U-3 Design of IIR Filters

Digital Filters

→ 2 Types

i) IIR Filters (Infinite Impulse Response) or Recursive Filters

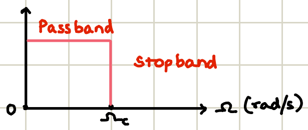
ii) FIR Filters (Finite Impulse Response) or Non-Recursive Filters

Design of IIR Filters

i) Design Analog Filter (Get $H(s)$)

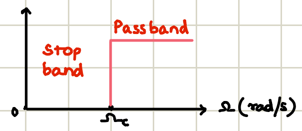
1) Low pass filter

→ Used to pass low frequency components & reject high frequencies

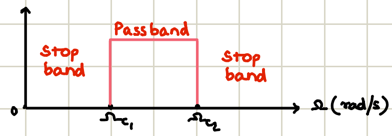


2) High pass filter

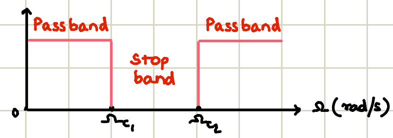
→ Used to pass high frequency components & reject low frequencies



3) Band pass filter

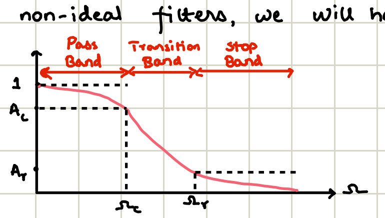


4) Band stop / Reject filter

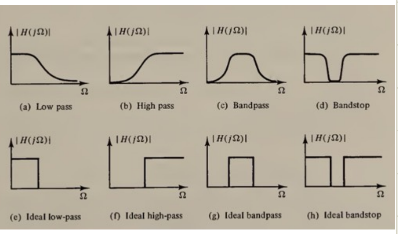


→ The above 4 are ideal filters.

For non-ideal filters, we will have a transition band



ii) Transform to Digital Filter



Butterworth Filters

→ Smooth passband & wider transition band

→ Magnitude square of frequency response is :

$$|H_n(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$$

→ **Frequency response characteristics of Butterworth Filter**

→ $|H_n(j\Omega)|^2_{\Omega=0} = 1$

→ $|H_n(j\Omega)|^2_{\Omega=\Omega_c} = 0.5$

$|H_n(j\Omega)|_{\Omega=\Omega_c} = 0.707$

$20\log\left(|H_n(j\Omega)|_{\Omega=\Omega_c}\right) = -3.0103$

→ $|H_n(j\Omega)|^2$ is monotonically decreasing function of Ω

→ As n increases, $|H_n(j\Omega)|^2$ approaches ideal low-pass frequency response

→ $|H_n(j\Omega)|^2$ is maximally flat at origin (Because all order derivatives exists and are 0)

→ $G_n(\Omega) = 20\log|H_n(j\Omega)|$

$= 10\log|H_n(j\Omega)|^2$

$= 10\log\left|\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}\right|$

$= -10\log\left|1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}\right|$

Case (i) : $\Omega \ll \Omega_c$

$G_n(\Omega) = -10\log|1+0| = 0\ dB$

case (ii) : $\Omega \gg \Omega_c$

$G_n(\Omega) = -10\log\left|\left(\frac{\Omega}{\Omega_c}\right)^{2n}\right| = -20n\log\left|\frac{\Omega}{\Omega_c}\right|$

Case (iii) : $\Omega = 10\Omega_c$

$G_n(\Omega) = -10\log(10)^{2n} = -20n\log 10 = -20n\ dB/decade$

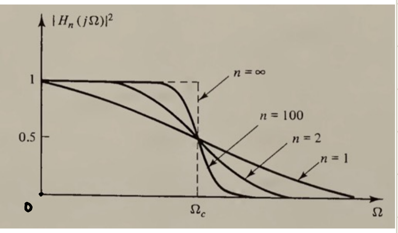
$G_1(\Omega) = -20\ dB/decade$

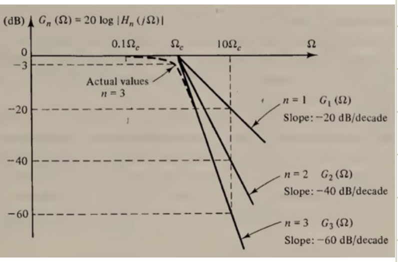
$G_2(\Omega) = -40\ dB/decade$

$G_3(\Omega) = -60\ dB/decade$

Normalized / Prototype Filter

↳ $\Omega_c = 1\ rad/sec$





Filter gain plot for analog Butterworth filters of various order n

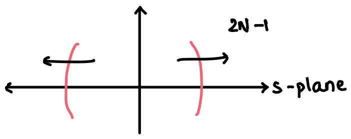
→ Consider Mean square freq. response

$$H_n(s) \cdot H_n(s) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$$

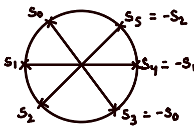
at $\Omega = 1\,rad/s$, $\left(s = j\Omega \Rightarrow \Omega = \frac{s}{j}\right)$

$$= \frac{1}{1+\Omega^{2n}}$$

$$= \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$



For $n=3$



To find poles,

$$1 + (s/j)^{2n} = 0$$

$$s_k = (-1)^{1/2n} \cdot j$$

$$= \left(e^{j\pi(2k+1)}\right)^{1/2n} e^{j\pi/2}$$

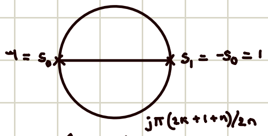
$$= e^{\frac{j\pi(2k+1)}{2n} + j\frac{\pi}{2}}$$

$$= e^{j\frac{\pi}{2n}(2k+n+1)} \qquad k = 0 \text{ to } 2N-1$$

If the filter $H_n(s)$ to be stable & causal filter,

Then $H_n(s) = \dfrac{1}{\prod\limits_{\substack{LHP\\Poles}} (s - s_k)} = \dfrac{1}{B(s)}$

Q. Design a normalised low pass filter system function for n=1 (or) order 1



$$H_n(s) = \frac{1}{\prod(s-s_k)}$$

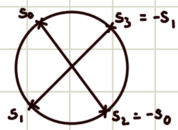
$s_k = \pm e^{j\pi(2k+1+n)/2n}$ $\qquad k = 0$ to $N-1$

$s_0 = \pm e^{j\pi(0+1+1)2}$ $\qquad k = 0$

$= \pm e^{j\pi} = \pm 1$ $\qquad$ Consider the poles on left hand side of s-plane

$$H_n(s) = \frac{1}{s - s_0} = \frac{1}{s-(-1)} = \frac{1}{s+1} = \frac{1}{B_n(s)}$$

Q. Design a prototype butterworth filter system function for n=2



$$s_0 = \pm e^{j\pi(0+1+2)/(2\times 2)} = \pm e^{j\frac{3\pi}{4}} = \pm(-0.707 + j0.707)$$

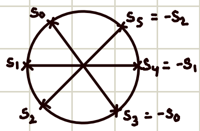
$$s_1 = \pm e^{j\pi(2(1)+1+2)/(2\times 2)} = \pm e^{j\frac{5\pi}{4}} = \pm(-0.707 - j0.707)$$

$s_1 = -0.707 + j0.707 \qquad s_2 = -0.707 - j0.707$

$$H(s) = \frac{1}{(s-(-0.707-0.707j))(s-(-0.707+0.707j))} = \frac{1}{s^2+\sqrt{2}s+1}$$

Q. Design a normalised butterworth filter system function for $n=3$

A.
$$S_K = \pm e^{j\pi(2k+1+n)/2n}$$
$$S_0 = \pm e^{j\pi(0+1+3)/6} = \pm e^{j4\pi/6} = \pm\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}j\right) \Rightarrow -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}j$$
$$S_1 = \pm e^{j\pi(2+1+3)/6} = \pm e^{j\pi} = \pm(-1) \Rightarrow -1$$
$$S_2 = \pm e^{j\pi(4+1+3)/6} = \pm e^{j8\pi/6} = \pm\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}j\right) \Rightarrow -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}j$$



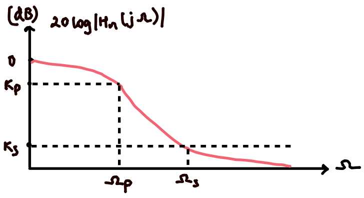
$$H_n(s) = \frac{1}{\left(s-\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}j\right)\right)\left(s-(-1)\right)\left(s-\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}j\right)\right)} = \frac{1}{(s^2+s+1)(s+1)} = \frac{1}{s^3+s^2+s+s^2+s+1} = \frac{1}{s^3+2s^2+2s+1}$$

$$B_n(s) = s^3+2s^2+2s+1$$

Q. Design a normalised butterworth filter system function for $n=4$

A.
$$S_K = \pm e^{j\pi(2k+1+n)/2n}$$
$$S_0 = \pm e^{j\pi(0+1+4)/8} = \pm e^{j\frac{5\pi}{8}} = \pm(-0.382 + j\,0.923)$$
$$S_1 = \pm e^{j\pi(2+1+4)/8} = \pm e^{j\frac{7\pi}{8}} = \pm(-0.923 + j\,0.382)$$
$$S_2 = \pm e^{j\pi(4+1+4)/8} = \pm e^{j\frac{9\pi}{8}} = \pm(-0.923 - j\,0.382)$$
$$S_3 = \pm e^{j\pi(6+1+4)/8} = \pm e^{j\frac{11\pi}{8}} = \pm(-0.382 - j\,0.923)$$
$$H_n(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$

Designing a butterworth Filter



→ Filter requirements are normally given in terms of set of critical frequencies Ω_p & Ω_s and gains K_p & K_s.

Then,
$$0 \geq 20\log|H(j\Omega)| \geq K_p \quad \forall\ \Omega \leq \Omega_p \longrightarrow (1)$$
$$20\log|H(j\Omega)| \leq K_s \quad \forall\ \Omega \geq \Omega_s \longrightarrow (2)$$

From (1), at $\Omega = \Omega_p$,
$$20\log|H_n(j\Omega)| = K_p$$
$$10\log|H_n(j\Omega)|^2 = K_p$$
$$10\log\left|\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2n}}\right| = K_p$$
$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2n} = 10^{-K_p/10} - 1 \longrightarrow (3)$$

From (2), at $\Omega = \Omega_s$
$$20\log_{10}|H_n(j\Omega)| = K_s$$
$$10\log_{10}|H_n(j\Omega)|^2 = K_s$$
$$10\log_{10}\left[\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2n}}\right] = K_s$$
$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2n} = 10^{\frac{-K_s}{10}} - 1 \longrightarrow (4)$$

Summary

n	$B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.765365s+1)(s^2+1.847765s+1)$
5	$(s+1)(s^2+0.61805s+1)(s^2+1.61805s+1)$

Dividing ③ by ④,

$$\frac{\left(\frac{\Omega_p}{\Omega_c}\right)^{2n}}{\left(\frac{\Omega_s}{\Omega_c}\right)^{2n}} = \frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_s}{10}} - 1} \Rightarrow \left(\frac{\Omega_p}{\Omega_s}\right)^{2n} = \frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_s}{10}} - 1}$$

$$2n \log\left(\frac{\Omega_p}{\Omega_s}\right) = \log\left(\frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_s}{10}} - 1}\right)$$

$$n = \frac{\log\left(\frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_s}{10}} - 1}\right)}{2 \log\left(\frac{\Omega_p}{\Omega_s}\right)}$$

Using this, From ③,

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2n} = 10^{-K_p/10} - 1$$

$$\Omega_{c_p} = \frac{\Omega_p}{\left(10^{-K_p/10} - 1\right)^{1/2n}}$$

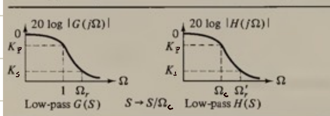
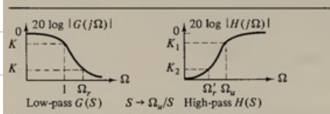
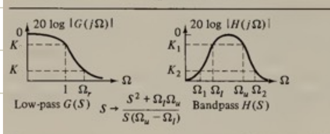
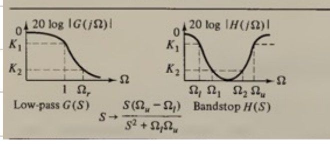
From ④

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2n} = 10^{-K_s/10} - 1$$

$$\Omega_{c_s} = \frac{\Omega_s}{\left(10^{-K_s/10} - 1\right)^{1/2n}}$$

$$\Omega_c = \frac{\Omega_{cp} + \Omega_{cs}}{2}$$

Analog to Analog Transformation

	Prototype response / Transformed filter response	Design equations
1)	 Low-pass $G(S)$; $S \to S/\Omega_c$; Low-pass $H(S)$	Forward: $\Omega_r' = \Omega_r\Omega_c$ Backward: $\Omega_r = \Omega_r'/\Omega_c$ $\Omega_r = \frac{\Omega_s'}{\Omega_p'}$
2)	 Low-pass $G(S)$; $S \to \Omega_u/S$; High-pass $H(S)$	Forward: $\Omega_r' = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega_r'$
3)	 Low-pass $G(S)$; $S \to \frac{S^2 + \Omega_l\Omega_u}{S(\Omega_u - \Omega_l)}$; Bandpass $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{av}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} + \Omega_{av}\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
4)	 Low-pass $G(S)$; $S \to \frac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_l\Omega_u}$; Bandstop $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{av}/\Omega_r$ $\Omega_2 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{av}/\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

