# UNIT - 3

A complex number $z$ is an ordered pair of real numbers $x$ & $y$ written as $z = x + iy$ where $x, y \in R$ & $i^2 = -1$

$i = (0, 1)$ , $x = Re\{z\}$

$y = Im\{z\}$

1) $z = (x, 0) \Rightarrow z = x$
2) $z_1 = x_1 + iy_1$, & $z_2 = x_2 + iy_2$ are equal iff $x_1 = x_2$ & $y_1 = y_2$
3) $x_1 + iy_1 \neq y_1 + ix_1$
4) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$= (x_1 + x_2, y_1 + y_2)$

5) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$= (x_1 - x_2, y_1 - y_2)$

6) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

$= (x_1, y_1)(x_2, y_2)$

7) $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \times \dfrac{x_2 - iy_2}{x_2 - iy_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2}$

→ $\mathbb{C} = \{x + iy : x, y \in R\}$

→ Commutative property : $z_1 + z_2 = z_2 + z_1$

$z_1 z_2 = z_2 z_1$

→ Associative property : $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$z_1 (z_2 z_3) = (z_1 z_2) z_3$

→ Distributive property : $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

→ Additive Inverse : If $z = x + iy$

$-z = -x - iy$

such that $z + (-z) = 0$

→ Multiplicative Inverse : If $z = x + iy$

there exists $\frac{1}{z}$

such that $z \times \left(\frac{1}{z}\right) = 1$

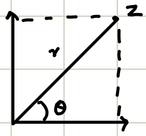
where $z \neq 0$

→ Complex Conjugate : If $z = x + iy$

then $\bar{z} = x - iy$

(diagonally opposite in graph when plotted)

Polar coordinates : $x = r\cos\theta$




$y = r\sin\theta$

$|r| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

$z = (r, \theta)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ , $\theta$ not defined when $r = 0$ & $\theta$ is not unique

$\theta = \theta + 2n\pi \quad \forall\ n \in \mathbb{Z}$

If $\theta$ is restricted to 0 to $2\pi$, then its called principal arguement

Properties : 1) $|z|$ is euclidian distance from origin to point $z$

2) $|z_1 - z_2|$ denotes distance b/w $z_1$ & $z_2$

3) $|Re(z)| \leq |z|$ x

$|Im(z)| \leq |z|$ y

4) $r^2 = z\bar{z}$

5) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangular Inequality)

→ $z = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

$= re^{i(\theta + 2n\pi)}$

→ Geometrical Interpretation of multiplication of complex domain :

If $z_1$ at $\theta_1$ & $z_2$ at $\theta_2$

Then $z_1 z_2$ at $\theta_1 + \theta_2$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$|z_1 z_2| = r_1 r_2$

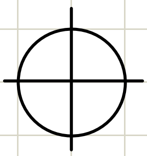
$\angle z_1 z_2 = \angle z_1 + \angle z_2$

If $z = r(\cos\theta + i\sin\theta)$

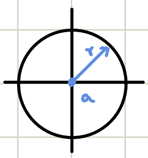
$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$ ⟶ De Moivre Theorem

$\{z \in \mathbb{C} , |z| = 1\}$



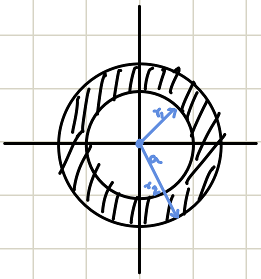
$|z - a| = r$




$|z - a| < r \Rightarrow$ represents an open circular disc

$|z - a| \leq r \Rightarrow$ represents an closed circular disc

any point inside the circle is in neighbourhood of a



$r_1 < |z_1 - z_2| < r_2 \Rightarrow$ open annulus

$r_1 \leq |z_1 - z_2| \leq r_2 \Rightarrow$ closed annulus

## Complex Functions :

→ A function defined on a complex number is a mapping

$f : \mathbb{C} \to \mathbb{C} \quad \Rightarrow \quad \omega = f(z)$

ex: $f(z) = z^2 + 1$

all polynomial, rational fns $\in \omega$

$f(z) = z + b$

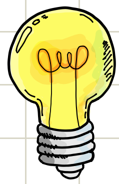
→ $f(z) = az$

$a \in \mathbb{C}$

and let $a = re^{j\theta}$

$z = pe^{j\phi}$

$f(z) = rpe^{j(\theta+\phi)}$ ⇒ function either $\boxed{\text{stretches, rotates or translates}}$ in complex domain



→ Limit ⇒ $\lim_{z \to z_0} f(z) = l$

→ Continuity ⇒ $\lim_{z \to z_0} f(z) = f(z_0)$

→ Limit Rules

1) $\lim_{z \to z_0} f(z) = l_1$ & $\lim_{z \to z_0} g(z) = l_2$

Then, $\lim_{z \to z_0} (f(z) \pm g(z)) = l_1 \pm l_2$

$\lim_{z \to z_0} (f(z) \cdot g(z)) = l_1 \cdot l_2$

$\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{l_1}{l_2}$ $(l_2 \neq 0)$

2) Derivative of $f(z)$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$z = z_0 + \Delta z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{z - z_0}$$

Q. Find $f'(z)$ when $f(z) = z^2$

A. $f'(z) = \lim_{\Delta z \to 0} \dfrac{(z+\Delta z)^2 - z^2}{\Delta z}$

$= \lim_{\Delta z \to 0} \dfrac{\Delta z^2 + 2z\,\Delta z}{\Delta z}$

$= \lim_{\Delta z \to 0} \Delta z + 2z$

$= 2z$

Q. Find $f'(z)$ when $f(z) = \bar{z}$

A. $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \dfrac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \dfrac{\Delta \bar{z}}{\Delta z} = \lim \dfrac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$

When $\Delta x \to 0 \Rightarrow f'(z) = -1$

when $\Delta y \to 0 \Rightarrow f'(z) = 1$

Hence it is not differentiable

Q. Find $f'(z)$ where $f(z) = \dfrac{3z}{z-i}$

A. $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\dfrac{3(z+\Delta z)}{z+\Delta z - i} - \dfrac{3z}{z-i}}{\Delta z} = \dfrac{3(z+\Delta z)(z-i) - 3z(z+\Delta z - i)}{(z+\Delta z - i)(z-i)(\Delta z)}$

$= \dfrac{3(\cancel{z^2} + \cancel{z\Delta z} - \cancel{iz} - i\Delta z) - 3(\cancel{z^2} + \cancel{z\Delta z} - \cancel{iz})}{(z^2 - iz + z\Delta z - i\Delta z - iz - 1)(\Delta z)}$

$= \dfrac{-3i\cancel{\Delta z}}{(z^2 - iz - iz - 1)\cancel{\Delta z}} \qquad = \dfrac{-3i}{(z-i)^2}$

→ Note : All polynomial & rational functions are differentiable

Q. $f(z) = z\,\mathrm{Re}(z)$

Show that it is differentiable at $z = 0$

https://math.stackexchange.com/questions/451427/differentiate-fz-zrez

→ In a complex plane, not all functions are differentiable and we focus on functions which are differentiable & look at the conditions under which derivative exists. They are called analytic functions

→ $f(z)$ is analytic in domain D, if $f(z)$ is defined & differentiable at all points $z \in D$

→ $f(z)$ is analytic at a point $z = z_0$ in D, if $f(z)$ is analytic in the neighbourhood of $z_0$

→ Analytic funcⁿ is also called holomorphic

→ Let $f(z) = u(x,y) + iv(x,y)$, we define it to be continuous in some neighbourhood of a point $z = x+iy$ & differentiable at $z$,

then recall that $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$
$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - \big(u(x,y) + iv(x,y)\big)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\big(v(x+\Delta x, y+\Delta y) - v(x,y)\big)}{\Delta x + i\Delta y}$$

1) $$\lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\big(v(x+\Delta x, y+\Delta y) - v(x,y)\big)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i\big(v(x+\Delta x, y) - v(x,y)\big)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x$$

