2+1} \quad ; \text{ROC} \Rightarrow |z| > 1.$$

we are aware that

$$Z\{\sin(\omega_0 n)u[n]\} = \frac{z \sin \omega_0}{z^2 - 2z\cos\omega_0 + 1} \quad ; |z|>1$$

Comparing both the denominators.

$$-2z\cos\omega_0 = 0 \Rightarrow \omega_0 = \pi/2$$

Therefore $h[n] = \sin\left(\frac{\pi}{2}n\right)u[n]$

(d) Find the difference equation relating $y[n]$ & $x[n]$.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z^2+1}$$ . For causal system both the numerator & denominator should have negative powers of $z$.

$$\therefore \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1+z^{-2}} \Rightarrow Y(z)(1+z^{-2}) = X(z)z^{-1}$$

$$\boxed{y[n] + y[n-2] = x[n-1]}$$

(e) Is it linear? Yes it is linear since $H(z)$ is defined.

(f) Is it shift Invariant? From (d) we see the coefficients are constants & hence it is shift invariant.

**H.W. 9** Using ZT find $y[n] = x[n] * h[n]$ where $x[n] = \alpha^n u[n]$ & $h[n] = \beta^n u[n]$.

(12.11)



Find $X(z)$ if $x[n] = \left(\frac{1}{3}\right)^{n-4} u[n-2]$

**Solution** $x[n]$ can be re-written as

$$x[n] = \left(\frac{1}{3}\right)^{n-4} u[n-2] = \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{n-2} u[n-2]$$

Let $x_1[n] = \left(\frac{1}{3}\right)^{n} u[n] \Rightarrow X_1(z) = \dfrac{z}{(z-1/3)}$ ; $|z| > 1/3$

Using Time shift Property

$$x_2[n] = x_1[n-2] \overset{ZT}{\longleftrightarrow} X_2(z) = z^{-2} X_1(z)$$
$$= z^{-2}\left\{\frac{z}{z-1/3}\right\}$$

Non change in RoC.

Therefore

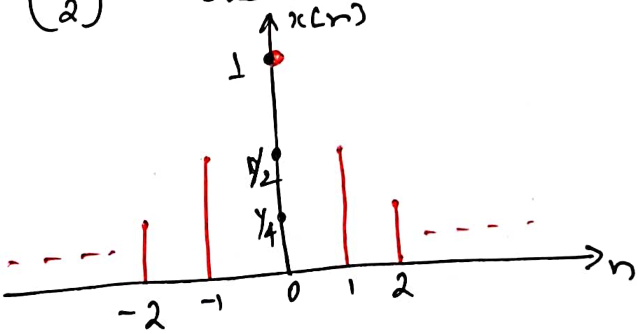
$$x[n] = \left(\frac{1}{3}\right)^{-2} x_2[n] \longrightarrow X(z) = \left(\frac{1}{3}\right)^{-2} X_2(z)$$

$$X(z) = \frac{9}{z(z-1/3)} = 9z^{-2}\left\{\frac{z}{z-1/3}\right\} : \text{RoC } |z| > 1/3$$

---

A.P. 12 Let $x[n] = \left(\frac{1}{2}\right)^{|n|}$. Sketch $x[n]$ & evaluate $X(z)$.

**Solution**:




It is a two-sided sequence & can be represented as

$$x[n] = \left(\frac{1}{2}\right)^{n} u[n] + \left(\frac{1}{2}\right)^{-n} u[-n-1]$$

$$= \left(\frac{1}{2}\right)^{n} u[n] - \left\{-2^{n} u[-n-1]\right\} \quad (*)$$

Therefore, if

$$x_1[n] = \left(\frac{1}{2}\right)^{n} u[n] \Rightarrow X_1(z) = \frac{z}{z-1/2} \; ; \; |z| > 1/2$$

$$x_2[n] = -2^{n} u[-n-1] \Rightarrow X_2(z) = \frac{z}{z-2} \; ; \; |z| < 2$$

(23A)

Hence $X(z) = X_1(z) - X_2(z)$ by linearity + ROC → Intersection.

$$\therefore X(z) = \frac{z}{z-1/2} - \frac{z}{z-2} = \frac{-3z}{2(z-1/2)(z-2)} \quad ; \text{ROC } 1/2 < |z| < 2$$

---

**H.W-10** Find $X(z)$ if $x[n] = 2^n \sin\left(\frac{\pi}{4}\right) n\, u[n]$

**Answer:** $X(z) = \dfrac{\sqrt{2}\, z}{z^2 - 2\sqrt{2}\, z + 4} \quad ; |z| > 2.$

---

**PP-13** Find the ZT of $x[n] = n a^n u[-n]$

**Solution** Recall $a^n u[+n] \longleftrightarrow \dfrac{z}{z-a} \quad ; |z| > |a|$

Let $x_1[n] = a^{-n} u[n] = \left(\frac{1}{a}\right)^n u[n] \longleftrightarrow \dfrac{z}{(z-1/a)} \quad ; |z| > |1/a|$

Applying Time Reversal property

$$x_2[n] = x_1[-n] \longleftrightarrow X_2(z) = X_1(z^{-1})$$

$$= \frac{z^{-1}}{z^{-1} - 1/a} \quad ; |z^{-1}| > \frac{1}{|a|} \text{ or } |z| < |a|$$

Using the multiplication property by $n$, we have

$$x[n] = n\, x_2[n] \longleftrightarrow X(z) = -z \frac{d X_2(z)}{dz}$$

No change in ROC

$$\therefore X(z) = -z \frac{d}{dz}\left\{\frac{z^{-1}}{z^{-1} - 1/a}\right\}$$

$$= \frac{+a^{-1} z}{(1 - a^{-1} z)^2} \quad ; \text{ROC} \rightarrow |z| < |a|$$