Summary,

$$K_p = 10 \log\left(\frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2n}}\right) \qquad \Omega_{c_p} = \frac{\Omega_p}{\left(10^{-K_p/10} - 1\right)^{1/2n}}$$

$$K_s = 10 \log\left(\frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2n}}\right) \qquad \Omega_{c_s} = \frac{\Omega_s}{\left(10^{-K_s/10} - 1\right)^{1/2n}}$$

$$n = \frac{\log\left(\frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_s}{10}} - 1}\right)}{2 \log\left(\frac{\Omega_p}{\Omega_s}\right)} \qquad \Omega_c = \frac{\Omega_{c_s} + \Omega_{cp}}{2}$$

Steps:

1) convert given filter into normalized filter specification
2) Design normalized filter

Q. Design analog butterworth filter to meet following requirements
i) passband freq, $\Omega_p = 20$ rad/s with 2dB passband ripple
ii) stopband edge freq, $\Omega_s = 30$ rad/s with 10dB atleast stopband attenuation

A. ① $\Omega_p = 20 \quad K_p = -2$; $\Omega_s = 30 \quad K_s = -10$ $\xrightarrow{\text{normalized}}$ $\Omega_p = 1 \quad \Omega_r = \frac{30}{20} = 1.5$; $K_p = -2 \quad K_s = -10$

$$n = \left[\frac{\log\left[\left(10^{-(-2)/10} - 1\right)/\left(10^{-(-10)/10} - 1\right)\right]}{2\log_{10}(1/1.5)}\right] = 3.37 = 4$$

$$\Omega_c = \frac{1}{\left(10^{\frac{-(-2)}{10}} - 1\right)^{\frac{1}{8}}} = 1.0693$$

$$\Omega_{cp} = \Omega_p \times \Omega_c = 20 \times 1.0693 = 21.38 \text{ rad/sec}$$

$$H_4(s) = \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.84776s + 1)}$$

$$s \to \frac{s}{21.38} \Rightarrow H_4\left(s \to \frac{s}{21.38}\right) = \frac{1}{\left(\left(\frac{s}{21.38}\right)^2 + \frac{0.7653s}{21.38}\right) + 1)} \cdot \frac{1}{\left(\left(\frac{s}{21.38}\right)^2 + \left(\frac{1.84776s}{21.38}\right) + 1\right)}$$

$$= \frac{1}{\frac{s^2}{457.1044} + 0.0358s + 1} \cdot \frac{1}{\frac{s^2}{457.1044} + 0.0864s + 1}$$

$$= \frac{208944.4325}{(s^2 + 1.636s + 457.1044)(s^2 + 39.494s + 457.1044)}$$

Q. Design analog lowpass filter which is maximally flat at origin to meet following requirements
i) passband freq, $\Omega_p = 20$ rad/s with 3 dB passband ripple
ii) stopband edge freq, $\Omega_s = 40$ rad/s with 20 dB atleast stopband attenuation

A. $\Omega_r = \frac{\Omega_s}{\Omega_p} = 2 \qquad K_p = -3 \qquad K_s = -20$

$$n = \left[\frac{\log\left[\left(10^{-(-3)/10} - 1\right)/\left(10^{-(-20)/10} - 1\right)\right]}{2\log_{10}(1/2)}\right] = 3.318 \simeq 4$$

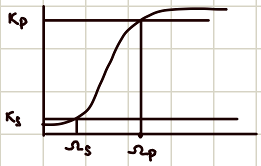
$$\Omega_c = \frac{1}{\left(10^{-(-3)/10} - 1\right)^{1/2(4)}} = 1.00059$$

$$\Omega_{cp} = 20.0118 \text{ rad/s}$$

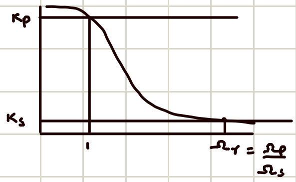
$$H_4(s) = \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.84776s + 1)}$$

$$H_4\left(s \to \frac{s}{20.0118}\right) = \frac{1}{\left(\left(\frac{s}{20.0118}\right)^2 + \frac{0.7653s}{20.0118} + 1\right)\left(\left(\frac{s}{20.0118}\right)^2 + \frac{1.84776s}{20.0118} + 1\right)}$$

High Pass to Low Pass



$$\Omega_r = \frac{\Omega_p}{\Omega_s} \longrightarrow$$



$$s \to \frac{\Omega_c}{s}$$

Q Design High pass filter where $\Omega_p = 400$ rad/s $\Omega_s = 100$ rad/s $K_p = -2$dB $K_s = -20$dB

A. $\Omega_r = \dfrac{\Omega_p}{\Omega_s} = 4$

$$n = \frac{\log\left(\dfrac{10^{\frac{-K_p}{10}}-1}{10^{-K_s/10}-1}\right)}{2\log\left(1/\Omega_r\right)} = \frac{\log\left(\dfrac{10^{0.2}-1}{10^{2}-1}\right)}{2\log\left(1/4\right)} = 1.85 = 2$$

$$\Omega_c = \frac{1}{\left(10^{0.2}-1\right)^{1/4}} = 1.14 \text{ rad/s}$$

$$\Omega_{cp} = \frac{400}{1.14} = 349.8$$

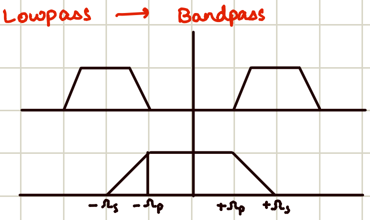
(For High pass, $\Omega_{cp} = \dfrac{\Omega_p}{\Omega_c}$

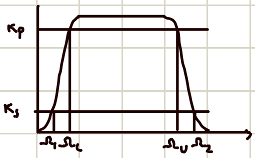
For Low pass, $\Omega_{cp} = \Omega_c \times \Omega_p$)

$$H_2(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

$$H_2\left(\frac{\Omega_{cp}}{s}\right) = \frac{1}{\left(\dfrac{\Omega_{cp}}{s}\right)^2+\dfrac{\sqrt{2}\,\Omega_{cp}}{s}+1} = \frac{1}{\left(\dfrac{349.8}{s}\right)^2+\dfrac{349.8\sqrt{2}}{s}+1}$$

Bandpass → Lowpass ⟶ Bandpass





Ω_1 : 1st stopband freq

Ω_L : lower cutoff freq

Ω_U : upper cutoff freq

Ω_2 : 2nd stopband freq

$$A = \frac{-\Omega_1^2+\Omega_U\Omega_L}{\Omega_1(\Omega_U-\Omega_L)} \qquad B = \frac{\Omega_2^2-\Omega_U\Omega_L}{\Omega_2(\Omega_U-\Omega_L)}$$

$$s \longrightarrow \frac{s^2+\Omega_U\Omega_L}{s(\Omega_U-\Omega_L)} \qquad \Omega_r = \min(A, B)$$

Q. Design analog butterworth bandpass filter

a) at $-3dB$, it has lower & upper freq of $50\ Hz$ & $20\ KHz$

b) stopband attenuation of atleast $20dB$ at $20\ Hz$ & $50\ KHz$

A. $\Omega_1 = 2\pi f_1 = 2\pi \times (20) = 40\pi$ rad/s $\qquad K_p = -3$

$\Omega_2 = 2\pi f_2 = 2\pi \times (50K) = 100\pi K$ rad/s $\qquad K_s = -20$

$\Omega_l = 2\pi f_l = 2\pi \times (50) = 100\pi$ rad/s

$\Omega_u = 2\pi f_u = 2\pi \times (20K) = 40\pi K$ rad/s

$$A = \frac{-\Omega_1^2 + \Omega_u\Omega_l}{\Omega_1(\Omega_u - \Omega_l)} = 2.505$$

$$B = \frac{\Omega_2^2 - \Omega_u\Omega_l}{\Omega_2(\Omega_u - \Omega_l)} = 2.505$$

$$\Omega_r = \min(A, B) = 2.505$$

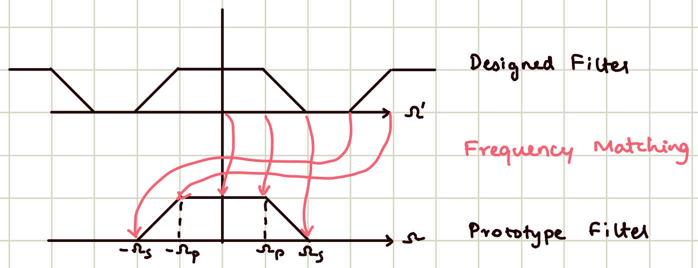
$$n = \left\lceil \frac{\log\left(\frac{10^{0.3} - 1}{10^2 - 1}\right)}{2\log\left(\frac{1}{2.255}\right)} \right\rceil = \lceil 2.5 \rceil = 3$$

$$H_3(s) = \frac{1}{(s^2+s+1)(s+1)} = \frac{1}{\left(\frac{s^2+4\pi^2\times10^6}{39900\pi s}\right)^2 + \left(\frac{s^2+4\pi^2\times10^6}{39900\pi s}\right) + 1} \cdot \frac{1}{\left(\frac{s^2+4\pi^2\times10^6}{39900\pi s}\right) + 1}$$

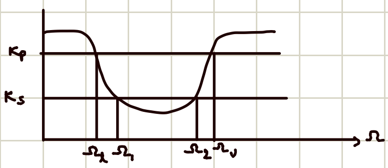
$$s \rightarrow \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s^2 + (4\pi^2\times10^6)}{39900\pi s}$$

Band Stop Filter

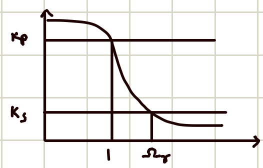
$\rightarrow$



$\rightarrow$ Given Band Stop Filter



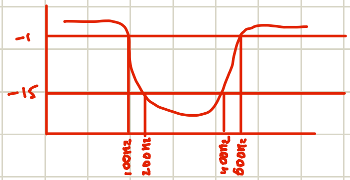
$\Rightarrow$



$$\Omega_r = \min(|A|, |B|)$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u\Omega_l} \qquad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l} \qquad H_{BS}(s) = H_n(s)\Big|_{s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}}$$

Q. Design a Band stop filter for the frequency shown below



A. $\omega = 2\pi f$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u\Omega_l}$$

$$= \frac{400\pi\,(1200\pi - 200\pi)}{-160000\pi^2 + 240000\pi^2} = \frac{400000}{80000} = 5$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l}$$

$$= \frac{800\pi\,(1200\pi - 200\pi)}{-640000\pi^2 + 240000\pi^2} = -2$$

$$\Omega_r = \min(5, 2) = 2$$

$$n = \frac{\log\left(\dfrac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}\right)}{2\log(1/\Omega_r)} = \frac{\log\left(\dfrac{10^{-0.1} - 1}{10^{1.5} - 1}\right)}{2\log(1/2)} = 4$$

$$\Omega_c = \frac{1}{(10^{0.1} - 1)^{1/8}} = 1.18$$

$$H_4(s) = \left.\frac{(1.18)^4}{(s^2 + (0.76533 \times 1.18) + 1.18)(s^2 + (1.84776 \times 1.18)s + 1.18)}\right|_{s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}} = \frac{1.94}{(s^2 + 0.903054s + 1.18)(s^2 + 2.18035s + 1.18)}$$

$$\frac{s(1200\pi - 200\pi)}{s^2 + (1200\pi \times 200\pi)} = \frac{1000s\pi}{s^2 + 240000\pi^2} = \frac{3141.6s}{s^2 + 2368705.1}$$

$$= \frac{1.94}{\left(\left(\dfrac{3141.6s}{s^2+2368705.1}\right)^2 + \dfrac{2837.03}{s^2+2368705.1} + 1.18\right)} \cdot \frac{1}{\left(\left(\dfrac{3141.6s}{s^2+2368705.1}\right)^2 + \dfrac{6849.78s}{s^2+2368705.1} + 1.18\right)}$$

Q. Design an analog butterworth filter with the following specifications

a) −2 dB passband ripple at the upper & lower cutoff freq of 200π rad/s & 500π rad/s

b) −20dB stopband attenuation at 50π rad/s & 800π rad/s

A. $K_p = -2 \Rightarrow \Omega_u = 500\pi \qquad \Omega_l = 200\pi$

$K_s = -20 \Rightarrow \Omega_1 = 50\pi \qquad \Omega_2 = 800\pi$

$$A = \frac{-\Omega_1^2 + \Omega_u \Omega_l}{\Omega_1(\Omega_u - \Omega_l)} = 6.5$$

$$B = \frac{\Omega_2^2 - \Omega_u \Omega_l}{\Omega_2(\Omega_u - \Omega_l)} = 2.25$$

$$\Omega_r = \min(A, B) = 2.25$$

$$n = \frac{\log\left(\dfrac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}\right)}{2\log\left(1/\Omega_r\right)} = 3.16 \simeq 4$$

$$H_4(s) = \frac{1}{(s^2 + 0.7653s + 1)(s^2 + 1.84776s + 1)} \qquad \left(\frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 100000\pi^2}{s(300\pi)}\right)$$

$$H_4\left(s \rightarrow \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}\right) = \frac{1}{\left(\dfrac{s^2 + 100000\pi^2}{300\pi s}\right)^2 + 0.7653\left(\dfrac{s^2 + 100000\pi^2}{300\pi s}\right) + 1} \cdot \frac{1}{\left(\dfrac{s^2 + 100000\pi^2}{300\pi s}\right)^2 + 1.84776\left(\dfrac{s^2 + 100000\pi^2}{300\pi s}\right) + 1}$$

Q. Design an analog bandstop .

a) −3dB passband ripple at the upper & lower cutoff freq of 20 rad/s & 1200 rad/s

b) −25dB stopband attenuation at 50 rad/s & 90 rad/s

A. $K_p = -3 \Rightarrow \Omega_u = 1200$ rad/s $\qquad \Omega_l = 20$ rad/s

$K_s = -25 \Rightarrow \Omega_1 = 50$ rad/s $\qquad \Omega_2 = 90$ rad/s

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u\Omega_l} = 2.74$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l} = 6.68$$

$$\Omega_r = \min(A, B) = 2.74$$

$$n = \frac{\log\left(\dfrac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}\right)}{2\log\left(1/\Omega_r\right)} = 2.85 \simeq 3$$