2) $$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\big(v(x+\Delta x, y+\Delta y) - v(x,y)\big)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i\big(v(x, y+\Delta y) - v(x,y)\big)}{\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -iu_y + v_y$$

If the limit exists, 1 & 2 must give same results which implies $\boxed{\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}}$

Cauchy's Riemann Equations

→ Analytic functions must satisfy CRE

→ Let $f(z) = u(x,y) + iv(x,y)$ be defined & continuous at the neighbourhood of some point $z = x+iy$ & differentiable at $z$, then at that point, first order partial derivative of $u$ & $v$ exist & satify CRE

Q. Show that $f(z) = z^2$ is analytic at all $c \in \mathbb{C}$

A. $f(z) = z^2$

$= (x+iy)^2$

$= x^2 + y^2 - i2xy$

$u = x^2 - y^2$

$v = 2xy$

$u_x = 2x \qquad u_y = -2y$

$v_x = 2y \qquad v_y = 2x$

$\Rightarrow \quad u_x = v_y$

$u_y = -v_x$

Q. $f(z) = e^x [\cos y + i \sin y]$

A. $u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

$u_x = v_y$

$u_y = -v_x$

→ If the funcⁿ is differentiable over the entire complex plane, we call it an entire funcⁿ

$z = r(\cos\theta + i\sin\theta)$

$f(z) = u(r,\theta) + i\,v(r,\theta)$

- If $f(z)$ is analytic

$u_r = \frac{1}{r} v_\theta \quad , \quad v_r = -\frac{1}{r} u_\theta \qquad$ for $r > 0$

Q. 1) $z + e^z$ 2) $z e^z$

3) $\sin z$ 4) $\cosh z$

5) $\log z$

A. 1) $z + e^z = x + iy + e^{x+iy} = x + iy + e^x . e^{iy} = \underbrace{x + iy}_{1} + \underbrace{e^x(\cos y + i \sin y)}_{2}$

$u_1 = x \qquad v_1 = y \qquad\qquad u_2 = e^x \cos y \qquad v_2 = e^x \sin y$

$u_{1x} = 1 \qquad v_{1x} = 0 \qquad\qquad u_{2x} = e^x \cos y \qquad v_{2x} = e^x \sin y$

$u_{1y} = 0 \qquad v_{1y} = 1 \qquad\qquad u_{2y} = -e^x \sin y \qquad v_{2y} = e^x \cos y$

$u_{1x} = v_{1y} \qquad\qquad u_{2x} = v_{2y}$

$u_{1y} = -v_{1x} \qquad\qquad u_{2y} = -v_{2x}$

5) $\log z = \log(r\cos\theta + ir\sin\theta) = \log(re^{i\theta}) = \log r + i\theta \quad$ (Undefined for $z = 0$) ($z \in \mathbb{R}^-$)

$u = \log r \qquad v = \theta$

$u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$

$\Rightarrow \quad u_r = \frac{v_\theta}{r} \qquad v_r = -\frac{u_\theta}{r}$

2) $ze^z = (x+iy)e^{x+iy} = (x+iy)(e^x)(e^{iy}) = (xe^x + iye^x)(\cos y + i\sin y)$

$= xe^x\cos y - ye^x\sin y + i(xe^x\sin y + ye^x\cos y)$

$u = xe^x\cos y - ye^x\sin y \qquad v = xe^x\sin y + ye^x\cos y$

$u_x = e^x\cos y + xe^x\cos y - ye^x\sin y \qquad v_x = e^x\sin y + xe^x\sin y + ye^x\cos y$

$u_y = -xe^x\sin y - e^x\sin y - ye^x\cos y \qquad v_y = xe^x\cos y + e^x\cos y - ye^x\sin y$

$u_x = v_y \qquad u_y = -v_x$

3) $\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy)$

$= \sin x\cosh y + i\cos x\sinh y$

$u = \sin x\cosh y \qquad v = \cos x\sinh y$

$u_x = \cos x\cosh y \qquad v_x = -\sin x\sinh y$

$u_y = \sin x\sinh y \qquad v_y = \cos x\cosh y$

$u_x = v_y$ & $u_y = -v_x$

4) $\cosh z = \dfrac{e^z + e^{-z}}{2} = \dfrac{e^{x+iy} + e^{-x-iy}}{2} = \dfrac{e^x . e^{iy} + e^{-x} . e^{-iy}}{2} = \dfrac{e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)}{2}$

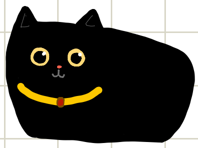
$= \cos y\left(\dfrac{e^x + e^{-x}}{2}\right) + i\sin y\left(\dfrac{e^x - e^{-x}}{2}\right) = \cosh x\cos y + i\sinh x\sin y$

$u = \cosh x\cos y \qquad v = \sinh x\sin y$

$u_x = \sinh x\cos y \qquad v_x = \cosh x\sin y$

$u_y = -\cos x\sin y \qquad v_y = \sinh x\cos y$

$u_x = v_y$ & $u_y = -v_x$



Q. Find an analytic $f^n$ $f(z)$ whose Real part is $e^{2x}\{x\cos 2y - y\sin 2y\}$

A. $u = e^{2x}\{x\cos 2y - y\sin 2y\}$

we know $u_x = v_y$

$u_x = 2e^{2x}\{x\cos 2y - y\sin 2y\} + e^{2x}\cos 2y = v_y$

$\int v_y\, dy = \dfrac{2e^{2x}x\sin 2y}{2} + \dfrac{e^{2x}\sin 2y}{2} - 2e^{2x}\int y\sin 2y\, dy$

$= e^{2x}x\sin 2y + \dfrac{e^{2x}\sin 2y}{2} - 2e^{2x}\left(-\dfrac{y\cos 2y}{2} - \int \dfrac{\sin 2y}{2}\, dy\right)$

$= e^{2x}x\sin 2y + \cancel{\dfrac{e^{2x}\sin 2y}{2}} + e^{2x}y\cos 2y - \cancel{\dfrac{e^{2x}\sin 2y}{2}}$

$= e^{2x}(x\sin 2y + y\cos 2y)$

Q. $u = -r^3 \sin 3\theta$
Find $f(z)$

A. $u_r = \frac{v_\theta}{r}$

$u_r = -3r^3 \sin 3\theta = \frac{v_\theta}{r}$

$v_\theta = -3r^3 \sin 3\theta$

$v = -3r^3 \int \sin 3\theta \, d\theta = r^3 \cos 3\theta + C$

$f(z) = u + iv = r^3(i\cos 3\theta - \sin 3\theta) + iC$

→ It is interesting to know, complex analytic functions has derivatives of all orders unlike real functions, where the function once differentiable is not guaranteed to be differentiable again & higher order derivative may not exist
So, this property of complex analytic function results in $f'(z)$ also to be analytic and in this way, complex functions are much simpler than real functions which is seen later in cauchy's integral formula

## Harmonic Function

→ Consider an analytic function, $f(z)$
Recall CRE, $u_x = v_y$ ① & $u_y = -v_x$ ②
Differentiating ① wrt $x$ & ② wrt $y$

$u_{xx} = v_{yx}$ & $u_{yy} = -v_{xy}$

Given that derivative of analytic funⁿ is also analytic & partial derivative exists
then $v_{yx} = v_{xy}$
and $u_{xx} = -u_{yy} \Rightarrow \boxed{u_{xx} + u_{yy} = 0} \Rightarrow \Delta^2 u$

Differentiating ① wrt $y$ & ② wrt $x$

$u_{xy} = v_{yy}$ & $u_{yx} = -v_{xx}$

So similarly, $\boxed{v_{xx} + v_{yy} = 0} \Rightarrow \Delta^2 v$

→ Solution of Laplace eq, having continuous $2^{nd}$ order derivatives are called harmonic equations
Therefore $\text{Re}(z)$ & $\text{Im}(z)$ of analytic functions are harmonic functions
→ Function $v$ is said to harmonic conjugate of $u$

