Solution to Homework 1

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1. Ordering the Expressions

Question:

Order the following expressions using their related asymptotic group functions.

Answer:

To order functions by their growth rates:

- Compare the exponents of nn in polynomial terms.
- Logarithmic functions grow slower than any polynomial function.
- Factorial functions grow faster than exponential functions.

Example ordering:

log⁡n, n, n2, 2n, n!\log n, \ n, \ n^2, \ 2^n, \ n!

This ordering assumes the typical hierarchy of asymptotic growth rates.

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2. Expressing the Function

Question:

Express the following function in terms of Big-O notation:

f(n)=n31000−100n2−100n+3f(n) = \frac{n^3}{1000} - 100n^2 - 100n + 3

Answer:

1. Identify the term with the highest growth rate:
   - The dominant term is n31000\frac{n^3}{1000}, as it grows faster than n2n^2, nn, and the constant.
2. Simplify the Big-O expression:

f(n)=Θ(n3)f(n) = \Theta(n^3)

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3. Recursive Insertion Sort

Question:

Write a recurrence for the running time of the recursive version of insertion sort.

Answer:

1. Insertion sort recursively sorts the first n−1n-1 elements, then inserts the nn-th element.

2. The recurrence relation:

T(n)=T(n−1)+Θ(n)T(n) = T(n-1) + \Theta(n)

   - T(n−1)T(n-1): Time to sort n−1n-1 elements.
   - Θ(n)\Theta(n): Time to insert the last element into its correct position.

3. Solve using the summation method:

T(n)=∑i=1ni=n(n+1)2=Θ(n2)T(n) = \sum_{i=1}^n i = \frac{n(n+1)}{2} = \Theta(n^2)

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4. Iterative Method (Recurrence 1)

Question:

Use the iterative method to give asymptotic bounds for the following recurrence relation:

T(n)=7T(n2)+n2T(n) = 7T\left(\frac{n}{2}\right) + n^2

Answer:

1. Expand the recurrence:
   1. First substitution: T(n)=7T(n2)+n2T(n) = 7T\left(\frac{n}{2}\right) + n^2
   2. Second substitution: T(n)=7(7T(n4)+(n2)2)+n2T(n) = 7\left(7T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right) + n^2 T(n)=72T(n4)+7(n24)+n2T(n) = 7^2T\left(\frac{n}{4}\right) + 7\left(\frac{n^2}{4}\right) + n^2
   3. General pattern: T(n)=7kT(n2k)+∑i=0k−17i(n24i)T(n) = 7^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 7^i \left(\frac{n^2}{4^i}\right)
2. Stopping condition: n2k=1 ⟹ k=log2n\frac{n}{2^k} = 1 \implies k = \log_2 n.
3. Solve for T(n)T(n):
   - First term: 7log2n=nlog277^{\log_2 n} = n^{\log_2 7}
   - Second term (sum): ∑i=0log2n−17i(n24i)=Θ(n2⋅nlog27−2)=Θ(nlog27)\sum_{i=0}^{\log_2 n - 1} 7^i \left(\frac{n^2}{4^i}\right) = \Theta(n^2 \cdot n^{\log_2 7 - 2}) = \Theta(n^{\log_2 7})
4. Final result:

T(n)=Θ(nlog27)T(n) = \Theta(n^{\log_2 7})

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5. Iterative Method (Recurrence 2)

Question:

Use the iterative method to give asymptotic bounds for the following recurrence relation:

$T(n) = 8T\left(\frac{n}{2}\right) + n^2$

Answer:

1. Expand the recurrence:
   - First substitution: $T(n) = 8T\left(\frac{n}{2}\right) + n^2$
   - Second substitution: $T(n) = 8\left(8T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right) + n^2$
     $T(n) = 8^2T\left(\frac{n}{4}\right) + 8 \cdot \frac{n^2}{4} + n^2$
   - General pattern: $T(n) = 8^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 8^i \frac{n^2}{4^i}$
2. Stopping condition: $\frac{n}{2^k} = 1 \implies k = \log_2 n$.
3. Solve for $T(n)$:
   - First term: $8^{\log_2 n} = n^{\log_2 8} = n^3$
   - Second term (sum): $\sum_{i=0}^{\log_2 n - 1} 8^i \frac{n^2}{4^i} = \Theta(n^2 \cdot n^{\log_2 8 - 2}) = \Theta(n^3)$
4. Final result:

$T(n) = \Theta(n^3)$

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6. Master Theorem

Question:

Use the Master Theorem to give asymptotic bounds for the following recurrence relation:

$T(n) = 8T\left(\frac{n}{2}\right) + n^2$

Answer:

1. Compare to the standard form:

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Here:

   - $a = 8$,
   - $b = 2$,
   - $f(n) = n^2$.

2. Calculate $p = \log_b a$:

p=log⁡28=3p = \log_2 8 = 3

3. Compare f(n)f(n) with npn^p:
    - f(n)=n2f(n) = n^2,
    - np=n3n^p = n^3.

Since f(n)f(n) grows slower than npn^p (2<32 < 3), the solution is determined by npn^p:

T(n)=Θ(n3)T(n) = \Theta(n^3)