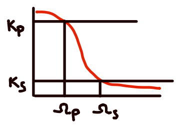
$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)} \Rightarrow H_3(s)\Big|_{s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}}$$

$$H_3(s)\Big|_{s \rightarrow \frac{1180s}{s^2 + 24000}} = \frac{1}{\left(\left(\dfrac{1180s}{s^2+24000}\right) + 1\right)\left(\left(\dfrac{1180s}{s^2+24000}\right)^2 + \left(\dfrac{1180s}{s^2+24000}\right) + 1\right)}$$

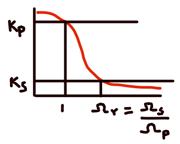
Desired Filter Frequency Response

Normalised Filter Frequency Response

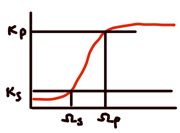
1)



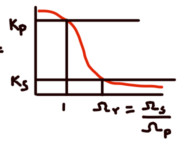
$$\xrightarrow{H_{LP}(s) = H_n(s)\big|_{s \rightarrow \frac{s}{\Omega_c}}}$$



2)



$$H_{HP}(s) = H_n(s)\big|_{s \rightarrow \frac{\Omega_c}{s}}$$



Chebyshev Type - 1

→ $|H_n(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2 T_n^2\left(\frac{\Omega}{\Omega_p}\right)}$ where $\Omega_p = 1$ rad/s normalised

Magnitude squared

$|H_n(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2 T_n^2(\Omega)}$

ε : ripple factor

$T_n(\Omega)$ or $T_n(x)$: Chebyshev polynomial

$T_n(x) = \cos nT|_{x=\cos t}$

For $n=0$, $T_n(x) = 1$

For $n=1$, $T_n(x) = \cos t = x$

For $n=2$, $T_n(x) = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$

For $n>2$, recursive method

$T_{n+1}(x) = \cos(n+1)t = \cos nt \cos t - \sin t \sin t$

$T_{n-1}(x) = \cos(n-1)t = \cos nt \cos t + \sin t \sin t$

$T_{n+1}(x) + T_{n-1}(x) = 2\cos nt \cos t$

$T_{n+1}(x) = 2\cos nt \cos t - T_{n-1}(x)$

$= 2T_n(x).x - T_{n-1}(x)$

$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$ ⇒ Chebyshev Polynomial

For $n=3$, $T_3(x) = 2x\,T_2(x) - T_1(x)$

$= 2x(2x^2-1) - x$

$= 4x^3 - 3x$

For $n=4$, $T_4(x) = 2x\,T_3(x) - T_2(x)$

$= 2x(4x^3-3x) - (2x^2-1)$

$= 8x^4 - 6x^2 - 2x^2 + 1$

$= 8x^4 - 8x^2 + 1$

For $n=5$, $T_5(x) = 2x\,T_4(x) - T_3(x)$

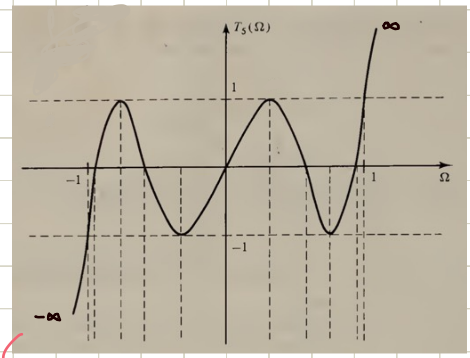
$= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$

$= 16x^5 - 16x^3 + 2x - 4x^3 + 3x$

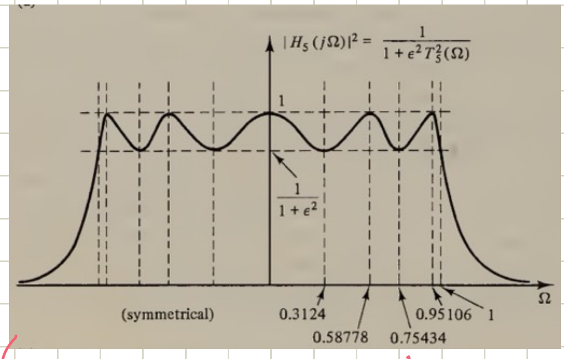
$= 16x^5 - 20x^3 + 5x$

n	$T_n(x)$	$T_n(0)$
0	1	1
1	x	0
2	$2x^2-1$	−1
3	$4x^3-3x$	0
4	$8x^4-8x^2+1$	1
5	$16x^5-20x^3+5x$	0
6	$32x^6-48x^4+18x^2-1$	−1
7	$64x^7-112x^5+56x^3-7x$	0

Frequency response of chebyshev filters



Plot of 5^{th} order Chebyshev polynomial $T_5(\Omega)$



Plot of corresponding $|H_5(j\Omega)|^2$ for type I Chebyshev filter

→ when n is odd, $T_n(\Omega) = 0$

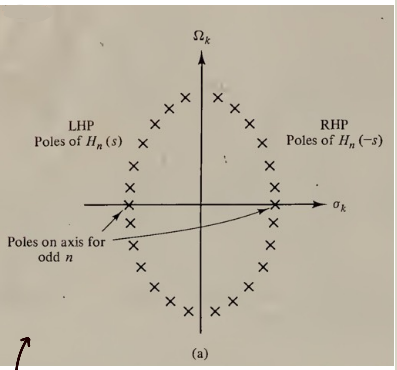
$\Omega = 0 \Rightarrow |H_n(j\Omega)|^2 = \dfrac{1}{1+0} = 1$

when n is even, $T_n(\Omega) = 1$

$\Omega = 1\,\text{rad/s} \Rightarrow |H_n(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2}$ (Passband edge frequency)

$\Omega = \Omega_r \Rightarrow |H_n(j\Omega)|^2 = \dfrac{1}{A^2}$ (Stopband edge frequency)

$T_N(\Omega)$	$\lvert H_N(\Omega)\rvert^2$
→ The polynomial oscillates b/w 1 & −1 in the range $\Omega = [-1, 1]$	→ There will be equal magnitude ripples in the polynomial $\Omega = [-1, 1]$ Period of oscillation is unequal Amplitude of $\lvert H_n(j\Omega)\rvert^2$ oscillates b/w 1 & $\frac{1}{1+\varepsilon^2}$ when Ω goes from −1 to 1
→ Outside the range $[-1, 1]$ it goes towards $-\infty$ & ∞ respectively	→ $\lvert H_n(j\Omega)\rvert^2$ goes towards zero outside $[-1, 1]$
→ At $\Omega = 0$, when $n = $ odd $\Rightarrow T_n(\Omega) = 0$ At $\Omega = 0$, when $n = $ even $\Rightarrow T_n(\Omega) = 1$	→ $\lvert H_n(j\Omega)\rvert^2 = 1$ at $\Omega = 0$ when n is odd $\lvert H_n(j\Omega)\rvert^2 = \frac{1}{1+\varepsilon^2}$ at $\Omega = 0$ when n is even → At $\Omega = 1$ rad/sec, $\lvert H_n(j\Omega)\rvert^2 = \frac{1}{1+\varepsilon^2}$ (Passband) At $\Omega = \Omega_r$, $\lvert H_n(j\Omega)\rvert^2 = \frac{1}{A^2}$ (Stopband)



Poles of $H_n(s)\,H_n(-s)$ for the normalized low pass chebyshev filter

K: normalizing factor

Design of a chebyshev type-1 filter

→ $|H_n(j\Omega)|^2 = \dfrac{1}{1+\varepsilon^2 T_n^2(\Omega)}$ $\Omega_p = 1$ rad/sec → Normalised filter

→ $s = j\Omega \Rightarrow \Omega = s/j$

$$H_n(s)\,H_n(-s) = \frac{1}{1+\varepsilon^2 T_n^2(s/j)}$$

Take denomitor & equate to 0 $\Rightarrow$ $1+\varepsilon^2 T_n^2(s/j) = 0$ → Poles of $H_n(s)\,H_n(-s)$

→ $s_k = \sigma_k + j\Omega_k$ represents a pole

Then σ_k & Ω_k satisfy $\dfrac{\sigma_k^2}{a^2} + \dfrac{\Omega_k^2}{b^2} = 1$

where $a = \sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$ & $b = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2k+1}{2n}\right)\pi\right) \qquad k = 0 \text{ to } n-1$$

$$\Omega_k = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2k+1}{2n}\right)\pi\right) \qquad k = 0 \text{ to } n-1$$

$$H_n(s) = \frac{K}{\prod_{LHS}(s-s_k)}$$

K → normalising factor

$\prod_{LHS}(s-s_k)$ → Poles of system on LHS

$$= \frac{K}{V_n(s)} = \frac{K}{s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_0}$$

When n = odd, $K = b_0$

n = even, $K = \dfrac{b_0}{\sqrt{1+\varepsilon^2}}$

To find the order, $n = \left[\dfrac{\log_{10}\left(g + (g^2-1)^{1/2}\right)}{\log_{10}\left(\Omega_r + (\Omega_r^2 - 1)^{1/2}\right)}\right]$

$A = \dfrac{1}{|H_n(j\Omega_r)|}$ and $g = \sqrt{\dfrac{A^2-1}{\varepsilon^2}}$

Design Procedure

1) Convert desired filter specifications to normalised filter specifications
2) Design a normalised filter
3) Convert normalised System function $H_n(s)$ to desired system function $H_D(s)$

(D can be lowpass, highpass, bandpass, bandstop)

Butterworth Summary!

LP → LP ⇒ $\Omega_r = \dfrac{\Omega_s}{\Omega_p}$, $s \to \dfrac{s}{\Omega_c}$

$\Omega_c = \Omega_p$

HP − LP ⇒ $\Omega_r = \dfrac{\Omega_p}{\Omega_s}$, $s \to \dfrac{\Omega_c}{s}$

$\Omega_c = \Omega_p$

BP − LP ⇒ $\Omega_r = \min(|A|,|B|)$

$A = \dfrac{-\Omega_1^2 + \Omega_u\Omega_l}{\Omega_1(\Omega_u - \Omega_l)}$

$B = \dfrac{\Omega_2^2 - \Omega_u\Omega_l}{\Omega_2(\Omega_u - \Omega_l)}$

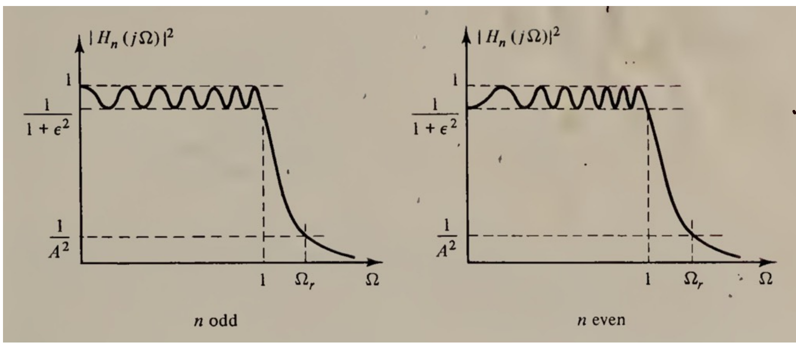
$s \to \dfrac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l)}$

BS → LP ⇒ $\Omega_r = \min(|A|,|B|)$

$A = \dfrac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u\Omega_l}$

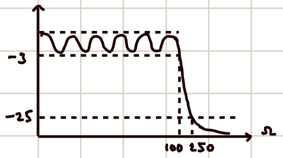
$B = \dfrac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l}$

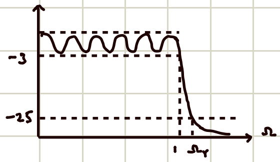
$s \to \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}$



Q. Design analog chebyshev type-I Low pass filter where
a) acceptable passband ripple of -3 dB
b) passband edge frequency of 100 rad/s
c) stopband attenuation of 25dB at 250 rad/s

A. To find n, Ω_c, $H_n(s)$ & $H_{LP}(s)$





$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \frac{250}{100} = 2.5$$

To find ε,

$$20\log|H_n(j1)| = 20\log\left(\frac{1}{1+\varepsilon^2}\right)^{1/2} = 10\log\left(\frac{1}{1+\varepsilon^2}\right) = -3 \Rightarrow \log(1+\varepsilon^2) = 0.3$$

$$\varepsilon = \sqrt{10^{0.3}-1} = 0.9976$$

(or) $\boxed{\varepsilon = \sqrt{10^{-K_p/10}-1}}$

↳ Passband ripple factor

To find A,

$$20\log|H_n(j\Omega_r)| = 20\log\left(\frac{1}{A^2}\right)^{1/2} = 20\log\left(\frac{1}{A}\right) = -25 \Rightarrow \log A = \frac{25}{20} \Rightarrow A = 10^{1.25}$$

$$= 17.782$$

(or) $A = 10^{-K_{\Omega r}/20}$

$$g = \sqrt{\frac{A^2-1}{\varepsilon^2}} = \frac{\sqrt{17.78^2-1}}{0.9976} = 17.809$$

$\Omega_r = 2.5$

$$n = \left\lceil \frac{\log_{10}\left(g+\sqrt{g^2-1}\right)}{\log_{10}\left(\Omega_r+\sqrt{\Omega_r^2-1}\right)} \right\rceil = \lceil 2.27 \rceil = 3$$