Q. Verify that $u = x^2 - y^2 - y$ is harmonic in a whole complex plane and find the harmonic conjugate func$^n$ $v$

A. $u_x = 2x$

$u_{xx} = 2$

$u_y = -2y - 1$

$u_{yy} = -2$

$u_{xx} + u_{yy} = 0 \Rightarrow$ Harmonic

$u_x = v_y = 2x$

$v = 2xy + c_1$

$u_y = -v_x = -2y - 1$

$v_x = 2y + 1$

$v = 2xy + x + c_2$

Q. $f(z) = (x-y)^2 + 2i(x+y)$

A. $u = (x-y)^2$ $\quad v = 2x + 2y$

$= x^2 + y^2 - 2xy$

$u_x = 2x - 2y$ $\quad v_x = 2$

$u_y = 2y - 2x$ $\quad v_y = 2$

$u_x \neq v_y$

$u_y \neq -v_x$

Q. Find the value of $a$ such that $f(z) = r^2 \cos 2\theta + i r^2 \sin a\theta$ is analytic

A. $u = r^2 \cos 2\theta$ $\quad v = r^2 \sin a\theta$

$u_r = 2r \cos 2\theta$ $\quad v_r = 2r \sin a\theta$

$u_\theta = -2r^2 \sin 2\theta$ $\quad v_\theta = ar^2 \cos a\theta$

$u_r = \frac{v_\theta}{r} \Rightarrow 2r\cos 2\theta = \frac{ar^{\cancel{2}} \cos a\theta}{\cancel{r}} \Rightarrow \boxed{a = 2}$

Q. Find the analytic func$^n$ such that $u = 2x - x^3 + 3xy^2$

A. $u_x = 2 - 3x^2 + 3y^2$

$u_y = 6xy$

$u_x = v_y \Rightarrow v_y = 2 - 3x^2 + 3y^2$

$v = 2y - 3x^2y + y^3 + c(x)$

$u_y = 6xy = -v_x$

$v_x = -6xy$

$v = -3x^2y + c(y)$

$v_y = -3x^2 + c'(y)$

$c'(y) = 2 + 3y^2$

$c(y) = 2y + y^3$

$z = u + iv = (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3)$

Q. Verify that $f(z) = 2z - z^3 + C$ is analytic

A. $f(z) = 2z - z^3 + c$

$= 2(x+iy) - (x+iy)^3 + c$

$= 2x + i2y - (x^3 - iy^3 + i3x^2y - 3xy^2) + c$

$= 2x + i2y - x^3 + iy^3 - 3ix^2y + 3xy^2 + c$

$= (2x - x^3 + 3xy^2) + i(2y + y^3 - 3x^2y) + c$

$u = 2x - x^3 + 3xy^2$ $\qquad$ $v = 2y + y^3 - 3x^2y$

$u_x = 2 - 3x^2 + 3y^2$ $\qquad$ $v_x = -6xy$

$u_y = 6xy$ $\qquad$ $v_y = 2 + 3y^2 - 3x^2$

$u_x = v_y$ & $u_y = -v_x$

Hence $f(z)$ is analytic

Q. $v = e^{-x}(x\sin y - y\cos y)$

A. $v_x = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y)$

$= e^{-x}(y\cos y - x\sin y + \sin y)$

Ans: $f(z) = -ze^{-z} + C$

$u_y = v_x = e^{-x}(y\cos y - x\sin y + \sin y)$

$u = \int u_y\, dy = e^{-x}\int(y\cos y - x\sin y + \sin y)$

$= e^{-x}(y\sin y + \cancel{\cos y} + x\cos y - \cancel{\cos y})$

$= e^{-x}(y\sin y + x\cos y)$

$f(z) = u + iv$

$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(x\sin y - y\cos y)$

$= -e^{-x}x(\cos y + i\sin y) - e^{-x}y(-i\cos y + \sin y)$

$= -e^{-x}x(e^{iy}) - ie^{-x}y e^{iy}$

$f(z) = -ze^{-z} + c$

Q. Construct the analytic func$^n$ where $v = r^2\cos 2\theta - r\cos\theta + 2$

A. $v = r^2\cos 2\theta - r\cos\theta + 2$

$v_r = 2r\cos 2\theta - \cos\theta$

$v_r = -\frac{u_\theta}{r} \Rightarrow u_\theta = r\cos\theta - 2r^2\cos 2\theta$

$\int u_\theta\, d\theta = \int(r\cos\theta - 2r^2\cos 2\theta)d\theta = r\sin\theta - r^2\sin 2\theta + c$

$f(z) = u + iv$

$= r\sin\theta - r^2\sin 2\theta + i(r^2\cos 2\theta - r\cos\theta + 2)$

$= -ir(\cos\theta + i\sin\theta) + ir^2(\cos 2\theta + i\sin 2\theta) + 2i + c$

$= -iz + iz^2 + 2i + c$

$= i(z^2 - z + 2) + c$

# Laplace Equation in Polar Form

$\rightarrow$ $u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2} = 0$

$v_{rr} + \dfrac{v_r}{r} + \dfrac{v_{\theta\theta}}{r^2} = 0$

Q. $f(z) = \left(r + \dfrac{K^2}{r}\right)\cos\theta + i\left(r - \dfrac{K^2}{r}\right)\sin\theta$ where $r \neq 0$

Find $f'(z)$, show that if it is analytic, $u = y + e^{x}\cos y$

A. $u = \left(r + \dfrac{K^2}{r}\right)\cos\theta$ $\qquad v = \left(r - \dfrac{K^2}{r}\right)\sin\theta$

$u_r = \left(1 - \dfrac{K^2}{r^2}\right)\cos\theta$ $\qquad v_r = \left(1 + \dfrac{K^2}{r^2}\right)\sin\theta$

$u_\theta = -\left(r + \dfrac{K^2}{r}\right)\sin\theta$ $\qquad v_\theta = \left(r - \dfrac{K^2}{r}\right)\cos\theta$

$u_r = \dfrac{v_\theta}{r}$ $\qquad v_r = -\dfrac{u_\theta}{r}$ $\Rightarrow$ Hence, Analytic

$f(z) = r(\cos\theta + i\sin\theta) + \dfrac{K^2}{r}(\cos\theta - i\sin\theta)$

$= re^{i\theta} + \dfrac{K^2}{re^{i\theta}} = z + \dfrac{K^2}{z}$

$f'(z) = 1 - \dfrac{K^2}{z}$

Q. $u = \left(r + \dfrac{1}{r}\right)\cos\theta$, $r \neq 0$. Show that $u$ is a harmonic function

Also find harmonic conjugate of $u$ & Find corresponding analytic func$^n$

A. $u = \left(r + \dfrac{1}{r}\right)\cos\theta$

$u_r = \left(1 - \dfrac{1}{r^2}\right)\cos\theta$ $\qquad u_\theta = -\left(r + \dfrac{1}{r}\right)\sin\theta$

$u_r = \dfrac{v_\theta}{r} \Rightarrow v_\theta = \left(r - \dfrac{1}{r}\right)\cos\theta \Rightarrow v = \left(r - \dfrac{1}{r}\right)\sin\theta + c$

To verify if harmonic, $u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2} = 0$

$u_{rr} = \dfrac{2}{r^3}\cos\theta$ ; $\dfrac{u_r}{r} = \left(\dfrac{1}{r} - \dfrac{1}{r^3}\right)\cos\theta$ ; $u_{\theta\theta} = -\left(r + \dfrac{1}{r}\right)\cos\theta$

$\dfrac{2}{r^3}\cos\theta + \left(\dfrac{1}{r} - \dfrac{1}{r^3}\right)\cos\theta - \left(r + \dfrac{1}{r}\right)\cos\theta = 0$ $\Rightarrow$ Hence, harmonic

$v = \left(r - \dfrac{1}{r}\right)\sin\theta + c$ $\Rightarrow$ Harmonic Conjugate

$f(z) = u + iv = \left(r + \dfrac{1}{r}\right)\cos\theta + i\left(r - \dfrac{1}{r}\right)\sin\theta + c = re^{i\theta} + \dfrac{1}{re^{i\theta}} + C = z + z^{-1} + c$

Q. Determine the analytic function if $u = r^2\cos 2\theta$

A $u = r^2\cos 2\theta$

$u_r = 2r\cos 2\theta$

$u_r = \frac{v_\theta}{r}$

$v_\theta = 2r^2\cos 2\theta \Rightarrow v = \int 2r^2\cos 2\theta \, d\theta = r^2\sin 2\theta + c$

$f(z) = u + iv$

$= r^2\cos 2\theta + i r^2\sin 2\theta + c$

$= r^2(\cos 2\theta + i\sin 2\theta) + c$

$= z^2 + c$

Q. $u = x^2 - y^2 \qquad v = \frac{-y}{x^2+y^2}$

Prove that they are harmonic

But $u + iv$ is not an analytic func$^n$

A. $u = x^2 - y^2 \qquad v = \frac{-y}{x^2+y^2}$

$u_x = 2x \qquad v_x = \frac{0 - 2x(-y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$

$u_{xx} = 2$

$u_y = -2y \qquad v_y = \frac{-1(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$u_{yy} = -2$

$u_{xx} + u_{yy} = 0 \qquad v_{xx} = \frac{2y(x^2+y^2)^2 - 2xy(2)(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$

But $u_x \neq v_y$

$v_{yy} = \frac{2y(x^2+y^2)^2 - (y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{2y(x^2+y^2) - 4y^3 + 4x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

$v_{xx} + v_{yy} = 0$ but $u_y \neq -v_x$ , Hence harmonic but not analytic

Q. If $f(z)$ is an analytic func$^n$ with constant modulus function

Show that $f(z)$ is a constant

A. $|f(z)| = k \;, \quad k \in \mathbb{C}$

$|f(z)|^2 = |u+iv|^2 = u^2 + v^2 = k^2$

$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \longrightarrow$ ⓐ

$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \longrightarrow$ ⓑ

Using $u_x = v_y$ in ⓑ , $v_x = -u_y$ in ⓐ,

$uu_x - vu_y = 0$

$uu_y + vu_x = 0$

Solving for $u_x$ & $u_y$ $\Rightarrow (u^2+v^2)u_x = 0 \;, (u^2+v^2)u_y = 0 \Rightarrow k^2u_x = 0$ & $k^2u_y = 0$

If $k^2 = u^2 + v^2 = 0 \Rightarrow u_x = u_y = 0$

If $k^2 = u^2 + v^2 \neq 0$ , even then $u_x = u_y = 0$ , So, $u_x = u_y = 0$

Q. $u = xy^3 - x^3y$

Show that it is a harmonic function & find the harmonic conjugate of u

SOLVE ON OWN!

Q. Let $\phi + i\psi$ represents a complex potential of an electrostatic field where $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$. Find the complex potential as a function of variable z and show that $\phi$ is harmonic

SOLVE ON OWN!

Q. Verify that $u(x,y) = \sinh x \sin y$ is harmonic and find v and $f(z)$

SOLVE ON OWN!

# Line Integral

→ First we consider integrals which are functions of form $f : \mathbb{R} \longrightarrow \mathbb{C}$

Let $f : [a,b] \longrightarrow \mathbb{C}$ be a continuous function

→ Also called complex definite integrals & are written as $\int_c f(z)\,dz$ where c is the curve called as path of integration which is represented as

$$f(t) = \mu(t) + i\vartheta(t) \qquad a \leq t \leq b \quad (\text{or}) \quad t \in [a,b]$$

In this case, we define the definite integral as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Q. $f(t) = t\cos t + i\,t\sin t \qquad [2\pi, 4\pi]$

A. $$\int_{2\pi}^{4\pi} f(t)\,dt = \int_{2\pi}^{4\pi} t\cos t\,dt + i\int_{2\pi}^{4\pi} t\sin t\,dt$$

$$= \left(t\sin t + \cos t\right)_{2\pi}^{4\pi} + i\left(-t\cos t + \cos t\right)_{2\pi}^{4\pi}$$

$$= \left(0 - 0 + 1 - 1\right) + i\left(-4\pi - (-2\pi) + 1 - 1\right)$$

$$= -2\pi i$$

Q. $e^{-i(m+n)2\pi} - 1$

A. $$e^{-i(m+n)2\pi} - 1 = \cos(m+n)2\pi - i\sin(m+n)2\pi - 1$$

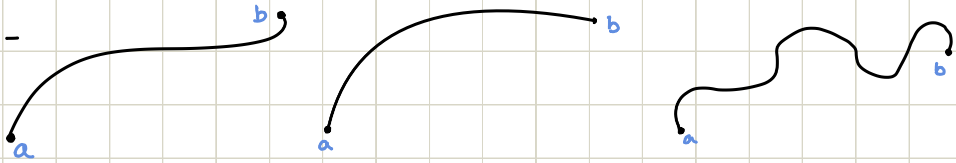
$$= \cos 2\pi x - i\sin 2\pi x - 1$$

$$= 0$$

→ Now we look into the integrals of the form $f: \mathbb{C} \to \mathbb{C}$. This is typically evaluated over the curve on a complex plane which has some properties, such a curve is called a **contour** and the method is called **contour integration**

1) **Curve** – A curve ($\gamma$) with a parameter integral $[a,b]$ is a function $\gamma : [a,b] \longrightarrow \mathbb{C}$ and it has direction

ex –

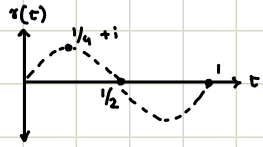



2) **Image of a curve** – It is represented as $\gamma^* = \{\gamma(t) : a \le t \le b\}$

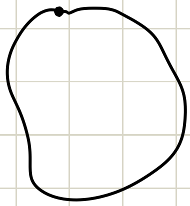
We say that curve $c$ contains $D$ where $D$ is called domain

Then $D \subseteq C$ if $\gamma^* \subseteq D$

ex – $\gamma(t) = t + i\sin 2\pi t$, $t \in [0,1]$




3) **Closed Curve** – A curve $\gamma(t)$ is said to be closed if $\gamma(a) = \gamma(b)$



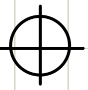
4) **Simple Curve** – A curve $\gamma(t)$ is simple if $\gamma(t_1) \neq \gamma(t_2)$ $\forall\ t_1, t_2 \in [a,b]$ & $t_1 \neq t_2$

→ A closed curve is simple if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = a$ & $t_2 = b$
in other words, simple curve doesn't have intersections within its parameter range

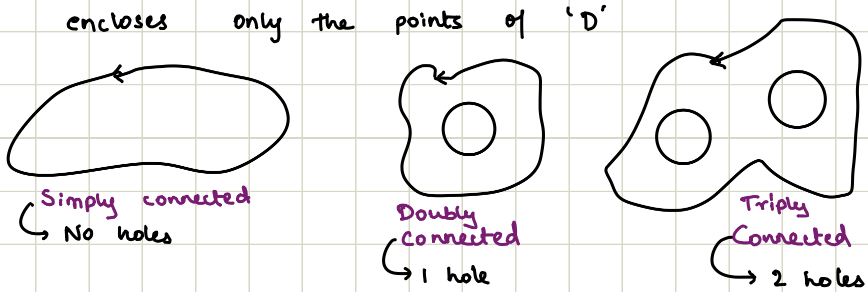
5) **Smooth Curve** – A curve $\gamma : [a,b] \to \mathbb{C}$ is said smooth if $\gamma$ has continuous at all $t \in [a,b]$ also called as **regular curve**

6) **Path / Contour** – It is a join of finitely many smooth curves also called as **piece-wise smooth**

ex – $\gamma(t) = e^{it}$ is a circular contour which is a unit circle



7) **Simply Connected sets** – A simple connected set 'D' in a complex plain is such that every simple closed path in 'D' encloses only the points of 'D'