Then, $H_3(s) = \dfrac{K}{\prod (s-s_k)}$ $k=0$ to $n-1$ ($3-1=2$)

$$s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

$$\Omega_k = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

K	σ_k	Ω_k
0	-0.1491	0.9037
1	-0.2983	0
2	-0.1491	-0.9037

$$H_3(s) = \frac{K}{\prod(s-s_k)} = \frac{K}{(s-s_0)(s-s_1)(s-s_2)} = \frac{K}{(s-(-0.1491+j0.9037))(s-(-0.2987))(s-(-0.1491-0.9037))}$$

Since n is odd, $K = b_0 = (0.1491-j0.9037)(0.2987)(0.1491+j0.9037) = 0.25$

$$H_{LP}(s) = H_N(s)\Big|_{s\to\frac{s}{100}} = \frac{0.25}{\left(\frac{s}{100}-(-0.1491+j0.9037)\right)\left(\frac{s}{100}-(-0.2987)\right)\left(\frac{s}{100}-(-0.1491-0.9037)\right)}$$

Q. Design an analog chebyshev type-1 high pass filter of order $n=2$ and
1) allowed pass band ripple is 1dB
2) passband edge frequency is 1rad/s

A. $n=2$

$K_p = -1\,dB$

$\Omega_p = 1\,rad/s$

$\varepsilon = \sqrt{10^{-K_p/10}-1} = \sqrt{10^{0.1}-1} = 0.508$

$H_2(s) = \dfrac{K}{(s-s_0)(s-s_1)}$

K	σ_K	Ω_K
0	−0.549	0.895
1	−0.549	−0.895

$\sigma_K = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

$\Omega_K = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

$$H_2 = \frac{K}{(s-(-0.549+j0.895))(s-(-0.549-j0.895))}$$

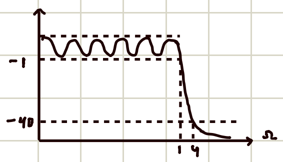
$$K = \frac{b_0}{\sqrt{1+\varepsilon^2}} = \frac{(0.549-j0.895)(0.549+j0.895)}{\sqrt{1+(0.501)^2}} = 0.98$$

$$H_2 = \frac{0.98}{s^2+1.0977s+1.1025}$$

$$H_{HP}(s) = H_2(s)\Big|_{s\to\frac{\Omega}{s}=\frac{1}{s}} = \frac{0.98s^2}{1.1025s^2+1.0977s+1}$$

Q. Find the order & magnitude square function of the normalised chebyshev type-1 filter, where allowed passband ripple is 1dB and stopband attenuation is 40dB for $\Omega \geq 4$ rad/s

A.



$A = 10^{-K_s/20} = 10^{40/20} = 100$

$\varepsilon = \sqrt{10^{-K_p/10}-1} = 0.509$

$g = \dfrac{\sqrt{A^2-1}}{\varepsilon} = 196.45$

$$n = \left\lceil \frac{\log_{10}\left(g+\sqrt{g^2-1}\right)}{\log_{10}\left(\underbrace{\Omega_s}_{4}+\sqrt{\Omega_s^2-1}\right)} \right\rceil = \lceil 2.89 \rceil = 3$$

$T_3^2(\Omega) = (4x^3-3x)^2 = 16x^6-24x^4+9x^2$

$$|H_3(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_3^2(\Omega)} = \frac{1}{1+0.259(16x^6-24x^4+9x^2)}$$

$$= \frac{1}{4.14x^6-6.217x^4+2.331x^2+1}$$

Q. Design a low pass analog filter to meet the following requirements

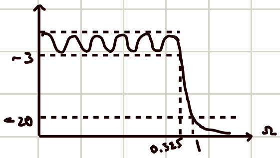
$0.707 \leq |H(j\Omega)| \leq 1 \qquad \Omega \leq 0.325$

$|H(j\Omega)| \leq 0.1 \qquad \Omega \geq 1 \text{ rad/s}$

A. $0.707 = 1 - \delta_p$

$K_p = 20 \log_{10}(1-\delta_p) = -3\text{dB}$

$K_s = 20 \log(0.1) = -20\text{dB}$



$A = 10^{-K_s/20} = 10$

$\varepsilon = \sqrt{10^{-K_p/10} - 1} = \sqrt{10^{0.3} - 1} = 0.997$

$\Omega_r = \dfrac{1}{0.325} = 3.077$

$g = \dfrac{\sqrt{10^2 - 1}}{0.997} = 9.97$

$n = \left\lceil \dfrac{\log_{10}(g + \sqrt{g^2 - 1})}{\log_{10}(\Omega_r + \sqrt{\Omega_r^2 - 1})} \right\rceil = \lceil 1.67 \rceil = 2$

$H_n(s) = \dfrac{K}{(s-s_0)(s-s_1)}$

K	σ_k	Ω_k
0	-0.32	0.77
1	-0.32	-0.77

$b_0 = 0.6953$

$K = \dfrac{b_0}{\sqrt{1+\varepsilon^2}} = 0.492$

$H_n(s) = \dfrac{0.492}{s^2 + 0.64s + 0.707}$

$H_{LP}(s) = H_n(s)\Big|_{s \to \frac{s}{0.325}} = \dfrac{0.492}{\left(\frac{s}{0.325}\right)^2 + \frac{0.64s}{0.325} + 0.707}$

Q. An analog L.P filter is to be designed that has a passband cutoff of 0.668 rad/s with $\delta_p = 0.01$ and stopband cutoff frequency of 1 rad/s with $\delta_s = 0.01$. What order of butterworth & chebyshev type-1 filters are necessary to meet the specifications

A. $K_p = 20 \log(1-\delta_p) = 20 \log(0.99) = -0.08 \text{ dB}$

$K_s = 20 \log(\delta_s) = 20 \log(0.01) = -40 \text{ dB}$

$A = 10^{-K_s/20} = 100 \qquad \Omega_r = \dfrac{1}{0.668} = 1.5$

$\varepsilon = \sqrt{10^{-K_p/10} - 1} = 0.136$

$g = \dfrac{\sqrt{A^2 - 1}}{\varepsilon} = 735.25$

$n_c = \left\lceil \dfrac{\log(g + \sqrt{g^2 - 1})}{\log(\Omega_r + \sqrt{\Omega_r^2 - 1})} \right\rceil = \lceil 7.57 \rceil = 8 \quad ; \quad n_B = \left\lceil \dfrac{\log\left(\frac{10^{-K_p/10} - 1}{10^{-K_s/10} - 1}\right)}{2\log(1/\Omega_r)} \right\rceil = \lceil 16.247 \rceil = 17$

(If desired order = x, normalised order = $\frac{x}{2}$) For bandpass & bandstop

Q. Design a 4th order chebyshev type - I bandpass filter where passband is between 200 to 300 Hz with allowed passband ripple = 0.5dB

A.

$N_{desired} = 4$

$N_{normalised} = 2$

$\Omega_l = 200 \times 2\pi = 400\pi$ rad/s

$\Omega_u = 300 \times 2\pi = 600\pi$ rad/s

$K_p = -0.5$

$\varepsilon = \sqrt{10^{-K_p/10} - 1} = \sqrt{10^{0.05} - 1} = 0.349$

K	σ_K	Ω_K
0	-0.713	1
1	-0.713	-1

$\sigma_K = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

$\Omega_K = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

$$H_2(\Omega) = \frac{K}{(s-(-0.713+j))(s-(-0.713-j))}$$

$$K = \frac{b_0}{\sqrt{1+\varepsilon^2}} = \frac{1.508}{\sqrt{1+0.349^2}} = 1.424$$

$$H_2(s) = \frac{K}{\prod(s-s_K)} = \frac{1.424}{(s-(-0.713+j))(s-(-0.713-j))}$$

$$H_{BP}(s) = H_2(s)\Big|_{s \to \frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}} = \frac{s^2+240000\pi^2}{s(200\pi)}$$

$$= \frac{1.424}{\left(\frac{s^2+240000\pi^2}{s(200\pi)}-(-0.713+j)\right)\left(\frac{s^2+240000\pi^2}{s(200\pi)}-(-0.713-j)\right)}$$

Q. Design a normalised low pass chebyshev filter with $\Omega_p = 1$ rad/s & $K_p = -2$dB, $\Omega_s = 1.3$ rad/s with $K_s = -20$dB atleast. Sketch the poles of the system

A.

$A = 10^{-K_s/20} = 10$

$\Omega_r = 1.3$

$\varepsilon = \sqrt{10^{-K_p/10} - 1} = 0.764$

$g = \frac{\sqrt{A^2-1}}{\varepsilon} = 13.01$

$$n = \left[\frac{\log\left(g+\sqrt{g^2-1}\right)}{\log\left(\Omega_r+\sqrt{\Omega_r^2-1}\right)}\right] = [4.3] = 5$$

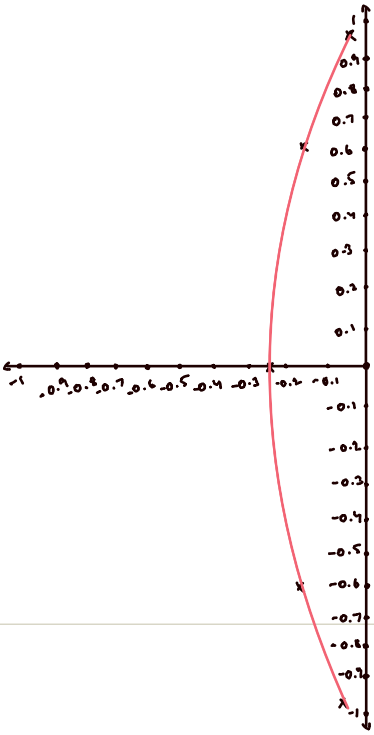
$\Rightarrow \sigma_K = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

$\Omega_K = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2K+1}{2n}\right)\pi\right)$

K	σ_K	Ω_K
0	-0.067	0.973
1	-0.176	0.601
2	-0.218	0
3	-0.176	-0.601
4	-0.067	-0.973

$K = (0.067-j0.973)(0.176-j0.601)(+0.218)(0.067+j0.973)(0.176+j0.601)$

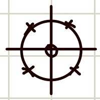
$$H_5(s) = \frac{0.081}{(s+0.067-j0.973)(s+0.176-j0.601)(s+0.218)(s+0.067+j0.973)(s+0.176+j0.601)}$$



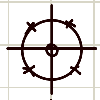
Q. For the following transfer functions, sketch the poles and zeros and identify the type of filter for $n = 2$. Assume $\Omega_0 = \Omega_c$.

a) $H(s) = \dfrac{\Omega_0^2}{s^2 + \left(\frac{\Omega_0}{Q}\right)s + \Omega_0^2}$ $\Rightarrow$ LPF (2 poles, ∞ zeros)

b) $H(s) = \dfrac{s^2}{s^2 + \frac{\Omega_0 s}{Q} + \Omega_0^2}$ $\Rightarrow$ HPF (2 poles, 1 zero)

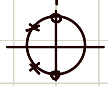


c) $H(s) = \dfrac{\left(\frac{\Omega_0}{Q}\right)s}{s^2 + \frac{\Omega_0 s}{Q} + \Omega_0^2}$ $\Rightarrow$ BPF (2 poles, 1 zero)



$s \rightarrow \dfrac{s^2 + \Omega_u \Omega_\ell}{s(\Omega_u - \Omega_\ell)}$

d) $H(s) = \dfrac{s^2 + \Omega_0^2}{s^2 + \frac{\Omega_0 s}{Q} + \Omega_0^2}$ $\Rightarrow$ BSF (2 poles, ∞ zeros)



$s \rightarrow \dfrac{s(\Omega_u - \Omega_\ell)}{s^2 + \Omega_u \Omega_\ell}$

Q. Given the following specifications,

i) Passband ripple of 0.5 dB at a passband edge freq. of 1 rad/s

ii) Stopband attenuation of 22 dB at a stopband edge freq. of 2.33 rad/s

Obtain $H_a(s)$ of butterworth LPF & Chebyshev type-1 low pass filter. Also sketch the poles

A. Butterworth

$$n = \left\lceil \frac{\log\left(\frac{10^{\frac{0.5}{10}} - 1}{10^{22/10} - 1}\right)}{2\log\left(\frac{1}{2.33}\right)} \right\rceil = \lceil 4.23 \rceil = 5$$

$$\Omega_c = \frac{1}{\left(10^{\frac{-(-0.5)}{10}} - 1\right)^{\frac{1}{10}}} = 1.234 \quad \Rightarrow \Omega_{cp} = 1 \times 1.234 = 1.234$$

$$H_5(s) = \frac{1}{(s+1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)}$$

$$H_{LP}(s) = H_5(s)\Big|_{s \to \frac{s}{1.234}} = \frac{1}{\left(\frac{s}{1.234} + 1\right)\left(\left(\frac{s}{1.234}\right)^2 + \frac{0.618s}{1.234} + 1\right)\left(\left(\frac{s}{1.234}\right)^2 + \frac{1.618s}{1.234} + 1\right)}$$

Chebyshev

$$A = 10^{\frac{-K_s}{20}} = 10^{+1.1} = 12.59$$

$$\varepsilon = \sqrt{10^{-K_p/10} - 1} = 0.35$$

$$g = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 34.956 \qquad \Omega_r = \frac{2.33}{1}$$

$$n = \frac{\log\left(g + \sqrt{g^2 - 1}\right)}{\log\left(\Omega_r + \sqrt{\Omega_r^2 - 1}\right)} = 5$$

K	σ_K	Ω_K
0	-0.111	1.011
1	-0.293	0.625
2	-0.362	0
3	-0.293	-0.625
4	-0.111	-1.011

$H_5(s)$

Butterworth

1) Smooth passband

2) Wide transition Region

3) $|H_n(j\omega)|^2 = \dfrac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2n}}$

4) All poles lie on a circle of radius Ω_c

5) $s_K = \pm\, \Omega_c e^{j\pi(2K+1+n)/2n}$, $K = 0$ to $n-1$

6) $H_n(s) = \dfrac{1}{\prod (s - s_K)}$

7) $\Omega_c = -3\,dB$

Chebyshev

1) Ripples in the passband

2) Narrow transition region

3) $|H_n(j\Omega)|^2 = \dfrac{1}{1 + \varepsilon^2 T_n^2\left(\frac{\Omega}{\Omega_p}\right)}$

4) All poles lie on an ellipse

5) $s_K = \sigma_K + j\Omega_K$, $K = 0$ to $n-1$

6) $H_n(s) = \dfrac{K}{\prod (s - s_K)}$

7) $\Omega_p = \Omega_c$

For given specification, $n_{Butterworth} > n_{Chebyshev}$

IIR Filters (Infinite Impulse Response)

→ Recursive filter

$y(n) = x(n) + x(n-1) - y(n-1) \ldots$

→ More effective filter compared to filter

→ Non-linear phase

→ Analog filter ⟶ Digital IIR Filter

→ Analog filter can be described by its system function

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

$\{\alpha_k\}$ & $\{\beta_k\}$ are filter co-efficients or by its impulse response which is related as

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

and $H(s)$ described in linear constant-coefficient differential equation as

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

We have 4 methods to transform from analog to digital

i) Backward difference method

ii) Impulse invariance Transform

iii) Matched z-Transform

iv) Bilinear Transform

Design Procedure

1) Select the appropriate transformation method

2) Convert the given digital filter specification to analog filter specification

3) Design the analog filter

4) Convert the analog system function $H_a(s)$ ⟶ Digital filter system function $H(z)$

→ Any method must satisfy the 2 properties

i) The $j\omega$ axis on s-plane must be mapped onto unit circle in z-plane
(Direct relationship b/w 2 frequency variable in 2 domains)

ii) Left-half plane of s-plane must be mapped inside of unit circle in z-plane
(Stable analog filter converted to stable digital filter)

1) Backward Difference Method

→

$$y(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{dy(t)}{dt}$$

$$y(n) \longrightarrow \boxed{H(z) = ?} \longrightarrow \frac{y(n) - y(n-1)}{T}$$

By approximating, $\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$

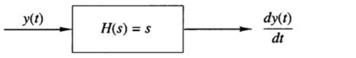
$= \frac{y(n) - y(n-1)}{T}$ $\quad (y(n) \equiv y(nT))$

T: sampling interval

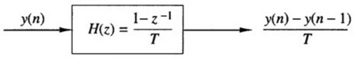
Apply z-Transform on RHS, $= \frac{Y(z) - z^{-1}Y(z)}{T}$

Apply Laplace Transform on LHS, $sY(s)$

Comparing, $s = \frac{1-z^{-1}}{T}$



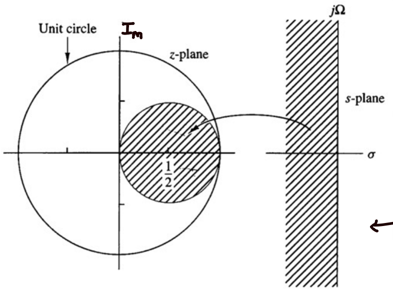
(a)



(b)

$$z = \frac{1}{1-sT} = \frac{1}{1-(\sigma+j\Omega)T} = \frac{1}{1-\sigma T - j\Omega T}$$

$$|z| = \frac{1}{\sqrt{(1-\sigma T)^2 + (\Omega T)^2}}$$



case i) $\sigma < 0$, $|z| < 1$ ⇒ left hand is mapped onto unit circle, stability maintained

Only LPF & BPF of low resonant frequency

case ii) $\sigma = 0$, $z = \frac{1}{1-\sigma T - j\Omega T} = \frac{1}{1-j\Omega T} = \frac{0.5 + 0.5j\Omega T + 0.5 - 0.5j\Omega T}{1-j\Omega T} = 0.5\left(\frac{1+j\Omega T}{1-j\Omega T}\right) + 0.5$

$$z - 0.5 = 0.5\left[\frac{1+j\Omega T}{1-j\Omega T}\right] \Rightarrow |z-0.5| = 0.5$$

case iii) $\sigma > 0$, $|z-0.5| > 0.5$

→ For second derivative

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{t=nT} = \frac{\left[\frac{y(nT)-y(nT-T)}{T}\right] - \left[\frac{y(nT-T)-y(nT-2T)}{T}\right]}{T}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

then $s^2 Y(s) = \left(\frac{1 - 2z^{-1} + z^{-2}}{T^2}\right) Y(z)$ $\quad s^2 = \left(\frac{1-z^{-1}}{T}\right)^2 \Rightarrow s^k = \left(\frac{1-z^{-1}}{T}\right)^k$

Q. Convert the given analog system function $H_a(s) = \frac{1}{s^2+16}$ into digital filter system function $H(z)$ using Backward Difference method (Assume $T = 1s$)

A. $H(z) = H_a(s)\Big|_{s = \frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)^2 + 16} = \frac{1}{1+z^{-2}-2z^{-1}+16} = \frac{1}{z^{-2}-2z^{-1}+17}$

2) Impulse invariance Transform

→ We design an IIR Filter from unit sample response $h(n)$ which is sampled version of impulse response of analog filter $h_a(t)$

$$H_a(s) = \sum_{K=1}^{N} \frac{C_K}{s - s_K} \longrightarrow ①$$

$$h_a(t) = \sum_{K=1}^{N} C_K e^{s_K t}, \quad t \geq 0 \longrightarrow ②$$

Impulse response

$$h(n) = h_a(t)\Big|_{t=nT} = \sum_{K=1}^{N} C_K e^{s_K(nT)}$$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty}\left[\sum_{K=1}^{N} C_K e^{s_K(nT)}\right] z^{-n}$$

$$= \sum_{K=1}^{N} C_K \sum_{n=0}^{\infty}\left(e^{s_K T} z^{-1}\right)^n$$

$$= \sum_{K=1}^{N} C_K \left(\frac{1}{1 - e^{s_K T} z^{-1}}\right) \longrightarrow ③ \qquad \left(\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right)$$

Comparing ① & ③, $\dfrac{1}{s - s_K} \xrightarrow{\text{IIT}} \dfrac{1}{1 - e^{s_K T} z^{-1}}$

For a pole $s = s_K$, $z_K = e^{s_K T}$

$= e^{sT}$

$z = re^{jw}$ & $s = \sigma + j\Omega$

$z = e^{sT}$

$re^{jw} = e^{(\sigma + j\Omega)T}$

$re^{jw} = e^{\sigma T} \cdot e^{j\Omega T}$

$r = e^{\sigma T}$ & $w = \Omega T$

Taking $r = e^{\sigma T}$

i) $\sigma < 0 \Rightarrow r < 1 \Rightarrow$ left side of s-plane mapped inside unit circle $\Rightarrow$ stability verified

ii) $\sigma > 0 \Rightarrow r > 1 \Rightarrow$ right side of s-plane mapped outside unit circle

iii) $\sigma = 0 \Rightarrow r = 1 \Rightarrow j\omega$ axis mapped on unit circle circumfrence

→ $(2K-1)\frac{\pi}{T} < \Omega < (2K+1)\frac{\pi}{T}$, $K \in \mathbb{Z}$

$-\pi < w < \pi$

→ aliasing

* To stop aliasing, we bandlimit ($f_s \geq 2f_h$)

But we can't avoid aliasing like this

So we use LPF & BPF to avoid it (Ω_s doesn't overlap with it)

HPF & BSF can't avoid aliasing (Ω_p overlaps with it)

Q. Given $H_a(s) = \frac{b}{(s+a)^2+b^2}$. Convert into digital filter system using IIT

A. $$H_a(s) = \frac{b}{(s+a)^2+b^2} = \frac{b}{(s-(-a-jb))(s-(-a+jb))} = \frac{b}{(s+a+jb)(s+a-jb)} = \frac{A}{s+a+jb} + \frac{B}{s+a-jb}$$

Solving A & B, $A = \frac{-1}{2j}$ & $B = \frac{1}{2j}$

$$\Rightarrow \frac{-1}{2j(s+a+jb)} + \frac{1}{2j(s+a-jb)}$$

Apply ZT, $$\frac{-1}{2j}\left(\frac{1}{1-e^{(-a-jb)T}z^{-1}}\right) + \frac{1}{2j}\left(\frac{1}{1-e^{(-a+jb)T}z^{-1}}\right)$$

$$= \frac{\frac{-1}{2j}\left(1-e^{(-a+jb)T}z^{-1}\right) + \frac{1}{2j}\left(1-e^{(-a-jb)T}z^{-1}\right)}{\left(1-e^{(-a-jb)T}z^{-1}\right)\left(1-e^{(-a+jb)T}z^{-1}\right)}$$

$$= \frac{z^{-1}e^{-aT}\left(\frac{e^{jb}-e^{-jb}}{2j}\right)}{1-2e^{-aT}\cos bT\, z^{-1} + z^{-2}e^{-2aT}} = \frac{z^{-1}e^{-aT}\sin bT}{1-2e^{-aT}\cos bT\, z^{-1} + z^{-2}e^{-2aT}}$$

Q. Given $H_a(s) = \frac{s+a}{(s+a)^2+b^2}$. Convert into digital filter system using IIT

A. $$H_a(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{s+a}{(s-(-a-jb))(s-(-a+jb))} = \frac{s+a}{(s+a+jb)(s+a-jb)} = \frac{A}{s+a+jb} + \frac{B}{s+a-jb}$$

Solving A & B, $A = \frac{1}{2}$ & $B = \frac{1}{2}$

$$\Rightarrow \frac{1}{2(s+a+jb)} + \frac{1}{2(s+a-jb)}$$

Apply ZT, $$\frac{1}{2}\left(\frac{1}{1-e^{(-a-jb)T}z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-e^{(-a+jb)T}z^{-1}}\right)$$

$$= \frac{\frac{1}{2}\left(1-e^{(-a+jb)T}z^{-1}\right) + \frac{1}{2}\left(1-e^{(-a-jb)T}z^{-1}\right)}{\left(1-e^{(-a-jb)T}z^{-1}\right)\left(1-e^{(-a+jb)T}z^{-1}\right)} = \frac{1 - z^{-1}e^{-aT}\cos bT}{1-2e^{-aT}\cos bT\, z^{-1} + z^{-2}e^{-2aT}}$$

Q. Given $H_a(s) = \frac{s}{(s+a)^2+b^2}$. Convert into digital filter system using IIT

A. $$H_a(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{s+a-a}{(s+a)^2+b^2}$$

$$= \frac{1-(e^{-at}\cos bT)z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}z^{-2}} + \frac{a}{b}\cdot\frac{b}{(s+a)^2+b^2}$$

$$= \frac{1-(e^{-at}\cos bT)z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}z^{-2}} + \frac{a}{b}\left(\frac{e^{-at}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}z^{-2}}\right)$$

Q. Given $H_a(s) = \frac{1}{(s+1)(s+2)}$. Find equivalent $H(z)$ using IIT. Assume $T = 0.3s$

A. $H_a(s) = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$

Solving A & B, $A = 1$, $B = -1$

$$\frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}} = \frac{1}{1-0.74z^{-1}} - \frac{1}{1-0.54z^{-1}}$$

Q. Design LP IIR Butterworth & Chebyshev filter using IIT for
i) $\omega_p = 0.162$ rad/sample. $K_p = -3$
ii) $\omega_s = 1.63$ rad/sample. $K_s = -30$
iii) $f_{sampling} = 8K\ Hz$

A. $T = \frac{1}{f} = \frac{1}{8000}$

$$\Omega_p = \frac{\omega_p}{T} = 0.162 \times 8000 = 1296 \text{ rad/sec}$$

$$\Rightarrow \Omega_r = \frac{\Omega_s}{\Omega_p} = 10.061$$

$$\Omega_s = \frac{\omega_s}{T} = 1.63 \times 8000 = 13040 \text{ rad/sec}$$

Butterworth,

$$n = \left[\frac{\log_{10}\left[\frac{10^{-K_p/10}-1}{10^{-K_s/10}-1}\right]}{2\log_{10}\left(\frac{1}{\Omega_r}\right)}\right] = \left[\frac{\log\left(\frac{10^{0.3}-1}{10^{3}-1}\right)}{2\log\left(\frac{1}{10.06}\right)}\right] = [1.496] = 2$$

$$\Omega_c = \frac{1}{(10^{-K_p/10}-1)^{2n}} = \frac{1}{(10^{0.3}-1)^{4}} = 1$$

$$\Omega_{cp} = 1296 \text{ rad/s}$$

$$H_n(s) = \frac{1}{\Pi(s-s_k)} = \frac{1}{B_n(s)} = \frac{1}{s^2+\sqrt{2}s+1}$$

$$H_{LP}(s) = H_n(s)\Big|_{s\to\frac{s}{1296}} = \frac{(1296)^2}{s^2+\sqrt{2}\times1296s+(1296)^2} = (1296)^2\left(\frac{1}{s-(-916.27+j916.27)}\cdot\frac{1}{s-(-916.27-j916.27)}\right)$$

$$= \frac{(1296)^2}{916.27}\left[\frac{e^{-916.27}\sin 916.27T\,z^{-1}}{1-2e^{-916.27}\cos 916.27T z^{-1} + e^{-1832.54T}z^{-2}}\right]$$