→ There are 2 ways to evaluate contour integrals on a complex plane

1) Substitution of Limits (Valid only for analytic func$^n$)

→ Let $f(z)$ be analytic in a simply connected domain, then there exists an indefinite integral of $f(z)$ in domain D which is also an analytic function $F(z)$ such that $F'(z) = f(z)$

→ In D & ∀ points in D joining the points $z_0, z_1 \in D$

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

Note: The above integral depends only on the initial & final points $z_0, z_1$ & not on the shape of the contour in D hence we don't use $\int_C$

ex -
$$\int_0^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3 - 0}{3} = \frac{(1+i)^3}{3}$$

2) Representation of a path

→ This method of contour integration is generic & valid for any continuous func$^n$ over a piece-wise smooth path

let C be a piece-wise smooth path represented as $z(t)$, $t \in [a,b]$

Let $f: z^* \longrightarrow c$ be a continuous function on c, then $\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

then RHS is a Riemann Integral

Q. Find the integral of $f(z) = z^2$ over the unit circle

A.
$$\int_C f(z)\,dz = \int_C f(z(t))\,z'(t)\,dt$$

$$f(z) = z^2 \quad \& \quad z(t) = e^{it}$$

$$\int_C f(z)\,dz = \int_0^{2\pi} (e^{it})^2 \cdot ie^{it}\,dt = i\int_0^{2\pi} e^{3it}\,dt = \frac{\cancel{i}}{3\cancel{i}}\left[e^{3it}\right]_0^{2\pi} = \left[\frac{\cos 3t + i\sin 3t}{3}\right]_0^{2\pi} = 0$$

Q. Let $f(z) = z^n$ where $n \in \mathbb{Z}$ with $z \neq 0$ for $n < 0$ over $z(t) = re^{it}$ for $t \in [0, 2\pi]$. Compute the contour integral over $z(t)$

A.
$$\int_C f(z)\,dz = \int f(z(t))\,z'(t)\,dt$$

$$= \int_0^{2\pi} (re^{it})^n \cdot (ire^{it})\,dt = ir^{n+1}\int_0^{2\pi} e^{(n+1)it}\,dt = ir^{n+1}\left(\int_0^{2\pi}\cos(n+1)t\,dt + i\int_0^{2\pi}\sin(n+1)t\,dt\right)$$

when $n = -1$, $i\left(\int_0^{2\pi} dt + 0\right) = 2\pi i$

when $n \neq -1$, $ir^{n+1}\left(\left(\cancel{\frac{\sin(n+1)t}{n+1}}\right)_0^{2\pi} - i\left(\frac{\cos(n+1)t}{n+1}\right)_0^{2\pi}\right)$

$$= \frac{r^{n+1}}{n+1}\left(\cos(n+1)2\pi - 1\right)$$

$$= 0$$

Q. Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along

i) the line $x = 2y$

ii) the real axis to 2 & then vertically to $2+i$

A. $I = \int_0^{2+i} (\bar{z})^2 \, dz$

$(\bar{z})^2 = (x - iy)^2 = x^2 - y^2 - i2xy$

$dz = dx + idy$

i) $x = 2y \Rightarrow dx = 2dy$

Now, $z = 0$ to $2+i$

$(x, y)$ varies from $(0,0)$ to $(2,1)$

$y$ varies from $0$ to $1$

$$\int_{y=0}^{1} \left[(x^2 - y^2) - i(2xy)\right]\left[dx + idy\right]$$

$$= \int_{y=0}^{1} \left[(4y^2 - y^2) - i(4y^2)\right]\left[2dy + idy\right]$$

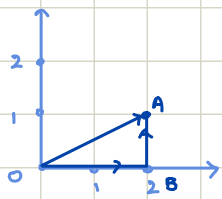
$$I = \int_0^{2+i} (\bar{z})^2 dz = \int_{y=0}^{1} \left[(4y^2 - y^2) - i(4y^2)\right]\left[2dy + idy\right]$$

$$= \int_{y=0}^{1} (3 - 4i)\, y^2 (2 + i)\, dy$$

$$= \int_{y=0}^{1} (6 + 4 + 3i - 8i)\, y^2 dy = \int_{y=0}^{1} (10 - 5i) y^2 dy = (10 - 5i)\left(\frac{y^3}{3}\right)_0^1 = \frac{10}{3} - \frac{5}{3}i = \frac{5}{3}(2 - i)$$

ii) $x \rightarrow 0$ to $2$

$y \rightarrow 0$ to $1$




$$I = \int_{OB} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz$$

$$\int_{OB} (\bar{z})^2 \, dz = \int_{OB} (x - iy)^2 (dx + idy) = \int_{x=0}^{2} (x^2 - y^2 - 2ixy)(dx + idy)\Big|_{y=0}$$

$$= \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$\int_{AB} (\bar{z})^2 dz = \int_{AB} (x - iy)^2 (dx + idy) = \int_{y=0}^{1} (x^2 + y^2 - 2ixy)(dx + idy)\Big|_{x=2}$$

$$= \int_0^1 (4 - y^2 - 4iy)(idy) = \int_0^1 (4idy - iy^2dy + 4ydy) = (4iy)_0^1 - i\left(\frac{y^3}{3}\right)_0^1 + \left(\frac{4y^2}{2}\right)_0^1$$

$$= 4i - \frac{i}{3} + 2 = 2 + \frac{11i}{3}$$

$$I = \frac{8}{3} + 2 + \frac{11i}{3} = \frac{14 + 11i}{3}$$

Q. Show that

a) $\int_c \frac{dz}{z-a} = 2\pi i$

b) $\int_c (z-a)^n dz = 0 \qquad (n \neq -1)$

where C is the circle $|z-a| = r$

A. a) $|z-a| = r$

$z - a = re^{i\theta}$

$z = a + re^{i\theta}$

$dz = ire^{i\theta} d\theta$

$$\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = \int_0^{2\pi} i\, d\theta = 2\pi i$$

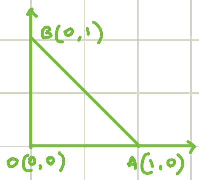
b) $$\int_c (z-a)^n dz = \int_0^{2\pi} (re^{i\theta})^n (ire^{i\theta} d\theta) = \int_0^{2\pi} i (re^{i\theta})^{n+1} d\theta = \frac{i r^{n+1}}{(i)(n+1)} \left( (e^{i\theta})^{n+1} \right)_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1} \left( e^{i2\pi(n+1)} - 1 \right)$$

$$= \frac{r^{n+1}}{n+1} \left( \cos(\cancelto{1}{2\pi(n+1)}) + i \sin(\cancelto{0}{2\pi(n+1)}) - 1 \right)$$

$= 0 \qquad \in n \neq -1$

Q. Verify Cauchy's Theorem for the function $f(z) = z^2$ over the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ → Ans = 0




3 parts : i) $x = 0$

ii) $y = 0$

iii) $x + y = 1$

A. $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$dz = dx + idy$

i) $x = 0 \Rightarrow dx = 0$

$dz = idy$ & $f(z) = -y^2$

$$\int f(z)\,dz = \int_{y=0}^{1} (-y^2)\, dy = \left(\frac{-y^3}{3}\right)_0^1 = \frac{-1^3}{3} = -1/3$$

ii) $y = 0 \Rightarrow dy = 0$

$dz = dx$ & $f(z) = x^2$

$$\int f(z)\,dz = \int_{x=0}^{1} x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$

iii) $x + y = 1 \Rightarrow dx + dy = 0 \Rightarrow dx = -dy$

$f(z) = x^2 + y^2 + 2ixy = (1-y)^2 + y^2 + 2i(1-y)y = 1 + y^2 - 2y + y^2 + 2i(y - y^2) = (1 + 2y^2 - 2y) + 2i(y - y^2)$

$$\int f(z)\,dz = \int_{y=0}^{1} \left( (1+2y^2-2y) + 2i(y-y^2) \right)(-dy + idy) = \int_0^1 \left( (1+2y^2-2y)(i-1) + 2i(y-y^2)(i-1) \right) dy = \int \left( (-1) + i(1+4y^2-4y) \right) dy = 0$$

① + ② + ③ $= 0$

Integral not only depends on end points, but even path of integration.
This makes integration complex
under certain conditions, the contour of the integration doesn't matter & always gives the same result. This is possible when $f(z)$ is analytic & simply connected in a domain D.