Chebyshev,

$$A = 10^{-K_s/20} = 31.622 \quad ; \quad \varepsilon = \sqrt{10^{-K_p/10}-1} = \sqrt{10^{0.3}-1} = 0.997$$

$$g = \frac{\sqrt{A^2-1}}{\varepsilon} = 31.702$$

$$\sigma_K = -\sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

$$\Omega_K = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)\cos\left(\left(\frac{2k+1}{2n}\right)\pi\right)$$

$$n = \left[\frac{\log\left(g+\sqrt{g^2-1}\right)}{\log\left(\Omega_r+\sqrt{\Omega_r^2-1}\right)}\right] = [1.38] = 2$$

K	σ_K	Ω_K
0	-0.322	0.777
1	-0.322	-0.777

$$H_2(s) = \frac{\frac{b_0}{\sqrt{1+\varepsilon^2}}}{(s+0.322-j0.777)(s+0.322+j0.777)} = \frac{\frac{0.707}{\sqrt{1+0.997^2}}}{(s+0.322-j0.777)(s+0.322+j0.777)} = \frac{0.5}{(s+0.322)^2+(0.777)^2}$$

$$H_{LP}(s) = H_2(s)\Big|_{s\to\frac{s}{1296}} = \frac{0.5\times1296^2}{\left(\frac{s}{1296}+0.322\right)^2+(0.777)^2} = \frac{0.5\times1296^2}{(s-(-417.312+j1006.992))(s-(-417.312-j1006.992))}$$

$$= \frac{0.5}{1006.992}\left(\frac{1006.992}{(s+417.312)^2+(1006.992)^2}\right) = \frac{0.5\times1296^2}{1006.992}\left(\frac{e^{-417.312T}\sin 1006.992T\,z^{-1}}{1-2e^{-417.312T}\cos 1006.992T z^{-1}+e^{-834.624}z^{-2}}\right)$$

substitute $T = \frac{1}{8000}$

$=$

$$\left[\frac{a}{(s+a)^2+b^2} \longrightarrow \frac{a}{b}\left(\frac{e^{-at}\sin bT\,z^{-1}}{1-2e^{-aT}\cos bT z^{-1}+e^{-2aT}z^{-2}}\right)\right]$$

Q. $H(s) = \dfrac{1}{(s+0.5)(s^2+0.5s+2)}$. Given T = 1 sec. Convert the analog filter system function to a digital filter system function using IIT

A. $H(s) = \dfrac{A}{s+0.5} + \dfrac{Bs+C}{s^2+0.5s+2}$

$$As^2 + 0.5As + 2A + Bs^2 + Cs + 0.5Bs + 0.5C = 1$$

$$A + B = 0$$

$$0.5A + C + 0.5B = 0 \Rightarrow C = 0$$

$$2A + 0.5C = 1 \Rightarrow A = \tfrac{1}{2} \quad B = -\tfrac{1}{2}$$

$$H(s) = \frac{\frac{1}{2}}{s+0.5} + \frac{-\frac{s}{2}}{s^2+0.5s+2}$$

$$= \frac{1}{2(s+0.5)} - \frac{s}{2(s+0.25-j1.392)(s+0.25+j1.392)}$$

$$= \frac{1}{2}\left(\frac{1}{s+0.5}\right) - 0.5\left[\frac{s+0.25}{(s+0.25)^2+1.392^2} - \frac{0.25}{(s+0.25)^2+1.392^2}\right]$$

$$= \frac{1}{2}\frac{1}{(1-e^{0.5T}z^{-1})} - 0.5\left[\frac{1-(e^{-0.25T}\cos 1.392T)z^{-1}}{1-2e^{-0.25T}\cos 1.392T z^{-1} + e^{-0.5T}z^{-2}} + \frac{0.25}{1.392}\left(\frac{e^{-0.25T}\sin 1.392T z^{-1}}{1-2e^{-0.25T}\cos 1.392T z^{-1} + e^{-0.5T}z^{-2}}\right)\right.$$

$$\frac{A}{s+0.25-j1.392} + \frac{B}{s+0.25+j1.392}$$

$$As + 0.25A + j1.392A + Bs + 0.25B - j1.392 = \tfrac{s}{2}$$

$$A + B = \frac{1}{2}$$

$$0.25A + j1.392A + 0.25B - j1.392B =$$

$$j(1.392A - 1.392B) = -0.125$$

$$1.392(A-B) = j0.125$$

$$A - B = j0.089$$

$$A = 0.25 + j0.0445$$

$$B = 0.25 - j0.0445$$

Q. For the given specification, design low pass IIR butterworth filter using IIT

a) Passband is from 0 to 400 Hz → Passband ripple : 2dB

b) Stopband is from 2100 to 4000 Hz → Stopband attenuation : 20dB

Assume sampling frequency of 10KHz

A. $\omega_p = \dfrac{2\pi f_p}{F_T} = \dfrac{2\pi(400)}{10000} = 0.251$ rad/sample

$\omega_s = \dfrac{2\pi f_s}{F_T} = \dfrac{2\pi(2100)}{10000} = 1.319$ rad/sample

$\Omega_p = \dfrac{0.251}{T} = 2513.27$ rad/sec

$\Omega_s = \dfrac{1.319}{T} = 13194.68$ rad/sec

$\Omega_r = 5.25$

$$\Rightarrow \quad n = \frac{\log\left(\dfrac{10^{-k_p/10}-1}{10^{-k_s/10}-1}\right)}{2\log\left(1/\Omega_r\right)} = 2 \qquad \Omega_c = \frac{1}{(10^{-k_p/10}-1)^{\frac{1}{2n}}} = 1.143$$

$$\Omega_{cp} = 2872.66$$

$$\Rightarrow H_2(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

$$H_{cp}(s) = H_2(s)\Big|_{s \to \frac{s}{2872.66}} = \frac{1}{\left(\frac{s}{2872.66}\right)^2 + \frac{\sqrt{2}s}{2872.66} + 1} = \frac{1}{(s+2031.27-j2031.27)(s+2031.27+j2031.27)}$$

$$= \frac{1}{(s+2031.27)^2+(2031.27)^2}$$

$$\Rightarrow \frac{1}{1779.95}\left(\frac{e^{-2031.27T}\sin 2031.27T z^{-1}}{1-2e^{-2031.27T}\cos 2031.27T z^{-1} + e^{-4062.54T}z^{-2}}\right)$$

substitute $T = \dfrac{1}{10000}$

Q. For the previous question, do Chebyshev

A.
$$A = 10^{-K_s/20} = 10$$
$$\varepsilon = \sqrt{10^{-K_s/10} - 1} = 0.765$$
$$g = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 13.01$$
$$n = \left\lceil \frac{\log\left(g + \sqrt{g^2 - 1}\right)}{\log\left(\Omega_r + \sqrt{\Omega_r^2 - 1}\right)} \right\rceil = 2$$

	σ_K	Ω_K
0	-0.027	0.707
1	-0.027	-0.707

$$H_2(s) = \frac{\frac{b_0}{\sqrt{1+\varepsilon^2}}}{(s + 0.027 - j0.707)(s + 0.027 + j0.707)} = \frac{0.653}{(s + 0.027)^2 + 0.707^2}$$

$$H_{LP}(s) = H_2(s)\Big|_{s \to \frac{s}{2500}} = \frac{0.653 \times 2500^2}{\left(\frac{s}{2500} + 0.027\right)^2 + 0.707^2} = \frac{0.653 \times 2500^2}{(s + 67.5 - j1767.5)(s + 67.5 + j1767.5)} = \frac{4081250}{(s+67.5)^2 + 1767.5^2}$$

$$\Rightarrow \frac{4081250}{1767.5}\left(\frac{e^{-67.5T} \sin 1767.5\, T z^{-1}}{1 - 2e^{-67.5T} \cos 1767.5\, T z^{-1} + e^{-135T} z^{-2}}\right)$$

$$= 2309.05\left(\frac{e^{-67.5T} \sin 1767.5\, T z^{-1}}{1 - 2e^{-67.5T} \cos 1767.5\, T z^{-1} + e^{-135T} z^{-2}}\right)$$

3) Bilinear Transform

→ Consider $\frac{dy(t)}{dt} = x(t)$ ⟶ ① $\xrightarrow{L.T}$ $sY(s) = X(s)$ ⟶ ②

Integrate ① b/w $(n-1)T$ & nT where 'T' is sampling interval,

$$\int_{(n-1)T}^{nT} \frac{dy}{dt}\,dt = \int_{(n-1)T}^{nT} x(t)\,dt \Rightarrow y(nT) - y((n-1)T) = \int_{(n-1)T}^{nT} x(t)\,dt$$

According to trapezoidal rule, Area = mean height × width

Then, $y(nT) - y((n-1)T) \equiv \left[\frac{x(nT) + x((n-1)T)}{2}\right] T$ where T is sampling interval & is small

$y(nT) \equiv y(n)$ $\quad x(nT) \equiv x(n)$

$y((n-1)T) = y(n-1)$ $\quad x((n-1)T) \equiv x(n-1)$

Then, $y(n) - y(n-1) \equiv \left(\frac{x(n) + x(n-1)}{2}\right) T$

Apply Z-Transform, $Y(z) - z^{-1}Y(z) = \left(\frac{X(z) + z^{-1}X(z)}{2}\right) T$

$$(1 - z^{-1})Y(z) = \left(\frac{(1 + z^{-1})X(z)}{2}\right) T$$

$$X(z) = \frac{2(1 - z^{-1})}{T(1 + z^{-1})} Y(z) \longrightarrow ③$$

Compare ② & ③, $\boxed{s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})}}$ ⟶ Bilinear Transformation eqⁿ

$$s = \frac{2(z-1)}{T(z+1)}$$

$$\sigma + j\omega = \frac{2(re^{j\omega}-1)}{T(re^{j\omega}+1)}$$

$$= \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) + 1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega - 1) + jr\sin\omega}{(r\cos\omega + 1) + jr\sin\omega}\right] \times \frac{(r\cos\omega + 1) - jr\sin\omega}{(r\cos\omega + 1) - jr\sin\omega}$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega)^2 - 1 - jr\sin\omega(r\cos\omega - 1 - r\cos\omega - 1) + (r\sin\omega)^2}{(r\cos\omega + 1)^2 + (r\sin\omega)^2}\right]$$

$$= \frac{2}{T}\left[\frac{r^2 - 1 + j\,2r\sin\omega}{(r\cos\omega + 1)^2 + (r\sin\omega)^2}\right] \Rightarrow \sigma = \underbrace{\frac{2(r^2-1)}{T(r^2+1+2r\cos\omega)}}_{A} \quad \& \quad \omega = \underbrace{\frac{2r\sin\omega}{T(r^2+1+2r\cos\omega)}}_{B}$$

case (i) $\Rightarrow$ $r < 1 \rightarrow \sigma < 0$

Left side of s-plane is mapped inside unit circle on z-plane

Stability maintained

case (ii) $\Rightarrow$ $r > 1 \rightarrow \sigma > 0$

Right side of s-plane is mapped outside unit circle on z-plane

case (iii) $\Rightarrow$ $r = 1 \rightarrow \sigma = 0$

jw axis of s-plane is mapped onto unit circle on z-plane

$$\Omega = \frac{2}{T}\frac{2(1)\sin\omega}{2+2\cos\omega} = \frac{2\sin\omega}{T(1+\cos\omega)} = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)} \longrightarrow \text{Digital Freq. eq}^{n}$$

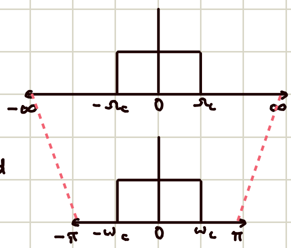
→ No aliasing

→ one-to-one mapping

→ Frequency compression / warping

→ All types of filters can be designed

→ Stability maintained



$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

Prewarp $\Rightarrow$ $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$

warp $\Rightarrow$ $\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$

→ Procedure

1) Convert digital to analog

2) Design analog filter

3) Transform analog to digital using Bilinear Transform

Q. Design second order L.P digital butterworth filter $f_c = 1\text{KHz}$ & sampling freq is $F_T = 10^4$ samples/sec

A. $n = 2$, $F_T = 10^4$ $f_c = 1000$

$$\omega_c = \frac{2\pi f_c}{F_T} = 0.2\pi \text{ rad/sample}$$

1) $$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = \frac{2}{1/10^4}\tan\left(\frac{0.2\pi}{2}\right) = 6498.4 \text{ rad/sec}$$

2) $$n = 2 \Rightarrow H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

$$H_{LP}(s) = H(s)\Big|_{s\to\frac{s}{6498.4}} = \frac{1}{\left(\frac{s}{6498.4}\right)^2 + \frac{\sqrt{2}s}{6498.4} + 1} = \frac{4218336.32}{s^2 + 9190.125s + 4218336.32}$$

$$H(z) = H_{LP}(s)\Big|_{s\to\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} = \frac{4218336.32}{4\times10^8\frac{(1-2z^{-1}+z^{-2})}{(1+2z^{-1}+z^{-2})} + 36760.5\frac{(1-z^{-1})}{(1+z^{-1})} + 4218336.32}$$

$$= \frac{0.0676\left(1+2z^{-1}+z^{-2}\right)}{1-1.432z^{-1}+0.04128z^{-2}}$$

Q. Design LP chebyshev type 1 where $\omega_p = 0.2\pi$ rad/sample & $\omega_s = 0.3\pi$ rad/sample. $K_p = 1\text{dB}$ $K_s = 15\text{dB}$. Use bilinear transform. assume $T = 2s$

A. 1) $T = 2s$

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 0.325 \text{ rad/s}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 0.509 \text{ rad/s}$$

2) $$A = 10^{\frac{-K_s}{20}} = 5.62 \qquad \Omega_r = 1.566$$

$$\varepsilon = \sqrt{10^{-K_p/10}-1} = 0.508$$

$$g = \frac{\sqrt{A^2-1}}{\varepsilon} = 10.886$$

$$n = \frac{\log\left(g+\sqrt{g^2-1}\right)}{\log\left(\Omega_r+\sqrt{\Omega_r^2-1}\right)} = 3.01 \simeq 3$$

$$\sigma_k = -\sinh^{-1}\left(\frac{1}{n}\cosh\left(\frac{1}{\varepsilon}\right)\right)\sin\left(\frac{2k+1}{2n}\right)\pi$$

$$\Omega_k = \cosh^{-1}\left(\frac{1}{n}\cosh\left(\frac{1}{\varepsilon}\right)\right)\cosh\left(\frac{2k+1}{2n}\right)\pi$$

σ_k	Ω_k
-0.247	$+0.965$
-0.494	0
-0.247	-0.965

$$K = \left((-0.247)^2+(0.965)^2\right)(0.494) = 0.491$$

$$H_3(s) = \frac{0.491}{(s+0.247-j0.965)(s+0.247+j0.965)(s+0.494)}$$

$$H_{LP}(s) = H_3(s)\Big|_{s\to\frac{s}{0.325}} = \frac{0.491}{\left(\left(\frac{s}{0.325}+0.247\right)^2+(0.965)^2\right)\left(\frac{s}{0.325}+0.494\right)} = \frac{0.491}{\left(9.467s^2+1.523+0.992\right)\left(\frac{s}{0.325}+0.494\right)}$$

$$= \frac{0.491}{29.129s^3+9.353s^2+3.80s+0.491} = \frac{0.016}{s^3+0.321s^2+0.13s+0.016}$$

$$H(z) = H_{LP}(s)\Big|_{s\to\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{0.016}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3+0.321\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.13\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.016}$$

$$= \frac{0.016\left(1+3z^{-1}+3z^{-2}+z^{-3}\right)}{\left(1-3z^{-1}+3z^{-2}+z^{-3}\right)+0.321\left(1-z^{-1}-z^{-2}+z^{-3}\right)+0.13\left(1+z^{-1}-z^{-2}-z^{-3}\right)+0.016}$$

Q. Design 1st order HPF with $f_c = 30\,Hz$ & sampling freq $= 150\,Hz$ using B.L.T

A. $n = 1$, $f_c = 30\,Hz$, $F_T = 150\,Hz$

$$\omega_c = \frac{2\pi f_c}{F_T} = \frac{2\pi \times 30}{150} = 1.256$$

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) = 300\tan\left(\frac{1.25}{2}\right) = 217.96 \text{ rad/s}$$

$$n = 1 \Rightarrow H(s) = \frac{1}{s+1}$$

$$H_{HP}(s) = H_1(s)\Big|_{s \to \frac{217.96}{s}} = \frac{1}{\frac{217.96}{s}+1} = \frac{s}{s+217.96}$$

$$H(z) = H_{HP}(s)\Big|_{s \to \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{300\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+217.96} = \frac{300(1-z^{-1})}{300(1-z^{-1})+217.96(1+z^{-1})} = \frac{300(1-z^{-1})}{517.96 - 82.04z^{-1}}$$

Q. Design Bandstop Butterworth filter using B.L.T where stopband b/w 200 & 300 Hz and lower passband edge freq is 100 Hz & upper passband edge freq is 600 Hz, $K_p = -3$ & $K_s = -20$. Sampling freq $= 2\,KHz$

A. $$\omega_1 = \frac{2\pi f_1}{f_T} = \frac{2\pi \times 200}{2000} = 0.628 \text{ rad/sample} \Rightarrow \Omega_1 = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = 1298.97$$

$$\omega_2 = \frac{2\pi f_2}{f_T} = \frac{2\pi \times 300}{2000} = 1.256 \text{ rad/sample} \Rightarrow \Omega_2 = \frac{2}{T}\tan\left(\frac{\omega_2}{2}\right) = 2904.22$$

$$\omega_l = \frac{2\pi f_l}{f_T} = \frac{2\pi \times 100}{2000} = 0.314 \text{ rad/sample} \Rightarrow \Omega_l = \frac{2}{T}\tan\left(\frac{\omega_l}{2}\right) = 633.21$$

$$\omega_u = \frac{2\pi f_u}{f_T} = \frac{2\pi \times 600}{2000} = 1.885 \text{ rad/sample} \Rightarrow \Omega_u = \frac{2}{T}\tan\left(\frac{\omega_u}{2}\right) = 5500$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u\Omega_l} = 3.521$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_u\Omega_l} = -2.85$$

$$\Omega_r = \min(A, B) = 2.85$$

$$n = \left\lceil \frac{\log\left(\frac{10^{-K_p/10}-1}{10^{-K_s/10}-1}\right)}{2\log(1/\Omega_r)} \right\rceil = \left\lceil \frac{\log\left(\frac{10^{0.3}-1}{10^2-1}\right)}{2\log(1/2.85)} \right\rceil = \lceil 2.2 \rceil = 3$$

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)} = \frac{1}{s^3+2s^2+s+1}$$

$$\Rightarrow H_3(s)\Big|_{s \to \frac{s(\Omega_u - \Omega_l)}{s^2+\Omega_u\Omega_l}} = \frac{1}{\left(\frac{4866.79s}{s^2+3.483\times10^6}+1\right)\left(\left(\frac{4866.79s}{s^2+3.483\times10^6}\right)^2+\frac{4866.79s}{s^2+3.483\times10^6}+1\right)}$$

$$\frac{s(4866.79)}{s^2+3482655}$$

$$= \frac{(s^2+3.483\times10^6)^3}{(s^2+4866.79s+3.483\times10^6)(2.368\times10^7 s^2 + 4866.79s^3 + 1.695\times10^{10}s + s^2 + 3.483\times10^6)}$$

$$H(z) = H_{BS}(s)\Big|_{s \to 4000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{0.1667z^6 - 0.642z^5 + 1.324z^4 - 1.637z^3 + 1.324z^2 - 0.642z + 0.1667}{z^6 - 1.926z^5 + 1.374z^4 - 0.782^3 +}$$

4) Matched z - Transform

Q. $H(s) = \dfrac{s+2}{(s+1)(s+3)}$ $T = 0.1\,s$

Find equivalent $H(z)$ using z - Transform

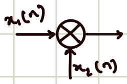
A. $H(z) = \dfrac{(1-e^{-2T}z^{-1})}{(1-e^{-T}z^{-1})(1-e^{-3T}z^{-1})}$

Q. $H(s) = \dfrac{4s(s+1)}{(s+2)(s+3)}$. $T = \dfrac{1}{4}\,s$

$$H(z) = \frac{4(1-z^{-1})(1-e^{-T}z^{-1})}{(1-e^{-2T}z^{-1})(1-e^{-3T}z^{-1})} = \frac{4(1-z^{-1})(1-e^{-\frac{1}{4}}z^{-1})}{(1-e^{-\frac{1}{2}}z^{-1})(1-e^{-\frac{3}{4}}z^{-1})}$$

Realisation of IIR Filter

Basic Operations :

1) 

$x_1(n) + x_2(n)$

2) $x(n) \xrightarrow{a} a\,x(n)$

3) $x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow x(n-1)$

$x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{z^{-1}} \longrightarrow x(n-2)$

IIR Filter Realisation Methods

1) Direct Form - I
2) Direct Form - II
3) Cascaded
4) Parallel
5) Latice Realisation
~~6)~~ Ladder Realisation (Not in Syllabus)

1) Direct Form - I

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

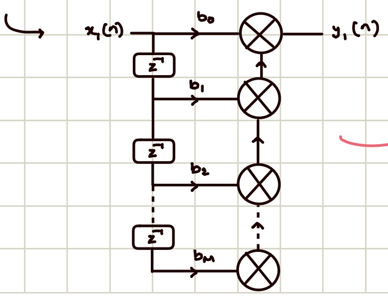
$$H(z) = H_1(z) \cdot H_2(z)$$

$$= \underbrace{\left[\sum_{k=0}^{M} b_k z^{-k}\right]}_{\text{All zero System}} \underbrace{\left[\frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right]}_{\text{All Pole System}}$$

$$H_1(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] \Rightarrow \frac{Y_1(z)}{X_1(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}$$

$$Y_1(z) = b_0 X_1(z) + b_1 X_1(z) z^{-1} + b_2 X_2(z) z^{-2} + b_3 X_3(z) z^{-3} + \dots + b_M X_1(z) z^{-M}$$

By inverse z-Transform, $y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + b_2 x_1(n-2) + \dots + b_M x(n-M)$



All zero System

$$H_2(z) = \left[\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \right] \Rightarrow \frac{Y_2(z)}{X_2(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

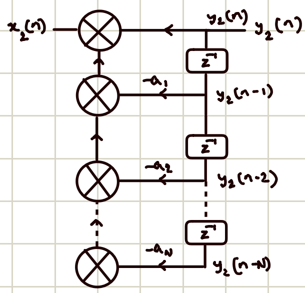
$$Y_2(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X_2(z)$$

$$Y_2(z) = X_2(z) - Y_2(z)\sum_{k=1}^{N} a_k z^{-k}$$

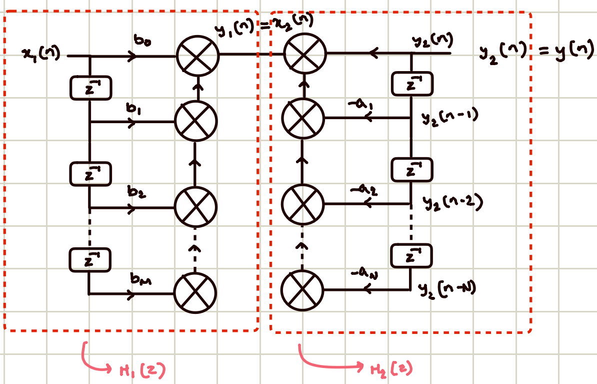
$$= X_2(z) - a_1 z^{-1} Y_2(z) - a_2 z^{-2} Y_2(z) \ldots - a_N z^{-N} Y_2(z)$$

Apply inverse z-transform,

$$y_2(n) = x_2(n) - a_1 y_2(n-1) - a_2 y_2(n-2) \ldots - a_N y_2(n-N)$$



All pole system



N → Order ⇒ we need 'N+M' no. of delay elements

2) Direct Form - II

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{Y(z)}{V(z)} \cdot \frac{V(z)}{X(z)}$$

$$= \left[\sum_{k=0}^{M} b_k z^{-k}\right] \cdot \left[\frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right]$$

$$\frac{V(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \Rightarrow V(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)$$

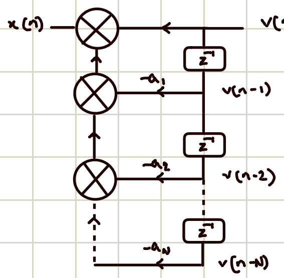
$$V(z) + V(z) \sum_{k=1}^{N} a_k z^{-k} = X(z)$$

$$V(z) = X(z) - \left[\sum_{k=1}^{N} a_k z^{-k}\right] V(z)$$

$$= X(z) - a_1 z^{-1} V(z) - a_2 z^{-2} V(z) \; . \; . \; . - a_N z^{-N} V(z)$$

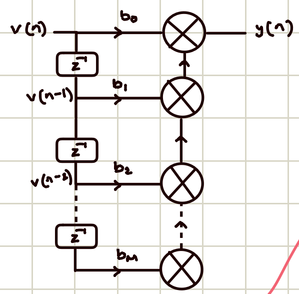
Applying inverse z-transform

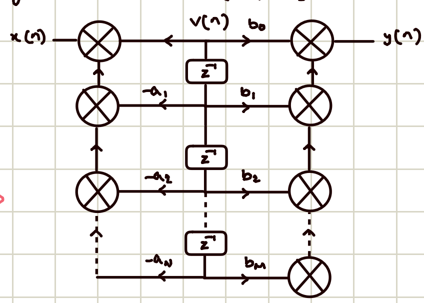
$$v(n) = x(n) - a_1 v(n-1) - a_2 v(n-2) \ldots - a_N v(n-N)$$



$$\frac{Y(z)}{V(z)} = \sum_{k=0}^{M} b_k z^{-k} \Rightarrow Y(z) = V(z) \sum_{k=0}^{M} b_k z^{-k} = V(z)\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}\right)$$

Applying inverse z-transform, $y(n) = b_0 v(n) + b_1 v(n-1) + b_2 v(n-2) + \ldots + b_M v(n-N)$





Combining both

N → Order ⇒ we need max(N,M) no. of delay elements

↳ Reducing complexity compared to direct form-1

Q. Realise the given system function

$$H(z) = \frac{7z^2 - 5.25z + 1.325}{z^2 - 0.75z + 0.125}$$

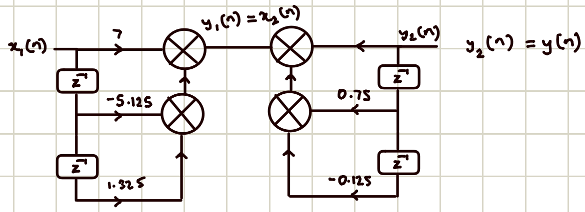
using direct form I & II

Comment on complexity of system

A. $$H(z) = \frac{7z^2 - 5.25z + 1.325}{z^2 - 0.75z + 0.125} \times \frac{z^{-2}}{z^{-2}} = \frac{7 - 5.25z^{-1} + 1.325z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Direct form I, $H_1(z) = 7 - 5.25z^{-1} + 1.325z^{-2} \Rightarrow y_1(n) = 7x_1(n) - 5.25x_1(n-1) + 1.325x_1(n-2)$