## Cauchy's Integral Theorem

→ If $f[z]$ is analytic in a simply connected domain D, then for every closed path $C \in D$ then

$$\oint f(z)\,dz = 0$$

The above theorem is called Cauchy - Gaursat Theorem

## Green's Theorem

→ Let C be a positively oriented smooth simple closed curve in a plane & let D be region bounded by C

If u & v are the functions of $x, y$ defined on an open region containing D which has half partial derivative Then

$$\oint_C (u\,dx + v\,dy) = \iint_D \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy$$

Let $f(z) = u + iv$ be analytic

$dz = dx + i\,dy$

$$\oint_C f(z)dz = \oint (u + iv)(dx + i\,dy) = \oint_C (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

By Green's Theorem ⇒ $\displaystyle\oint_C (u\,dx - v\,dy) = \iint_D \left(-\frac{dv}{dx} - \frac{du}{dy}\right) dx\,dy$

& $\displaystyle\oint_C (v\,dx + u\,dy) = \iint_D \left(\frac{du}{dx} - \frac{dv}{dy}\right) dx\,dy$

Following CR eqn$^s$, RHS of both integrals are 0

Cauchy's Integral theorem holds good for entire function

ex: $\oint \sin z = 0$ , $\oint z^n = 0$

→ Exception :

1) z is a unit circle $= re^{i\theta} = e^{i\theta}$

$$\Rightarrow \int_0^{2\pi} \bar{e}^{i\theta}\, i e^{i\theta} d\theta \Rightarrow 2\pi i \neq 0$$

This isn't a violation of Integral theorem since $\bar{z}$ isn't analytic

2) $z^{-2}$

$$\int_0^{2\pi} z^{-2} dz = \int_0^{2\pi} (e^{i\theta})^{-2}\, i e^{i\theta} d\theta = i\int_0^{2\pi} e^{-i\theta} d\theta = 0$$

This is not due to integral formula since $\frac{1}{z^2}$ is not analytic when $z = 0$

# Cauchy's Integral Formula

→ This is a consequence of cauchy's integral theorem. Let $f(z)$ be analytic is a simply connected domain $z$. For any point $z \in D$, any simple closed path $C \in D$ that encloses $z_0$

Alternatively $f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}$ is true

ex: $\oint \frac{e^z}{z-2} dz = 2\pi i \underbrace{f(2)}_{e^2}$

Q. $\oint \frac{z^2+5}{z-3} dz \qquad |z| = 4$

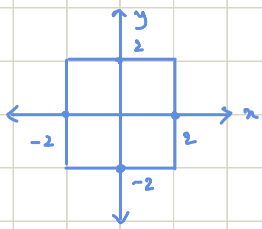
A. $\oint \frac{z^2+5}{z-3} dz = 2\pi i\, f(3) = 2\pi i\,(9+5) = 28\pi i$

Q. $\oint \frac{z^3-6}{2z-i} dz$

A. $\oint \frac{z^3-6}{2z-i} dz = \oint \frac{z^3-6}{2\left(z-\frac{i}{2}\right)} dz = 2\pi i\, f\left(\frac{i}{2}\right) = 2\pi i \left(\frac{\frac{i^3}{8}-6}{2}\right) = \pi i\left(-6-\frac{i}{8}\right) = \frac{\pi}{8} - 6\pi i$

Q. $\oint \frac{\cos z}{z(z^2+8)} dz \quad , \quad x = \pm 2 \quad y = \pm 2$

A.




For $z^2 + 8 = 0 \Rightarrow z = 2\sqrt{2}\, i \Rightarrow$ outside the domain

$\oint \frac{\frac{\cos z}{z^2+8}}{z} = 2\pi i\, f(0) = 2\pi i\left(\frac{\cos 0}{0+8}\right) = \frac{\pi i}{4}$

Q. $\oint \frac{z+4}{z^2+2z+5} dz$ inside $|z+1+i| = 2$

A $\oint \frac{z+4}{(z+1-2i)(z+1+2i)} dz$

$|z+1+i| \leq 2$

$|-1+2i+1+i| \leq 2 \Rightarrow |3i| \nleq 2$

$|-1-2i+1+i| \leq 2 \Rightarrow |-i| \leq 2$

$\oint \frac{\frac{z+4}{(z+1-2i)}}{(z+1+2i)} dz = 2\pi i\, f(-1-2i) = 2\pi i\left(\frac{3-2i}{-1-2i+1-2i}\right) = \frac{(3-2i)\pi}{-2} = \frac{(2i-3)\pi}{2}$

Q. $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ $\quad c: |z| = 3$

$x = \pm 2 \quad , \quad y = \pm 2$

A. $\oint \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)} dz = 2\pi i\, f(2) = 2\pi i \left( \frac{\sin 4\pi + \cos 4\pi}{1} \right) = 2\pi i$

$\oint \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)} dz = 2\pi i\, f(1) = 2\pi i \left( \frac{\sin \pi + \cos \pi}{-1} \right) = 2\pi i$

Sum $= 4\pi i$

→ If $f(z)$ is an analytic func$^n$, and 'a' is a point in the curve, then $\oint \frac{f(z)}{z-a}$ can be found using cauchy's integral formula & hence will have all the derivatives

Q. $\oint \frac{\sin^2 z}{(z - \frac{\pi}{6})^3} dz \quad c: |z| = 1$

A. $\oint \frac{\sin^2 z}{(z - \frac{\pi}{6})^3} dz = \frac{2\pi i\, f''(z)}{2!} \Big|_{z = \frac{\pi}{6}} = \pi i \times 2(\cos^2 z - \sin^2 z) \Big|_{z = \frac{\pi}{6}} = 2\pi i \left( \frac{3}{4} - \frac{1}{4} \right) = \pi i$

Q. $\oint \frac{e^{2z}}{(z+1)^4} dz \quad c: |z| = 2$

A. $\oint \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(z) \Big|_{z=-1} = \frac{2\pi i}{6} \times 8 e^{2(-1)} = \frac{8\pi i}{3e^2}$

Q. $\oint \frac{z^2+z+1}{(z-2)^3} dz \quad c: |z| = 3$

A. $\oint \frac{z^2+z+1}{(z-2)^3} dz = \frac{2\pi i}{2!} f''(z) \Big|_{z=2} = \pi i\, (2) = 2\pi i$

Q. $\oint \frac{e^z}{(z^2+\pi^2)^2} dz \quad c: |z| = 4$

A. $\oint \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{2\pi i}{1!} f'(z) \Big|_{z = \pm i\pi} = 2\pi i\, (e^{\pm i\pi}) = 2\pi i\, (1) = 2\pi i$

# Taylor Series

→ If a function is analytic at a point, it can be exponential using power series expansion

→ If a complex function $f(z)$ is analytic inside a circle $c : |z-a| = r$, then $\forall\, z \in C$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} f^n(a)$$

→ If $a = 0$, taylor series becomes Maclaurin series

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0)$$

→ Let $z = z_0$,

In general, $f(z) = \sum_{n=1}^{\infty} a_n (z - z_n)^n$

Q. $f(z) = \frac{1}{1-z}$, $z = 0$, Converges for $|z| < 1$

A. $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \frac{1}{n!} f^n(0) = \frac{1}{n!} \left( \frac{n!}{(1-z)^{n+1}} \right)_{z=0}$