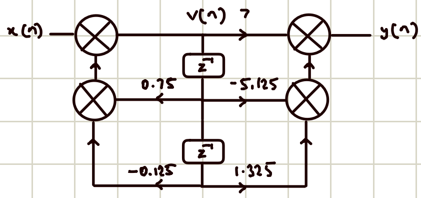
$H_2(z) = \dfrac{1}{1 - 0.75z^{-1} + 0.125z^{-2}} \Rightarrow y_2(n) = x_2(n) + 0.75y_2(n-1) - 0.125y_2(n-2)$



We need 4 delay elements

Direct form II, $\dfrac{V(z)}{X(z)} = \dfrac{1}{1 - 0.75z^{-1} + 0.125z^{-2}} \Rightarrow v(n) = x(n) + 0.75v(n-1) - 0.125v(n-2)$

$\dfrac{Y(z)}{V(z)} = 7 - 5.25z^{-1} + 1.325z^{-2} \Rightarrow y(n) = 7v(n) - 5.125v(n-1) + 1.325v(n-2)$



We need only 2 delay elements

Q. $y(n) = 0.75\,y(n-1) - 0.33\,y(n-2) + x(n) + 0.5\,x(n-1)$

Realise the given function using DF - I & II

A. $y(n) - 0.75y(n-1) + 0.33y(n-2) = x(n) + 0.5x(n-1)$

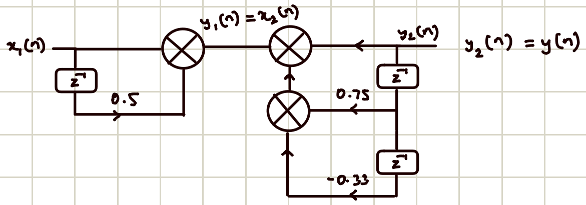
Apply z - Transform

$$Y(z) - 0.75\,Y(z)z^{-1} + 0.33\,Y(z)z^{-2} = X(z) + 0.5\,X(z)z^{-1}$$

$$\frac{Y(z)}{X(z)} = \frac{1+0.5z^{-1}}{(1-0.75z^{-1}+0.33z^{-2})}$$

Direct form I, $H_1(z) = 1+0.5z^{-1} \Rightarrow y_1(n) = x_1(n) + 0.5\,x_1(n-1)$

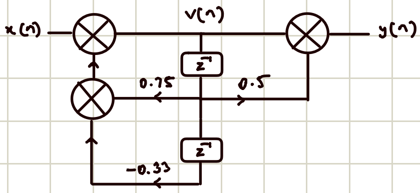
$$H_2(z) = \frac{1}{1-0.75z^{-1}+0.33z^{-2}} \Rightarrow y_2(n) = x_2(n) + 0.75\,y_2(n-1) - 0.33\,y_2(n-2)$$



We need 3 delay elements

Direct form II, $\frac{V(z)}{X(z)} = \frac{1}{1-0.75z^{-1}+0.33z^{-2}} \Rightarrow v(n) = x(n) + 0.75v(n-1) - 0.33\,v(n-2)$

$$\frac{Y(z)}{V(z)} = 1+0.5z^{-1} \Rightarrow y(n) = v(n) + 0.5\,v(n-1)$$



We need only 2 delay elements

3) Cascaded Realisation

Q. $H(z) = \dfrac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$

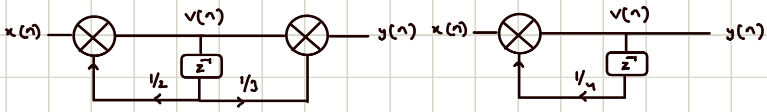
A. $H_1(z) = \dfrac{\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}$ & $H_2(z) = \dfrac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow v(n) = x(n) + \frac{1}{2}v(n-1)$

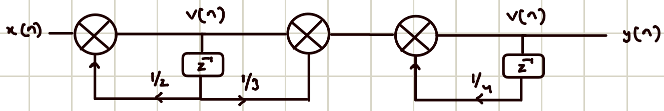
$\dfrac{Y(z)}{V(z)} = 1 + \frac{1}{3}z^{-1} \Rightarrow y(n) = v(n) + \frac{1}{3}v(n-1)$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1-\frac{1}{4}z^{-1}} \Rightarrow v(n) = x(n) + \frac{1}{4}v(n-1)$

$\dfrac{Y(z)}{V(z)} = 1 \Rightarrow y(n) = v(n)$



Cascade both



Q. $H(z) = \dfrac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$

A. $H(z) = \dfrac{3\left(1+0.2z^{-1}\right)\left(1+z^{-1}\right)}{\left(1+0.5z^{-1}\right)\left(1-0.4z^{-1}\right)}$

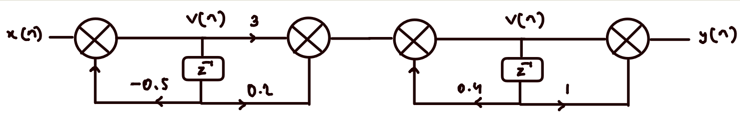
$H_1(z) = \dfrac{3\left(1+0.2z^{-1}\right)}{\left(1+0.5z^{-1}\right)}$ $\quad H_2(z) = \left(\dfrac{1+z^{-1}}{1-0.4z^{-1}}\right)$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1+0.5z^{-1}} \Rightarrow v(n) = x(n) - 0.5(n-1)$

$\dfrac{Y(z)}{V(z)} = 3\left(1+0.2z^{-1}\right) \Rightarrow y(n) = 3\left(v(n) + 0.2\,v(n-1)\right)$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1-0.4z^{-1}} \Rightarrow v(n) = x(n) + 0.4\,v(n-1)$

$\dfrac{Y(z)}{V(z)} = 1+z^{-1} \Rightarrow y(n) = v(n) + v(n-1)$



4) Parallel Realization

Q. $H(z) = \dfrac{1 - z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}}$

A. $\dfrac{H(z)}{z} = \dfrac{z-1}{z^2 - 0.2z - 0.15} = \dfrac{z-1}{(z-0.5)(z+0.3)} = \dfrac{A}{z-0.5} + \dfrac{B}{z+0.3} = \dfrac{-0.625}{z-0.5} + \dfrac{1.625}{z+0.3}$

$$\left.\begin{aligned} A + B &= 1 \\ 0.3A - 0.5B &= -1 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -0.625 \\ B &= 1.625 \end{aligned}$$

$$\frac{H(z)}{\cancel{z}} = \frac{1}{\cancel{z}}\left(\frac{-0.625}{1-0.5z^{-1}} + \frac{1.625}{1+0.3z^{-1}}\right)$$

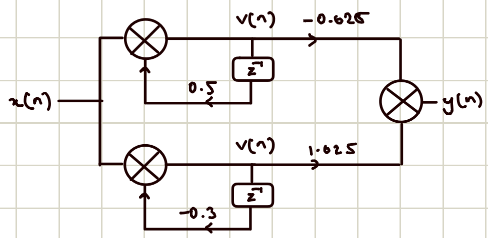
$H_1(z) = \dfrac{-0.625}{1-0.5z^{-1}}$ $\qquad$ $H_2(z) = \dfrac{1.625}{1+0.3z^{-1}}$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1-0.5z^{-1}} \Rightarrow v(n) = x(n) + 0.5\,v(n-1)$

$\dfrac{Y(z)}{V(z)} = -0.625 \Rightarrow y(n) = -0.625\,v(n)$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1+0.3z^{-1}} \Rightarrow v(n) = x(n) - 0.3\,v(n-1)$

$\dfrac{Y(z)}{V(z)} = 1.625 \Rightarrow y(n) = 1.625\,v(n)$



Q. $H(z) = \dfrac{1-\frac{2}{3}z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} \cdot \dfrac{1+\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$

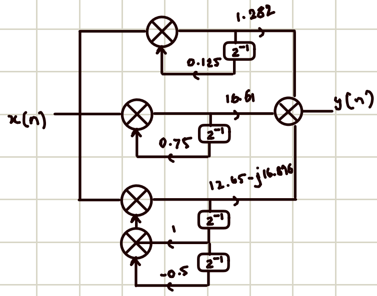
A. $\dfrac{H(z)}{z} = \dfrac{z-\frac{2}{3}}{(z-0.75)(z-0.125)} \dfrac{z^2+\frac{7}{4}z+\frac{1}{2}}{\left(z-\left(\frac{1+i}{2}\right)\right)\left(z-\left(\frac{1-i}{2}\right)\right)}$

$$= \frac{A_1}{(z-0.75)} + \frac{A_2}{(z-0.125)} + \frac{A_3}{\left(z-\left(\frac{1+i}{2}\right)\right)} + \frac{A_4}{\left(z-\left(\frac{1-i}{2}\right)\right)}$$

$$= \frac{z^3+\frac{13}{2}z^2-\frac{2}{3}z-\frac{1}{3}}{(z-0.75)(z-0.125)\left(z-\left(\frac{1+i}{2}\right)\right)\left(z-\left(\frac{1-i}{2}\right)\right)}$$

$$= -\left(\frac{8.448+4.202i}{z-\left(\frac{1+i}{2}\right)}\right) - \left(\frac{8.448-4.202i}{z-\left(\frac{1-i}{2}\right)}\right) + \frac{1.282}{z-0.125} + \frac{16.61}{z-0.75}$$

$$= \frac{12.65-16.896i}{z^2-z+0.5} + \frac{1.282}{z-0.125} + \frac{16.61}{z-0.75}$$



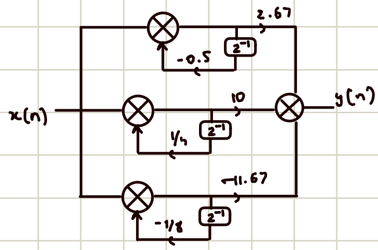
Q. $H(z) = \dfrac{(1+z^{-1})(1+2z^{-1})}{(1+0.5z^{-1})(1-\frac{1}{4}z^{-1})(1+\frac{1}{8}z^{-1})}$ Parallel Realization

A. $\dfrac{H(z)}{z} = \dfrac{(z+1)(z+2)}{(z+0.5)(z-\frac{1}{4})(z+\frac{1}{8})}$

$$= \frac{A}{z+0.5} + \frac{B}{z-\frac{1}{4}} + \frac{C}{z+\frac{1}{8}}$$

$$A\left(z^2-\frac{1}{8}z-\frac{1}{32}\right) + B\left(z^2+\frac{5}{8}z+\frac{1}{16}\right) + C\left(z^2+\frac{1}{4}z-\frac{1}{8}\right)$$

$$\left.\begin{aligned} A+B+C &= 1 \\ -\frac{A}{8}+\frac{5B}{8}+\frac{C}{4} &= 3 \\ -\frac{A}{32}+\frac{B}{16}-\frac{C}{8} &= 2 \end{aligned}\right] \rightarrow \begin{aligned} A &= 2.67 \\ B &= 10 \\ C &= -11.67 \end{aligned}$$



Q. $H(z) = \dfrac{6z^2 + 7z + 1}{z^2 - 0.75z + 0.125}$

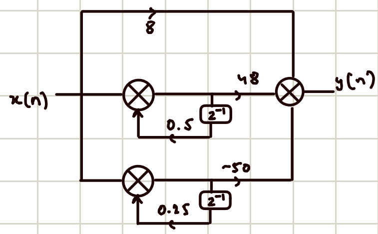
A. $\dfrac{H(z)}{z} = \dfrac{6z^2 + 7z + 1}{z(z-0.5)(z-0.25)}$

$$= \frac{A}{z} + \frac{B}{z-0.5} + \frac{C}{z-0.25}$$

$$= \frac{A(z^2-0.75z+0.125) + B(z^2-0.25z) + C(z^2-0.5z)}{z(z-0.5)(z-0.25)}$$

$$\left.\begin{aligned} A+B+C &= 6 \\ -0.75A - 0.25B - 0.5C &= 7 \\ 0.125A = 1 \quad &\Rightarrow A = 8 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 8 \\ B &= 48 \\ C &= -50 \end{aligned}$$

$$\Rightarrow H(z) = 8 + \frac{48}{1-0.5z^{-1}} - \frac{50}{1-0.25z^{-1}}$$



Q. $H(z) = \dfrac{(z-1)(z-2)(z+1)\,z}{(z-0.5-j0.5)(z-0.5+j0.5)(z-j\frac{1}{4})(z+j\frac{1}{4})}$ Cascaded Realization

A. $H(z) = \dfrac{(z^2-2z)(z^2-1)}{(z^2-z+0.5)(z^2+\frac{1}{16})}$

$H_1(z) = \dfrac{z^2-1}{z^2+\frac{1}{16}}$ $\qquad H_2(z) = \dfrac{z^2-2z}{z^2-z+0.5}$

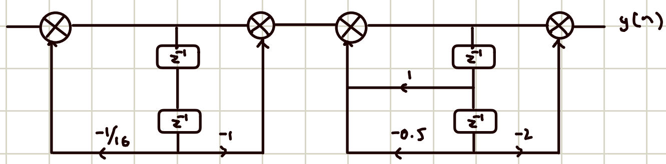
$= \left(\dfrac{1-z^{-2}}{1+\frac{z^{-2}}{16}}\right)$ $\qquad = \dfrac{1-2z^{-1}}{1-z^{-1}+0.5z^{-2}}$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1+\frac{1}{16}z^{-2}}$ $\Rightarrow v(n) = x(n) - \frac{1}{16}x(n-2)$

$\dfrac{Y(z)}{V(z)} = 1 - z^{-2}$ $\Rightarrow y(n) = v(n) - v(n-2)$

$\dfrac{V(z)}{X(z)} = \dfrac{1}{1-z^{-1}+0.5z^{-2}}$ $\Rightarrow v(n) = x(n) + x(n-1) - 0.5x(n-2)$

$\dfrac{Y(z)}{V(z)} = 1-2z^{-1}$ $\Rightarrow y(n) = v(n) - 2v(n-1)$



Last Topic Lattice Realisation done after FIR Filter (unit-4)