$a_0 = \frac{1}{0!} f(0) = 1$

$a_1 = \frac{1}{1!} f'(0) = 1$

$a_2 = \frac{1}{2!} f''(0) = 1$

$f(z) = 1 + z + z^2 + \dots$, $|z| < 1$

→ Circle of convergence → $|z| = 1$

Radius of convergence → $|z| < 1$

Q. $f(z) = e^z$ about the point $z = 0$

A. $f(z) = e^z$

$a_n = \frac{f^n(z_0)}{n!} = \frac{f^n(0)}{n!}$

$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

→ When $z = iy$, $e^{iy} = \cos y + i \sin y$

$f(z) = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

→ sin z expansion at $z = iy$

→ cos z expansion at $z = iy$

Q. $\cosh z$

A. $\cosh z = \dfrac{e^z + e^{-z}}{2}$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots$$

$$\cosh z = \frac{1}{2}\left(1 + 1 + \cancel{z} - \cancel{z} + \frac{z^2}{2!} + \frac{z^2}{2!} + \cancel{\frac{z^3}{3!}} - \cancel{\frac{z^3}{3!}} + \dots\right)$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

Q. $\sinh z$

A. $\sinh z = \dfrac{e^z - e^{-z}}{2}$

$$= \frac{1}{2}\left(\cancel{1} - \cancel{1} + z + z + \cancel{\frac{z^2}{2!}} - \cancel{\frac{z^2}{2!}} + \frac{z^3}{3!} + \frac{z^3}{3!} + \dots\right)$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Q. Find Taylor Series Expansion at $z=0$ for $f(z) = \dfrac{1}{1+z^2}$

A. $f(z) = \dfrac{1}{1+z^2} = \dfrac{1}{1-(-z^2)}$

$$= 1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + (-z^2)^4 + \dots$$

$$= 1 - z^2 + z^4 - z^6 + \dots$$

Q. $f(z) = e^z$ at $z = a$

A. $f(a) = e^a$

$f'(z) = e^z \Rightarrow f'(a) = e^a$

$f''(z) = e^z \Rightarrow f''(a) = e^a$

$f'''(z) = e^z \Rightarrow f'''(a) = e^a$

$f^n(z) = e^z \Rightarrow f^n(a) = e^a$

By Taylor Series,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots + \frac{(z-a)^n}{n!}f^n(a)$$

$$e^z = e^a + (z-a)e^a + \frac{(z-a)^2}{2!}e^a + \dots + \frac{(z-a)^n}{n!}e^a$$

Q. $f(z) = \tan^{-1} z$ around $z = 0$

A. $f(z) = \tan^{-1} z$

$f'(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots$

$\int f'(z)\,dz = \int (1 - z^2 + z^4 - z^6 + \ldots)\,dz$

$= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots$

$\tan^{-1} z = u + iv$ is defined for all values of $|u| < \frac{\pi}{2}$

Q. Expand $f(z) = \cos z$ about point $z = \frac{\pi}{4}$

A. $f\left(\frac{\pi}{4}\right) = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z = \frac{-1}{\sqrt{2}}$

$f''(z) = -\cos z = \frac{-1}{\sqrt{2}}$

$f'''(z) = \sin z = \frac{1}{\sqrt{2}}$

$f(z) = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\cdot\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + \ldots$

Q. Expand $f(z) = \frac{1}{z^2 - 3z + 2}$ as Taylor's in region $|z| < 1$

A. $f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$A = (z-1)\cdot f(z)\big|_{z=1} = \frac{1}{z-2}\Big|_{z=1} = -1$

$B = (z-2)\, f(z)\big|_{z=2} = \frac{1}{z-1}\Big|_{z=2} = 1$

$= \frac{1}{z-2} - \frac{1}{z-1}$

$= \frac{1}{-2\left(1 - \frac{z}{2}\right)} + \frac{1}{1-z}$

$= \frac{1}{-2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\right) + \left(1 + z + z^2 + \ldots\right), \quad |z| < 1$

Q. $f(z) = \frac{z+1}{(z-3)(z-4)}$ about $z = 2$

A. $f(z) = \frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$

$A = (z-3)\, f(z)\big|_{z=3} = \frac{z+1}{z-4}\Big|_{z=3} = -4$

$B = (z-4)\, f(z)\big|_{z=4} = \frac{z+1}{z-3}\Big|_{z=4} = 5$

But $z = 3$ & $4$ lie outside $z = 2$, So $f(z)$ is completely analytic within $C$

$f(z) = \frac{+4}{1-(z-2)} - \frac{5}{1-(z-3)}$

$= 4\left(1 + z - 2 + \frac{(z-2)^2}{2!} + \ldots\right) - 5\left(1 + z - 3 + \frac{(z-3)^2}{2!} + \ldots\right)$

Q. Let $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$. Find Taylor Series in $|z| < 2$

A. $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$

$= 1 + \dfrac{-5z-7}{(z+2)(z+3)}$ $\Rightarrow z = -2, -3$ both lie outside $|z| < 2$, $f(z)$ is analytic within C

$= \dfrac{3}{z+2} - \dfrac{8}{z+3}$

$= \dfrac{3}{2\left(1-\left(\frac{-z}{2}\right)\right)} - \dfrac{8}{3\left(1-\left(\frac{-z}{3}\right)\right)}$

$= \dfrac{3}{2}\left(1 + \left(\frac{-z}{2}\right) + \left(\frac{-z}{2}\right)^2 + \dots\right) - \dfrac{8}{3}\left(1 + \left(\frac{-z}{3}\right) + \left(\frac{-z}{3}\right)^2 + \dots\right)$

→ Some binomial expansions:

i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

v) $(1+b)^n = {}^nC_0 b^0 + {}^nC_1 b^1 + {}^nC_2 b^2 + \dots + {}^nC_n b^n$

$= 1 + nb + \dfrac{n(n-1)}{2!} b^2 + \dots + b^n$

Q. Find maclaurin for $f(z) = \dfrac{e^z}{1-z}$ at $z = 0$

A. $e^z = \left(1 + z + \dfrac{z^2}{2!} + \dots\right)$

$\dfrac{1}{1-z} = (1 + z + z^2 + \dots)$

$f(z) = \left(1 + z + \dfrac{z^2}{2!} + \dots\right)(1 + z + z^2 + \dots)$

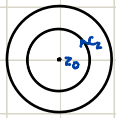
## Laurent's Theorem

→ The taylor series for an analytic func$^n$ $f(z)$ is defined around a point $z = 0$.
If $f(z_0)$ is not defined then we can't have taylor series expansion.

→ In such cases, laurent series is helpful

**Laurent Theorem →** Let $f(z)$ be analytic in an annulus (ring) with 2 circles $C_1$ & $C_2$ centred around $z_0$, then $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$




**Q.** Find Laurent's series for $f(z) = \dfrac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$

**A.** $f(z) = \dfrac{1}{(z-1)(z-2)}$ and $1 < |z| < 2 \Rightarrow \dfrac{1}{|z|} < 1$ & $\dfrac{|z|}{2} < 1$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2\left(1 - \frac{z}{2}\right)} - \frac{1}{z\left(1 - \frac{1}{z}\right)}$$

$$= \frac{-1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right)$$

**Q.** Find Laurent's series, $f(z) = \dfrac{z-1}{(z-2)(z-3)^2}$ for i) $|z| > 3$ & $|z| > 2$
ii) $2 < |z| < 3$

**A.** $f(z) = \dfrac{z-1}{(z-2)(z-3)^2} = \dfrac{A}{z-2} + \dfrac{Bz+C}{(z-3)^2}$

$$\frac{A}{z-2} + \frac{Bz+C}{(z-3)^2}$$

$$\frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}$$

$$\frac{A(z-3)^2 + B(z-3)(z-2) + C(z-2)}{(z-2)(z-3)^2} = \frac{z-1}{(z-2)(z-3)^2}$$

$A(z^2 - 6z + 9) + B(z^2 - 5z + 6) + C(z-2) = z - 1$

$A + B = 0$
$-6A - 5B + C = 1$
$9A + 6B - 2C = -1$

$A = 1$
$B = -1$
$C = 2$

$$\frac{1}{z-2} - \frac{1}{z-3} + \frac{2}{(z-3)^2}$$

i) $|z| > 3 \Rightarrow \dfrac{|z|}{3} > 1$, $|z| > 2 \Rightarrow \dfrac{|z|}{2} > 1$

$$f(z) = \frac{1}{z\left(1 - \frac{2}{z}\right)} - \frac{1}{z\left(1 - \frac{3}{z}\right)} + \frac{2}{z^2\left(1 - \frac{3}{z}\right)^2} = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \dots\right) - \frac{1}{z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \dots\right) + \frac{2}{z^2}\left(1 + \frac{6}{z} + \frac{18}{z^2} + \frac{54}{z^3} + \dots\right)$$

ii) $2 < |z| < 3$

$\dfrac{2}{|z|} < 1$ & $\dfrac{|z|}{3} < 1$

$$f(z) = \frac{1}{z\left(1 - \frac{2}{z}\right)} - \frac{1}{-3\left(1 - \frac{z}{3}\right)} + \frac{2}{\left(-3\left(1 - \frac{z}{3}\right)\right)^2} = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \dots\right) + \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \dots\right) + \frac{2}{9}\left(1 + \frac{2z}{3} + \frac{2z^2}{9} + \frac{2z^3}{27} + \dots\right)$$

Q. $f(z) = \frac{z+1}{z}$ around $z = 0$

A. $f(z) = \frac{z+1}{z} = 1 + \frac{1}{z}$

Q. $f(z) = \frac{z}{z^2+1}$ around $z = i$

A. $f(z) = \frac{z}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i}$

$A = (z-i).f(z)\big|_{z=i} = \frac{z}{z+i} = \frac{1}{2}$

$B = (z+i).f(z)\big|_{z=-i} = \frac{z}{z-i} = \frac{1}{2}$

$f(z) = \frac{1}{2(z-i)} + \frac{1}{2(z+i)}$

Not analytic

Should be in terms of $z-i$

$f(z) = \frac{1}{2\left(1+\frac{z-i}{2i}\right)2i} = \frac{1}{4i}\left(1+\frac{z-i}{2i}\right)^{-1} = \frac{1}{4i}\left(1-\left(\frac{z-i}{2i}\right)+\left(\frac{z-i}{2i}\right)^2-\dots\right)$

Q. $f(z) = \frac{z+1}{z^3(z^2+1)}$ at $z = 0$

A. At $z = 0$, $(z^2+1)$ is analytic

$f(z) = \frac{z+1}{z^3}\left(1+z^2\right)^{-1}$

$= \frac{z+1}{z^3}\left(1 - z^2 + z^4 - z^6 + \dots\right)$

Q. Find Laurent Series for $f(z) = \frac{1}{z^3 - z^4}$ around $z = 0$

A. $f(z) = \frac{1}{z^3-z^4}$

$= \frac{1}{z^3(1-z)}$

$= \frac{1}{z^3}(1-z)^{-1}$

$= \frac{1}{z^3}\left(1+z-z^2+z^3-\dots\right)$

# Theory of Residues

→ If $f(z)$ is analytic at a point $z_0 \in C$, then $z_0$ is called **regular point**

If $f(z)$ isn't analytic at $z_0$, then $z_0$ is called **singular point**

→ Singularity are of 2 types

- Isolated
  - Removable
  - Pole
  - Essential
- Non-Isolated

## Removable Singularity

→ $f(z)$ has removable singularity at $z_0$ if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ for $z_0 \in C$ ↳ ①

& $b_n = 0 \ \forall n$

This implies that $f(z)$ can be made analytic at $z_0$

ex: $f(z) = \frac{\sin z}{z}$ has a removable isolated singularity at $z = 0$

ex: $f(z) = \frac{z^2 - 1}{z - 1}$ has a removable isolated singularity at $z = 1$

## Pole

→ $f(z)$ is said to have a pole of order $n$ at $z_0$ if $b_1 \dots b_m \neq 0$ and $b_n = 0 \ \forall \ n > m$

The principle part of ① has finite no. of terms

Pole of order 1 are called simple pole

ex: $f(z) = \frac{\sin z}{z^2}$ has a simple pole at $z = 0$ of order 1

$$\lim_{z \to 0} \frac{\sin z}{z^2} = \infty$$

ex: $f(z) = \frac{(z-3)^2}{z(z-4)^3}$ has a pole at $z = 0$ of order 1 & $z = 4$ of order 3

ex: $f(z) = \frac{e^z}{z^3}$ has a pole at $z = 0$ of order 3

## Essential Singularity

→ $f(z)$ has essential singularity at $z = z_0$ if the principle part of ① has infinite terms

ex: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots$ has an essential singularity at $z = 0$

ex: $f(z) = \sin\left(\frac{1}{z}\right)$ has an essential singularity at $z = 0$

→ A zero of an analytic function in a domain D for $z_0 \in D$, the func$^n$ $f(z_0) = 0$

ex: $f(z) = (1 - z^4)^2$ has second order zeroes at $z = \pm 1$ & $\pm i$

ex: $f(z) = e^z \Rightarrow \lim_{z \to 0} f(z) = 1$, no zeroes

ex: $f(z) = \sin z \Rightarrow f(z) = 0$ at $n = 0, \pm\pi, \pm 2\pi, \dots$

# Residues

→ Suppose $f(z)$ is analytic in the domain D, $z_0 \in D$, be a pole of $f(z)$, residue of $f(z)$ at $z_0$ is defined as the coefficient $b_1$ of lauren series expansion $f(z)$ around $z_0$

→ Significance of residue

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$\oint_c f(z) = \sum_{n=0}^{\infty} a_n \oint_c (z-z_0)^n \, dz + \sum_{n=1}^{\infty} b_n \oint \frac{1}{(z-z_0)^n} dz$$

$$\oint_c f(z)\, dz = 2\pi i \left[ \operatorname{Res}_{z=z_0} f(z) \right] \quad \Rightarrow \quad \operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_0)^m f(z) \right\}$$

Q. $f(z) = \frac{4-3z}{z^2-z}$. evaluate the integral i) at both 0 & 1 within c

ii) only 0 within c

iii) only 1 within c

iv) 0 & 1 outside c

A. i) $2\pi i \left[ \text{Residue at } z=0 + \text{Residue at } z=1 \right]$ ii)

$$\frac{4-3z}{z(z-1)}$$

$$z \times \frac{4-3z}{z(z-1)} \Big|_{z=0} = -4$$

$$(z-1) \times \frac{4-3z}{z(z-1)} \Big|_{z=1} = 1$$

$$= 2\pi i \left[ -4 + 1 \right] = -6\pi i$$

Q. Integrate $\frac{\tan z}{z^2-1}$ contour - clockwise around the circle $|z| = \frac{3}{2}$

A. $2\pi i \left[ \text{Res at } z=1 + \text{Res at } z=-1 \right]$

$$= 2\pi i \left[ \frac{\tan z}{z+1} \Big|_{z=1} + \frac{\tan z}{z-1} \Big|_{z=-1} \right]$$

$$= 2\pi i \left[ 0.7787 + 0.7787 \right]$$

$$= 3.1148 \pi i$$

$$= 9.785 i$$

Q. $\int_c \frac{8-z}{z(4-z)} dz$. C is a circle of radius 7 & centre = 0

A. $\to 2\pi i$

Q. $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$     $c : |z| = 3 \longrightarrow 2\pi i$

Q. $f(z) = \dfrac{1-e^{2z}}{z^4}$     $c : |z| = 1 \longrightarrow -\dfrac{8\pi i}{3}$

Q. $\displaystyle\int_c \frac{dz}{z^3(z-1)}$     $c : |z| = 2 \longrightarrow 0$

Q. $\displaystyle\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$     $: |z| = 3 \longrightarrow 4\pi(1+\pi)i$